The sutured Floer homology polytope

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In this paper, we extend the theory of sutured Floer homology developed by the author [13; 14]. We first prove an adjunction inequality and then define a polytope $P(M,\gamma)$ in $H^2(M,\partial M;\mathbb{R})$ that is spanned by the Spin^c–structures which support nonzero Floer homology groups. If $(M,\gamma)\rightsquigarrow(M',\gamma')$ is a taut surface decomposition, then an affine map projects $P(M',\gamma')$ onto a face of $P(M,\gamma)$; moreover, if $H_2(M)=0$, then every face of $P(M,\gamma)$ can be obtained in this way for some surface decomposition. We show that if (M,γ) is reduced, horizontally prime and $H_2(M)=0$, then $P(M,\gamma)$ is maximal dimensional in $H^2(M,\partial M;\mathbb{R})$. This implies that if $\mathrm{rk}(SFH(M,\gamma))<2^{k+1}$, then (M,γ) has depth at most $2k$. Moreover, SFH acts as a complexity for balanced sutured manifolds. In particular, the rank of the top term of knot Floer homology bounds the topological complexity of the knot complement, in addition to simply detecting fibred knots.

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1 Introduction

Heegaard Floer homology is an invariant of closed oriented three-manifolds defined by Ozsváth and Szabó in [19]. It comes in four different flavors: $\widehat{HF}$, HF^+, HF^- and HF^∞. This was extended to an invariant HFK of knots by Ozsváth and Szabó in [18], and independently by Rasmussen in [23]. Later, Ozsváth and Szabó [21] generalized $\widehat{HFK}$ to an invariant $\widehat{HFL}$ of links in S^3. For a knot K in S^3, the group $\widehat{HFK}(K)$ splits as a direct sum $\bigoplus_{i,j\in\mathbb{Z}}\widehat{HFK}_j(K,i)$, and has a homological $\mathbb{Z}$–grading. For each $i\in\mathbb{Z}$, the Euler characteristic of $\widehat{HFK}_*(K,i)$ is equal to the i–th coefficient a_i of the Alexander–Conway polynomial $\Delta_K(t)$ of K.

It is a classical result that for a knot K in S^3 the polynomial $\Delta_K(t)$ gives a lower bound on the genus of K in the following sense:

$$g(K)\geq\max\{\,i\in\mathbb{Z}:a_i\neq 0\,\}.$$

Ozsváth and Szabó in [17] showed that knot Floer homology actually detects the genus of K:

$$g(K)=\max\{\,i\in\mathbb{Z}:\widehat{HFK}(K,i)\neq 0\,\}.$$

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The proof of this striking result uses Gabai's theory of sutured manifolds [4; 7; 8], the Eliashberg–Thurston theory of confoliations [2], the contact invariant and cobordism maps in Heegaard Floer homology, symplectic semifillings and Lefschetz pencils.

The theory of sutured manifolds was developed by Gabai in [4] in order to study the existence of taut foliations on 3–manifolds. Sutured manifolds are oriented 3–manifolds with boundary, together with a set of oriented simple closed curves, called sutures, that divide the boundary into a plus and a minus part. They can be thought of as cobordisms between compact oriented surfaces with boundary. Gabai also defined an operation on sutured manifolds, called sutured manifold decomposition. It consists of cutting the manifold along a properly embedded oriented surface R, and adding one side of R to the plus, and the other side to the minus part of the boundary. He showed that a sutured manifold carries a taut foliation if and only if there is a sequence of decompositions that results in a product sutured manifold (essentially a trivial cobordism). The theory of sutured manifold decompositions was generalized by Honda, Kazez and Matić [11] to study tight contact structures on 3–manifolds, and was called convex decomposition theory.

In [13], I introduced sutured Floer homology, in short SFH, which is an invariant of balanced sutured manifolds. SFH is an invariant of three-manifolds with boundary and generalizes $\widehat{HF}$, $\widehat{HFK}$ and $\widehat{HFL}$. The balanced condition is not very restrictive, since in Proposition 3.14 we show that every open *taut* sutured manifold that has at least one suture on each boundary component is balanced.

I used SFH in [14] to give a more elegant and direct proof of the fact that knot Floer homology detects the genus of a knot. That proof only relies on Gabai's theory of sutured manifolds and the following two results. First, if R is a Seifert surface of a knot K in S^3, then

$$\widehat{HFK}(K, g(R)) \cong SFH(S^3(R)),$$

where $S^3(R)$ is the sutured manifold complementary to R; see [14, Theorem 1.5]. Secondly, by [14, Theorem 1.3], if we decompose a sutured manifold (M, γ) along a "nice" surface and get the sutured manifold (M', γ'), then $SFH(M', \gamma')$ is a direct summand of $SFH(M, \gamma)$. We will refer to this as "the decomposition formula". If R is of minimal genus, then $S^3(R)$ is taut, so by [4] there is a sequence of nice decompositions that ends in a product. The SFH of a product is $\mathbb{Z}$, so the decomposition formula implies that $SFH(S^3(R))$ contains a $\mathbb{Z}$ direct summand.

For a Seifert surface R, even though $SFH(S^3(R))$ is isomorphic to the top term of knot Floer homology, it carries an extra Spin^c–grading. Note that for a sutured manifold (M, γ), the set of Spin^c–structures $\mathrm{Spin}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M) \cong H_1(M)$. In the present paper, we study this extra grading on SFH, and

how it behaves under sutured manifold decompositions. Using our results, we show that the top term of knot Floer homology carries deep topological information about the knot complement. In particular, we have the following, which is a special case of Corollary 7.8.

Theorem 1 *Suppose that K is a knot in S^3, and*

$$\mathrm{rk}\big(\widehat{HFK}(K, g(K))\big) < 2^{k+1}.$$

Then the sutured manifold $S^3(K)$ complementary to K has depth $d(S^3(K)) \leq 2k+1$. In particular, if $k = 0$, then K is fibred.

Here the depth of a sutured manifold is the minimal number of decompositions needed to get a product sutured manifold.

Ozsváth and Szabó conjectured that knot Floer homology detects fibred knots in the sense that $\widehat{HFK}(K, g(K)) \cong \mathbb{Z}$ if and only if K is fibred. This was proved by Ghiggini in [9] for genus one knots, and proceeds along the lines of the Ozsváth–Szabó proof of $\widehat{HFK}(K, g(K)) \neq 0$, using deep symplectic and contact topology. Building on Ghiggini's work, Ni proposed a proof of the general case in [15], using an alternative version of sutured Floer homology (without the Spin^c–grading), and a restricted version of the decomposition formula for horizontal surfaces and separating product annuli. Shortly after this, in [14] I presented a more direct proof of the fibred knot conjecture, only using SFH and the general decomposition formula. This starts out with an observation of Gabai [6] that a knot K is fibred if and only if $S^3(R)$ is a product sutured manifold, where R is a minimal genus Seifert surface for K. So the problem can be reduced to the question whether SFH detects product sutured manifolds. Later, it turned out that the last part of the proof in [15] had a gap due to an incorrect reference to Cooper and Long [1] concerning characteristic product regions. In [14], I borrowed Ni's last argument to conclude my proof, so [14] has the same gap. Ni filled in this gap in [16]. In the present paper, I correct and generalize [14] by eliminating the use of characteristic product regions. My approach is completely different from that of [16]. Instead, I only use reduced sutured manifolds, ones in which every product annulus is parallel to a suture. Since in [14] I also proved the decomposition formula for nonseparating product annuli, it is enough to work with reduced sutured manifolds. The introduction of the SFH polytope makes the proof very transparent, and makes it possible to get a much sharper result, namely Theorem 1.

For a sutured manifold (M, γ), the Spin^c–structures that support nonzero Floer homology groups span a polytope $P(M, \gamma)$ in $H^2(M, \partial M; \mathbb{R})$. This polytope is well defined up to translations. A major tool in this paper is the adjunction inequality, Theorem 4.1.

Theorem 2 (Adjunction inequality) *Suppose that the sutured manifold (M,γ) is strongly balanced, and fix a trivialization $t \in T(M,\gamma)$. Let $S \subset M$ be a nice decomposing surface. If a Spin^c–structure $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ satisfies*

$$\langle c_1(\mathfrak{s},t),[S]\rangle < c(S,t),$$

then $SFH(M,\gamma,\mathfrak{s}) = 0$.

Here $c(S,t)$ is a purely topological quantity, and we show that the above inequality can be rearranged to get a Thurston–Bennequin type inequality. In Theorem 4.5, we use the adjunction inequality to extend the decomposition formula [14, Theorem 3.11] to disconnected decomposing surfaces.

In Proposition 4.13 and Corollary 4.15, we establish a relationship between decompositions of (M,γ) and faces of $P(M,\gamma)$, and show that if $H_2(M)=0$, then every face of $P(M,\gamma)$ corresponds to a well-groomed surface decomposition. More concretely, Theorem 5.11 implies that if $(M,\gamma)\overset{S}{\rightsquigarrow}(M',\gamma')$ is a taut surface decomposition, then there is an affine map from $H^2(M',\partial M';\mathbb{R})$ to $H^2(M,\partial M;\mathbb{R})$ which projects $P(M',\gamma')$ onto a face of $P(M,\gamma)$. This map is a translate of the dual of the map $H_1(M')\to H_1(M)$ induced by the embedding $M'\hookrightarrow M$. If S is a disk, then this projection is actually an isomorphism. So we see how Spin^c–structures split under surface decompositions. This, for example, implies a result of Gabai [5] that if a sutured manifold is disk decomposable, then it can be decomposed into a product using a single (not necessarily connected) surface, and if γ is connected, then it carries a taut foliation of depth at most one.

From now on, we are going to suppose that (M,γ) is a taut balanced sutured manifold which satisfies the condition $H_2(M)=0$. The condition $H_2(M)=0$ is not very restrictive, since it is satisfied by any sutured manifold complementary to a connected surface in a rational homology 3–sphere; furthermore, it is preserved by nice surface decompositions. And the most studied sutured manifolds are exactly the ones which are complementary to a Seifert surface of a knot or a link.

Theorem 6.1 is one of the main results of this paper.

Theorem 3 *Suppose that $H_2(M)=0$, and the sutured manifold (M,γ) is balanced, taut, reduced and horizontally prime. Then*

$$\dim P(M,\gamma) = \dim H^2(M,\partial M;\mathbb{R}) = b_1(M) = b_1(\partial M)/2.$$

In particular,

$$\mathrm{rk}(SFH(M,\gamma)) \geq b_1(\partial M)/2+1.$$

This result fills in the gap in [14], and makes further generalizations possible. Using this, we prove Proposition 7.6:

Theorem 4 *Suppose that (M,γ) is a taut balanced sutured manifold such that $H_2(M)=0$ and $\mathrm{rk}(SFH(M,\gamma))<2^{k+1}$ for some integer $k\geq 0$. Then the depth of (M,γ) is at most $2k$.*

The proof proceeds by induction on k. One has to first decompose (M,γ) along a maximal set of product annuli to make it reduced. Then $P(M,\gamma)$ becomes maximal dimensional in its ambient space. There is a Spin^c–structure $\mathfrak{s}$ that is a vertex of $P(M,\gamma)$, and such that $\mathrm{rk}(SFH(M,\gamma,\mathfrak{s}))<2^k$. Furthermore, we saw that there is a decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that $SFH(M',\gamma')\cong SFH(M,\gamma,\mathfrak{s})$. So we can apply the induction hypotheses to (M',γ') to see that it has depth at most $2k-2$. In particular, this illustrates how the rank of SFH can be used to measure the complexity of balanced sutured manifolds, and to perform inductive proofs using it. It is worth comparing it to the complexity defined by Gabai in [4] to show the existence of sutured manifold hierarchies.

Theorem 4 implies Theorem 1, since if we decompose the knot complement $S^3(K)$ along a minimal genus Seifert surface R, then we get the taut balanced sutured manifold $S^3(R)$ with $\mathrm{rk}(SFH(M,\gamma))<2^{k+1}$.

The sutured manifold (M,γ) constructed in Example 7.5 has the following surprising property: the polytope $P(M,\gamma)$ consists of a single point, even though (M,γ) is horizontally prime and is not a product. This example also illustrates that decompositions along product annuli can change the sutured Floer homology polytope, while the rank of SFH remains unchanged. So we always need to make our sutured manifold reduced by cutting along product annuli before we can decrease the rank of SFH by another sutured manifold decomposition.

In Section 8, we define a function y on $H_2(M,\partial M;\mathbb{R})$ that is a seminorm, except that $y(c)$ and $y(-c)$ might be different. The dual unit norm polytope of y is exactly $-P(M,\gamma)$. Moreover, y is nondegenerate if (M,γ) is reduced and horizontally prime. Using this norm, we can view Corollary 4.11 as an extension of a theorem of Ozsváth and Szabó [22] that link Floer homology detects the Thurston norm of the link complement. We will prove in [3] that if we symmetrize y, we get a seminorm that gives a lower bound on the seminorm defined by Scharlemann in [24], but is different from it.

Finally, we compute the sutured Floer homology of any sutured manifold (M,γ) such that $M\approx S^1\times D^2$. This illustrates some of the techniques developed in this paper,

and will be used in future computations. For further examples of $P(M,\gamma)$, including whole families where M is a genus two handlebody, we refer the reader to [3]. In [10], we show how the sutured Floer homology polytope can be used to distinguish Seifert surfaces up to isotopy.

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2 Sutured manifolds

To get an in depth introduction to the theory of sutured manifolds and surface decompositions, we recommend reading Gabai's original papers [4; 7; 8]. For the reader's convenience, let us review the most important definitions and results here.

Notation 2.1 Throughout this paper, we are going to use the following notation. If K is a submanifold of the manifold M, then $N(K)$ denotes a regular neighborhood of K in M and $[K]$ is the homology class represented by K. If A is a set, then $|A|$ is the cardinality of A. If X is a topological space, then $|X|$ is the number of components of X.

Definition 2.2 A *sutured manifold* (M,γ) is a compact oriented 3–manifold M with boundary together with a set $\gamma\subset\partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, ie, a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by $s(\gamma)$.

Finally every component of $R(\gamma)=\partial M\setminus \mathrm{Int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M\setminus\mathrm{Int}(\gamma)$ whose normal vectors point out of (into) M. The orientation on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, ie, if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

Definition 2.3 A sutured manifold (M,γ) is called *balanced* if M has no closed components, $\chi(R_+(\gamma))=\chi(R_-(\gamma))$, and the map $\pi_0(A(\gamma))\to\pi_0(\partial M)$ is surjective.

Definition 2.4 A sutured manifold (M,γ) is *taut* if M is irreducible and $R(\gamma)$ is incompressible and Thurston norm minimizing in $H_2(M,\gamma)$. For the definition of the Thurston norm, we refer the reader to Thurston's paper [25].

Definition 2.5 Let (M,γ) be a sutured manifold. A *decomposing surface* is a properly embedded oriented surface S in M such that no component of ∂S bounds a disk in $R(\gamma)$ and no component of S is a disk D with $\partial D\subset R(\gamma)$. Moreover, for every component λ of $S\cap\gamma$ one of (1)–(3) holds:

(1) λ is a properly embedded nonseparating arc in γ such that $|\lambda\cap s(\gamma)|=1$.

(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A\cap s(\gamma)$.

(3) λ is a homotopically nontrivial curve in a torus component T of γ, and if δ is another component of $T\cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Then S defines a *sutured manifold decomposition*

$$(M,\gamma)\rightsquigarrow^S(M',\gamma'),$$

where $M'=M\setminus\mathrm{Int}(N(S))$ and

$$\begin{aligned}\gamma' &= (\gamma\cap M')\cup N(S'_+\cap R_-(\gamma))\cup N(S'_-\cap R_+(\gamma)),\\ R_+(\gamma') &= ((R_+(\gamma)\cap M')\cup S'_+)\setminus\mathrm{Int}(\gamma'),\\ R_-(\gamma') &= ((R_-(\gamma)\cap M')\cup S'_-)\setminus\mathrm{Int}(\gamma'),\end{aligned}$$

where S'_+ (S'_-) is the component of $\partial N(S)\cap M'$ whose normal vector points out of (into) M'.

Definition 2.6 Let (M,γ) be a balanced sutured manifold. A surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ is called *groomed* if for each component V of $R(\gamma)$ one of the following holds:

- $S\cap V$ is a union of parallel, coherently oriented, nonseparating simple closed curves.
- $S\cap V$ is a union of arcs such that for each component δ of ∂V we have $|\delta\cap\partial S|=|\langle\delta\cap\partial S\rangle|$, where $\langle\delta\cap\partial S\rangle$ is the algebraic intersection number of δ and ∂S.

A surface decomposition is called *well groomed* if for each component V of $R(\gamma)$ the intersection $S \cap V$ is a union of parallel, coherently oriented, nonseparating simple closed curves or arcs.

The following definition is motivated by [4, Lemma 3.8].

Definition 2.7 Let (M,γ) be a balanced sutured manifold. We say that a class $z \in H_2(M,\partial M)$ is *well groomed* if $\partial z \neq 0$ in $H_1(\partial M)$ and the following hold.

(1) For each nonplanar component V of $R(\gamma)$ and each component λ of ∂V we have $\langle z, \lambda \rangle = 0$.

(2) For each planar component V of $R(\gamma)$ there exist at most two components λ_1 and λ_2 of ∂V such that $\langle z, \lambda_i \rangle \neq 0$ for $i = 1, 2$.

Note that $z \in H_2(M,\partial M)$ is well groomed if and only if $-z$ is well groomed. Using this terminology [4, Lemma 3.8] can be stated as follows.

Lemma 2.8 *Let (M,γ) be a balanced sutured manifold. Then there exists a well groomed class in $H_2(M,\partial M)$.*

Lemma 2.9 *Let (M,γ) be a taut balanced sutured manifold and $z \in H_2(M,\partial M)$ a well groomed homology class. Then there is a well groomed surface decomposition $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ such that $[S] = z$ and (M',γ') is taut.*

Proof This lemma follows from the last argument in the proof of [8, Lemma 3.4] which goes as follows. By [7, Lemma 0.7], there is a groomed surface S which gives a taut decomposition and $[S] = z$. Now [4, Lemma 3.9] implies that if V is a component of $R(\gamma)$, then $S \cap V$ is homologous to a set of parallel curves. Finally, an application of [7, Lemma 0.6] yields the desired well groomed surface. □

Definition 2.10 Let (M,γ) be a sutured manifold. A *product annulus* in (M,γ) is an annulus A properly embedded in M such that $\partial A \subset R(\gamma)$, $\partial A \cap R_+(\gamma) \neq \varnothing$, and $\partial A \cap R_-(\gamma) \neq \varnothing$. A *product disk* is a disk D properly embedded in M such that $\partial D \cap \gamma$ consists of two essential arcs in γ. Product disks and product annuli detect where a sutured manifold is locally a product. (M,γ) is a *product sutured manifold* if $M = R \times I$, $\gamma = \partial R \times I$, $R_+(\gamma) = R \times \{1\}$, and $R_-(\gamma) = R \times \{0\}$.

Lemma 2.11 *Let (M,γ) be a sutured manifold such that M is irreducible and $R(\gamma)$ is incompressible. If A is a compressible product annulus in (M,γ), then there is a cylinder $D^2 \times I \subset M$ such that $A = \partial D^2 \times I$ and $D^2 \times \partial I \subset R(\gamma)$.*

Proof Suppose that a simple closed curve in A bounds a disk in M but it does not bound a disk in A. Then both $a_- = A \cap R_-(\gamma)$ and $a_+ = A \cap R_+(\gamma)$ bound disks in M. Since $R(\gamma)$ is incompressible, both a_- and a_+ bound disks D_- and D_+ in $R(\gamma)$, respectively. But M is irreducible, hence the embedded sphere $D_- \cup A \cup D_+$ bounds a 3–ball in M. This 3–ball can be identified with $D^2 \times I$ such that it satisfies the stated properties. □

Definition 2.12 We say that a balanced sutured manifold (M, γ) is *reduced* if every incompressible product annulus A in (M, γ) is ambient isotopic to a component of γ such that ∂A stays in $R(\gamma)$ throughout. Call a product disk D *inessential* if there is an ambient isotopy of D into γ which fixes $D \cap \gamma$, and *essential* otherwise.

Lemma 2.13 *Let (M, γ) be a* reduced *sutured manifold such that M is irreducible and $R(\gamma)$ is incompressible. Then exactly one of the following holds.*

(1) *Every product disk in (M, γ) is* inessential.

(2) *(M, γ) is homeomorphic to $(\Sigma \times I, \partial\Sigma \times I)$, where Σ is a sphere with either two or three open disks removed.*

Proof In this proof we implicitly use the following observation several times. Suppose that the product annulus $A \subset M \setminus \gamma$ is ambient isotopic to a component γ' of γ such that ∂A stays in $R(\gamma)$ throughout. Then there is a submanifold $C \times I$ inside M such that C is an annulus, $\partial C \times I = A \cup \gamma'$ and $C \times \partial I \subset R(\gamma)$.

Suppose that (M, γ) contains an essential product disk D. We distinguish two cases depending on whether the two arcs of $D \cap \gamma$ lie in the same component of γ. First suppose that there is a single component γ_0 of γ which contains $D \cap \gamma$. Denote the closures of the components of $\partial N(\gamma_0 \cup D) \cap \mathrm{Int}(M)$ by A_1 and A_2. Then the product annuli A_1 and A_2 both have to be incompressible, otherwise by Lemma 2.11 the product disk D would by inessential. Furthermore, neither A_1 nor A_2 can be ambient isotopic to γ_0, else again D would be inessential. Thus there are components γ_1 and γ_2 of γ, both distinct from γ_0 and from each other, such that A_i is ambient isotopic to γ_i for $i = 1, 2$. It follows that (M, γ) is the product $(\Sigma \times I, \partial\Sigma \times I)$, where Σ is a sphere with three open disks removed.

Now suppose that there are components γ_0 and γ_1 of γ such that $D \cap \gamma_i \neq \varnothing$ for $i = 0, 1$. Let A be the closure of $\partial N(\gamma_0 \cup \gamma_1 \cup D) \cap \mathrm{Int}(M)$. If the product annulus A is compressible, then by Lemma 2.11 we are in case (2) with Σ being a sphere with two open disks removed. Otherwise A is ambient isotopic to a component γ_2 of γ different from γ_0 and γ_1, and hence we are again in case (2) with Σ being a sphere with three open disks removed.

On the other hand, if (2) holds, then (M,γ) is reduced, but it contains an essential product disk. □

Definition 2.14 A 3–*manifold pair* is a pair (M,T), where M is a 3–manifold with boundary and $T\subset\partial M$ is a subsurface with boundary. A compact 3–manifold pair (M,T) is said to be *irreducible* if M is irreducible and T is incompressible.

Next we recall [12, Proposition V.1.6].

Proposition 2.15 *For each compact, irreducible 3–manifold pair (M,T), there is a number $h(M,T)$ with the following property. Let $W\subset M$ be a two-sided, incompressible surface having more than $h(M,T)$ components and such that $\partial W\subset T$. Then either*

(1) *W has a T–parallel component, or*

(2) *W has two components which are parallel in (M,T).*

Proposition 2.16 *Let (M,γ) be a sutured manifold such that M is irreducible and $R(\gamma)$ is incompressible. Then there is a decomposition $(M,\gamma)\rightsquigarrow^{A}(M',\gamma')$ such that A is a union of pairwise disjoint incompressible product annuli and (M',γ') is reduced.*

Proof Using the terminology of [12] the 3–manifold pair $(M,R(\gamma))$ is irreducible, thus we can apply Proposition 2.15 to get a number $h(M,R(\gamma))$. Note that a product annulus cannot be $R(\gamma)$–parallel since its two boundary components lie in different components of $R(\gamma)$. So we can recursively construct a maximal set of pairwise disjoint incompressible product annuli $A_1,\dots,A_n$ such that for $A=A_1\cup\dots\cup A_n$ no two components of $\gamma\cup A$ are parallel in $(M,R(\gamma))$. Indeed, $n\leq h(M,R(\gamma))-|\gamma|$ for such an A, thus the recursion has to terminate in finitely many steps.

Decomposing (M,γ) along a maximal A the resulting (M',γ') is reduced. Indeed, an incompressible product annulus C' in (M',γ') which is not parallel to γ' gives rise to a product annulus C in (M,γ) which is not parallel to any component of $\gamma\cup A$. We show that C is incompressible. Indeed, if C was compressible, then $\partial C\cap R_+(\gamma)$ would bound a disk D in $R_+(\gamma)$. Since C' is incompressible $\partial A\cap D\neq\varnothing$, thus A would also be compressible, a contradiction. But the existence of such a C contradicts the maximality of A. □

Definition 2.17 Let (M,γ) be a balanced sutured manifold. A decomposing surface $S\subset M$ is called a *horizontal surface* if

(i) S is open and incompressible,

(ii) $\partial S\subset\gamma$ and ∂S is isotopic to $\partial R_+(\gamma)$,

(iii) $[S]=[R_+(\gamma)]$ in $H_2(M,\gamma)$,

(iv) $\chi(S)=\chi(R_+(\gamma))$.

We say that (M,γ) is *horizontally prime* if every horizontal surface in (M,γ) is parallel to either $R_+(\gamma)$ or $R_-(\gamma)$.

Proposition 2.18 *Let (M,γ) be a balanced sutured manifold. Then there is a surface decomposition $(M,\gamma)\overset{H}{\rightsquigarrow}(M',\gamma')$ such that every component of H is a horizontal surface and (M',γ') is horizontally prime.*

Proof Apply Proposition 2.15 to the 3–manifold pair (M,γ). □

3 Sutured Floer homology and Spin^c–structures

Sutured Floer homology is an invariant of balanced sutured manifolds defined in [13]. It is constructed in a way analogous to ordinary Heegaard Floer homology.

Definition 3.1 A *sutured Heegaard diagram* is a tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$, where Σ is a compact oriented surface with boundary and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two sets of pairwise disjoint simple closed curves in $\mathrm{Int}(\Sigma)$.

Every sutured Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ uniquely *defines* a sutured manifold (M,γ) using the following construction. Let $\boldsymbol{\alpha}=\{\alpha_1,\dots,\alpha_m\}$ and $\boldsymbol{\beta}=\{\beta_1,\dots,\beta_n\}$. Let M be the 3–manifold obtained from $\Sigma\times I$ by attaching 3–dimensional 2–handles along the curves $\alpha_i\times\{0\}$ and $\beta_j\times\{1\}$ for $i=1,\dots,m$ and $j=1,\dots,n$. The sutures are defined by taking $\gamma=\partial\Sigma\times I$ and $s(\gamma)=\partial\Sigma\times\{1/2\}$.

Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be an admissible sutured Heegaard diagram defining a balanced sutured manifold (M,γ). (For the definition of admissibility see [13, Definition 3.11], it means that every nonzero periodic domain has both positive and negative coefficients.) Then $|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|$, denote this number by d. After an appropriate choice of a generic almost complex and a symplectic structure on $\mathrm{Sym}^d(\Sigma)$, we can apply the Lagrangian Floer homology machinery to the Lagrangian submanifolds $\mathbb{T}_\alpha=\alpha_1\times\cdots\times\alpha_d$ and $\mathbb{T}_\beta=\beta_1\times\cdots\times\beta_d$ of $\mathrm{Sym}^d(\Sigma)$. This way we obtain a chain complex whose homology $SFH(M,\gamma)$ depends only on the homeomorphism type of (M,γ). For the details see my paper [13]. Now we recall [13, Corollary 3.12].

Lemma 3.2 *Let (M,γ) be a balanced sutured manifold such that $H_2(M)=0$. Then every balanced diagram defining (M,γ) is admissible.*

Next we review the definition of a Spin^c structure on a balanced sutured manifold (M,γ), which was introduced in [13]; also see my paper[14]. Note that in a balanced sutured manifold none of the sutures are tori. Fix a Riemannian metric on M.

Notation 3.3 Let v_0 be a nowhere vanishing vector field along ∂M that points into M along $R_-(\gamma)$, points out of M along $R_+(\gamma)$, and on γ it is the gradient of a height function $s(\gamma)\times I\to I$. The space of such vector fields is contractible.

Definition 3.4 Let v and w be nowhere vanishing vector fields on M that agree with v_0 on ∂M. We say that v and w are *homologous* if there is an open ball $B\subset \mathrm{Int}(M)$ such that $v|(M\setminus B)$ is homotopic to $w|(M\setminus B)$ through nowhere vanishing vector fields rel ∂M. We define Spin$^c(M,\gamma)$ to be the set of homology classes of nowhere vanishing vector fields v on M such that $v|\partial M=v_0$.

Remark 3.5 By obstruction theory, Spin$^c(M,\gamma)$ forms an affine space over

$$H^2(M,\partial M)\cong H_1(M).$$

For $\mathfrak{s}_1,\mathfrak{s}_2\in \mathrm{Spin}^c(M,\gamma)$, their difference $\mathfrak{s}_1-\mathfrak{s}_2\in H^2(M,\partial M)$ can be described as follows. Choose a triangulation $\mathcal{T}$ of M, and pick a Riemannian metric on M. For $i=1,2$, represent $\mathfrak{s}_i$ by a section w_i of the unit tangent bundle STM of M such that $w_i|\partial M=v_0$. Let $\mathrm{sk}_j(M)$ denote the j–skeleton of M. Since $\pi_j(S^2)=0$ for $j=0,1$, we can homotope w_1 rel ∂M such that $w_1|\mathrm{sk}_1(M)=w_2|\mathrm{sk}_1(M)$.

Let Δ be a two-simplex of the triangulation $\mathcal{T}$. Fix an arbitrary trivialization of the S^2–bundle $STM|\Delta$. Then $w_1|\Delta$ and $w_2|\Delta$ can be viewed as maps from Δ to S^2. By gluing Δ and $-\Delta$ along their boundary, we get a map $w_1-w_2\colon S^2\to S^2$. Consider the cochain o defined by the formula

$$o(\Delta)=[w_1-w_2]\in\pi_2(S^2)\cong\mathbb{Z}.$$

If $\Delta\subset\partial M$, then $o(\Delta)=0$ as $w_1|\Delta=w_2|\Delta$, so $o\in C^2(M,\partial M)$. In fact, o is a cocycle, and its cohomology class is $\mathfrak{s}_1-\mathfrak{s}_2$. This is the first obstruction to homotoping w_1 to w_2 rel ∂M. If it is zero, w_1 and w_2 are homotopic over $\mathrm{sk}_2(M)$ rel ∂M.

Alternatively, we can describe $\mathfrak{s}_1-\mathfrak{s}_2$ as follows. Fix a trivialization of TM. In this trivialization, we can view w_1 and w_2 as maps from M to S^2. Let $p\in S^2$ be a common regular value of w_1 and w_2. Then $w_1^{-1}(p)-w_2^{-1}(p)$ is a one-cycle since $w_1|\partial M=w_2|\partial M$, and it represents the Poincaré dual of $\mathfrak{s}_1-\mathfrak{s}_2$ in $H_1(M)$.

Proposition 3.6 *Let (M,γ) be a sutured manifold such that M is open and the map $\pi_0(\gamma)\to\pi_0(\partial M)$ is surjective. Then $\mathrm{Spin}^c(M,\gamma)\neq\varnothing$ if and only if every component of M is balanced.*

Proof It is sufficient to prove the proposition for M connected. First suppose that $\mathrm{Spin}^c(M,\gamma)\neq\varnothing$, and let $\mathfrak{s}$ be an arbitrary element. Recall from [13, Definition 4.4] that $c_1(\mathfrak{s})$ is defined as the Euler class of the oriented 2–plane field $v^\perp$, where v is an arbitrary vector field representing $\mathfrak{s}$. Let $i\colon \partial M\hookrightarrow M$ denote the embedding and let $\delta=c_1(v_0^\perp)$. Then $i^*(c_1(\mathfrak{s}))=\delta$, thus

$$\langle\delta,[\partial M]\rangle=\langle i^*(c_1(\mathfrak{s})),[\partial M]\rangle=\langle c_1(\mathfrak{s}),i_*([\partial M])\rangle=0$$

since the cycle ∂M represents zero in $H_2(M;\mathbb{Z})$. On the other hand, $v_0^\perp|R_+(\gamma)=TR_+(\gamma)$ and $v_0^\perp|R_-(\gamma)=-TR_-(\gamma)$, so

$$\langle\delta,[\partial M]\rangle=\chi(R_+(\gamma))-\chi(R_-(\gamma)).$$

Thus (M,γ) is balanced.

Now suppose that (M,γ) is balanced. Let f be a Morse function as in the proof of [13, Proposition 2.13]. Then the vector field $\mathrm{grad}(f)|\partial M=v_0$, the number d of index 1 and 2 critical points of f agree, and f has no index 0 or 3 critical points. Choose d pairwise disjoint balls in M, each containing exactly one index 1 and one index 2 critical point of f. Then we can modify $\mathrm{grad}(f)$ on these balls so that we obtain a nowhere zero vector field on M such that $v|\partial M=v_0$. This shows that $\mathrm{Spin}^c(M,\gamma)\neq\varnothing$. □

Definition 3.7 Let (M,γ) be a balanced sutured manifold and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ a balanced diagram defining it. To each $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ we assign a Spin^c structure $\mathfrak{s}(\mathbf{x})\in\mathrm{Spin}^c(M,\gamma)$ as follows. Choose a Morse function f on M compatible with the given balanced diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. Then $\mathbf{x}$ corresponds to a multitrajectory $\gamma_{\mathbf{x}}$ of $\mathrm{grad}(f)$ connecting the index one and two critical points of f. In a regular neighborhood $N(\gamma_{\mathbf{x}})$ we can modify $\mathrm{grad}(f)$ to obtain a nowhere vanishing vector field v on M such that $v|\partial M=v_0$. We define $\mathfrak{s}(\mathbf{x})$ to be the homology class of this vector field v.

Definition 3.8 We call a sutured manifold (M,γ) *strongly balanced* if for every component F of ∂M the equality $\chi(F\cap R_+(\gamma))=\chi(F\cap R_-(\gamma))$ holds.

The following is [14, Proposition 3.4].

Proposition 3.9 *The vector bundle $v_0^\perp$ over ∂M is trivial if and only if (M,γ) is strongly balanced.*

Notation 3.10 If the sutured manifold (M,γ) is strongly balanced, then let $T(M,\gamma)$ denote the set of trivializations of $v_0^\perp$.

Definition 3.11 Suppose that (M,γ) is a strongly balanced sutured manifold. Let $t \in T(M,\gamma)$ and $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$. Then we define

$$c_1(\mathfrak{s},t) \in H^2(M,\partial M;\mathbb{Z})$$

to be the relative Euler class of the vector bundle $v^\perp$ with respect to the trivialization t. In other words, $c_1(\mathfrak{s},t)$ is the obstruction to extending t from ∂M to a trivialization of $v^\perp$ over M.

Lemma 3.12 *Suppose that (M,γ) is a strongly balanced sutured manifold. Let $d\colon H^1(\partial M) \to H^2(M,\partial M)$ be the coboundary map in the cohomology long exact sequence of the pair $(M,\partial M)$. If $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ and $t_1,t_2 \in T(M,\gamma)$, then*

$$c_1(\mathfrak{s},t_1) - c_1(\mathfrak{s},t_2) = d(t_1 - t_2).$$

Proof Fix a nowhere vanishing vector field v on M representing the Spin^c–structure $\mathfrak{s}$ and also fix a triangulation of M. The circle bundle $Sv^\perp$ over M will be denoted by E. A trivialization $t \in T(M,\gamma)$ can be considered to be a section of $E|\partial M$, and by definition $c_1(\mathfrak{s},t)$ is the obstruction to extending t from ∂M to M. More precisely, choose an arbitrary extension of t to the one-skeleton of M. Then the value of a cocycle $o(E,t)$ representing $c_1(\mathfrak{s},t)$ on a two-simplex Δ is the homotopy class of $t|\partial\Delta$ in $\pi_1(S^1) \cong \mathbb{Z}$ obtained after trivializing $E|\Delta$.

Given the sections t_1 and t_2 of E over ∂M, we can homotope them to coincide on the zero-skeleton of ∂M. Then we can choose a common extension of t_1 and t_2 to $\mathrm{sk}_1(M)\setminus \mathrm{sk}_1(\partial M)$, where sk_i denotes the i–skeleton. The cohomology class $t_1 - t_2 \in H^1(\partial M;\mathbb{Z})$ is represented by the cocycle $o(t_1,t_2)$. The value of $o(t_1,t_2)$ on an edge ϵ of $\mathrm{sk}_1(\partial M)$ is the homotopy class of t_1 in $\pi_1(S^1)$ in the trivialization of $E|\epsilon$ given by t_2.

Let Δ be a two-simplex of $\mathrm{sk}_2(M)$. Then $\langle o(E,t_1) - o(E,t_2), \Delta\rangle$ is the difference of the sections $t_1|\partial\Delta$ and $t_2|\partial\Delta$ in a trivialization of $E|\Delta$. But t_1 and t_2 agree on $\partial\Delta \setminus \partial M$, so

$$\langle o(E,t_1) - o(E,t_2), \Delta\rangle = \langle o(t_1,t_2), \partial\Delta \cap \partial M\rangle.$$

Thus for a relative 2–chain $c \in C_2(M,\partial M)$ we have

$$\langle o(E,t_1) - o(E,t_2), c\rangle = \langle o(t_1,t_2), \partial c \cap \partial M\rangle,$$

proving that $c_1(\mathfrak{s},t_1) - c_1(\mathfrak{s},t_2) = d(t_1 - t_2)$. □

Lemma 3.13 *Let (M,γ) be a strongly balanced sutured manifold, and pick a trivialization $t\in T(M,\gamma)$. If $\mathfrak{s}_1,\mathfrak{s}_2\in \mathrm{Spin}^c(M,\gamma)$, then*

$$c_1(\mathfrak{s}_1,t)-c_1(\mathfrak{s}_2,t)=2(\mathfrak{s}_1-\mathfrak{s}_2).$$

Proof We use the description of $\mathfrak{s}_1-\mathfrak{s}_2$ given in Remark 3.5, and the description of $c_1(\mathfrak{s},t)$ in the proof of Lemma 3.12. So fix a triangulation of M, and representatives w_i of $\mathfrak{s}_i$ for $i=1,2$ that agree on $\mathrm{sk}_1(M)$. Furthermore, fix an extension of t over $\mathrm{sk}_1(M)$. Then the cocycle $o\in C^2(M,\partial M)$ represents $\mathfrak{s}_1-\mathfrak{s}_2$, while $o_i=o(Sw_i^{\perp},t)\in C^2(M,\partial M)$ represents $c_1(\mathfrak{s}_i,t)$.

Let Δ be a two-simplex of the triangulation, and fix a trivialization of $STM|\Delta$. Notice that $w_i^*(TS^2)=w_i^{\perp}$. When we glue Δ and $-\Delta$ along their boundary, we get a map $w_1-w_2\colon S^2\to S^2$ such that

$$(w_1-w_2)^*(TS^2)=(w_1^{\perp}|\Delta)\cup(w_2^{\perp}|-\Delta).$$

Let $\xi=(w_1-w_2)^*(TS^2)$. Recall that $o(\Delta)=[w_1-w_2]\in\pi_2(S^2)\cong\mathbb{Z}$. Then the map $(w_1-w_2)_*\colon H_2(S^2)\to H_2(S^2)$ is multiplication by $o(\Delta)$. Furthermore, the Euler class $e(\xi)=(w_1-w_2)^*(e(TS^2))$, hence

$$\langle e(\xi),[S^2]\rangle=\langle e(TS^2),(w_1-w_2)_*([S^2])\rangle=o(\Delta)\langle e(TS^2),[S^2]\rangle=2o(\Delta).$$

Now $o_1(\Delta)$ is the rotation of $t|\partial\Delta$ with respect to a trivialization u_+ of $\xi|\Delta$, while $o_2(\Delta)$ is the rotation of $t|\partial\Delta$ with respect to a trivialization u_- of $\xi|(-\Delta)$. So $o_1(\Delta)-o_2(\Delta)$ is the rotation of $u_+|\partial\Delta$ with respect to $u_-|\partial\Delta$, which is exactly $\langle e(\xi),[S^2]\rangle=2o(\Delta)$. So $o_1-o_2=2o$, as required. □

Proposition 3.14 *Suppose that (M,γ) is a taut sutured manifold, M is open (ie, it is compact and has no closed components), and the map $\pi_0(A(\gamma))\to\pi_0(\partial M)$ is surjective. Then (M,γ) is balanced.*

Proof Since $R_+=R_+(\gamma)$ and $R_-=R_-(\gamma)$ are both norm minimizing representatives of their homology class in $H_2(M,\gamma)$ and $[R_+]=[R_-]$, we see that $x(R_+)=x(R_-)$. Let V be a component of $R(\gamma)$. Then V is open, so $x(V)=-\chi(V)$, except when V is a disk. Suppose that V is a disk component of say R_+. Then if we push ∂V into R_- we get a curve C in R_- which bounds a disk in M. Since R_- is incompressible, C has to be inessential in R_-, so it bounds a disk in R_-. This argument shows that R_+ and R_- have the same number of disk components. Thus $\chi(R_+)=\chi(R_-)$. □

Remark 3.15 Suppose that (M,γ) is a balanced sutured manifold and $H_2(M)=0$. Then ∂M is connected, and so (M,γ) is strongly balanced.

This, together with Proposition 3.14, shows that if (M,γ) is taut, M is open, $H_2(M)=0$, and ∂M is not a torus which belongs to γ, then (M,γ) is strongly balanced.

Definition 3.16 Let S be a decomposing surface in a balanced sutured manifold (M,γ) such that the positive unit normal field ν_S of S is nowhere parallel to v_0 along ∂S. This holds for generic S. We endow ∂S with the boundary orientation. Let us denote the components of ∂S by $T_1,\dots,T_k$.

Let w_0 denote the projection of v_0 into TS, this is a nowhere zero vector field. Moreover, let f be the positive unit tangent vector field of ∂S. For $1\leq i\leq k$, we define the *index* $I(T_i)$ to be the number of times w_0 rotates with respect to f as we go around T_i. Then define

$$I(S)=\sum_{i=1}^{k} I(T_k).$$

Let $p(\nu_S)$ be the projection of ν_S into $v^{\perp}$. Observe that $p(\nu_S)|\partial S$ is nowhere zero. For $1\leq i\leq k$, we define $r(T_i,t)$ to be the rotation of $p(\nu_S)|\partial T_i$ with respect to the trivialization t as we go around T_i. Moreover, let

$$r(S,t)=\sum_{i=1}^{k} r(T_i,t).$$

We introduce the notation

$$c(S,t)=\chi(S)+I(S)-r(S,t).$$

The following is [14, Lemma 3.9].

Lemma 3.17 *Let (M,γ) be a balanced sutured manifold and let S be a decomposing surface as in Definition 3.16.*

(1) *If T is a component of ∂S such that $T\not\subset\gamma$, then*

$$I(T)=-\frac{|T\cap s(\gamma)|}{2}.$$

(2) *Suppose that $T_1,\dots,T_a$ are components of ∂S such that $\mathcal{T}=T_1\cup\dots\cup T_a\subset\gamma$ is parallel to $s(\gamma)$ and ν_S points out of M along $\mathcal{T}$. Then $I(T_j)=0$ for $1\leq j\leq a$; moreover,*

$$\sum_{j=1}^{a} r(T_j,t)=\chi(R_+(\gamma)).$$

Remark 3.18 Observe that if the components of the decomposing surface S are $S_1, \dots, S_k$, and ν_S is nowhere parallel to v_0, then

$$c(S,t) = c(S_1,t) + \cdots + c(S_k,t).$$

Definition 3.19 Let (M,γ) be a balanced sutured manifold, and let $(S,\partial S) \subset (M,\partial M)$ be a properly embedded oriented surface. An element $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ is called *outer* with respect to S if there is a unit vector field v on M whose homology class is $\mathfrak{s}$ and $v_p \neq -(\nu_S)_p$ for every $p \in S$. Here ν_S is the unit normal vector field of S. Let O_S denote the set of outer Spin^c structures.

The following is [14, Lemma 3.10]

Lemma 3.20 *Suppose that (M,γ) is a strongly balanced sutured manifold. Let $t \in T(M,\gamma)$, choose $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$, and let S be a decomposing surface in (M,γ) as in Definition 3.16. Denote the components of S by $S_1, \dots, S_k$. Then $\mathfrak{s}$ is outer with respect to S if and only if*

$$\langle c_1(\mathfrak{s},t), [S_i] \rangle = c(S_i,t) \text{ for every } 1 \le i \le k. \tag{3-1}$$

In particular, if $\mathfrak{s} \in O_S$, then

$$\langle c_1(\mathfrak{s},t), [S] \rangle = \sum_{i=1}^{k} c(S_i,t) = c(S,t). \tag{3-2}$$

Definition 3.21 Suppose that R is a compact, oriented, and open surface. Let C be an oriented simple closed curve in R. If $[C] = 0$ in $H_1(R;\mathbb{Z})$, then $R \setminus C$ can be written as $R_1 \cup R_2$, where R_1 is the component of $R \setminus C$ that is disjoint from ∂R and satisfies $\partial R_1 = C$. We call R_1 the *interior* and R_2 the *exterior* of C.

We say that the curve C is *boundary-coherent* if either $[C] \neq 0$ in $H_1(R;\mathbb{Z})$, or if $[C] = 0$ in $H_1(R;\mathbb{Z})$ and C is oriented as the boundary of its interior.

Definition 3.22 A decomposing surface S in (M,γ) is called *nice* if S is open, ν_S is nowhere parallel to v_0, and for each component V of $R(\gamma)$ the set of closed components of $S \cap V$ consists of parallel, coherently oriented, and boundary-coherent simple closed curves.

Remark 3.23 Note that every open and groomed decomposing surface becomes nice if we put it into generic position along the boundary.

The following theorem is one of the main technical results of the paper [14] – see Theorem 3.11 there.

Theorem 3.24 *Let (M,γ) be a strongly balanced sutured manifold; furthermore, let $(M,\gamma)\overset{S}{\rightsquigarrow}(M',\gamma')$ be a sutured manifold decomposition along a nice decomposing surface S. Denote the components of S by $S_1,\dots,S_k$ and choose a trivialization $t\in T(M,\gamma)$. Then*

$$SFH(M',\gamma') = \bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma):\\ \langle c_1(\mathfrak{s},t),[S_i]\rangle = c(S_i,t)\ \forall 1\le i\le k}} SFH(M,\gamma,\mathfrak{s}).$$

4 Adjunction inequality

Theorem 4.1 (Adjunction inequality) *Suppose that the sutured manifold (M,γ) is strongly balanced, and fix a trivialization $t\in T(M,\gamma)$. Let $S\subset M$ be a nice decomposing surface. If a Spin^c–structure $\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)$ satisfies*

$$\langle c_1(\mathfrak{s},t),[S]\rangle < c(S,t),$$

then $SFH(M,\gamma,\mathfrak{s})=0$.

Proof Let $S_1,\dots,S_k$ denote the components of S. Then

$$\sum_{i=1}^{k}\langle c_1(\mathfrak{s},t),[S_i]\rangle = \langle c_1(\mathfrak{s},t),[S]\rangle < c(S,t) = \sum_{i=1}^{k} c(S_i,t),$$

which implies that there is an i for which $\langle c_1(\mathfrak{s},t),[S_i]\rangle < c(S_i,t)$.

Now we are going to show that $c(S_i,t)-\langle c_1(\mathfrak{s},t),[S_i]\rangle$ is even. From the proof of [14, Lemma 3.10] it follows that $O_{S_i}\neq\varnothing$, and for $\mathfrak{s}_0\in O_{S_i}$ we have $\langle c_1(\mathfrak{s}_0,t),[S_i]\rangle = c(S_i,t)$. Thus

$$c(S_i,t)-\langle c_1(\mathfrak{s},t),[S_i]\rangle = \langle c_1(\mathfrak{s}_0,t)-c_1(\mathfrak{s},t),[S_i]\rangle = 2\langle \mathfrak{s}_0-\mathfrak{s},[S_i]\rangle,$$

where we used Lemma 3.13 in the second equality.

Add $(c(S_i,t)-\langle c_1(\mathfrak{s},t),[S_i]\rangle)/2$ compressible 1–handles to S_i to obtain a decomposing surface S_i' in (M,γ). Since S_i and S_i' agree in a neighborhood of $\partial S_i=\partial S_i'$, we have $I(S_i')=I(S_i)$ and $r(S_i',t)=r(S_i,t)$. Moreover,

$$\chi(S_i') = \chi(S_i)-c(S_i,t)+\langle c_1(\mathfrak{s},t),[S_i]\rangle.$$

Thus $c(S_i',t)=\langle c_1(\mathfrak{s},t),[S_i]\rangle=\langle c_1(\mathfrak{s},t),[S_i']\rangle$. Decomposing (M,γ) along S_i' we get a sutured manifold (M',γ') which is not taut since $R(\gamma')$ is compressible. Thus $SFH(M',\gamma')=0$ by [13, Proposition 9.18]. Using Theorem 3.24,

$$SFH(M',\gamma')=\bigoplus_{\substack{\mathfrak{s}'\in \mathrm{Spin}^c(M,\gamma):\\ \langle c_1(\mathfrak{s}',t),[S_i']\rangle=c(S_i',t)}} SFH(M,\gamma,\mathfrak{s}'),$$

so from $\langle c_1(\mathfrak{s},t),[S_i']\rangle=c(S_i',t)$ we get that $SFH(M,\gamma,\mathfrak{s})=0$. □

Definition 4.2 Let (M,γ) be a balanced sutured manifold. We say that a closed, oriented, one-dimensional submanifold $L\subset\partial M$ is an *$\mathcal{L}$–link* if it is transverse to $v_0^{\perp}|\gamma$. Given an $\mathcal{L}$–link L, a homology class $\alpha\in H_2(M,\partial M)$ such that $\partial\alpha=[L]$ in $H_1(\partial M)$, and a Spin^c–structure $\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)$, we can define the *rotation number* $\mathrm{rot}_{\alpha,\mathfrak{s}}(L)$ as follows. Choose a properly embedded, oriented, open surface $S\subset M$ such that $\partial S=L$ and $[S]=\alpha$. Furthermore, pick a nowhere zero vector field v on M with $v|\partial M=v_0$ whose homology class is $\mathfrak{s}$. Then $v^{\perp}|S$ is trivial, let t_S be an arbitrary trivialization. We let $\mathrm{rot}_{\alpha,\mathfrak{s}}(L)$ be the sum over all components of L of the rotation of $p(v_S)$ with respect to t_S. Finally, define $x_\alpha(L)$ to be the minimum of $-\chi(S)$ for all surfaces S as above.

It is straightforward to check that $\mathrm{rot}_{\alpha,\mathfrak{s}}(L)$ is independent of the various choices. In some sense, the notion of an $\mathcal{L}$–link is analogous to the notion of a Legendrian link in contact topology, and our rotation number corresponds to the classical rotation number of a Legendrian link. This analogy will be justified by the following Thurston–Bennequin type inequality.

Corollary 4.3 *Let (M,γ) be a strongly balanced sutured manifold, and suppose that $L\subset\partial M$ is an $\mathcal{L}$–link such that for each component V of $R(\gamma)$ the set of closed components of $L\cap V$ consists of parallel, coherently oriented, nonseparating simple closed curves. If for a* Spin^c*–structure $\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)$ we have $SFH(M,\gamma,\mathfrak{s})\neq 0$, and $\alpha\in H_2(M,\partial M)$ satisfies $\partial\alpha=[L]$, then*

$$x_\alpha(L)\geq|\mathrm{rot}_{\alpha,\mathfrak{s}}(L)|-\frac{|L\cap\gamma|}{2}.$$

Proof If S is a properly embedded, oriented, open surface $S\subset M$ such that $\partial S=L$, then we can perturb S slightly fixing ∂S to get a nice decomposing surface. So we can apply Theorem 4.1 to get

$$\langle c_1(S,t),[S]\rangle\geq c(S,t)=\chi(S)+I(S)-r(S,t).$$

Since L is transverse to $v_0^\perp|\gamma$, no component of L lies in γ, hence by Lemma 3.17 we have $I(S) = -|L \cap \gamma|/2$. Observe that we have three trivializations of $v_0^\perp|L$, namely t, t_S, and $p(v_S)$. Furthermore, $\langle c_1(S,t), [S]\rangle$ is the rotation of t with respect to t_S, while $r(S,t)$ is the rotation of $p(v_S)$ with respect to t. Hence

$$\langle c_1(S,t), [S]\rangle + r(S,t) = \mathrm{rot}_{\alpha,\mathfrak{s}}(L),$$

and so

$$-\chi(S) \geq -\mathrm{rot}_{\alpha,\mathfrak{s}}(L) - \frac{|L \cap \gamma|}{2}.$$

Taking the minimum of the left hand side over all such S, we get

$$x_\alpha(L) \geq -\mathrm{rot}_{\alpha,\mathfrak{s}}(L) - \frac{|L \cap \gamma|}{2}.$$

Replacing L by $-L$ and α by $-\alpha$ in the above inequality gives

$$x_{-\alpha}(-L) \geq -\mathrm{rot}_{-\alpha,\mathfrak{s}}(-L) - \frac{|L \cap \gamma|}{2}.$$

Note that we had to flip the sign of α since $\partial(-\alpha) = [-L]$. Since $-\mathrm{rot}_{-\alpha,\mathfrak{s}}(-L) = \mathrm{rot}_{\alpha,\mathfrak{s}}(L)$, while $x_{-\alpha}(-L) = x_\alpha(L)$, we obtain

$$x_\alpha(L) \geq |\mathrm{rot}_{\alpha,\mathfrak{s}}(L)| - \frac{|L \cap \gamma|}{2}. \qquad \square$$

Remark 4.4 Note that if ξ is a contact structure on M such that ∂M is convex with dividing set $s(\gamma)$ and L is a Legendrian link on ∂M, then the Thurston–Bennequin number of L is precisely $-|L \cap s(\gamma)|/2$. In the above version of the Thurston–Bennequin inequality, the role of the contact structure ξ is played by the Spin^c–structure $\mathfrak{s}$, and instead of tightness of ξ we have the condition $SFH(M,\gamma,\mathfrak{s}) \neq 0$. However, if ξ is a tight and $\mathfrak{s}_\xi$ is the Spin^c–structure of ξ, we might have $SFH(M,\gamma,\mathfrak{s}_\xi) = 0$. In the other direction, if $SFH(M,\gamma,\mathfrak{s}) \neq 0$, it is not clear whether there is a tight contact structure ξ with $\mathfrak{s}_\xi = \mathfrak{s}$.

Theorem 4.5 *Suppose that (M,γ) is a strongly balanced sutured manifold and fix a trivialization $t \in T(M,\gamma)$. Let $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ be a surface decomposition along a nice decomposing surface. Then*

$$SFH(M',\gamma') = \bigoplus_{\substack{\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma):\\ \langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t)}} SFH(M,\gamma,\mathfrak{s}).$$

Proof By Theorem 3.24 we know that if the components of S are $S_1,\dots,S_k$, then

$$SFH(M',\gamma') = \bigoplus_{\substack{\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma):\\ \langle c_1(\mathfrak{s},t),[S_i]\rangle = c(S_i,t)\ \forall 1\leq i\leq k}} SFH(M,\gamma,\mathfrak{s}).$$

If $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ satisfies $\langle c_1(\mathfrak{s},t),[S_i]\rangle = c(S_i,t)$ for every $1 \le i \le k$, then it also satisfies $\langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t)$.

Now suppose that

$$\langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t) \tag{4-1}$$

and $SFH(M,\gamma,\mathfrak{s}) \neq 0$. If we apply Theorem 4.1 to S_i, we get that $\langle c_1(\mathfrak{s},t),[S_i]\rangle \ge c(S_i,t)$. Summing these inequalities for $1 \le i \le k$ gives the equality (4-1). Thus $\langle c_1(\mathfrak{s},t),[S_i]\rangle = c(S_i,t)$ for every $1 \le i \le k$.

So we have shown that

$$\bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma):\\ \langle c_1(\mathfrak{s},t),[S_i]\rangle = c(S_i,t)\ \forall 1\le i\le k}} SFH(M,\gamma,\mathfrak{s}) = \bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma):\\ \langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t)}} SFH(M,\gamma,\mathfrak{s}). \qquad \square$$

Corollary 4.6 *Suppose that (M,γ) is a strongly balanced sutured manifold. If $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ is a decomposition along a nice surface such that $[S] = 0$ in $H_2(M,\partial M)$, then either*

$$SFH(M',\gamma') \cong SFH(M,\gamma),$$

or $SFH(M',\gamma') = 0$, in which case (M',γ') is not taut.

Proof Note that $\langle c_1(\mathfrak{s},t),[S]\rangle = 0$ since $[S] = 0$. Then by Theorem 4.5 we see that $SFH(M',\gamma') = SFH(M,\gamma)$ if $c(S,t) = 0$, and $SFH(M',\gamma') = 0$ if $c(S,t) \neq 0$. By [14, Theorem 1.4], the condition $SFH(M',\gamma') = 0$ implies that (M',γ') is not taut. $\square$

Corollary 4.7 *Let $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ be a surface decomposition of a strongly balanced sutured manifold. If there is a surface S' disjoint from S such that $S \cup S'$ is nice, $[S \cup S'] = 0$ in $H_2(M,\partial M)$, and $S \cup S'$ gives a taut decomposition, then $SFH(M',\gamma') \cong SFH(M,\gamma)$.*

Proof Suppose that S' gives a decomposition $(M',\gamma') \rightsquigarrow^{S'} (M'',\gamma'')$. Then

$$SFH(M'',\gamma'') \le SFH(M',\gamma') \le SFH(M,\gamma).$$

If we apply Corollary 4.6 to $S \cup S'$, then we get that $SFH(M'',\gamma'') \cong SFH(M,\gamma)$. Thus $SFH(M',\gamma') \cong SFH(M,\gamma)$. $\square$

In the following lemma, we define $SFH(M,\gamma)$ to be zero for any sutured manifold such that $\chi(R_+(\gamma)) \neq \chi(R_-(\gamma))$.

Lemma 4.8 *Suppose that the balanced sutured manifold (M,γ) is the disjoint union of the sutured manifolds (M_1,γ_1) and (M_2,γ_2). Then*

$$\mathrm{rk}(SFH(M,\gamma)) = \mathrm{rk}(SFH(M_1,\gamma_1))\cdot\mathrm{rk}(SFH(M_2,\gamma_2)).$$

Proof We work with $\mathbb{Z}_2$ coefficients throughout. Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be a balanced diagram defining (M,γ). This is the disjoint union of the sutured diagrams $(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1)$ and $(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2)$ defining (M_1,γ_1) and (M_2,γ_2), respectively.

Suppose that (M_i,γ_i) is not balanced for $i=1$ or $i=2$. Since (M,γ) is balanced, this can only happen when $\chi(R_+(\gamma_i))\neq\chi(R_-(\gamma_i))$. Then $|\boldsymbol{\alpha}_i|\neq|\boldsymbol{\beta}_i|$, hence $\mathbb{T}_\alpha\cap\mathbb{T}_\beta=\varnothing$. So $SFH(M,\gamma)=0$, and by definition $SFH(M_i,\gamma_i)=0$, which proves the lemma in this case.

Now suppose that both (M_1,γ_1) and (M_2,γ_2) are balanced. Then

$$CF(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})\cong CF(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1)\otimes CF(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2).$$

Using the Künneth formula,

$$SFH(M,\gamma)\cong SFH(M_1,\gamma_1)\otimes SFH(M_2,\gamma_2).$$

The result follows. □

Definition 4.9 Let (M,γ) be a balanced sutured manifold. The *support* of the sutured Floer homology of (M,γ) is

$$S(M,\gamma)=\{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma) : SFH(M,\gamma,\mathfrak{s})\neq 0\}.$$

Since $SFH(M,\gamma)$ is a finitely generated Abelian group, $S(M,\gamma)$ is a finite set. Moreover, if (M,γ) is taut, then $S(M,\gamma)\neq\varnothing$ by [14, Theorem 1.4].

Let $i\colon H^2(M,\partial M;\mathbb{Z})\to H^2(M,\partial M;\mathbb{R})$ be the map induced by the embedding $\mathbb{Z}\hookrightarrow\mathbb{R}$. If (M,γ) is strongly balanced and $t\in T(M,\gamma)$, then we define

$$C(M,\gamma,t)=\{i(c_1(\mathfrak{s},t)) : \mathfrak{s}\in S(M,\gamma)\}\subset H^2(M,\partial M;\mathbb{R}).$$

Let $P(M,\gamma,t)$ be the convex hull of $C(M,\gamma,t)$ inside $H^2(M,\gamma;\mathbb{R})$, this is a finite polytope. Thus if (M,γ) is taut and $\alpha\in H_2(M,\partial M)$, then

$$c(\alpha,t)=\min\{\langle c,\alpha\rangle : c\in C(M,\gamma,t)\}$$

is a well-defined number.

Remark 4.10 We call $P(M,\gamma,t)$ the *sutured Floer homology polytope* of the sutured manifold (M,γ). It follows from Lemma 3.12 that for $t_1, t_2 \in T(M,\gamma)$ the relationship

$$P(M,\gamma,t_2) = P(M,\gamma,t_1) + i \circ d(t_2 - t_1)$$

holds. So the sutured Floer homology polytope is well defined up to translations in the vector space $H^2(M,\partial M;\mathbb{R})$.

Corollary 4.11 *Let the sutured manifold (M,γ) be taut and strongly balanced. Choose a trivialization $t \in T(M,\gamma)$ and let $\alpha \in H_2(M,\partial M)$ be a nonzero element. Then*

$$\max\{\, c(S,t) : S \text{ is a nice decomposing surface}, [S] = \alpha \,\} \le c(\alpha,t). \tag{4-2}$$

Moreover, for S in the above set $c(S,t) = c(\alpha,t)$ if and only if S gives a taut decomposition. If $H_2(M) = 0$ also holds, then inequality (4-2) is an equality.

Proof We introduce the notation

$$M = \max\{\, c(S,t) : S \text{ is a nice decomposing surface}, [S] = \alpha \,\}.$$

First suppose that S is a nice decomposing surface such that $[S] = \alpha$. If $c \in C(M,\gamma,t)$, then there is an $\mathfrak{s} \in S(M,\gamma)$ such that $i(c_1(\mathfrak{s},t)) = c$. Since $\mathfrak{s} \in S(M,\gamma)$, we know that $SFH(M,\gamma,\mathfrak{s}) \neq 0$. Thus by Theorem 4.1 we get that

$$c(S,t) \le \langle\, c_1(\mathfrak{s},t), [S] \,\rangle = \langle\, c, \alpha \,\rangle.$$

Taking the minimum over all $c \in C(M,\gamma,t)$, we see that $c(S,t) \le c(\alpha,t)$. Since this holds for every nice S, this implies that $M \le c(\alpha,t)$.

Let $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ be a decomposition along a nice decomposing surface S such that $[S] = \alpha$. We saw above that $c(S,t) \le c(\alpha,t)$. Suppose that $c(S,t) < c(\alpha,t)$. Then for every $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ such that $\langle\, c_1(\mathfrak{s},t), [S] \,\rangle = c(S,t)$, we have $SFH(M,\gamma,\mathfrak{s}) = 0$. Thus by Theorem 4.5 we get $SFH(M',\gamma') = 0$, and so [14, Theorem 1.4] implies that (M',γ') is not taut.

Now assume that $c(S,t) = c(\alpha,t)$. Then by the definition of $c(\alpha,t)$, there exists an $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ such that $\langle\, c_1(\mathfrak{s},t), [S] \,\rangle = c(\alpha,t)$ and $SFH(M,\gamma,\mathfrak{s}) \neq 0$. Since (M,γ) is taut, it is irreducible, thus (M',γ') is also irreducible. Using Theorem 4.5 again, we see that $SFH(M',\gamma') \neq 0$, and so (M',γ') is taut by [13, Proposition 9.18].

Suppose that $H_2(M) = 0$. By [7, Lemma 0.7], for every $\alpha \neq 0$ in $H_2(M,\partial M)$ there is a nice surface decomposition $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ such that (M',γ') is taut and

$[S]=\alpha$. We can assume that S is open since $H_2(M)=0$. Using Theorem 4.5,

$$SFH(M',\gamma') = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma):\langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t)} SFH(M,\gamma,\mathfrak{s}).$$

Since (M',γ') is taut, by [14, Theorem 1.4] we see that $SFH(M',\gamma')\neq 0$. Thus there exists an $\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)$ such that $\langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t)$ and $SFH(M,\gamma,\mathfrak{s})$ is nonzero, ie, $\mathfrak{s}\in S(M,\gamma)$. This implies that $c(\alpha,t)\leq M$. So indeed we have $c(\alpha,t)=M$. □

Definition 4.12 For $\alpha\in H_2(M,\partial M)$, let

$$H_\alpha = \{x\in H^2(M,\partial M;\mathbb{R}) : \langle x,\alpha\rangle = c(\alpha,t)\},$$

moreover, $$C_\alpha(M,\gamma,t) = H_\alpha \cap C(M,\gamma,t)$$

and similarly $$P_\alpha(M,\gamma,t) = H_\alpha \cap P(M,\gamma,t).$$

Finally, we introduce the notation

$$SFH_\alpha(M,\gamma) = \bigoplus \{SFH(M,\gamma,\mathfrak{s}) : i(c_1(\mathfrak{s},t))\in C_\alpha(M,\gamma,t)\}.$$

Note that this is independent of t by Lemma 3.12.

Proposition 4.13 *Let the sutured manifold (M,γ) be taut and strongly balanced. Fix an element $\alpha\in H_2(M,\partial M)$. Then $P_\alpha(M,\gamma,t)$ is the convex hull of $C_\alpha(M,\gamma,t)$ and it is a face of the polytope $P(M,\gamma,t)$. If S is a nice decomposing surface that gives a taut decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ and $[S]=\alpha$, then*

$$SFH(M',\gamma') \cong SFH_\alpha(M,\gamma). \tag{4-3}$$

Proof First we prove Equation (4-3). If S gives a taut decomposition, then $c(S,t)=c(\alpha,t)$ by Corollary 4.11. Thus Equation (4-3) follows from Theorem 4.5.

If $\alpha=0$, then $c(\alpha,t)=0$. So $C_\alpha(M,\gamma,t)=C(M,\gamma,t)$ and $P_\alpha(M,\gamma,t)=P(M,\gamma,t)$, hence Proposition 4.13 is true for $\alpha=0$. Now suppose that $\alpha\neq 0$. Then H_α is a hyperplane in $H^2(M,\partial M;\mathbb{R})$. Using the definition of $c(\alpha,t)$, we see that $H_\alpha\cap C(M,\gamma,t)\neq\varnothing$ and $\langle c,\alpha\rangle\geq c(\alpha,t)$ for every $c\in C(M,\gamma,t)$. Thus $P_\alpha(M,\gamma,t)$ is the convex hull of $C_\alpha(M,\gamma,t)$ and is a face of $P(M,\gamma,t)$. □

Remark 4.14 It follows from Proposition 4.13 that if $i(c_1(\mathfrak{s},t))$ lies in the interior of the polytope $P(M,\gamma,t)$, then $SFH(M,\gamma,\mathfrak{s})$ dies under any nice surface decomposition that strictly decreases $SFH(M,\gamma)$. However, we might still be able to obtain information about the interior of the polytope using decomposing surfaces that are null-homologous in $H_2(M,\partial M)$.

Corollary 4.15 *Let the sutured manifold (M,γ) be taut and balanced, and suppose that $H_2(M)=0$. Then the following hold.*

(1) *For every $\alpha\in H_2(M,\partial M)$, there exists a groomed surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that (M',γ') is taut, $[S]=\alpha$, and*

$$SFH(M',\gamma')\cong SFH_\alpha(M,\gamma).$$

If, moreover, α is well groomed, then S can be chosen to be well groomed.

(2) *For every face F of $P(M,\gamma,t)$, there exists an $\alpha\in H_2(M,\partial M)$ such that $F=P_\alpha(M,\gamma,t)$.*

Proof First we prove (1). In the case $\alpha=0$ we can choose $S=\varnothing$, so suppose that $\alpha\neq 0$. Then by [7, Lemma 0.7] there exists a groomed surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that (M',γ') is taut and $[S]=\alpha$. If α is well groomed, then the existence of a well groomed S follows from Lemma 2.9. Since $H_2(M)=0$, we can assume that S has no closed components, so S is nice. Thus the first part of Corollary 4.15 follows from Equation (4-3).

Now we prove (2). Let $P=P(M,\gamma,t)$. If $F=P$, then $\alpha=0$ works. So suppose that F is a proper face of P. Recall that P is spanned by points lying in the lattice $L=i(H^2(M,\partial M;\mathbb{Z}))$. Thus there exists an affine hyperplane of the form $H=H_0+v_0$, where $v_0\in L$ and H_0 is a linear hyperplane spanned by elements of L, and such that $F=H\cap P$.

Consider the following commutative diagram.

$$\begin{array}{ccc} L & \longrightarrow & H^2(M,\partial M;\mathbb{R}) \\ \downarrow{\scriptstyle u|_L} & & \downarrow{\scriptstyle u} \\ \mathrm{Hom}(H_2(M,\partial M),\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_2(M,\partial M),\mathbb{R}) \end{array}$$

Here the horizontal arrows are embeddings. Moreover, u is given by $u(c)(\lambda)=\langle c,\lambda\rangle$ for $c\in H^2(M,\partial M;\mathbb{R})$ and $\lambda\in H_2(M,\partial M;\mathbb{R})$. Both u and $u|_L$ are isomorphisms because of the universal coefficient theorem. Let $b_1,\dots,b_n$ be a basis of the free Abelian group L. Then $b_1,\dots,b_n$ is also a basis of $H^2(M,\partial M;\mathbb{R})$. Consider the scalar product $\cdot$ on $H^2(M,\partial M;\mathbb{R})$ such that $b_1,\dots,b_n$ is an orthonormal basis. Since $u|_L$ is an isomorphism, and because $H_2(M,\partial M)$ is torsion free, there are unique elements $\beta_1,\dots,\beta_n\in H_2(M,\partial M)$ that satisfy the condition $\langle b_i,\beta_j\rangle=\delta_{ij}$, where δ_{ij} is the Kronecker delta.

Since H_0 is spanned by elements of L, there is a vector $a\in L$ which is perpendicular to H_0 and such that v_0+a and P lie on the same side of H. In other words,

$a \cdot H_0 = a \cdot (H - v_0) = 0$ and $a \cdot (P - v_0) \geq 0$. Thus $a \cdot H = a \cdot v_0$ and $a \cdot P \geq a \cdot v_0$. Let $A_1, \dots, A_n$ be the coordinates of a in the basis $b_1, \dots, b_n$, these are all integers. Define $\alpha = A_1\beta_1 + \cdots + A_n\beta_n \in H_2(M, \partial M)$. Then for any $c \in H^2(M, \partial M; \mathbb{R})$, we have $\langle c, \alpha \rangle = a \cdot c$. Thus $\langle H, \alpha \rangle = \langle v_0, \alpha \rangle$ and $\langle P, \alpha \rangle \geq \langle v_0, \alpha \rangle$. This implies that $\langle v_0, \alpha \rangle = c(\alpha, t)$, and so $H = H_\alpha$. Thus $F = H \cap P = P_\alpha(M, \gamma, t)$. □

Proposition 4.16 *Let the sutured manifold (M, γ) be taut and balanced, and suppose that $H_2(M) = 0$. If the polytope $P(M, \gamma, t)$ has k vertices, then*

$$\mathrm{rk}(SFH(M, \gamma)) \geq k,$$

and there exists a groomed surface decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ such that (M', γ') is taut and

$$\mathrm{rk}(SFH(M', \gamma')) \leq \mathrm{rk}(SFH(M, \gamma))/k.$$

Proof Let $v_1, \dots, v_k$ be the vertices of $P(M, \gamma, t)$. By Corollary 4.15, for every $1 \leq j \leq k$ there is a groomed surface decomposition $(M, \gamma) \rightsquigarrow^{S_j} (M'_j, \gamma'_j)$ such that (M'_j, γ'_j) is taut; furthermore, for $\alpha_j = [S_j]$ we have $SFH(M'_j, \gamma'_j) = SFH_{\alpha_j}(M, \gamma)$ and $P_{\alpha_j}(M, \gamma, t) = \{v_j\}$. By [14, Theorem 1.4], we see that $\mathrm{rk}(SFH(M'_j, \gamma'_j)) \geq 1$ for $1 \leq j \leq k$. Since the faces $P_{\alpha_j}(M, \gamma, t) = C_{\alpha_j}(M, \gamma, t)$ are pairwise disjoint for $j = 1, \dots, k$, we get that

$$SFH(M, \gamma) \geq \bigoplus_{j=1}^{k} SFH_{\alpha_j}(M, \gamma).$$

Thus $\mathrm{rk}(SFH(M, \gamma)) \geq k$, and for some $1 \leq l \leq k$ the inequality

$$\mathrm{rk}(SFH_{\alpha_l}(M, \gamma)) \leq \mathrm{rk}(SFH(M, \gamma))/k$$

holds. So we can choose $S = S_l$. □

5 How the polytope changes under surface decompositions

In what follows $b_i(X)$ denotes the i–th Betti number of a topological space X.

Lemma 5.1 *Suppose that (M, γ) is a sutured manifold and let $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ be a surface decomposition along the nice decomposing surface S. If $H_2(M) = 0$ or $S = D^2$, then*

$$b_1(M') = b_1(M) + b_1(S) - b_0(S) + b_0(M') - b_0(M).$$

Furthermore, $H_2(M) = 0$ implies that $H_2(M') = 0$.

Proof Let $N(S)$ be a regular neighborhood of S. Since $M' \cap N(S)$ is homotopy equivalent to $S \sqcup S$, the Mayer–Vietoris sequence of the pair $(M', N(S))$ looks like:

$$\cdots \to H_j(S) \oplus H_j(S) \to H_j(M') \oplus H_j(S) \to H_j(M) \to \cdots \tag{5-1}$$

Note that $H_2(S) = 0$ because S is open. Thus if we write down the sequence (5-1) for $j = 2$, then we see that $H_2(M) = 0$ implies $H_2(M') = 0$.

Suppose that $H_2(M) = 0$. If we look at the portion of sequence (5-1) starting at $H_2(M)$ and we take the alternating sum of the ranks of the groups that appear, then we get that

$$2b_1(S) - (b_1(M') + b_1(S)) + b_1(M) - 2b_0(S) + (b_0(M') + b_0(S)) - b_0(M) = 0.$$

The result follows. If $S = D^2$, then $b_2(M) = b_2(M')$, and we obtain the same conclusion. □

Let us review the relative Maslov grading on Sutured Floer homology; see Definitions 8.1 and 8.2 of [13].

Definition 5.2 For $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$, the *divisibility* of $\mathfrak{s}$ is

$$\mathfrak{d}(\mathfrak{s}) = \gcd_{\xi \in H_2(M;\mathbb{Z})} \langle c_1(\mathfrak{s}), \xi \rangle.$$

Suppose that $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$, and let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be an admissible balanced diagram for (M, γ). Then we define a relative $\mathbb{Z}_{\mathfrak{d}(\mathfrak{s})}$ grading on $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s})$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y}) = \mathfrak{s}$ we have

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) \mod \mathfrak{d}(\mathfrak{s}),$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is an arbitrary homotopy class and μ is the Maslov index of ϕ.

Definition 5.3 Let $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ be a surface decomposition. If $e\colon M' \hookrightarrow M$ denotes the embedding, then we define

$$F_S = \mathrm{PD} \circ e_* \circ (\mathrm{PD}')^{-1}\colon H^2(M', \partial M'; \mathbb{R}) \to H^2(M, \partial M; \mathbb{R}),$$

hence the following diagram is commutative.

$$\begin{array}{ccc} H_1(M'; \mathbb{R}) & \xrightarrow{e_*} & H_1(M; \mathbb{R}) \\ \downarrow \mathrm{PD}' & & \downarrow \mathrm{PD} \\ H^2(M', \partial M'; \mathbb{R}) & \xrightarrow{F_S} & H^2(M, \partial M; \mathbb{R}) \end{array}$$

Here PD and PD′ are Poincareé duality maps and e_* is the map induced by e. We will use the same symbol F_S to denote the map $H^2(M', \partial M') \to H^2(M, \partial M)$ defined over $\mathbb{Z}$ in a completely analogous way.

Next we study how the Spin^c–grading changes under a sutured manifold decomposition $(M,\gamma) \rightsquigarrow^S (M',\gamma')$. Intuitively, a Spin^c–structure $\mathfrak{s}' \in \text{Spin}^c(M',\gamma')$ can be glued along S to get an outer Spin^c–structure $f_S(\mathfrak{s}') \in O_S$ on (M,γ). Recall from Remark 3.5 that $\text{Spin}^c(M,\gamma)$ is an affine space over $H^2(M,\partial M) \cong H_1(M)$. Then the gluing map f_S is an affine version of the map $e_*\colon H_1(M') \to H_1(M)$ induced by the embedding $e\colon M' \hookrightarrow M$. Furthermore, for every outer Spin^c–structure $\mathfrak{s} \in O_S$ the group $SFH(M,\gamma,\mathfrak{s})$ is isomorphic to the direct sum of $SFH(M',\gamma',\mathfrak{s}')$ for all $\mathfrak{s}'$ that glue to give $\mathfrak{s}$, and the relative Maslov grading is preserved by this gluing isomorphism.

Proposition 5.4 *Let $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ be a nice surface decomposition of a strongly balanced sutured manifold (M,γ), and fix $t \in T(M,\gamma)$ and $t' \in T(M',\gamma')$. Then there are an affine map*

$$f_S\colon \text{Spin}^c(M',\gamma') \to \text{Spin}^c(M,\gamma)$$

and an element $c(t,t') \in H^2(M,\partial M;\mathbb{R})$ satisfying the following three conditions.

(1) *f_S maps onto O_S, and for any $\mathfrak{s} \in O_S$ we have*

$$SFH(M,\gamma,\mathfrak{s}) \cong \bigoplus_{\mathfrak{s}' \in \text{Spin}^c(M',\gamma')\colon f_S(\mathfrak{s}')=\mathfrak{s}} SFH(M',\gamma',\mathfrak{s}').$$

Furthermore, there is an isomorphism

$$\hat{p}\colon SFH(M',\gamma') \to \bigoplus_{\mathfrak{s} \in O_S} SFH(M,\gamma,\mathfrak{s})$$

such that for every $\mathfrak{s}' \in \text{Spin}^c(M',\gamma')$ and for every $x', y' \in SFH(M',\gamma',\mathfrak{s}')$ we have $\hat{p}(SFH(M',\gamma',\mathfrak{s}')) \subset SFH(M,\gamma,f_S(\mathfrak{s}'))$ and

$$\text{gr}(x',y') = \text{gr}(\hat{p}(x'),\hat{p}(y')).$$

(2) *If $\mathfrak{s}'_1, \mathfrak{s}'_2 \in \text{Spin}^c(M',\gamma')$, then*

$$F_S(\mathfrak{s}'_1 - \mathfrak{s}'_2) = f_S(\mathfrak{s}'_1) - f_S(\mathfrak{s}'_2) \in H^2(M,\partial M).$$

(3) *For every $\mathfrak{s}' \in \text{Spin}^c(M',\gamma')$ we have*

$$F_S(i(c_1(\mathfrak{s}',t'))) = i(c_1(f_S(\mathfrak{s}'),t)) + c(t,t').$$

Proof We improve the proof of [14, Theorem 1.3] by also taking into consideration the Spin^c and the relative Maslov gradings on $SFH(M',\gamma')$. First we need to recall Definitions 4.3 and 5.1 of [14].

Definition 5.5 A balanced diagram *adapted* to the decomposing surface S in (M,γ) is a quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},P)$ that satisfies the following conditions. $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a balanced diagram of (M,γ); furthermore, $P\subset\Sigma$ is a quasipolygon (a closed subsurface of Σ whose boundary is a union of polygons) such that $P\cap\partial\Sigma$ is exactly the set of vertices of P. We are also given a decomposition $\partial P = A\cup B$, where both A and B are unions of pairwise disjoint edges of P. This decomposition has to satisfy the property that $\alpha\cap B=\varnothing$ and $\beta\cap A=\varnothing$ for every $\alpha\in\boldsymbol{\alpha}$ and $\beta\in\boldsymbol{\beta}$. Finally, S is given up to equivalence by smoothing the corners of the surface $(P\times\{1/2\})\cup(A\times[1/2,1])\cup(B\times[0,1/2])\subset(M,\gamma)$. The orientation of S is given by the orientation of $P\subset\Sigma$. We call a tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},P)$ satisfying the above conditions a *surface diagram*.

Definition 5.6 Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},P)$ be a surface diagram. Then we can uniquely associate to it a tuple $D(P)=(\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}',P_A,P_B,p)$, where $(\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')$ is a balanced diagram, $p\colon\Sigma'\to\Sigma$ is a smooth map, and $P_A,P_B\subset\Sigma'$ are two closed subsurfaces (see Figure 1).

To define Σ', take two disjoint copies of P that we call P_A and P_B, together with diffeomorphisms $p_A\colon P_A\to P$ and $p_B\colon P_B\to P$. Cut Σ along ∂P and remove P. Then glue A to P_A using p_A^{-1} and B to P_B using p_B^{-1} to obtain Σ'. The map $p\colon\Sigma'\to\Sigma$ agrees with p_A on P_A and with p_B on P_B, and it maps $\Sigma'\setminus(P_A\cup P_B)$ to $\Sigma\setminus P$ using the obvious diffeomorphism. Finally, let $\boldsymbol{\alpha}'=\{\,p^{-1}(\alpha)\setminus P_B:\alpha\in\boldsymbol{\alpha}\,\}$ and $\boldsymbol{\beta}'=\{\,p^{-1}(\beta)\setminus P_A:\beta\in\boldsymbol{\beta}\,\}$.

Since S is nice, by [14, Lemma 4.5] it is isotopic to a decomposing surface S' such that each component of $\partial S'$ intersects both $R_+(\gamma)$ and $R_-(\gamma)$, decomposing (M,γ) along S' also gives (M',γ'), and $O_S=O_{S'}$. Thus we can suppose that each component of ∂S intersects both $R_+(\gamma)$ and $R_-(\gamma)$. Then by [14, Proposition 4.8] and [14, Theorem 6.4] there is a nice and admissible surface diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},P)$ adapted to S. By [14, Proposition 5.2], if $D(P)=(\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}',P_A,P_B,p)$, then $(\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')$ is an admissible balanced diagram defining (M',γ'). Moreover, [14, Proposition 7.6] says that p gives an isomorphism $CF(\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')\cong(O_P,\partial|O_P)$. Here O_P is the subcomplex of $CF(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ generated by

$$\{\,\mathbf{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}:\mathbf{x}\cap P=\varnothing\,\}.$$

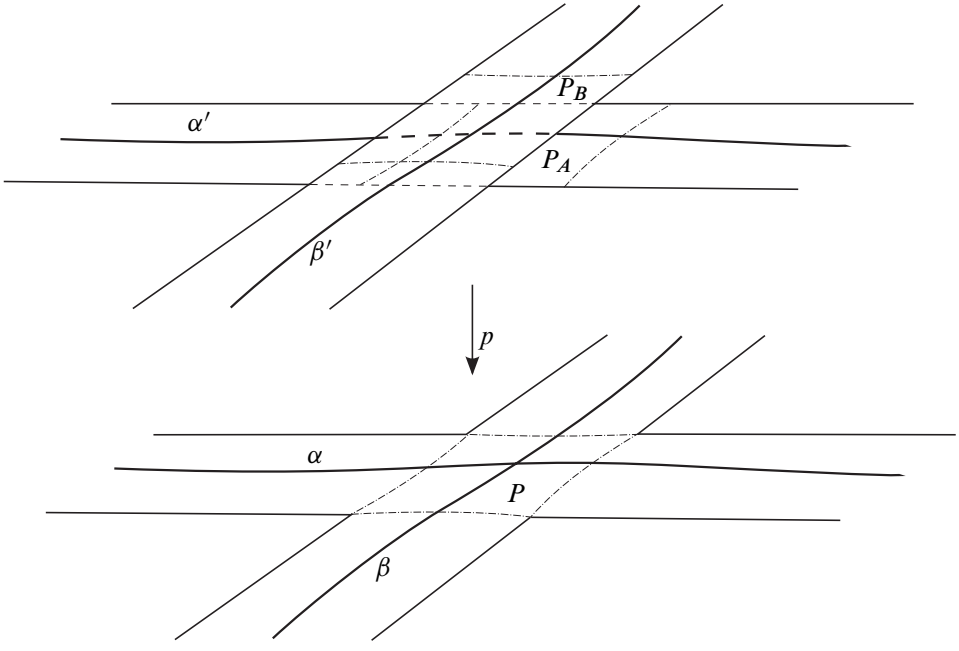

Figure 1: The quasipolygon P in the lower sutured diagram defines a decomposing surface S. The upper sutured diagram corresponds to the manifold obtained by decomposing along S.

However, [14, Lemma 5.5] implies that $\mathbf{x} \in O_P$ if and only if $s(\mathbf{x}) \in O_S$. Thus p induces an isomorphism

$$\widehat{p}\colon SFH(M',\gamma') \to \bigoplus_{\mathfrak{s}\in O_S} SFH(M,\gamma,\mathfrak{s}).$$

We can now define f_S. If $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} \neq \varnothing$, then fix an element $\mathbf{x}'_0 \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ and let $\mathbf{x}_0 = p(\mathbf{x}'_0)$. Put $\mathfrak{s}'_0 = \mathfrak{s}(\mathbf{x}'_0)$ and $\mathfrak{s}_0 = \mathfrak{s}(\mathbf{x}_0)$. Otherwise, let $\mathfrak{s}'_0 \in \mathrm{Spin}^c(M',\gamma')$ and $\mathfrak{s}_0 \in O_S$ be arbitrary elements. Then for any $\mathfrak{s}' \in \mathrm{Spin}^c(M',\gamma')$ define

$$f_S(\mathfrak{s}') = \mathfrak{s}_0 + F_S(\mathfrak{s}' - \mathfrak{s}'_0).$$

(2) is immediate from the definition of f_S.

Next we show that f_S maps onto O_S. It is sufficient to prove the following claim.

Claim 5.7 *Let $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$. Then $\mathfrak{s} \in O_S$ if and only if $\mathfrak{s} - \mathfrak{s}_0 \in \mathrm{im}(F_S)$.*

Proof Since $\mathbf{x}_0 \in O_P$, by [14, Lemma 5.5] we see that $\mathfrak{s}_0 \in O_S$. Note that $\mathfrak{s} - \mathfrak{s}_0 \in \mathrm{im}(F_S)$ if and only if $\mathrm{PD}^{-1}(\mathfrak{s} - \mathfrak{s}_0)$ can be represented by a 1–cycle disjoint from S.

Using cut-and-paste techniques, this is equivalent to the statement that $\langle \mathfrak{s}-\mathfrak{s}_0, [S_*]\rangle = 0$ for every component S_* of S. Finally, this happens if and only if $\mathfrak{s}$ and $\mathfrak{s}_0$ can be represented by nowhere zero vector fields that are homotopic (through nowhere zero vector fields) over each component of S rel ∂S. Since $\mathfrak{s}_0 \in O_S$, this is equivalent to saying that $\mathfrak{s} \in O_S$. □

To prove (1), recall that we have an isomorphism $CF(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}') \cong (O_P, \partial|O_P)$ induced by the projection $p\colon \Sigma' \to \Sigma$. Moreover, $p(\alpha') = \alpha$ and $p(\beta') = \beta$ for every $\alpha \in \boldsymbol{\alpha}$ and $\beta \in \boldsymbol{\beta}$. Furthermore, $H_1(M) \cong H_1(\Sigma)/\langle \boldsymbol{\alpha} \cup \boldsymbol{\beta}\rangle$ and $H_1(M') = H_1(\Sigma')/\langle \boldsymbol{\alpha}' \cup \boldsymbol{\beta}'\rangle$. There is a commutative diagram

$$\begin{array}{ccc} H_1(\Sigma') & \xrightarrow{p_*} & H_1(\Sigma) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ H_1(M') & \xrightarrow{e_*} & H_1(M), \end{array}$$

where π and π' are factor homomorphisms. Now we recall [13, Definition 4.6].

Definition 5.8 For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we define $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(M)$ as follows. Choose paths $a\colon I \to \mathbb{T}_\alpha$ and $b\colon I \to \mathbb{T}_\beta$ with $\partial a = \partial b = \mathbf{x} - \mathbf{y}$. Then $a - b$ can be viewed as a one-cycle in Σ whose homology class in M is $\epsilon(\mathbf{x}, \mathbf{y})$. This is independent of the choices of a and b.

In [13, Lemma 4.7] we showed that $\mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y}) = \mathrm{PD}[\epsilon(\mathbf{x}, \mathbf{y})]$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Now pick elements $\mathbf{x}', \mathbf{y}' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$, and let $\mathbf{x} = p(\mathbf{x}')$ and $\mathbf{y} = p(\mathbf{y}')$. Choose paths $a'\colon I \to \mathbb{T}_{\alpha'}$ and $b'\colon I \to \mathbb{T}_{\beta'}$ such that $\partial a' = \partial b' = \mathbf{x}' - \mathbf{y}'$. Let $a = p(a')$ and $b = p(b')$. Since $p(a' - b') = a - b$, using the above commutative diagram, we get $e_*(\epsilon(\mathbf{x}', \mathbf{y}')) = \epsilon(\mathbf{x}, \mathbf{y})$. Hence

$$\mathrm{PD}[e_*(\epsilon(\mathbf{x}', \mathbf{y}'))] = \mathrm{PD}[\epsilon(\mathbf{x}, \mathbf{y})] = \mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y}).$$

Another application of [13, Lemma 4.7] gives

$$\mathrm{PD}[e_*(\epsilon(\mathbf{x}', \mathbf{y}'))] = \mathrm{PD} \circ e_* \circ (\mathrm{PD}')^{-1}(\mathfrak{s}(\mathbf{x}') - \mathfrak{s}(\mathbf{y}')) = F_S(\mathfrak{s}(\mathbf{x}') - \mathfrak{s}(\mathbf{y}')).$$

So we see that $F_S(\mathfrak{s}(\mathbf{x}') - \mathfrak{s}(\mathbf{y}')) = \mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y})$, and by definition $f_S(\mathfrak{s}(\mathbf{x}'_0)) = \mathfrak{s}(\mathbf{x}_0)$. Thus $f_S(\mathfrak{s}(\mathbf{x}')) = \mathfrak{s}(\mathbf{x})$. Hence

$$\widehat{p}(SFH(M', \gamma', \mathfrak{s}')) \subset SFH(M, \gamma, f_S(\mathfrak{s}'))$$

for every $\mathfrak{s}' \in \mathrm{Spin}^c(M', \gamma')$.

If $\mathbf{x}', \mathbf{y}' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ satisfy $\mathfrak{s}(\mathbf{x}') = \mathfrak{s}(\mathbf{y}')$, then choose a domain $\mathcal{D}'$ connecting $\mathbf{x}'$ and $\mathbf{y}'$. Then $\mathcal{D} = p(\mathcal{D}')$ is a domain connecting $\mathbf{x} = p(\mathbf{x}')$ and $\mathbf{y} = p(\mathbf{y}')$. Using

Lipshitz's Maslov index formula (cf [14, Proposition 7.3]) we see that $\mu(\mathcal{D}) = \mu(\mathcal{D}')$ since the local diffeomorphism p preserves both the Euler and the point measures. Thus $\mathrm{gr}(x', y') = \mathrm{gr}(\hat{p}(x'), \hat{p}(y'))$ if $x', y' \in SFH(M', \gamma', \mathfrak{s}')$. This concludes the proof of (1).

Finally, we prove (3). Given $t \in T(M, \gamma)$ and $t' \in T(M', \gamma')$, let

$$c(t, t') = F_S(i(c_1(\mathfrak{s}'_0, t'))) - i(c_1(\mathfrak{s}_0, t)).$$

If $\mathfrak{s}' \in \mathrm{Spin}^c(M', \gamma')$ is arbitrary and $\mathfrak{s} = f_S(\mathfrak{s}')$, then using (2) we have

$$\begin{aligned} F_S(i(c_1(\mathfrak{s}', t'))) &= F_S(i(c_1(\mathfrak{s}'_0, t'))) + F_S(i(2(\mathfrak{s}' - \mathfrak{s}'_0))) \\ &= i(c_1(\mathfrak{s}_0, t)) + c(t, t') + 2i(\mathfrak{s} - \mathfrak{s}_0) = i(c_1(\mathfrak{s}, t)) + c(t, t'). \quad \square \end{aligned}$$

Remark 5.9 Actually f_S depends only on S and is independent of the choice of a surface diagram representing S. It can be defined geometrically as follows. Let v' be a vector field representing $\mathfrak{s}' \in \mathrm{Spin}^c(M', \gamma')$. After homotoping v' over S'_- and S'_+ (see Definition 2.5) one can glue $v'|S'_+$ and $v'|S'_-$ to obtain a nowhere zero vector field v on M, which is unique up to homotopy. Then v represents $f_S(\mathfrak{s}')$. To verify this claim, one has to trace through the identifications in the proof of [14, Proposition 5.2] to see that $\mathfrak{s}(\mathbf{x}')$ and $\mathfrak{s}(p(\mathbf{x}'))$ are related by the above "gluing" operation. This is straightforward but tedious, and we will make no use of it in the rest of the paper.

On the other hand, the isomorphism $\hat{p}$ might possibly depend on the choice of a surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ representing S. To show independence, one either needs a different proof of the decomposition formula, or one must show invariance of $\hat{p}$ under a sequence of moves relating two different surface diagrams.

Remark 5.10 If $H^2(M) = 0$, then for every $t' \in T(M', \gamma')$ there is a $t \in T(M, \gamma)$ such that $c(t, t') = 0$ in Proposition 5.4. Indeed, fix an $\mathfrak{s}'_0 \in \mathrm{Spin}^c(M', \gamma')$ and let $\mathfrak{s}_0 = f_S(\mathfrak{s}'_0)$. The map $H^1(\partial M) \to H^2(M, \partial M)$ is surjective since $H^2(M) = 0$. Thus by Lemma 3.12 in the proof of part (3) of Proposition 5.4, we can choose a $t \in T(M, \gamma)$ which satisfies $F_S(i(c_1(\mathfrak{s}'_0, t'))) = i(c_1(\mathfrak{s}_0, t))$.

Note that if we only suppose that $H_2(M) = 0$, then such a t might not exist. For example, if K is the knot in $S^1 \times S^2$ which goes around twice monotonically in the S^1 direction and $M = (S^1 \times S^2) \setminus N(K)$, then M fibres over S^1 with annulus fibers. Actually M deformation retracts onto a Klein bottle. Thus $H_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}$ and $H_2(M) = 0$. So $H^2(M) = \mathbb{Z}_2$. The map $H^1(\partial M) \to H^2(M, \partial M)$ is Poincaré dual to $H_1(\partial M) \to H_1(M)$. This map is $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$. It takes a pair (a, b) to $(a \mod 2, 2b)$. Here a corresponds to the meridional and b to the longitudinal component. Thus the map d in Lemma 3.12 is not surjective onto the $\mathbb{Z}$ component of $H^2(M, \partial M)$.

Theorem 5.11 *Let $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ be a taut surface decomposition, where (M,γ) is strongly balanced, and fix $t\in T(M,\gamma)$ and $t'\in T(M',\gamma')$. Suppose that S is nice and let $\alpha=[S]$. Then the following hold.*

(i) *The map*

$$F_S-c(t,t')\colon H^2(M',\partial M';\mathbb{R})\to H^2(M,\partial M;\mathbb{R})$$

projects the polytope $P(M',\gamma',t')$ onto the face $P_\alpha(M,\gamma,t)$ of $P(M,\gamma,t)$.

(ii) *If, moreover, S is connected and nonseparating, then the image of F_S is the hyperplane $H_\alpha+c(t,t')$.*

(iii) *If, in addition to the assumptions of (2), we have $H_2(M)=0$ or $S=D^2$, then* $\dim\ker(F_S)=b_1(S)$.

Proof Let $\mathfrak{S}=S(M,\gamma)$ and $\mathfrak{S}'=S(M',\gamma')$. First we show that $f_S(\mathfrak{S}')=\mathfrak{S}\cap O_S$. Indeed, if $\mathfrak{s}'\in\mathfrak{S}'$, then by definition $SFH(M',\gamma',\mathfrak{s}')\neq 0$. So from part (1) of Proposition 5.4 it follows that $f_S(\mathfrak{s}')\in O_S$ and $SFH(M,\gamma,f_S(\mathfrak{s}'))\neq 0$, ie, $f_S(\mathfrak{s}')\in\mathfrak{S}\cap O_S$. Similarly, if $\mathfrak{s}\in\mathfrak{S}\cap O_S$, then by part (1) of Proposition 5.4 there exists an $\mathfrak{s}'\in S(M',\gamma')$ such that $f_S(\mathfrak{s}')=\mathfrak{s}$.

Let $C=C(M,\gamma,t)$ and $C'=C(M',\gamma',t')$. Using part (3) of Proposition 5.4,

$$F_S(C')-c(t,t')=F_S(i(c_1(\mathfrak{S}',t')))-c(t,t')=i(c_1(f_S(\mathfrak{S}'),t))=i(c_1(\mathfrak{S}\cap O_S,t)).$$

By Lemma 3.20, for $\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)$ we have $\mathfrak{s}\in O_S$ if and only if

$$\langle c_1(\mathfrak{s},t),[S_i]\rangle=c(S_i,t)$$

for every component S_i of S. However, we saw in the proof of Theorem 4.5 that $\mathfrak{s}\in\mathfrak{S}\cap O_S$ if and only if $\mathfrak{s}\in\mathfrak{S}$ and $\langle c_1(\mathfrak{s},t),[S]\rangle=c(S,t)$. Since S gives a taut decomposition, Corollary 4.11 gives that $c(S,t)=c(\alpha,t)$. Hence

$$i(c_1(\mathfrak{S}\cap O_S,t))=C\cap H_\alpha=C_\alpha(M,\gamma,t),$$

which concludes the proof of (i).

Now we prove (ii). Part (1) of Proposition 5.4 states that the image of f_S is O_S, thus by part (3) the image of F_S is $i(c_1(O_S,t))+c(t,t')$. Using Lemma 3.20, the fact that S is connected, and $c(S,t)=c(\alpha,t)$, we conclude that $i(c_1(O_S,t))=H_\alpha$. Since S is nonseparating, H_α is a hyperplane, ie, $\dim H_\alpha=b_1(M)-1$.

To see (iii), first note that $b_0(M)=b_0(M')$. If $H_2(M)=0$ or $S=D^2$, then we can apply Lemma 5.1 to conclude that $b_1(M')=b_1(M)+b_1(S)-1$. Thus the kernel of the map F_S has dimension $b_1(S)$. □

Proposition 5.12 *Suppose that $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ is a taut surface decomposition of a strongly balanced sutured manifold, such that S is a disk and $[S]=\alpha\neq 0$. Fix $t\in T(M,\gamma)$ and $t'\in T(M',\gamma')$. Then the map $c'\mapsto F_S(c')-c(t,t')$ is an affine isomorphism between the polytope $P(M',\gamma',t')$ and the face $P_\alpha(M,\gamma,t)$ of $P(M,\gamma,t)$.*

Proof We use Theorem 5.11. As $b_1(S)=0$, the map $F_S-c(t,t')$ is an affine isomorphism between $H^2(M',\partial M')$ and H_α, and maps $P(M',\gamma',t')$ onto $P_\alpha(M,\gamma,t)$. □

The next result also follows from [5].

Corollary 5.13 *Suppose that the balanced sutured manifold (M,γ) is disk decomposable. Then there is a single groomed surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that (M',γ') is a product. Moreover, if γ is connected then (M,γ) has a depth at most one taut foliation.*

Proof First note that M has to be a handlebody, thus $H_2(M)=0$. In this proof we suppress the trivialization t in the notation $P(M,\gamma,t)$, this is justified by Remark 4.10. Suppose that

$$(M,\gamma)=(M_0,\gamma_0)\rightsquigarrow^{S_1}(M_1,\gamma_1)\rightsquigarrow^{S_2}\dots\rightsquigarrow^{S_n}(M_n,\gamma_n)$$

is a sutured manifold hierarchy such that each S_i is a disk and (M_n,γ_n) is a product. Then $SFH(M_k,\gamma_k)\neq 0$ for $0\leq k\leq n$, since $SFH(M_n,\gamma_n)\leq SFH(M_k,\gamma_k)$ by Theorem 4.5. Furthermore, (M_k,γ_k) is irreducible because (M_n,γ_n) is. Together with [13, Proposition 9.18] these imply that every (M_k,γ_k) is taut.

Let $\alpha_i=[S_i]$. Then $P(M_{i+1},\gamma_{i+1})$ is isomorphic to the face $P_{\alpha_i}(M_i,\gamma_i)$ of $P(M_i,\gamma_i)$ by Proposition 5.12. Furthermore, $SFH(M_{i+1},\gamma_{i+1})\cong SFH_{\alpha_i}(M_i,\gamma_i)$ by Proposition 4.13. Since (M_n,γ_n) is a product, $P(M_n,\gamma_n)$ is a single point and $SFH(M_n,\gamma_n)\cong\mathbb{Z}$. So $P(M_n,\gamma_n)$ corresponds to a vertex $v=i(c_1(\mathfrak{s},t))$ of $P(M,\gamma)$ such that $SFH(M,\gamma,\mathfrak{s})\cong\mathbb{Z}$.

Hence Corollary 4.15 implies that there is a groomed and taut surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that for $\alpha=[S]$ we have $P_\alpha(M,\gamma)=\{v\}$ and

$$SFH(M',\gamma')\cong SFH_\alpha(M,\gamma)\cong\mathbb{Z}.$$

Since (M',γ') is taut, we can use [14, Theorem 9.7] to conclude that (M',γ') is a product. If γ is connected, then S is necessarily well groomed, thus by [4] there is a depth at most one taut foliation on (M,γ). □

Proposition 5.14 *Suppose that $(M,\gamma)\overset{S}{\rightsquigarrow}(M',\gamma')$ is a decomposition of a balanced sutured manifold along a disk S such that $I(S)=-2$, ie, $|\partial S\cap s(\gamma)|=4$. If we decompose (M,γ) along $-S$, we get (M',γ''). Then*

$$SFH(M,\gamma)\cong SFH(M',\gamma')\oplus SFH(M',\gamma''). \tag{5-2}$$

Proof By Theorem 4.1, if for some $\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)$ we have $SFH(M,\gamma,\mathfrak{s})\neq 0$, then $\langle c_1(\mathfrak{s},t),[S]\rangle \geq c(S,t)$ and $\langle c_1(\mathfrak{s},t),[-S]\rangle \geq c(-S,t)$. Thus

$$c(S,t)\leq \langle c_1(\mathfrak{s},t),[S]\rangle \leq -c(-S,t).$$

Note that $-c(-S,t)-c(S,t)=-\chi(S)-\chi(-S)-I(S)-I(-S)=2$.

Furthermore, $\langle c_1(\mathfrak{s},t),[S]\rangle$ is always congruent to $c(S,t)$ modulo 2. Indeed, for $\mathfrak{s}_0\in O_S$ we have $\langle c_1(\mathfrak{s}_0,t),[S]\rangle = c(S,t)$ and $c_1(\mathfrak{s},t)-c_1(\mathfrak{s}_0,t)=2(\mathfrak{s}-\mathfrak{s}_0)$.

So for any $\mathfrak{s}\in S(M,\gamma)$ either $\langle c_1(\mathfrak{s},t),[S]\rangle = c(S,t)$ or $\langle c_1(\mathfrak{s},t),[-S]\rangle = c(-S,t)$. Together with Theorem 4.5, this implies Equation (5-2). □

Corollary 5.15 *With the assumptions of Proposition 5.14, if (M,γ) is taut, then at least one of (M',γ') and (M',γ'') is taut.*

Remark 5.16 Corollary 5.15 can also be proven using simple cut-and-paste methods. The following, yet unpublished argument was communicated to me by David Gabai. If x denotes the Thurston norm, then $x(R_+(\gamma'))=x(R_+(\gamma))-1$ and $x(R_-(\gamma''))=x(R_-(\gamma))-1$. If neither (M',γ'), nor (M',γ'') are taut then we have $x([R_+(\gamma')])\leq x(R_+(\gamma))-3$ and $x([R_-(\gamma'')])\leq x(R_-(\gamma))-3$. Let T' and T'' be properly embedded, norm minimizing representatives of $[R_+(\gamma')]$ and $[R_-(\gamma'')]$, respectively. Recall that S'_+ and S'_- were introduced in Definition 2.5. Let T be the oriented surface obtained from $T'\cup T''$ (viewed as an immersed surface in M) by gluing $T'\cap S'_+$ to $T''\cap S'_-$ and $T'\cap S'_-$ to $T''\cap S'_+$ (each intersection consists of two arcs), and then doing oriented cut-and-paste along the double curves. We can assume that T has no S^2 components since M is irreducible. Then

$$x(T)=x([R_+(\gamma')])+x([R_-(\gamma'')])+4\leq x(R_+(\gamma))+x(R_-(\gamma))-2=x(R(\gamma))-2.$$

Since $[T]=[R(\gamma)]$ in $H_2(M,\gamma)$, we get that $R(\gamma)$ is not norm minimizing in its homology class in $H_2(M,\gamma)$, contradicting the assumption that (M,γ) is taut. So at least one of (M',γ') and (M',γ'') is taut.

6 Dimension of the sutured Floer homology polytope

Theorem 6.1 *Suppose that $H_2(M)=0$ and the sutured manifold (M,γ) is balanced, taut, reduced and horizontally prime. Let $t\in T(M,\gamma)$. Then*

$$\dim P(M,\gamma,t)=\dim H^2(M,\partial M;\mathbb{R})=b_1(M)=b_1(\partial M)/2.$$

In particular,

$$\mathrm{rk}(SFH(M,\gamma))\geq b_1(\partial M)/2+1.$$

Proof We improve on and simplify the proof of [14, Theorem 9.7]. By [7, Lemma 0.7] for any nonzero element $\alpha\in H_2(M,\partial M)$ there is a groomed surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that (M',γ') is taut and $[S]=\alpha$. We can assume that S is open since $H_2(M)=0$, and we can make S nice by putting it into generic position.

We are now going to show that $c(\alpha,t)+c(-\alpha,t)<0$. Since S gives a taut decomposition, $c(\alpha,t)=c(S,t)=\chi(S)+I(S)-r(S,t)$ by Corollary 4.11. Using [24, Theorem 2.5] and the fact that $H_2(M)=0$, we can find a nice decomposing surface S' that gives a taut decomposition, $[S']=-\alpha$, and $\partial S\cap R(\gamma)=-\partial S'\cap R(\gamma)$. We briefly outline the construction of S' here. Let $U=N(\partial S\subset\partial M)\cup\gamma$ be a regular neighborhood of ∂S inside ∂M, together with γ. Denote by ρ the relative homology class represented by $R(\gamma)$ in $H_2(M,U)$. Similarly, let $\sigma=[S]\in H_2(M,U)$. Consider the Thurston norm x_U on $H_2(M,U)$. It is shown in [24] that there is an integer k_0 such that for every $k>k_0$ we have

$$x_U(-\sigma+(k+1)\rho)=x_U(-\sigma+k\rho)+x_U(\rho). \tag{6-1}$$

Then the surface S' is a Thurston norm minimizing representative of $-\sigma+k\rho$ for some fixed $k>k_0$. Since $H_2(M)=0$, we can remove all closed components of S' without changing the homology class it represents in $H_2(M,U)$, and without increasing its Thurston norm. Equation (6-1) ensures that S' gives a taut decomposition.

Note that $c(-\alpha,t)=c(S',t)=\chi(S')+I(S')-r(S',t)$. So we have to show that

$$\chi(S)+\chi(S')+I(S)+I(S')<r(S,t)+r(S',t).$$

Since $\partial S\cap R(\gamma)=-\partial S'\cap R(\gamma)$, and by the construction of S' described above, the one-cycle $\partial S+\partial S'\subset\gamma$ is homologous to $ks(\gamma)$ in $H_1(\gamma)$ for some nonnegative integer k; see Figure 2.

Recall that $r(S,t)$ is defined as the rotation of $p(\nu_S)$ with respect to the trivialization t as we go around ∂S, where p is orthogonal projection onto $v^\perp$. Note that $r(S',t)$ can also be computed as the rotation of $-p(\nu_{S'})$ with respect to t. Moreover, $p(\nu_S)=-p(\nu_{S'})$ over $S\cap R(\gamma)=S'\cap R(\gamma)$.

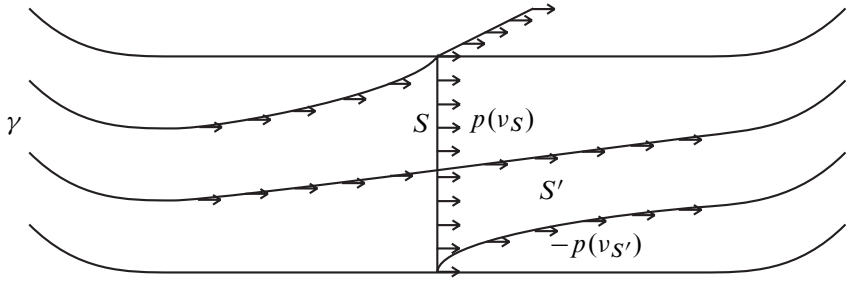

Figure 2: The decomposing surfaces S and S'

Let C be a component of $\partial S \cap \gamma$ or $\partial S' \cap \gamma$. We can assume that if C is closed then $\nu_S|C$ (or $-\nu_{S'}|C$) points out of M; and if C is an arc then it is monotonic between $R_-(\gamma)$ and $R_+(\gamma)$ and $p(\nu_S)|C$ (or $p(\nu_{S'})|C$) is nonzero and parallel to $s(\gamma)$. Observe that $p(\nu_S) \cup -p(\nu_{S'})$ is a continuous vector field along $\partial S + \partial S'$, and it can be homotoped inside $v_0^\perp \setminus 0$ such that it points out of M everywhere. Using the Poincaré–Hopf index formula we get that

$$r(S,t) + r(S',t) = k\chi(R_+(\gamma)).$$

See Lemma 3.17.

Suppose that C and C' are components of $\partial S \cap \gamma$ and $\partial S' \cap \gamma$, respectively, such that $C \approx [0,1] \approx C'$ and $\partial C = \partial C'$; see Figure 2. Let s_0 be the component of $s(\gamma)$ containing $C \cap s(\gamma)$. If $C + C'$ is null-homologous in γ, then we can achieve by an isotopy of S' that $C = C'$. If $C + C'$ is homologous to ms_0 in $H_1(\gamma)$ for some $m > 0$, then we can achieve that $|C \cap C'| = m + 1$.

Make S and S' transverse by perturbing them in the interior of M. Then take the double curve sum P of S and S'. We are now going to see how $\chi(S) + \chi(S')$ changes when doing the cut-and-paste. The number of components of $\partial S \cap \partial S'$ homeomorphic to $[0,1]$ is at most $|S \cap s(\gamma)|$. (It is strictly smaller if there are $[0,1]$ components C and C' of $S \cap \gamma$ and $S' \cap \gamma$, respectively, such that $C = C'$.) Now let K be a component of $S \cap S'$. If $\operatorname{int}(K) \subset \operatorname{int}(S) \cap \operatorname{int}(S')$, then doing cut-and-paste along K doesn't change the Euler characteristic (we remove two circles or intervals and glue them back in a different way). On the other hand, if $K \subset \partial S \cap \partial S'$ is homeomorphic to $[0,1]$, then cut-and-paste along K decreases the Euler characteristic by one (we glue two surfaces together along two arcs in their boundaries). According to Lemma 3.17, we have $I(S) = I(S') = -|S \cap s(\gamma)|/2$, so

$$\chi(S) + \chi(S') + I(S) + I(S') \le \chi(P).$$

Thus it is sufficient to prove that $\chi(P) < k\chi(R_+(\gamma))$.

Let $r = [R_+(\gamma)] = [R_-(\gamma)] \in H_2(M, \gamma)$. From $H_2(M) = 0$, and by looking at the exact sequence of the pair (M, γ), we see that the map $\partial\colon H_2(M, \gamma) \to H_1(\gamma)$ is injective. Thus

$$\partial[P] = \partial(k\,r) = k[s(\gamma)] \in H_1(\gamma)$$

implies that $[P] = k\,r$ in $H_2(M, \gamma)$. Let x denote the Thurston seminorm on $H_2(M, \gamma)$. As in [14, Claim 9.10], using the fact that M is irreducible we can suppose that P has no S^2 and T^2 components.

Suppose that P has a D^2 component. Since $H_2(M) = 0$, the boundary ∂M is connected. Using the fact that $R(\gamma)$ is incompressible and $\partial P \subset \gamma$, we get that $\partial M = S^2$ and γ is connected. But M is irreducible, so $M = D^3$. The sutured manifold $(D^2 \times I, \partial D^2 \times I)$ obviously satisfies the theorem, so we can suppose from now on that P has no D^2 component. Similarly, we can assume that $R_+(\gamma)$ has no D^2 component.

Using the above assumptions, $x(P) = -\chi(P)$ and $x(R_+(\gamma)) = -\chi(R_+(\gamma))$. Since (M, γ) is taut,

$$-\chi(P) = x(P) \geq x([P]) = x(k\,r) = k\,x(r) = -k\chi(R_+(\gamma)).$$

So we only have to exclude the possibility $x(P) = k\,x(r)$. In this case, P is norm minimizing in $k\,r$. Thus P cannot have genus > 1 closed components, for if it did we could remove them without changing $[P]$ (as $H_2(M) = 0$) and decrease $x(P)$.

Fix a point $z_0 \in R_+(\gamma)$. We define a function $\varphi\colon M \setminus P \to \mathbb{Z}$ by setting $\varphi(z)$ to be the algebraic intersection number of P with a path connecting z_0 and z. This is well defined because $[P] = [S] + [S'] = \alpha - \alpha = 0$ in $H_2(M, \partial M)$, and thus any closed curve in M intersects P algebraically zero times. There is a well defined homological pairing between $H_1(M, R(\gamma))$ and $H_2(M, \gamma)$. Thus if $z \in R(\gamma)$, then $\varphi(z)$ can be computed by taking the intersection number of a path connecting z_0 and z with the cycle $k\,R_-(\gamma)$. So $\varphi|R_+(\gamma) \equiv 0$ and $\varphi|R_-(\gamma) \equiv k$. Since P has no closed components, every component of $M \setminus P$ intersects γ. Thus by considering paths on γ, we see that $0 \leq \varphi \leq k$.

As in [14, Claim 9.10], let $J_i = \mathrm{cl}((\varphi^{-1})(i))$ for $0 \leq i \leq k$ and let $P_i = J_{i-1} \cap J_i$ for $1 \leq i \leq k$. Then P is the disjoint union of the surfaces $P_1, \ldots, P_k$, and $\bigcup_{l=0}^{i-1} J_l$ is a homology between $R_+(\gamma)$ and P_i in $H_2(M, \gamma)$. Thus $[P_i] = r$, and hence $x(P_i) \geq x(r)$. Since

$$\sum_{i=1}^{k} x(P_i) = x(P) = k\,x(r),$$

we must have $x(P_i) = x(r)$ for $1 \le i \le k$. The boundary ∂P consists of k parallel copies of $s(\gamma)$, hence computing $\varphi|\gamma$ using curves in γ implies that ∂P_i is isotopic to $\partial R_+(\gamma)$ for $1 \le i \le k$. So each P_i is a horizontal surface. Since (M,γ) is horizontally prime, for some $1 \le j \le k$ the surfaces $P_1, \dots, P_j$ are parallel to $R_+(\gamma)$ and $P_{j+1}, \dots, P_k$ are parallel to $R_-(\gamma)$.

Let $\gamma_j = \gamma \cap J_j$. Then the sutured manifold (J_j, γ_j) is homeomorphic to (M,γ). Thus (J_j, γ_j) is reduced. Observe that the closure of each component of $S \cap \mathrm{Int}(J_j)$ is either a product disk or product annulus in (J_j, γ_j), which in turn lies in a product neighborhood $N(\gamma_j)$ of γ_j. Indeed, by Lemma 2.13 every product disk is inessential in (M_j, γ_j), and every product annulus is either ambient isotopic to a component of γ_j or bounds a $D^2 \times I$ by Lemma 2.11. The rest of S, ie, $S \setminus J_j$ lies in a product neighborhood of $R(\gamma)$ since $N_+ = J_1 \cup \cdots \cup J_{j-1}$ is a regular neighborhood of $R_+(\gamma)$ and $N_- = J_{j+1} \cup \cdots \cup J_k$ is a regular neighborhood of $R_-(\gamma)$. Thus S lies in a product neighborhood $N = N(\gamma) \cup N_+ \cup N_-$ of ∂M. Let $r\colon N \to \partial M$ be a retraction. Then $r(S)$ represents a 2–chain in ∂M whose boundary is ∂S. The map $\partial\colon H_2(M, \partial M) \to H_1(\partial M)$ is injective because $H_2(M) = 0$. Moreover, $[S] = \alpha \ne 0$, thus $[\partial S] = \partial\alpha \ne 0$ in $H_1(\partial M)$. Consequently, ∂S cannot be a boundary in ∂M, a contradiction.

So indeed $c(\alpha, t) + c(-\alpha, t) < 0$. By Definition 4.9, this means that the interval $\langle \alpha, P(M,\gamma,t) \rangle = [c(\alpha,t), -c(-\alpha,t)]$ is not a single point. Since this holds for every $\alpha \ne 0$ in $H_2(M, \partial M)$, the dimensions of $P(M,\gamma,t)$ has to be at least $b_2(M, \partial M)$. Since $P(M,\gamma,t)$ sits inside $H^2(M, \partial M, \mathbb{R})$, the dimension has to be equal to $b_2(M, \partial M)$. By Poincaré duality $b_2(M, \partial M) = b_1(M)$, and this is equal to $b_1(\partial M)/2$ because $H_2(M) = 0$.

The last statement follows from the fact that a d–dimensional polytope has at least $d+1$ vertices and from Proposition 4.16. □

Proposition 6.2 *Suppose that S is a nice decomposing surface in the strongly balanced sutured manifold (M,γ) whose components are $S_1, \dots, S_k$. Let $\alpha = [S]$ and $\alpha_j = [S_j]$ for $1 \le j \le k$. Suppose that $\alpha \ne 0$,*

$$\dim P(M,\gamma,t) = \dim H^2(M, \partial M; \mathbb{R}) = b_1(M)$$

and $\dim P_\alpha(M,\gamma,t) = b_1(M) - 1$. Then there is a nonzero class $\sigma \in H_2(M, \partial M)$ and integers $a_1, \dots, a_k$ such that $\alpha_j = a_j \cdot \sigma$ for $1 \le j \le k$. Moreover, $P_{\alpha_j}(M,\gamma,t) = P_\alpha(M,\gamma,t)$ for every $1 \le j \le k$ such that $\mathrm{sgn}(a_j) = \mathrm{sgn}(a_1 + \cdots + a_k)$.

Proof Note that $\alpha_1+\cdots+\alpha_k=\alpha\neq 0$, so there is a nonzero α_j. Since

$$P_\alpha(M,\gamma,t)\subset\bigcap_{j=1}^{k}H_{\alpha_j},$$

we must have

$$\dim\left(\bigcap_{j=1}^{k}H_{\alpha_j}\right)=b_1(M)-1.$$

Thus $\alpha_1,\dots,\alpha_k$ are pairwise linearly dependent. The existence of σ and $a_1,\dots,a_k$ follows. If $\operatorname{sgn}(a_j)=\operatorname{sgn}(a_1+\cdots+a_k)$, then α and α_j are parallel and point in the same direction, thus $H_\alpha=H_{\alpha_j}$, and consequently $P_\alpha(M,\gamma,t)=P_{\alpha_j}(M,\gamma,t)$. □

Corollary 6.3 *Let (M,γ) be strongly balanced. If $H_2(M)=0$, then every face of $P(M,\gamma,t)$ whose dimension is $b_1(M)-1$ is of the form $P_{[R]}(M,\gamma,t)$ for some* connected *groomed surface R.*

Proof This follows from Proposition 6.2 and Corollary 4.15. □

7 Depth of a sutured manifold

Proposition 7.1 *Suppose that $H_2(M)=0$ and the sutured manifold (M,γ) is balanced, taut, reduced, horizontally prime, and not a product. Then there is always a well-groomed surface decomposition $(M,\gamma)\rightsquigarrow^S(M',\gamma')$ such that (M',γ') is taut and $\operatorname{rk}(SFH(M',\gamma'))\leq\operatorname{rk}(SFH(M,\gamma))/2$.*

Proof Note that ∂M is connected because $H_2(M)=0$. We show that $\partial M\neq S^2$. Indeed, otherwise by the irreducibility of M we have $M=D^3$, and since (M,γ) is taut, γ would have to be a single annulus. But this would contradict that (M,γ) is not a product. So $b_1(\partial M)\geq 2$. By Theorem 6.1,

$$\dim P(M,\gamma,t)=\dim H^2(M,\partial M;\mathbb{R})=b_1(\partial M)/2\geq 1.$$

Lemma 2.8 implies that there is a well groomed homology class $\alpha\in H_2(M,\partial M)$. Then $-\alpha$ is also well groomed. Thus using Corollary 4.15 we get well groomed and taut surface decompositions $(M,\gamma)\rightsquigarrow^{S_1}(M_1,\gamma_1)$ and $(M,\gamma)\rightsquigarrow^{S_2}(M_2,\gamma_2)$ such that $[S_1]=\alpha$ and $[S_2]=-\alpha$; furthermore, $SFH(M_1,\gamma_1)\cong SFH_\alpha(M,\gamma)$ and $SFH(M_2,\gamma_2)\cong SFH_{-\alpha}(M,\gamma)$. Since the dimension of $P(M,\gamma,t)$ is the same as the dimension of the ambient space $H^2(M,\partial M;\mathbb{R})$, we have

$$P_\alpha(M,\gamma,t)\cap P_{-\alpha}(M,\gamma,t)=\varnothing.$$

Thus

$$SFH_\alpha(M,\gamma)\oplus SFH_{-\alpha}(M,\gamma)\leq SFH(M,\gamma),$$

and consequently either S_1 or S_2 satisfies the requirements of the proposition. □

Remark 7.2 Proposition 7.1 implies that the number $\mathrm{rk}(SFH(M,\gamma))$ acts as a complexity of taut balanced sutured manifolds with $H_2(M)=0$ in the following sense. If (M,γ) is not a product, then Proposition 2.18 and Proposition 2.16 imply that we can perform finitely many horizontal and product annulus decompositions to get an (M,γ) which is horizontally prime, reduced, and $H_2(M)$ is still zero. By Proposition 7.1, now there is a taut decomposition which strictly decreases $\mathrm{rk}(SFH(M,\gamma))$.

Compare this with the complexity defined in [4] to show the existence of sutured manifold hierarchies. Note that we used the existence of sutured manifold hierarchies to prove that $SFH(M,\gamma)\geq\mathbb{Z}$ if (M,γ) is taut, which in turn is implicitly needed in the proof of Proposition 7.1.

Definition 7.3 Let (M,γ) be a taut sutured manifold. By [4], (M,γ) has a sutured manifold hierarchy

$$(M,\gamma)\overset{S_1}{\rightsquigarrow}(M_1,\gamma_1)\overset{S_2}{\rightsquigarrow}\cdots\overset{S_n}{\rightsquigarrow}(M_n,\gamma_n),$$

where (M_n,γ_n) is a product. We define the *depth* $d(M,\gamma)$ of (M,γ) to be the minimal such n. In particular, $d(M,\gamma)=0$ if and only if (M,γ) is a product.

Remark 7.4 It is important to note that in the above definition $S_1,\dots,S_n$ can be arbitrary decomposing surfaces, they are not necessarily connected.

Example 7.5 This example illustrates that Theorem 6.1 and Proposition 7.1 do not hold without the assumption that the sutured manifold (M,γ) is reduced. Furthermore, it also shows that even though annulus decompositions do not change $SFH(M,\gamma)$, they might change the sutured Floer homology polytope $P(M,\gamma,t)$.

Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be the balanced diagram shown in Figure 3, and let (M,γ) be the balanced sutured manifold defined by it. Here Σ is a genus one surface with three boundary components, each represented by a little circle. There is one α and one β curve; moreover, $\alpha\cap\beta$ consists of two points denoted by x and y. Since there are no periodic domains, $H_2(M)=0$. It is a simple exercise to check that ∂M is a surface of genus two.

The chain complex $CF(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is generated by the points x and y. They lie in the same Spin^c–structure $\mathfrak{s}_0$ because the component D of $\Sigma\setminus(\alpha\cup\beta)$ containing the one-handle gives a homology class of Whitney disks connecting x and y (if we stabilize the diagram there is even a topological Whitney disc in the symmetric product). There are no holomorphic disks connecting x and y, thus $SFH(M,\gamma)\cong\mathbb{Z}^2$, which lies in $\mathfrak{s}_0$. Thus $P(M,\gamma,t)$ is a single point. On the other hand, (M,γ) is not a product

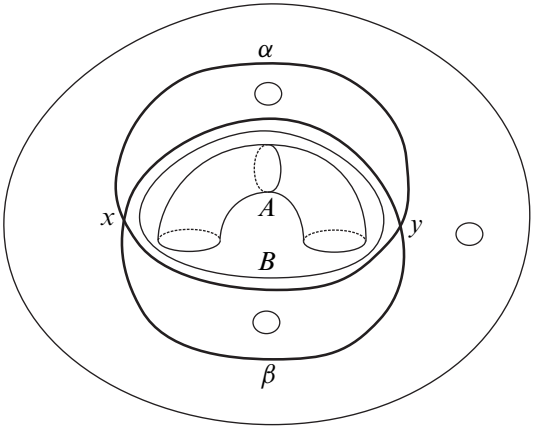

Figure 3: A nonproduct, horizontally prime, taut sutured manifold such that the polytope $P(M,\gamma)$ is a single point

because $SFH(M,\gamma) \ncong \mathbb{Z}$, and it is taut since $SFH(M,\gamma) \neq 0$ and M is irreducible. Moreover, (M,γ) is horizontally prime. Indeed, suppose that

$$(M,\gamma) \rightsquigarrow^{S} (M_1,\gamma_1) \sqcup (M_2,\gamma_2)$$

is a horizontal decomposition. Since S gives a taut decomposition and $[S]=0$ in $H_2(M,\partial M)$, we can apply Corollary 4.6 and conclude that

$$SFH(M,\gamma) \cong SFH((M_1,\gamma_1) \sqcup (M_2,\gamma_2)).$$

By Lemma 4.8,

$$2 = \mathrm{rk}(SFH(M,\gamma)) = \mathrm{rk}(SFH(M_1,\gamma_1)) \cdot \mathrm{rk}(SFH(M_2,\gamma_2)).$$

So $\mathrm{rk}(SFH(M_i,\gamma_i)) = 1$ for $i=1$ or $i=2$, and this means that (M_i,γ_i) is a product. That is, S is parallel to either $R_+(\gamma)$ or $R_-(\gamma)$, and so (M,γ) is horizontally prime. This shows that Theorem 6.1 fails if (M,γ) is not reduced. In fact, there is no nice surface decomposition that would change $SFH(M,\gamma)$. Thus (M,γ) cannot be decomposed into a product using a single nice surface decomposition.

Let A denote the core of the handle in D, and let B be a simple closed curve in D parallel to ∂D. Then $A\times I$ is a nonseparating product annulus, and $B\times I$ is a separating product annulus in (M,γ). Both of them are nice decomposing surfaces.

If we decompose (M,γ) along $A\times I$, then we get a sutured manifold (M_A,γ_A) which is defined by the diagram $(\Sigma_A,\boldsymbol{\alpha},\boldsymbol{\beta})$, where Σ_A is the completion of $\Sigma \setminus A$. Here x and y lie in different Spin^c–structures. So $SFH(M_A,\gamma_A) \cong \mathbb{Z}^2$, and the two $\mathbb{Z}$–summands lie in different Spin^c–structures. Thus $S(M_A,\gamma_A)$ consists of two points, so

$P(M_A, \gamma_A, t_A)$ is a nondegenerate line segment for any trivialization t_A. By Corollary 4.15, there is a well-groomed surface decomposition $(M_A, \gamma_A) \rightsquigarrow^S (M', \gamma')$ such that $SFH(M', \gamma') \cong \mathbb{Z}$, and thus (M', γ') is a product. This shows that $d(M, \gamma) \leq 2$.

Decomposing (M, γ) along $B \times I$, we get the disjoint union of a sutured manifold (M_B, γ_B) and the product sutured manifold $(D \times I, \partial D \times I)$. Note that (M_B, γ_B) can be obtained from (M_A, γ_A) by decomposing along a product disk (which corresponds to an arc connecting the feet of the handle in D). As above, $SFH(M_B, \gamma_B) \cong \mathbb{Z}^2$ and $P(M_B, \gamma_B, t_B)$ consists of a nondegenerate line segment. Thus even a separating product annulus can change the sutured Floer homology polytope.

It is not hard to see that (M_B, γ_B) is a solid torus with four longitudinal sutures. We can obtain (M, γ) from this by attaching $(D \times I, \partial D \times I)$ along $\partial D \times I$ to one of the components of γ_B. Of course D is a punctured torus. This again shows that (M, γ) is taut. And we can directly see that (M_B, γ_B) can be reduced to a product, namely by decomposing along a disk which intersects $s(\gamma_B)$ in four points.

The following proposition contains [14, Theorem 9.7], which claims that SFH detects product sutured manifolds, as the special case $k = 0$. The proof presented here is independent of the proof of [14, Theorem 9.7], which refers to an erroneous result in [15] that has been corrected in [16].

Proposition 7.6 *Suppose that (M, γ) is a taut balanced sutured manifold such that $H_2(M) = 0$ and $\mathrm{rk}(SFH(M, \gamma)) < 2^{k+1}$ for some integer $k \geq 0$. Then*

$$d(M, \gamma) \leq 2k.$$

Proof We proceed by induction on k. First suppose that $k = 0$. By Proposition 2.18, after a finite sequence of horizontal decompositions we get a taut sutured manifold (M', γ') which is horizontally prime. Using Lemma 5.1, we see that $H_2(M') = 0$. Furthermore, $SFH(M', \gamma') \cong SFH(M, \gamma)$ by Corollary 4.6. Then, using Proposition 2.16, we can decompose (M', γ') along product annuli to get a reduced, horizontally prime, and taut sutured manifold (M'', γ''). Now [14, Lemma 8.10] and Lemma 5.1 imply that $SFH(M'', \gamma'') \leq SFH(M', \gamma')$ and $H_2(M'') = 0$. So $\mathrm{rk}(SFH(M'', \gamma'')) \leq 1$. Then, by the second part of Theorem 6.1, each component of $\partial M''$ has to be a sphere. But (M'', γ'') is taut (in particular irreducible), so it is necessarily a disjoint union of product sutured manifolds of the form $(D^2 \times I, \partial D^2 \times I)$. Consequently, the sutured manifold (M, γ) is a product, and hence $d(M, \gamma) = 0$.

Now suppose that $k > 0$ and that the result holds for $i < k$. We show that this implies the result for k. It suffices to consider the case when (M, γ) is not a product. First

assume that (M,γ) is not horizontally prime. Then Proposition 2.18 gives a taut decomposition

$$(M,\gamma)\rightsquigarrow^{H}(M',\gamma'),$$

such that each component of H is a horizontal surface and (M',γ') is horizontally prime. Let $(M'_1,\gamma'_1),\dots,(M'_l,\gamma'_l)$ denote the components of (M',γ'). Then $l\geq 2$, and we can suppose that for each $1\leq i\leq l$ the sutured manifold (M'_i,γ'_i) is not a product. Moreover, $H_2(M'_i)=0$ by Lemma 5.1. If we apply the $k=0$ case to (M'_i,γ'_i), we get that $\mathrm{rk}(SFH(M'_i,\gamma'_i))\geq 2$ for $1\leq i\leq l$. By Corollary 4.6 and Lemma 4.8,

$$\mathrm{rk}(SFH(M,\gamma))=\mathrm{rk}(SFH(M'_1,\gamma'_1))\dots\mathrm{rk}(SFH(M'_l,\gamma'_l)).$$

So $\mathrm{rk}(SFH(M'_i,\gamma'_i))<2^k$ for every $1\leq i\leq l$. Hence we can apply the induction hypothesis to each (M'_i,γ'_i) separately to obtain $d(M'_i,\gamma'_i)\leq 2k-2$ for every $1\leq i\leq l$. But $(M'_1,\gamma'_1),\dots,(M'_l,\gamma'_l)$ are pairwise disjoint, hence $d(M',\gamma')\leq 2k-2$. So $d(M,\gamma)\leq 2k-1$.

Consequently, we can assume that (M,γ) is horizontally prime. Using Proposition 2.16, there is a decomposition $(M,\gamma)\rightsquigarrow^{A}(M_1,\gamma_1)$, where A is a union of pairwise disjoint product annuli $A_1,\dots,A_r$ and (M_1,γ_1) is reduced, horizontally prime, taut, $H_2(M_1)=0$, and is not a product. If we apply [14, Lemma 8.10] to $A_1,\dots,A_r$, then we get $SFH(M_1,\gamma_1)\leq SFH(M,\gamma)$. So we can use Proposition 4.16 to get a taut decomposition $(M_1,\gamma_1)\rightsquigarrow^{S}(M'_1,\gamma'_1)$ such that

$$\mathrm{rk}(SFH(M'_1,\gamma'_1))\leq\mathrm{rk}(SFH(M_1,\gamma_1))/2<2^k.$$

Thus, using the induction hypothesis for (M'_1,γ'_1), we see that $d(M'_1,\gamma'_1)\leq 2k-2$, and hence $d(M,\gamma)\leq 2k$. □

Remark 7.7 In the above proof, every decomposition can be chosen to be well groomed, except possibly the one along A, which is a disjoint union of product annuli. If every decomposition were well groomed, then we could even claim the existence of a depth at most $2k$ taut foliation on (M,γ). Unfortunately, I have overlooked this point in the proof of [14, Theorem 1.8], which leaves [14, Question 9.14] unanswered. If one could make a sutured manifold reduced using a groomed decomposition, that would give a positive answer to [14, Question 9.14].

Corollary 7.8 *Suppose that K is a null-homologous knot in the rational homology 3–sphere Y, and*

$$\mathrm{rk}\big(\widehat{HFK}(Y,K,g(K))\big)<2^{k+1}.$$

Then the sutured manifold $Y(K)$ complementary to K has depth $d(Y(K))\leq 2k+1$. In particular, if $k=0$, then K is fibred.

Proof Let R be a minimal genus Seifert surface for K. By [14, Theorem 1.5],

$$SFH(Y(R)) \cong \widehat{HFK}(Y, K, g(K)).$$

So Proposition 7.6 implies that $d(Y(R)) \leq 2k$. Since we have the sutured manifold decomposition $Y(K) \rightsquigarrow^R Y(R)$, we get $d(Y(K)) \leq 2k+1$. Finally, if $k = 0$, then $Y(R)$ is a product, so K is fibred. □

8 A seminorm on the homology of a sutured manifold

In this section, we are going to define a seminorm on $H_2(M, \partial M; \mathbb{R})$ for a strongly balanced sutured manifold (M, γ). Then we will show that it is nondegenerate if (M, γ) is taut, reduced, horizontally prime, and $H_2(M) = 0$. Note that $H_2(M, \partial M)$ is torsion free, and hence can be considered to be a subgroup of $H_2(M, \partial M; \mathbb{R})$.

Definition 8.1 Let (M, γ) be taut and strongly balanced. For $t \in T(M, \gamma)$, let $p_t \in H^2(M, \partial M; \mathbb{R})$ denote the center of mass of $P(M, \gamma, t)$. Then the polytope $P(M, \gamma) = P(M, \gamma, t) - p_t$ is independent of t because of Lemma 3.12. Of course the center of mass of $P(M, \gamma)$ is 0.

Proposition 8.2 *Let (M, γ) be taut and strongly balanced. Then for a homology class $\alpha \in H_2(M, \partial M; \mathbb{R})$ the formula*

$$y(\alpha) = \max\{ \langle -c, \alpha \rangle : c \in P(M, \gamma) \}$$

defines a seminorm on $H_2(M, \partial M; \mathbb{R})$. It is nondegenerate if and only if

$$\dim P(M, \gamma) = b_1(M).$$

Remark 8.3 If $t \in T(M, \gamma)$, then $y(\alpha) = -c(\alpha, t) + \langle p_t, \alpha \rangle$. Furthermore, note that for every $k \geq 0$ we have $y(k\alpha) = k\, y(\alpha)$, but in general $y(\alpha) \neq y(-\alpha)$ can happen. Indeed, in [3, Example 8.5] we exhibit a family of sutured manifolds whose sutured Floer homology polytopes are all centrally asymmetric. So in those examples y is not symmetric.

Proof Since 0 is the center of mass of $P(M, \gamma)$, and because $P(M, \gamma)$ is convex, we have $0 \in P(M, \gamma)$. So $y(\alpha) \geq 0$ for every $\alpha \in H_2(M, \partial M; \mathbb{R})$. If $y(\alpha) = 0$ for some $\alpha \neq 0$, then $P(M, \gamma)$ lies in the hyperplane

$$\{c \in H^2(M, \partial M; \mathbb{R}) : \langle c, \alpha \rangle = 0\},$$

thus
$$\dim P(M, \gamma) < \dim H^2(M, \partial M; \mathbb{R}) = b_1(M).$$

On the other hand, if $\dim P(M,\gamma) < b_1(M)$, then there is a hyperplane H containing $P(M,\gamma)$. There is also a nonzero homology class $\alpha \in H_2(M,\partial M;\mathbb{R})$ for which $\langle H,\alpha\rangle = 0$, ie, $y(\alpha)=0$ and y is degenerate.

Suppose that $\alpha,\beta \in H_2(M,\partial M;\mathbb{R})$. Then

$$\begin{aligned} y(\alpha+\beta) &= \max\{\langle -c,\alpha+\beta\rangle : c\in P(M,\gamma)\} \\ &= \max\{\langle -c,\alpha\rangle + \langle -c,\beta\rangle : c\in P(M,\gamma)\} \\ &\le \max\{\langle -c,\alpha\rangle : c\in P(M,\gamma)\} + \max\{\langle -c,\beta\rangle : c\in P(M,\gamma)\} \\ &= y(\alpha)+y(\beta). \end{aligned}$$

□

Remark 8.4 Notice that by construction $-P(M,\gamma)$ is the dual unit norm ball of the seminorm y. That is, the polytope $-P(M,\gamma)$ is the unit ball of the dual seminorm y^* on $H^2(M,\partial M;\mathbb{R})$ defined by the formula

$$y^*(c) = \max\{\langle c,\alpha\rangle : \alpha\in H_2(M,\partial M;\mathbb{R}),\ y(\alpha)\le 1\}.$$

Proposition 8.5 *Suppose that (M,γ) is taut, balanced, reduced, horizontally prime, and $H_2(M)=0$. Then y is a norm on $H_2(M,\partial M;\mathbb{R})$.*

Proof Theorem 6.1 implies that $\dim P(M,\gamma) = b_1(M)$, thus by Proposition 8.2 the seminorm y is nondegenerate. □

Remark 8.6 In [24], another seminorm is defined on $H_2(M,\partial M;\mathbb{R})$, which we will denote by x^s. Given a properly embedded, compact, oriented, and connected surface $S\subset M$, let

$$x^s(S) = \max\{0, -\chi(S) - I(S)\},$$

and we extend x^s to disconnected surfaces by taking the sum over the components. For $\alpha\in H_2(M,\partial M)$, we define $x^s(\alpha)$ as the minimum of $x^s(S)$ for all properly embedded, compact, oriented surfaces S that represent the homology class α. Finally, it is straightforward to show that x^s extends to $H^2(M,\partial M;\mathbb{R})$.

As opposed to y, the Scharlemann norm x^s is always symmetric. Hence it makes sense to compare x^s with the symmetrized seminorm

$$z(\alpha) = \tfrac{1}{2}(y(\alpha)+y(-\alpha)).$$

In [3, Theorem 7.7], we show that $z\le x^s$. Somewhat surprisingly, in general $z\ne x^s$ by [3, Proposition 7.12].

9 Sutured manifolds with $M = S^1 \times D^2$

In this section, we will compute the sutured Floer homology of every sutured manifold (M,γ) with $M = S^1 \times D^2$. This will illustrate some of the techniques we have just developed.

First, note that if such an (M,γ) is taut, then $s(\gamma)$ has to be a collection of n parallel torus knots of type $T_{p,q}$. Here p denotes the number of times the curve on ∂M goes around in the longitudinal direction. Furthermore, if $p = 0$, then $R(\gamma)$ is compressible, hence (M,γ) is not taut. Since M is irreducible, if (M,γ) is not taut, then $SFH(M,\gamma) = 0$. Also note that n is necessarily even. We will denote this sutured manifold by $T(p,q;n)$.

Proposition 9.1 *Let $T(p,q;n)$ be the sutured manifold defined above, and suppose that $n = 2k+2$ for $k \geq 0$. Then there is an identification $\mathrm{Spin}^c(T(p,q;n)) \cong \mathbb{Z}$ such that the following holds.*

$$SFH(T(p,q;n),i) \cong \begin{cases} \mathbb{Z}^{\binom{k}{\lfloor i/p \rfloor}} & \text{if } 0 \leq i < p(k+1), \\ 0 & \text{otherwise.} \end{cases} \tag{9-1}$$

Moreover, in each Spin^c–structure any two elements of SFH lie in the same relative Maslov grading.

Proof We saw in Example 7.5 that $SFH(T(1,0;4)) \cong \mathbb{Z}^2$, where the two $\mathbb{Z}$ summands lie in Spin^c–structures whose difference is a generator l of $H_1(S^1 \times D^2;\mathbb{Z})$. Figure 4 shows a sutured diagram for $T(1,0;4)$.

Let $(M_1,\gamma_1) = T(1,0;n)$ and $(M_2,\gamma_2) = T(p,q;m)$, and suppose that A_i is a component of γ_i for $i = 1,2$. Now glue the annuli $A_1 \subset M_1$ and $A_2 \subset M_2$ such that $A_1 \cap R_-(\gamma_1)$ is identified with $A_2 \cap R_-(\gamma_2)$. Then we obtain the sutured manifold $T(p,q;n+m-2)$. If we decompose $T(p,q;n+m-2)$ along the separating product annulus $A = A_1 = A_2$, then we get the disjoint union of $T(1,0;n)$ and $T(p,q;m)$. Since A is a nice decomposing surface in $T(p,q;n+m-2)$, an application of Corollary 4.6 gives that

$$SFH(T(p,q;n+m-2)) \cong SFH(T(1,0;n)) \otimes SFH(T(p,q;m)). \tag{9-2}$$

Using the above formula repeatedly for $T(p,q;m) = T(1,0;4)$, together with the fact that $SFH(T(1,0;4)) \cong \mathbb{Z}^2$, we get that

$$SFH(T(1,0;n)) \cong \bigotimes_{j=1}^{k} \mathbb{Z}^2,$$

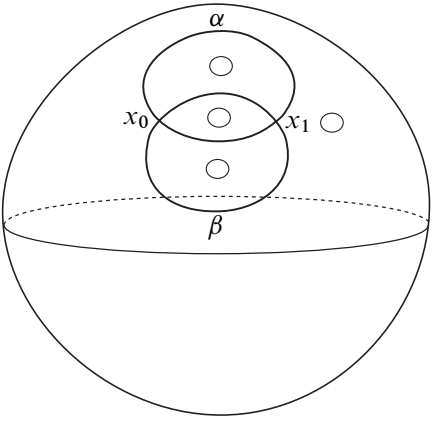

Figure 4: A sutured diagram for $T(1,0;4)$

where $n=2k+2$. It follows from Proposition 5.4 that there are generators x_0^j and x_1^j of the j–th $\mathbb{Z}^2$ factor in the above expression such that if $(\varepsilon_1,\dots,\varepsilon_k),(\nu_1,\dots,\nu_k)\in\{0,1\}^k$, then

$$\mathfrak{s}(x^1_{\varepsilon_1}\otimes\cdots\otimes x^k_{\varepsilon_k})-\mathfrak{s}(x^1_{\nu_1}\otimes\cdots\otimes x^k_{\nu_k})=\sum_{j=1}^{k}(\varepsilon_j-\nu_j)\cdot l.$$

In other words, there is an identification between $\mathrm{Spin}^c(T(1,0;n))$ and $\mathbb{Z}$ such that $\mathfrak{s}(x^1_{\varepsilon_1}\otimes\cdots\otimes x^k_{\varepsilon_k})=\sum_{j=1}^k\varepsilon_j$, which proves Equation (9-1) for $(p,q)=(1,0)$. Indeed, for every $0\le i\le k$ the number of generators whose Spin^c–structure is i equals the number of length k sequences of zeros and ones whose sum is i, which is $\binom{k}{i}$.

In light of formula (9-2), we only have to determine $SFH(T(p,q;2))$. The lens space $L(p,q)$ is obtained from $(M,\gamma)=T(p,q)$ by gluing an $S^1\times D^2$ to M such that the meridian $\{1\}\times D^2$ maps to one component, say α, of $s(\gamma)$. Let the knot K be the image of $S^1\times\{0\}$ in $L(p,q)$. Then the sutured manifold $L(p,q)(K)$ (see [13, Example 2.4]) is exactly $T(p,q;2)$. Hence we only have to find a knot diagram for the knot $K\subset L(p,q)$. But this has already been done in [20, Proof of Proposition 1.8]. First, observe that K is isotopic to a curve on the Heegaard surface $T^2=S^1\times S^1$ that intersects α in a single point. Let β be a meridian of M that intersects α in exactly p points. Then (T^2,α,β) is a Heegaard diagram for $L(p,q)$. As we go around α, label the points of $\alpha\cap\beta$ with $y_0,\dots,y_{p-1}$. For $0\le s\le p-1$, let A_s be the segment of $\alpha\setminus\{y_0,\dots,y_{p-1}\}$ connecting y_s and y_{s+1}, where y_p is by definition the same as y_0. Choose basepoints z and w on the two sides of A_{p-1}. Then I claim that (T^2,α,β,z,w) is a knot diagram defining K. Indeed, if we connect z to w in

$T^2 \setminus \alpha$ with an arc, then w to z in $T^2 \setminus \beta$ with a short arc that intersects α in a single point, then we obtain a simple closed curve on T^2 that intersects α in a single point, and hence is isotopic to the knot K. Let Σ be T^2 with two small open disks removed around z and w. Then the previous argument implies that (Σ, α, β) is a sutured diagram defining $T(p,q;2)$.

It is immediate that

$$SFH(T(p,q;2)) \cong \mathbb{Z}^p,$$

which is generated by $y_0, \dots, y_{p-1}$. Indeed, if we connect y_s and y_{s+1} along α using A_s and then on β with an arbitrary curve, then we get a curve on Σ whose homology class in $H_1(M)$ is l if $0 \le s < p-1$, and is $-(p-1)l$ if $s = p$ (this is because components of $\partial\Sigma$ represent $\pm pl$ in $H_1(M)$). So $\mathfrak{s}(y_{s+1}) - \mathfrak{s}(y_s) = l$ for $0 \le s < p-1$. This verifies Equation (9-1) for $n = 2$ and (p,q) arbitrary.

To get Equation (9-1) in general, glue $(M_1, \gamma_1) = T(1,0;n)$ and $(M_2, \gamma_2) = T(p,q;2)$ using formula (9-2), and apply Proposition 5.4 to see what happens to the Spin^c grading. If l_i denotes a generator of $H_1(M_i)$ for $i = 1, 2$, then l_1 is identified with pl_2 when we glue M_1 and M_2 along one of their sutures. Consider the decomposition

$$T(p,q;n) \rightsquigarrow^A (M_1, \gamma_1) \sqcup (M_2, \gamma_2).$$

Using the above, $SFH((M_1, \gamma_1) \sqcup (M_2, \gamma_2))$ is supported on the rectangle

$$\{0, \dots, k\} \times \{0, \dots, p-1\}$$

inside the lattice $\mathbb{Z}^2$ generated by l_1 and l_2. For the Spin^c–structure corresponding to the point (i,j) in the support, the Künneth formula gives

$$SFH((M_1, \gamma_1) \sqcup (M_2, \gamma_2), (i,j)) \cong \mathbb{Z}^{\binom{k}{i}} \otimes \mathbb{Z} \cong \mathbb{Z}^{\binom{k}{i}}.$$

Now we apply Proposition 5.4. The affine map

$$f_A\colon \text{Spin}^c((M_1, \gamma_1) \sqcup (M_2, \gamma_2)) \to \text{Spin}^c(T(p,q;n))$$

can be identified with the projection from $\mathbb{Z}^2$ to $\mathbb{Z}$ given by $f_A(1,0) = p$ and $f_A(0,1) = 1$. So for $0 \le i \le k$ and $0 \le j \le p-1$, we have $f_A(i,j) = pi + j$. This map is injective when restricted to the rectangle $\{0, \dots, k\} \times \{0, \dots, p-1\}$, hence

$$SFH(T(p,q;n), f_A(i,j)) \cong \mathbb{Z}^{\binom{k}{i}}.$$

This implies that $$SFH(T(p,q;n), s) \cong \mathbb{Z}^{\binom{k}{\lfloor s/p \rfloor}}$$

if $0 \le s \le p(k+1)$, and is zero otherwise.

The above argument actually tells us how to obtain an explicit sutured diagram for $T(p,q;n)$. Let $(\Sigma_1,\alpha_1,\beta_1),\dots,(\Sigma_k,\alpha_k,\beta_k)$ be k identical copies of the sutured diagram shown in Figure 4, and introduce the notation $(\Sigma_0,\alpha_0,\beta_0)$ for the diagram defining $T(p,q;2)$ described above. For $0\le j<k$, let c_j be a fixed component of $\partial\Sigma_j$, and for $1\le h\le k$, let d_h be a component of $\partial\Sigma_h$ distinct from c_h. Then we obtain Σ from $\coprod_{j=0}^{k}\Sigma_j$ by identifying c_j with d_{j+1} for $0\le j<k$. Finally, let $\boldsymbol{\alpha}=\{\alpha_0,\alpha_1,\dots\alpha_k\}$ and $\boldsymbol{\beta}=\{\beta_0,\beta_1,\dots,\beta_k\}$. Then $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a sutured diagram defining $T(p,q;n)$.

Every intersection point in $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ is of the form $y_s\times x^1_{\varepsilon_1}\times\cdots\times x^k_{\varepsilon_k}$, where $0\le s\le p-1$ and $(\varepsilon_1,\dots,\varepsilon_k)\in\{0,1\}^k$. There is an identification $\mathrm{Spin}^c(T(p,q;n))\cong\mathbb{Z}$ such that

$$\mathfrak{s}(y_s\times x^1_{\varepsilon_1}\times\cdots\times x^k_{\varepsilon_k})=s+\varepsilon_1+\cdots+\varepsilon_k.$$

Therefore $\mathfrak{s}(y_s\times x^1_{\varepsilon_1}\times\cdots\times x^k_{\varepsilon_k})=\mathfrak{s}(y_{s'}\times x^1_{\nu_1}\times\cdots\times x^k_{\nu_k})$ if and only if $s=s'$ and $\varepsilon_1+\cdots+\varepsilon_k=\nu_1+\cdots+\nu_k$. Since we already know the total rank of $SFH(T(p,q;n))$ by Equation (9-2), this gives another, more direct proof of formula (9-1). To show that

$$\mu(y_s\times x^1_{\varepsilon_1}\times\cdots\times x^k_{\varepsilon_k},\, y_s\times x^1_{\nu_1}\times\cdots\times x^k_{\nu_k})=0 \tag{9-3}$$

for any two intersection points lying in the same Spin^c structure, it suffices to check the following. If $1\le t<k$ is fixed and $\nu_j\equiv\varepsilon_j+1 \mod 2$ for $j=t$ and $j=t+1$; furthermore, $\nu_j=\varepsilon_j$ for every other $1\le j\le k$, then Equation (9-3) holds. To see this, look at the region D in $\Sigma\setminus(\bigcup\boldsymbol{\alpha}\cup\bigcup\boldsymbol{\beta})$ whose corners are x^t_0,x^t_1,x^{t+1}_0, and x^{t+1}_1. This is obtained from the punctured bigons in Σ_t and Σ_{t+1} containing c_t and d_{t+1}, respectively, by gluing c_t to d_{t+1}. Let $\mathcal{D}$ be the domain whose multiplicity in D is one and is zero everywhere else. Then $\mathcal{D}$ connects $y_s\times x^1_{\varepsilon_1}\times\cdots\times x^k_{\varepsilon_k}$ and $y_s\times x^1_{\nu_1}\times\cdots\times x^k_{\nu_k}$; moreover, Lipshitz's Maslov index formula [14, Proposition 7.3] tells us that $\mu(\mathcal{D})=0$. This concludes the proof of our claim about the relative Maslov index being zero within a given Spin^c structure. □

The same way as we proved formula (9-2), we can obtain the following.

Proposition 9.2 *Suppose that (M,γ) is a balanced sutured manifold, and let γ_0 be a component of γ. If (M,γ_1) is obtained from (M,γ) by adding two sutures parallel to γ_0, then*

$$SFH(M,\gamma_1)\cong SFH(M,\gamma)\otimes\mathbb{Z}^2.$$

Remark 9.3 In [3, Chapter 8], we compute $SFH(M,\gamma,\mathfrak{s})$ for every $\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)$ when (M,γ) is a sutured manifold complementary to a pretzel surface, so M is a genus two handlebody. These examples illustrate well how complicated the support $S(M,\gamma)$ of sutured Floer homology can be in general.

References

[1] **D Cooper**, **D D Long**, *Virtually Haken Dehn-filling*, J. Differential Geom. 52 (1999) 173–187 MR1743462

[2] **Y M Eliashberg**, **W P Thurston**, *Confoliations*, Univ. Lecture Ser. 13, Amer. Math. Soc. (1998) MR1483314

[3] **S Friedl**, **A Juhász**, **J Rasmussen**, *The decategorification of sutured Floer homology* arXiv:0903.5287

[4] **D Gabai**, *Foliations and the topology of 3–manifolds*, J. Differential Geom. 18 (1983) 445–503 MR723813

[5] **D Gabai**, *Foliations and genera of links*, Topology 23 (1984) 381–394 MR780731

[6] **D Gabai**, *Detecting fibred links in S^3*, Comment. Math. Helv. 61 (1986) 519–555 MR870705

[7] **D Gabai**, *Foliations and the topology of 3–manifolds. II*, J. Differential Geom. 26 (1987) 461–478 MR910017

[8] **D Gabai**, *Foliations and the topology of 3–manifolds. III*, J. Differential Geom. 26 (1987) 479–536 MR910018

[9] **P Ghiggini**, *Knot Floer homology detects genus-one fibred knots*, Amer. J. Math. 130 (2008) 1151–1169 MR2450204

[10] **M Hedden**, **A Juhász**, **S Sarkar**, *On sutured Floer homology and the equivalence of Seifert surfaces* arXiv:0811.0178

[11] **K Honda**, **W H Kazez**, **G Matić**, *Convex decomposition theory*, Int. Math. Res. Not. (2002) 55–88 MR1874319

[12] **W H Jaco**, **P B Shalen**, *Seifert fibered spaces in 3–manifolds*, Mem. Amer. Math. Soc. 21 (1979) viii+192 MR539411

[13] **A Juhász**, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. 6 (2006) 1429–1457 MR2253454

[14] **A Juhász**, *Floer homology and surface decompositions*, Geom. Topol. 12 (2008) 299–350 MR2390347

[15] **Y Ni**, *Knot Floer homology detects fibred knots*, Invent. Math. 170 (2007) 577–608 MR2357503

[16] **Y Ni**, *Erratum: Knot Floer homology detects fibred knots* [MR2357503], Invent. Math. 177 (2009) 235–238 MR2507641

[17] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR2023281

[18] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR2065507

[19] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) 159 (2004) 1027–1158 MR2113019

[20] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR2168576

[21] **P Ozsváth**, **Z Szabó**, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. 8 (2008) 615–692 MR2443092

[22] **P Ozsváth**, **Z Szabó**, *Link Floer homology and the Thurston norm*, J. Amer. Math. Soc. 21 (2008) 671–709 MR2393424

[23] **J A Rasmussen**, *Floer homology and knot complements*, PhD Thesis, Harvard University, ProQuest LLC, Ann Arbor, MI (2003) MR2619970

[24] **M Scharlemann**, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom. 29 (1989) 557–614 MR992331

[25] **W P Thurston**, *A norm for the homology of* 3*–manifolds*, Mem. Amer. Math. Soc. 59 (1986) i–vi, 99–130 MR823443

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