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Polynomial Bridgeland stability conditions and the large volume limit

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We introduce the notion of a polynomial stability condition, generalizing Bridgeland stability conditions on triangulated categories. We construct and study a family of polynomial stability conditions for any normal projective variety. This family includes both Simpson-stability and large volume limits of Bridgeland stability conditions.

We show that the PT/DT–correspondence relating stable pairs to Donaldson–Thomas invariants (conjectured by Pandharipande and Thomas) can be understood as a wall-crossing in our family of polynomial stability conditions. Similarly, we show that the relation between stable pairs and invariants of one-dimensional torsion sheaves (proven recently by the same authors) is a wall-crossing formula.

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1 Introduction

In this article, we introduce polynomial stability conditions on triangulated categories. They are a generalization of Bridgeland's notion of stability in triangulated categories. The generalization is motivated by trying to understand limits of Bridgeland's stability conditions; it allows for the central charge to have values in complex polynomials rather than complex numbers.

While Bridgeland stability conditions have been constructed only in dimension ≤ 2 and some special cases, we construct a family of polynomial stability conditions on the derived category of any normal projective variety. This family includes both Simpson-stability of coherent sheaves and stability conditions that we expect to be the large volume limit of Bridgeland stability conditions.

We interpret both the PT/DT–correspondence conjectured by Pandharipande and Thomas [30], and the relation between stable pair invariants and one-dimensional torsion sheaves which they proved in [31], as a wall-crossing phenomenon in our family of polynomial stability conditions.

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1.1 Bridgeland's stability conditions

Since their introduction by Bridgeland [9], stability conditions for triangulated categories have drawn an increasing amount of interest from various perspectives. They generalize the concept of stability from abelian categories to triangulated categories.

Originally, Bridgeland introduced the concept as an attempt to mathematically understand Douglas' construction [13] of Π–stability of D–branes. Following Douglas' ideas, Bridgeland showed that the set of stability conditions on $D^b(X)$ has a natural structure as a smooth manifold. There are also various purely mathematical reasons to study the space of stability conditions.

Definition 1.1.1 [9] A stability condition on $D^b(X)$ is a pair $(Z, \mathcal{P})$ where Z is a group homomorphism from $K(X) \cong K(D^b(X))$ to $\mathbb{C}$, and $\mathcal{P}$ is a collection of extension-closed subcategories $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$, such that

(a) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(b) $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ for all $\phi_1 > \phi_2$,
(c) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$, and
(d) for every $0 \neq E \in D^b(X)$ there is a sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ of real numbers and a sequence of exact triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$.

Objects of $\mathcal{P}(\phi)$ are called semistable of phase ϕ, and the group homomorphism Z is called the central charge. We now restrict our attention to "numerical" stability conditions: these are stability conditions for which $Z(E)$ is given by numerical invariants of E, ie where Z factors via the projection $K(D^b(X)) \to \mathcal{N}(X) := \mathcal{N}(D^b(X))$ to the numerical K–group.[1]

1.2 The space of stability conditions

The role of $\mathcal{P}$ (called "slicing") is easily understood, as it naturally generalizes the notion of semistable objects in an abelian category, together with the ordering of their slopes and the existence of Harder–Narasimhan filtrations. The role of Z is less obvious; we will explain two aspects in the following paragraphs.

[1] The numerical K–group $\mathcal{N}(D^b(X))$ is the quotient of $K(D^b(X))$ by the zero-space of the bilinear form $\chi(E, F) = \chi(\mathbf{R}\mathrm{Hom}(E, F))$.

It seems unsatisfactory that semistable objects in the derived category have to be given explicitly, rather than characterized intrinsically by a slope function. This deficiency is somewhat corrected by the following observation:

Given a slicing $\mathcal{P}$, consider the category $\mathcal{A} = \mathcal{P}((0,1])$ generated by all semistable objects of phase $0 < \phi \le 1$ and extensions. It can be seen that $\mathcal{A}$ is the heart of a bounded t–structure (and in particular an abelian category); the slicing is thus a refinement of the datum of a bounded t–structure. Bridgeland showed that this refinement is uniquely determined by Z:

Proposition 1.2.1 [9, Proposition 5.3] *To give a stability condition $(Z, \mathcal{P})$ is equivalent to giving the heart $\mathcal{A} \subset D^b(X)$ of a bounded t–structure, and a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ with the following properties:*

(a) *For every object $E \in \mathcal{A}$, we have $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi(E)}$ with $0 < \phi(E) \le 1$.*

(b) *We say an object is Z–semistable if it has no subobjects $A \hookrightarrow E$ with $\phi(A) > \phi(E)$. We require that every object has a Harder–Narasimhan filtration with Z–semistable filtration quotients.*

Given $\mathcal{A}$ and Z, the semistable objects in the derived category are the shifts of the Z–semistable objects in $\mathcal{A}$. The positivity property (a) is somewhat delicate; for example, it can't be satisfied for the category of coherent sheaves on a projective surface.

There is a natural topology on the space of slicings. However, only together with the central charge does the topological space of stability conditions become a smooth manifold. One can paraphrase Bridgeland's result as follows: *One can equip the space* $\mathrm{Stab}(X)$ *of "locally finite"*[2] *numerical stability conditions with the structure of a smooth manifold, such that the forgetful map* $\mathcal{Z}\colon \mathrm{Stab}(X) \to \mathcal{N}(X)^*$, $(Z, \mathcal{P}) \mapsto Z$ *gives local coordinates at every point.* In other words, a stability condition can be deformed by deforming its central charge.

The space $\mathrm{Stab}(X)$ is closely related to the moduli space of $N = 2$ superconformal field theories; see Bridgeland [8]. The existence of $\mathcal{Z}$ has interesting implications for the group of auto-equivalences of $D^b(X)$, as one can study its induced action on $\mathcal{Z}$; see eg Bridgeland [10] and Huybrechts, Macrì and Stellari [17].

[2] [9, Definition 5.7]

1.3 Reconstruction of X from $D^b(X)$

If the canonical bundle ω_X, or its inverse ω_X^{-1}, of a smooth variety X is ample, then the variety can be reconstructed from its bounded derived category; see Bondal and Orlov [7]. Without the assumption of ampleness, this statement is wrong, and the proof already breaks down at its first step: the intrinsic characterization of point-like objects in $D^b(X)$ (the shifts $\mathcal{O}_x[j]$ of skyscraper sheaves for closed points $x \in X$) by the action of the Serre-functor.

However, the mathematical translation of ideas by Aspinwall, originally suggested in [2], suggests that a stability condition provides exactly the missing data to characterize the point-like objects. Inside the space of stability conditions, there should be a special chamber, which we will call the ample chamber, with the following property: When $(Z, \mathcal{P})$ is a stability condition in the ample chamber, and $E \in D^b(X)$ an object with class $[E] = [\mathcal{O}_x]$ in the numerical K–group, then E is $(Z, \mathcal{P})$–stable if and only if E is isomorphic to the shift of a skyscraper sheaf $[\mathcal{O}_x]$. One could then reconstruct X as the moduli space of $(Z, \mathcal{P})$–stable objects.

Moving to a chamber of the space of stability conditions adjacent to the ample chamber, the moduli space $\widetilde{X}$ of semistable objects of the same class $[\mathcal{O}_x]$ comes with a fully faithful functor Φ: $D^b(\widetilde{X}) \to D^b(X)$ induced by the universal family. This suggests that $\widetilde{X}$ could be a birational model of X with isomorphic derived category (eg a flop), it could be isomorphic to X with Φ being a nontrivial auto-equivalence of X, or it could be a birational contraction or a flip of X. It seems an intriguing question to what extent the birational geometry of X can be captured by this phenomenon.

This suggestion is consistent with many of the known examples of Bridgeland stability conditions. Maybe most convincing is the case of a crepant resolution $Y \to \mathbb{C}^3/G$ of a three-dimensional Gorenstein quotient singularity. The results of Craw and Ishii [12] can be reinterpreted as saying that every other crepant resolution $Y' \to \mathbb{C}^3/G$ can be constructed as a moduli space of Bridgeland-stable objects in $D^b(Y)$; see also Toda [37] for the local construction of a flop along these lines.

1.4 Examples of stability conditions

The existence of stability conditions on $D^b(X)$ for X a smooth, projective variety has only been shown in very few cases:

- For a smooth curve C, stability conditions were constructed by Bridgeland [9], and $\mathrm{Stab}(C)$ has been described by Macrì [25] and Okada [29]; in Burban and Kreußler [11] the case of singular curves of genus one was considered.

- For the case of a K3 surface S, Bridgeland completely described one connected component of $\mathrm{Stab}(S)$ in [10] (including a complete description of the ample chamber). Macrì, Mehrotra and Stellari [26] studied the space of stability conditions on Kummer and Enriques surfaces. For arbitrary smooth projective surfaces, stability conditions have recently been constructed by Arcara and Bertram [1].
- If $D^b(X)$ has a complete exceptional collection, then stability conditions exist by Macrì [24].

For complex nonprojective tori, stability conditions were studied by Meinhardt [28].

1.5 Stability conditions related to σ–models

Let X be a smooth projective variety. Following ideas in the physical literature (see Douglas [13], Aspinwall and Douglas [3], Aspinwall [2] and Aspinwall and Lawrence [4]), it should be possible to construct stability conditions on $D^b(X)$ coming from the nonlinear σ–model associated to X. At least for an open subset of these stability conditions, skyscraper sheaves of points should be stable. Further, it is known how the central charge should depend on the complexified Kähler moduli space: if $\beta \in H^2(X)$ is an arbitrary class, and $\omega \in H^2(X)$ and ample class, then the central charge should be given as

$$Z_{\beta,\omega}(E) = -\int_X e^{-\beta - i\omega} \cdot \mathrm{ch}(E)\sqrt{\mathrm{td}\, X}. \tag{1}$$

However, in general not even a matching t–structure whose heart $\mathcal{A}$ would satisfy the positivity property (a) of Proposition 1.2.1 is known; in fact no example of a stability condition on a projective Calabi–Yau threefold is known.

1.6 Polynomial stability conditions

However, if we replace ω by $m\omega$ and let $m \to +\infty$ (this is the large volume limit), then a matching t–structure can be constructed: If E is a coherent sheaf and d its dimension of support, then $Z(E)(m) \to -(-i)^d \cdot \infty$ as $m \to \infty$. Thus the central charge $Z(E[\lfloor \frac{d}{2} \rfloor])(m)$ of the shift of E will go to $-\infty$ or $i\infty$; this suggests that a t–structure can be constructed by a filtration of dimension of support, ie a t–structure of perverse coherent sheaves. However, the limit of the phase $\phi(E)(m)$ is too coarse as an information to characterize semistable objects; instead, it is more natural to consider the central charge $Z_{\beta,m\omega}$ given by Equation (1) as a polynomial in m: then we can say a perverse coherent sheaf E is semistable if there is no perverse coherent subsheaf $A \hookrightarrow E$ with $\phi(A)(m) > \phi(E)(m)$ for m being large.

Motivated by this observation, we introduce a notion of polynomial stability condition in Definition 2.3.1. It allows the central charge to have values in polynomials $\mathbb{C}[m]$ instead of $\mathbb{C}$; accordingly, the slicing $\mathcal{P}$ has to depend not on real numbers, but on phases of polynomials (considered for $m \gg 0$). It gives a precise meaning to the notion of a "stability condition in the limit of $m \to \infty$".

1.7 Results

Our main result is Theorem 3.2.2. It shows the existence of a family of polynomial stability conditions for every normal projective variety. Its associated bounded t–structure is a t–structure of perverse coherent sheaves. The family contains stability conditions corresponding to Simpson stability (see Section 2.1) and stability conditions that should be the large volume limit of Bridgeland stability conditions (see Section 4).

In the case of surfaces, Proposition 4.1 makes the last statement precise: the polynomial stability condition $(Z, \mathcal{P})$ at the large volume limit is the limit of Bridgeland stability conditions $(Z_m, \mathcal{P}_m)$, depending on m, in the sense that objects are $\mathcal{P}$–stable if and only if they are $\mathcal{P}_m$–stable for $m \gg 0$, and the Harder–Narasimhan filtration with respect to $\mathcal{P}$ is the same as the Harder–Narasimhan filtration with respect to $\mathcal{P}_m$ for $m \gg 0$.

The polynomial stability conditions provide many new t–structures on the derived category of a projective variety.[3] They might help to construct Bridgeland stability conditions on higher-dimensional varieties.

With Proposition 5.1, we observe that the polynomial stability conditions constructed in Theorem 3.2.2 are "ample" in the sense of Section 1.3: X can be reconstructed from $D^b(X)$, the stability condition and the class of $[\mathcal{O}_x] \in \mathcal{N}(X)$ as a moduli space of semistable objects.

1.8 PT/DT—correspondence as a wall-crossing

Pandharipande and Thomas [30] introduced new invariants of stable pairs on smooth projective threefolds. In the Calabi–Yau case, they conjecture a simple relation between their generating function and the generating function of Donaldson–Thomas invariants (introduced in Maulik, Nekrasov, Okounkov and Pandharipande [27]). With Proposition 6.1.1, we show that this relation can be understood as a wall-crossing phenomenon (in the sense of Joyce [19]) in a family of polynomial stability conditions.

[3] The t–structures used in the construction are those described in [6], but tilting with respect to different phase functions yields new torsion pairs, and thus new t–structures, in the same way that Gieseker- or slope-stability yield new t–structures by tilting the category of coherent sheaves.

Similarly, we show in Section 6.2 that the relation between stable pair invariants and invariants counting one-dimensional torsion sheaves can be understood as a wall-crossing formula.

1.9 The space of polynomial stability conditions

In Section 8, we discuss to what extent the deformation result by Bridgeland carries over to our situation. We introduce a natural topology on the set of polynomial stability conditions and show that the forgetful map

$$\mathcal{Z}\colon \mathrm{Stab}_{\mathrm{Pol}}(X) \to \mathrm{Hom}(\mathcal{N}(X), \mathbb{C}[m]), \quad (Z, \mathcal{P}) \mapsto Z$$

is continuous and locally injective. Under a strong local finiteness assumption, we can also show that it is a local homeomorphism.

1.10 Notation

If Σ is a set of objects in a triangulated category $\mathcal{D}$ (resp. a set of subcategories of $\mathcal{D}$), we write $\langle \Sigma \rangle$ for the full subcategory generated by Σ and extensions; ie the smallest full subcategory of $\mathcal{D}$ that is closed under extensions and contains Σ (resp. contains all subcategories in Σ).

We will write $\mathbb{H} \subset \mathbb{C}$ for the semiclosed upper half plane

$$\mathbb{H} = \left\{ z \in \mathbb{C} \;\middle|\; z \in \mathbb{R}_{>0} \cdot e^{i\pi\phi(z)}, 0 < \phi(z) \leq 1 \right\},$$

and $\phi(z)$ for the phase of $z \in \mathbb{H}$.

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Some of the stability conditions constructed in this article have also been constructed independently by Yukinobu Toda [34], namely the stability conditions at the large volume limit of Calabi–Yau threefolds. In particular, Toda also explains the key formula (8) as wall-crossing formula in his family of stability conditions. The complete family of stability conditions considered by Toda lives on a wall of the space of polynomial stability conditions considered here.

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2 Polynomial stability conditions

2.1 Example: Simpson/Rudakov stability as a polynomial stability condition

Before giving the precise definition of polynomial stability conditions, we give an example that is more easily constructed than the large volume limit considered in the introduction, which will hopefully motivate the definition.

Let $\mathcal{A} = \mathrm{Coh}\, X \subset D(X)$ be the standard heart in the derived category of a projective variety X with a chosen ample line bundle $\mathcal{L}$. Pick complex numbers $\rho_0, \rho_1, \ldots, \rho_n$ in the open upper half plane H with $\phi(\rho_0) > \phi(\rho_1) > \cdots > \phi(\rho_n)$ as in Figure 1. For any

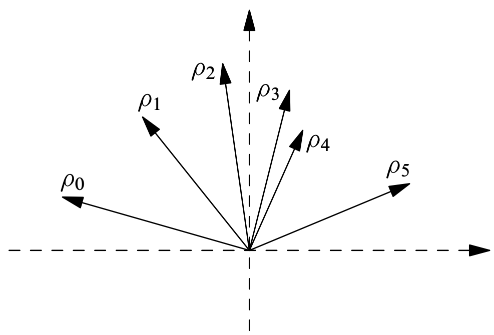


Figure 1: Stability vector for Simpson stability

coherent sheaf $E \in \mathrm{Coh}\, X$, let $\chi_E(m) = \sum_{i=0}^{n} a_i(E)m^i$ be the Hilbert polynomial with respect to $\mathcal{L}$. We define the central charge by

$$Z(E)(m) = \sum_{i=0}^{n} \rho_i a_i(E) m^i.$$

Then $Z(E)(m) \in \mathbb{H}$ for E nontrivial and $m \gg 0$, and we can consider the phase $\phi(E)(m) \in (0,1]$. We say that a sheaf if Z–stable if for every subsheaf $A \hookrightarrow E$, we have $\phi(A)(m) \leq \phi(E)(m)$ for $m \gg 0$.

Then a sheaf E is Z–stable if and only if it is a Simpson-stable sheaf; this is most easily seen by using Rudakov's reformulation in [33]. In particular, stability does not depend on the particular choice of the ρ_i. In order not to lose any information, we should consider the phase of a stable object to be the function $\phi(E)(m)$ defined for $m \gg 0$ rather than the limit $\lim_{m\to\infty} \phi(E)(m)$; in other words, we consider its phase to be the function germ

$$\phi(E)\colon (\mathbb{R} \cup \{+\infty\}) \to \mathbb{R}.$$

Then we can define an object $E \in D^b(X)$ to be stable if and only if it is isomorphic to the shift $F[n]$ of Z–stable sheaf; its phase is given by the function germ $\phi(F) + n$.

Combining the Harder–Narasimhan filtrations of arbitrary sheaves with respect to Simpson stability with the filtration of a complex by its cohomology sheaves, we obtain a filtration of an arbitrary complex similar to the filtration in part (d) of Definition 1.1.1.

2.2 Slicings

Definition 2.2.1 Let $(S, \succeq)$ be a linearly ordered set, equipped with an order-preserving bijection $S \to S, \phi \mapsto \phi + 1$ (called the shift) satisfying $\phi + 1 \succeq \phi$. An S–valued slicing of a triangulated category $\mathcal{D}$ is given by full additive extension-closed subcategories $\mathcal{P}(\phi)$ for all $\phi \in S$, such that the following properties are satisfied:

(a) For all $\phi \in S$, we have $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.

(b) If $\phi \succ \psi$ for $\phi, \psi \in S$, and $A \in \mathcal{P}(\phi)$, $B \in \mathcal{P}(\psi)$, then $\mathrm{Hom}(A, B) = 0$.

(c) For all nonzero objects $E \in \mathcal{D}$, there is a finite sequence $\phi_1 \succ \phi_2 \succ \cdots \succ \phi_n$ of elements in S, and a sequence of exact triangles

(2)
$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$.

This was called "stability data" or "t-stability" in [14]. If $S = \mathbb{Z}$, this notion is equivalent to a bounded t–structure (see Bridgeland [10, Lemma 3.1]), and for $S = \mathbb{R}$, it is a "slicing" as defined in [9]. The objects in $\mathcal{P}(\phi)$ are called semistable of phase ϕ. The sequence of exact triangles in part (c) is also called the Harder–Narasimhan filtration of E. If a Harder–Narasimhan filtration exists, then Definition (b) forces it to be unique.

Definition 2.2.2 The set S of polynomial phase functions is the set of continuous function germs

$$\phi\colon (\mathbb{R} \cup \{+\infty\}, +\infty) \to \mathbb{R}$$

such that there exists a polynomial $Z(m) \in \mathbb{C}[m]$ with $Z(m) \in \mathbb{R}_{>0} \cdot e^{\pi i \phi(m)}$ for $m \gg 0$. It is linearly ordered by setting

$$\phi \prec \psi \quad \Leftrightarrow \quad \phi(m) < \psi(m) \quad \text{for } 0 \ll m < +\infty,$$

and its shift $\phi \mapsto \phi + 1$ is given by point-wise addition.

The condition that ϕ, ψ can be written as arguments of polynomial functions guarantees that either $\phi \succ \psi$ or $\phi \preceq \psi$ holds; given $Z(m)$, the function $\phi(m)$ is of course determined up to an even integer constant.

From now on, S will be the set of polynomial phase functions. In our construction, S–valued slicings will play the role of $\mathbb{R}$–valued slicings in Bridgeland's construction.

The following easy lemma is implicitly used in both [9] and [14], but we will make it explicit:

Lemma 2.2.3 *Let S_1, S_2 be two linearly ordered sets equipped with shifts τ_1, τ_2, and let $\pi\colon S_1 \to S_2$ be a morphism of ordered sets commuting with τ_1, τ_2. Then π induces a pushforward of stability conditions as follows: If $\mathcal{P}$ is an S_1–valued slicing, then $\pi_*\mathcal{P}(\phi_2)$ for some $\phi_2 \in S_2$ is defined as $\langle \{\mathcal{P}(\phi_1) \mid \pi(\phi_1) = \phi_2\}\rangle$.*

The proof is an exercise in the use of the octahedral axiom.

We will make use of the following pushforwards: By the projection $\pi\colon S \to \mathbb{R}, \phi \mapsto \phi(\infty)$, we obtain an $\mathbb{R}$–valued slicing from every S–valued slicing. Further, for each $\phi_0 \in S$ we get a projection $\pi^{\phi_0}\colon S \to \mathbb{Z}, \phi \mapsto \max_{n\in\mathbb{Z}} \phi_0 + n \preceq \phi$ (we could also choose $\phi \mapsto \max_{n\in\mathbb{Z}} \phi_0 + n \prec \phi$). This produces a bounded t–structure from every S–valued slicing; in other words, an S–valued slicing is a refinement of a bounded t–structure, breaking up the category into even smaller slices.

For any interval I in the set of phases, we get an extension-closed subcategory $\mathcal{P}(I) = \langle\{\mathcal{P}(\phi) \mid \phi \in I\}\rangle$. In the case of an S–valued slicing, the categories $\mathcal{P}([\phi, \phi+1))$ and $\mathcal{P}((\phi, \phi+1])$ are abelian, as they are the hearts of the t–structures constructed in the last paragraph. The proof for these statements carries over literally from the one given by Bridgeland: we can include these categories into the abelian category $\mathcal{P}([\phi, \phi+1))$. The slices $\mathcal{P}(\phi)$ are abelian.

2.3 Central charge

We now come to the main definition:

Definition 2.3.1 A polynomial stability condition on a triangulated category $\mathcal{D}$ is given by a pair $(Z, \mathcal{P})$, where $\mathcal{P}$ is an S–valued slicing of $\mathcal{D}$, and Z is a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}[m]$, with the following property: if $0 \neq E \in \mathcal{P}(\phi)$, then

$$Z(E)(m) \in \mathbb{R}_{>0} \cdot e^{\pi i \phi(m)}$$

for $m \gg 0$.

In the case where Z maps to constant polynomials $\mathbb{C} \subset \mathbb{C}[m]$, this is equivalent to Bridgeland's notion of a stability condition. Similarly to that case, a polynomial stability condition can be constructed from a bounded t–structure and a compatible central charge Z:

Definition 2.3.2 A polynomial stability function on an abelian category $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}[m]$ such that there exists a polynomial phase function $\phi_0 \in S$ with the following property:

For any $0 \neq E \in \mathcal{A}$, there is a polynomial phase function $\phi(E)$ with $\phi_0 \prec \phi(E) \preceq \phi_0 + 1$ and $Z(E)(m) \in \mathbb{R}_{>0} \cdot e^{\pi i \phi_E(m)}$ for $m \gg 0$.

This definition allows slightly bigger freedom than requiring $Z(E)(m) \in \mathbb{H}$ for $m \gg 0$.

We call $\phi(E) \in S$ the phase of E; the function $\operatorname{Ob} \mathcal{A} \setminus \{0\} \to S$, $E \mapsto \phi(E)$ is a slope function in the sense that it satisfies the see-saw property on short exact sequences (cf [33]). An object $0 \neq E$ is called *semistable with respect to* Z if for all subobjects $0 \neq A \subset E$, we have $\phi(A) \preceq \phi(E)$; equivalently, if for every quotient $E \twoheadrightarrow B$ in $\mathcal{A}$ we have $\phi(E) \preceq \phi(B)$. We say that a stability function has the Harder–Narasimhan property if for all $E \in \mathcal{A}$, there is a finite filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_n = E$ such that E_i/E_{i-1} are semistable with slopes $\phi(E_1/E_0) \succ \phi(E_2/E_1) \succ \cdots \succ \phi(E_n/E_{n-1})$.

Finally, note that the set of polynomials $Z(E)$ for which a polynomial phase function $\phi(E)$ as in the above definition exist forms a convex cone in $\mathbb{C}[m]$. Its only extremal ray is the set of polynomials with $\phi(E) = \phi_0 + 1$. This is an important reason why many of the proofs of [9] carry over to our situation.

We restate two propositions by Bridgeland in our context; the proofs are identical to the ones given by Bridgeland:

Proposition 2.3.3 [9, Proposition 5.3] *Giving a polynomial stability condition on $\mathcal{D}$ is equivalent to giving a bounded t–structure on $\mathcal{D}$ and a polynomial stability function on its heart with the Harder–Narasimhan property.*

The following proposition shows that the Harder–Narasimhan property can be deduced from a finiteness assumption of $\mathcal{A}$ with respect to Z:

Proposition 2.3.4 [9, Proposition 2.4] *Let $\mathcal{A}$ be an abelian category, $Z\colon K(\mathcal{A}) \to \mathbb{C}[m]$ a polynomial stability function, and assume that they satisfy the following chain conditions:*

- (Z–Artinian) *There are no infinite chains of subobjects*
$$\ldots \hookrightarrow E_{j+1} \hookrightarrow E_j \hookrightarrow \ldots \hookrightarrow E_2 \hookrightarrow E_1$$
with $\phi(E_{j+1}) \succ \phi(E_j)$ for all j.
- (Z–Noetherian) *There are no infinite chains of quotients*
$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \ldots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \ldots$$
with $\phi(E_j) \succ \phi(E_{j+1})$ for all j.

Then $\mathcal{A}, Z$ have the Harder–Narasimhan property.

3 The standard family of polynomial stability condition

In this section, we will construct a standard family of stability conditions on the bounded derived category $D^b(X)$ of an arbitrary normal projective variety X. Let n be the dimension of X.

3.1 Perverse coherent sheaves

The t–structures relevant for our stability conditions are t–structures of perverse coherent sheaves. The theory of perverse coherent sheaves is apparently originally due to Deligne, and has been developed by Bezrukavnikov [6] and Kashiwara [20]. We will need only a special case of perverse coherent sheaves, which are given by filtrations of dimension.

Definition 3.1.1 A function $p\colon \{0, 1, \ldots, n\} \to \mathbb{Z}$ is called a perversity function if p is monotone decreasing, and if $\overline{p}\colon \{0, 1, \ldots, n\} \to \mathbb{Z}$ (called the dual perversity) given by $\overline{p}(d) = -d - p(d)$ is also monotone decreasing.

In other words we require that $p(d) \geq p(d+1) \geq p(d) - 1$. Given a perversity function in the above sense, the function $X^{\mathrm{top}} \to \mathbb{Z}, x \mapsto p(\dim x)$ is a *monotone and comonotone* perversity function in the sense of [6].

Let $\mathcal{A}^{p,\leq k}$ be the following increasing filtration of $\operatorname{Coh} X$ by abelian subcategories:
$$\mathcal{A}^{p,\leq k} = \{\mathcal{F} \in \operatorname{Coh} X \mid p(\dim \operatorname{supp} \mathcal{F}) \geq -k\}$$

Theorem 3.1.2 [6; 20] *If p is a perversity function, then the following pair defines a bounded t–structure on $D^b(X)$:*

(3) $D^{p,\le 0} = \{E \in D^b(X) \mid H^{-k}(E) \in \mathcal{A}^{p,\le k} \text{ for all } k \in \mathbb{Z}\}$

(4) $D^{p,\ge 0} = \{E \in D^b(X) \mid \operatorname{Hom}(A, E) = 0 \text{ for all } k \in \mathbb{Z} \text{ and } A \in \mathcal{A}^{p,\le k}[k+1]\}$

This description is slightly different to the one given in [6; 20] but easily seen to be equivalent. Once $D^{p,\le 0}$ is given, $D^{p,\ge 0}$ is of course determined as the right-orthogonal complement of $D^{p,\le -1}$. Our notation is somewhat intuitive as $\mathcal{A}^{p,\le k}$ can be recovered as $\mathcal{A} \cap D^{p,\le k}$, which completely determines the t–structure.

Objects in the heart $\mathcal{A}^p = D^{p,\ge 0} \cap D^{p,\le 0}$ are called perverse coherent sheaves.

3.2 Construction of polynomial stability conditions

Definition 3.2.1 A stability vector ρ is a sequence $(\rho_0, \rho_1, \dots, \rho_n) \in (\mathbb{C}^*)^{n+1}$ of nonzero complex numbers such that ρ_d / ρ_{d+1} is in the open upper half plane for $0 \le d \le n-1$.

Given a stability vector ρ, we call $p\colon \{0, 1, \dots, n\} \to \mathbb{Z}$ a perversity function associated to ρ if it is a perversity function satisfying $(-1)^{p(d)} \rho_d \in \mathbb{H}$ for all $0 \le d \le n$.

Such p is uniquely determined by $p(0)$, and given $p(0)$ such a perversity function exists if $p(0)$ is of the correct parity; see Figure 2 for an example on a 5–fold. The

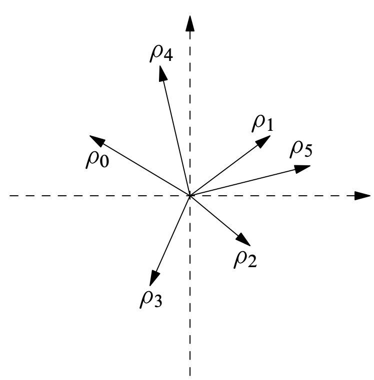


Figure 2: A stability vector with associated perversity function $p(0) = p(1) = 0$, $p(2) = p(3) = -1$, $p(4) = p(5) = -2$

number $p(0) - p(d)$ counts how often the piecewise linear path $\rho_0 \to \rho_1 \to \cdots \to \rho_d$ crosses the real line. We will construct stability conditions by giving a polynomial stability functions on $\mathcal{A}^p$.

In the following, a Weil divisor $\omega \in A^1(X)_{\mathbb{R}}$ is called ample if for any effective class $\alpha \in A_d(X)$, we have $\omega^d \cdot \alpha > 0$.

Theorem 3.2.2 *Let the data $\Omega = (\omega, \rho, p, U)$ be given, consisting of*

- *an ample class $\omega \in A^1(X)_{\mathbb{R}}$,*
- *a stability vector $\rho = (\rho_0, \dots, \rho_n)$,*
- *a perversity function p associated to ρ, and*
- *a unipotent operator $U \in A^*(X)_{\mathbb{C}}$ (ie $U = 1 + N$ where N is concentrated in positive degrees).*

Let $Z_\Omega \colon K(X) \to \mathbb{C}[m]$ be the following central charge:

$$Z_\Omega(E)(m) = \int_X \sum_{d=0}^{n} \rho_d \omega^d m^d \cdot \mathrm{ch}(E) \cdot U$$

Then $Z_\Omega(E)(m)$ is a polynomial stability function for $\mathcal{A}^p$ with the Harder–Narasimhan property.

By Proposition 2.3.3, this gives a polynomial stability condition $(Z_\Omega, \mathcal{P}_\Omega)$ on $D^b(X)$.

We will drop the subscript Ω from the notation. In this section we will just prove that Z is a polynomial stability function according to Definition 2.3.2 with $\phi_0 = \epsilon$ for some small constant $\epsilon \geq 0$. In other words, we have to prove that for every $E \in \mathcal{A}^p$, we have $Z(E)(m) \in e^{i\epsilon} \cdot \mathbb{H}$ for $m \gg 0$. The proof of the existence of Harder–Narasimhan filtrations will be postponed until Section 7.

We start the proof with the following immediate observation:

Lemma 3.2.3 *Given a nonzero object $E \in \mathcal{A}^p$, let k be the largest integer such that $H^{-k}(E) \neq 0$, and let d be the dimension of support of $H^{-k}(E)$. Then $p(d) = -k$, the sheaf $H^{-k}(E)$ has no torsion in dimension d' whenever $p(d') > -k$, and all other cohomology sheaves of E are supported in smaller dimension.*

We call d the dimension of support of E.

Proof By $E \in D^{p,\leq 0}$ we have $p\big(\dim \operatorname{supp} H^{-k(E)}(E)\big) \geq -k$. The claim follows from $E \in D^{p,\geq 0}$ and

$$\begin{aligned}\mathrm{Hom}(\mathcal{A}^{p,\leq k-1}, H^{-k}(E)) &= \mathrm{Hom}(\mathcal{A}^{p,\leq k-1}[k], H^{-k}(E)[k]) \\ &= \mathrm{Hom}(\mathcal{A}^{p,\leq k-1}[k], E) = 0.\end{aligned}$$

□

Choose $\epsilon > 0$ such that $(-1)^{p(d)}\rho_d$ is in the interior of $\mathbb{H}_\epsilon = e^{i\epsilon}\cdot\mathbb{H}$ for all d; we will first show that $Z(E)(m) \in \mathbb{H}_\epsilon$ for $m \gg 0$.

Let k be as in the lemma, and $d = \dim\operatorname{supp} H^{-k}(E)$. Since all other cohomology sheaves of E are supported in lower dimension, we have

$$(\mathrm{ch}(E)\cdot U)_{n-d} = (-1)^k \,\mathrm{ch}_{n-d}(H^{-k}(E)).$$

Since ω is ample and $\mathrm{ch}_{n-d}(H^{-k}(E))$ is effective, the intersection product $a := \int_X \omega^d \cdot \mathrm{ch}_{n-d}(H^{-k}(E))$ is positive. Therefore the leading term of $Z(E)(m)$ is $a(-1)^d\rho_d m^d$. Since $a(-1)^d\rho_d \in \mathbb{H}_\epsilon$, the same must hold for $Z(E)(m)$ and large m.

3.3 Dual stability condition

Let ω_X be a local dualizing complex of X, and let

$$\mathbb{D}\colon D^b(X) \to D(X), \quad E \mapsto \mathbf{R}\mathcal{H}om(E, \omega_X)$$

be the associated dualizing functor. Let D be such that $\omega_X|_{X^{\text{smooth}}}$ is the shift of a line bundle by D.

To every polynomial stability condition $(Z_\Omega, \mathcal{P}_\Omega)$ of Theorem 3.2.2 one can explicitly construct a stability condition dual to $(Z, \mathcal{P})$ under $\mathbb{D}$. In the case where X is not smooth, this will be a stability condition on $\mathbb{D}(D^b(X))$ rather than $D^b(X)$; however, its associated heart is still given by a category of perverse coherent sheaves as described earlier.

Let $P\colon A_*(X) \to A_*(X)$ be the parity operator acting by $(-1)^{n-d}$ on $A_d(X)$. Given the data $\Omega = (\omega, \rho, p, U)$ as in Theorem 3.2.2, we define the dual data $\Omega^* = (\omega, \rho^*, \bar{p}, U^*)$ by $\rho_d^* = (-1)^{D+d}\overline{\rho_d}$, $U^* = (-1)^D\,\mathrm{ch}(\omega_X)^{-1}\cdot P(\bar{U})$. Consider the central charge $Z_{\Omega^*}\colon K(X) \to \mathbb{C}[m]$ defined by the same formula as Z_Ω in Theorem 3.2.2.

Proposition 3.3.1 *The central charge Z_{Ω^*} induces a polynomial stability function on $\mathbb{D}(\mathcal{A}^p)$. The induced polynomial stability condition $(Z_{\Omega^*}, \mathcal{P}_{\Omega^*})$ is dual to $(Z_\Omega, \mathcal{P}_\Omega)$ in the following sense:*

(a) *An object E is $(Z_\Omega, \mathcal{P}_\Omega)$–stable if and only if $\mathbb{D}(E)$ is $(Z_{\Omega^*}, \mathcal{P}_{\Omega^*})$–stable.*

(b) *If E, F are $(Z_\Omega, \mathcal{P}_\Omega)$–stable, then*

$$\phi(E) \succ \phi(F) \quad \Leftrightarrow \quad \phi(\mathbb{D}(E)) \prec \phi(\mathbb{D}(F)).$$

(c) *The Harder–Narasimhan filtration of $\mathbb{D}(E)$ with respect to $(Z_{\Omega^*}, \mathcal{P}_{\Omega^*})$ is obtained from that of E with respect to $(Z_\Omega, \mathcal{P}_\Omega)$ by dualization.*

By the uniqueness of HN filtrations, (a) and (b) imply (c). The proof of (a) and (b) will also be postponed until Section 7.

4 The large volume limit

Fix $\beta \in A^1(X)_{\mathbb{R}}$ and an ample class $\omega_0 \in A^1(X)_{\mathbb{R}}$. Let $\rho_d = -(-i)^d/d!$ and let

$$U = e^{-\beta} \cdot \sqrt{\mathrm{td}(X)}.$$

Then $p(d) = -\lfloor d/2 \rfloor$ is a perversity function associated to $\rho = (\rho_0, \dots, \rho_n)$, and the central charge $Z = Z_\Omega$ of Theorem 3.2.2 for $\Omega = (\omega, \rho, p, U)$ is given by

$$Z(E)(m) = -\int_X e^{-\beta - im\omega} \cdot \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$$

This is the central charge $Z_{\beta,m\omega}$ discussed in Section 1.6 as the central charge at the large-volume limit.

This stability condition has many of the properties predicted by physicists for the large volume limit. For example, both skyscraper sheaves of points and μ–stable vector bundles are stable; the prediction that their phases differ by $n/2$ is reflected by $\phi(\mathcal{O}_x)(+\infty) = 1$ and $\phi(\mathcal{E})(+\infty) = 1 - n/2$.

It may be worth mentioning that even for vector bundles and $\beta = 0$, stability at the large volume limit does not coincide with Gieseker-stability. Both stability conditions are refinements of slope stability, but they are different refinements.

If X is a smooth Calabi–Yau variety, and if $2\beta = c_1(\mathcal{L})$ is the first Chern class of a line bundle $\mathcal{L}$, then the stability condition is self-dual in the sense of Proposition 3.3.1, with respect to $\mathcal{L}^{-1}[n]$ as dualizing complex.

Now consider the case of a smooth projective surface. Then $\mathcal{A}^p$ is the category of two-term complexes E with $H^{-1}(E)$ being torsion-free, and $H^0(E)$ being a torsion sheaf.

In the case of a K3 surface, the ample chamber is described completely by [10, Proposition 10.3]; and for an arbitrary smooth projective surface, the stability condition constructed in [1, Section 2] are also part of the ample chamber. The following proposition gives a precise meaning to the catch phrase "polynomial stability conditions at the large volume limit are limits of Bridgeland stability conditions in the ample chamber":

Proposition 4.1 *Let S be a surface, $\beta \in A^1(X)_{\mathbb{R}}$ be a divisor class, $\omega \in A^1(X)_{\mathbb{Q}}$ a* rational *ample class, and let ρ, p be as above. Consider either of the following situations:*

(a) *S is a K3 surface; let $(Z_m, \mathcal{P}_m)$ be the stability condition constructed in [10] from β and $\omega = n \cdot \omega_0$ (assuming $\omega^2 > 2$), and let $(Z, \mathcal{P})$ be the polynomial stability condition constructed from the data Ω.*

(b) *S is a smooth projective surface; let $(Z_m, \mathcal{P}_m)$ be the stability condition constructed in [1] from β and $\omega = n \cdot \omega_0$, and let $(Z, \mathcal{P})$ be the polynomial stability condition constructed from $\Omega' = (\omega, \rho, p, U = e^{-\beta})$.*

Then $E \in D^b(S)$ is $(Z_m, \mathcal{P}_m)$–stable for $m \gg 0$ if and only if it is $(Z, \mathcal{P})$–stable. If $E \in D^b(S)$ is an arbitrary object, then the HN-filtration of E with respect to $(Z, \mathcal{P})$ is identical to the HN-filtration with respect to $(Z_m, \mathcal{P}_m)$ for $m \gg 0$.

In either case, the stability function is of the form

$$Z(E)(m) = \mathrm{ch}_0(E)\omega^2 \cdot \frac{m^2}{2} + i\big(\omega\, \mathrm{ch}_1(E) - \mathrm{ch}_0(E)\beta\omega\big)m + c(E) \tag{5}$$

for some real constant $c(E)$. Let $\mu_\omega = \mathrm{ch}_1(E) \cdot \omega / \mathrm{ch}_0(E)$ be the slope function for torsion-free sheaves on S defined by ω.

Lemma 4.2 *Let $E \in D^b(S)$ be a $(Z, \mathcal{P})$–semistable object with $0 \prec \phi(E) \preceq 1$. Then E satisfies one of the following conditions:*

(a) *E is a μ_ω–semistable torsion sheaf.*

(b) *E is a torsion-free μ_ω–semistable sheaf with $\mu_\omega(E) > \beta \cdot \omega$.*

(c) *$H^{-1}(E)$ is torsion-free μ_ω–semistable sheaf of slope $\mu_\omega(H^{-1}(E)) \leq \beta \cdot \omega$, $H^0(E)$ is zero-dimensional, and all other cohomology sheaves vanish.*

Proof Note that such an E satisfies $E \in \mathcal{A}^p$ or $E \in \mathcal{A}^p[-1]$, as $\mathcal{A}^p = \mathcal{P}((\frac{1}{4}, \frac{5}{4}])$.

If $E \in \mathcal{A}^p[-1]$, then $H^1(E)$ is a torsion sheaf by the definition of $\mathcal{A}^p$. In fact, $H^1(E)$ has to vanish: otherwise $\phi(H^1(E)) \preceq 1$, and because of $\phi(E[1]) = \phi(E) + 1 \succ 1$ the surjection $E[1] \twoheadrightarrow H^1(E)$ would destabilize $E[1]$ in $\mathcal{A}^p$. Hence E is a torsion-free sheaf. Further, E must be μ_ω–semistable: for any surjection $E \twoheadrightarrow B$ with B torsion-free and $\mu_\omega(E) > \mu_\omega(B)$, the surjection that $E[1] \twoheadrightarrow B[1]$ would destabilize $E[1]$ in $\mathcal{A}^p$. Since $\phi(E[1]) \succ 1$, we must have $\Im(Z(E)(m) < 0$ for $m \gg 0$; this is equivalent to $\omega\, \mathrm{ch}_1(E[1]) - \mathrm{ch}_0(E[1])\beta\omega < 0$ or $\mu_\omega(E) > \beta \cdot \omega$.

Similarly, one shows that if $E \in \mathcal{A}^p$ and $H^{-1}(E)$ does not vanish, then it is torsion-free and μ_ω–semistable of slope $\mu_\omega(H^{-1}(E)) \leq \beta \cdot \omega$. Also, $H^0(E)$ is of dimension zero: otherwise $\phi(H^0(E))(+\infty) = \frac{1}{2}$, in contradiction to $\phi(E)(+\infty) = 1$ and the surjection $E \twoheadrightarrow H^0(E)$ in $\mathcal{A}^p$.

Finally, if $E \in \mathcal{A}^p$ and $H^{-1}(E)$ vanishes, then E is a torsion sheaf, which is easily seen to be μ_ω–semistable. $\square$

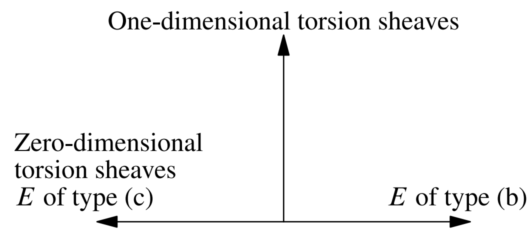


Figure 3: Asymptotic directions of $Z(E)$ for Z–stable objects $E \in \mathcal{A}(\beta, \omega)$

Proof of Proposition 4.1 Let $\mathcal{A} = \mathcal{P}((0,1])$. We first show that $\mathcal{A}$ is identical to the heart $\mathcal{A}(\beta, \omega)$ defined in [10, Lemma 6.1], respectively $\mathcal{A}^\sharp_{(D,F)}$ defined in [1, Section 2]. Recall that $\mathcal{A}(\beta, \omega)$ is characterized as the extension-closed subcategory of $D^b(S)$ generated by torsion sheaves, by μ_ω–semistable sheaves F of slope $\mu_\omega(F) > \beta \cdot \omega$, and by the shifts $F[1]$ of μ_ω–semistable sheaves F of slope $\mu_\omega(F) \leq \beta \cdot \omega$.

Since $\mathcal{A}(\beta, \omega)$ is extension-closed and every E in the above list is an element of $\mathcal{A}(\beta, \omega)$, it follows that $\mathcal{A} \subset \mathcal{A}(\beta, \omega)$. As both categories are the heart of a bounded t–structure, they must be equal.

The first statement of the Proposition thus simplifies to the claim that an object $E \in \mathcal{A}$ is Z–stable if and only if E is Z_m–stable for $m \gg 0$. By definition, we have $\phi(E) \succ \phi(F)$ if and only if $\phi_m(E) = \phi(E)(m) > \phi_m(F) = \phi(F)(m)$ for $m \gg 0$; in particular, if $E \in \mathcal{A}$ is Z–unstable, then it will be Z_m–unstable for $m \gg 0$.

Conversely, assume that E is Z–semistable. In case (a) of the lemma, E is Z_m–stable for all m. We now assume case (c); case (b) can be dealt with similarly. We need to show the following: *Given E, there is a constant M such that whenever $A \hookrightarrow E \twoheadrightarrow B$ is a short exact sequence in $\mathcal{A}$, then $\phi(E)(m) \leq \phi(B)(m)$ for all $m \geq M$.*

If B is a zero-dimensional torsion sheaf, the claim is evidently satisfied. Otherwise write $\mathcal{E} := H^{-1}(E)$, $\mathcal{B} := H^{-1}(B)$; let $\mathcal{F}$ be the image of the induced map $\mathcal{E} \to \mathcal{B}$, and $\mathcal{G}$ the cokernel. The induced map $\mathcal{G} \hookrightarrow H^0(A)$ has zero-dimensional cokernel; hence

$\beta \cdot \omega < \mu_\omega(H^0(A)) = \mu_\omega(\mathcal{G})$. Since $\mathcal{E}$ surjects onto $\mathcal{F}$, we have $\mu_\omega(\mathcal{E}) \le \mu_\omega(\mathcal{F})$. Combined with the definition of $\mathcal{A}(\beta,\omega)$, we obtain

$$\mu_\omega(\mathcal{E}) \le \mu_\omega(\mathcal{B}) \le \omega \cdot \beta. \tag{6}$$

Since $Z(B)(m)$ and $Z(E)(m)$ are in the semiclosed upper half plane $\mathbb{H}$ for all m, the assertion is equivalent to $\Im\big(\overline{Z(E)(m)}Z(B)(m)\big) \le 0$. Using Equation (5) with $\mathrm{ch}_0(E) = -\mathrm{rk}(\mathcal{E})$ and $\mathrm{ch}_1(E) = -\mathrm{ch}_1(\mathcal{E})$ etc., this can be simplified to:

$$\frac{\omega^2 m^2}{2}\big(\mu_\omega(\mathcal{B}) - \mu_\omega(\mathcal{E})\big) \ge \frac{c(B)}{\mathrm{rk}(\mathcal{B})}\big(\beta\omega - \mu_\omega(\mathcal{E})\big) - \frac{c(E)}{\mathrm{rk}(\mathcal{E})}\big(\beta\omega - \mu_\omega(\mathcal{B})\big)$$

By inequality (6), all the expressions in parentheses are nonnegative. Since E is Z–semistable, the inequality is satisfied for $m \gg 0$; in particular, in the case $\mu_\omega(\mathcal{B}) = \mu_\omega(\mathcal{E})$ it holds for all m. Excluding this case, the claim follows if we can bound $\mu_\omega(\mathcal{B}) - \mu_\omega(\mathcal{E})$ from below by a positive constant and $c(B)/\mathrm{rk}(\mathcal{B})$ from above.

If $\mathcal{G} = 0$, then the rank of $\mathcal{B}$ is bounded above. By the rationality of ω, the set of possible values of $\omega \cdot \mathrm{ch}_1(\mathcal{B})$ is discrete, giving a positive lower bound for $\mu_\omega(\mathcal{B}) - \mu_\omega(\mathcal{E})$. Otherwise, the lower bound follows from $\mu_{\omega(\mathcal{G})} > \beta \cdot \omega$ and the upper bound on the rank of $\mathcal{F}$.

To prove the upper bound of $c(B)/\mathrm{rk}(\mathcal{B})$, we restrict to the case (b) of the proposition. Case (a) can be proved similarly (and similarly to the proof of [10, Proposition 14.2]); the argument is similar to the proof of the existence of stability conditions in [1, Section 2].

It is sufficient to bound the number $c(B_j)/\mathrm{rk}(\mathcal{B}_j)$ for every HN filtration quotient B_j of B with respect to Z, and $\mathcal{B}_j = H^{-1}(B_j)$. Then $\mathcal{B}_j$ is μ_ω–semistable, and its slope still satisfies the inequality

$$\mu_\omega(\mathcal{E}) \le \mu_\omega(\mathcal{B}_j) \le \beta \cdot \omega. \tag{7}$$

Using the Bogomolov–Gieseker inequality for $\mathrm{ch}_2(\mathcal{B}_j)$, we get:

$$\begin{aligned} c(B_j) &= -e^{-\beta} \cdot \mathrm{ch}(B_j) = \mathrm{ch}_2(\mathcal{B}_j) - \mathrm{ch}_2(H^0(B_j)) - \beta \cdot \mathrm{ch}_1(\mathcal{B}_j) + \mathrm{rk}(\mathcal{B}_j)\frac{\beta^2}{2} \\ &\le \frac{\mathrm{ch}_1(\mathcal{B}_j)^2}{2 \cdot \mathrm{rk}(\mathcal{B}_j)} - \beta \cdot \mathrm{ch}_1(\mathcal{B}_j) + \mathrm{rk}(\mathcal{B}_j)\frac{\beta^2}{2}, \\ \frac{c(B_j)}{\mathrm{rk}(\mathcal{B}_j)} &\le \frac{1}{2}\left(\frac{\mathrm{ch}_1(\mathcal{B}_j)}{\mathrm{rk}(\mathcal{B}_j)} - \beta\right)^2 \end{aligned}$$

Due to inequality (7) and the Hodge index theorem, this number is bounded from above.

It remains to show the statement about the Harder–Narasimhan filtrations. It is enough to show this for $E \in \mathcal{A}$, as we already showed $\mathcal{P}((0,1]) = \mathcal{A} = \mathcal{P}_m((0,1])$. Let $A_1, A_2, \dots, A_n$ be the Harder–Narasimhan filtration quotients of E with respect to $\mathcal{P}$. Then the claim follows if m is big enough such that every A_j is Z_m–semistable, and such that $\phi(A_1)(m) > \phi(A_2)(m) > \cdots > \phi(A_n)(m)$. □

5 X as the moduli space of stable point-like objects

As a toy example of moduli problems in the derived category we will show that in the smooth case, the moduli space of stable point-like objects is given by X itself. It shows that all our polynomial stability conditions are "ample" in the sense of the ample chamber in the introduction.

Given a polynomial stability condition $(Z, \mathcal{P})$ on X, a family of stable objects over S is an object $E \in D^b(X \times S)$ such that for every closed point $s \in S$, the object $\mathbb{L}i_s^* E \in D^b(X)$ is $(Z, \mathcal{P})$–stable. Since $\mathrm{Ext}^{<0}(E, E) = 0$ for any stable object, it is known that the moduli problem of stable objects is an abstract stack (see Lieblich [23, Proposition 2.1.10] for a precise statement and references). However, in general it is not known whether this stack is an algebraic Artin stack; see Toda [35] for a proof in a large class of examples.

Let c be a class in the numerical K–group. By some abuse of notation, we denote by $M_c(Z, \mathcal{P})$ the substack of $(Z, \mathcal{P})$–stable objects such that $\mathbb{L}i_s^* E$ is an element of $\mathcal{A}^p$ and of class c.

Proposition 5.1 *Assume that X is a smooth projective variety over $\mathbb{C}$. Let $(Z, \mathcal{P})$ be any of the polynomial stability conditions constructed in Theorem 3.2.2 that has $p(0) = 0$. The moduli stack $M_{[\mathcal{O}_x]}(Z, \mathcal{P})$ of stable objects of the class of a point is isomorphic to the trivial $\mathbb{C}^*$–gerbe $X/\mathbb{C}^*$ over X.*

The assumption ensures that every skyscraper sheaf $\mathcal{O}_x$ is an objects of $\mathcal{A}^p$ (otherwise the same would be true after a shift, and we might have to replace $[\mathcal{O}_x]$ by $-[\mathcal{O}_x]$ in the proposition).

Proof If $A \hookrightarrow \mathcal{O}_x \twoheadrightarrow B$ is a short exact sequence in $\mathcal{A}^p$, then the long exact cohomology sequence combined with Lemma 3.2.3 shows that $H^k(A) = 0 = H^k(B)$ for $k \neq 0$, and so $A \cong \mathcal{O}_x$ or $B \cong \mathcal{O}_x$. Hence every $\mathcal{O}_x$ is stable.

Conversely, let $E \in \mathcal{A}^p$ be any object with $[E] = [\mathcal{O}_x]$. From Lemma 3.2.3 it follows that $H^k(E) = 0$ for $k \neq 0$, and hence $E \cong \mathcal{O}_x$ for some $x \in X$.

Hence the map $X \to M_{[\mathcal{O}_x]}(Z, \mathcal{P})$ given by the structure sheaf of the diagonal in $X \times X$ is bijective on closed points. By the deformation theory of complexes (see Lieblich [23, Section 3] or Inaba [18]) and $T_x X \cong \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$, it induces an isomorphism on tangent spaces. Since X is smooth, the map is surjective. □

6 Wall-crossings: The PT/DT–correspondence and 1–dimensional torsion sheaves

In [30], Pandharipande and Thomas introduced new invariants of smooth projective threefolds. They are obtained from moduli spaces of stable pairs constructed by Le Potier in [22]; in their context, a stable pair is a section $s\colon \mathcal{O}_X \to \mathcal{F}$ of a pure one-dimensional sheaf $\mathcal{F}$ that generically generates $\mathcal{F}$.

In the Calabi–Yau case, the authors conjecture that the generating function of stable pairs invariants equals the reduced generating function of Donaldson–Thomas invariants introduced in [27]. A heuristic justification of the conjecture was given in [30, Section 3.3] by interpreting the formula as a wall-crossing formula under a change of Bridgeland stability conditions, assuming the existence of certain stability conditions.

With Proposition 6.1.1, we will show that this wall-crossing can actually be achieved in a family of polynomial stability conditions, thus making the heuristic justification one step more rigorous.

Further, in the subsequent article [31], the authors give a new geometric definition of BPS state counts. It relies on a relation between invariants of stable pairs and invariants of one-dimensional torsion sheaves (see [31, Proposition 2.2]). In Section 6.2, we show that this relation can similarly be interpreted as a wall-crossing in our family of polynomial stability condition; in fact the wall-crossing formula is much simpler than in the case of the PT/DT–correspondence.

We refer to [36] for a similar use of a wall-crossing to relate (differently defined) BPS state counts on birational Calabi–Yau threefolds.

6.1 PT/DT–correspondence

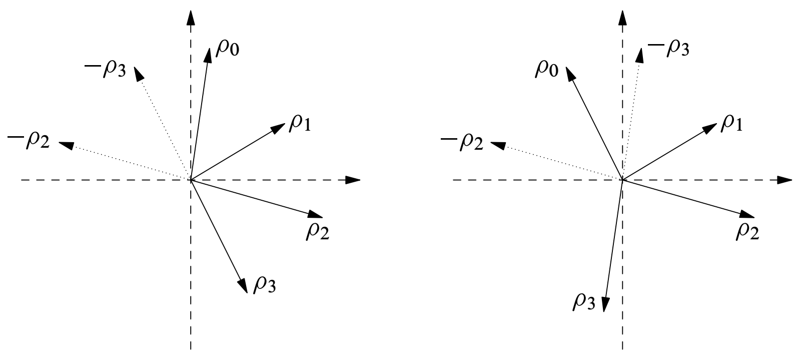
Let X be a smooth complex threefold. Fix an ample class $\omega \in A^1_{\mathbb{R}}$, and let p be the perversity function $p(d) = -\lfloor \frac{d}{2} \rfloor$. Then the category of perverse coherent sheaves $\mathcal{A}^p$ can be described explicitly: a complex $E \in D^b(X)$ is an element of $\mathcal{A}^p$ if

- $H^i(E) = 0$ for $i \neq 0, -1$,
- $H^0(E)$ is supported in dimension ≤ 1, and
- $H^{-1}(E)$ has no torsion in dimension ≤ 1.

Consider stability vectors ρ such that p is an associated perversity function, ie $\rho_0, \rho_1 \in \mathbb{H}$ and $\rho_2, \rho_3 \in -\mathbb{H}$. Let U be arbitrary, and consider the polynomial stability functions given by

$$Z(E)(m) = \sum_{d=0}^{3} \rho_d m^d \omega^d \cdot \mathrm{ch}(E) \cdot U.$$

We further assume $\phi(-\rho_3) > \phi(\rho_1)$. We call it a *DT–stability function* if $\phi(-\rho_3) > \phi(\rho_0)$ and a *PT–stability function* if $\phi(\rho_0) > \phi(-\rho_3)$; see Figure 4.



(a) DT–stability (b) PT–stability

Figure 4: PT/DT wall-crossing

If ω is the class of an ample line bundle $\mathcal{L}$, $U = \mathrm{td}\, X$ and $h(E)(m) = \sum_{d=0}^{3} a_d m^d$ is the Hilbert polynomial of E with respect to $\mathcal{L}$, then the central charges can also be written as the complexified Hilbert polynomial $Z(E)(m) = \sum_{d=0}^{3} d! \rho_d a_d m^d$.

Fix numerical invariants $\beta \in A_1^{\mathrm{num}}$ and $n \in A_0 \cong \mathbb{Z}$. We consider the moduli problem $M^0_{(-1,0,\beta,n)}(Z, \mathcal{P})$ of Z–stable objects in $\mathcal{A}^p$ with trivialized determinant and numerical invariants in A_*^{num} given by $\mathrm{ch}(E) = (-1, 0, \beta, n)$.

Proposition 6.1.1 *Let S be of finite type over $\mathbb{C}$, and $I \in D^b(X \times S)$ be an object with $\mathrm{ch}(I_s) = (-1, 0, \beta, n)$ for every closed point $s \in S$, and with trivialized determinant.*

If Z is a DT–stability function, then I is a Z–stable family of objects in $\mathcal{A}^p$ if and only if it is quasi-isomorphic to the shift $\mathcal{J}[1]$ of a flat family of ideal sheaves of one-dimensional subschemes.

If Z is a PT–stability function and $\beta \neq 0$, then $I \in D^b(X \times S)$ is a Z–stable family of objects in $\mathcal{A}^p$ if and only if it is quasi-isomorphic to the complex $\mathcal{O}_{X \times S} \to \mathcal{F}$ (with $\mathcal{F}$ in degree zero) of a family of stable pairs as defined in [22; 30].

Thus in both cases we get an isomorphism of moduli spaces of ideal sheaves/stable pairs with the moduli space $M^0_{(-1,0,\beta,n)}(Z,\mathcal{P})$ of Z–stable objects with trivialized determinant.

If $\beta = 0$ and Z is a PT–stability function, then the only semistable object is $\mathcal{O}_X[1]$ of class $(-1,0,0,0)$. This does not agree with the definition of stable pairs, but does give the correct generating function, so that the conjectured wall-crossing formula of [30] holds for all β.

Proof Let Z be a DT–stability function, and let assume that I is a family of stable objects. If for any closed point $s \in S$, we would have both $H^{-1}(I_s) \neq 0$ and $H^0(I_s) \neq 0$, then the short exact sequence

$$H^{-1}(I_s)[1] \to I_s \to H^0(I_s)$$

would destabilize I_s: for large m, the phase of $Z(H^{-1}(I_s)[1])(m)$ is approaching $\phi(-\rho_2)$ or $\phi(-\rho_3)$ (depending on the dimension of support of I_s); while the phase of $Z(H^0(I_s))(m)$ is approaching $\phi(\rho_1)$ or $\phi(\rho_0)$. Hence I is the shift of a flat family $\mathcal{J}$ of sheaves of rank one.[4] To be both stable and an element of $\mathcal{A}^p$, it has to be torsion-free. Its double dual is locally free by [21, Lemma 6.13]. Since $\mathcal{J}$ has trivialized determinant, the double dual $\mathcal{J}^{**}$ is the structure sheaf $\mathcal{O}_{X\times S}$; the natural inclusion $\mathcal{J} \hookrightarrow \mathcal{J}^{**}$ exhibits $\mathcal{J}$ as a flat family of ideal sheaves. Conversely, any such flat family of ideal sheaves gives a family of stable objects in $\mathcal{A}^p$.

(In fact, the DT–stability conditions are obtained from the stability conditions of Section 2.1 corresponding to Simpson stability by a rotation of the complex plane and accordingly tilting the heart of the t–structure. Hence the stable objects are exactly the shifts of Simpson-stable sheaves; their moduli space is well-known to be isomorphic to the Hilbert scheme.)

Now let Z be a PT–stability function. We have to show that I_s is stable for all s if and only if I is quasi-isomorphic to a family of stable pairs: a complex $\mathcal{O}_{X\times S} \to \mathcal{F}$ such that

(1) $\mathcal{F}$ is flat over S, and

(2) $\mathcal{O}_X \to \mathcal{F}_s$ is a stable pair for all $s \in S$.

[4]Here, and again later in the proof of the PT-case, we are using the following standard fact (cf [16, Lemma 3.31]): If I is a complex on $X \times S$ such that for every closed point $s \in S$, the derived pullback I_s is a sheaf, then I is a sheaf, flat over S.

Given such a family of stable pairs, the associated complex is a family of objects in $\mathcal{A}^p$ with trivialized determinant.

First assume that I is Z–stable. By the same argument as in the DT-case, $H^0(I_s)$ must be zero-dimensional, and $H^{-1}(I_s)[1]$ torsion-free of rank one with trivialized determinant.

It follows that $Q := H^0(I)$ is zero-dimensional over S, and that $H^{-1}(I)$ is a torsion-free rank one sheaf with trivialized determinant. Let $U \subset X \times S$ be the complement of the support of Q, and let $I_U := I|_U[-1]$ be the restriction of $I[-1]$ to U. Then the derived pullback of I_U to every fiber over $s \in S$ is a sheaf; so $I_U \cong H^{-1}(I)|_U$ is itself a sheaf, flat over S. Hence $H^{-1}(I)$ is flat over S outside a set of codimension 3.

By the same arguments as in the proof of [30, Theorem 2.7] it follows that $H^{-1}(I)$ is a family of ideal sheaves $\mathcal{I}_Z$ of one-dimensional subschemes of X. The complex I is the cone of a map $Q \to \mathcal{I}_Z[2]$. Since Q is zero-dimensional over S, we have $\operatorname{Ext}^1(Q, \mathcal{O}_{X\times S}) = 0 = \operatorname{Ext}^2(Q, \mathcal{O}_{X\times S})$; combined with the short exact sequence $\mathcal{I}_Z \to \mathcal{O}_{X\times S} \to \mathcal{O}_Z$, we get a unique factorization $Q \to \mathcal{O}_Z[1] \to \mathcal{I}_Z[2]$. Using the octahedral axiom associated to this composition, we see that I is the cone of a map $\mathcal{O}_{X\times S} \to \mathcal{F}$, where $\mathcal{F}$ (in degree zero) is the extension of $\mathcal{O}_Z$ and Q given as the cone of the map $Q \to \mathcal{O}_Z[1]$ above.

It remains to prove that I_s is Z–stable if and only if $\mathcal{O}_X \to \mathcal{F}_s$ is a stable pair. Assume that I_s is Z–stable, and note that $\phi(I_s)(+\infty) = \phi(-\rho_3)$.

Since $\beta \neq 0$, the sheaf $\mathcal{F}_s$ is one-dimensional. It cannot have a zero-dimensional subsheaf $Q \hookrightarrow \mathcal{F}_s$, as this would induce an inclusion $Q \hookrightarrow I_s$ in $\mathcal{A}^p$, destabilizing I_s due to $\phi(Q) = \phi(\rho_0) > \phi(-\rho_3)$. Thus $\mathcal{F}_s$ is purely one-dimensional, and $\mathcal{O}_X \to \mathcal{F}_s$ is stable by [30, Lemma 1.3].

Conversely, assume that I_s is a stable pair. Consider any destabilizing short exact sequence $A \hookrightarrow I_s \twoheadrightarrow B$ in $\mathcal{A}^p$ with $\phi(A) \succ \phi(I_s) \succ \phi(B)$, and its long exact cohomology sequence

$$H^{-1}(A) \hookrightarrow H^{-1}(I_s) \to H^{-1}(B) \to H^0(A) \to H^0(I_s) \twoheadrightarrow H^0(B).$$

If $H^{-1}(A) \hookrightarrow H^{-1}(I_s)$ is a proper inclusion, then $H^{-1}(B)$ is supported in dimension 2, and we get the contradiction $\phi(B)(+\infty) = \phi(-\rho_2) > \phi(I_s)(+\infty)$. So either $H^{-1}(A) = H^{-1}(I_s)$ or $H^{-1}(A) = 0$. In the former case, $H^{-1}(B) = 0$; since $B = H^0(B)$ is supported in dimension zero, we get the contradiction $\phi(B) = \phi(\rho_0) > \phi(I_s)(+\infty)$. In the latter case, $A = H^0(A)$ must be zero-dimensional to destabilize I_s; by the purity of $\mathcal{F}_s$, this implies $\operatorname{Hom}(A, \mathcal{F}_s) = 0$. Together with $\operatorname{Ext}^1(A, \mathcal{O}_X) = 0$ and the exact triangle $\mathcal{F}_s \to I_s \to \mathcal{O}_X[1]$, this shows the vanishing of $\operatorname{Hom}(A, I_s)$. $\square$

The reason to expect a wall-crossing formula in a situation as above is the following: Denote by Z_{PT} a PT–stability function, and by Z_{DT} a DT–stability function. If E is Z_{PT}–semistable but Z_{DT}–unstable, then we can write E as an extension of Z_{DT}–semistable objects (by the existence of Harder–Narasimhan filtrations); and conversely for Z_{DT}–semistable but Z_{PT}–unstable objects. Hence one can expect an expression for the difference between the counting invariants of Z_{DT}– respectively Z_{PT}–semistable objects in terms of lower degree counting invariants. This observation (due to Joyce [19]) can be made more concrete and precise in the situation considered below.

6.2 Stable pairs and one-dimensional torsion sheaves

Let X be a Calabi–Yau threefold, and β, n as before. By a counting invariant we will always denote the signed weighted Euler characteristic (in the sense of [5]) of a moduli space of stable objects of some fixed numerical class, and with trivialized determinant.

In the very recent preprint [31], the authors give a new geometric description of BPS state counts for irreducible curve classes on X. They use the counting invariants $N_{n,\beta}$ of the moduli spaces $\mathcal{M}_n(X,\beta)$ of stable one-dimensional torsion sheaves of class $(0,0,\beta,n)$. At the core of their argument is the following relation: if β is an irreducible effective class and $P_n(X,\beta)$ denotes the counting invariant of stable pairs of class $(-1,0,\beta,n)$, they prove that

$$P_n(X,\beta) - P_{-n}(X,\beta) = (-1)^{n-1} n N_{n,\beta}. \tag{8}$$

To make the subsequent discussion more specific, we fix $\rho_0 \in \mathbb{R}_{>0}\cdot(-1)$, $\rho_1 \in \mathbb{R}_{>0}\cdot i$, $\rho_2 \in \mathbb{R}_{>0}$. We keep ω, p, and in particular continue to work with the same category of perverse coherent sheaves $\mathcal{A}^p$. Assume that $P(\bar{U}) = U$, eg $U \in A^{\mathrm{even}}(X)_{\mathbb{R}}$. For $a > 0$ write Z_a for the polynomial stability function on $\mathcal{A}^p$ obtained from $\rho_3 = -b\cdot i + a$ (for some $b > 0$), and similarly Z_{-a} for $\rho_3 = -b\cdot i - a$; see also Figure 5.

Then Z_a is a "PT–stability function" (in the terminology of the previous section), hence the stable objects of class $(-1,0,\beta,n)$ are the stable pairs $I \cong \mathcal{O}_X \to \mathcal{F}$. If we cross the wall $a = 0$ (the large volume limit), then short exact sequence $\mathcal{F} \to I \to \mathcal{O}_X[1]$ destabilizes I for $a < 0$.

Let $\mathbb{D}$ be the dualizing functor $E \mapsto \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X[2])$. Then the polynomial stability condition obtained from Z_a is dual to that of Z_{-a}; this can be seem from Proposition 3.3.1 and the fact that in our case $\mathcal{A}^p$ is a tilt of $\mathcal{A}^{p^*}$, compatible with the stability condition.

It follows that if $a < 0$, then the stable objects of same class are the derived duals of stable pairs of class $(-1,0,\beta,-n)$; their counting invariant is thus given by $P_{-n}(X,\beta)$.

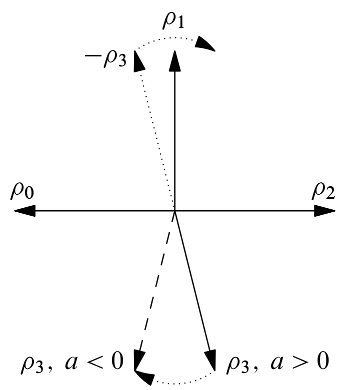


Figure 5: Wall-crossing between PT invariants and BPS state counts

If we additionally assume that β is irreducible, then $\mathcal{F}$ is stable for both Z_a and Z_{-a}, and the short exact sequence $\mathcal{F} \to I \to \mathcal{O}_X[1]$ is the HN filtration with respect to Z_{-a} of a stable pair I. Conversely, the dual short exact sequence $\mathcal{O}_X[1] = \mathbb{D}(\mathcal{O}_X[1]) \to \mathbb{D}(I) \to \mathbb{D}(F)$ will be the HN filtration of $\mathbb{D}(I)$ with respect to Z_a (where $\mathbb{D}(F)$ is a stable sheaf of class $(0,0,\beta,n)$). Hence the wall-crossing formula can be written schematically as

$$P_n(X,\beta) - P_{-n}(X,\beta) = \sharp\big(\text{Extensions of } \mathcal{O}_X[1] \text{ with } \mathcal{F}\big) \\ - \sharp\big(\text{Extensions of } \mathcal{F}' \text{ with } \mathcal{O}_X[1]\big),$$

where $\mathcal{F}, \mathcal{F}'$ can be any stable sheaf of class $(0,0,\beta,n)$.

If the dimensions of $\mathrm{Ext}^1(\mathcal{O}_X[1], \mathcal{F}) = H^0(\mathcal{F})$ and $\mathrm{Ext}^1(\mathcal{F}', \mathcal{O}_X[1]) = H^1(\mathcal{F}')^*$ were constant, then the moduli spaces of extensions would be projective bundles over $\mathcal{M}_n(X,\beta)$; in this case, formula (8) would follow immediately. Without this simplifying assumption, one can still hope to prove formulas such as (8) using a stratification of the moduli spaces and the formalism of [5]. In fact, the proof in [31] exactly follows this general principle, the key ingredient being a control of the constructible functions of [5] by [31, Theorem 3].

7 Existence of Harder–Narasimhan filtrations

In this section we will prove that the category of perverse coherent sheaves has the Harder–Narasimhan property for the polynomial stability function Z defined in Theorem 3.2.2. The proof is complicated by the fact that $\mathcal{A}^p$ is in general neither Z–Artinian nor Z–Noetherian.

7.1 Perverse coherent sheaves and tilting

Essential for the proof is a more detailed understanding of the category of perverse coherent sheaves, more precisely the existence of certain torsion pairs in that category. We recall briefly the notion of a torsion pair and a tilt of a t–structure:

Definition 7.1.1 A torsion pair in an abelian category $\mathcal{A}$ is a pair of full subcategories $\mathcal{T}, \mathcal{F}$ such that

(a) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and

(b) for every $E \in \mathcal{A}$ there is a short exact sequence $T \hookrightarrow E \twoheadrightarrow F$ in $\mathcal{A}$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ satisfy both conditions, then T, F are uniquely determined by E. They depend functorially on E, and the functors $E \mapsto T$ and $E \mapsto F$ are left-exact and right-exact, respectively.

Now assume $\mathcal{A}$ is the heart of a bounded t–structure in a triangulated category $\mathcal{D}$, with associated cohomology functors $H^i_{\mathcal{A}}\colon \mathcal{D} \to \mathcal{A}$. Given a torsion pair $\mathcal{T}, \mathcal{F}$ in $\mathcal{A}$, then the following defines the heart $\mathcal{A}^\sharp$ of a related t–structure (called the tilt of $\mathcal{A}$; see Happel, Reiten and Smalø [15]): An object A is in $\mathcal{A}^\sharp$ if

$$H^0_{\mathcal{A}}(A) \in \mathcal{T}, \quad H^{-1}_{\mathcal{A}}(A) \in \mathcal{F} \quad \text{and} \quad H^i_{\mathcal{A}}(A) = 0 \quad \text{if } i \neq 0, -1.$$

The new heart $\mathcal{A}^\sharp$ evidently satisfies $\mathcal{A}^\sharp \subset \langle \mathcal{A}, \mathcal{A}[1] \rangle$, and on the other hand every heart of a bounded t–structure with this property is obtained as a tilt. This is shown by the following lemma, which is a slight reformulation of a lemma in [32]:

Lemma 7.1.2 *Let $\mathcal{A}, \mathcal{A}^\sharp$ be the hearts of bounded t–structures in a triangulated category $\mathcal{D}$. If they satisfy $\mathcal{A}^\sharp \subset \langle \mathcal{A}, \mathcal{A}[1] \rangle$ (or, equivalently, $\mathcal{A} \subset \langle \mathcal{A}^\sharp, \mathcal{A}^\sharp[-1] \rangle$), then*

$$\mathcal{T} := \mathcal{A} \cap \mathcal{A}^\sharp, \quad \mathcal{F} := \mathcal{A} \cap \mathcal{A}^\sharp[-1]$$

defines a torsion pair in $\mathcal{A}$, the heart $\mathcal{A}^\sharp$ is obtained from $\mathcal{A}$ by tilting at this torsion pair, and $\mathcal{F}[1], \mathcal{T}$ is a torsion pair in $\mathcal{A}^\sharp$.

Proof If $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ and $(\mathcal{D}^{\sharp, \geq 0}, \mathcal{D}^{\sharp, \leq 0})$ are the two t–structures, either assumption is equivalent to either of the following equivalent assumptions:

$$\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\sharp, \geq 0} \subset \mathcal{D}^{\geq -1} \quad \text{or} \quad \mathcal{D}^{\leq 0} \supset \mathcal{D}^{\sharp, \leq 0} \supset \mathcal{D}^{\leq -1}$$

This is the assumption of [32, Lemma 1.1.2]. $\square$

Now consider a perversity function p and any $k \in \mathbb{Z}$ such that $k = -p(d)$ for some $0 \le d \le n$. Consider the function $p^k : \{0, \dots, n\}$ defined by

$$p^k(d) = \begin{cases} p(d) & \text{if } p(d) \ge -k, \\ p(d) + 1 & \text{if } p(d) < -k. \end{cases}$$

Then $p^k(d)$ is a perversity function, and the hearts of perverse coherent sheaves $\mathcal{A}^{p^k}, \mathcal{A}^p$ satisfy the assumptions of the lemma. Hence

$$\mathcal{F}_k = \mathcal{A}^p \cap \mathcal{A}^{p^k}, \quad \mathcal{T}_k = \mathcal{A}^p \cap \mathcal{A}^{p^k}[1]$$

defines a torsion pair in $\mathcal{A}^p$.

From the definition of the t–structures in Theorem 3.1.2, and from Lemma 3.2.3, it can easily be seen that the torsion pairs can be described as below:

Proposition 7.1.3 *Let p be a perversity function and $k \in \mathbb{Z}$ such that $p(0) \ge -k > p(n)$. There is a torsion pair $(\mathcal{T}_k, \mathcal{F}_k)$ in $\mathcal{A}^p$ defined as follows:*

$$\mathcal{F}_k = \left\{E \in \mathcal{A}^p \,\middle|\, H^{-k'}(E) = 0 \quad \text{for } k' > k\right\}$$
$$\mathcal{T}_k = \left\{E \in \mathcal{A}^p \,\middle|\, H^{-k'}(E) \in \mathcal{A}^{p, \le k'-1} \quad \text{for } k' \le k\right\}$$

The subcategory $\mathcal{F}_k$ is closed under subobjects and quotients.

The only thing left to prove is the statement about $\mathcal{F}_k$. It is always the case for a torsion pair that $\mathcal{F}$ is closed under subobjects and $\mathcal{T}$ under quotients. That $\mathcal{F}_k$ is additionally closed under quotients follows easily from the long exact cohomology sequence.

We denote by $\tau^k_{\mathcal{T}}\colon \mathcal{A}^p \to \mathcal{T}_k$ and $\tau^k_{\mathcal{F}}\colon \mathcal{A}^p \to \mathcal{F}_k$ the associated functors; then for any short exact sequence $A \hookrightarrow E \twoheadrightarrow B$ in $\mathcal{A}^p$ there is a (not very) long exact sequence

$$\tau^k_{\mathcal{T}} A \hookrightarrow \tau^k_{\mathcal{T}} E \to \tau^k_{\mathcal{T}} B \to \tau^k_{\mathcal{F}} A \to \tau^k_{\mathcal{F}} E \twoheadrightarrow \tau^k_{\mathcal{F}} B. \tag{9}$$

Note that $\tau^k_{\mathcal{F}}$ will in general not coincide with the truncation functor $\tau_{\ge -k}$ of the standard t–structure; in fact, given $E \in \mathcal{A}^p$ there is no reason why $\tau_{\ge -k}(E)$ should also be an object of $\mathcal{A}^p$.

7.2 Dual stability condition

The proof in the following section is substantially simplified by the use of the dual stability condition constructed in Proposition 3.3.1. To use it, we need a partial proof of the duality here.

It is constructed from the dual t–structure. Let ω_X, D, $\mathbb{D}$ be as in Section 3.3.

Proposition 7.2.1 [6] *Let p be a perversity function, and $\overline{p}$ the dual perversity function (cf Definition 3.1.1); let $p^* = \overline{p} + D - n$ be the dual perversity normalized according to the choice of ω_X. Define $D^{p^*,\geq 0}, D^{p^*,\leq 0} \subset \mathbb{D}(D^b(X))$ by the analogues of Equations (3) and (4), respectively. Then the t–structures associated to p, p^* are dual to each other with respect to $\mathbb{D}$:*

$$\mathbb{D}(D^{p,\leq 0}) = D^{p^*,\geq 0} \quad \text{and} \quad \mathbb{D}(D^{p,\geq 0}) = D^{p^*,\leq 0}$$

By abuse of notation, we write $\mathcal{A}^{p^*}$ for the intersection $D^{p^*,\geq 0} \cap D^{p^*,\leq 0} \subset \mathbb{D}(D^b(X))$.

Lemma 7.2.2 *Given Ω and Ω^* as in Proposition 3.3.1, Z_{Ω^*} is a polynomial stability function for the category of perverse sheaves $\mathcal{A}^{p^*}$ of the dual perversity. If ϕ, ϕ^* are the polynomial phase functions of $\mathcal{A}^p, Z_\Omega$ and $\mathcal{A}^{p^*}, Z_{\Omega^*}$, respectively, then*

$$\phi(E_1) \prec \phi(E_2) \quad \Leftrightarrow \quad \phi(\mathbb{D}(E_2)) \prec \phi(\mathbb{D}(E_1)). \tag{10}$$

An object $E \in \mathcal{A}^p$ is Z_Ω–stable if and only if $\mathbb{D}(E) \in \mathcal{A}^{p^*}$ is Z_{Ω^*}–stable.

Proof Since $\operatorname{ch}(\mathbb{D}(E)) = P(\operatorname{ch}(E)) \cdot \operatorname{ch}(\omega_X)$, we have

$$\begin{aligned} Z_{\Omega^*}(\mathbb{D}(E)))(m) &= \int_X \sum_{d=0}^{n} (-1)^{d+D} \overline{\rho_d} \omega^d m^d \cdot P(\operatorname{ch}(E)) \operatorname{ch}(\omega_X) \\ &\qquad\qquad\qquad \cdot (-1)^D \operatorname{ch}(\omega_X)^{-1} P(\bar{U}) \\ &= \int_X \sum_{d=0}^{n} m^d P(\overline{\rho_d}\omega^d) P(\operatorname{ch}(E)) = (-1)^n \overline{Z(E)(m)}. \end{aligned}$$

This shows that Z_{Ω^*} is a polynomial stability function, as $Z_{\Omega^*}(\mathbb{D}(E)(m))$ is in the interior of $(-1)^{n+1} e^{-i\epsilon} \cdot \mathbb{H}$ whenever $Z_\Omega(E(m))$ is in the interior of $e^{i\epsilon} \cdot \mathbb{H}$; it also shows the equivalence (10).

Since $\mathbb{D}$ turns inclusions $E_1 \hookrightarrow E_2$ in $\mathcal{A}^p$ into quotients $\mathbb{D}(E_2) \twoheadrightarrow \mathbb{D}(E_1)$ in $\mathcal{A}^{p^*}$, and vice versa, this also implies the claim about stable objects. □

The lemma yields part (a) and (b) of Proposition 3.3.1.

7.3 Induction proof

Lemma 7.3.1 *Consider the quotient category $\mathcal{A}^{p,=k} = \mathcal{A}^{p,\leq k} / \mathcal{A}^{p,\leq k-1} \cong \mathcal{F}_k / \mathcal{F}_{k-1}$ and let $Z' \colon K(\mathcal{A}^{p,=k}) \to \mathbb{C}[m]$ be the restricted stability function defined by*

$$Z'(E)(m) = \int_X \sum_{\substack{d \in \{0,\dots,n\} \\ p(d)=-k}} \rho_d \omega^d m^d \cdot \operatorname{ch}(E) \cdot U.$$

Then $\mathcal{A}^{p,=k}$ is Noetherian and strongly Z'–Artinian.

Here "strongly Z'–Artinian" says that there is no sequence of inclusions

$$\ldots \hookrightarrow E_{j+1} \hookrightarrow E_j \hookrightarrow \ldots \hookrightarrow E_2 \hookrightarrow E_1$$

as in Proposition 2.3.4 with the weaker assumption $\phi(E_{j+1}) \succeq \phi(E_j)$ for all j.

Proof For both statements, the proof is almost identical to the proof of the same statement for $\mathcal{A} = \operatorname{Coh} X$ and Simpson stability. We will prove that the category is strongly Z'–Artinian.

Consider an infinite sequence of inclusions as above. Since the dimension of the support of E_j is decreasing, we may assume it is constant, equal to d. Similarly, we may assume that the lengths of E_j at the generic points of the (finitely many) d–dimensional components of its support are constant. In particular the leading term of $Z'(E_j)(m)$ given by $\omega^d \cdot \mathrm{ch}_{n-d}(E_j)\rho_d \cdot m^d$ is constant. The quotient $B_j = E_j / E_{j+1}$ is supported in strictly smaller dimension $d' < d$. Hence the leading term of $Z'(B_j)(m)$ is a positive linear multiple of $\rho_{d'} m^{d'}$. This implies $\phi(E)(+\infty) = \phi(\rho_d) < \phi(\rho_{d'}) = \phi(B)(+\infty)$, since $p(d') = p(d)$ and p is a perversity function associated to ρ. Thus $\phi(E_j) \prec \phi(B_j)$, in contradiction to $\phi_{E_{j+1}} \succeq \phi_{E_j}$ and the see-saw property. □

We now come to the main proof. As mentioned before, we can't apply Proposition 2.3.4. Nevertheless, our proof follows Bridgeland's proof of the corresponding statement [9, Proposition 5.3] quite closely:

Step 1 Every nonsemistable $E \in \mathcal{A}^p$ has a semistable subobject $A \hookrightarrow E$ such that $\phi(A) \succ \phi(E)$, and a semistable quotient $E \twoheadrightarrow B$ with $\phi(E) \succ \phi(B)$.

Step 2 Every object E has a maximal destabilizing quotient (mdq) $E \twoheadrightarrow B$.

Step 3 Let $E_{j+1} \hookrightarrow E_j \hookrightarrow \ldots \hookrightarrow E_1$ be the sequence of inclusions in $\mathcal{A}^p$ determined by B_j being the mdq of E_j, and E_{j+1} being the kernel of the surjection $E_j \twoheadrightarrow B_j$. Then this sequence terminates.

A mdq is a quotient $E \twoheadrightarrow B$ such that for every other quotient $E \twoheadrightarrow B'$, we have $\phi(B') \succeq \phi(B)$, and such that equality holds if and only if the quotient factors via $E \twoheadrightarrow B \twoheadrightarrow B'$. The proof of [9, Proposition 5.3] shows that the existence of Harder–Narasimhan filtrations is equivalent to the existence of a mdq for every object, and the termination of the sequence defined in Step 3.

Step 1 Define a sequence of inclusions as follows: *If E_j is not semistable, then among all subobjects $A \hookrightarrow E_j$ with $\phi(A) \succ \phi(E_j)$, let E_{j+1} be one such that the dimension of support $d(B_j)$ of B_j is maximal, where B_j is the cokernel of $E_{j+1} \hookrightarrow E_j$.* It suffices to prove that this sequence terminates.

By the definition of E_{j+1}, the sequence $d(B_j)$ of dimension of support is monotone decreasing. By induction, we just need to show that any such sequence with $d(B_j)=d$ for all j must terminate.

Let $k=-p(d)$, and consider the functors $\tau_{\mathcal{T}}^k, \tau_{\mathcal{F}}^k$ of Proposition 7.1.3. Since $B_j \in \mathcal{F}_k$, we have $\tau_{\mathcal{T}}^k(B_j)=0$. By the exact sequence (9), this shows that $\tau_{\mathcal{T}}^k(E_{j+1})=\tau_{\mathcal{T}}^k T(E_j)$ and that

$$0 \to \tau_{\mathcal{F}}^k(E_{j+1}) \to \tau_{\mathcal{F}}^k(E_j) \to B_j \to 0$$

is exact. Taking cohomology, we get an induced short exact sequence

$$0 \to H^{-k}(\tau_{\mathcal{F}}^k(E_{j+1})) \to H^{-k}(\tau_{\mathcal{F}}^k(E_j)) \to H^{-k}(B_j) \to 0$$

in $\mathcal{A}^{p,=k}$. From the lemma it follows that there must be a j_0 with

$$\phi(\tau_{\mathcal{F}}^k(E_{j_0+1})) \prec \phi(\tau_{\mathcal{F}}^k(E_{j_0})) \prec \phi(B_{j_0}).$$

By the see-saw property, $\tau_{\mathcal{T}}^k E_{j_0}$ is another subobject of E_{j_0} with $\phi(\tau_{\mathcal{T}}^k E_{j_0}) \succ \phi(E_{j_0})$. By the definition of E_{j_0+1}, this implies $d(\tau_{\mathcal{F}}^k(E_j)) = d(B_j)$ for $j = j_0$, and thus also for all $j > j_0$, which is impossible.

This shows that every object has a semistable subobject as desired. By applying the same arguments to the dual perversity and dual stability function, this also shows that every object has a semistable quotient as claimed.

Step 2 We will prove Steps 2 and 3 in a two-step induction: To prove Step 2 for an object supported in dimension d, we assume that Steps 2 and 3 have been proven for objects supported in dimension at most $d-1$. To prove Step 3, we will assume that Step 2 has been proven in dimension d, and that Step 3 has been proven in dimension $d-1$. The reason this induction works well is that the subcategory of $\mathcal{A}^p$ of objects supported in dimension at most d is closed under subquotients.

To prove Step 2, we will instead show the dual statement: Every object has a *minimal destabilizing subobject* (mds), ie a subobject $A \hookrightarrow E$ such that for every $A' \hookrightarrow E$ we have $\phi(A) \preceq \phi(A')$, with equality if and only if there is a factorization $A' \hookrightarrow A \hookrightarrow E$.

Let $E_1 \in \mathcal{A}^p$ be supported in dimension d, and let $k=-p(d)$. Define the sequence of objects E_j as follows:

(1) If E_j is semistable, stop.

(2) If there is a semistable quotient $E_j \twoheadrightarrow B_j$ with $\phi(E_j) \succ \phi(B_j)$ and $H^{-k}(B_j) \neq 0$, then let E_{j+1} be its kernel.

(3) Otherwise, let B_j be the maximal destabilizing quotient of $\tau_{\mathcal{F}}^{k-1} E_j$, which exists by induction; and E_{j+1} be the kernel of the composition $E_j \twoheadrightarrow \tau_{\mathcal{F}}^{k-1} E_j \twoheadrightarrow B_j$.

If neither case (1) nor (2) applies, there must be a semistable quotient $E \twoheadrightarrow B$ with $\phi(E) \succ \phi(B)$ and $H^{-k}(B) = 0$. Then the quotient must factor as $E \twoheadrightarrow \tau_{\mathcal{F}}^{k-1} E_j \twoheadrightarrow B$. Then the mdq B_j of $\tau_{\mathcal{F}}^{k-1} E_j$ satisfies $\phi(B) \succ \phi(B_j)$ by definition.

Hence both in case (2) and (3), we have a short exact sequence $E_{j+1} \hookrightarrow E_j \twoheadrightarrow B_j$ with B_j semistable and $\phi(E_j) \succ \phi(B_j)$. By the arguments dual to those given by Bridgeland, a mds of E_{j+1} is also be a mds of E_j, and if E_j is semistable it is its own mds. So we just need to prove that the above algorithm terminates.

By the lemma, case (2) will only happen a finite number of times. However, in case (3) we get a short exact sequence

$$\tau_{\mathcal{F}}^{k-1} E_{j+1} \hookrightarrow \tau_{\mathcal{F}}^{k-1} E_j \twoheadrightarrow B_j,$$

where B_j is the mdq of $\tau_{\mathcal{F}}^{k-1} E_j$. By the induction assumption about Step 3, this sequence must terminate as well.

Finally, note that if E is supported in dimension d, then so is $\mathbb{D}(E)$. Again we can use the same arguments in the dual setting and prove the existence of an mdq for objects supported in dimension d as well.

Step 3 Let $k = -p(\dim E_1)$. Again, by Lemma 7.3.1, the sequence of inclusions $H^{-k}(E_{j+1}) \hookrightarrow H^{-k}(E_j)$ will become an isomorphism in the quotient category $\mathcal{A}^{p,=k}$ after a finite number of steps. Then $H^{-k}(B_j)$ is in $\mathcal{A}^{0,\leq k-1}$; by Lemma 3.2.3 it must be zero. So $B_j \in \mathcal{F}_{k-1}$, and the quotient must factor via $E_j \twoheadrightarrow \tau_{\mathcal{F}}^{k-1} E_j \twoheadrightarrow B_j$. Then B_j must be the mdq of $\tau_{\mathcal{F}}^{k-1} E_j$, and by induction we know that the sequence of inclusions will terminate.

This finishes the proof of Theorem 3.2.2.

8 The space of polynomial stability conditions

In this section, we will describe to what extent Bridgeland's deformation result for stability conditions carries over to polynomial stability conditions. We will first introduce a natural topology on the space of polynomial stability conditions (with respect to which the stability conditions of Theorem 3.2.2 form a "family").

We will also briefly discuss what assumptions are necessary to proof a deformation result comparable to [9, Theorem 1.2].

We will omit most proofs; after having adjusted all necessary definitions, they carry over almost literally from Bridgeland's proofs.

8.1 The topology

We continue with the following translations of definitions of [9] to our situation:

Definition 8.1.1 If the triangulated category $\mathcal{D}$ is linear over a field, a polynomial stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ is called *numerical* if $Z\colon K(\mathcal{D}) \to \mathbb{C}[m]$ factors via $\mathcal{N}(\mathcal{D})$, the numerical Grothendieck group.

Let $\mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$ be the set of stability conditions on $\mathcal{D}$, and $\mathrm{Stab}_{\mathrm{Pol}}{}^{\mathcal{N}}(\mathcal{D})$ the subset of numerical ones.

By a *semimetric* on a set Σ we denote a function $d\colon \Sigma \times \Sigma \to [0, \infty]$ that satisfies the triangle inequality and $d(x, x) = 0$, but is not necessarily finite or nonzero for two distinct elements. Similarly, we call a function $\|\cdot\|\colon V \to [0, \infty]$ on a vector space a *seminorm* if it satisfies subadditivity and linearity with respect to multiplication with scalars.

Bridgeland introduced the following semimetric on the space of $\mathbb{R}$–valued slicings:

For any $X \in \mathcal{D}$ and an $\mathbb{R}$–valued slicing, let $\phi^-_{\mathcal{P}}(X)$ and $\phi^+_{\mathcal{P}}(X)$ be the smallest and highest phase appearing in the Harder–Narasimhan filtration of X according to Definition 2.2.1(c), respectively. Then $d(\mathcal{P}, \mathcal{Q}) \in [0, \infty]$ is defined as

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq X \in \mathcal{D}} \left\{ \left|\phi^-_{\mathcal{P}}(X) - \phi^-_{\mathcal{Q}}(X)\right|, \left|\phi^+_{\mathcal{P}}(X) - \phi^+_{\mathcal{Q}}(X)\right| \right\}.$$

Via the projection $\pi\colon S \to \mathbb{R}, \phi \mapsto \phi(+\infty)$, we can pull back d to get a semimetric d_S on the space of S–valued slicings.

Following [9, Section 6], we introduce a seminorm on the infinite-dimensional linear space $\mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[m])$ for all $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$:

$$\|\cdot\|_\sigma\colon \mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[m]) \to [0, \infty]$$
$$\|U\|_\sigma = \sup \left\{ \limsup_{m \to \infty} \frac{|U(E)(m)|}{|Z(E)(m)|} \;\middle|\; E \text{ semistable in } \sigma \right\}$$

The next step is to show that [9, Lemma 6.2] carries over: For $0 < \epsilon < \frac{1}{4}$, and $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$ define $B_\epsilon(\sigma) \subset \mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$ as

$$B_\epsilon(\sigma) = \left\{\tau = (\mathcal{Q}, W) \;\middle|\; \|W - Z\|_\sigma < \sin(\pi\epsilon) \text{ and } d_S(\mathcal{P}, \mathcal{Q}) < \epsilon\right\}.$$

Lemma 8.1.2 *If $\tau = (\mathcal{Q}, W) \in B_\epsilon(\sigma)$, then the seminorms $\|\cdot\|_\sigma, \|\cdot\|_\tau$ of σ and τ are equivalent, ie there are constants k_1, k_2 such that $k_1\|U\|_\sigma < \|U\|_\tau < k_2\|U\|_\sigma$ for all $U \in \mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[m])$.*

The proof is identical to that of [9, Lemma 6.2].

On $\mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[m])$ we have the natural topology of point-wise convergence; via the forgetful map $(Z, \mathcal{P}) \mapsto Z$ we can pull this back to get a system of open sets in $\mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$. Now equip $\mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$ with the topology generated, in the sense of a subbasis[5], by this system of open sets and the sets $B_\epsilon(\sigma)$ defined above.

By the definition of the topology and Lemma 8.1.2, the subspace

$$\{U \in \mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[m]) \bigm| \|U\|_\sigma < \infty\}$$

is locally constant in $\mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$ and hence constant on a connected component Σ, denoted by $V(\Sigma)$. It is equipped with the topology generated by the topology of point-wise convergence and the seminorms $\|\cdot\|_\sigma$ for $\sigma \in \Sigma$ (which are equivalent by Lemma 8.1.2); we have obtained:

Proposition 8.1.3 *For each connected component of $\Sigma \subset \mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$ there is a topological vector space $V(\Sigma)$, which is a subspace of $\mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C}[m])$, such that the forgetful map $\Sigma \to V(\Sigma)$ given by $(Z, \mathcal{P}) \mapsto Z$ is continuous.*

Let E be stable in some polynomial stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$. Then for any $Z' \in V(\Sigma)$, the degree of $Z'(E)$ is bounded by the degree of $Z(E)$. In particular, if $K(\mathcal{D})$ is finite dimensional, then $V(\Sigma)$ is finite-dimensional. Further, Bridgeland's space $\mathrm{Stab}(\mathcal{D})$ is a union of connected components of $\mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D})$.

Proposition 8.1.4 *Suppose that $\sigma = (Z, \mathcal{P})$ and $\tau = (Z, \mathcal{Q})$ are polynomial stability conditions with identical central charge Z and $d_S(\mathcal{P}, \mathcal{Q}) < 1$. Then they are identical.*

Again, the proof of [9, Lemma 6.4] carries over literally.

Combining the two previous propositions, we obtain a natural continuous and locally injective map

$$\mathrm{Stab}_{\mathrm{Pol}}(\mathcal{D}) \supset \Sigma \to V(\Sigma) \subset \mathrm{Hom}(K(\mathcal{D}), \mathbb{C}[m]).$$

The discussion applies equally to numerical polynomial stability conditions: for every connected component $\Sigma \subset {\mathrm{Stab}_{\mathrm{Pol}}}^{\mathcal{N}}(\mathcal{D})$ there is a subspace $V(\Sigma) \subset \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C}[m])$ with the structure of a topological vector space, such that the forgetful map $(Z, \mathcal{P}) \mapsto Z$ induces a locally injective continuous map

$$\Sigma \to V(\Sigma).$$

[5] A topology T on a set S is generated by a subbasis Π of subsets of S if open sets in T are exactly the (infinite) unions of finite intersections of sets in Π.

8.2 Deformations of a polynomial stability condition

Definition 8.2.1 A polynomial stability condition $(Z, \mathcal{P})$ is called *locally finite* if there exists a real number $\epsilon > 0$ such that for all $\phi \in S$, the quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length.

Under this strong finiteness assumption, an analogue of Bridgeland's deformation result can be proven:

Theorem 8.2.2 *Let $\sigma = (Z, \mathcal{P})$ be a locally finite polynomial stability condition. Then there is an $\epsilon > 0$ such that if a group homomorphism $W\colon \mathcal{K}(\mathcal{D}) \to \mathbb{C}[m]$ satisfies $\|W - Z\|_\sigma < \sin(\pi\epsilon)$, there is a locally finite stability condition $\tau = (W, \mathcal{Q})$ with $d_{\mathcal{S}}(\mathcal{P}, \mathcal{Q}) < \epsilon$.*

In other words, a locally finite polynomial stability condition in the connected component Σ can be deformed uniquely by deforming its central charge in the subspace $V(\Sigma) \subset \mathrm{Hom}(K(\mathcal{D}), \mathbb{C}[m])$, and the space of locally finite polynomial stability conditions is a smooth manifold.

The theorem can be shown exactly along the lines of Bridgeland's proof. Since we are not using the result in this paper, we omit the proof.

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