

Quotient groups of the fundamental groups of certain strata of the moduli space of quadratic differentials

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In this paper, we study fundamental groups of strata of the moduli space of quadratic differentials. We use certain properties of the Abel–Jacobi map, combined with local surgeries on quadratic differentials, to construct quotient groups of the fundamental groups for a particular family of strata.

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1 Introduction

Little is currently known about the fundamental groups of strata of the moduli spaces of either abelian or quadratic differentials. In this paper we construct a quotient group of the fundamental group for a certain family of strata of quadratic differentials. We do so by mapping a stratum into a larger configuration space of points on surfaces and showing that the image of the fundamental group of the stratum under this map is in the kernel of a version of the Abel–Jacobi map. We then construct a set of generators for the kernel of the Abel–Jacobi map and show that in some cases the image of the fundamental group of the stratum in the fundamental group of the configuration space is equal to this kernel.

More specifically, let \mathcal{Q}_g be the space of quadratic differentials over Teichmüller space \mathcal{T}_g and let $\lambda = (k_1, \dots, k_n)$ be a partition of $4g - 4$. Define $\mathcal{Q}_g(k_1, \dots, k_n) = \mathcal{Q}_\lambda$ to be the subset of \mathcal{Q}_g of quadratic differentials with n zeroes of order k_1, \dots, k_n . Let $\bar{\mathcal{Q}}_g$ and $\bar{\mathcal{Q}}_\lambda$ be the analogous spaces over moduli space, \mathcal{M}_g . We are interested in $\pi_1(\bar{\mathcal{Q}}_\lambda)$; however, when \mathcal{Q}_λ and $\bar{\mathcal{Q}}_\lambda$ are both connected we have a short exact sequence

$$1 \rightarrow \pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\bar{\mathcal{Q}}_\lambda) \rightarrow \Gamma_g \rightarrow 1$$

where Γ_g is the genus g mapping class group. In many cases (although not all) both \mathcal{Q}_λ and $\bar{\mathcal{Q}}_\lambda$ are connected, so we focus on proving results about $\pi_1(\mathcal{Q}_\lambda)$.

To do this we first embed \mathcal{Q}_λ into a larger configuration space. In particular, to any partition λ we associate a generalized symmetric group S_λ that allows points of equal weights to be exchanged. For $M \in \mathcal{T}_g$ and λ of length n , let $M^{[n]}$ denote the space of

n ordered distinct marked points on M , and let $\text{Sym}^\lambda(M)$ denote $M^{[n]}/S_\lambda$. Define Sym_g^λ to be the associated bundle over \mathcal{T}_g , and Pic_g^{4g-4} the bundle over \mathcal{T}_g with fiber $\text{Pic}^{4g-4}(M)$, the Picard variety parametrizing line bundles on M of degree $4g - 4$. Then we have the following maps:

$$(1) \quad \mathcal{Q}_\lambda \xrightarrow{i} \text{Sym}_g^\lambda \xrightarrow{\text{AJ}} \text{Pic}_g^{4g-4}$$

The first map is given by considering the zeroes of a quadratic differential as weighted marked points. The second map is the Abel–Jacobi map, given by mapping a divisor to its associated line bundle. The maps in (1) induce a sequence of maps:

$$(2) \quad \pi_1(\mathcal{Q}_\lambda) \xrightarrow{i_*} \pi_1(\text{Sym}_g^\lambda) \xrightarrow{\text{AJ}_*} \pi_1(\text{Pic}_g^{4g-4}) \cong H_1(M, \mathbb{Z})$$

We show that $\text{AJ}_* \circ i_*: \pi_1(\mathcal{Q}_\lambda) \rightarrow H_1(\Sigma, \mathbb{Z})$ is trivial, so the image of i_* will be in the kernel of AJ_* . In the case where there are at least $O(\sqrt{g})$ zeroes of order 1 in λ , we are able to construct a set of generators for the kernel of AJ_* .

Theorem 1.1 *Let $\lambda = (k_1^a, k_2^{b_2}, \dots, k_m^{b_m})$, with exponents satisfying $\sum_{i=2}^m b_i = b$ and $a \geq (3 + \sqrt{9 + 8(2g + b - 2)})/2$. Then the kernel of $\text{AJ}_*: \pi_1(\text{Sym}_g^\lambda) \rightarrow H_1(M, \mathbb{Z})$ is generated by transpositions of zeroes of equal weight, squares of transpositions of zeroes of unequal weight, moving sets of points of equal weight opposite ways around generators of $\pi_1(M)$, and in some cases moving single points around homologically trivial curves.*

A more precise statement of [Theorem 1.1](#) is given in [Section 4](#). To show that the elements detailed in [Theorem 1.1](#) are contained in $\pi_1(\mathcal{Q}_\lambda)$ we first follow a variety of authors, including Eskin, Masur and Zorich [\[3\]](#), Laneeau [\[8\]](#) and Kontsevich and Zorich [\[6\]](#), to create local surgeries on surfaces with quadratic differentials that affect the flat metric induced by the quadratic differential in only a small area around a zero. We then take advantage of the fact that for any genus 0 stratum, $\mathcal{Q}_\lambda \cong \text{Sym}^\lambda(\mathbb{P}^1)$ (in other words, a quadratic differential on \mathbb{P}^1 may have zeroes at any set of points) to create explicit curves of quadratic differentials in the hyperelliptic loci of certain strata. This leads to the following theorem:

Theorem 1.2 *Let $\lambda = (1^a, k_1, \dots, k_n)$ with $a > \max\{g + 5, k_1, \dots, k_n\}$, all k_i even, and some $k_i = k_j$, $1 \leq i, j \leq n$. Then $\text{im}(i_*: \pi_1(\mathcal{Q}_\lambda, (M, q)) \rightarrow \pi_1(\text{Sym}_g^\lambda)) = \text{ker}(\text{AJ}_*)$.*

[Theorem 1.2](#) states that for certain λ we can describe the image of $\pi_1(\mathcal{Q}_\lambda)$ in $\pi_1(\text{Sym}_g^\lambda)$. However, the kernel of $\pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\text{Sym}_g^\lambda)$ may be nontrivial, so we have created a quotient group of $\pi_1(\mathcal{Q}_\lambda)$.

The structure of the paper is as follows. In [Section 2](#) we give some general background. In [Section 3](#) we collect some results about $\pi_1(\text{Sym}_g^\lambda)$, and in [Section 4](#) we construct a set of generators for the kernel of AJ_* when λ has sufficiently many zeroes of the same order. In [Section 5](#) we recall some local surgeries that allow us alter the flat metric associated to a quadratic differential near a particular zero. In [Section 6](#) and [Section 7](#) we use the results of [Section 5](#) to construct explicit elements in the $\ker(\text{AJ}_*)$. [Section 8](#) summarizes when we have the image of $\pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\text{Sym}_g^\lambda)$ equal to the kernel of AJ_* : $\pi_1(\text{Sym}_g^\lambda) \rightarrow H_1(M, \mathbb{Z})$. [Theorem 1.1](#) and [Theorem 1.2](#) are proved in [Section 4](#) and [Section 8](#), respectively.

2 Preliminary definitions

A meromorphic quadratic differential, q , on a Riemann surface, M , is a meromorphic section of the square of the canonical bundle, K_M , of M . In local coordinates q assigns to each (U_α, z_α) a meromorphic function f_α such that

$$f_\beta(z_\beta) \left(\frac{dz_\beta}{dz_\alpha} \right)^2 = f_\alpha(z_\alpha), \quad dz_\beta = \frac{d}{dz_\alpha} z_\beta dz_\alpha$$

on $U_\alpha \cap U_\beta$.

A horizontal trajectory, or simply a trajectory, of a quadratic differential q on M is a smooth curve $\gamma: [0, 1] \rightarrow M$ such that $f(\gamma(t))(\gamma'(t)dt)^2$ is real and positive for all t . Similarly, a vertical trajectory of q is $\gamma: [0, 1] \rightarrow M$ such that $f(\gamma(t))(\gamma'(t)dt)^2$ is real and negative, and a θ -trajectory is γ such that the argument of $f(\gamma(t))(\gamma'(t)dt)^2$ is 2θ . (For $q = dz^2$ on \mathbb{C} these correspond to straight lines of angle θ .) Through every regular point of q there exist unique horizontal and vertical trajectories that are transverse. Near a zero of q of order n , p_0 , we can choose a local coordinate, ζ , such that $\zeta = 0$ at p_0 and $q = \zeta^n d\zeta^2$. Then the $n + 2$ curves $\gamma(t) = te^{i(2\pi/(n+2))k}$ for $k = 0, 1, \dots, n + 1$, are all horizontal trajectories that dead-end into p_0 . A θ -trajectory between two critical points of q is called a saddle connection.

Any nonzero quadratic differential on M gives us a metric on M

$$|\gamma|_q = \int_\gamma |f(z)|^{1/2} |dz|$$

for γ a real curve on M . Geodesics in this metric are unions of θ -trajectories, with vertices at critical points of q .

Through much of this paper we will be concerned not just with individual quadratic differentials but also their moduli spaces. Let \mathcal{T}_g be the Teichmüller space of a closed

genus g surface and \mathcal{M}_g the genus g moduli space. We consider the bundle \mathcal{E} over \mathcal{T}_g with fiber $H^0(M, K_M^2)$ over $M \in \mathcal{T}_g$, and define \mathcal{Q}_g to be the subspace of the total space of \mathcal{E} consisting of quadratic differentials that are not zero sections of K_M^2 . Define $\mathcal{Q}_g(k_1, \dots, k_n)$ to be the subspace of \mathcal{Q}_g of quadratic differentials with zeroes of order k_1, \dots, k_n , $\sum_1^n k_i = 4g - 4$ and $k_i \in \mathbb{N}$. The set of integer partitions of $4g - 4$ give a natural stratification of \mathcal{Q}_g and a single $\mathcal{Q}_g(k_1, \dots, k_n)$ is often called *stratum* of \mathcal{Q}_g . We also occasionally consider an analog of \mathcal{E} where each fiber is the space of meromorphic sections of K^2 with up to some fixed number of single poles, and then we can consider $\mathcal{Q}_g(k_1, \dots, k_n)$ where some of the $k_i = -1$. All of these spaces are well known to be manifolds (see Veech [12], for example). \mathcal{E} is also well-known to be the cotangent bundle of \mathcal{T}_g , and from this we see \mathcal{Q}_g is a complex manifold of dimension $6g - 6$. Unless $n = 1$ $\mathcal{Q}_g(k_1, \dots, k_n)$ is not closed in \mathcal{Q}_g , as zeroes may collide to form higher order zeroes.

One can construct the analog of \mathcal{E} over \mathcal{M}_g , $\bar{\mathcal{E}} \rightarrow \mathcal{M}_g$, again with fiber $H^0(\bar{M}, K_{\bar{M}}^2)$ over $\bar{M} \in \mathcal{M}_g$. Let $\bar{\mathcal{Q}}_g$ be the total space of $\bar{\mathcal{E}}$ minus zero sections. Alternatively, the mapping class group, Γ_g , acts on \mathcal{Q}_g by a lift of its action on \mathcal{T}_g . We can also define $\bar{\mathcal{Q}}_g$ as the quotient of \mathcal{Q}_g by Γ_g . Similarly define $\bar{\mathcal{Q}}_g(k_1, \dots, k_n)$ as the quotient of $\mathcal{Q}_g(k_1, \dots, k_n)$ by Γ_g . Since Γ_g does not act freely, the $\bar{\mathcal{Q}}_g(k_1, \dots, k_n)$ will be complex orbifolds.

In general we do not require the k_i to be distinct, but if a stratum has multiple zeroes of the same order we will sometimes use the notation $\mathcal{Q}_g(k_1^n, k_2, \dots, k_n)$ to indicate $\mathcal{Q}_g(k_1, k_1, \dots, k_1, k_2, \dots, k_n)$. We denote elements of $\mathcal{Q}_g(k_1, \dots, k_n)$ as (M, q) where M is a Riemann surface together with a homeomorphism from a fixed genus g surface to M , defined up to isotopy. When the specific orders of the zeroes are not important we will sometimes let $\lambda = (k_1, \dots, k_n)$ denote a partition of $4g - 4$ and let $\mathcal{Q}_\lambda = \mathcal{Q}_g(k_1, \dots, k_n)$. The *length* of λ will be the number of k_i in the partition.

The following sums up the structure of various strata:

Theorem 2.1 (Masur, Smillie, Veech) *Every $\mathcal{Q}_g(k_1, \dots, k_n)$ is nonempty, with four exceptions: $\mathcal{Q}_1(\emptyset)$, $\mathcal{Q}_1(-1, 1)$, $\mathcal{Q}_2(3, 1)$, $\mathcal{Q}_2(4)$. With the exception of these four strata, the $\mathcal{Q}_g(k_1, \dots, k_n)$ are complex manifolds of dimension $2g - 2 + n$.*

The same is true of the $\bar{\mathcal{Q}}_g(k_1, \dots, k_n)$, except that they are orbifolds instead of manifolds. Proofs of the above may be found in Masur and Smillie [9] and Veech [12]. Genus 0 is special in that it is possible to construct a rational function, $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, with zeroes and poles wherever desired, and $f(z)dz^2$ is then a quadratic differential on \mathbb{P}^1 . This implies that $\mathcal{Q}_0(k_1, \dots, k_n)$ is simply the configuration space of n points on the sphere, which is well-known to be connected.

Proposition 2.2 Any $\mathcal{Q}_0(k_1, \dots, k_n)$ is connected.

If one has a (ramified) cover of some M , one can pull back a quadratic differential on M to get one on its cover.

Lemma 2.3 Let $\pi: \tilde{M} \rightarrow M$ be a ramified double cover, q a quadratic differential on M and \tilde{q} its pullback under π . Let \tilde{p} be a ramification point of π and $p = \pi(\tilde{p})$. Then, if p is a singularity of order k of q , \tilde{p} will be a singularity of order $2k + 2$ on \tilde{q} .

One can see this by noticing that a singularity of order k corresponds to a cone angle of $(k + 2)\pi$; doubling the cone angle at this singularity gives a cone angle of $2k + 4\pi$ or a singularity of order $2k + 2$. More detailed proofs of Proposition 2.2 and Lemma 2.3 may be found in Lanneau [8].

Using double covers one may construct a continuous map between two different strata of quadratics differentials.

Construction 2.4 Let $\sum_1^n k_i = -4$, $n \geq 2g + 2$. We can construct a local map $\mathcal{Q}_0(k_1, \dots, k_n) \rightarrow \mathcal{Q}_g(2k_1 + 2, \dots, 2k_{2g+2} + 2, k_{2g+3}^2, \dots, k_n^2)$ by taking $(M', q') \in \mathcal{Q}_0(k_1, \dots, k_n)$ to its double cover ramified at its zeroes of order k_1, \dots, k_{2g+2} . This gives a surface (M, q) of genus g such that each of the first $2g + 2$ zeroes of order k_i goes to one with order $2k_i + 2$, and we get 2 copies of each remaining zero.

For the above map to be global one would need to be able to distinguish between zeroes of the same order, so as to consistently choose ramification points.

Definition 2.5 Define a quadratic differential $(M, q) \in \mathcal{Q}_g(k_1, \dots, k_n)$ to be *hyperelliptic* if M is a hyperelliptic Riemann surface and q is invariant under the hyperelliptic involution. Define $\mathcal{Q}_g(k_1, \dots, k_n)$ to be *hyperelliptic* if it contains a hyperelliptic quadratic differential.

This definition of a hyperelliptic stratum is somewhat different from one frequently used in the literature, where a connected component of a stratum is called hyperelliptic if every quadratic differential in it is hyperelliptic.

Finally, since $1 \rightarrow \Gamma_g \rightarrow \mathcal{Q}_\lambda \rightarrow \bar{\mathcal{Q}}_\lambda \rightarrow 1$ is a fibration, if \mathcal{Q}_λ and $\bar{\mathcal{Q}}_\lambda$ are both connected then $1 \rightarrow \pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\bar{\mathcal{Q}}_\lambda) \rightarrow \pi_0(\Gamma_g) \cong \Gamma_g \rightarrow 1$ will be short exact. Interestingly, not all of the $\bar{\mathcal{Q}}_\lambda$ are actually connected. For those that are disconnected, the connected components are classified by whether or not hyperelliptic quadratic differentials form a full-dimensional subset of the stratum. Lanneau proves this in [8].

Theorem 2.6 (Lanneau) *For $g \geq 3$ the following strata have two connected components:*

- (1) $\bar{Q}_g(4(g-k) - 6, 4k + 2)$, $0 \leq k \leq g - 2$
- (2) $\bar{Q}_g((2(g-k) - 3)^2, 4k + 2)$, $0 \leq k \leq g - 1$
- (3) $\bar{Q}_g((2(g-k) - 3)^2, 2k + 1^2)$, $-1 \leq k \leq g - 2$

where one is hyperelliptic and the other is not. For $g \geq 5$ all other strata have one component; for $g = 3, 4$ all other strata are connected with the following four exceptions: $\bar{Q}_3(-1, 9)$, $\bar{Q}_3(-1, 3, 6)$, $\bar{Q}_3(-1, 3, 3, 3)$, and $\bar{Q}_4(12)$. These four sporadic strata have two connected components, but neither is hyperelliptic. For $g = 0, 1$ all strata are connected, and for $g = 2$ $\bar{Q}_2(3, 3, -1, -1)$ and $\bar{Q}_2(6, -1, -1)$ have two components, but all others are connected.

In [13] we showed the following:

Theorem 2.7 *Let $g \geq 2$. Then any stratum of the form $Q_g(1^g, k_1, \dots, k_n)$ is connected.*

For λ as in Theorem 2.7 both Q_λ and \bar{Q}_λ are connected, and $1 \rightarrow \Gamma_g \rightarrow \pi_1(Q_\lambda) \rightarrow \pi_1(\bar{Q}_\lambda) \rightarrow 1$ is short exact.

3 Surface braid groups

In this section we collect some results about surface braid groups, to use in analyzing the kernel of AJ_* .

Let S_n be the standard symmetric group on n letters. To any partition, λ , of $4g - 4$ we associate a symmetric group, S_λ , which allows equal values to be exchanged. For example, to $(1^4, 2, 5^2)$ we associate $S_4 \times S_2$. The length of λ will be the number of elements it contains. For a particular $M \in \mathcal{T}_g$ and partition λ of length n , let $M^{[n]}$ denote the space of n ordered distinct marked points on M , and let $\text{Sym}^\lambda(M)$ be $M^{[n]}/S_\lambda$.

$\pi_1(M^{[n]})$ is well-known as the pure or special braid group of n elements on a genus g surface, which we will denote by $SB_n(M)$ or simply SB_n . Similarly, $\pi_1(M^{[n]}/S_n)$ is the full braid group on M , $B_n(M)$ or B_n . The generators of both SB_n and B_n are well-known. In particular, let l_1, \dots, l_{2g} be $2g$ standard generators of $\pi_1(M)$, let $(p_1, \dots, p_n) \in M^{[n]}$, and let ρ_{ij} , $1 \leq i \leq n$, $1 \leq j \leq 2g$, denote an element of SB_n such that p_i follows a path that is homotopic to l_j . Let D be any disk containing

$p_k, p_l, 1 \leq k < l \leq n$, and let κ_{kl} be either generator of $\pi_1(D^{[2]}, (p_k, p_l))$. (We may extend this to an element of $\pi_1(M^{[n]})$ by letting the other $n - 2$ points move along constant paths.) Similarly, let σ_{st} be either generator of $\pi_1(D^{[2]}/S_2, (p_s, p_t))$. The following theorem is classical.

Theorem 3.1 SB_n is generated by the $\rho_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2g$, and the $\kappa_{kl}, 1 \leq k < l \leq n$. B_n is generated by the ρ_{ij} and the $\sigma_{s(s+1)}, 1 \leq s < n$.

It should be noted that there are multiple nonequivalent ways to define each of the ρ_{ij}, κ_{kl} , and $\sigma_{s(s+1)}$; however, any choice yields a generating set.

Let M_m denote $M - \{p_{n+1}, \dots, p_{n+m}\}$, where the p_{n+i} are any distinct points on M , and let $SB_{n,m}$ denote $SB_n(M_m)$ and $B_{n,m}$ denote $B_n(M_m)$. (We will only be concerned with the topology of M_m , which does not depend on the choice of $\{p_{n+1}, \dots, p_{n+m}\}$.) Let $\kappa_{kl}, 1 \leq k \leq n, n < l \leq n + m$, denote p_k moving in a simple loop around p_l . Again the generators of both $SB_{n,m}$ and $B_{n,m}$ are well-known.

Theorem 3.2 $SB_{n,m}$ is generated by the ρ_{ij} and the $\kappa_{kl}, 1 \leq j \leq 2g, 1 \leq i, k \leq n, 1 \leq l \leq n + m$. $B_{n,m}$ is generated by the ρ_{ij} , the $\sigma_{s(s+1)}, 1 \leq s < n$, and the $\kappa_{kl}, 1 \leq k \leq n, n < l \leq n + m$.

A reference for [Theorem 3.1](#) and [Theorem 3.2](#) is Birman [1].

We will primarily be interested in $\pi_1(\text{Sym}^\lambda(M))$, which we will denote by $B_\lambda(M)$ or B_λ . For $\lambda = (k^n)$, B_λ is just B_n . For more complicated λ we note that the covering map $M^{[n]} \rightarrow \text{Sym}^\lambda(M)$ is normal and thus

$$SB_n \rightarrow B_\lambda \rightarrow S_\lambda$$

is a short exact sequence. This tells us the generators of B_λ :

Proposition 3.3 Let λ be a partition of $4g - 4$ of length n . Then B_λ is generated by the $\rho_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2g$, and for each pair $1 \leq k < l \leq n$, either σ_{kl} if p_k and p_l are of the same weight, or κ_{kl} if p_k and p_l are of different weights.

In fact it is possible to generate B_λ with fewer transpositions; however, this generating set will suffice for our purposes. Although it will not be explicitly used in this paper, it is also worth noting that for particular choices of generating sets, the relations among the generators of $SB_m, SB_{n,m}$ and B_n are well-known (see Scott [11], for example), and thus the same will be true for any of their subgroups.

Another classical result about surface braid groups is the following, from [4]:

Theorem 3.4 (Fadell–Neuwirth) The map $(M_m)^{[n]} \rightarrow (M_m)^{[r]}$ is a fibration, with fiber $(M_{m+r})^{[n-r]}$.

This fibration induces a long exact sequence of homotopy groups. All higher homotopy groups are trivial, so

$$1 \rightarrow SB_{n-r,m+r}(M) \rightarrow SB_{n,m}(M) \rightarrow SB_{r,m}(M) \rightarrow 1$$

is a short exact sequence.

Finally, we will sometimes want to distinguish the many different transpositions of two points on a surface. Let $p, p' \in M$ and define an *edge*, e , to be an embedding of the interval $[0, 1]$ in M with endpoints p and p' . Let $U \subset M$ be a contractible neighborhood of e . Then we define σ_e to be either of the two generators of $\pi_1(U^{[2]}/S_2, (p, p'))$, with σ_e^{-1} its inverse, and κ_e to be either of the two generators of $\pi_1(U^{[2]}, (p, p'))$ with κ_e^{-1} its inverse. For a particular (M, q) , let $P = \{p_1, \dots, p_n\}$ be the zeroes of q and define $\bar{E}_{M,q} = \{e: I \hookrightarrow M \mid e(I) \cap P = e(0) \cup e(1)\}$ to be the set of all edges on (M, q) . We say $e \sim e'$ if e and e' are homotopic in $M \setminus P$ relative to their endpoints, and define $E_{M,q} = \bar{E}_{M,q} / \sim$. We index the set of all transpositions associated to (M, q) by $E_{M,q}$.

4 The kernel of the Abel–Jacobi map

In this section we first define the Abel–Jacobi map, and then construct a set of generators for its kernel.

Define Sym_g^λ to be the bundle over \mathcal{T}_g with fiber $\text{Sym}^\lambda(M)$. Similarly let $\text{Pic}^{4g-4}(M)$ be the Picard variety parametrizing line bundles of degree $4g - 4$, and define Pic_g^{4g-4} to be the bundle over \mathcal{T}_g with fiber $\text{Pic}^{4g-4}(M)$. Let Λ_g be the set of all integer partitions of $4g - 4$.

Then we have a sequence of maps:

$$(3) \quad \mathcal{Q}_\lambda \xrightarrow{i} \text{Sym}_g^\lambda \xrightarrow{\text{AJ}} \text{Pic}_g^{4g-4}$$

The first map is given by considering the zeroes of a quadratic differential as marked points, weighted by the order of the zero. The second map is the Abel–Jacobi map, given by mapping a divisor to its associated line bundle, $\text{AJ}(D) = |D|$. The composition of these maps is fiber-wise trivial because every element of \mathcal{Q}_λ over a particular $M \in \mathcal{T}_g$ maps to $K_M^2 \in \text{Pic}_g^{4g-4}$.

Using the fact that $\text{Pic}(M) \cong \text{Jac}(M)$, the Jacobian variety of M , AJ may alternatively be defined (on each fiber of Sym_g^λ) as a map $\text{Sym}_g^\lambda(M) \rightarrow \text{Jac}(M)$, defined as follows.

Choose holomorphic 1-forms ω_i , $1 \leq i \leq g$ such that the ω_i form a basis for $H^0(M, \Omega_M^1)$, and choose a basepoint $p_0 \in M$. Then for $D = \sum_1^n k_i p_i$,

$$AJ(D) = \left(\sum_1^n k_i \int_{p_0}^{p_i} \omega_1, \dots, \sum_1^n k_i \int_{p_0}^{p_i} \omega_g \right).$$

See Griffiths and Harris [5], for example, for the equivalence of these definitions of AJ. Since $Jac(M) = H^0(M, K_M)^*/H_1(M, \mathbb{Z})$, there is a natural isomorphism $H_1(Jac(M), \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$. The Hurewicz map, $\pi_1(Jac(M)) \rightarrow H_1(Jac(M), \mathbb{Z})$, is also an isomorphism in this case. From this perspective one can see that

$$AJ_*: \pi_1(\text{Sym}_g^\lambda(M)) \rightarrow H_1(Jac(M), \mathbb{Z}) \cong H_1(M, \mathbb{Z})$$

takes a braid in $\pi_1(\text{Sym}_g^\lambda(M))$ to a union of homology classes of loops in M , where the loop followed by p_i is weighted by k_i .

As previously noted it is possible that Q_λ is not connected, and in general we will denote a connected component of Q_λ by Q_λ^0 . Then the maps in (3) induce a sequence of maps:

$$(4) \quad \pi_1(Q_\lambda^0) \xrightarrow{i_*} \pi_1(\text{Sym}_g^\lambda) \xrightarrow{AJ_*} \pi_1(\text{Pic}_g^{4g-4}) \cong H_1(M, \mathbb{Z})$$

($\pi_1(\text{Pic}_g^{4g-4}(M)) \cong \pi_1(Jac(M)) \cong H_1(M, \mathbb{Z})$, and Pic_g^{4g-4} is a bundle over a contractible space, so $\pi_1(\text{Pic}_g^{4g-4}) \cong H_1(M, \mathbb{Z})$ as well.) $AJ_* \circ i_*$ will take the sum of the paths travelled by the zeroes of quadratic differentials to their corresponding homology class, where the paths are weighted by the order of the zero.

Since \mathcal{T}_g is simply connected, if the map $Q_\lambda \rightarrow \mathcal{T}_g$ has no empty fibers, then the image of Q_λ under $AJ \circ i$ is also simply connected. In this case $AJ_* \circ i_*$ will again be trivial.

It is not always true that $Q_\lambda \rightarrow \mathcal{T}_g$ has no empty fibers; for example, there exist λ for which the dimension of Q_λ is less than the dimension of \mathcal{T}_g . In this case we consider the inclusion $\iota: Q_\lambda \hookrightarrow Q_g$, and the induced map of their fundamental groups.

$$\begin{array}{ccc}
 Q_\lambda^0 & \xrightarrow{i} \text{Sym}_g^\lambda & \xrightarrow{AJ} \text{Pic}_g^{4g-4} \\
 & \searrow \iota & \uparrow AJ \\
 & & Q_g
 \end{array}
 \qquad
 \begin{array}{ccc}
 \pi_1(Q_\lambda^0) & \xrightarrow{i_*} \pi_1(\text{Sym}_g^\lambda) & \xrightarrow{AJ_*} H_1(M, \mathbb{Z}) \\
 & \searrow \iota_* & \uparrow AJ_* \\
 & & \pi_1(Q_g)
 \end{array}$$

The diagram on the left clearly commutes, so the diagram on the right does as well. Since the image of Q_g under $Q_g \rightarrow \mathcal{T}_g$ is indeed simply connected, we have $AJ_* \circ \iota_*$ is trivial, and thus $AJ_* \circ i_*$ is as well. The combination of these two cases gives us the following.

Proposition 4.1 For all $\lambda \in \Lambda_g$ and any connected component \mathcal{Q}_λ^0 of \mathcal{Q}_λ

$$\text{AJ}_* \circ i_*: \pi_1(\mathcal{Q}_\lambda^0) \rightarrow H_1(M, \mathbb{Z})$$

is trivial.

Proposition 4.1 implies that the image of $\pi_1(\mathcal{Q}_\lambda^0) \rightarrow \pi_1(\text{Sym}_g^\lambda)$ will be in the kernel of AJ_* . Since \mathcal{T}_g is contractible, $\pi_1(\text{Sym}_g^\lambda) \cong B_\lambda$. Thus, we now use the generators of B_λ constructed in the previous section to create a set of generators for $\ker(\text{AJ}_*: \pi_1(\text{Sym}_g^\lambda) \rightarrow H_1(M, \mathbb{Z}))$. In Section 4.1 we consider $\lambda = (k^n)$, and in Section 4.2 more general λ .

4.1 Strata with zeroes of only one weight

We wish to calculate the kernel of AJ_* for $\lambda = (k^n)$ and n reasonably large. Recall that we defined M_f to be the surface M with f punctures. We have the following from [2]:

Theorem 4.2 (Copeland) *If the surface M has the structure of a polyhedron (a two-dimensional cell complex) of genus g with n vertices and f faces such that the associated 1-skeleton has no double edges and no loops (edges with both ends at the same vertex), then $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M_f)) \rightarrow H_1(M, \mathbb{Z}))$ is generated by the edge set. Specifically the base point of $\text{Sym}^n(M_f)$ may be chosen to be the vertices of the cell complex, each face may be viewed as having a puncture, and each edge may be viewed as a transposition of its two vertices.*

Using the natural projection $\pi_1(\text{Sym}^n(M_f)) \rightarrow \pi_1(\text{Sym}^n(M))$ one can show that the kernel of $\pi_1(\text{Sym}^n(M)) \rightarrow H_1(M, \mathbb{Z})$ is a quotient group of the kernel of the map $\pi_1(\text{Sym}^n(M_f)) \rightarrow H_1(M, \mathbb{Z})$ and is also generated by transpositions. Then Theorem 4.2 implies that if we can construct a graph with any number of faces on M of the form described in the theorem, the kernel of $\text{AJ}_*: \pi_1(\text{Sym}^n(M)) \rightarrow H_1(M, \mathbb{Z})$ is generated by transpositions.

Copeland shows in [2] that it is possible to construct such a graph for any $g > 2$ and $n = 4g - 4$. We would like to show the same is true for smaller n , since $\lambda = (k^n)$ implies $n \leq 4g - 4$. In general, the best bound for n we can hope to achieve will be on the order of \sqrt{g} . This is because a graph with n vertices, no double edges, and no loops can have a maximum of $\binom{n}{2}$ edges. The Euler characteristic then implies that

$$2 - 2g \leq n - \binom{n}{2} + f.$$

Since f must be ≥ 1 , at most g grows at the rate of n^2 . More specifically, solving the equation above we get $n \geq (3 + \sqrt{9 + 8(2g + f - 2)})/2$. To show that graphs of the required form exist for n close to this bound we will need some standard results from graph theory. Through the remainder of the section we will assume all graphs are connected, with no double edges or loops.

A 2-cell embedding of a graph, G , into a genus g topological surface, M_g , is an embedding, $\varphi: G \rightarrow M_g$, such that each component of $M_g \setminus \varphi(G)$ is a 2-cell. The genus of G , $\gamma(G)$, is the minimal g for which G embeds into M_g (such an embedding will always be a 2-cell embedding). The maximal genus of G , $\gamma_M(G)$, is the maximal g for which G has a 2-cell embedding into M_g . Let $v(G)$ and $e(G)$ denote the number of vertices and edges of G . For a 2-cell embedding of G , let $f(G, \varphi)$ denote the number of components of $M_g \setminus \varphi(G)$, which we will call faces. Finally, let K_n denote the complete graph on n vertices. The following are well-known results in graph theory:

Proposition 4.3 For any integer $n \geq 3$:

- (1) $\gamma_M(K_n) = \lfloor (e(K_n) - v(K_n) + 1)/2 \rfloor = \lfloor (n - 2)(n - 1)/4 \rfloor$.
- (2) $\gamma(K_n) = \lceil (n - 3)(n - 4)/12 \rceil$.
- (3) K_n has a 2-cell embedding into M_g if and only $\gamma(K_n) \leq g \leq \gamma_M(K_n)$.

See Kronk, Ringelsen and White [7], for example, for a survey of these results. Note that the maximal genus of G is simply the largest genus for which $v(G) - \binom{v(G)}{2} + f = 2 - 2g$ has a positive solution for f . A 2-cell embedding of K_n into M_g has $\binom{n}{2} - n + 2 - 2g$ faces. We would like to show that graphs with a wider range of faces embed into M_g , and in fact knowing that K_n 2-cell embeds into M_g we can also show that “almost complete” graphs on n vertices have 2-cell embeddings into M_g .

Lemma 4.4 If K_n has a 2-cell embedding into M_g , then for any f such that $1 \leq f \leq \binom{n}{2} - n + 2 - 2g$ there exists a graph with n vertices that has a 2-cell embedding into M_g with f faces.

Proof We prove the lemma by induction on the number of faces. First, K_n embeds into M_g with $\binom{n}{2} - n + 2 - 2g$ faces. Now suppose we have a connected graph G that has a 2-cell embedding $\varphi: G \rightarrow M_g$ with $f(G, \varphi)$ faces, $2 \leq f(G, \varphi) \leq \binom{n}{2} - n + 2 - 2g$. Each edge of a graph is adjacent to two faces (it may be adjacent to the same face twice), and since G is connected any face must share at least one edge with some other face. Call this edge e . $G \setminus e$ will still be connected and $\varphi|_{G \setminus e}$ will be a 2-cell embedding of $G \setminus e$ with one face fewer than G . □

Proposition 4.5 *Let $1 \leq f \leq 4g - 4$. Then for any $n \geq (3 + \sqrt{9 + 8(2g + f - 2)})/2$ there exists a graph embedded into a surface of genus g with n vertices, f faces, no loops and no double edges.*

Proof First, it suffices to show that for $n = \lceil (3 + \sqrt{9 + 8(2g + f - 2)})/2 \rceil$ there exists such a graph, G ; to construct such a graph for $n' > n$ we simply subdivide the edges of G with the required number of additional vertices.

To prove the proposition for $n = \lceil (3 + \sqrt{9 + 8(2g + f - 2)})/2 \rceil$ note that

$$\frac{3 + \sqrt{9 + 8(2g + f - 2)}}{2} \leq n \leq \frac{3 + \sqrt{9 + 8(2g + f - 2)}}{2} + 1$$

which implies

$$2g + f \leq \frac{(2n - 3)^2 - 9}{8} + 2, \quad 2g + f \geq \frac{(2n - 5)^2 - 9}{8} + 2.$$

Since $1 \leq f \leq 4g - 4$,

$$g \leq \frac{((2n - 3)^2 - 9)/8 + 2 - 1}{2}, \quad g \geq \frac{((2n - 5)^2 - 9)/8 + 2 + 4}{6}.$$

Simplifying, we see that if $n = \lceil (3 + \sqrt{9 + 8(2g + f - 2)})/2 \rceil$ and $1 \leq f \leq 4g - 4$,

$$\frac{n^2 - 5n + 16}{12} \leq g \leq \frac{(n - 1)(n - 2)}{4}.$$

Since n is positive, $(n^2 - 7n + 12)/12 \leq (n^2 - 5n + 16)/12$, and by [Proposition 4.3](#) K_n has a 2-cell embedding into M_g . If $n = (3 + \sqrt{9 + 8(2g + f - 2)})/2$, the embedding of K_n has f faces and is the desired graph. Otherwise the embedding of K_n has f' faces, $f' > f$, but then by [Lemma 4.4](#) there exists a graph with n vertices and an embedding into M_g with f faces. □

Corollary 4.6 *For $g \geq 2$, $1 \leq f \leq 4g - 4$ and $n \geq (3 + \sqrt{9 + 8(2g + f - 2)})/2$, $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M_f)) \rightarrow H_1(M, \mathbb{Z}))$ is generated by transpositions.*

Proof This is an immediate consequence of [Theorem 4.2](#) and [Proposition 4.5](#). □

Corollary 4.7 *For $g \geq 2$, $n \geq (3 + \sqrt{1 + 16g})/2$ and $\lambda = (k^n)$,*

$$\ker(\text{AJ}_*: \pi_1(\text{Sym}_g^\lambda) \rightarrow H_1(M, \mathbb{Z}))$$

is generated by transpositions.

Proof Corollary 4.6 implies that $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M_1)) \rightarrow H_1(M, \mathbb{Z}))$ is generated by transpositions. There is a natural projection $p: \pi_1(\text{Sym}^n(M_1)) \rightarrow \pi_1(\text{Sym}^n(M))$; hence, $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M)) \rightarrow H_1(M, \mathbb{Z}))$ may be viewed as a quotient group of $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M_1)) \rightarrow H_1(M, \mathbb{Z}))$. This implies $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M)) \rightarrow H_1(M, \mathbb{Z}))$ is generated by *products* of transpositions. However, any single transposition in $\pi_1(\text{Sym}^n(M_1))$ is not in the kernel of p ; thus, $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M)) \rightarrow H_1(M, \mathbb{Z}))$ is generated by transpositions. \square

Corollary 4.7 gives us the structure of the kernel of AJ_* for $\lambda = (k^n)$ and n reasonably large.

When the length of λ is small, it is more difficult to enumerate a set of generators for the kernel of AJ_* . It is not true in general that the kernel will be generated by transpositions; for example, consider $i_*\mathcal{Q}_g(4g-4) \subset \text{Sym}_g^1$. We may move the single marked point around a curve that is homologically but not homotopically trivial. This will be in the kernel of AJ_* but is not a product of transpositions. However, for λ with sufficiently many zeroes of the same order we may make some generalizations to the results of this subsection. In particular, a combination of Corollary 4.6 and the Fadell–Neuwirth fibration will allow us to generalize Corollary 4.7 to a larger class of λ , and this is what we will do in the next subsection.

4.2 Strata with zeroes of more than one weight

We would like to generalize Corollary 4.7 to $\lambda = (k_1^{n_1}, \dots, k_m^{n_m})$ with n_1 large. Let λ be of length n and let $(p_1, \dots, p_n) \in \text{Sym}_g^\lambda(M)$ be the basepoint of B_λ . From Section 3 we know that $B_\lambda \cong \pi_1(\text{Sym}_g^\lambda)$ is generated by the ρ_{ir} , $1 \leq i \leq n$, $1 \leq r \leq 2g$, and for each pair $1 \leq i < j \leq n$, either σ_{ij} if p_i and p_j are of the same weight, or κ_{ij} if p_i and p_j are of different weights. We can immediately show that some of these generators are in $\ker(\text{AJ}_*: \pi_1(B_\lambda) \rightarrow H_1(M, \mathbb{Z}))$:

Lemma 4.8 *Any transposition or square transposition in B_λ , σ_{ij} or κ_{ij} , is in the kernel of AJ_* .*

Proof A transposition of two points of equal weight consists of moving them in opposite directions along homotopic paths. The sum of these paths is then homotopic (and therefore homologous) to zero. A square transposition of two points of unequal weight moves each point along some path, and then back along a homologous path. Thus both points follow paths that are homologous to zero. \square

An individual ρ_{ir} will not be in the kernel of AJ_* ; however, there are two cases when it is easy to see that a product of them will be. First, if two sets of points of equal total weight follow l_r and l_r^{-1} respectively, then their paths will cancel each other out in $H_1(M, \mathbb{Z})$. Second, a single p_i may follow a path that is homologically trivial but not homotopically trivial. More precisely, we have the following two definitions.

Definition 4.9 Fix r , $1 \leq r \leq 2g$. A null ρ_r is a product $\prod_{n=1}^m \rho_{i_n r}^{\pm 1}$ such that $\sum_{n=1}^m \pm k_{i_n} = 0$, where the sign in front of k_{i_n} is given by the sign of the exponent of $\rho_{i_n r}$.

Definition 4.10 Let $\lambda = (k_1, \dots, k_n)$, fix i , $1 \leq i \leq n$, and let M_{n-i} be M punctured at p_{i+1}, \dots, p_n . An i -commutator is a product of ρ_{ir} and κ_{ij} , $1 \leq r \leq 2g$, $i + 1 \leq j \leq n$, such that the path followed by p_i is in $[\pi_1(M_{n-i}), \pi_1(M_{n-i})]$.

In the remainder of the section we show that for $\lambda = (k_1^{n_1}, \dots, k_m^{n_m})$ with n_1 large, transpositions, square transpositions, null ρ_r , and i -commutators for p_i not of order k_1 suffice to generate the kernel of AJ_* . For n_2, \dots, n_m all sufficiently large, transpositions, square transpositions, and null ρ_r suffice to generate.

To show this first recall that for any λ of length n

$$(5) \quad 1 \rightarrow SB_n \rightarrow \pi_1(\text{Sym}_g^\lambda) \cong B_\lambda \xrightarrow{\text{pr}} S_\lambda \rightarrow 1$$

is a short exact sequence. Thus for the remainder of the section we regard SB_n as a natural subgroup of B_λ . From Section 3 we have that SB_n is generated by ρ_{ir} and κ_{ij} , $1 \leq i < j \leq n$, $1 \leq r \leq 2g$.

Lemma 4.11 Any $Z \in B_\lambda$ can be written as $Y \cdot X$ where Y is a product of transpositions and $X \in SB_n$.

Proof Let $\text{pr}(Z) = \bar{Y} \in S_\lambda$. For every p_i, p_j of the same weight, pick a transposition, $\sigma_{ij} \in B_\lambda$. Let $\text{pr}(\sigma_{ij}) = \bar{\sigma}_{ij}$ and note that S_λ is generated by the $\bar{\sigma}_{ij}$. Then we can write \bar{Y} as a product of the $\bar{\sigma}_{ij}$, and we construct $Y \in B_\lambda$ by writing Y as a product of the corresponding σ_{ij} . Then $\text{pr}(Y) = \text{pr}(Z) = \bar{Y}$. Let $X = Y^{-1}Z$. Since $\text{pr}(X) = 1$ and (5) is exact, $X \in SB_n$. □

Let $K = \ker(AJ_*: B_\lambda \rightarrow H_1(M, \mathbb{Z}))$. Then $K \cap SB_n = \ker(AJ_*: SB_n \rightarrow H_1(M, \mathbb{Z}))$. By Lemma 4.11, to prove that any $Z \in K$ is a product of null $\rho_r, \sigma_{ij}, \kappa_{ij}$, and i -commutators it suffices to show that this is true for all $X \in SB_n$. To show this we use the short exact sequence of Theorem 3.4:

$$(6) \quad 1 \rightarrow SB_{a,b} \rightarrow SB_{a+b} = SB_n \xrightarrow{\text{pr}'} SB_b \rightarrow 1$$

for any a, b such that $a + b = n$. Again we regard $SB_{a,b}$ as a natural subgroup of B_λ for the remainder of the section. $K \cap SB_{a,b} = \ker(\text{AJ}_*: SB_{a,b} \rightarrow H_1(M, \mathbb{Z}))$. **Lemma 4.13** will prove that words in $K \cap SB_{a,b}$ can be written as a product of the desired elements, and **Theorem 4.14** will then prove the same for words in $K \cap SB_n$ (and therefore K).

First we need a technical lemma, and in it we break up $SB_{a,b}$ further, using **Theorem 3.4**,

$$(7) \quad 1 \rightarrow SB_{a-1,b+1} \rightarrow SB_{a,b} \xrightarrow{\text{pr}''} SB_{1,b} \rightarrow 1$$

for the purpose of analyzing it.

Lemma 4.12 *Let $\lambda = (k_1^{n_1}, \dots, k_m^{n_m})$, with $\sum_1^m n_i = n$. Pick any $a, 1 \leq a \leq n$, let $b = n - a$, and let p_a have weight $k_l, 1 \leq l \leq m$. Then $\text{im}(\text{pr}'': K \cap SB_{a,b} \rightarrow SB_{1,b})$ lies in $\{S \in SB_{1,b} \mid \text{AJ}_*(S) \in dH_1(M, \mathbb{Z})\}$, where d is the smallest positive integer such that there exist $c_1, c_2, \dots, \widehat{c}_l, \dots, c_m \in \mathbb{Z}$ such that $c_1 \cdot k_1 + c_2 \cdot k_2 + \dots + d \cdot k_l + \dots + c_m \cdot k_m = 0$ has a solution.*

Proof $SB_{a,b}$ is generated by $\rho_{ir}, \kappa_{ij}, 1 \leq i \leq a, 1 \leq j \leq a + b, 1 \leq r \leq 2g$. We need not consider the κ_{ij} as they all go to zero under AJ_* . Let $w(i)$ be the weight of $p_i, w(i) \in \{k_1, \dots, k_m\}$. Any word W in $K \cap SB_{a,b}$ has the property that for a fixed r the product of $\rho_{ir}^{\pm 1}$ in W must be such that $\sum \pm w(i) = 0$. (The sign in front of $w(i)$ corresponds to the sign in the exponent of ρ_{ir} .) This implies that the number of times ρ_{ar} occurs in a particular word must be a multiple of d . \square

Lemma 4.13 *Let λ, a and b be as in **Lemma 4.12**. For all $X_b \in K \cap SB_{a,b}$ there exists $W \in K \cap SB_{a,b}$ such that $\text{pr}''(X_b) = \text{pr}''(W)$ and W is a product of null ρ_r, κ_{aj} or σ_{aj} , and a -commutators.*

Proof As in **Lemma 4.12** let p_a be of weight k_l and also let $c_1, c_2, \dots, \widehat{c}_l, \dots, c_m$, and d be as in **Lemma 4.12**. $SB_{1,b} = \pi_1(M_b)$ is generated by $\bar{\rho}_{ar}, \bar{\kappa}_{aj}, a + 1 \leq j \leq a + b, 1 \leq r \leq 2g$, where $\rho_{ar}, \kappa_{aj} \in SB_{a,b}$ project to their corresponding barred elements. (In $SB_{a,b}$ the points p_1, \dots, p_a move around punctures p_{a+1}, \dots, p_{a+b} ; under pr'' , p_a moves around p_{a+1}, \dots, p_{a+b} .) Let $G = [SB_{1,b}, SB_{1,b}]$ be the commutator of $SB_{1,b}$.

Since G abelianizes $SB_{1,b}$, for all $X_b \in K \cap SB_{a,b}$ there exists $\bar{h} \in G$ such that $\bar{h} \cdot \text{pr}''(X_b)$ is a product of $\bar{\rho}_{ar}, \bar{\kappa}_{aj}$, such that all $\bar{\kappa}_{aj}$ are on the right of all $\bar{\rho}_{ar}$ and if $r_1 < r_2, \bar{\rho}_{ar_1}$ is to the left of $\bar{\rho}_{ar_2}$. In other words, $\bar{h} \cdot \text{pr}''(X_b)$ is a word such that for fixed r all $\bar{\rho}_{ar}$ are adjacent. By **Lemma 4.12**, the power of any $\bar{\rho}_{ar}$ in $\bar{h} \cdot \text{pr}''(X)$ must be a multiple of d .

Now we construct $W \in K \cap SB_{a,b}$ by inserting c_i elements of weight k_i moving around l_r adjacent to each set of d elements of weight k_l moving around the l_r in $\bar{h} \cdot \text{pr}''(X_b)$, and then multiplying by h^{-1} , $\text{pr}''(h) = \bar{h}$. For example, let $\lambda = (k_1, k_2)$ with p_1 of weight k_1 , p_2 of weight k_2 , and $ck_1 - dk_2 = 0$. If $\bar{h} \cdot \text{pr}''(X) = \bar{\rho}_{2r_1}^{n_1} \bar{\rho}_{2r_2}^{n_2} \bar{\kappa}_{12}$, then $W = h^{-1}(\rho_{2r_1}^{n_1} \rho_{1r_1}^{-(c/d)n_1})(\rho_{2r_2}^{n_2} \rho_{1r_2}^{-(c/d)n_2})\kappa_{12}$, where Lemma 4.12 implies n_1 and n_2 are divisible by d . Both of the elements in parentheses are null ρ_r . W is thus made up of null ρ_r , κ_{aj} , and a -commutators. By construction $\text{pr}''(W) = \text{pr}''(X_b)$. \square

If there are sufficiently many points of weight k_l (if $n_l \geq (3 + \sqrt{9 + 8(2g + b - 2)})/2$), then by Corollary 4.6 $\ker(\text{AJ}_*: \pi_1(\text{Sym}^n(M_b) \rightarrow H_1(M, \mathbb{Z}))$ is generated by transpositions. This implies that the a -commutators that make up h^{-1} above can be written as products of transpositions.

Theorem 4.14 *Let $\lambda = (k_1^a, k_2^{b_2}, \dots, k_m^{b_m})$, with exponents satisfying $\sum_{k=2}^m b_k = b$ and $a \geq (3 + \sqrt{9 + 8(2g + b - 2)})/2$ (as in Corollary 4.6). Then any $Z \in K$ can be written as a product of null ρ_r , σ_{ij} , κ_{ij} , $1 \leq i, j \leq a + b$, $1 \leq r \leq 2g$, and i -commutators for p_i not of weight k_1 .*

Proof By Lemma 4.11 it suffices to prove the theorem for all $X \in K \cap SB_{a+b}$. By Corollary 4.6 the theorem is true for all $X_b \in K \cap SB_{a,b}$.

For $0 \leq k \leq b$ we have the following version of the short exact sequence in (7):

$$1 \rightarrow SB_{a+k-1, b-k+1} \rightarrow SB_{a+k, b-k} \xrightarrow{\text{pr}''} SB_{1, b-k} \rightarrow 1$$

Using this and Lemma 4.13 we show that if elements of $K \cap SB_{a+k-1, b-k+1}$ are of the desired form, then so are elements of $K \cap SB_{a+k, b-k}$.

Let X_k be an element of $K \cap SB_{a+k, b-k}$. By Lemma 4.13 we construct $W_k \in K \cap SB_{a+k, b-k}$ such that $\text{pr}''(W_k) = \text{pr}''(X_k)$ and W_k is a product of the desired elements. Using the above exact sequence $W_k^{-1} \cdot X_k \in K \cap SB_{a+k-1, b-k+1}$. If $W_k^{-1} \cdot X_k$ and W_k are both of the desired form, then so is X_k .

Therefore by induction elements of $K \cap SB_{a+b}$ are of the desired form. \square

Corollary 4.15 *Let $\lambda = (k_1^a, k_2^{b_2}, \dots, k_n^{b_n})$ with exponents satisfying $\sum_2^m b_k = b$ and $a \geq (3 + \sqrt{9 + 8(2g + b - 2)})/2$. Additionally assume that for $2 \leq k \leq m$, we have $b_k \geq (3 + \sqrt{9 + 8(2g + b_{k+1} + \dots + b_m - 2)})/2$. Then any $Z \in \ker(\pi_1(\text{Sym}_g^\lambda \rightarrow H_1(M, \mathbb{Z}))$ may be written as a product of σ_{ij} , κ_{ij} , and null ρ_r , $1 \leq i, j \leq a + b$, $1 \leq r \leq 2g$.*

Proof This differs from [Theorem 4.14](#) only in that we eliminate the i -commutators, $a < i \leq a + b$. We may do this because any i -commutator is contained in the kernel of $\pi_1(\text{Sym}^{b_i}(M_{b_{i+1}+\dots+b_m})) \rightarrow H_1(M, \mathbb{Z})$ and may thus by [Corollary 4.6](#) be written as products of transpositions. \square

This gives us a description of $\ker(\text{AJ}_*: \text{Sym}_g^\lambda \rightarrow H_1(M, \mathbb{Z}))$ for λ with at least one reasonably large set of points of the same weight (on the order of \sqrt{g}). The rest of the paper will be spent in determining which of the elements of this kernel are also elements of $\pi_1(\mathcal{Q}_\lambda)$. To do so we first present some methods for surgering existing quadratic differentials to create new ones.

5 Some local surgeries

We denote by \mathcal{Q}_λ^0 a connected component of \mathcal{Q}_λ , and by $\text{cl}(\mathcal{Q}_\lambda)$ the closure of \mathcal{Q}_λ . Choose λ_1, λ_2 such that $\mathcal{Q}_{\lambda_2}^0 \subset \text{cl}(\mathcal{Q}_{\lambda_1}^0)$. In this section we present a method whereby, for certain $(M, q) \in \mathcal{Q}_{\lambda_1}^0$ sufficiently near to an element of $\mathcal{Q}_{\lambda_2}^0$, we may construct some transpositions and square transpositions of zeroes of q in $\pi_1(\mathcal{Q}_{\lambda_1}^0, (M, q))$. For $\lambda = (k_1, \dots, k_n)$ we refer to the zeroes of q as p_1, \dots, p_n with p_i of order k_i , and we let $P := \{p_1, \dots, p_n\}$. We also sometimes refer to a (square) transposition of p_i and p_j – by this we will mean a transposition if $k_i = k_j$ and a square transposition otherwise.

First we consider the following definition and two lemmas, which closely follow lemmas in [\[3\]](#) and [\[8\]](#), amongst others. The lemmas allow us in certain cases to construct an element of $\mathcal{Q}_{\lambda_1}^0$ from an element of $\mathcal{Q}_{\lambda_2}^0$. For a fixed (M, q) and $p \in M$ let $D_\varepsilon(p)$ denote the disk of radius ε centered at p in the q metric.

Definition 5.1 A zero p of (M, q) of order k_1 can be broken into zeroes of order l_1, \dots, l_m , $\sum_1^m l_i = k_1$, if there exists ε_0 such that for every $\varepsilon < \varepsilon_0$ there exists a quadratic differential (M', q') and a disk $D' \subset M'$ of radius ε in the q' metric such that $M' \setminus D_\varepsilon(p)$ is isometric to $M' \setminus D'$ and D' contains zeroes of order l_1, \dots, l_m . If $(M, q) \in \mathcal{Q}_g(k_1, k_2, \dots, k_n)$, then $(M', q') \in \mathcal{Q}_g(l_1, \dots, l_m, k_2, \dots, k_n)$.

Lemma 5.2 Let $(M, q) \in \mathcal{Q}_g(k_1, \dots, k_n)$. Pick $l_1, l_2 \in \mathbb{N}$ such that $l_1 + l_2 = k_i$, where if k_i is even then l_1 and l_2 are even as well. Let p_i be a zero of q of order k_i . Then p_i can be broken into zeroes of order l_1, l_2 .

Lemma 5.3 Let $(M, q) \in \mathcal{Q}_g(k_1, \dots, k_n)$. Pick $l_1, l_2, l_3 \in \mathbb{N}$ such that $l_1 + l_2 + l_3 = k_i$. Then p_i can be broken into zeroes of order l_1, l_2, l_3 . Similarly, given any $l_1, l_2, l_3, l_4 \in \mathbb{N}$ such that $l_1 + l_2 + l_3 + l_4 = k_i$, p_i can be broken into zeroes of order l_1, l_2, l_3, l_4 .

Sketch of Proof A proof of all but the 4 point case may be found in [8]; we prove the two point case with both zeroes even here. Consider a disk D_ε , centered at p_i , of radius ε in the flat metric given by q , where ε is small enough that D_ε contains no other zeroes of q . Since p_i is of order k_i , $k_i + 2$ horizontal trajectories will dead-end into p_i . Cut along these to make $k_i + 2$ half-disks, each with a marked point given by p_i half way along the cut. Pick some δ with $0 < \delta < \varepsilon$. Construct two special half-disks by splitting the marked point given by p_i into two marked points 2δ apart. Also shift the marked point given by p_i by δ on the remaining disks, as in Figure 1. The trajectory structures on the individual half-disks do not change, and the half-disks

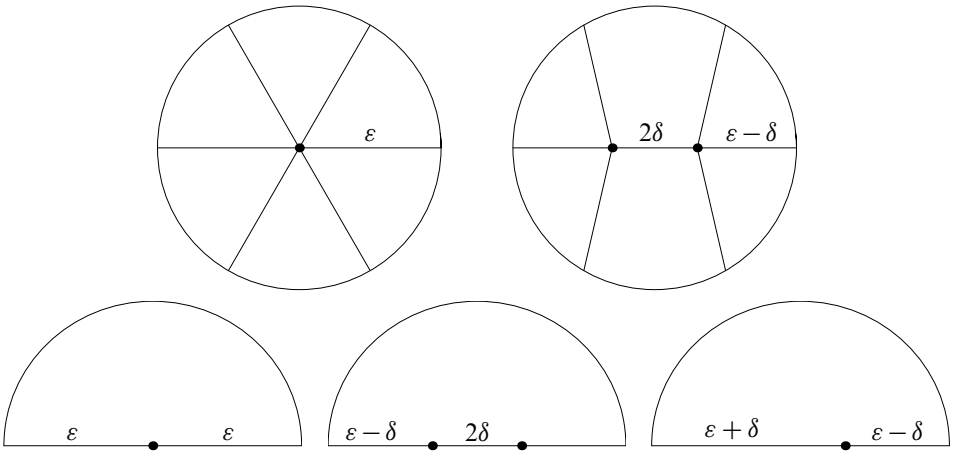


Figure 1: Taking a zero of order 4 to two zeroes of order 2 via splitting/moving the zero on individual half-disks. These diagrams are purely schematic and not to scale; the angle between any two straight lines in the disks is π .

can be glued back together, again as in Figure 1, to give an ε disk containing zeroes of order l_1, l_2 , with the same trajectory structure at the boundary as the original D_ε . We have a choice of the direction of the newly formed saddle connection: if we label one of the $k_i + 2$ horizontal trajectories that dead-end into p_i as 0° , then the new saddle connection can be in any direction θ , $\theta \in [0, (k_i + 2)\pi)$. The surgered surface is completely determined by δ and θ . If $l_1 = l_2 = k_i/2$ then the surface given by δ, θ is the same as the surface given by $\delta, \theta + (l_1 + 1)\pi$, so we restrict ourselves to $\theta \in [0, (l_1 + 1)\pi)$.

The remaining two and three point cases are proved similarly to the one above. For the case of four zeroes, if any of the l_j are even then we may apply so combination of two and three point surgeries to get the desired result. If l_1, l_2, l_3, l_4 are all odd then we

cut D_ε around p_i into $k_i + 2$ half-disks and glue as in Figure 2. Again, in all of these surgeries we have a choice of both δ and $\theta \in [0, (k_i + 2)\pi)$. \square

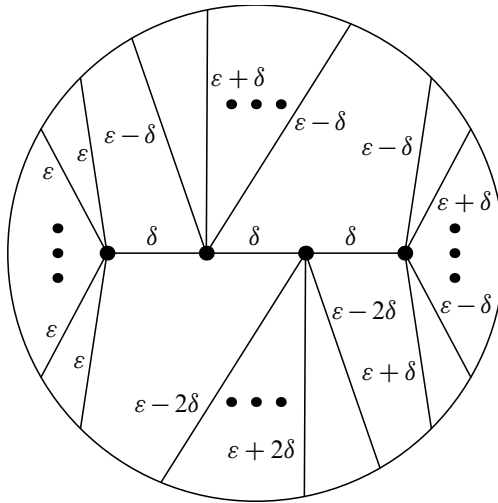


Figure 2: Splitting an even zero of high order into 4 odd zeroes of lower order

Define $\mathcal{Q}_{\lambda_2}^0$ to be a *degeneration* of $\mathcal{Q}_{\lambda_1}^0$ if it is possible to obtain an element in $\mathcal{Q}_{\lambda_1}^0$ by (possibly repeatedly) applying the surgeries of Lemma 5.2 and Lemma 5.3 to an element in $\mathcal{Q}_{\lambda_2}^0$. This puts a partial ordering on Λ , the set of partitions of $4g - 4$: we say $\lambda_1 > \lambda_2$ if there exists a component of \mathcal{Q}_{λ_2} that is a degeneration of a component of \mathcal{Q}_{λ_1} . Note that if $\mathcal{Q}_{\lambda_2}^0$ is a degeneration of $\mathcal{Q}_{\lambda_1}^0$, then $\mathcal{Q}_{\lambda_2}^0 \subset \text{cl}(\mathcal{Q}_{\lambda_1}^0)$, but the converse is not necessarily true. For example, $\mathcal{Q}_g(1^{4g-6}, 2)$ is in the closure of $\mathcal{Q}_g(1^{4g-4})$ ($\text{cl}(\mathcal{Q}_g(1^{4g-4})) = \mathcal{Q}_g$, so every stratum is in its closure), but $\mathcal{Q}_g(1^{4g-6}, 2)$ is not a degeneration of $\mathcal{Q}_g(1^{4g-4})$.

Lemma 5.2 and Lemma 5.3 detail surgeries that allow us to break up zeroes of quadratic differentials; however, again following [3] we may *collapse* zeroes in the reverse process. We say three points, p_1, p_2 , and p_3 , are *colinear* if for a fixed θ there exist θ -trajectories between p_1, p_2 and p_2, p_3 .

Lemma 5.4 Suppose $(M', q') \in \mathcal{Q}_g(k_1, \dots, k_n)$ is such that at least one of k_1 and k_2 is even, there exists a saddle connection between p_1 and p_2 of length δ , and all other saddle connections from p_1 or p_2 are of length greater than δ for both even or $(3/2)\delta$ for one odd. Then there exists $(M, q) \in \mathcal{Q}_g(k_1 + k_2, k_3, \dots, k_n)$ such that (M', q') may be obtained from (M, q) via the surgery of Lemma 5.2.

Suppose $(M', q') \in \mathcal{Q}_g(k_1, \dots, k_n)$ is such that there exist saddle connections between p_1, p_2 and p_2, p_3 of length δ , p_1, p_2 , and p_3 are colinear, and all saddle connections from p_2 are of length at least 3δ . Then there exists $(M, q) \in \mathcal{Q}_g(k_1 + k_2 + k_3, k_4, \dots, k_n)$ such that (M', q') may be obtained from (M, q) via the surgery of [Lemma 5.2](#).

Proof We first prove the two point case. Label the points in each of the two top left disks of [Figure 3](#) by p_1 and p_2 . The top left disc (representing the one odd case) is contained in $D_{\delta/2+\varepsilon}(p_1) \cup D_{\delta/2+\varepsilon}(p_2)$. Then there exists $\varepsilon > \delta$ such that the disc contains no other zeroes of q' . Since $\varepsilon > \delta$ we may reverse the cutting and pasting process of [Lemma 5.2](#) to obtain (M, q) . In the left of the second pair of discs, the disc is again contained in $D_{\delta/2+\varepsilon}(p_1) \cup D_{\delta/2+\varepsilon}(p_2)$. Then there exists $\varepsilon > \delta/2$ such that the disc contains no other zeroes of q' , and since $\varepsilon > \delta/2$ we may reverse the cutting and pasting process of [Lemma 5.2](#) to obtain (M, q) .

For the three point case, label the points in the two bottom left discs of [Figure 3](#) by p_1, p_2, p_3 , with p_2 in the middle. $D_{\delta+\varepsilon}(p_2)$ contains either of these two discs. There exists $\varepsilon > 2\delta$ such that these disks contain no other zeroes of q' and we may reverse the cutting and pasting of [Lemma 5.3](#) to obtain (M, q) . \square

In the next proposition we construct (square) transpositions of zeroes of $(M', q') \in \mathcal{Q}_{\lambda_1}$ by colliding two zeroes to get (M, q) in a degeneration of \mathcal{Q}_{λ_1} , and then breaking up the newly formed zero of q with respect to varying θ . Breaking up a single zero into two zeroes in this manner gives us a transposition; however, breaking a single zero into three gives us something slightly different. Define a *transposition of p_i and p_j mod p_k* to be an element of $\pi_1(\mathcal{Q}_{\lambda_1}, (M, q))$ such that p_i, p_j follow paths that are not homotopic on $M \setminus P$, but are homotopic on $(M \setminus P) \cup p_k$. If e is a path between p_i, p_j through p_k we refer to a transposition of p_i, p_j mod p_k along e ; by this we mean that p_i and p_j follow small deformations of e on opposite sides of p_k . Similarly if we refer to a transposition of p_i, p_j along e we mean their transposition along a slight deformation of e to one side of p_k .

Proposition 5.5 Let $\lambda_1 = (k_1, k_2, \dots, k_n)$, $\lambda_2 = (k_1 + k_2, k_3, \dots, k_n)$, and suppose $(M', q') \in \mathcal{Q}_{\lambda_1}$ satisfies the conditions of [Lemma 5.4](#) in the two point case. Then there exists a trajectory of length δ between p_1 and p_2 (specified by [Lemma 5.4](#)), and an element of $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$ corresponding to the (square) transposition of p_1, p_2 along the specified trajectory.

Suppose $\lambda_1 = (k_1, k_2, \dots, k_n)$, $\lambda_3 = (k_1 + k_2 + k_3, k_4, \dots, k_n)$, and $(M', q') \in \mathcal{Q}_{\lambda_1}$ satisfies the conditions of [Lemma 5.4](#) in the three point case. Then there exists a transposition of p_1, p_2 mod p_3 along the union of two trajectories specified by [Lemma 5.4](#).

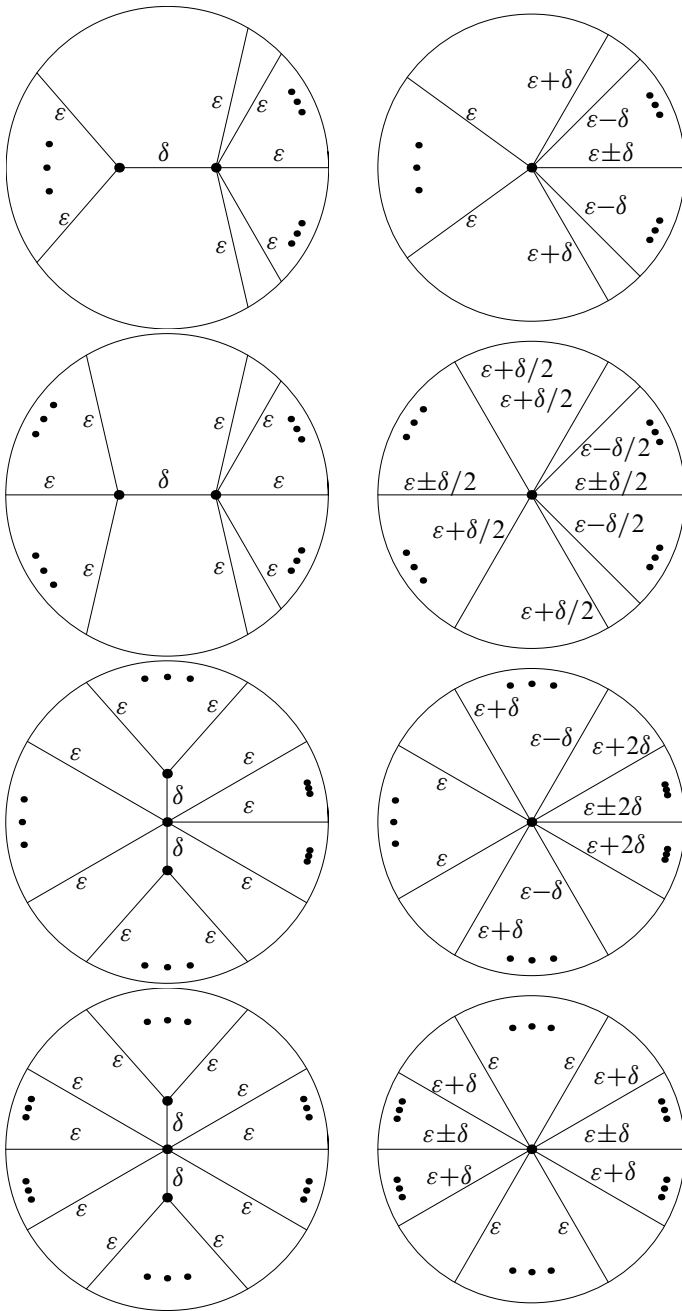


Figure 3: (i) Colliding two zeroes, one of odd order and one of even order. (ii) Colliding two zeroes of even order. (iii) Colliding three zeroes of odd order. (iv) Colliding three zeroes, two of any order and one even.

Proof We first prove the proposition for the two point case. Suppose (M', q') is obtained from $(M, q) \in \mathcal{Q}_{\lambda_2}$ by surgering with respect to δ_0, θ_0 . Let e_0 be the new trajectory between p_1, p_2 of length δ_0 . Create a curve $\eta: [0, \delta_0] \rightarrow CI(\mathcal{Q}_{\lambda_1})$ by surgering (M, q) with respect to $t, \theta_0, t \in (0, \delta_0]$. Now, there exists a ball of some radius α around the zero of q of order $k_1 + k_2$ containing no other zeroes of q , so we apply the surgery of Lemma 5.2 to (M, q) with respect to some fixed $\delta_1 < \min\{\delta_0, \alpha\}$ and vary θ between 0 and either $(k_1 + 1)\pi$ or $(k_1 + k_2 + 2)\pi$ (depending on whether we transpose or square transpose p_1 and p_2) to get a loop of surfaces, $\eta' \subset \mathcal{Q}_{\lambda_1}$, centered at (M, q) . η intersects η' at $\eta(\delta_1)$, and we define $\tilde{\eta}$ to be the subcurve of η from δ_0 to δ_1 . Then $\tilde{\eta} \circ \eta' \circ \tilde{\eta}^{-1}$ gives us the desired loop in $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$.

Following the same procedure for three points, we get a (square) transposition of $p_1, p_2 \bmod p_3$ along the union of the two trajectories between them in $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$. \square

Proposition 5.5 allows us to create (square) transpositions of various zeroes of a quadratic differential (possibly mod a third zero); however, the (square) transpositions created this way may be based at different elements of \mathcal{Q}_λ . In the next section we show that one way of obtaining elements based at the same (M, q) is to consider hyperelliptic quadratic differentials.

6 (Square) transpositions in certain strata

In the next two sections we show that there exists a family of λ for which $i_*(\pi_1(\mathcal{Q}_\lambda)) = \ker(\text{AJ}_*)$. We do so by explicitly constructing loops of quadratic differentials corresponding to the generators of $\ker(\text{AJ}_*)$; in this section we construct the necessary (square) transpositions, and in the next section the null ρ_r and i -commutators. For simplicity of notation we refer to the elements we construct as being in $\pi_1(\mathcal{Q}_\lambda)$.

We say that $\pi_1(\mathcal{Q}_\lambda^0, (M, q))$ contains all transpositions between p_i, p_j if for all edges $e \in E_{M,q}$ between p_i and p_j , σ_e or $\kappa_e \in i_*(\pi_1(\mathcal{Q}_\lambda^0, (M, q)))$ (depending on whether p_i and p_j are of the same weight or not). We say that $\pi_1(\mathcal{Q}_\lambda^0, (M, q))$ contains all transpositions if the above is true for all pairs of zeroes of q . In Lemmas 6.1–6.4 we construct a variety of transpositions in hyperelliptic strata, and in the final proposition of the section we show that for some λ the transpositions constructed in the four lemmas suffice to generate all transpositions in $\pi_1(\mathcal{Q}_\lambda^0, (M, q))$.

For any hyperelliptic quadratic differential, (M, q) , we let τ denote the hyperelliptic involution on M and π the degree 2 projection of M to \mathbb{P}^1 . We will use $\pi(M, q)$ to denote the projection of q to \mathbb{P}^1 . Notice that any zero of a hyperelliptic quadratic differential at a branch point of π must be of even order. Unless specified otherwise we do not assume that the k_1, \dots, k_n in $\mathcal{Q}_g(k_1^{n_1}, k_2^{n_2}, \dots, k_m^{n_m})$ are distinct.

Lemma 6.1 Let $(M_0, q_0) \in \mathcal{Q}_\lambda^0 = \mathcal{Q}_g^0(k_1^2, \dots, k_m^2, k_{m+1}, \dots, k_n)$ be a hyperelliptic quadratic differential, and label the zeroes of q_0 by $p_1^\pm, \dots, p_m^\pm, p_{m+1}, \dots, p_n$. Suppose exactly p_{m+1}, \dots, p_n are at branch points of M_0 and at least one of k_1, \dots, k_m , say k_l , is even. Let e be an edge between p_i^+, p_j^+ , $1 \leq i < j \leq m$, such that $e \cap \tau(e) = \emptyset$. Then $\pi_1(\mathcal{Q}_\lambda^0, (M_0, q_0))$ contains an element corresponding to the (square) transposition of p_i^+, p_j^+ along e . Analogous statements are true for p_i^+, p_j^- , and p_i^-, p_j^- .

Sketch of proof To prove the lemma we construct paths in the hyperelliptic locus of \mathcal{Q}_λ , from (M_0, q_0) to quadratic differentials with discs that satisfy the conditions of Lemma 5.4 (as in Figure 3), and we then apply Proposition 5.5. In the case where at least one of k_i, k_j is even, this proves the lemma. In the case where both k_i and k_j are odd the path we create moves p_i, p_j near an even zero, p_l , and we create a (square) transposition of $p_i, p_j \bmod p_l$. However, since k_l is even we also have a square transposition of p_j and p_l , and the composition of the two gives us the desired (square) transposition of p_i and p_j (see Figure 4). □

Proof Let $e: [0, 1] \hookrightarrow M_0$, $e(0) = p_i^+, e(1) = p_j^+$, and let $\tilde{e} = \pi \circ e: [0, 1] \rightarrow \mathbb{P}^1$ be the projection of e to \mathbb{P}^1 . By abuse of notation we also use e, \tilde{e} to mean the images of $[0, 1]$ under the maps e and \tilde{e} .

We first assume at least one of k_i, k_j is even. Define

$$\begin{aligned} (\mathbb{P}^1, \tilde{q}_0) &= \pi(M_0, q_0) \in \mathcal{Q}_\lambda^- \\ &= \mathcal{Q}_0 \left(k_1, \dots, k_m, \frac{k_{m+1} - 2}{2}, \dots, \frac{k_n - 2}{2}, -1^{2g+2-(n-m)} \right) \end{aligned}$$

and label the singularities of \tilde{q}_0 by $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{2g+2+m}$, such that $\pi(p_s^\pm) = \tilde{p}_s$ for $1 \leq s \leq m$ and $\pi(p_t) = \tilde{p}_t$ for $m < t \leq n$. By Proposition 2.2 we may construct a quadratic differential on \mathbb{P}^1 with arbitrary zeroes; thus, define $(\mathbb{P}^1, \tilde{q}_t) \in \mathcal{Q}_\lambda^-$ to be the quadratic differential with zeroes at $\tilde{p}_1, \dots, \tilde{p}_{i-1}, \tilde{e}(t), \tilde{p}_{i+1}, \dots, \tilde{p}_{2g+2+m}$. Let $(M_t, q_t) \in \mathcal{Q}_\lambda^0$ be the double cover of $(\mathbb{P}^1, \tilde{q}_t)$, ramified at $\tilde{p}_{m+1}, \dots, \tilde{p}_{2g+2+m}$.

As t approaches 1, $e(t)$ approaches p_j^+ , and there exists t_0 close to 1 such that (M_{t_0}, q_{t_0}) satisfies the conditions of Lemma 5.4. Then we may apply Proposition 5.5 to obtain a (square) transposition, $T \in \pi_1(\mathcal{Q}_\lambda^0, (M_{t_0}, q_{t_0}))$, of p_j and $e(t_0)$ along a trajectory homotopic to $e([t_0, 1])$. Let E be the image of $[0, t_0]$ in \mathcal{Q}_λ^0 under $t \mapsto (M_t, q_t)$. Along $E^{-1} \circ T \circ E$ p_i^+ and p_j^+ (square) transpose along e . Further, p_i^- follows a trivial path (since $e \cap \tau(e) = \emptyset$) and all other zeroes stay constant, so this gives us the desired (square) transposition.

Now suppose both k_i and k_j are odd. Let e be as above and recall that k_l is even. let $\tilde{e}': [0, 1] \hookrightarrow \mathbb{P}^1$ be an edge from \tilde{p}_l to \tilde{p}_j such that $\tilde{e}' \cap \tilde{e} = \tilde{p}_j$, and let $e': [0, 1] \hookrightarrow M_0$ be the edge between p_j^+ and either p_l^+ or p_l^- that projects to \tilde{e}' . Since \tilde{e}' does not intersect itself $e' \cap \tau(e') = \emptyset$, and also by construction $e' \cap e = p_j^+$.

Define $(\mathbb{P}^1, \tilde{q}'_t) \in \mathcal{Q}_{\tilde{\lambda}}$ to be the quadratic differential with zeroes at $\tilde{p}_1, \dots, \tilde{p}_{i-1}, \tilde{e}(t), \tilde{p}_{i+1}, \dots, \tilde{p}_{l-1}, \tilde{e}'(t), \tilde{p}_{l+1}, \dots, \tilde{p}_{2g+2+m}$, and let $(M'_t, q'_t) \in \mathcal{Q}_{\lambda}^0$ be the double cover of $(\mathbb{P}^1, \tilde{q}'_t)$, ramified at $\tilde{p}_{m+1}, \dots, \tilde{p}_{2g+2+m}$. Again there will be some t_0 such that (M'_{t_0}, q'_{t_0}) satisfies the conditions of Lemma 5.4. (Some slight deformation of e and e' may be required to get $e(t), e'(t)$ and p_j^+ colinear). Then by Proposition 5.5, $\pi_1(\mathcal{Q}_{\lambda}, (M'_{t_0}, q'_{t_0}))$ contains a transposition of $e(t_0)$ and p_j mod $e'(t_0)$ along $e([t_0, 1])$. However, since $e'(t_0)$ is a zero of even order there exists a square transposition of $e'(t_0)$ and p_j along $e'([t_0, 1])$, also in $\pi_1(\mathcal{Q}_{\lambda}, (M'_{t_0}, q'_{t_0}))$. The composition of this square transposition and the transposition of $e(t_0)$ and p_j mod $e'(t_0)$ is shown in Figure 4, and gives a true transposition of $e(t_0)$ and p_j along $e([t_0, 1])$. The rest of the proof follows as in the even case. \square

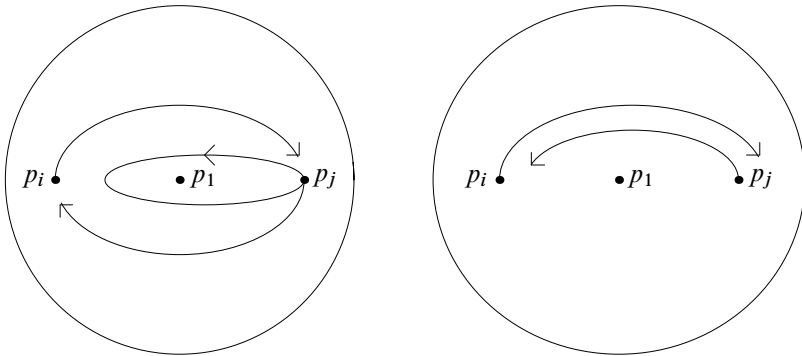


Figure 4: Homotopic paths along which to transpose p_i, p_j , of odd order when p_l is of even order

Notice that any saddle connection, e , between two branch points of a hyperelliptic quadratic differential will have a “twin” saddle connection, $\tau(e)$, parallel and of the same length. We would like to prove a lemma similar to Lemma 6.1 for branch zeroes; however, colliding two branch zeroes along e will also cause the length of $\tau(e)$ to go to zero and a homology cycle to collapse. We deal with this by leaving the hyperelliptic locus of \mathcal{Q}_{λ} shortly before colliding the two zeroes, and showing that for a generic element outside the hyperelliptic locus, the saddle connections corresponding to e and $\tau(e)$ will be of different lengths.

In particular, for any (M, q) pick $x \in M \setminus P$ and notice that for any loop, γ on $M \setminus P$ based at x and $v \in T_x M$, parallel transport along γ takes a vector v either to v or $-v$. This gives us a homomorphism

$$\text{hol}: \pi_1(M \setminus P, x) \rightarrow \mathbb{Z}_2.$$

Following Masur and Zorich in [10], this gives us a double cover of M , \widehat{M} , endowed with an induced flat metric that by construction has trivial holonomy and thus defines an abelian differential, $(\widehat{M}, \widehat{q}) \in \mathcal{H}_{\widehat{\lambda}}$ of genus \widehat{g} , with zeroes \widehat{P} . \widehat{M} is the *canonical double cover* of M , and has a natural involution, φ . Let e be an oriented saddle connection (either closed or not) on M , and let \widehat{e}' , \widehat{e}'' be its lifts to \widehat{M} . The lifts \widehat{e}' and \widehat{e}'' both determine homology classes $[\widehat{e}'], [\widehat{e}''] \in H_1(\widehat{M}, \widehat{P}; \mathbb{Z})$. Again following [10], if $[\widehat{e}'] = -[\widehat{e}'']$ we define $[\widehat{e}] := [\widehat{e}']$; otherwise, we define $[\widehat{e}] := [\widehat{e}'] - [\widehat{e}'']$. We then say two saddle connections, e_1 and e_2 , are *homologous* if $[\widehat{e}_1] = \pm[\widehat{e}_2]$ in $H_1(\widehat{M}, \widehat{P}; \mathbb{Z})$.

Now pick $(M_0, \omega) \in \mathcal{H}_{\widehat{\lambda}}$ with zeroes P_0 , and let $\gamma \in H_1(M', P'; \mathbb{Z})$. The *period* of γ is $|\gamma| := \int_{\gamma} \omega'$. Fix a basis $(\gamma_1, \dots, \gamma_{2\widehat{g}+n-1})$ of $H_1(M_0, P_0; \mathbb{Z})$. Any nearby element, $(M', \omega') \in \mathcal{H}_{\widehat{\lambda}}$ has the same underlying topological surface and singular points, and therefore a well-defined notion of $(\gamma_1, \dots, \gamma_{2\widehat{g}+n-1})$. Then we have a local map $\mathcal{H}_{\widehat{\lambda}} \rightarrow H^1(M_0, P_0; \mathbb{C}) \cong \mathbb{C}^{2\widehat{g}+n-1}$,

$$(M', \omega') \mapsto (|\gamma_1|, \dots, |\gamma_{2\widehat{g}+n-1}|).$$

This is called the period map and is a local homeomorphism of a neighborhood of (M', ω') with a domain of $\mathbb{C}^{2\widehat{g}+n-1}$. This gives us a local coordinates on $\mathcal{H}_{\widehat{\lambda}}$ and an induced Lebesgue measure on $\mathcal{H}_{\widehat{\lambda}}$. Back to $(\widehat{M}, \widehat{q})$, let $H^{1-}(\widehat{M}, \widehat{P}; \mathbb{C})$ be the subspace of $H^1(\widehat{M}, \widehat{P}; \mathbb{Z})$ that is anti-invariant with respect to φ^* . Notice that locally elements of \mathcal{Q}_{λ} will have double covers in the same stratum of abelian differentials. Abelian differentials whose images lie in $H^{1-}(\widehat{M}, \widehat{P}; \mathbb{C})$ are precisely those that arise from quadratic differentials in \mathcal{Q}_{λ} . This gives a local homeomorphism between a neighborhood of \mathcal{Q}_{λ} and an open domain of \mathbb{C}^{2g+n-2} , and again a corresponding Lebesgue measure on \mathcal{Q}_{λ} . See Lanneau [8], for example. If e_1 and e_2 are not *homologous*, the homology classes of \widehat{e}_1 , \widehat{e}_2 are independent in $H_1^-(\widehat{M}, \widehat{P}, \mathbb{Z})$, and $||[\widehat{e}_1]|| = ||[\widehat{e}_2]||$ only on a set of measure zero. Since $||[e]|| := |\int_e q| = |\int_{\widehat{e}'} \widehat{q}| = (1/2) |\int_{\widehat{e}} \widehat{q}|$, the same will be true of e_1 and e_2 – if they are non-*homologous* $||[e_1]|| = ||[e_2]||$ only on a set of measure zero.

Lemma 6.2 *Let (M_0, q_0) and λ be as in Lemma 6.1, with some k_i odd, $1 \leq i \leq n$, at least two branch zeroes, and $g > 0$. Let e^+ be a saddle connection between two of the branch zeroes of q_0 , and let $e^- := \tau(e^+)$. Then there exists some open set, $U \subset \mathcal{Q}_{\lambda}$, containing (M_0, q_0) , such that for almost every element of U these saddle connections are not of the same length.*

Proof As noted above, if e^+ and e^- are not \hat{h} homologous in $H_1^-(\hat{M}_0, \hat{P}_0; \mathbb{Z})$ then they are generically not of the same length. Thus, to prove the lemma we suppose that $[\hat{e}] := [\hat{e}^+] - [\hat{e}^-] = 0$ and show a contradiction.

If $[\hat{e}] = 0$ then $\hat{e} \in [\hat{e}]$ is a separating curve on \hat{M}_0 . Let \tilde{M}_0 be the surface with boundary obtained by cutting M_0 along $e^+ \cup e^-$, which together form a nontrivial cycle, and let $p \in \tilde{M}_0 \setminus P_0$. Since $g > 0$, \tilde{M}_0 is connected. If any element of $\pi_1(\tilde{M}_0 \setminus P_0, p)$ mapped to -1 under $\text{hol}: \pi_1(\tilde{M}_0 \setminus P_0, p) \rightarrow \mathbb{Z}_2$ then \hat{M}_0 cut at \hat{e} would still be connected, which implies that every element of $\pi_1(\tilde{M}_0 \setminus P_0, p)$ maps to 1 under hol . But by assumption one of the zeroes of P_0 is odd and a small loop around it will map to -1 under hol , giving us the desired contradiction. \square

Now we can prove the analog of [Lemma 6.1](#) for branch zeroes.

Lemma 6.3 *Let (M_0, q_0) , λ and $\tilde{\lambda}$ be as in [Lemma 6.1](#), with at least one k_i odd, $1 \leq i \leq n$, and let e be any edge between two branch zeroes of q_0 , p_i and p_j , $m < i < j \leq n$. Then $\pi_1(Q_\lambda^0, (M_0, q_0))$ contains a (square) transposition of p_i, p_j along e .*

Proof Let $e: [0, 1] \hookrightarrow M_0$, $e(0) = p_i$, $e(1) = p_j$, be the edge between p_i, p_j , and let $\tilde{e} := \pi \circ e$ be its projection to \mathbb{P}^1 . Let

$$\tilde{\lambda} = \left(k_1, \dots, k_m, \frac{k_{m+1} - 2}{2}, \dots, \frac{k_n - 2}{2}, -1^{2g+2-(n-m)} \right).$$

Let $(\mathbb{P}^1, \tilde{q}_0) = \pi(M_0, q_0) \in Q_{\tilde{\lambda}}$ and label the singularities of \tilde{q}_0 as $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{2g+2+m}$. Define $(\mathbb{P}^1, \tilde{q}_t)$ to be the element of $Q_{\tilde{\lambda}}$ with singularities at $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{i-1}, \tilde{e}(t), \tilde{p}_{i+1}, \dots, \tilde{p}_{2g+2+m}$. Taking a double cover of $(\mathbb{P}^1, \tilde{q}_t)$ branched at its last $2g + 2$ singularities we get an element of Q_λ , which we call (M_t, q_t) .

As t approaches 1, $e(t)$ approaches p_j and all other p_k are fixed. As in [Lemma 6.1](#) we wish to show there exists a subset of some M_t containing $e(t)$ and p_j , and satisfying the conditions of [Lemma 5.4](#). In particular, if $e(t)$ and p_j are δ apart it suffices to show that all other saddle connections on M_t are of length greater than δ . This condition is not initially satisfied because on (M_t, q_t) there are two short saddle connections of the same length running between $e(t)$ and p_j . However, [Lemma 6.2](#) implies that there exists (M, q) arbitrarily close to any (M_t, q_t) such that the saddle connection in the homotopy classes of $e([t, 1])$ and $\tau(e([t, 1]))$ are of different lengths. Because of the involution on (M_t, q_t) we may assume the saddle connection on (M, q) in the homotopy class of $e([t, 1])$ is shorter and thus (M, q) satisfies the conditions of [Lemma 5.4](#). Then there exists a (square) transposition of $e(t), p_j$ along $e([t, 1])$, based at (M, q) . The rest of the argument follows as in [Lemma 6.1](#). \square

Lemma 6.4 *Let (M_0, q_0) and λ be as in Lemma 6.1, and let b be a branch point of M_0 that is also a regular point of q_0 . Let e_b be an edge between b and p_i^+ , $1 \leq i \leq m$, such that $\tau(e_b) \cap e_b = b$. Then $e := e_b \cup \tau(e_b)$ is an edge between p_i^+ and p_i^- , and $\pi_1(\mathcal{Q}_\lambda^0, (M_0, q_0))$ contains an element corresponding to the transposition of p_i^+, p_i^- along e .*

Proof Let e' be a small deformation of e to one side of b that is still an edge. We can always choose e' so that $e' \sim e \sim \tau(e')$ and $e' \cap \tau(e') = p_i^\pm$. As in the previous two lemmas, let $(\mathbb{P}^1, \tilde{q}_0) = \pi(M_0, q_0)$, with singularities $\tilde{p}_1, \dots, \tilde{p}_{2g+2+m}$, and let $\tilde{e}' = \pi(e')$. Define $(\mathbb{P}^1, \tilde{q}_t)$ to be the quadratic differential with zeroes at $\tilde{p}_1, \dots, \tilde{p}_{i-1}, \tilde{e}'(t), \tilde{p}_{i+1}, \dots, \tilde{p}_{2g+2+m}$, and let (M_t, q_t) be its double cover ramified at $\tilde{p}_{m+1}, \dots, \tilde{p}_{2g+2+m}$. Then $(M_0, q_0) = (M_1, q_1)$ and $T: [0, 1] \rightarrow \mathcal{Q}_\lambda, t \mapsto (M_t, q_t)$ is the desired element of $\pi_1(\mathcal{Q}_\lambda, (M_0, q_0))$. □

The previous four lemmas will allow us to show that certain hyperelliptic strata contain all of their transpositions, but first we need one more technical lemma. Recall that the maximal number of faces a planar graph with n vertices can have is $2n - 4$. (Such a graph will be a triangulation, with $3n - 6$ edges.) By removing edges from such a graph it is always possible to construct a planar graph with n vertices and fewer than $2n - 4$ faces. (As earlier in the paper, we do not allow graphs to have double edges or loops.)

Lemma 6.5 *Let Γ be a planar graph with n vertices and f faces, $f \leq 2n - 4$. Then we may associate to each face of Γ a pair of vertices adjacent to the face such that the same pair is not associated to more than 1 face.*

Proof It suffices to associate a unique adjacent edge to each face of Γ , since this is equivalent to associating the two vertices adjacent to the edge to the face. Pick any face, F_1 , of Γ and any edge, e_1 , adjacent to F_1 . The edge e_1 is adjacent to one other face, which we call F_2 . F_2 has at least two other possible adjacent edges, so again pick any edge $e_2 \neq e_1$ to associate to F_2 , and let F_3 be the other face adjacent to e_2 . Continue this process until one of two things happens. Either an edge is associated to each face, or at stage k $F_k = F_i$ for $1 \leq i \leq k$ and $k \neq f$. In the first case we are done, and in the second the remaining faces of the graph do not have any adjacent edges that have been associated to any other face, so we pick any face, call it F_{k+1} and resume the process. □

Proposition 6.6 *Suppose that $\mathcal{Q}_\lambda = \mathcal{Q}_g(1^{2n}, k_1, \dots, k_m)$ is such that $2n \geq g + 5$, k_1, \dots, k_m are all even, and there exists $i, j, 1 \leq i < j \leq m$, such that $k_i = k_j$. Then for any $(M, q) \in \mathcal{Q}_\lambda, \pi_1(\mathcal{Q}_\lambda, (M, q))$ contains all transpositions.*

Sketch of Proof Our plan is to construct a hyperelliptic (M, q) with higher order zeroes at branch points, and then put a graph on M with the single zeroes of q at the vertices and at most one higher order zero in each face. We use [Lemma 6.1](#) and [Lemma 6.4](#) to construct the transpositions associated to each edge of the graph, which via [Theorem 4.2](#) gives us all (square) transpositions of single zeroes with each other and with higher order zeroes. We then apply [Lemma 6.3](#) to get (square) transpositions of the higher order zeroes with each other. \square

Proof By [Theorem 2.7](#) Q_λ is connected so we need not consider connected components. Without loss of generality suppose $k_1 = k_2$. Put a graph $\tilde{\Gamma}$ on \mathbb{P}^1 with n vertices and $g + 1$ faces. Since $n \geq (g + 5)/2$, $g + 1 \leq 2n - 4$, and it is always possible to construct such a graph. Mark 2 points in each of g faces of $\tilde{\Gamma}$, and 3 points in the $(g + 1)$ -st. By [Proposition 2.2](#) there exists \tilde{q} on \mathbb{P}^1 such that the vertices of $\tilde{\Gamma}$ are zeroes of \tilde{q} of order 1, $m - 1$ marked points are zeroes of \tilde{q} of order $k_1, (k_3 - 2)/2, (k_4 - 2)/2, \dots, (k_m - 2)/2$, and the $2g + 3 - m$ remaining marked points are poles of \tilde{q} of order 1. Since $m < 2g + 2$, \tilde{q} has at least 2 poles and we assume the $(g + 1)$ -st face contains the zero of order k_1 and 2 poles. Denote the n single zeroes of \tilde{q} by p_1, \dots, p_n . To each face of $\tilde{\Gamma}$ we associate a pair of vertices as in [Lemma 6.5](#).

Take a double cover of $(\mathbb{P}^1, \tilde{q})$ ramified at the $2g + 2$ marked points in the faces of $\tilde{\Gamma}$, minus the zero of order k_1 . Assume each branch cut is between two points in the same face and is contained in that face. This gives us $(M, q) \in Q_g(1^{2n}, k_1, k_2, \dots, k_m)$. (M, q) has two copies of $\tilde{\Gamma}$ embedded into it, $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$; denote their vertices by p_1^+, \dots, p_n^+ and p_1^-, \dots, p_n^- , with $\tau(p_l^+) = p_l^-$, $1 \leq l \leq n$. By associating a pair of vertices to each face of $\tilde{\Gamma}$ we have associated a quadruplet of vertices, p_i^\pm and p_j^\pm , to each of the $g + 1$ branch cuts of M . Construct a pair of edges, e_k and $\tau(e_k)$, between p_i^+, p_j^- and p_j^+, p_i^- through the k -th branch cut, $1 \leq k \leq g$, as in [Figure 5](#). Notice that we may always construct e_k and $\tau(e_k)$ so that they do not intersect. For the $(g + 1)$ -st branch cut, both branch points are regular and we construct edges, e_{g+1} and e'_{g+1} between p_i^\pm and p_j^\pm , as in [Lemma 6.4](#). An example of this is shown in [Figure 6](#). The union $\tilde{\Gamma}^+ \cup \tilde{\Gamma}^- \cup e_1 \cup \tau(e_1) \cup \dots \cup e_g \cup \tau(e_g) \cup e_{g+1} \cup e'_{g+1}$ gives us a connected graph Γ on M with $2g + 2$ faces and two faces associated to each branch cut. This graph will be a 2-cell embedding and by construction satisfies all of the hypotheses of [Theorem 4.2](#). Exactly one branch point is in each of faces associated to the first g branch cuts. For the two faces, F and F' , associated to the $(g + 1)$ -st branch cut, $F' = \tau(F)$ so again there is one zero of order $k_1 (= k_2)$ in each of F, F' .

By [Lemma 6.1](#) we may construct an element of $\pi_1(Q_\lambda, (M, q))$ corresponding to the transposition of p_i^+, p_j^- along both e_k and $\tau(e_k)$ for $1 \leq k \leq g$. Further, each edge e^+

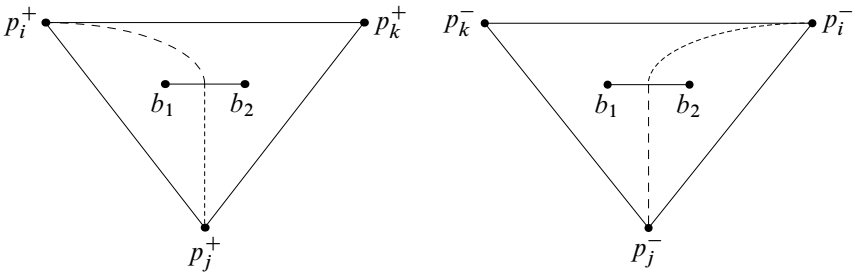


Figure 5: The edges e_k and $\tau(e_k)$ between p_i^+, p_j^- and p_i^-, p_j^+ . b_1 and b_2 denote branch points, and the line between them denotes a branch cut.

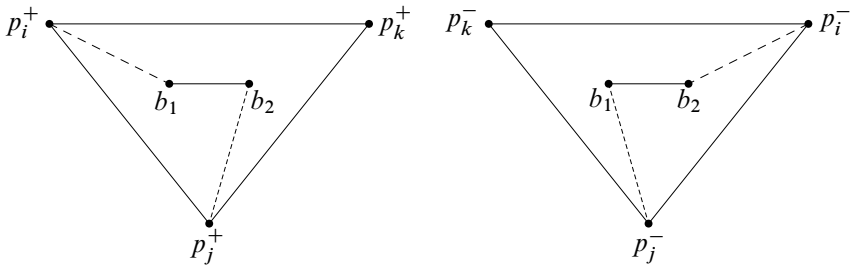


Figure 6: The edges e_{g+1} and e'_{g+1} between p_i^+, p_i^- and p_j^-, p_j^+

of $\tilde{\Gamma}^+$ is contained entirely on one sheet of M , so $e^+ \cap \tau(e^+) = \emptyset$ and again by Lemma 6.1 $\sigma_{e^+}, \sigma_{e^-} \in \pi_1(\mathcal{Q}_\lambda, (M, q))$. Lemma 6.4 gives us transpositions associated to e_{g+1} and e'_{g+1} . Consequently for every edge e of Γ , $\sigma_e \in \pi_1(\mathcal{Q}_\lambda, (M, q))$. Since there is at most one higher order zero in each face, Theorem 4.2 implies $\pi_1(\mathcal{Q}_\lambda, (M, q))$ contains all transposition of the $2n$ zeroes of order 1 with each other, and all square transpositions of zeroes of order 1 with zeroes of order k_1, \dots, k_m .

Now pick any $(\mathbb{P}^1, \tilde{q}') \in \mathcal{Q}_0(1^n, (k_1 - 2)/2, \dots, (k_m - 2)/2, -1^{2g+2-m})$, and let $(M', q') \in \mathcal{Q}_\lambda$ be its double cover, ramified at the zeroes of \tilde{q}' not of order 1. Then all of the even zeroes of q' are at branch points of M' and by Lemma 6.3 $\pi_1(\mathcal{Q}_\lambda, (M', q'))$ contains all (square) transpositions of zeroes of order k_1, \dots, k_m with each other. But $\pi_1(\mathcal{Q}_\lambda, (M', q'))$ is isomorphic to $\pi_1(\mathcal{Q}_\lambda, (M, q))$ so the same is then true for $\pi_1(\mathcal{Q}_\lambda, (M, q))$. This proves the proposition. \square

7 Constructing the remaining generators

We have shown that in some cases all transpositions and square transpositions of zeroes of q are contained in $\pi_1(\mathcal{Q}_\lambda^0, (M, q))$. However, for λ where many but not all points are of equal weight, Theorem 4.14 and Corollary 4.15 imply that the

kernel $\ker(\pi_1(\text{Sym}_g^\lambda \rightarrow H_1(M, \mathbb{Z}))$ is generated by transpositions, null ρ_r and i -commutators. In some cases we can again use techniques of colliding and breaking apart points from Section 5 to show that these last two types of elements are in $\pi_1(\mathcal{Q}_\lambda^0, (M, q))$.

Let l_1, \dots, l_{2g} be standard generators of $\pi_1(M)$, λ of length n , and (p_1, \dots, p_n) a base point for B_λ . Recall that in Section 3 and Section 4 we did not explicitly define the ρ_{ir} , $1 \leq i \leq n$, $1 \leq r \leq 2g$, we only stated that one of the generators of B_λ must be a loop corresponding to p_i moving around l_r . There are infinitely many choices of such a loop, differing by various products of transpositions.

Lemma 7.1 *Let $\mathcal{Q}_{\lambda_1} = \mathcal{Q}_g(k_1, k_2, \dots, k_n)$ contain all of its transpositions, and suppose $\sum_{i=1}^l k_i = \sum_{j=l+1}^m k_j$. Further suppose there exists a degeneration of $\mathcal{Q}_{\lambda_1}^0, \mathcal{Q}_{\lambda_2}^0 := \mathcal{Q}_g^0(k_1 + \dots + k_l, k_{l+1} + \dots + k_m, k_{m+1}, \dots, k_n) \ni (M, q)$, and $\pi_1(\mathcal{Q}_{\lambda_2}^0, (M, q))$ contains all transpositions of the two newly formed points. Define $\alpha(1), \alpha(2), \dots, \alpha(l)$ to all equal 1, and $\alpha(l+1), \dots, \alpha(m)$ to all equal -1 . Then for any $(M', q') \in \mathcal{Q}_{\lambda_1}$, for any choice of $\rho_{1r}, \dots, \rho_{nr}$, for any i_1, \dots, i_m such that $k_{i_s} = k_s$, $1 \leq s \leq m$, and for any $\sigma \in S_m$, $\rho_{i_{\sigma(1)r}}^{\alpha(\sigma(1))} \dots \rho_{i_{\sigma(m)r}}^{\alpha(\sigma(m))} \in \pi_1(\mathcal{Q}_{\lambda_1}^0, (M', q'))$.*

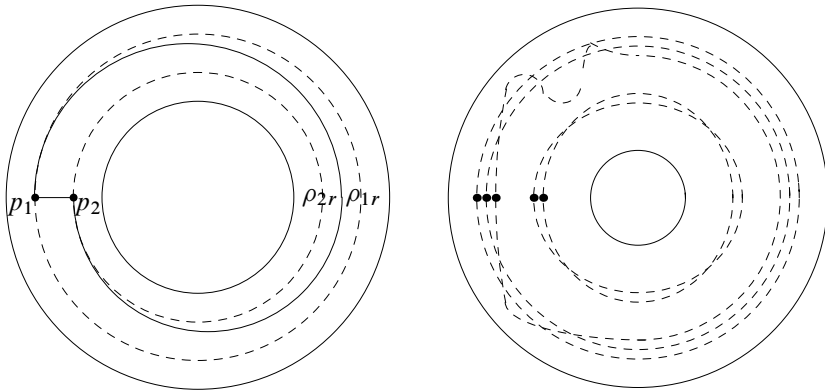


Figure 7: On the left, transposing p_1, p_2 along both the solid curves is equivalent to $\rho_{1r}^\pm \rho_{2r}^\mp$. On the right, surgering give us one possible choice of ρ_{ir} from the original two.

Proof To prove the lemma we first show that there exist explicit $\rho_{1r}, \dots, \rho_{nr}$ such that $\rho := \rho_{1r} \dots \rho_{lr} \rho_{(l+1)r}^{-1} \dots \rho_{m'r}^{-1} \in \pi_1(\mathcal{Q}_\lambda, (M', q'))$. We then pick any other choice of generators, $\rho'_{1r}, \dots, \rho'_{nr}$, any i_1, \dots, i_m (not necessarily distinct) such that $k_{i_s} = k_s$, and any permutation $\sigma \in S_m$. We define $\rho' := \rho'_{i_{\sigma(1)r}}^{\alpha(\sigma(1))} \dots \rho'_{i_{\sigma(m)r}}^{\alpha(\sigma(m))}$. We show that ρ' differs from ρ only by a product of transpositions and is thus also in $\pi_1(\mathcal{Q}_\lambda, (M', q'))$.

Label the two zeroes of q of order $k_1 + k_2 + \dots + k_l$ as p_1 and p_2 . For any r pick two edges, e and e' , between p_1, p_2 such that $e \cup e'$ is homotopic to l_r and $e \cap e' = \{p_1, p_2\}$, as in the left of Figure 7. By assumption $\sigma_e, \sigma_{e'} \in \pi_1(\mathcal{Q}_{\lambda_2}^0, (M, q))$. Further, defining ρ_{1r} and ρ_{2r} as in the left of Figure 7, $\sigma_e \sigma_{e'}$ is homotopic to $\rho_{1r} \rho_{2r}^{-1}$. Thus $\rho_{1r} \rho_{2r}^{-1} \in \pi_1(\mathcal{Q}_{\lambda_2}^0, (M, q))$.

Now pick an explicit loop of surfaces in the homotopy class $\rho_{1r} \rho_{2r}^{-1}$, $\eta: [0, 1] \rightarrow \mathcal{Q}_{\lambda_2}^0$, such that $\eta(0) = \eta(1) = (M, q)$.

Surger every $\eta(t)$ by one of the surgeries of Lemmas 5.2–5.3 such that the new trajectory created is horizontal, and δ_t varies continuously, with $\delta_0 = \delta_1$. This creates a new curve of surfaces, $\gamma: [0, 1] \rightarrow \mathcal{Q}_g$, ($\gamma([0, 1])$ is either in \mathcal{Q}_{λ_1} or in some “intermediate” stratum between \mathcal{Q}_{λ_1} and \mathcal{Q}_{λ_2}) such that either $\gamma(0) = \gamma(1)$ or $\gamma(0)$ differs from $\gamma(1)$ in that the newly formed trajectories’ directions differ by a multiple of π . In the latter case, label the direction of the trajectory formed by $\gamma(0)$ as 0° , so the one formed by $\gamma(1)$ will be in the direction $k\pi$, $1 \leq k \leq k_1 + k_2 + \dots + k_l + 2$. Consider the curve of surfaces $\alpha: [0, k\pi] \rightarrow \mathcal{Q}_g$ created by surgering $\eta(0)$ with respect to δ_0, t . Concatenating γ with α gives us a closed curve of surfaces, again either in \mathcal{Q}_{λ_1} or an intermediate stratum.

Repeating the above process multiple times if necessary, we get a loop of surfaces $\eta': [0, 1] \rightarrow \mathcal{Q}_{\lambda_1}$, and define $(M', q') = \eta'(0)$. Let $p_1(t), \dots, p_m(t)$ be the zeroes of $\eta'(t)$ of order k_1, \dots, k_m formed from the surgery. We have created an element of $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$ such that each of p_1, \dots, p_m follows a path homotopic to $l_r^{\pm 1}$. Thus we may choose to define $\rho_{1r}, \dots, \rho_{mr}$ in such a way that $\rho := \rho_{1r} \cdots \rho_{lr} \rho_{(l+1)r}^{-1} \cdots \rho_{mr}^{-1} \in \pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$.

Now let σ be an element of S_m and consider $\rho_\sigma := \rho_{\sigma(1)r}^{\alpha(\sigma(1))} \cdots \rho_{\sigma(m)r}^{\alpha(\sigma(m))}$. The product $\rho_\sigma \rho^{-1}$ is an element of B_λ , but since $p_1(0), \dots, p_m(0)$ all return to themselves under $\rho_\sigma \rho^{-1}$ and the other zeroes of (M', q') follow constant paths, we may view $\rho_\sigma \rho^{-1}$ as an element of SB_m . It is a theorem of Goldberg (see Birman [1, Section 1.4]) that for any genus g topological surface, M_g , the following sequence is short exact:

$$1 \rightarrow SB_m(D) \rightarrow SB_m(M_g) \xrightarrow{\psi} \prod_{i=1}^m \pi_1(M_g) \rightarrow 1$$

where $D \subset M_g$ is an open disc containing the m marked points that are the basepoint of SB_m . Since the path traveled by each of the zeroes of (M', q') under $\rho_\sigma \rho^{-1}$ is trivial in $\pi_1(M')$, $\rho_\sigma \rho^{-1}$ is in the kernel of ψ . Thus, it may be written as a product of the generators of $SB_m(D)$, which are square transpositions. We have assumed $\pi_1(\mathcal{Q}_\lambda, (M', q'))$ contains all (square) transpositions, so $\rho_\sigma \rho^{-1} \in \pi_1(\mathcal{Q}_\lambda, (M', q'))$ and therefore $\rho_\sigma \in \pi_1(\mathcal{Q}_\lambda, (M', q'))$.

Similarly define $\rho'_{1r}, \rho'_{2r}, \dots, \rho'_{nr}$ to be a different choice of generators, and define $\rho' := \rho'_{1r} \cdots \rho'_{lr} \rho'^{-1}_{(l+1)r} \cdots \rho'^{-1}_{mr}$. Again $\rho' \rho^{-1}$ is in the kernel of ψ and thus $\rho' \in \pi_1(\mathcal{Q}_\lambda, (M', q'))$.

Finally suppose we have p_{i_1}, \dots, p_{i_m} (not necessarily distinct) such that $k_{i_s} = k_s$, $1 \leq s \leq m$, and define $\rho_i := \rho_{i_1} \cdots \rho_{i_l} \rho_{i_{l+1}}^{-1} \cdots \rho_{i_m}^{-1}$. Either $s = i_s$ and $\rho_{sr}^\pm \rho_{i_s r}^\mp = 1$ or $\rho_{sr}^\pm \rho_{i_s r}^\mp$ may be written as a product of two transpositions, as in the first paragraph of this proof. Then the following is in $\pi_1(\mathcal{Q}_\lambda, (M', q'))$:

$$\beta := \rho_{1r} \rho_{i_1 r}^{-1} \rho_{2r} \rho_{i_2 r}^{-1} \cdots \rho_{lr} \rho_{i_l r}^{-1} \rho_{(l+1)r}^{-1} \rho_{i_{l+1} r} \cdots \rho_{mr}^{-1} \rho_{i_m r}$$

As above $\rho_i \beta \rho^{-1}$ is in the kernel of ψ and is therefore in $\pi_1(\mathcal{Q}_\lambda, (M', q'))$. This then implies $\rho_i \in \pi_1(\mathcal{Q}_\lambda, (M', q'))$.

A combination of the above implies the lemma for the specific choice of (M', q') , surgered from (M, q) . However, since fundamental groups with different base points are isomorphic, the same will be true for any element of \mathcal{Q}_{λ_1} . □

Lemma 7.1 shows that under certain degeneracy conditions it is possible to construct any null ρ_r involving points of certain weights. Notice that when $l, m = 1$ the lemma implies that if $\pi_1(\mathcal{Q}_{\lambda_1}^0, (M', q'))$ contains all of its transpositions, it also contains all null ρ_r involving two points of equal weights.

Proposition 7.2 *Suppose that $g > 2$ and $\lambda_1 = (1^a, k_1, \dots, k_n)$ with exponent $a > \max\{g + 4, k_1, \dots, k_n\}$, and additionally assume that all k_i are even and there exists $s, t, 1 \leq s, t \leq n$ such that $k_s = k_t$. Then for any (M', q') , all null ρ_r are contained in $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$.*

Proof By **Theorem 2.7** there is only one connected component of \mathcal{Q}_{λ_1} .

Let $(M', q') \in \mathcal{Q}_{\lambda_1}$ with p_1, \dots, p_a the zeroes of q of order 1. By **Proposition 6.6** $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$ contains all of its transpositions; by **Lemma 7.1** it contains any $\rho_{jr} \rho_{lr}^{-1}, 1 \leq j, l \leq a, 1 \leq r \leq 2g$.

For each $k_i, 1 \leq i \leq n$ and each r we construct a null ρ_r consisting of p_{a+i} and k_i zeroes of order 1. These, combined with the $\rho_{jr} \rho_{lr}^{-1}$ where $k_j = k_l = 1$, generate all null ρ_r . We do this in two cases, when $k_i \neq 2$ and when $k_i = 2$.

For $k_i \neq 2$, let $\mathcal{Q}_{\lambda_2} = \mathcal{Q}_g(1^{a-k_i}, k_1, \dots, k_i^2, \dots, k_n)$ and notice that by **Lemma 5.2** and **Lemma 5.3**, $\lambda_1 > \lambda_2$ (this is not true if $k_i = 2$). Since k_i is even there exists a component of $\mathcal{Q}_{\lambda_2}, \mathcal{Q}_{\lambda_2}^0$, that contains a hyperelliptic element, (M, q) , for which both zeroes of order k_i are at branch points. By **Lemma 6.3**, $\pi_1(\mathcal{Q}_{\lambda_2}^0, (M, q))$ contains all transpositions of the two zeroes of order k_i . Therefore **Lemma 7.1** implies any null ρ_r

consisting of a single zero of order k_i moving around l_r and k_i zeroes of order 1 moving around l_r^{-1} is in $\pi_1(\mathcal{Q}_{\lambda_1}, (M', q'))$.

If $k_i = 2$ we let $\mathcal{Q}_{\lambda_2} = \mathcal{Q}_g(1^{a-6}, 4^2, k_1, \dots, \widehat{k_i}, \dots, k_n)$ and note that $\lambda_1 > \lambda_2$. By the same argument as in the previous paragraph there exists $(M, q) \in \mathcal{Q}_{\lambda_2}$ such that $\pi_1(\mathcal{Q}_{\lambda_2}, (M, q))$ contains each of the $2g$ null ρ_r involving only the two zeroes of order 4. Thus we get the null ρ_r involving a zero of order 2 and 2 zeroes of order 1 moving one way around l_r , and 4 zeroes of order 1 moving the other. We compose and cancel with null ρ_r involving points of order 1 to get a null ρ_r with a zero of order 2 moving around l_r and two zeroes of order 1 moving around l_r^{-1} .

This gives us any null ρ_r consisting of a single zero of higher order moving one way around l_r and zeroes of order 1 moving the other. These and the $\rho_{jr}\rho_{l_r}^{-1}$ where $k_j = k_l = 1$ generate all null ρ_r . □

Notice that there are strata of the form specified in [Proposition 6.6](#) for which there exist null ρ_r to which [Lemma 7.1](#) does not apply. For example, in $\mathcal{Q}_{10}(1^{16}, 20)$ we can have a null ρ_r consisting of the point of order 20 moving one way around l_r and 20 points of order 1 moving the other, but since there are not 20 distinct points of order 1 we cannot collide them to use the technique of [Lemma 7.1](#).

Finally we would like to consider when i -commutators are contained in $\pi_1(\mathcal{Q}_\lambda)$.

Proposition 7.3 *Let $\lambda = (1^a, k_1, \dots, k_n)$, where a and the k_i are as in [Proposition 7.2](#), $1 \leq i \leq n$. Then for any $(M, q) \in \mathcal{Q}_\lambda$, any i -commutator of a point not of weight one is in $\pi_1(\mathcal{Q}_\lambda, (M, q))$.*

Proof Let p_1, p_2, \dots, p_a be the zeroes of q of weight 1, and p_{a+1}, \dots, p_{a+n} the zeroes of weight k_1, \dots, k_n , where k_j is by assumption is less than a , $1 \leq j \leq n$. Recall we defined an i -commutator to be an element of the commutator of $\langle \rho_{a+i,r}, \kappa_{a+i,j} \rangle$, $1 \leq r \leq 2g$, $a+i < j \leq n$. Thus it suffices to show that $\pi_1(\mathcal{Q}_\lambda, (M, q))$ contains all elements of the form $g^{-1}h^{-1}gh$, where g and h are arbitrary words in the $\rho_{a+i,r}$ and $\kappa_{a+i,j}$.

By [Proposition 7.2](#) $\pi_1(\mathcal{Q}_\lambda, (M, q))$ contains all of its null ρ_r . Thus for any i , $1 \leq i \leq n$, the following is also in $\pi_1(\mathcal{Q}_\lambda, (M, q))$:

$$(8) (\rho_{1r} \cdots \rho_{k_i r} \rho_{a+i,r}^{-1})(\rho_{1s} \cdots \rho_{k_i s} \rho_{a+i,s}^{-1})(\rho_{1r}^{-1} \cdots \rho_{k_i r}^{-1} \rho_{a+i,r})(\rho_{1s}^{-1} \cdots \rho_{k_i s}^{-1} \rho_{a+i,s})$$

For $k \neq j$ and $l_r \cap l_s = \emptyset$, ρ_{kr} commutes with ρ_{js} , $1 \leq j, k \leq a+n$. If $k \neq j$ but $l_r \cap l_s = 1$ then $\rho_{kr}\rho_{js} = \rho_{js}\rho_{kr}\kappa_e$ where κ_e is a square transposition of p_k, p_j , defined appropriately with respect to ρ_{js}, ρ_{kr} . For example, in [Figure 8](#) we have two

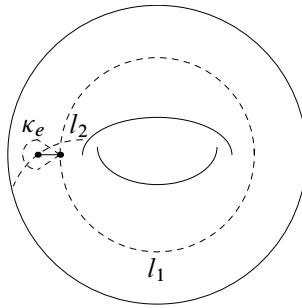


Figure 8: ρ_{11}, ρ_{22} , and κ_e defined on a torus with two marked points. The solid line is e .

points moving around the cycles l_1, l_2 on a torus and an explicitly defined κ_e , with $\rho_{11}\rho_{22} = \rho_{22}\rho_{11}\kappa_e$.

Thus if $l_r \cap l_s = \emptyset$, (8) is equal to

$$(9) \quad (\rho_{1r} \cdots \rho_{k_i r} \rho_{1s} \cdots \rho_{k_i s} \rho_{1r}^{-1} \cdots \rho_{k_i r}^{-1} \rho_{1s}^{-1} \cdots \rho_{k_i s}^{-1})(\rho_{a+i,r}^{-1} \rho_{a+i,s}^{-1} \rho_{a+i,r} \rho_{a+i,s}).$$

Notice that the maximal value for n is $(4g - 4 - (g + 5))/2$, and this implies that $a \geq g + 5 \geq (3 + \sqrt{9 + 8(2g + n - 2)})/2$. The first of the two elements in parentheses in (9) is in the kernel of AJ_* : $\pi_1(\text{Sym}^a(M_n)) \rightarrow H_1(M, \mathbb{Z})$, so we may apply Corollary 4.6 to show that it can be written as a product of transpositions. Proposition 6.6 says that $\pi_1(\mathcal{Q}_\lambda, (M, q))$ contains all of its transpositions; thus the element on the left is in $\pi_1(\mathcal{Q}_\lambda, (M, q))$. This in turn implies $\rho_{a+i,r} \rho_{a+i,s} \rho_{a+i,r}^{-1} \rho_{a+i,s}^{-1} \in \pi_1(\mathcal{Q}_\lambda, (M, q))$.

If $l_r \cap l_s = 1$ then (8) is equal to (9) except that the first element in parentheses will contain some additional κ_e 's. However it will still be in the kernel of AJ_* and again $\rho_{a+i,r} \rho_{a+i,s} \rho_{a+i,r}^{-1} \rho_{a+i,s}^{-1} \in \pi_1(\mathcal{Q}_\lambda, (M, q))$. The above argument is easily generalized to show that the commutators of arbitrary words in the $\rho_{a+i,r}$, $1 \leq r \leq 2g$, are in $\pi_1(\mathcal{Q}_\lambda, (M, q))$.

Similarly, for p_l a zero of higher order and e' an edge between p_i and p_l , we would like to show that $\kappa_{e'}^{-1} \rho_{a+i,r}^{-1} \kappa_{e'} \rho_{a+i,r}$ is in $\pi_1(\mathcal{Q}_\lambda, (M, q))$, so we consider the following:

$$(10) \quad \kappa_{e'}^{-1} (\rho_{1r} \cdots \rho_{k_i r} \rho_{a+i,r}^{-1}) \kappa_{e'} (\rho_{1r}^{-1} \cdots \rho_{k_i r}^{-1} \rho_{a+i,r})$$

By assumption both $\kappa_{e'}^{\pm 1}$ and the elements in parentheses are in $\pi_1(\mathcal{Q}_\lambda, (M, q))$, so the whole element in (10) is. Since $\kappa_{e'}$ commutes with $\rho_{1r}, \dots, \rho_{kr}$, commuting the $\rho_{1r}, \dots, \rho_{kr}$ as from (8) to (9) gives us the desired result. A combination of this argument and the above shows that the commutators of arbitrary words in the $\rho_{a+i,r}$ and $\kappa_{a+i,j}$ are in $\pi_1(\mathcal{Q}_\lambda, (M, q))$. \square

8 Conclusion

We summarize by answering the question of when $\ker(\text{AJ}_*: \pi_1(\text{Sym}_g^\lambda) \rightarrow H_1(M, \mathbb{Z}))$ is equal to $\text{im}(i_*: \pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\text{Sym}_g^\lambda))$.

Theorem 8.1 *Let $\lambda = (1^a, k_1, \dots, k_n)$ with $a > \max\{g + 5, k_1, \dots, k_n\}$, all k_i even, and some $k_i = k_j$, $1 \leq i < j \leq n$. Then $\text{im}(i_*) = \ker(\text{AJ}_*)$.*

Proof Proposition 4.1 implies $\text{im}(i_*: \pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\text{Sym}_g^\lambda)) \subset \ker(\text{AJ}_*)$. By Theorem 4.14 and Corollary 4.15 $\ker(\text{AJ}_*)$ is generated by transpositions, square transpositions, null ρ_r and in some cases i -commutators. Proposition 6.6, Proposition 7.2, and Proposition 7.3 show that all of these elements are in $\text{im}(i_*: \pi_1(\mathcal{Q}_\lambda) \rightarrow \pi_1(\text{Sym}_g^\lambda))$. \square

In [2], Copeland shows a similar result for $g > 2$ and $\lambda = (1^{4g-4})$. His techniques are somewhat different and rely on the fact that in the top stratum one may interpolate two quadratic differentials and expect the result to be in the same stratum.

Thus, for certain λ we have constructed $i_*(\pi_1(\mathcal{Q}_\lambda))$. Of course, we are actually interested in $\pi_1(\mathcal{Q}_\lambda)$ and would thus like to determine the kernel of i_* . However, it may be difficult to say anything about this kernel.

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