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# The asymptotic behavior of least pseudo-Anosov dilatations

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For a surface $S$ with $n$ marked points and fixed genus $g \geq 2$, we prove that the logarithm of the minimal dilatation of a pseudo-Anosov homeomorphism of $S$ is on the order of $(\log n)/n$. This is in contrast with the cases of genus zero or one where the order is $1/n$.

37E30; 57M99, 30F60

## 1 Introduction

Let $S = S_{g,n}$ be an orientable surface with genus $g$ and $n$ marked points. The *mapping class group* of $S$ is defined to be the group of homotopy classes of orientation preserving homeomorphisms of $S$. We denote it by $\mathrm{Mod}(S)$. Given a pseudo-Anosov element $f \in \mathrm{Mod}(S)$, let $\lambda(f)$ denote the *dilatation* of $f$ (see Section 2.1). We define

$$\mathcal{L}(S_{g,n}) := \{\log \lambda(f) \mid f \in \mathrm{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

This is precisely the length spectrum of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points with respect to the Teichmuller metric; see Ivanov [8]. There is a shortest closed geodesic and we denote its length by

$$l_{g,n} = \min\{\log \lambda(f) \mid f \in \mathrm{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

Our main theorem is the following:

**Theorem 1.1** *For any fixed $g \geq 2$, there is a constant $c_g \geq 1$ depending on $g$ such that*

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n},$$

*for all $n \geq 3$.*

To contrast with known results, recall that in [13] Penner proves that for $2g-2+n > 0$,

$$l_{g,n} \geq \frac{\log 2}{12g - 12 + 4n},$$

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and for closed surfaces with genus $g \geq 2$,

$$\frac{\log 2}{12g-12} \leq l_{g,0} \leq \frac{\log 11}{g}.$$

The bounds on $l_{g,0}$ have been improved by a number of authors; see Bauer [1], McMullen [10], Minakawa [11]and Hironaka and Kin [7].

In [13], Penner suggests that there may be an "analogous upper bound for $n \neq 0$". In [7], Hironaka and Kin use a concrete construction to prove that for genus $g = 0$,

$$l_{0,n} < \frac{\log(2+\sqrt{3})}{\lfloor \frac{n-2}{2} \rfloor} \leq \frac{2\log(2+\sqrt{3})}{n-3},$$

for all $n \geq 4$. The inequality is proven for even $n$ in [7], but it follows for odd $n$ by letting the fixed point of their example be a marked point. Combining this with Penner's lower bound, one sees for $n \geq 4$,

$$\frac{\log 2}{4n-12} \leq l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3},$$

which shows that the upper bound is on the same order as Penner's lower bound for $g = 0$. A similar situation holds for $g = 1$; see Section 5.1 of the Appendix.

Inspired by the construction of Hironaka and Kin, we tried to find examples of pseudo-Anosov $f_{g,n} \in \mathrm{Mod}(S_{g,n})$ with

$$\log \lambda(f_{g,n}) = O\left(\frac{1}{|\chi(S_{g,n})|}\right),$$

for $\chi(S_{g,n}) = 2-2g-n < 0$. However for any fixed $g \geq 2$, all attempts resulted in $f_{g,n} \in \mathrm{Mod}(S_{g,n})$ pseudo-Anosov with

$$\log \lambda(f_{g,n}) = O_g\left(\frac{\log|\chi(S_{g,n})|}{|\chi(S_{g,n})|}\right) \quad \text{and not} \quad O\left(\frac{1}{|\chi(S_{g,n})|}\right).$$

This led us to prove Theorem 1.1.

The preceding discussion suggests that the asymptotic behavior of $l_{g,n}$ while varying both $g$ and $n$ can be quite complicated, in general. Hence, we will focus on understanding what happens along different $(g,n)$–rays. In addition to the results discussed above, there are other rays in which the asymptotic behavior of $l_{g,n}$ can be understood via examples (see Section 5.2 of the Appendix) and Penner's lower bound. Table 1 summarizes these behaviors for $\chi(S_{g,n}) < 0$.

**Question** What are asymptotic behaviors of $l_{g,n}$ along different $(g,n)$–rays in the $(g,n)$ plane?

| $(g,n)$–*rays* | *The asymptotic behavior of* $l_{g,n}$ |
|---|---|
| $g=0$ | $1/\lvert\chi(S_{g,n})\rvert$ |
| $g=1$ and $n$ is even | $1/\lvert\chi(S_{g,n})\rvert$ |
| $g=\text{constant}\geq 2$ | $\log\left(\lvert\chi(S_{g,n})\rvert\right)/\lvert\chi(S_{g,n})\rvert$ |
| $n=0,1,2,3,$ or $4$ | $1/\lvert\chi(S_{g,n})\rvert$ |
| $n=g, g+1,$ or $g+2$ | $1/\lvert\chi(S_{g,n})\rvert$ |
| $n=g-1$ or $2(g-1)$ | $1/\lvert\chi(S_{g,n})\rvert$ |

Table 1

### 1.1 Outline of the paper

We will first recall some definitions and properties in Section 2. In Section 3 we prove the lower bound of Theorem 1.1. We construct examples in Section 4 which give an upper bound for the genus 2 case, and we extend the example to arbitrary genus $g \geq 2$ to obtain the upper bound of Theorem 1.1. Finally, we construct a pseudo-Anosov element in $\mathrm{Mod}(S_{1,2n})$ and obtain an upper bound on $l_{1,2n}$ in the Appendix.

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## 2 Preliminaries

### 2.1 Homeomorphisms of a surface

We say that a homeomorphism $f\colon S \to S$ is *pseudo-Anosov* if there are transverse singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ together with transverse measures $\mu^s$ and $\mu^u$ such that for some $\lambda > 1$,

$$f(\mathcal{F}^s,\mu^s) = (\mathcal{F}^s,\lambda\mu^s),$$
$$f(\mathcal{F}^u,\mu^u) = (\mathcal{F}^u,\lambda^{-1}\mu^u).$$

The number $\lambda = \lambda(f)$ is called the *dilatation* of $f$. We call $f$ *reducible* if there is a finite disjoint union $U$ of simple essential closed curves on $S$ such that $f$ leaves $U$ invariant. If there exists $k > 0$ such that $f^k$ is the identity, then $f$ is *periodic*.

A mapping class $[f]$ is pseudo-Anosov, reducible or periodic (respectively) if $f$ is homotopic to a pseudo-Anosov, reducible or periodic homeomorphism (respectively). The following is proved in Fathi, Laudenbach and Poenaru [4].

**Theorem 2.1** (Nielsen–Thurston) *A mapping class* $[f] \in \mathrm{Mod}(S)$ *is either periodic, reducible, or pseudo-Anosov.*

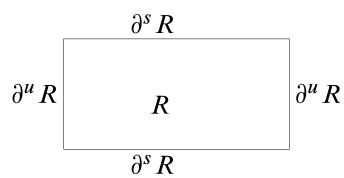
As a slight abuse of notation, we sometimes refer to a mapping class $[f]$ by one of its representatives $f$.

## 2.2 Markov partitions

Suppose $f\colon S \to S$ is pseudo-Anosov with stable and unstable measured singular foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$. We define a rectangle $R$ to be a map

$$\rho\colon I \times I \to S,$$

such that $\rho$ is an embedding on the interior, $\rho(\text{point} \times I)$ is contained in a leaf of $\mathcal{F}^u$, and $\rho(I \times \text{point})$ is contained in a leaf of $\mathcal{F}^s$. We denote $\rho(\partial I \times I)$ by $\partial^u R$ and $\rho(I \times \partial I)$ by $\partial^s R$.




As a standard abuse of notation, we will write $R \subset S$ for the image of a rectangle map $\rho\colon I \times I \to S$.

**Definition 2.2** A Markov partition for $f\colon S \to S$ is a decomposition of $S$ into a finite union of rectangles $\{R_i\}_{i=1}^k$, such that:

1. $\mathrm{Int}(R_i) \cap \mathrm{Int}(R_j)$ is empty, when $i \neq j$,
2. $f(\bigcup_{j=1}^k \partial^u R_j) \subset \bigcup_{j=1}^k \partial^u R_j$,
3. $f^{-1}(\bigcup_{i=1}^k \partial^s R_i) \subset \bigcup_{i=1}^k \partial^s R_i$.

Given a pseudo-Anosov homeomorphism $f\colon S \to S$, a Markov partition is constructed in Bestvina and Handel [2] from a train track map for $f$. The advantage of this construction over Fathi, Laudenbach and Poenaru [4], for example, is that the number of rectangles is substantially smaller. From [2], one has the following:

**Theorem 2.3** *For any pseudo-Anosov homeomorphism $f\colon S \to S$ of a surface $S$ with at least one marked point, there exists a Markov partition for $f$ with at most $-3\chi(S)$ rectangles.*

We say that a matrix is *positive* (respectively, *nonnegative*) if all the entries are positive (respectively, nonnegative).

We can define a *transition matrix* $M$ associated to the Markov partition with rectangles $\{R_i\}_{i=1}^k$. The entry $m_{i,j}$ of $M$ is the number of times that $f(R_j)$ wraps over $R_i$, so $M$ is a nonnegative integral $k \times k$ matrix. In Bestvina and Handel's construction, $M$ is the same as the transition matrix of the train track map and they show it is an integral Perron–Frobenius matrix (ie it is irreducible with nonnegative integer entries); see Gantmacher [5]. Furthermore, the Perron–Frobenius eigenvalue $\mu(M) = \lambda(f)$ is the dilatation of $f$. The width (respectively, height) of $R_i$ is the $i$–th entry of the corresponding Perron–Frobenius eigenvector of $M$ (respectively, $M^T$), where the eigenvectors are both positive by the irreducibility of $M$.

The following proposition will be used in proving the lower bound.

**Proposition 2.4** *Let $M$ be a $k \times k$ integral Perron–Frobenius matrix. If there is a nonzero entry on the diagonal of $M$, then $M^{2k}$ is a positive matrix and its Perron–Frobenius eigenvalue $\mu(M^{2k})$ is at least $k$.*

**Proof** We construct a directed graph $\Gamma$ from $M$ with $k$ vertices $\{i\}_{i=1}^k$ such that the number of the directed edge from $i$ to $j$ in $\Gamma$ equals $m_{i,j}$. We observe that for any $r > 0$ the $(i,j)$–th entry $m_{i,j}^{(r)}$ of $M^r$ is the number of directed edge paths from $i$ to $j$ of length $r$ in $\Gamma$.

Since $M$ is a Perron–Frobenius matrix, we know that $\Gamma$ is path-connected by directed paths. Suppose $M$ has a nonzero entry at the $(l,l)$–th entry, then we will see at least one corresponding loop edge at the vertex $l$. For any $i$ and $j$ in $\Gamma$, path-connectivity ensures us that there are directed edge paths of length $\leq k$ from $i$ to $l$ and from $l$ to $j$. This tells us that there is a directed edge path $P$ of length $\leq 2k$ from $i$ to $j$ passing through $l$. Since we can wrap around the loop edge adjacent to $l$ to increase the length of $P$, there is always a directed edge path of length $2k$ from $i$ to $j$. In other words, $m_{i,j}^{(2k)}$ is at least 1 for all $i$ and $j$, so $M^{2k}$ is a positive matrix.

Let $v$ be a corresponding Perron–Frobenius eigenvector, so that we have $M^{2k}v = \mu(M^{2k})v$. This implies that if $v = [v_1 \cdots v_k]^T$, for all $i$,

$$\sum_{j=1}^{k} m_{i,j}^{(2k)} v_j = \mu(M^{2k}) v_i,$$

or equivalently, $$\mu(M^{2k}) = \sum_{j=1}^{k} m_{i,j}^{(2k)} \frac{v_j}{v_i}.$$

Choosing $i$ such that $v_i \leq v_j$ for all $j$, we obtain

$$\mu(M^{2k}) \geq \sum_{j=1}^{k} m_{i,j}^{(2k)} \geq \sum_{j=1}^{k} 1 = k. \qquad \square$$

The following proposition will be used in proving the upper bound.

**Proposition 2.5** *Let $\Gamma$ be the induced directed graph of an integral Perron–Frobenius matrix $M$ with Perron–Frobenius eigenvalue $\mu(M) = \mu$. Let $P_\Gamma(i,d)$ be the total number of paths of length $d$ emanating from vertex $i$ in $\Gamma$. Then, for all $i$,*

$$\sqrt[d]{P_\Gamma(i,d)} \longrightarrow \mu(M) \quad \text{as } d \to \infty.$$

**Proof** Let $M$ be an integral $k \times k$ Perron–Frobenius matrix with Perron–Frobenius eigenvalue $\mu$ and Perron–Frobenius eigenvector $v$. As above

$$\sum_{j=1}^{k} m_{i,j}^{(d)} v_j = \mu(M^d) v_i = \mu^d v_i.$$

Let $v_{\max} = \max_i\{v_i\}$ and $v_{\min} = \min_i\{v_i\}$. According to the Perron–Frobenius theory, the irreducibility of $M$ implies that $v_i > 0$ for all $i$. For all $i$ we have

$$\frac{v_{\min}\left(\sum_j m_{i,j}^{(d)}\right)}{\mu^d} \leq \frac{\sum_j m_{i,j}^{(d)} v_j}{\mu^d} \leq \frac{v_{\max}\left(\sum_j m_{i,j}^{(d)}\right)}{\mu^d},$$

hence $$\frac{v_i}{v_{\max}} \leq \frac{\sum_j m_{i,j}^{(d)}}{\mu^d} \leq \frac{v_i}{v_{\min}}.$$

We are done, since $\sum_j m_{i,j}^{(d)} = P_\Gamma(i,d)$ and for all $i$,

$$\sqrt[d]{\frac{v_i}{v_{\max}}} \to 1 \quad \text{and} \quad \sqrt[d]{\frac{v_i}{v_{\min}}} \to 1, \quad \text{as } d \text{ tends to } \infty. \qquad \square$$

## 2.3 Lefschetz numbers

We will review some definitions and properties of Lefschetz numbers. A more complete discussion can be found in Guillemin and Pollack [6] and Bott and Tu [3].

Let $X$ be a compact oriented manifold, and $f\colon X \to X$ be a map. Define

$$\mathrm{graph}(f) = \{(x, f(x)) | x \in X\} \subset X \times X$$

and let $\Delta$ be the diagonal of $X \times X$. The algebraic intersection number $I(\Delta, \mathrm{graph}(f))$ is an invariant of the homotopy class of $f$, called the *(global) Lefschetz number* of $f$ and it is denoted $L(f)$. As in [3], this can be alternatively described by

$$L(f) = \sum_{i \geq 0} (-1)^i \,\mathrm{trace}(f_*^{(i)}), \tag{1}$$

where $f_*^{(i)}$ is the matrix induced by $f$ acting on $H_i(X) = H_i(X; \mathbb{R})$. The Euler characteristic is the self-intersection number of the diagonal $\Delta$ in $X \times X$,

$$\chi(X) = I(\Delta, \Delta) = L(\mathrm{id}).$$

As seen in [6], if $f$ has isolated fixed points, we can compute the *local Lefschetz number* of $f$ at a fixed point $x$ in local coordinates as

$$L_x(f) = \deg\left( z \mapsto \frac{f(z) - z}{|f(z) - z|} \right),$$

where $z$ is on the boundary of a small disk centered at $x$ which contains no other fixed points. Moreover we can compute the Lefschetz number by summing the local Lefschetz numbers of fixed points,

$$L(f) = \sum_{f(x) = x} L_x(f).$$

This description of $L_x(f)$ is given for smooth $f$ in [6], but it is equally valid for continuous $f$ since such a map is approximated by smooth maps. We will be computing the Lefschetz number of a homeomorphism $f\colon S_{g,n} \to S_{g,n}$, *ignoring the marked points*.

**Proposition 2.6** *If a homeomorphism $f\colon S_{g,n} \to S_{g,n}$ is homotopic (not necessarily fixing the marked points) to the identity or a multitwist, then*

$$L(f) = \chi(S_{g,0}) = 2 - 2g.$$

A *multitwist* is a composition of powers of Dehn twists on pairwise disjoint simple essential closed curves.

**Proof** If $f$ is homotopic to the identity, the homotopy invariance of the Lefschetz number tells us $L(f) = L(\mathrm{id}) = I(\Delta, \Delta)$ which is $\chi(S_{g,0})$.

Suppose $f$ is homotopic to a multitwist. We will use (1) to compute $L(f)$. Note that $H_i(S_{g,0})$ is 0 for $i \geq 3$, $H_0(S_{g,0}) \cong H_2(S_{g,0}) \cong \mathbb{R}$ and $f_*^{(i)}$ is the identity when $i = 0$ or 2, so this implies $L(f) = 2 - \mathrm{trace}(f_*^{(1)})$.

There exists a set $\{\gamma_i\}_{i=1}^k$ of disjoint simple essential closed curves with some integers $n_i \neq 0$ such that

$$f \simeq T_{\gamma_1}^{n_1} \circ \cdots \circ T_{\gamma_k}^{n_k},$$

where $T_{\gamma_i}^{n_i}$ is the $n_i$–th power of a Dehn twist along $\gamma_i$.

For any curve $\gamma$,

$$T_{\gamma_i *}^{n_i}([\gamma]) = [\gamma] + n_i \langle \gamma, \gamma_i \rangle [\gamma_i],$$

where $[\gamma]$ is the homology class of $\gamma$ and $\langle \gamma, \gamma_i \rangle$ is the algebraic intersection number of $[\gamma]$ and $[\gamma_i]$. If any $\gamma_i$ is a separating curve, then $[\gamma_i]$ is the trivial homology class and $T_{\gamma_i *}^{n_i}$ acts trivially on $H_1(S_{g,0})$. We may therefore assume that each $\gamma_i$ is nonseparating. After renaming the curves, we can assume that there is a subset $\{\gamma_1, \gamma_2, \dots, \gamma_s\}$ such that $\widehat{\gamma} = \bigcup_{i=1}^s \gamma_i$ is nonseparating and $\widehat{\gamma} \cup \gamma_j$ is separating for all $j > s$. Thus, for all $k \geq j > s$,

$$[\gamma_j] = \sum_{i=1}^{s} c_{ji} [\gamma_i],$$

for some constants $c_{ji} \in \mathbb{R}$. We can extend $\{[\gamma_i]\}_{i=1}^s$ to a basis of $H_1(S_{g,0})$,

$$\{\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \beta_2, \dots, \beta_g\},$$

where $[\gamma_i] = \alpha_i$ for $i \leq s \leq g$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$.

First suppose $s = k$, then $\langle \alpha_j, \gamma_i \rangle = \langle \alpha_j, \alpha_i \rangle = 0$ for all $i$ and $j$. Therefore, for all $j$,

$$f_*^{(1)}(\alpha_j) = \alpha_j$$

and $$f_*^{(1)}(\beta_j) = \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] = \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \alpha_i \rangle \alpha_i = \beta_j - n_j \alpha_j.$$

So we have

$$f_*^{(1)} = \left( \begin{array}{c|c} I_{g\times g} & * \\ \hline 0 & I_{g\times g} \end{array} \right)$$

and $L(f) = 2 - \mathrm{trace}(f_*^{(1)}) = 2 - 2g$.

For $s < k$, we will have

$$\begin{aligned}
f_*^{(1)}(\alpha_j) &= \alpha_j + \sum_{i=1}^{k} n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\
&= \alpha_j + \sum_{i=1}^{s} n_i \langle \alpha_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^{k} n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\
&= \alpha_j + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \alpha_j, \gamma_t \rangle [\gamma_t] \\
&= \alpha_j + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \alpha_j, \alpha_t \rangle \alpha_t \\
&= \alpha_j
\end{aligned}$$

and

$$\begin{aligned}
f_*^{(1)}(\beta_j) &= \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] \\
&= \beta_j + \sum_{i=1}^{s} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \beta_j, \gamma_t \rangle [\gamma_t] \\
&= \beta_j + \sum_{i=1}^{s} n_i \langle \beta_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \beta_j, \alpha_t \rangle \alpha_t \\
&= \begin{cases} \beta_j, & \text{if } j > s, \\ \beta_j - n_j \alpha_j - \sum_{i=s+1}^{k} n_i c_{ij} \alpha_j, & \text{if } j \le s. \end{cases}
\end{aligned}$$

Therefore, the diagonal of the matrix $f_*^{(1)}$ is still all 1's and

$$L(f) = 2 - \mathrm{trace}(f_*^{(1)}) = 2 - 2g. \qquad \square$$

# 3 Bounding the dilatation from below

**Lemma 3.1** *For any pseudo-Anosov element $f \in \mathrm{Mod}(S_{g,n})$ equipped with a Markov partition, if $L(f) < 0$, then there is a rectangle $R$ of the Markov partition, such that the interiors of $f(R)$ and $R$ intersect.*

**Proof** Since $f$ is a pseudo-Anosov homeomorphism, it has isolated fixed points. Suppose $x$ is an isolated fixed point of $f$ such that one of the following happens:

(1) $x$ is a nonsingular fixed point and the local transverse orientation of $\mathcal{F}^s$ is reversed.

(2) $x$ is a singular fixed point and no separatrix of $\mathcal{F}^s$ emanating from $x$ is fixed.

A *separatrix* of $\mathcal{F}^s$ is a maximal arc starting at a singularity and contained in a leaf of $\mathcal{F}^s$.

**Claim** $L_x(f) = +1$.

Let $B$ be a small disk centered at $x$ containing no other fixed point of $f$. First we show that (in local coordinates) for every $z \in \partial B$, $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

It is easy to verify this in case 1 by choosing local coordinates $(\xi_1, \xi_2)$ around $x$ so that $f$ is given by

$$f(\xi_1, \xi_2) = \left( -\lambda \xi_1, \frac{-1}{\lambda} \xi_2 \right).$$

In case 2, we choose local coordinates around $x$ such that the separatrices of $\mathcal{F}^s$ emanating from $x$ are sent to rays from 0 through the $k$–th roots of unity in $\mathbb{R}^2$. This means $f$ rotates each of the sectors bounded by these rays through an angle $2\pi j/k$ for some $j = 1, \dots, k-1$, and so for all $z \in \partial B$ $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

Define a smooth map $h_0$: $\partial B \to S^1$ by $h_0(z) = (f(z) - z)/|f(z) - z|$, so $L_x(f) = \deg(h_0)$ by definition. Let $g$: $\partial B \to S^1$ be defined by $g(z) = z/|z|$ and $h_1$: $S^1 \to S^1$ be defined by $h_1(z/|z|) = (f(z) - z)/|f(z) - z|$, so that $h_0 = h_1 g$. Then

$$L_x(f) = \deg(h_0) = \deg(h_1 g) = \deg(h_1) \deg(g) = \deg(h_1)$$

since $\deg(g) = 1$. Note that $h_1$ has no fixed point since for all $z \in \partial B$,

$$f(z) - z \neq \alpha z,$$

for all $\alpha > 0$. Therefore $L_x(f) = \deg(h_1) = (-1)^{(1+1)} = +1$.

The assumption of $L(f) < 0$ implies that there exists a fixed point $x$ of $f$ which is in neither of the cases above. In other words, it falls into one of the cases in Figure 1. As seen in Figure 1, there is a rectangle $R$ of the Markov partition such that the interiors of $f(R)$ and $R$ intersect. □

Let $\Gamma_S(3) \lhd \mathrm{Mod}(S)$ denote the kernel of the action on $H_1(S; \mathbb{Z}/3\mathbb{Z})$, where $S = S_{g,0}$. In [9], it is shown that $\Gamma_S(3)$ consists of pure mapping classes. Setting

$$\Theta(g) = [\mathrm{Mod}(S) : \Gamma_S(3)],$$

we conclude the following.

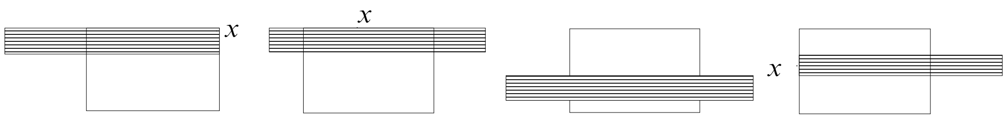



Figure 1: The intersection of $f(R)$ and $R$. $R$ is the underlying rectangle and $f(R)$ is the shaded rectangle.

**Lemma 3.2** *Let $f \in \mathrm{Mod}(S_{g,n})$ be a pseudo-Anosov element and $\hat{f} \in \mathrm{Mod}(S_{g,o})$ be the induced mapping class obtained by forgetting marked points. There exists a constant $1 \le \alpha \le \Theta(g)$ such that $\hat{f}^\alpha$ satisfies exactly one of the following:*

(1) *$\hat{f}^\alpha$ restricts to a pseudo-Anosov map on a connected subsurface.*

(2) *$\hat{f}^\alpha = \mathrm{Id}$.*

(3) *$\hat{f}^\alpha$ is a multitwist map.*

**Remark** For the first two cases of Lemma 3.2, one can find $\alpha$ bounded by a linear function of $g$, but in case 3, $\alpha$ may be exponential in $g$.

**Theorem 3.3** *For $g \ge 2$, given any pseudo-Anosov $f \in \mathrm{Mod}(S_{g,n})$, let $\alpha$ be as in Lemma 3.2. Then*

$$\log \lambda(f) \ge \min\left\{\frac{\log 2}{\alpha(12g-12)}, \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}\right\}.$$

**Proof** We will deal with case 1 of Lemma 3.2 first.

If $\hat{f}^\alpha$ restricts to a pseudo-Anosov homeomorphism on a connected subsurface $\sum_{g_0,n_0}$ of $S_{g,0}$ of genus $g_0$ with $n_0$ boundary components (we have $2g_0 + n_0 \le 2g$), then Penner's lower bound tells us

$$\log \lambda(\hat{f}^\alpha) \ge \frac{\log 2}{12g_0 - 12 + 4n_0} \ge \frac{\log 2}{12g - 12}.$$

Hence $\log \lambda(f) \ge \log \lambda(\hat{f}) > \log 2/\alpha(12g-12)$.

If $\hat{f}^\alpha$ is homotopic to the identity or a multitwist map, from Proposition 2.6, we have $L(f^\alpha) = L(\hat{f}^\alpha) = \chi(S_{g,0}) = 2-2g < 0$. Theorem 2.3 tells us that for any pseudo-Anosov $f$ there is a Markov partition with $k$ rectangles, where $k \le -3\chi(S)$. Recall that the transition matrix $M$ obtained from the rectangles is a $k \times k$ Perron–Frobenius matrix and the Perron–Frobenius eigenvalue $\mu(M)$ equals $\lambda(f)$.

By Lemma 3.1, there is a rectangle $R$ such that the interiors of $f^\alpha(R)$ and $R$ intersect. This implies that there is a nonzero entry on the diagonal of $M^\alpha$. Applying

Proposition 2.4, we obtain that $\mu((M^\alpha)^{2k}) = \mu(M^{2k\alpha})$ is at least $k$, so we have

$$(\lambda(f))^{2k\alpha} = \lambda(f^{2k\alpha}) = \mu(M^{2k\alpha}) \geq k.$$

One can easily check $(\log x)/x$ is monotone decreasing for $x \geq 3$. Since

$$3 \leq k \leq -3\chi(S) = 6g + 3n - 6,$$

hence
$$\log \lambda(f) \geq \frac{\log k}{2\alpha k} \geq \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}. \qquad \square$$

**Remark** Penner's proof in [13] does not use Lefschetz numbers which we used to conclude that $\mu(M^{2k\alpha})$ is at least $k$, so we obtain a sharper lower bound for $n \gg g$.

# 4 An example which provides an upper bound

## 4.1 For the genus two case

In this section, we will construct a pseudo-Anosov $f \in \mathrm{Mod}(S_{2,n})$ for all $n \geq 31$ then we compute its dilatation which gives us an upper bound for $l_{2,n}$.

Let $S_{0,m+2}$ be a genus 0 surface with $m+2$ marked points (ie a marked sphere), and recall an example of pseudo-Anosov $\phi \in \mathrm{Mod}(S_{0,m+2})$ in [7]. We view $S_{0,m+2}$ as a sphere with $s+1$ marked points $X$ circling an unmarked point $x$ and $t+1$ marked points $Y$ circling an unmarked point $y$, and a single extra marked point $z$. We can also draw this as a "turnover", as in Figure 2. Note that $|X \cap Y| = 1$, $|X| = s+1$, $|Y| = t+1$ and $m = s+t$.

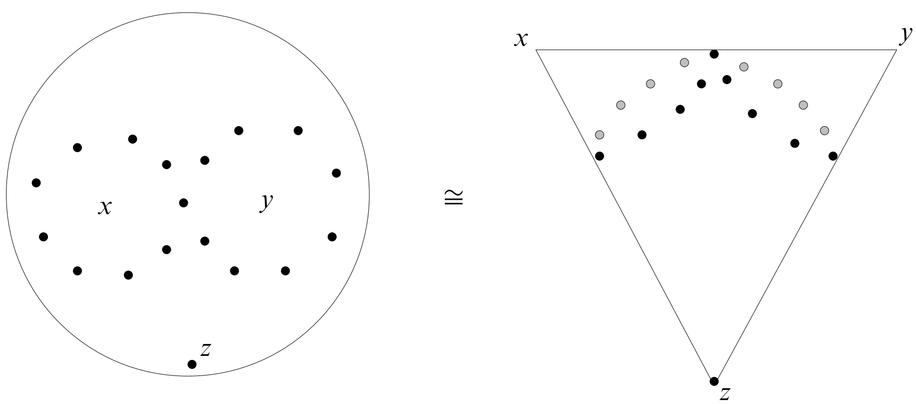



Figure 2: Two way of viewing a marked sphere. Black dots are marked points and the shaded dots on the right are marked points at the back.

We define homeomorphisms $\alpha_s$, $\beta_t\colon S_{0,m+2} \to S_{0,m+2}$ such that $\alpha_s$ rotates the marked points of $X$ counterclockwise around $x$ and $\beta_t$ rotates the marked points of $Y$ clockwise around $y$; see Figure 3. Define $\phi_{s,t} := \beta_t\alpha_s$. In [7], it is shown that $\phi_{s,t}$

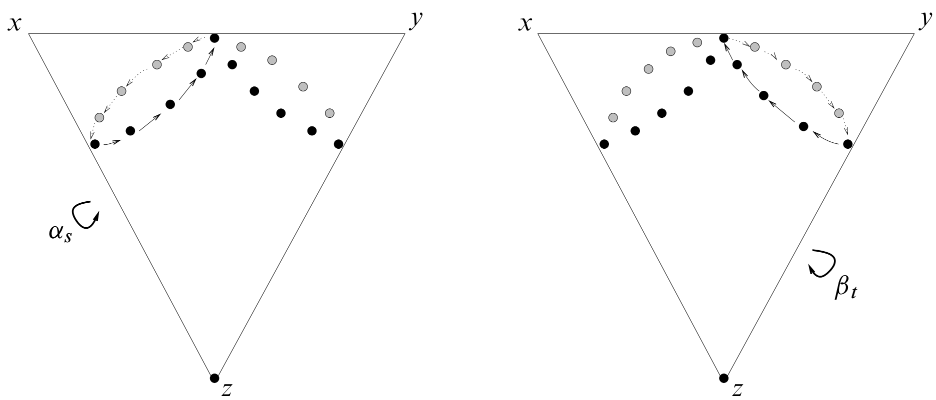



Figure 3: Homeomorphisms $\alpha_s$ and $\beta_t$

is pseudo-Anosov by checking it satisfies the criterion of [2]. We also note that from this one can check that $x$, $y$ and $z$ are fixed points of a pseudo-Anosov representative of $\phi_{s,t}$. Moreover, for $s$, $t \geq 1$ the dilatation of $\phi_{s,t}$ equals the largest root of the polynomial

$$\begin{aligned} T_{s,t}(x) &= x^{t+1}(x^s(x-1)-2) + x^{s+1}(x^{-s}(x^{-1}-1)-2) \\ &= (x-1)x^{(s+t+1)} - 2(x^{s+1}+x^{t+1}) - (x-1). \end{aligned}$$

The dilatation is minimized when $s = \lfloor m/2 \rfloor$ and $t = \lceil m/2 \rceil$. Let us define $\phi := \phi_{\lfloor m/2 \rfloor, \lceil m/2 \rceil}$ and its dilatation is the largest root of the polynomial

$$\begin{aligned} T_m(x) &:= T_{\lfloor m/2 \rfloor, \lceil m/2 \rceil}(x) \\ &= (x-1)x^{(m+1)} - 2\big(x^{\lfloor m/2 \rfloor+1} + x^{\lceil m/2 \rceil+1}\big) - (x-1). \end{aligned}$$

**Proposition 4.1** *If $m \geq 5$, then the largest real root of $T_m(x)$ is bounded above by $m^{3/m}$.*

**Proof** For all $m$, we have $T_m(1) = -4$. It is sufficient to show that for all $x \geq m^{3/m}$, we have $T_m(x) > 0$. Dividing the inequality by $x^{(m+1)}$, it is equivalent to show

$$(x-1) + x^{-(m+1)} > 2\big(x^{\lfloor m/2 \rfloor - m} + x^{\lceil m/2 \rceil - m}\big) + x^{-m}.$$

For $m \geq 5$, one can verify the following inequalities hold for all $x \geq m^{3/m}$:

(1) $x - 1 > (3 \log m)/m \geq 9/(2m)$,

(2) $x^{\lfloor m/2 \rfloor - m} \leq x^{\lceil m/2 \rceil - m} \leq 1/m$,

(3) $x^{-m} \leq 1/(25m)$.

Therefore,

$$(x-1) + x^{-(m+1)} > x - 1 > \frac{9}{2m} > \frac{101}{25m} = 2\left(\frac{1}{m} + \frac{1}{m}\right) + \frac{1}{25m}$$
$$\geq 2\big(x^{\lfloor m/2 \rfloor - m} + x^{\lceil m/2 \rceil - m}\big) + x^{-m}. \qquad \square$$

**Remark** Proposition 4.1 fails if we try to replace the bound with $c^{1/m}$ where $c$ is any constant.

**Remark** Hironaka and Kin [7] construct two infinite families of pseudo-Anosovs in $\mathrm{Mod}(S_{0,m})$, with $\phi_{s,t}$ being one of them. Unlike $\phi_{s,t}$, the other family provides the sharp bound on $l_{0,m}$.

Next, we take a cyclic branched cover $S_{2,n}$ of $S_{0,m+2}$ with branched points $x$, $y$, and $z$, where $n = 5(m+1)+1$ (See Figure 4.). Define $\widetilde{X} = \{\text{marked points around } \widetilde{x}\}$ and $\widetilde{Y} = \{\text{marked points around } \widetilde{y}\}$, so we have $|\widetilde{X} \cap \widetilde{Y}| = 5$, $|\widetilde{X}| = 5(s+1)$ and $|\widetilde{Y}| = 5(t+1)$.

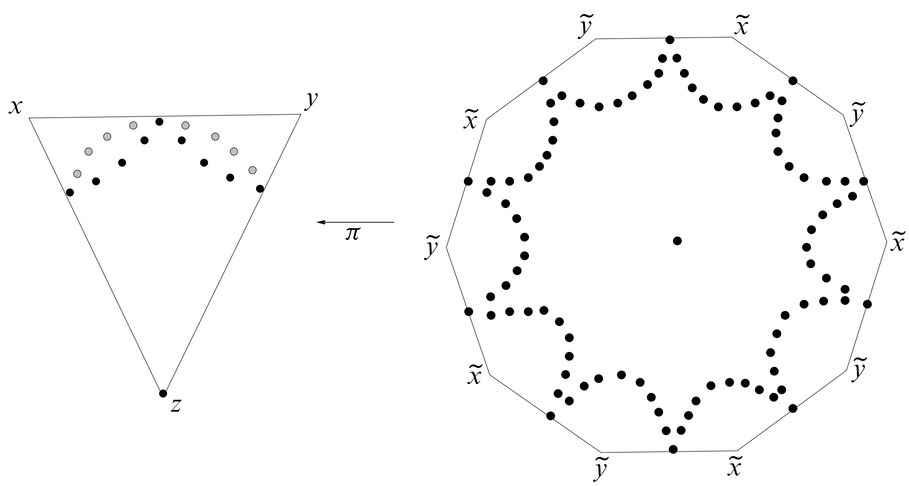



Figure 4: $\pi$ is the covering map. To form $S_{2,n}$ from the decagon, identify the opposite sides. Then $\pi$ is the quotient by the group generated by rotation of an angle $2\pi/5$.

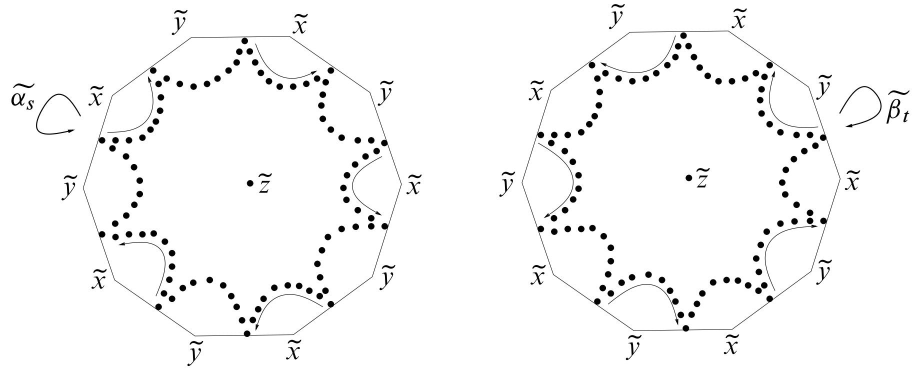



Figure 5: Homeomorphisms $\widetilde{\alpha_s}$ and $\widetilde{\beta_t}$

We lift $\alpha_s$, $\beta_t$ to $S_{2,n}$ and call them $\widetilde{\alpha_s}$, $\widetilde{\beta_t}$, so that $\widetilde{\alpha_s}$ rotates the marked points of $\widetilde{X}$ counterclockwise around $\widetilde{x}$ and $\widetilde{\beta_t}$ rotates the marked points of $\widetilde{Y}$ clockwise around $\widetilde{y}$; see Figure 5. We define $\psi_{s,t} := \widetilde{\beta_t}\,\widetilde{\alpha_s}$. It follows that $\psi_{s,t}$ is a lift of $\phi_{s,t}$, and so is pseudo-Anosov with $\lambda(\psi_{s,t}) = \lambda(\phi_{s,t})$. An invariant train track for $\psi_{s,t}$ is obtained by lifting the one constructed in [7], and is shown in Figure 6 for $s = t = 3$.

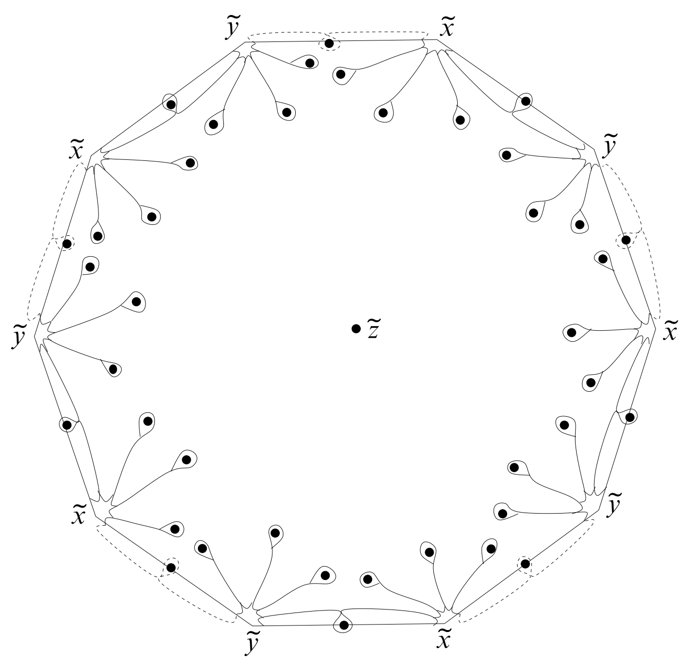



Figure 6: A train track for $\psi_{3,3}$

Hence for $n = 5(m+1)+1 \geq 31$, we have constructed a pseudo-Anosov $\psi = \psi_{\lfloor m/2\rfloor,\lceil m/2\rceil} \in \mathrm{Mod}(S_{2,n})$ with $\lambda(\psi) = \lambda(\phi) \leq m^{3/m}$ which implies

$$\log\lambda(\psi) \leq \frac{3\log m}{m} = \frac{15\log(n-6) - 15\log 5}{n-6}.$$

We will now extend $\psi$ so that $n$ can be an arbitrary number $\geq 31$. We add an extra marked point $p_1$ on $S_{2,n}$ between points in $\widetilde{X}$ or $\widetilde{Y}$ *except the places shown in Figure 7*.

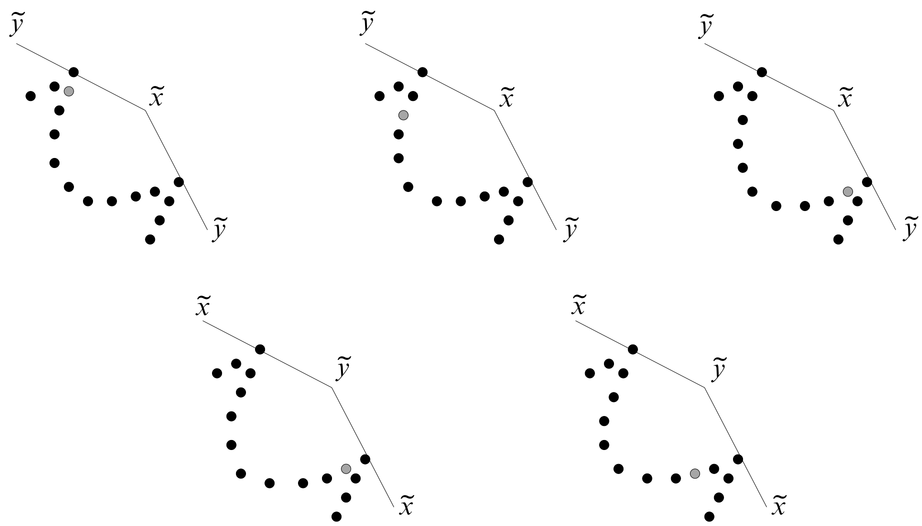



Figure 7: We are *not* allowed to add $p_1$ in the places indicated by a shaded point.

Without loss of generality we assume $p_1$ is added in $\widetilde{X}$ to obtain $S_{2,n+1}$ and we define $\psi_1 := \widetilde{\beta_t}\,\widetilde{\alpha_s}' \in \mathrm{Mod}(S_{2,n+1})$ where $\widetilde{\alpha_s}'$ is extended from $\widetilde{\alpha_s}$ in the obvious way; see Figure 8. One can check that $\psi_1$ is pseudo-Anosov via the techniques of [2]. An invariant train track for $\psi_1$ is shown in Figure 9 and is obtained by modifying the invariant train track for $\psi$ shown in Figure 6.

Next, we will show $\lambda(\psi_1) \leq \lambda(\psi)$. Let $H$ (respectively, $H_1$) be the associated transition matrix of the train track map for $\psi$ (respectively, $\psi_1$), and let $\Gamma$ (respectively, $\Gamma_1$) be the induced directed graph as constructed in Section 2.2.

From the construction above (ie adding $p_1$), the directed graph $\Gamma_1$ is obtained by adding a vertex on the edge going out from some vertex $i$ in $\Gamma$ (that is, subdividing the edge going out from $i$) where $i$ has exactly one edge coming in and exactly one

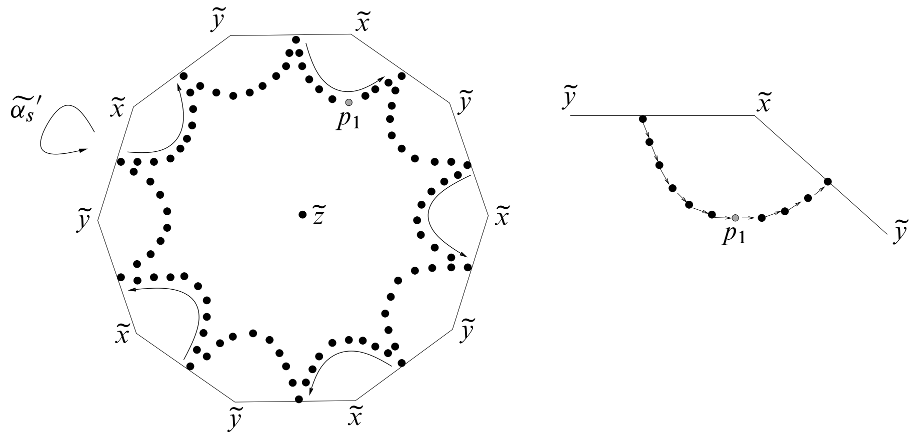



Figure 8: The homeomorphism $\widetilde{\alpha_s}'$. The figure on the right is a local picture near the added point $p_1$.

edge going out. This implies $P_{\Gamma_1}(i,k+1) = P_\Gamma(i,k)$ and

$$\sqrt[k+1]{P_{\Gamma_1}(i,k+1)} \leq \sqrt[k]{P_{\Gamma_1}(i,k+1)} = \sqrt[k]{P_\Gamma(i,k)}$$

for all $k$. Since $H$ and $H_1$ are Perron–Frobenius matrices with Perron–Frobenius eigenvalues corresponding to the dilatations of $\psi$ and $\psi_1$, and Proposition 2.5 tells us $\mu(H_1) \leq \mu(H)$, we have $\lambda(\psi_1) = \mu(H_1)$ is no greater than $\lambda(\psi) = \mu(H)$.

We can obtain $\psi_2$, $\psi_3$ and $\psi_4$ by repeating the construction above of adding more marked points without increasing dilatations (ie $\lambda(\psi_c) \leq \lambda(\psi)$ for $c = 1,2,3,4$). Since $(\log m)/m \geq (\log(m+1))/(m+1)$, we need not consider the cases with $c \geq 5$. Therefore, set $f\colon S_{2,n} \to S_{2,n}$ to be $\psi_c$, where $n = 5(m+1)+1+c$ with $c < 5$, and where $\psi_0 = \psi$. For $n \geq 31$, we have

$$\log \lambda(f) \leq \log \lambda(\psi) < \frac{3 \log m}{m} < \frac{3 \log\left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)},$$

where $m = \lfloor (n-6)/5 \rfloor$.

**Theorem 4.2** *There exists $\kappa_2 > 0$ such that*

$$l_{2,n} < \frac{\kappa_2 \log n}{n},$$

*for all $n \geq 3$.*

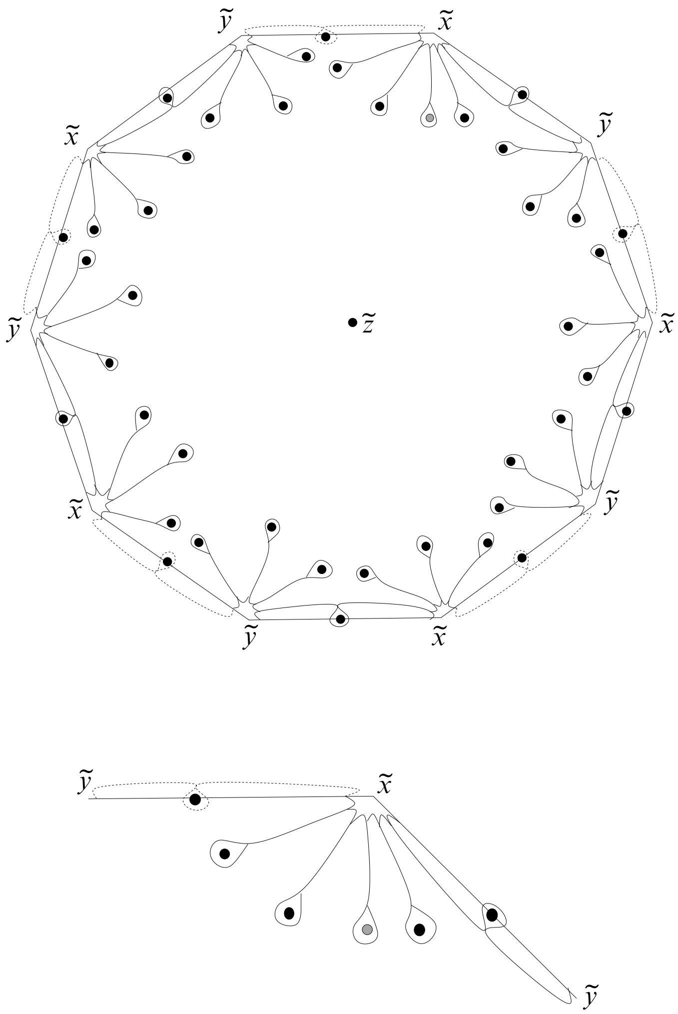



Figure 9: A train track for $\psi_1$. The figure on the bottom is a local picture.

**Proof** From the discussion above, for $n \geq 31$,

$$l_{2,n} < \frac{3\log\left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)} < \frac{\kappa_2' \log n}{n},$$

for some $\kappa_2'$. For $3 \leq n \leq 30$, let $\kappa_2'' = \max\{l_{2,3}, l_{2,4}, \dots, l_{2,30}\}$ then

$$l_{2,n} \leq \kappa_2'' = \left(\kappa_2'' \frac{31}{\log 31}\right)\frac{\log 31}{31} < \left(\kappa_2'' \frac{31}{\log 31}\right)\frac{\log n}{n}.$$

Let $\kappa_2 := \max\{\kappa_2', \kappa_2''(31/\log 31)\}$. □

## 4.2 Higher genus cases

We can generalize our construction and extend to any genus $g > 2$. For any fixed $g > 2$, we define $\psi$ to be a homeomorphism of $S_{g,n}$ in the same fashion with $n = (2g+1)(m+1)+1$ by taking an appropriate branched cover over $S_{0,m+2}$, and we can again extend to arbitrary $n$ by adding $c$ extra marked points and constructing $\psi_c$. Define $f\colon S_{g,n} \to S_{g,n}$ to be $\psi_c$ where $n = (2g+1)(m+1)+1+c$. If $n \geq 6(2g+1)+1$, then

$$\begin{aligned}\log\lambda(f) &< \frac{3\log m}{m}, \quad \text{where } m = \left\lfloor \frac{n-1}{2g+1} \right\rfloor - 1 \\ &< \frac{3\log\left(\frac{n-4g-3}{2g+1}\right)}{\left(\frac{n-4g-3}{2g+1}\right)}.\end{aligned}$$

**Theorem 4.3** *For any fixed $g \geq 2$, there exists $\kappa_g > 0$ such that*

$$l_{g,n} < \frac{\kappa_g \log n}{n},$$

*for all $n \geq 3$.*

**Proof** This is similar to the proof of Theorem 4.2, where $\kappa_g$ is defined to be

$$\kappa_g := \max\left\{\kappa'_g, \kappa''_g \frac{12g+7}{\log(12g+7)}\right\}. \qquad \square$$

**Proof of Theorem 1.1** We only need to prove that the lower bounds on $\log\lambda(f)$ of Theorem 3.3 are bounded below by $(\log n)/(\omega_g n)$ for some $\omega_g$ depending only on $g$, then let $c_g = \max\{\kappa_g, \omega_g\}$. We use the monotone decreasing property of $(\log n)/n$ for $n \geq 3$. Let

$$\omega'_g(\alpha) := \frac{\alpha(12g-12)}{\log 2}\frac{\log 3}{3} \geq \frac{\alpha(12g-12)}{\log 2}\frac{\log n}{n}$$

and so

$$\frac{\log 2}{\alpha(12g-12)} \geq \frac{\log n}{\omega'_g(\alpha) n}.$$

For $n \geq g-1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} \geq \frac{\log 9n}{2\alpha 9n} > \frac{1}{18\alpha}\frac{\log n}{n}.$$

For $3 \le n < g-1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} > \frac{\log(9(g-1))}{2\alpha 9(g-1)} > \frac{\log g}{18\alpha g}\frac{3}{\log 3}\frac{\log n}{n}.$$

Let $\omega_g := \max\{\omega'_g(\alpha), 18\alpha, (6\alpha g \log 3)/\log g\}$, where $0 \le \alpha \le \Theta(g)$. □

# 5 Appendix

## 5.1 Torus with marked points

We will construct an example to prove that $l_{1,2n}$ has an upper bound of the same order as Penner's lower bound in [13], ie $l_{1,2n} = O(1/n)$. The construction is analogous to the one given by Penner for $S_{g,0}$ in [13].

Let $S_{1,2n}$ be a marked torus of $2n$ marked points. Let $a$ and $b$ be essential simple closed curves as in Figure 10. Let $T_a^{-1}$ be the left Dehn twist along $a$ and $T_b$ be the

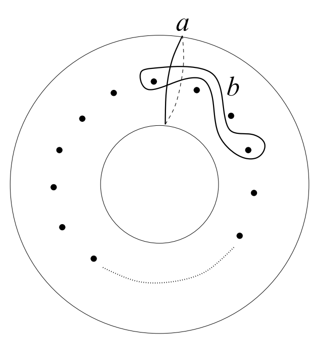



Figure 10: Essential simple closed curves $a$ and $b$ on a marked torus

right Dehn twist along $b$, then we define

$$f := \rho \circ T_b \circ T_a^{-1} \in \mathrm{Mod}(S_{1,2n})$$

where $\rho$ rotates the torus clockwise by an angle of $2\pi/n$, so it sends each marked point to the one which is two to the right. As in [12], $f^n$ is shown to be pseudo-Anosov, and thus so is $f$. Figure 11 shows a bigon track for $f^n$.

We obtain the $2n \times 2n$ transition matrix $M^n$ associated to the train track map of $f^n$ where $M^n$ is an integral Perron–Frobenius matrix and the Perron–Frobenius

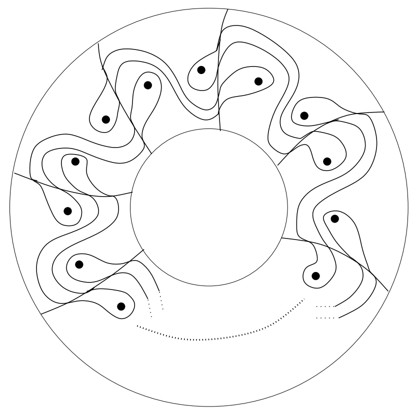


Figure 11: A bigon track for $f^n$

eigenvalues $\mu(M^n)$ is the dilatation $\lambda(f^n)$ of $f^n$. For $n \geq 5$, we have $M^n = N$, where

$$N = \begin{pmatrix} A_1 & B_1 & 0 & 0 & \cdots & 0 & 0 & D_1 \\ A_2 & B_2 & B_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_3 & B_2 & B_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & B_3 & B_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & B_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_2 & B_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_3 & B_2 & D_2 \\ A_3 & C & 0 & 0 & \cdots & 0 & B_3 & D_3 \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$
$$D_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

For $n \geq 5$, the greatest column sum of $M^n$ is 9 and the greatest row sum of $M^n$ is $11$. One can verify that both the greatest column sum and the greatest row sum are $\leq 11$ for $0 < n \leq 4$. Therefore, for $n \geq 1$,

$$11 \geq \mu(M^n) = \lambda(f^n) = (\lambda(f))^n$$
$$\Rightarrow l_{1,2n} \leq \log \lambda(f) \leq \frac{\log 11}{n}.$$

## 5.2 Higher genus with marked points

In all of the following examples we obtain a mapping class $\widetilde{f} \in \mathrm{Mod}(S_{g,n})$ from $f \in \mathrm{Mod}(S_{g,0})$ by adding marked points on the closed surface $S_{g,0}$, where $f$ is a composition of Dehn twists along some set $\mathcal{T}$ of closed geodesics. We can add one marked point in each of the complementary disks of the curves in $\mathcal{T}$ without creating essential reducing curves. By [12, Theorem 3.1], the induced mapping class $\widetilde{f} \in \mathrm{Mod}(S_{g,n})$ is pseudo-Anosov with dilatation $\lambda(\widetilde{f}) = \lambda(f)$.

**Example 1** Penner [13] constructed a pseudo-Anosov mapping class $f \in \mathrm{Mod}(S_{g,0})$ with dilatation $\lambda(f) \leq (\log 11)/g$ for $g \geq 2$, where

$$f := \rho \circ T_c \circ T_a^{-1} \circ T_b.$$

and $T_\alpha$ is the Dehn twist along $\alpha$. Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ with

$$\mathcal{A} = \bigsqcup_{i=1}^{g} a_i, \quad \mathcal{B} = \bigsqcup_{i=1}^{g} b_i \quad \text{and} \quad \mathcal{C} = \bigsqcup_{i=1}^{g} c_i.$$

We can add $g$ marked points as in the Figure 12 so that $\widetilde{f} \in \mathrm{Mod}(S_{g,g})$ is pseudo-Anosov. Therefore,

$$l_{g,g} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 11}{g}.$$

We can also add extra marked points at the fixed points of the rotation. For $g \geq 2$, we will have for $c = 0, 1$ and $2$,

$$l_{g,g+c} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 11}{g},$$

where $\widetilde{f} \in \mathrm{Mod}(S_{g,g+c})$.

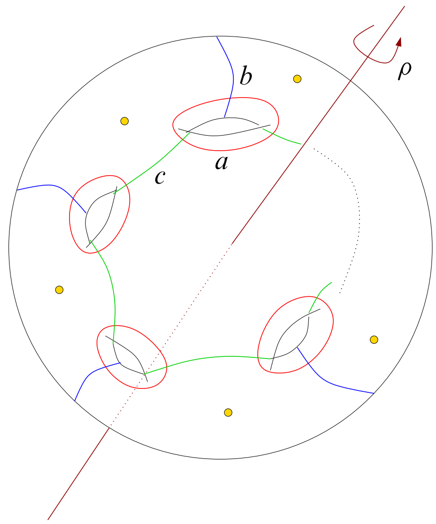



Figure 12: A pseudo-Anosov $\widetilde{f} \in \mathrm{Mod}(S_{g,g})$

**Example 2** For all $g \geq 3$, define $f\colon S_{g,0} \to S_{g,0}$ to be

$$f := \rho \circ T_{b_1} \circ T_{a_1}^{-1},$$

where

$$\rho(a_1) = a_{g+1}, \qquad \rho(b_1) = b_{g+1}$$

and

$$\rho(a_i) = a_{i-1}, \qquad \rho(b_i) = b_{i-1}, \qquad i = 2, \dots, g+1.$$

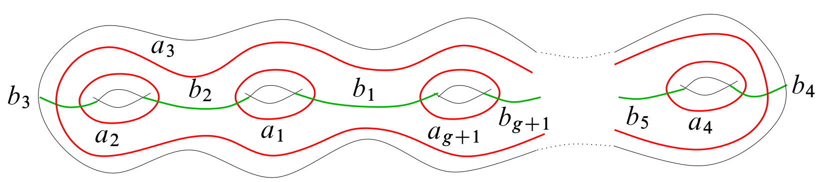



Figure 13: A pseudo-Anosov $f \in \mathrm{Mod}(S_{g,0})$

We construct the $(2g+2) \times (2g+2)$ transition matrix $M^{(g+1)}$ with respect to the spanning vectors associated with geodesics in $\mathcal{T}$. We will get $M^{(g+1)} = N$ for $g \geq 3$, where the matrices are the same as in the Appendix (Section 5.1). Therefore for $g \geq 3$ we have

$$\log \lambda(f) \leq \frac{\log 9}{g+1}.$$

Here $\mathcal{T}=\mathcal{A}\cup\mathcal{B}$ with

$$\mathcal{A}=\bigsqcup_{i=1}^{g} a_i \quad \text{and} \quad \mathcal{B}=\bigsqcup_{i=1}^{g} b_i.$$

For $g\geq 3$ and $c=0,1,2,3,4$, we have

$$l_{g,c}\leq \log\lambda(\tilde{f})\leq \frac{\log 9}{g+1},$$

where $\tilde{f}\in \mathrm{Mod}(S_{g,c})$.

**Example 3** For $g\geq 5$, define $f\colon S_{g,0}\to S_{g,0}$ by

$$f:=\rho\circ T_{d_1}\circ T_{c_1}^{-1}\circ T_{b_1}\circ T_{a_1},$$

where

$$\rho(a_1)=a_{g-1},\ \rho(b_1)=b_{g-1},\ \rho(c_1)=c_{g-1},\ \rho(d_1)=d_{g-1}$$

and $\rho(a_i)=a_{i-1},\ \ \rho(b_i)=b_{i-1},\ \ \rho(c_i)=c_{i-1},\ \ \rho(d_i)=d_{i-1},\ i=2,\dots,g-1.$

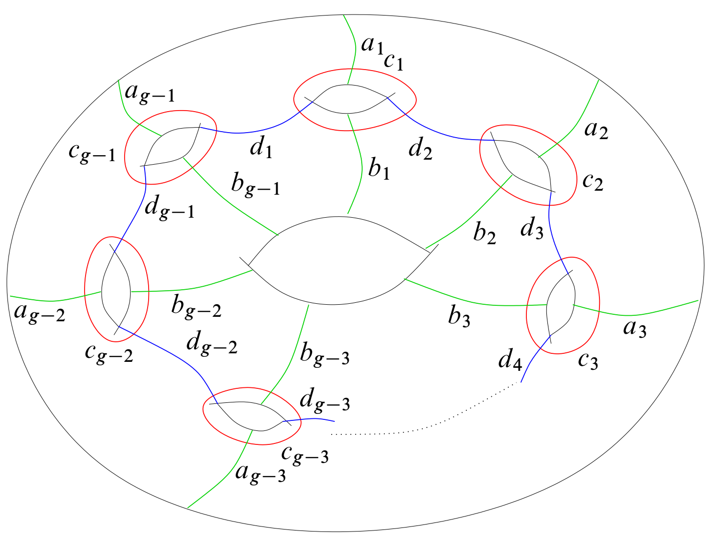


Figure 14: A pseudo-Anosov $f\in \mathrm{Mod}(S_{g,0})$

Similarly, we have the $(4g-4)\times(4g-4)$ transition matrix $M^{(g-1)}$ with respect to the spanning vectors associated with the geodesics in $\mathcal{T}$. For $g\geq 5$ we have $M^{(g-1)}=N$

where

$$A_1 = \begin{pmatrix} 1&0&1&0\\0&1&1&0\\1&1&3&1\\1&1&3&2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&3&1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&1&3&2\\1&1&3&0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1&0&1&0\\0&1&1&0\\1&1&3&1\\1&1&3&3 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&3&1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&1 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 1&0&1&0\\0&1&1&0\\1&1&4&1\\1&1&4&3 \end{pmatrix}.$$

For $g \geq 5$, the greatest column sum of $M^{(g-1)}$ is $17$ and the greatest row sum of $M^{(g-1)}$ is $21$, hence

$$\log \lambda(f) \leq \frac{\log 17}{g-1}.$$

Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ with

$$\mathcal{A} = \bigsqcup_{i=1}^{g} a_i, \quad \mathcal{B} = \bigsqcup_{i=1}^{g} b_i, \quad \mathcal{C} = \bigsqcup_{i=1}^{g} c_i \quad \text{and} \quad \mathcal{D} = \bigsqcup_{i=1}^{g} d_i.$$

For $c = 1$ and $2$, we can induce $\widetilde{f} \in \mathrm{Mod}(S_{g,c(g-1)})$ with

$$l_{g,c(g-1)} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 17}{g-1},$$

when $g \geq 5$.

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