Homology operations in the topological cyclic homology of a point

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We consider the commutative $\mathbb{S}$–algebra given by the topological cyclic homology of a point. The induced Dyer–Lashof operations in mod p homology are shown to be nontrivial for $p=2$, and an explicit formula is given. As a part of the calculation, we are led to compare the fixed point spectrum $\mathbb{S}^G$ of the sphere spectrum and the algebraic K–theory spectrum of finite G–sets, as structured ring spectra.

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Introduction

Let $A(\star)=K(\mathbb{S})$ denote Waldhausen's algebraic K–theory of a point [23]. It is a commutative $\mathbb{S}$–algebra, in the sense of Elmendorf, Kriz, Mandell and May [7], and the algebraic K–theory $A(X)$ of any space X, or more generally the algebraic K–theory $K(R)$ of any $\mathbb{S}$–algebra R, is a module spectrum over it. Hence it makes sense to carefully study the commutative $\mathbb{S}$–algebra structure of $A(\star)$, or equivalently its structure as an E_∞ ring spectrum. To the eyes of mod p homology, the primary incarnation of this structure is the Pontryagin algebra structure on $H_*(A(\star))$, together with the multiplicative Dyer–Lashof operations $Q^i\colon H_*(A(\star)) \to H_{*+i}(A(\star))$, as defined by Bruner, May, McClure and Steinberger [2]. Here and elsewhere we write $H_*(E)$ for the mod p homology $H_*(E;\mathbb{F}_p)$ of a spectrum E.

The additive structure of $H_*(A(\star))$ is known for $p=2$ and for p an odd regular prime, by the second author's papers [18; 19], but at present the Pontryagin product and Dyer–Lashof operations are not known for this E_∞ ring spectrum. There is, however, a very good approximation to Waldhausen's algebraic K–theory, given by the cyclotomic trace map to the topological cyclic homology of Bökstedt, Hsiang and Madsen [1]. This is a natural map trc: $K(R)\to \mathrm{TC}(R;p)$, which we write as trc: $A(\star)\to \mathrm{TC}(\star;p)$ in the special case when $R=\mathbb{S}$, where $\mathrm{TC}(\star;p)=\mathrm{TC}(\mathbb{S};p)$ is the topological cyclic

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homology of a point. By a theorem of Dundas [5], there is a homotopy cartesian square

$$\begin{tikzcd} A(\star) \arrow[r] \arrow[d, "\mathrm{trc}"] & K(\mathbb{Z}) \arrow[d, "\mathrm{trc}"] \\ \mathrm{TC}(\star; p) \arrow[r] & \mathrm{TC}(\mathbb{Z}; p) \end{tikzcd}$$

(after p–adic completion) of commutative $\mathbb{S}$–algebras (see Geisser and Hesselholt [9, Section 6]), and this square is the basis for our additive understanding of $H_*(A(\star))$.

We are therefore led to study the commutative $\mathbb{S}$–algebra structure of $\mathrm{TC}(\star; p)$, including the Pontryagin algebra structure and the Dyer–Lashof operations on its mod p homology. Like in the case of algebraic K–theory, the topological cyclic homology $\mathrm{TC}(X; p)$ of any space X, and more generally the topological cyclic homology $\mathrm{TC}(R; p)$ of any $\mathbb{S}$–algebra R, is a module spectrum over $\mathrm{TC}(\star; p)$, and this provides a second motivation for the study of $\mathrm{TC}(\star; p)$. In the present paper, we determine the Dyer–Lashof operations in $H_*(\mathrm{TC}(\star; p))$ in the case when $p = 2$, as explained in Theorem 0.2 and Corollary 0.3 below.

A third motivation stems from ideas of Jack Morava [17], to the effect that there may be a spectral enrichment of the algebro-geometric category of mixed Tate motives, given by A–theoretic (see Williams [24]) or TC–theoretic (see Dundas and Østvær [6]) correspondences, followed by stabilization. The trace map $A(\star) \to \mathrm{TC}(\star; p) \to \mathrm{THH}(\star) = \mathbb{S}$ defines a fiber functor to the category of $\mathbb{S}$–modules, with Tannakian automorphism group realized through its Hopf algebra of functions, which will be of the form $\mathbb{S} \wedge_{A(\star)} \mathbb{S}$ or $\mathbb{S} \wedge_{\mathrm{TC}(\star;p)} \mathbb{S}$. Rationally, this is well compatible with Deligne's results on the Tannakian group of mixed Tate motives over the integers [4]. A calculational analysis of the commutative $\mathbb{S}$–algebras $\mathbb{S} \wedge_{A(\star)} \mathbb{S}$ or $\mathbb{S} \wedge_{\mathrm{TC}(\star;p)} \mathbb{S}$ clearly depends heavily on a proper understanding of the commutative $\mathbb{S}$–algebra structures of $A(\star)$ and $\mathrm{TC}(\star; p)$.

Let $\mathbb{T}$ be the circle group and let $C_{p^n} \subset \mathbb{T}$ be the (cyclic) subgroup of order p^n. The spectrum $\mathrm{TC}(\star; p)$ is defined as the homotopy inverse limit of a diagram

$$\cdots \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S}^{C_{p^{n+1}}} \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S}^{C_{p^n}} \overset{R}{\underset{F}{\rightrightarrows}} \cdots \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S}^{C_p} \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S} \tag{0-1}$$

of E_∞ ring spectra, where $\mathbb{S}^{C_{p^n}}$ denotes the C_{p^n}–fixed points of the $\mathbb{T}$–equivariant sphere spectrum, the maps labeled R are restriction maps, and the maps labeled F are Frobenius maps. See Bökstedt, Hsiang and Madsen [1] or Hesselholt and Madsen [10] for the construction of these maps. Similarly, let $\mathrm{TC}^{(1)}(\star; p)$ denote the homotopy limit of the subdiagram

$$\mathbb{S}^{C_p} \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S} \quad , \tag{0-2}$$

that is, the homotopy equalizer of R and F. The canonical maps

$$\mathrm{TC}(\star; p) \xrightarrow{f_1} \mathrm{TC}^{(1)}(\star; p) \xrightarrow{g_1} \mathbb{S}^{C_p} \tag{0-3}$$

are then maps of E_∞ ring spectra.

The unit $\eta\colon \mathbb{S} \to \mathrm{TC}(\star; p)$ and the restriction $R\colon \mathbb{S}^{C_p} \to \mathbb{S}$ let us split off a copy of $\mathbb{S}$ from each term in (0-3). Let $\mathbb{C}P^\infty_{-1}$ be the Thom spectrum of the negative tautological complex line bundle $-\gamma^1_{\mathbb{C}}$ over $\mathbb{C}P^\infty$. Its suspension $\Sigma\mathbb{C}P^\infty_{-1}$ is equivalent to the homotopy fiber of the dimension-shifting $\mathbb{T}$–transfer map $t_{\mathbb{T}}\colon \Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}$; see Knapp [13, 2.9] or Lemma 1.1 below. We define the spectrum L^∞_{-1} to be the homotopy fiber of the C_p–transfer $t_p\colon \Sigma^\infty(BC_p)_+ \to \mathbb{S}$. For $p = 2$, there is an equivalence $L^\infty_{-1} \simeq \mathbb{R}P^\infty_{-1}$, where $\mathbb{R}P^\infty_{-1}$ is the Thom spectrum of the negative tautological real line bundle $-\gamma^1_{\mathbb{R}}$ over $\mathbb{R}P^\infty$. The mod p homology groups of these spectra are well known:

$$\begin{aligned}
H_*(\Sigma\mathbb{C}P^\infty_{-1}) &\cong \mathbb{F}_p\{\Sigma\beta_k \mid k \geq -1\}\\
H_*(L^\infty_{-1}) &\cong \mathbb{F}_p\{\alpha_k \mid k \geq -1\}\\
H_*(\Sigma^\infty(BC_p)_+) &\cong \mathbb{F}_p\{\alpha_k \mid k \geq 0\}
\end{aligned}$$

Here $\Sigma\beta_k$ has degree $2k+1$ and α_k has degree k.

Lemma 0.1 *After p–completion, diagram (0-3) is homotopy equivalent to a diagram*

$$\mathbb{S} \vee \Sigma\mathbb{C}P^\infty_{-1} \xrightarrow{1\vee f} \mathbb{S} \vee L^\infty_{-1} \xrightarrow{1\vee g} \mathbb{S} \vee \Sigma^\infty(BC_p)_+ \,.$$

In particular, the Pontryagin product on $H_*(\mathrm{TC}(\star; p))$ is trivial.

Applying homology gives a sequence

$$H_*(\mathbb{S}) \oplus H_*(\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{1\oplus f_*} H_*(\mathbb{S}) \oplus H_*(L^\infty_{-1}) \xrightarrow{1\oplus g_*} H_*(\mathbb{S}) \oplus H_*(\Sigma^\infty(BC_p)_+) \,.$$

Here f_* sends $\Sigma\beta_k$ to α_{2k+1} for $k \geq -1$, and g_* is the identity on α_k for $k \geq 0$, while α_{-1} maps to zero.

We now state our main result, which concerns the Dyer–Lashof operations in the mod p spectrum homology $H_*(\mathrm{TC}(\star; p))$ for $p = 2$. The calculations will be done in the auxiliary E_∞ ring spectra $\mathbb{S}^{C_2}$ and $\mathrm{TC}^{(1)}(\star; 2)$.

Theorem 0.2 *The Dyer–Lashof operations Q^i in $H_*(\mathrm{TC}^{(1)}(\star;2))$ and $H_*(\mathbb{S}^{C_2})$ are given by the formula*

$$Q^i(\alpha_j) = \binom{2^N+i-1}{2^N+j}\alpha_{i+j}\,,$$

where $j \ge -1$ and i is any integer, and N is sufficiently large.

Corollary 0.3 *The Dyer–Lashof operations Q^{2i} in $H_*(\mathrm{TC}(\star;2))$ are given by the formula*

$$Q^{2i}(\Sigma\beta_j) = \binom{2^N+i-1}{2^N+j}\Sigma\beta_{i+j}\,,$$

where $j \ge -1$ and i is any integer, and N is sufficiently large. The operations Q^{2i+1} are all zero for degree reasons.

Note that the binomial coefficients used in the theorem and corollary can be evaluated to

$$\binom{2^N+i-1}{2^N+j} \equiv \begin{cases} \binom{i-1}{j} & \text{for } i > j \ge 0, \\ 1 & \text{for } (i,j) = (0,-1), \\ 0 & \text{otherwise} \end{cases}$$

modulo 2, for all sufficiently large N. In particular $Q^0(\alpha_{-1}) = \alpha_{-1}$ and $Q^0(\Sigma\beta_{-1}) = \Sigma\beta_{-1}$.

We prove Lemma 0.1 in Section 1 and Theorem 0.2 in Section 3, after a homological comparison of E_∞ ring structures in Section 2. Corollary 0.3 follows immediately from the lemma and the theorem.

1 Topological cyclic homology of a point

In this preliminary section we review the calculation of $\mathrm{TC}(\star;p)$ from Bökstedt, Hsiang and Madsen [1, 5.17], in order to describe the map f_1 to $\mathrm{TC}^{(1)}(\star;p)$.

For each $n \ge 1$ the Segal–tom Dieck splitting tells us that the norm-restriction homotopy cofiber sequence

$$\Sigma^\infty(BC_{p^n})_+ \xrightarrow{N} \mathbb{S}^{C_{p^n}} \xrightarrow{R} \mathbb{S}^{C_{p^{n-1}}}$$

is canonically split. The homotopy limit $\mathrm{TR}(\star;p) = \mathrm{holim}_{n,R}\, \mathbb{S}^{C_{p^n}}$ of the R–maps in (0-1) thus factors as $\mathrm{TR}(\star;p) \simeq \prod_{n\ge 0} \Sigma^\infty(BC_{p^n})_+$. Let

$$\mathrm{pr}_n\colon \mathrm{TR}(\star;p) \to \Sigma^\infty(BC_{p^n})_+$$

denote the n–th projection, and let $\widetilde{\mathrm{TR}}(\star; p) \simeq \prod_{n\geq 1} \Sigma^\infty(BC_{p^n})_+$ be the homotopy fiber of pr_0. There is a vertical map of horizontal homotopy fiber sequences

$$\begin{tikzcd} \mathrm{TC}(\star; p) \arrow[r, "\pi"] \arrow[d, "f_1"'] & \mathrm{TR}(\star; p) \arrow[r, "F-R"] \arrow[d, "p_1"] & \mathrm{TR}(\star; p) \arrow[d, "\mathrm{pr}_0"] \\ \mathrm{TC}^{(1)}(\star; p) \arrow[r, "g_1"] & \mathbb{S}^{C_p} \arrow[r, "F-R"] & \mathbb{S} \end{tikzcd} \tag{1-1}$$

and the augmentation $\mathrm{TC}(\star; p) \to \mathbb{S}$ factors as $R \circ p_1 \circ \pi = \mathrm{pr}_0 \circ \pi$. Replacing the left hand square by the homotopy fibers of the augmentations to $\mathbb{S}$, we get a second vertical map of horizontal homotopy fiber sequences

$$\begin{tikzcd} \widetilde{\mathrm{TC}}(\star; p) \arrow[r] \arrow[d, "f_2"'] & \widetilde{\mathrm{TR}}(\star; p) \arrow[r, "T-I"] \arrow[d, "\mathrm{pr}_1"] & \mathrm{TR}(\star; p) \arrow[d, "\mathrm{pr}_0"] \\ \widetilde{\mathrm{TC}}^{(1)}(\star; p) \arrow[r, "g_2"] & \Sigma^\infty(BC_p)_+ \arrow[r, "t_p"] & \mathbb{S}\,. \end{tikzcd} \tag{1-2}$$

In the upper row we have used that $F-R$ restricted along the inclusion $I\colon \widetilde{\mathrm{TR}}(\star; p) \to \mathrm{TR}(\star; p)$ is homotopic to $T-I$, where T is the product of the C_p–transfer maps $\Sigma^\infty(BC_{p^n})_+ \to \Sigma^\infty(BC_{p^{n-1}})_+$ for all $n \geq 1$. See Bökstedt, Hsiang and Madsen [1, (5.18)]. In the lower row we have used that $F-R$ restricted along $N\colon \Sigma^\infty(BC_p)_+ \to \mathbb{S}^{C_p}$ is homotopic to the C_p–transfer map t_p.

There is a third vertical map of horizontal homotopy fiber sequences

$$\begin{tikzcd} \operatorname{holim}_n \Sigma^\infty(BC_{p^n})_+ \arrow[r] \arrow[d, "f_3"'] & \mathrm{TR}(\star; p) \arrow[r, "T-1"] \arrow[d, "p_1"] & \mathrm{TR}(\star; p) \arrow[d, "\mathrm{pr}_0"] \\ \mathrm{hofib}(F-2R) \arrow[r, "g_3"] & \mathbb{S}^{C_p} \arrow[r, "F-2R"] & \mathbb{S}\,. \end{tikzcd} \tag{1-3}$$

Replacing its left hand square by the homotopy fibers of the augmentations to $\mathbb{S}$, we also recover diagram (1-2).

Lemma 1.1 *There are equivalences*

$$\widetilde{\mathrm{TC}}(\star; p) \simeq \mathrm{hofib}\,(t_{\mathbb{T}}\colon \Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}) \simeq \Sigma\mathbb{C}P^\infty_{-1}$$

(after p–completion) and

$$\widetilde{\mathrm{TC}}^{(1)}(\star; p) \simeq \mathrm{hofib}\,(t_p\colon \Sigma^\infty(BC_p)_+ \to \mathbb{S}) = L^\infty_{-1}\,.$$

When $p = 2$, $L^\infty_{-1} \simeq \mathbb{R}P^\infty_{-1}$.

Proof The dimension-shifting $\mathbb{T}$–transfer maps for the bundles $BC_{p^n} \to B\mathbb{T}$ induce an equivalence $\Sigma^\infty \Sigma(\mathbb{C}P^\infty_+) \simeq \operatorname{holim}_n \Sigma^\infty(BC_{p^n})_+$ after p–completion [1, 5.15]. The augmentation $\operatorname{holim}_n \Sigma^\infty(BC_{p^n})_+ \to \mathbb{S}$ then gets identified with $t_\mathbb{T}$, which implies the first claim.

There is a $\mathbb{T}$–equivariant homotopy cofiber sequence $S^0 \xrightarrow{z} S^\mathbb{C} \xrightarrow{t} \mathbb{T}_+ \wedge S^1$, where z is the zero-inclusion. The right hand map t is the Pontryagin–Thom collapse associated to the standard embedding $\mathbb{T} \subset \mathbb{C}$, as in Lewis, May and Steinberger [14, II.5.1]. The dimension-shifting $\mathbb{T}$–transfer $t_\mathbb{T}\colon \Sigma^\infty \Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}$ for $E\mathbb{T} \to B\mathbb{T}$ is constructed as the balanced smash product

$$1 \wedge_\mathbb{T} \Sigma^{1-\mathbb{C}}(t)\colon E\mathbb{T}_+ \wedge_\mathbb{T} \Sigma^{1-\mathbb{C}}(S^\mathbb{C}) \to E\mathbb{T}_+ \wedge_\mathbb{T} \Sigma^{1-\mathbb{C}}(\mathbb{T}_+ \wedge S^1)$$

(see Lewis, May and Steinberger [14, II.7.5]). Hence its homotopy fiber is $E\mathbb{T}_+ \wedge_\mathbb{T} \Sigma^{1-\mathbb{C}}(S^0) \cong \Sigma\mathbb{C}P^\infty_{-1}$.

The proof that $\mathbb{R}P^\infty_{-1}$ is the homotopy fiber of t_p for $p = 2$ is essentially the same. □

Proof of Lemma 0.1 Under the identifications of Lemma 1.1, the maps $f\colon \Sigma\mathbb{C}P^\infty_{-1} \to L^\infty_{-1}$ and $g\colon L^\infty_{-1} \to \Sigma^\infty(BC_p)_+$ correspond to the maps f_2 and g_2 in diagram (1-2), respectively.

The C_p–transfer map t_p induces multiplication by p on π_0, and the zero map in mod p homology, so $\pi_{-1}(f)$ is surjective, f_* maps $\Sigma\beta_{-1}$ to α_{-1}, and g_* maps α_k to α_k for all $k \geq 0$. It remains to see that $g_* f_*$ maps $\Sigma\beta_k$ to α_{2k+1} for $k \geq 0$. This is clear from diagram (1-3), since $g_3 f_3$ agrees in positive degrees with the $\mathbb{T}$–transfer map $\Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \Sigma^\infty(BC_p)_+$, which has this behavior on homology. □

2 Algebraic K–theory of finite G–sets

In this section we will compare the algebraic K–theory spectrum of finite G–sets with the G–fixed points of the sphere spectrum, as structured ring spectra. Before we state the result we recall some of the definitions involved.

The K–theory construction we use is that of Elmendorf and Mandell [8]. When the input category is a bipermutative category $\mathcal{C}$, their machine produces a symmetric spectrum $K(\mathcal{C})$, in the sense of Hovey, Shipley and Smith [11], with an action of the simplicial Barratt–Eccles operad. We will use the same notation for the geometrically realized symmetric spectrum in topological spaces, which has an action

$$\kappa_j\colon E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j} \to K(\mathcal{C})$$

of the operad $E\Sigma$ consisting of the contractible Σ_j–free spaces $E\Sigma_j$. As usual, $E\Sigma_j$ can be defined as the nerve $N\widetilde{\Sigma}_j$ of the translation category $\widetilde{\Sigma}_j$, for $j \geq 0$. The K–theory construction itself is somewhat involved, but all we need to know is that the zeroth space $K(\mathcal{C})_0$ is the nerve $N\mathcal{C}$ of $\mathcal{C}$, so the zeroth space of $E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j}$ is the nerve of $\widetilde{\Sigma}_j \ltimes_{\Sigma_j} \mathcal{C}^j$, and the action of $E\Sigma$ on $K(\mathcal{C})_0$ is given by the maps $\lambda_j \colon E\Sigma_j \ltimes_{\Sigma_j} N\mathcal{C}^{\wedge j} \to N\mathcal{C}$ that are induced by the functors that take an object $(\sigma; a_1, \dots, a_j)$ in $\widetilde{\Sigma}_j \ltimes_{\Sigma_j} \mathcal{C}^j$ to the object $a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)}$ in $\mathcal{C}$ (see Elmendorf and Mandell [8, Section 8]). Here $\otimes$ denotes the product in the bipermutative structure on $\mathcal{C}$. Hence there is a commutative diagram

$$\begin{tikzcd}
E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j} \arrow[rr, "\kappa_j"] & & K(\mathcal{C}) \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty N\mathcal{C})^{\wedge j} \arrow[u] \arrow[r, "\cong"] & \Sigma^\infty(E\Sigma_j \ltimes_{\Sigma_j} N\mathcal{C}^{\wedge j}) \arrow[r, "\Sigma^\infty\lambda_j"] & \Sigma^\infty N\mathcal{C} \arrow[u]
\end{tikzcd}$$

for each $j \geq 0$.

Let G be a finite group, and let $\mathcal{E}^G$ denote the category of finite G–sets and G–equivariant bijections. This is a symmetric bimonoidal category under disjoint union and cartesian product, taking (X, Y) to $X \coprod Y$ and $X \times Y$, respectively. We give $X \times Y$ the diagonal G–action. There is a functorially defined bipermutative category $\Phi\mathcal{E}^G$, and a natural equivalence $\mathcal{E}^G \to \Phi\mathcal{E}^G$ [16, VI.3.5]. It follows that there is a homotopy commutative diagram

$$\begin{tikzcd}
E\Sigma_j \ltimes_{\Sigma_j} K(\Phi\mathcal{E}^G)^{\wedge j} \arrow[rr, "\kappa_j"] & & K(\Phi\mathcal{E}^G) \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty N\mathcal{E}^G)^{\wedge j} \arrow[u, "1\ltimes\epsilon^{\wedge j}"] \arrow[r, "\cong"] & \Sigma^\infty(E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j}) \arrow[r, "\Sigma^\infty\lambda_j"] & \Sigma^\infty N\mathcal{E}^G \arrow[u, "\epsilon"']
\end{tikzcd} \tag{2-1}$$

for each $j \geq 0$, where

$$\lambda_j \colon E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \to N\mathcal{E}^G$$

is induced by the functor $\widetilde{\Sigma}_j \ltimes_{\Sigma_j} (\mathcal{E}^G)^j \to \mathcal{E}^G$ that takes $(\sigma; X_1, \dots, X_j)$ to the cartesian product

$$X_{\sigma^{-1}(1)} \times (X_{\sigma^{-1}(2)} \times \cdots \times (X_{\sigma^{-1}(j-1)} \times X_{\sigma^{-1}(j)}) \cdots).$$

Let $\mathcal{U}$ be a complete G–universe, and let $\mathcal{L}$ denote the linear isometries operad with spaces $\mathcal{L}(j)$ consisting of linear isometries $\mathcal{U}^j \to \mathcal{U}$, where $\mathcal{U}^j$ denotes the direct sum of j copies of $\mathcal{U}$. There is an action of G on each $\mathcal{L}(j)$ given by conjugation, and this gives $\mathcal{L}$ the structure of an E_∞ G–operad in the sense of Lewis, May and

Steinberger [14, VII.1.1]. The E_∞ ring structure on the G–equivariant sphere spectrum $\mathbb{S}_G = \Sigma^\infty_G S^0$ is given by an action

$$\zeta_j \colon \mathcal{L}(j) \ltimes_{\Sigma_j} \mathbb{S}_G^{\wedge j} \to \mathbb{S}_G$$

of this operad (where, for once, $\ltimes$ denotes the twisted half-smash product in Lewis–May spectra). It is compatible with a corresponding action

$$\omega_j \colon \mathcal{L}(j) \ltimes_{\Sigma_j} Q_G(S^0)^{\wedge j} \to Q_G(S^0)$$

on the underlying infinite loop space $Q_G(S^0) = \Omega^\infty \mathbb{S}_G = \operatorname{colim}_{V \subset \mathcal{U}} \Omega^V S^V$, in the sense that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{L}(j) \ltimes_{\Sigma_j} (\Sigma^\infty Q_G(S^0))^{\wedge j} \arrow[r, "\cong"] \arrow[d, "1 \ltimes \epsilon^{\wedge j}"'] & \Sigma^\infty(\mathcal{L}(j) \ltimes_{\Sigma_j} Q_G(S^0)^{\wedge j}) \arrow[r, "\Sigma^\infty \omega_j"] & \Sigma^\infty Q_G(S^0) \arrow[d, "\epsilon"] \\
\mathcal{L}(j) \ltimes_{\Sigma_j} \mathbb{S}_G^{\wedge j} \arrow[rr, "\zeta_j"] & & \mathbb{S}_G
\end{tikzcd} \tag{2-2}$$

Here ω_j sends an element in $\mathcal{L}(j) \ltimes_{\Sigma_j} Q_G(S^0)^{\wedge j}$ represented by $(f; g_1, \dots, g_j)$, where $f \colon \mathcal{U}^j \to \mathcal{U}$ and $g_i \colon S^{V_i} \to S^{V_i}$, to the element represented by the composite of the following maps.

$$S^{f(V_1 \oplus \cdots \oplus V_j)} \xleftarrow[\cong]{f_*} S^{V_1 \oplus \cdots \oplus V_j} \xrightarrow{g_1 \wedge \cdots \wedge g_j} S^{V_1 \oplus \cdots \oplus V_j} \xrightarrow[\cong]{f_*} S^{f(V_1 \oplus \cdots \oplus V_j)}$$

By taking G–fixed points we get the nonequivariant E_∞ ring spectrum $\mathbb{S}^G = (\mathbb{S}_G)^G$ with an action

$$\xi_j \colon \mathcal{L}^G(j) \ltimes_{\Sigma_j} (\mathbb{S}^G)^{\wedge j} \to \mathbb{S}^G$$

of the nonequivariant E_∞ operad $\mathcal{L}^G$ of G–equivariant isometries. The corresponding infinite loop space $\Omega^\infty(\mathbb{S}^G)$ is the space $Q_G(S^0)^G = \operatorname{colim}_{V \subset \mathcal{U}} (\Omega^V S^V)^G$, with the inherited $\mathcal{L}^G$–action

$$\eta_j \colon \mathcal{L}^G(j) \ltimes_{\Sigma_j} (Q_G(S^0)^G)^{\wedge j} \to Q_G(S^0)^G .$$

Next we recall the definition of the Dyer–Lashof operations Q^i. Let $C_*(-)$ denote the cellular chains functor, from either CW complexes or CW spectra to chain complexes. Let E be a spectrum with an action of an E_∞ operad $\mathcal{O}$, and let W_* be the standard free C_p–resolution of $\mathbb{F}_p$ with basis elements e_i in degree i. There is a chain map $W_* \to C_*(\mathcal{O}(p))$ lifting the identity on $\mathbb{F}_p$, unique up to homotopy, and we also denote the image of e_i under this map by e_i. Let $x \in H_q(E)$ be represented by a cycle

$z \in C_q(E)$. Now consider the image of the cycle $e_i \otimes z^{\otimes p}$ under the map

$$C_*(\mathcal{O}(p)) \otimes_{\Sigma_p} C_*(E)^{\otimes p} \xrightarrow{\cong} C_*(\mathcal{O}(p) \ltimes_{\Sigma_p} E^{\wedge p}) \xrightarrow{\xi_{p*}} C_*(E),$$

and denote its image in homology by $Q_i(x)$. Here ξ_p is the E_∞ structure map. Then for $p = 2$ define $Q^i(x) = 0$ when $i < q$, and

$$Q^i(x) = Q_{i-q}(x)$$

when $i \geq q$. For $p > 2$ define $Q^i(x) = 0$ when $2i < q$, and

$$Q^i(x) = (-1)^i \nu(q) \cdot Q_{(2i-q)(p-1)}(x)$$

when $2i \geq q$, where $\nu(q) = (-1)^{q(q-1)(p-1)/4}((\frac{1}{2}(p-1))!)^q$. See Bruner, May, McClure and Steinberger [2, Chapter III] for more details.

The spectra $\mathbb{S}^G$ and $K(\Phi\mathcal{E}^G)$ should be equivalent as E_∞ ring spectra, but we will only need the following weaker result.

Lemma 2.1 *There is an equivalence $\mathbb{S}^G \simeq K(\Phi\mathcal{E}^G)$ of spectra such that the induced isomorphism $H_*(\mathbb{S}^G) \cong H_*(K(\Phi\mathcal{E}^G))$ commutes with the Dyer–Lashof operations.*

Proof Our first goal is to construct a commutative diagram:

$$\begin{tikzcd}
 & E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \arrow[r, "\lambda_j"] & N\mathcal{E}^G & \\
E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G_{\mathrm{tr}})^{\wedge j} \arrow[ur, "1\ltimes\phi^{\wedge j}"] \arrow[r] \arrow[ddr, "1\ltimes\psi^{\wedge j}"'] & (E\Sigma_j \times \mathcal{L}^G(j)) \ltimes_{\Sigma_j} D_{\mathcal{U}}^{\wedge j} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] \arrow[r, "\mu_j"] & D_{\mathcal{U}} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & N\mathcal{E}^G_{\mathrm{tr}} \arrow[l] \arrow[ul, "\phi"'] \arrow[ddl, "\psi"] \\
 & \mathcal{L}^G(j) \ltimes_{\Sigma_j} C_{\mathcal{U}}^{\wedge j} \arrow[r, "\nu_j"] \arrow[d] & C_{\mathcal{U}} \arrow[d] & \\
 & \mathcal{L}^G(j) \ltimes_{\Sigma_j} (Q_G(S^0)^G)^{\wedge j} \arrow[r, "\eta_j"] & Q_G(S^0)^G &
\end{tikzcd} \tag{2-3}$$

We start by describing the space $C_{\mathcal{U}}$. Let V be an indexing space in $\mathcal{U}$. For each finite G–set X, consider the space $E_V(X)$ of X–tuples of distance-reducing embeddings of V in V, closed under the action of G. More precisely, this is the space of G–equivariant maps $\coprod_X V \to V$ such that the restriction to each summand $g\colon V \to V$ is an embedding that satisfies $|g(v) - g(w)| \leq |v - w|$ for all $v, w \in V$. Let $K_V(X)$

be the space of paths $[0,1] \to E_V(X)$ such that the embeddings at the endpoint 0 are identities, and the embeddings at 1 have disjoint images. Now let

$$K_{\mathcal{U}}(X) = \operatorname*{colim}_{V \subset \mathcal{U}} K_V(X)\,.$$

These are G–equivariant versions of the spaces in the Steiner operad [22]. The group $\mathrm{Aut}^G(X)$ acts freely on $K_{\mathcal{U}}(X)$ by permuting the embeddings, and the space $C_{\mathcal{U}}$ is to be the disjoint union

$$C_{\mathcal{U}} = \coprod_{[X]} K_{\mathcal{U}}(X)/\,\mathrm{Aut}^G(X)$$

where X ranges over all isomorphism classes of finite G–sets.

The action of the operad $\mathcal{L}^G$ on $C_{\mathcal{U}}$ is defined as follows. Let $f\colon \mathcal{U}^j \to \mathcal{U}$ be a G–linear isometry, and let $[g_i]$, $1 \le i \le j$, be elements in $C_{\mathcal{U}}$, represented by paths of X_i–tuples of embeddings $g_i \in K_{\mathcal{U}}(X_i)$. Denote the component paths of embeddings that constitute g_i by g_{i,x_i}, where $x_i \in X_i$. The resulting element $\nu_j(f;[g_1],\dots,[g_j])$ in $C_{\mathcal{U}}$ is represented by an element in $K_{\mathcal{U}}(X_1 \times \cdots \times X_j)$, which on the summand indexed by $(x_1,\dots,x_j)$ is given by $f \circ (g_{1,x_1} \times \cdots \times g_{j,x_j}) \circ f^{-1}$.

There is a map $C_{\mathcal{U}} \to Q_G(S^0)^G$, given by evaluating a Steiner path in $E_V(X)$ at 1 to get a G–equivariant embedding $e\colon \coprod_X V \to V$, and then applying a folded Pontryagin–Thom construction to obtain a G–equivariant map $q\colon S^V \to S^V$, which is a point in $Q_G(S^0)^G$. Given the distance-reducing embedding e, let $S^V \to \bigvee_X S^V$ be the G–equivariant map that is given by e^{-1} on the image of e in $V \subset S^V$ and maps the remainder of S^V to the base point of $\bigvee_X S^V \supset \coprod_X V$. Let $\bigvee_X S^V \to S^V$ be the fold map that is the identity on each summand. The folded Pontryagin–Thom construction q is the composite of these two G–maps. If we permute the embeddings indexed by X we get the same element in $Q_G(S^0)^G$, so our map is well-defined. A comparison of definitions shows that this construction is compatible with the $\mathcal{L}^G$–actions on $C_{\mathcal{U}}$ and $Q_G(S^0)^G$, so the lower square in (2-3) commutes.

Let
$$D_{\mathcal{U}} = \coprod_{[X]} (E\,\mathrm{Aut}^G(X) \times K_{\mathcal{U}}(X))/\,\mathrm{Aut}^G(X)\,,$$

where $\mathrm{Aut}^G(X)$ acts diagonally on the product. The nerve $N\mathcal{E}^G$ splits as a sum of components

$$N\mathcal{E}^G \simeq \coprod_{[X]} B\mathrm{Aut}^G(X)\,, \tag{2-4}$$

where the disjoint union is over the isomorphism classes of G–sets X. Projection on the first factor in $D_{\mathcal{U}}$ followed by this homotopy equivalence gives the map

$D_{\mathcal{U}} \to N\mathcal{E}^G$ in (2-3), while the map $D_{\mathcal{U}} \to C_{\mathcal{U}}$ is the projection on the second factor. There is an induced action of the product operad $E\Sigma \times \mathcal{L}^G$ on $D_{\mathcal{U}}$, defined as follows. Let $(e, f) \in E\Sigma_j \times \mathcal{L}^G(j)$ and let $(e_i, f_i) \in E\operatorname{Aut}^G(X) \times C_{\mathcal{U}}(X)$ represent elements in $D_{\mathcal{U}}$, for $1 \le i \le j$. The image under μ_j is the element represented by $(\lambda_j(e; e_1, \dots, e_j), \nu_j(f; f_1, \dots, f_j))$. This makes the upper and middle squares in (2-3) commute.

Let
$$N\mathcal{E}^G_{\mathrm{tr}} = \Big(\coprod_{(H)} (E\operatorname{Aut}^G(G/H) \times K_{\mathcal{U}}(G/H)) / \operatorname{Aut}^G(G/H) \Big)_+ ,$$
where the coproduct is taken over the conjugacy classes of subgroups H of G. The map $N\mathcal{E}^G_{\mathrm{tr}} \to D_{\mathcal{U}}$ in (2-3) is the inclusion of the components indexed by the isomorphism classes of transitive G–sets.

The maps ϕ and ψ are defined by commutativity of the right hand triangles in the diagram. We claim that the adjoints

$$\Sigma^\infty N\mathcal{E}^G_{\mathrm{tr}} \to K(\mathcal{E}^G) \simeq K(\Phi\mathcal{E}^G) \tag{2-5}$$
$$\Sigma^\infty N\mathcal{E}^G_{\mathrm{tr}} \to \mathbb{S}^G \tag{2-6}$$

of the maps ϕ and ψ, respectively, are both equivalences. Here $K(\mathcal{E}^G)$ is the additive K–theory spectrum of $\mathcal{E}^G$, with zeroth space $K(\mathcal{E}^G)_0 = N\mathcal{E}^G$, which only depends on the additive symmetric monoidal structure of $\mathcal{E}^G$. There is an equivalence
$$\Sigma^\infty N\mathcal{E}^G_{\mathrm{tr}} \simeq \bigvee_{(H)} \Sigma^\infty BW_G H_+ ,$$
where $W_G H = N_G H / H \cong \operatorname{Aut}^G(G/H)$ is the Weyl group of H and the wedge sum is over the conjugacy classes of subgroups of G. By Waldhausen's additivity theorem [23, 1.3.2] applied to a suitable filtration of $\mathcal{E}^G$ according to stabilizer types, there is a splitting
$$K(\mathcal{E}^G) \simeq \bigvee_{(H)} K(\mathcal{E}(W_G H)) ,$$
where $\mathcal{E}(W_G H)$ is the category of finite free $W_G H$–sets and equivariant bijections. The map (2-5) is equivalent under these identifications to the wedge sum of the maps

$$\Sigma^\infty BW_G H_+ \to K(\mathcal{E}(W_G H)) \tag{2-7}$$

that are left adjoint to the inclusions $BW_G H_+ \to N\mathcal{E}(W_G H) = K(\mathcal{E}(W_G H))_0$.

The Barratt–Priddy–Quillen–Segal theorem [20, 3.6] says that each of the maps (2-7) is an equivalence, hence (2-5) is an equivalence. The map (2-6) is an equivalence by the Segal–tom Dieck splitting [14, V.11.2]. The composition of these two equivalences is the equivalence $\mathbb{S}^G \simeq K(\Phi\mathcal{E}^G)$ referred to in the statement of the lemma.

We apply the suspension spectrum functor Σ^∞ to the diagram (2-3), combine it with diagram (2-1) and the G–fixed part of (2-2), take homology, and end up with the following commutative diagram.

$$\begin{tikzcd}
 & H_*(\Sigma_j; H_*(K(\Phi\mathcal{E}^G))^{\otimes j}) \arrow[r, "\kappa_{j*}"] & H_*(K(\Phi\mathcal{E}^G)) & \\
H_*(\Sigma_j; H_*(N\mathcal{E}^G_{\mathrm{tr}})^{\otimes j}) \arrow[r] \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & H_*(\Sigma_j; \widetilde{H}_*(D_{\mathcal{U}})^{\otimes j}) \arrow[r, "\mu_{j*}"] \arrow[u] \arrow[d] & \widetilde{H}_*(D_{\mathcal{U}}) \arrow[u, "\epsilon_1"] \arrow[d, "\epsilon_2"] & H_*(N\mathcal{E}^G_{\mathrm{tr}}) \arrow[l] \arrow[ul, "\phi_*"', "\cong"] \arrow[dl, "\psi_*", "\cong"'] \\
 & H_*(\Sigma_j; H_*(\mathbb{S}^G)^{\otimes j}) \arrow[r, "\xi_{j*}"] & H_*(\mathbb{S}^G) &
\end{tikzcd} \tag{2-8}$$

We need the fact that ϵ_1 and ϵ_2 have the same kernel. In fact, all summands in $\widetilde{H}_*(D_{\mathcal{U}})$ indexed by G–sets with more than one orbit map to zero under both ϵ_1 and ϵ_2. This follows from the fact that Pontryagin products and additive Dyer–Lashof operations vanish after stabilization. More precisely, a decomposition of a G–set $X = \coprod_{i=1}^k n_i(G/H_i)$, where the H_i lie in distinct conjugacy classes, induces a factorization

$$B\mathrm{Aut}^G(X) \cong \prod_{i=1}^{k} B(\Sigma_{n_i} \wr W_G H_i)\,.$$

The homology group $H_*(B(\Sigma_{n_i} \wr W_G H_i)) \subset H_*(N\mathcal{E}^G)$ is generated by $H_*(BW_G H_i)$ under iterated Pontryagin products and Dyer–Lashof operations (see Cohen, Lada and May [3, I.4.1]), which all map to zero under ϵ_1 and ϵ_2 unless $k=1$ and $n_1 = 1$.

Let $x \in H_*(K(\Phi\mathcal{E}^G))$, and let $y \in H_*(\mathbb{S}^G)$ be the element corresponding to x under $\psi_* \circ \phi_*^{-1}$, via an element $z \in H_*(N\mathcal{E}^G_{\mathrm{tr}})$. We need to show that the image $Q_i(x)$ of $e_i \otimes x^{\otimes p}$ under the top map corresponds, via the isomorphism, to the image $Q_i(y)$ of $e_i \otimes y^{\otimes p}$ under the bottom map. The element $e_i \otimes z^{\otimes p} \in H_*(\Sigma_p; H_*(N\mathcal{E}^G_{\mathrm{tr}})^{\otimes p})$ maps to an element $Q_i(z) \in \widetilde{H}_*(D_{\mathcal{U}})$, which further maps to $Q_i(x)$ and $Q_i(y)$ under ϵ_1 and ϵ_2, respectively. Let $w \in H_*(N\mathcal{E}^G_{\mathrm{tr}})$ map to $Q_i(x)$ under ϕ_*. Since the maps ϵ_1 and ϵ_2 have the same kernel, the elements $Q_i(z)$ and w have the same image in $H_*(\mathbb{S}^G)$, which implies the result. □

Remark 2.2 The additive equivalence $\mathbb{S}^G \simeq K(\mathcal{E}^G) \simeq K(\Phi\mathcal{E}^G)$ of spectra can be realized as the G–fixed part of a G–equivalence $\mathbb{S}_G \simeq K_G(\mathcal{E})$ of G–spectra, for example using Shimakawa's construction [21] of G–equivariant K–theory spectra. Presumably this is a G–equivalence of E_∞ ring G–spectra.

3 Proof of the main theorem

Recall the E_∞ structure maps $\lambda_j\colon E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \to N\mathcal{E}^G$. We have inclusions $B\mathrm{Aut}^G(G) \to N\mathcal{E}^G$ and $\delta\colon B\mathrm{Aut}^G(G^j) \to N\mathcal{E}^G$, corresponding to the summands indexed by $X = G$ and $X = G^j = G \times \cdots \times G$, respectively, in the decomposition (2-4) of $N\mathcal{E}^G$. Restricting λ_j to these summands, we have a commutative diagram

(3-1)
$$\begin{tikzcd}
E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \arrow[rr, "\lambda_j"] & & N\mathcal{E}^G \\
E\Sigma_j \ltimes_{\Sigma_j} B\mathrm{Aut}^G(G)^j \arrow[u] \arrow[r, "\cong"] & B(\Sigma_j \ltimes \mathrm{Aut}^G(G)^j) \arrow[r, "B\phi"] & B\mathrm{Aut}^G(G^j) \arrow[u, "\delta"']
\end{tikzcd}$$

where the homomorphism ϕ sends an element $(\sigma; f_1, \dots, f_j)$ in $\Sigma_j \ltimes \mathrm{Aut}^G(G)^j$ to the G–automorphism $f_{\sigma^{-1}(1)} \times \cdots \times f_{\sigma^{-1}(j)}$ of G^j.

We write $\Sigma_j \wr \mathrm{Aut}^G(G) \cong \Sigma_j \wr G$ for the wreath product $\Sigma_j \ltimes \mathrm{Aut}^G(G)^j$. The free G–set G^j splits into $k = |G|^{j-1}$ orbits, and we fix a G–equivariant bijection $G^j \cong \coprod_k G$. This induces an isomorphism $\mathrm{Aut}^G(G^j) \cong \mathrm{Aut}^G(\coprod_k G)$, and we also have $\mathrm{Aut}^G(\coprod_k G) \cong \Sigma_k \wr G$. Thus we get a commutative diagram

(3-2)
$$\begin{tikzcd}
B(\Sigma_j \ltimes \mathrm{Aut}^G(G)^j) \arrow[r, "B\phi"] & B\mathrm{Aut}^G(G^j) \\
B(\Sigma_j \wr G) \arrow[u, "\cong"] \arrow[r, "B\phi"] & B(\Sigma_k \wr G) \arrow[u, "\cong"]
\end{tikzcd}$$

where we also write ϕ for the induced homomorphism $\Sigma_j \wr G \to \Sigma_k \wr G$.

Now we specialize to the case $p = 2$. First we study the Dyer–Lashof operation $Q^2\colon H_1(\mathbb{S}^{C_2}) \to H_3(\mathbb{S}^{C_2})$.

Lemma 3.1 *The operation Q^2 in $H_*(\mathbb{S}^{C_2})$ satisfies $Q^2(\alpha_1) = \alpha_3$.*

Proof Let $C = C_2$. By Lemma 2.1, we may instead compute Q^2 in $H_*(K(\Phi\mathcal{E}^C))$. We let $j = 2$, combine diagrams (2-1), (3-1) and (3-2), apply homology, and end up with the upper half of the diagram:

$$\begin{tikzcd}
H_*(E\Sigma_2 \ltimes_{\Sigma_2} K(\Phi\mathcal{E}^C)^{\wedge 2}) \arrow[r, "\kappa_{2*}"] & H_*(K(\Phi\mathcal{E}^C)) \\
H_*(B(\Sigma_2 \wr C)) \arrow[u] \arrow[r, "B\phi_*"] & H_*(B(\Sigma_2 \wr C)) \arrow[u, "\epsilon_* \circ \delta_*"'] \\
H_*(B(\Sigma_2 \times C)) \arrow[u, "Bd_*"] \arrow[r, "B\psi_*"] & H_*(B(C \times C)) \arrow[u, "B\iota_*"']
\end{tikzcd} \tag{3-3}$$

The vertical homomorphisms in the lower square are induced by the homomorphism $d = 1 \times \Delta$ that sends (σ, x) to $(\sigma; x, x)$, and the inclusion ι of the subgroup $C \times C = C^2$ in $\Sigma_2 \wr C = \Sigma_2 \ltimes C^2$. The homomorphism ψ is the restriction of ϕ to $\Sigma_2 \times C$. It is easily checked that ψ takes values in the subgroup $C \times C$ (since $p = 2$) and is given by $\psi(\sigma, x) = (x, \sigma x)$, using the description of ϕ given after diagram (3-1). We have $B\psi_*(e_i \otimes 1) = 1 \otimes \alpha_i$ and

$$B\psi_*(1 \otimes \alpha_j) = \Delta_*(\alpha_j) = \sum_{s+t=j} \alpha_s \otimes \alpha_t \,,$$

which combine to give

$$B\psi_*(e_i \otimes \alpha_j) = \sum_{s+t=j} \alpha_s \otimes (\alpha_i * \alpha_t) \,,$$

where $*$ denotes the Pontryagin product in $H_*(BC)$ induced by the topological group multiplication $BC \times BC \to BC$. We recall that $\alpha_i * \alpha_t = \binom{i+t}{i} \alpha_{i+t}$.

By May's paper [15, 9.1] the map Bd_* is given by

$$Bd_*(e_i \otimes \alpha_j) = \sum_k e_{i+2k-j} \otimes \mathrm{Sq}^k_*(\alpha_j) \otimes \mathrm{Sq}^k_*(\alpha_j) \,.$$

Recall that $\mathrm{Sq}^k_*(\alpha_j) = \binom{j-k}{k} \alpha_{j-k}$, where Sq^k_* denotes the dual of the Steenrod operation Sq^k. In particular $Bd_*(e_1 \otimes \alpha_2) = e_1 \otimes \alpha_1 \otimes \alpha_1$, which further maps to $Q_1(\alpha_1)$ in the upper right hand corner of (3-3). But now $Q_1(\alpha_1)$ is also the image of $e_1 \otimes \alpha_2$ under $\epsilon_* \circ \delta_* \circ B\iota_* \circ B\psi_*$. Using the description of $B\psi_*$ above, we see that $Q_1(\alpha_1)$ equals

$$\sum_{s+t=2} \epsilon_* \delta_* (1 \otimes \alpha_s \otimes (\alpha_1 * \alpha_t)) \,. \tag{3-4}$$

The map ϵ_* vanishes on decomposables with respect to the product in $H_*(N\mathcal{E}^C)$ which is induced by the additive symmetric monoidal structure on $\mathcal{E}^C$. The element

$\delta_*(1 \otimes \alpha_s \otimes (\alpha_1 * \alpha_t))$ is the image of

$$\alpha_s \otimes (\alpha_1 * \alpha_t) \in H_*(B\mathrm{Aut}^C(C) \times B\mathrm{Aut}^C(C))$$

in $H_*(B\mathrm{Aut}^C(C \coprod C)) \subset H_*(N\mathcal{E}^C)$ under the map induced by disjoint union, thus the only nonzero term in (3-4) is the one with $s = 0$ and $t = 2$, and $Q^2(\alpha_1) = Q_1(\alpha_1) = \epsilon_*(1 \otimes \alpha_0 \otimes (\alpha_1 * \alpha_2)) = \alpha_3$. $\square$

Proof of Theorem 0.2 We now turn to the operations in

$$H_*(\mathrm{TC}^{(1)}(\star; 2)) = \mathbb{F}_2 \oplus H_*(\mathbb{R}P^\infty_{-1}).$$

The general formula for the Q^i will follow from Lemma 3.1 and the Nishida relations, which say in particular (see Bruner et al [2, III.1.1]) that

$$\mathrm{Sq}_*^{i+j+1} Q^i(\alpha_j) = \sum_k \binom{2^N - j - 1}{2^N - i - 2j - 2 + 2k} Q^{k-j-1} \mathrm{Sq}_*^k(\alpha_j), \tag{3-5}$$

where N is sufficiently large. When $k \geq j+2$ the element $\mathrm{Sq}_*^k(\alpha_j)$ is zero for degree reasons, and when $k \leq j$ the fact that Q^{k-j-1} vanishes on classes in degree higher than $k-j-1$ implies that $Q^{k-j-1} \mathrm{Sq}_*^k(\alpha_j) = 0$. Hence the sum in (3-5) simplifies to the single term

$$\mathrm{Sq}_*^{i+j+1} Q^i(\alpha_j) = \binom{2^N - j - 1}{2^N - i} Q^0 \mathrm{Sq}_*^{j+1}(\alpha_j)$$

for $k = j+1$, where N is large.

Bob Bruner has observed that

$$\binom{2^N - j - 1}{2^N - i} \equiv \binom{2^N + i - 1}{2^N + j}$$

mod 2, for large N. Here is a quick proof. Let x_k denote the k–th bit in the binary expansion of a natural number x. Then

$$\binom{2^N - j - 1}{2^N - i} \equiv 1$$

if and only if $(2^N - i)_k = 1$ implies $(2^N - j - 1)_k = 1$ for all k, and

$$\binom{2^N + i - 1}{2^N + j} \equiv 1$$

if and only if $(2^N + j)_k = 1$ implies $(2^N + i - 1)_k = 1$ for all k. But for N large compared to i, j and k the bit $(2^N - i)_k$ is complementary to $(2^N + i - 1)_k$, and

$(2^N - j - 1)_k$ is complementary to $(2^N + j)_k$, so

$$\binom{2^N - j - 1}{2^N - i} \equiv 1$$

if and only if

$$\binom{2^N + i - 1}{2^N + j} \equiv 1.$$

The operations Sq_*^k in $H_*(\mathbb{R}P_{-1}^\infty)$ are given by the formula

$$\mathrm{Sq}_*^k(\alpha_j) = \binom{j-k}{k}\alpha_{j-k}\,.$$

This follows by the corresponding formula for $\mathbb{R}P^\infty$ and James periodicity. More precisely, a theorem of James [12] says that given $m \le n$, there is a positive integer M, depending only on $n-m$, such that $\mathbb{R}P_{m+\ell}^{n+\ell} \simeq \Sigma^\ell \mathbb{R}P_m^n$ when ℓ is a positive multiple of 2^M. The space $\mathbb{R}P_m^n$ is the stunted projective space $\mathbb{R}P^n/\mathbb{R}P^{m-1}$. If we now define the spectrum $\mathbb{R}P_{-1}^n$ to be $\Sigma^{-\ell}\Sigma^\infty \mathbb{R}P_{\ell-1}^{n+\ell}$ for such ℓ (depending on n), we have that $\mathbb{R}P_{-1}^\infty = \mathrm{colim}_n\, \mathbb{R}P_{-1}^n$. The Steenrod operations in $H_*(\mathbb{R}P_{-1}^\infty)$ can now be calculated from the operations in $H_*(\mathbb{R}P_{\ell-1}^{n+\ell})$, and the stated formula follows by noting that the relevant binomial coefficients are 2^M–periodic in the numerator.

In particular $\mathrm{Sq}_*^{j+1}(\alpha_j) = \alpha_{-1}$ for all $j \ge -1$, and we have

$$\mathrm{Sq}_*^{i+j+1}\, Q^i(\alpha_j) = \binom{2^N + i - 1}{2^N + j} Q^0(\alpha_{-1})\,.$$

If $Q^0(\alpha_{-1})$ were zero, it would follow that $Q^i(\alpha_j) = 0$ for all i and j, since Sq_*^{i+j+1} is an isomorphism to dimension -1. But this contradicts Lemma 3.1. Hence $Q^0(\alpha_{-1}) = \alpha_{-1}$, and the formula stated in the theorem follows. □

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