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Heegaard surfaces and the distance of amalgamation

TAO LI

Let M_1 and M_2 be orientable irreducible 3–manifolds with connected boundary and suppose $\partial M_1 \cong \partial M_2$. Let M be a closed 3–manifold obtained by gluing M_1 to M_2 along the boundary. We show that if the gluing homeomorphism is sufficiently complicated, then M is not homeomorphic to S^3 and all small-genus Heegaard splittings of M are standard in a certain sense. In particular, $g(M) = g(M_1) + g(M_2) - g(\partial M_i)$, where $g(M)$ denotes the Heegaard genus of M. This theorem is also true for certain manifolds with multiple boundary components.

57N10; 57M50

1 Introduction

One of the most useful ways of constructing a new 3–manifold is to glue two given 3–manifolds with boundary via a homeomorphism between their boundary surfaces. This construction is called amalgamation. Dehn filling and Heegaard splitting can be viewed as examples of such a construction. In this paper, we study Heegaard splittings of 3–manifolds obtained by amalgamation. Like Dehn filling, the 3–manifold obtained by amalgamation depends on the gluing homeomorphism. We will show that if the gluing homeomorphism is sufficiently complicated, then the small-genus Heegaard splittings of the resulting 3–manifold are standard.

The complexity of the gluing homeomorphism is defined using the curve complex. The curve complex of F, introduced by Harvey [6], is defined as follows. Let F be a closed orientable connected surface. The curve complex of F is the complex whose vertices are the isotopy classes of essential simple closed curves in F. If the genus of F is at least 2, then $k+1$ vertices in the curve complex determine a k–simplex if they are represented by pairwise disjoint curves. If F is a torus, then $k+1$ vertices determine a k–simplex if they are represented by curves that pairwise meet exactly once. Clearly the curve complex of the torus is the same as the Farey graph. We denote the curve complex of F by $\mathcal{C}(F)$. For any two vertices in $\mathcal{C}(F)$, the distance $d(x, y)$ is the minimal number of 1–simplices in a simplicial path jointing x to y. To simplify notation, unless necessary, we do not distinguish a vertex in $\mathcal{C}(F)$ from a simple closed curve in F representing this vertex.

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Let M_1 and M_2 be orientable irreducible 3–manifolds with boundary. Let F_i be a boundary component of M_i ($i = 1, 2$). In this paper, we suppose M_i is not a product $F_i \times I$ and $\partial M_i - F_i$ (if not empty) is incompressible in M_i. Suppose $F_1 \cong F_2 \cong F$. We can glue M_1 to M_2 via a homeomorphism ϕ: $F_1 \to F_2$ and obtain an orientable 3–manifold $M = M_1 \cup_\phi M_2$. We may view M_1 and M_2 as submanifolds of M and $F = M_1 \cap M_2$ as a closed nonperipheral surface embedded in M.

Definition 1.1 Let M_1, M_2, M and F be as above. If F is compressible in M_i, the *disk complex* of M_i is the set of vertices in $\mathcal{C}(F)$ represented by curves bounding compressing disks in M_i. If M_i is a twisted I–bundle over a closed nonorientable surface, the *annulus complex* of M_i is the set of vertices in $\mathcal{C}(F)$ represented by boundary curves of vertical annuli in M_i. If M_i has incompressible boundary and M_i is not a twisted I–bundle over a closed nonorientable surface, we fix a properly embedded essential surface Ω_i in M_i with $\partial\Omega_i \cap F \neq \varnothing$ and suppose the Euler characteristic $\chi(\Omega_i)$ is maximal among all such essential surfaces. We define $\mathcal{U}_i$ to be the set of vertices in $\mathcal{C}(F)$ as follows:

$$\mathcal{U}_i = \begin{cases} \text{the disk complex of } M_i & \text{if } F \text{ is compressible in } M_i, \\ \text{the annulus complex of } M_i & \text{if } M_i \text{ is a twisted } I\text{–bundle}, \\ \text{vertices represented by components of } \partial\Omega_i \cap F & \text{otherwise.} \end{cases}$$

We define the distance of the amalgamation to be $d(M) = d(\mathcal{U}_1, \mathcal{U}_2)$ in the curve complex $\mathcal{C}(F)$.

Note that the surface Ω_i in Definition 1.1 is not unique, but we will show in Section 3 that, if M_i has incompressible boundary and is not an I–bundle, then the diameter of the set of vertices in $\mathcal{C}(F)$ represented by boundary curves of such essential surfaces is bounded. Thus any different choice of Ω_i only changes $d(M)$ by an explicit small number. If both M_1 and M_2 are handlebodies or more generally if F is compressible in both M_1 and M_2, then $d(M)$ is the same as the Hempel distance; see Hempel [7] and Scharlemann and Thompson [26]. Schleimer informed the author that, similar to the disk complex, the annulus complex of a twisted I–bundle is also quasi-convex in $\mathcal{C}(F)$. So $d(M)$ is arbitrarily large if the gluing map ϕ is a sufficiently high power of a pseudo-Anosov map. Like the Hempel distance, $d(M)$ also provides a natural complexity measure for a one-sided Heegaard splitting, ie, a decomposition of M into a handlebody and a twisted I–bundle.

If one prefers, the following is a roughly equivalent way of defining $d(M)$ which does not involve a choice of Ω_i. Let k_i be the maximal Euler characteristic of essential orientable surfaces properly embedded in M_i and with at least one boundary component in F (we consider compressing disks as essential surfaces). Let $\mathcal{U}_i$ be the set of vertices in $\mathcal{C}(F)$ represented by boundary curves of such essential surfaces whose Euler characteristic is k_i. Then one can define $d(M) = d(\mathcal{U}_1, \mathcal{U}_2)$. If $k_i = 1$, then $\mathcal{U}_i$ is the disk complex of M_i. If M_i has incompressible boundary and is not a twisted I–bundle, then by Section 3, the diameter of $\mathcal{U}_i$ is bounded by a number depending only on k_i.

Theorem 1.2 *Let $M = M_1 \cup_F M_2$ be as above. Then there is a number K depending on M_1 and M_2 such that if $d(M) \geq K$ then*

(1) *M is irreducible and ∂–irreducible, and*

(2) *M is not homeomorphic to S^3.*

Similar to Dehn surgery, one can perform a surgery on a graph in S^3. An immediate corollary of Theorem 1.2 is that given a graph Γ in S^3, if one performs a complicated surgery on Γ, ie, gluing back a handlebody to $S^3 - N(\Gamma)$ via a high-distance map, then the resulting closed 3–manifold is irreducible and cannot be S^3.

Definition 1.3 Let N be a compression body and F a closed separating surface properly embedded in N. The surface F cuts N into two submanifolds N_1 and N_2 and we may view F as a boundary component of each N_i. Suppose F is not a 2–sphere and $\partial_+ N \subset \partial N_1$. We say F is a *middle surface* in N if both N_1 and N_2 are compression bodies, $\partial_+ N = \partial_+ N_1$, $\partial_- N_2 \subseteq \partial_- N$ and $F = \partial_+ N_2 \subseteq \partial_- N_1$. Note that if one views a compression body as a manifold obtained by adding 2–handles and 3–handles to $\partial_+ N \times I$ on the same side, then a middle surface is a middle level of this process. In particular, one can find a handle structure of N such that N_1 is a compression body obtained by adding a subset of the 2– and 3–handles to $\partial_+ N \times I$, and after adding the remaining 2– and 3–handles along F, we obtain the whole of N. Note that unless F is parallel to a component of $\partial_- N$, F is incompressible in N_1 but compressible in N_2.

Next we consider the untelescoping of a Heegaard splitting; see Scharlemann and Thompson [25] and Scharlemann [22]. Let $M = V \cup_S W$ be an irreducible Heegaard splitting. We may view the compression body V as the manifold obtained by attaching 1–handles to either a product neighborhood of $\partial_- V$ or to a 0–handle; and view W as the manifold obtained by attaching 2–handles and possibly a 3–handle to a product neighborhood of $S = \partial_+ W$. So a Heegaard splitting gives a natural handle-decomposition of M. The untelescoping of the Heegaard splitting is a rearrangement of the order in which these handles are attached. This rearrangement gives a decomposition of M into submanifolds $N_1, \dots, N_m$ along incompressible surfaces, and each N_i inherits a strongly irreducible Heegaard splitting from a subset of the original 1– and 2–handles; see Scharlemann and Thompson [25] for details. The decomposition is often called a generalized Heegaard splitting. We summarize this as the following theorem. Note that by part (1) of Theorem 1.2, if the gluing map is sufficiently complicated, then M has incompressible boundary. So in this paper we only consider the case that ∂M (if not empty) is incompressible, though untelescoping is also defined for manifolds with compressible boundary.

Theorem 1.4 (Scharlemann–Thompson [25]) *Let M be an irreducible and orientable 3–manifold with incompressible boundary. Let S be an unstabilized Heegaard surface of M. Then the untelescoping of the Heegaard splitting described above gives a decomposition of M as follows; see Figure 1 for a picture.*

(1) *$M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M.*

(2) *Each $N_i = A_i \cup_{P_i} B_i$, where each A_i and B_i is a union of compression bodies with $\partial_+ A_i = P_i = \partial_+ B_i$ and $\partial_- A_i = F_i = \partial_- B_{i-1}$.*

(3) *Each component of P_i is a strongly irreducible Heegaard surface of a component of N_i.*

(4) *No component of A_i and B_i is a trivial compression body (ie a product).*

(5) *The genus $g(F_i) < g(S)$ and $g(P_i) \leq g(S)$ for each i.*

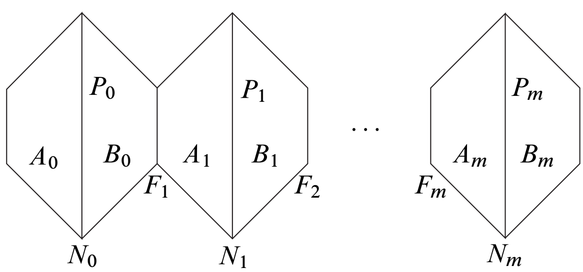

Figure 1

Let F be a closed connected surface embedded in M. We say F is a *canonical surface with respect to the untelescoping* if F is parallel to a middle surface in a component of A_i or B_i, for some i. Note that a component of P_i is a (trivial) middle surface for both A_i and B_i, and any component of F_i is a middle surface for both B_{i-1} and A_i. The main theorem of the paper is:

Theorem 1.5 *Let $M = M_1 \cup_F M_2$ be as above. In particular, suppose M_i is not a product $F \times I$ and suppose $\partial M_i - F$ (if not empty) is incompressible in M_i. Then for any integer g, there is a number K depending on M_1, M_2 and g, such that if $d(M) > K$, then for any unstabilized Heegaard surface S of M with $g(S) \leq g$, F is isotopic to a canonical surface with respect to any untelescoping of S.*

A corollary of Theorem 1.5 is a formula for the Heegaard genus.

Corollary 1.6 *Let $M = M_1 \cup_F M_2$ be as in Theorem 1.5. Then there is a number K depending on M_1 and M_2 such that if $d(M) > K$, $g(M) = g(M_1) + g(M_2) - g(F)$.*

It follows from the proof that the number K in Theorem 1.5 and Corollary 1.6 can be chosen to be an explicit quadratic function of g and $\chi(\Omega_i)$, where Ω_i is as in Definition 1.1. The bound K depends on several distance estimates in various places in the paper. Lemma 3.7 is the only place where the bound is quadratic and all other estimates are linear functions.

If both M_1 and M_2 are simple, ie, irreducible, ∂–irreducible, atoroidal and anannular, then Theorem 1.5 is a generalization of a theorem of Lackenby [9] and is proved by Souto [28] and the author [14]. Note that the complexity measure in [14] is defined using boundary curves of normal surfaces; see the author's paper [13] for a relation between Heegaard surfaces and normal surfaces.

Our motivation for the main theorem is to study the Heegaard genus of closed 3–manifolds. The following are special cases of Theorem 1.5.

Corollary 1.7 *Let M_1 and M_2 be orientable irreducible 3–manifolds with connected boundary and suppose $\partial M_1 \cong \partial M_2$. Let M be a closed 3–manifold obtained by gluing M_1 to M_2 along the boundary. Then for any integer g, there is a number K depending on M_1, M_2 and g, such that if $d(M) > K$, then for any unstabilized Heegaard surface S of M with $g(S) \leq g$, F is isotopic to a canonical surface with respect to any untelescoping of S. Moreover, $g(M) = g(M_1) + g(M_2) - g(F)$.*

Corollary 1.8 *Let M, M_1 and M_2 be as above. If M_1 is a handlebody, then there is a number K depending on M_2 and g such that if $d(M) > K$, any Heegaard surface of M with genus at most g is isotopic to a Heegaard surface of M_2. In particular, $g(M) = g(M_2)$.*

Corollary 1.8 says that if the gluing map is complicated, then there is no new small-genus Heegaard surface in the resulting 3–manifold M. In particular, if M_1 is a solid torus and M_2 is a knot manifold, Corollary 1.8 gives a weaker version of a known result on Heegaard structure and Dehn filling; see Moriah and Rubinstein [18], Moriah and Sedgwick [19] and Rieck and Sedgwick [20].

Theorem 1.5 and its proof give useful ways of constructing new 3–manifolds with certain control on the Heegaard genus. This may shed light on constructing counterexamples (or examples) for the rank conjecture, which asserts that for a closed (hyperbolic) 3–manifold, the rank of its fundamental group equals its Heegaard genus. For example, if one can construct an example of 3–manifold N with connected boundary whose rank is smaller than its Heegaard genus, then by Corollary 1.8, one can obtain a closed 3–manifold $\widehat{N}$ by capping off ∂N using a handlebody and via a sufficiently complicated gluing map, such that $\mathrm{rank}(\widehat{N}) < g(\widehat{N})$. Very recently, using hyperbolic JSJ pieces, the author has constructed examples of closed 3–manifolds with rank smaller than genus [10]. These are the first such examples having hyperbolic JSJ pieces and a main tool in the construction is Theorem 1.5. It is conceivable that this method may be generalized to give a hyperbolic counterexample to the rank conjecture.

The proof of Theorem 1.5 is also used by Bachman in [1] to study stabilization of Heegaard splittings.

In the proof of the main theorem, we study how the amalgamation surface F intersects the incompressible and strongly irreducible surfaces in the untelescoping of a Heegaard splitting. In particular, we will show that if the amalgamation distance $d(M)$ is sufficiently large, then any small-genus closed incompressible surfaces in M can be isotoped disjoint from F. So by our definition of $d(M)$, it is natural to study the distance between $\partial\Omega_i$ and the boundary curves of incompressible or strongly irreducible surfaces in M_i ($i = 1, 2$).

A basic observation in the proof is that the diameter in the curve complex of the vertices represented by the boundary curves of certain small-genus surfaces is bounded. More precisely, let N be a 3–manifold with incompressible boundary (which is not an I–bundle) and let F be a boundary component of N. We consider the set of surfaces in N that are either essential or strongly irreducible and ∂–strongly irreducible (see Definition 3.3) with bounded genus and bounded number of components in $\partial N - F$. The observation is that the diameter in the curve complex $\mathcal{C}(F)$ of the boundary curves of such surfaces is bounded. This is proved in Section 3.

In Section 4, we use this observation to prove Theorem 1.5 in the case that F is incompressible in both M_1 and M_2. The case that F is compressible in both M_1 and M_2 basically follows from a theorem of Scharlemann and Tomova [26] and this is discussed in Section 5. The most difficult case in Theorem 1.5 is that F is compressible on one side but incompressible on the other side. The last two sections are devoted to this case.

In Section 6, we show that if the distance of the amalgamation $d(M)$ is large, then F can be isotoped disjoint from all the incompressible surfaces (ie the F_i's in Theorem 1.4) in the untelescoping. In Section 6, we also discuss the case that F is disjoint from all the strongly irreducible Heegaard surfaces P_i's, and prove that in a certain generic situation, F is isotopic to a middle surface of a compression body in the Heegaard splitting of N_i (see Theorem 1.4). In Section 7, we study how F intersects the sweepout of the strongly irreducible Heegaard splitting of N_i in the untelescoping and finish the proof of the last case.

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2 A genus calculation

Notation 2.1 Throughout this paper, we denote the interior of X by $\mathrm{int}(X)$, the closure of X (under the path-metric) by $\overline{X}$, and the number of components of X by $|X|$ for any space X.

We first show that Corollary 1.6 follows from Theorem 1.5. For simplicity, we suppose M_1 and M_2 have only one boundary component, ie, M is a closed 3–manifold. If M has boundary, by Theorem 1.2, ∂M is incompressible in M and one may cap off each component of ∂M by a handlebody and calculate the Heegaard genus same as the case that M is closed.

Suppose M is closed. By Theorem 1.2, we may assume $M = M_1 \cup_F M_2$ is irreducible and is not S^3. Let S be an unstabilized Heegaard surface of M. Let $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$ and $N_i = A_i \cup_{P_i} B_i$ be the decompositions in an untelescoping of the Heegaard splitting; see Theorem 1.4 and Figure 1. As in [25; 22], one can rearrange the handle structure determined by the Heegaard splitting along S so that the sub-collection of 1– and 2–handles which occur in N_i determine the Heegaard splitting $N_i = A_i \cup_{P_i} B_i$.

Suppose S is a minimal genus Heegaard surface of M and let g be its genus. Suppose F is canonical with respect to the untelescoping of S as above. Without loss of generality, we may suppose F lies in the compression body B_j between P_j and F_{j+1} in the untelescoping; see Figure 1. We may assume B_j is connected. By the definition of middle surface, F separates B_j into two compression bodies and we can choose a handle structure for B_j so that the 2–handles in the two compression bodies are exactly the 2–handles for B_j. Next we count the handles in M_1 and M_2.

Let a_i, b_i, c_i and d_i $(i = 1, 2)$ be the numbers of 0–, 1–, 2– and 3–handles in M_i respectively in the handle decomposition determined by the Heegaard surface S as above. The total number of 0–handles is $a_1 + a_2$ and the total number of 1–handles is $b_1 + b_2$. So the Heegaard genus $g = (b_1 + b_2) - (a_1 + a_2) + 1 = (c_1 + c_2) - (d_1 + d_2) + 1$.

Since F and M_1 are connected, as in [25], one can rearrange the 0– and 1–handles in M_1 to form a connected handlebody and obtain a Heegaard splitting of M_1 with genus $g_1 = b_1 - a_1 + 1$. Hence $g(M_1) \leq b_1 - a_1 + 1$. Similarly, one can rearrange the 2– and 3–handles in M_2 to form a handlebody and obtain a Heegaard splitting of M_2 with genus $g_2 = c_2 - d_2 + 1$. Hence $g(M_2) \leq c_2 - d_2 + 1$. Moreover, an easy calculation of the Euler characteristic of M_1 yields $g(F) = 1 - a_1 + b_1 - c_1 + d_1$. Therefore, $g(M_1) + g(M_2) - g(F) \leq (b_1 - a_1 + 1) + (c_2 - d_2 + 1) - (1 - a_1 + b_1 - c_1 + d_1) = (c_1 + c_2) - (d_1 + d_2) + 1 = g = g(M)$.

Given two minimal-genus Heegaard splittings of M_1 and M_2, the amalgamation of the two splittings yields a Heegaard splitting of M with genus $g(M_1)+g(M_2)-g(F)$; see Lackenby [9], Li [14] and Scharlemann [27] for more detailed description. This means that $g(M) \leq g(M_1)+g(M_2)-g(F)$. So the equality $g(M) = g(M_1)+g(M_2)-g(F)$ holds.

3 Intersection of small surfaces

In this section, we prove several lemmas on the intersection of certain small-genus surfaces. These lemmas will be used in the later sections.

In this section, we fix an orientable irreducible compact connected 3–manifold N with incompressible boundary. We also fix a component of ∂N and denote it by F.

Definition 3.1 We define the annulus complex $\mathcal{A}_N(F)$ to be the subcomplex of $\mathcal{C}(F)$ consisting of vertices represented by boundary curves of essential annuli in N. Note that we only consider those essential annuli with at least one boundary component in F.

The following lemma is also proved by the author in [12].

Lemma 3.2 *Suppose N is not an I–bundle. Then the diameter of the annulus complex $\mathcal{A}_N(F)$ in $\mathcal{C}(F)$ is at most 2.*

Proof Let J be an I–bundle in N with its horizontal boundary $\partial_h J$ in ∂N and its vertical boundary consisting of essential annuli properly embedded in N. Suppose $\partial_h J \cap F \neq \varnothing$. Note that if N contains an essential annulus A with at least one boundary component in F, then a small neighborhood of A is such an I–bundle. We may suppose J is maximal up to isotopy. This is basically from the theory of characteristic submanifolds; see Jaco [8].

As N is not an I–bundle, $J \neq N$. By our assumption, $J \cap F \subset \partial_h J$ and any component of $\partial(J \cap F)$ is a boundary component of an essential annulus in N. Let A' be a vertical boundary component of J. So A' is an essential annulus in N and $\partial A' \cap F \subset \partial(J \cap F)$. Let A be any other essential annulus in N with at least one boundary component in F and we consider $A \cap A'$. Since A and A' are both essential annuli, no component of $A \cap A'$ can be essential in one annulus but trivial in the other annulus. If $A \cap A'$ contains a closed curve that is trivial in both annuli, then there is such a curve c that is innermost in A' and bounding disks $\Delta \subset A$ and $\Delta' \subset A'$. Since c is innermost in A', $\mathrm{int}(\Delta') \cap A = \varnothing$ and $\Delta \cup \Delta'$ is an embedded S^2 in N. Since N is irreducible, $\Delta \cup \Delta'$ must bound a 3–ball. Hence we can perform an isotopy on A, pushing Δ across the 3–ball and eliminate the intersection curve c. So after isotopy, we may assume $A \cap A'$ contains no trivial closed curve. If $A \cap A'$ contains an arc that is trivial in both annuli, then there is such an arc α that is outermost in A'. Since α is trivial in both A and A', α and subarcs of ∂A and $\partial A'$ bound bigon disks d and d' in A and A' respectively. Moreover, since α is outermost in A' and $A \cap A'$ contains no trivial closed curve, $\mathrm{int}(d') \cap A = \varnothing$ and $d \cup d'$ is a disk properly embedded in N. Since ∂N is incompressible, the disk $d \cup d'$ must be ∂–parallel in N. Hence an isotopy on A that pushes d across the 3–ball bounded by $d \cup d'$ and ∂N can eliminate the intersection arc α. Thus after some isotopies as above, every arc or closed curve in $A \cap A'$ is essential in both annuli and this means that either $\partial A \cap \partial A' = \varnothing$ or $A \cap A'$ consists of arcs vertical in both A and A'.

If $\partial A \cap \partial A' \neq \varnothing$ after isotopy, then the union of a small neighborhood of $J \cup A$ and possibly some 3–balls yields a larger I–bundle contradicting the assumption that J is maximal; see [11, Section 2] for a more detailed argument. So $\partial A \cap \partial A' = \varnothing$ after isotopy. This means that, for any component γ of $\partial(J \cap F)$, $d(\gamma, \partial A \cap F) \leq 1$ and the lemma holds. □

Definition 3.3 Let N be an orientable irreducible compact connected 3–manifold with incompressible boundary as above. Let Q be a surface properly embedded in N and suppose Q is not a disk or 2–sphere. We say Q is *essential* if it is incompressible and ∂–incompressible. A properly embedded disk in N is essential if its boundary is an essential curve in ∂N, and a 2–sphere in N is essential if it does not bound a 3–ball in N. Since N is irreducible and ∂N is incompressible, N contains no essential disk or 2–sphere. Let P be a properly embedded separating surface in N and we allow P to be disconnected. Suppose the surface P decomposes N into two submanifolds X and Y, where X and Y are on different sides of P (note that X and Y may be disconnected). We say P is *strongly irreducible* if P has compressing disks on both sides, and each compressing disk in X meets each compressing disk in Y. We say P is *∂–strongly irreducible* if

(1) every compressing and ∂–compressing disk in X meets every compressing and ∂–compressing disk in Y, and

(2) there is at least one compressing or ∂–compressing disk on each side of P.

If P is strongly irreducible, then ∂P consists of curves essential in ∂N. To see this, suppose a component of ∂P is trivial in ∂N. Then an innermost such component bounds a disk in ∂N that is disjoint from every compressing disk on the other side of P. This contradicts that P is strongly irreducible.

Let P be a strongly irreducible and ∂–strongly irreducible surface in N and $\partial P \neq \varnothing$. Let X and Y be the closure of the two submanifolds of $N - P$ on different sides of P as in Definition 3.3. Since P is compressible on both sides, we may compress P in both X and Y. Let P^X and P^Y be the possibly disconnected surfaces obtained by maximally compressing P in X and Y respectively and removing all possible 2–sphere components. Some components of P^X and P^Y may be closed surfaces. Let P^X_∂ (resp. P^Y_∂) be the union of the components of P^X (resp. P^Y) with boundary.

For any ∂–parallel surface R in N, we denote by $\pi(R)$ the subsurface of ∂N that is bounded by ∂R and isotopic to R relative to ∂R. We say a collection of pairwise disjoint ∂–parallel surfaces $R_1, \ldots, R_m$ in N are *non-nested* if $\pi(R_1), \ldots, \pi(R_m)$ are pairwise disjoint in ∂N.

Lemma 3.4 *Let P, P^X, P^Y, P^X_∂ and P^Y_∂ be as above. Then*

(a) *P^X and P^Y are incompressible in N,*

(b) *a component of P^X_∂ is either ∂–parallel or can be changed into an essential surface after some ∂–compressions in X and deleting any resulting ∂–parallel components, and*

(c) *the ∂–parallel components of P^X_∂ are non-nested in X.*

Proof Since P is strongly irreducible, part (a) of the lemma follows from [23, Lemma 5.5]. Our task is to prove parts (b) and (c).

As P is separating, we call the two sides of P plus and minus sides and suppose P^X is on the plus side and P^Y is on the minus side. Moreover, any surface obtained by compression or ∂–compression on P inherits plus and minus sides. So P^X is a surface obtained by compressing P on the plus side.

Let Q be either P^X or a surface obtained from P^X by some ∂–compressions on the plus side (ie in X), and let Q' be a component of Q with boundary. By part (a), P^X is incompressible, hence Q and Q' are incompressible in N.

Next prove that Q' is ∂–incompressible on the minus side. The main reason for this is that P is ∂–strongly irreducible and the proof is similar to [23, Lemma 5.5].

Suppose Q' is ∂–compressible on the minus side and let D be a ∂–compressing disk for Q' on the minus side. By viewing P as a surface obtained from Q by adding some tubes and possibly some half tubes on the minus side, we may view the arc $\alpha = \partial D \cap Q'$ as an arc in P and view D as a disk transverse to P with $\partial D = \alpha \cup \beta$, $\alpha \subset P$, $\beta \subset \partial N$, and $\partial\alpha = \partial\beta$. Moreover, a neighborhood of α in D lies on the minus side of P. Note that if $\mathrm{int}(D) \cap P = \varnothing$, then D is a ∂–compressing disk for P on the minus side disjoint from a compressing disk of P on the plus side, contradicting that P is ∂–strongly irreducible. So $\mathrm{int}(D) \cap P \neq \varnothing$. Let $\gamma_1, \dots, \gamma_n$ be the closed curves in $\mathrm{int}(D) \cap P$ and let $\alpha_1, \dots, \alpha_k$ be the arcs in $(D - \alpha) \cap P$ with $\partial\alpha_i \subset \beta$ for each i. After isotopy, we may assume the γ_i's and α_i's are essential curves and arcs in P.

Since P is strongly irreducible, it follows from the proof of Scharlemann's no-nesting lemma [21, Lemma 2.2] (also see [23, Lemma 5.5]), after some isotopy (one can also use the isotopy described below), we may assume the closed curves γ_i's are not nested in D. Let $\delta_1, \dots, \delta_n$ be the subdisks of D bounded by $\gamma_1, \dots, \gamma_n$ respectively. Each α_i and a subarc of β bound a subdisk D_i of D. Since P can be viewed as the surface obtained from Q by adding tubes and half tubes on the minus side corresponding to the compressions and ∂–compressions on the plus side, we may assume that (1) each δ_i is a compressing disk for P in X (ie on the plus side), and (2) if $\mathrm{int}(D_i) \cap P = \varnothing$, D_i is a ∂–compressing disk for P in X (ie on the plus side).

If some δ_i's lie inside some D_j, since the δ_i's are non-nested in D, there must be a disk D_j such that $\mathrm{int}(D_j) \cap P \neq \varnothing$ but those disks D_i's and δ_i's that lie inside D_j are pairwise disjoint (ie non-nested in D_j). This assumption implies that a small neighborhood of α_j in D_j lies on the minus side of P (since those δ_i's and D_i's in D_j are in X). Thus, after replacing D by this disk D_j in our argument if necessary, we may assume all the disks D_i's and δ_i's are non-nested in D. Furthermore, we may assume $|\mathrm{int}(D) \cap P|$, the number of components of $\mathrm{int}(D) \cap P$, is minimal among all such disks D. Note that the isotopies above eliminating nested closed curves and trivial intersection curves all reduce $|\mathrm{int}(D) \cap P|$. So each δ_i is a compressing disk for P on the plus side and each D_i is a ∂–compressing disk for P on the plus side.

Since P is compressible on both sides, P has a compressing disk D' on the minus side. Since P is strongly irreducible and ∂–strongly irreducible, $\partial D' \cap \partial\delta_i \neq \varnothing$ and $\partial D' \cap \partial D_i \neq \varnothing$ for each i. Thus $D' \cap D \neq \varnothing$. We may assume D' is transverse to D.

After some isotopies, we may also assume $D' \cap D$ does not contain any closed curve. We may assume $|D \cap D'|$ is minimal up to isotopy. Let κ be an arc in $D' \cap D$ that is outermost in D', ie, κ and a subarc of $\partial D'$ bound a subdisk Δ of D' and $\mathrm{int}(\Delta) \cap D = \varnothing$. We have the following 8 cases to consider.

Case (1) If κ is an arc connecting two different circles γ_i and γ_j in D, then a simple isotopy that pushes D across Δ will merge γ_i and γ_j into one closed curve. This contradicts the assumption that $|\mathrm{int}(D) \cap P|$ is minimal.

Case (2) If κ is an arc connecting a circle γ_i and an arc α_j, then the same isotopy above merges γ_i and α_j into a single arc. This again contradicts that $|\mathrm{int}(D) \cap P|$ is minimal.

Case (3) The third case is that $\partial\kappa$ lies in the same circle $\gamma_i = \partial\delta_i$, as shown in Figure 2(a). After the same isotopy pushing D across Δ, the disk δ_i becomes an annulus $A \subset D$; see Figure 2(b). Moreover, A is properly embedded in X on the plus side of P (since $\delta_i \subset X$).

We denote the two circles of ∂A by c_1 and c_2, as shown in Figure 2(b). Let d_i be the disk bounded by c_i in D and suppose $d_1 \subset d_2$ and $d_2 - \mathrm{int}(d_1) = A$. If c_i is a trivial curve in P, then a simple isotopy on P and D can eliminate c_i, and $|D \cap D'|$ is reduced after all these operations while $|\mathrm{int}(D) \cap P|$ is either reduced or unchanged. So we may assume both c_1 and c_2 are essential curves in P. If $\mathrm{int}(d_1) \cap P = \varnothing$, then d_1 is a compressing disk in Y (since $A \subset X$) and d_1 can be isotoped disjoint from δ_i, a contradiction to the hypothesis that P is strongly irreducible. Thus we may assume $\mathrm{int}(d_1) \cap P \neq \varnothing$.

Since the circles γ_i's are non-nested, c_1 and those γ_j's in $\mathrm{int}(d_1)$ bound a planar surface $R \subset d_1$ and R is properly embedded in Y; see Figure 2(b). By a theorem of Scharlemann [21, Theorem 2.1 and Lemma 2.2] (also see [23, Lemma 5.5]), one can perform an isotopy to eliminate the nested circles in d_2. In fact, it follows from [21, Theorem 2.1 and Lemma 2.2] and [23, Lemma 5.5] that R must be ∂–parallel in Y. This contradicts the minimality assumption on $|\mathrm{int}(D) \cap P|$.

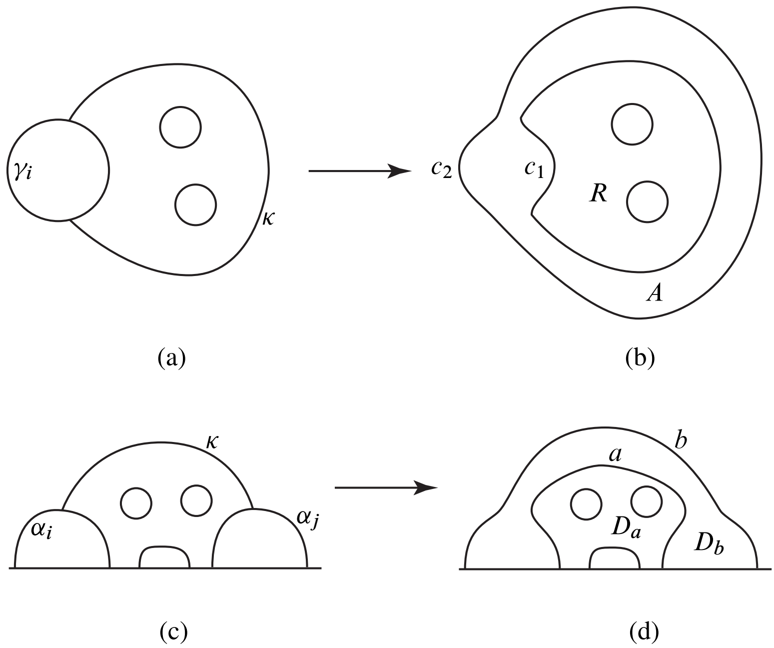

Figure 2

Case (4) Now we consider the case that $\partial\kappa$ lies in the same arc α_i. After the isotopy pushing D across Δ as above, α_i splits into an arc α_i' and a circle c. Let d be the subdisk of D bounded by c. By applying the same arguments for c_1 and d_1 in Case (3) to c and d, we get a contradiction to either the minimality of $|\mathrm{int}(D) \cap P|$ or the assumption that P is strongly irreducible.

Case (5) In this case we suppose κ is an arc connecting two different arcs α_i and α_j; see Figure 2(c). After the isotopy pushing D across Δ, α_i and α_j become a pair of arcs a and b with $\partial a \cup \partial b = \partial\alpha_i \cup \partial\alpha_j$, as shown in Figure 2(d). Let D_a and D_b be subdisks of D cut off by a and b respectively. By our construction, D_a and D_b are nested. Suppose $D_a \subset D_b$. If a is a trivial arc in P, then a simple isotopy on D can remove the intersection arc a and lead to a contradiction to the minimality of $|\mathrm{int}(D) \cap P|$. Suppose a is essential in P. If $\mathrm{int}(D_a) \cap P = \varnothing$, then D_a is a ∂–compressing disk for P on the minus side, since D_i and D_j are on the plus side. Moreover, we can perturb D_a to be disjoint from D_i and D_j, and this contradicts that P is ∂–strongly irreducible. Thus $\mathrm{int}(D_a) \cap P \neq \varnothing$. Since D_i and D_j lie on the plus side of P, a neighborhood of a in D_a is on the minus side of P. Since $|\mathrm{int}(D_a) \cap P| < |\mathrm{int}(D) \cap P|$, this again contradicts the minimality assumption on $|\mathrm{int}(D) \cap P|$.

Case (6) The final 3 cases deal with the situation that an endpoint of κ lies in $\alpha = \partial D \cap P$. We first suppose κ connects α and a circle γ_i in D. After pushing D across Δ as above, α and γ_i merge into a boundary arc α' of a new disk E, where $\partial\alpha' = \partial\alpha$, $\beta = \partial D \cap \partial N = \partial E \cap \partial N$ and $\partial E = \alpha' \cup \beta$. Note that after compressing P into P^X, the circle γ_i can be viewed as a trivial circle in Q' bounding a disk (in Q') corresponding to the compressing disk, hence α' can be viewed as an arc in Q' isotopic (in Q') to α. However, $|\text{int}(E) \cap P| < |\text{int}(D) \cap P|$ and we have a contradiction.

Case (7) If $\partial\kappa \subset \alpha$, then similar to Case (4), after pushing D across Δ, D splits into two disks E' and E'', and α splits into a closed curve $\gamma' = \partial E'$ and an arc $\alpha' \subset \partial E''$. Similar to Case (6), we may view γ' and α' as curves in Q'. The closed curve γ' must be trivial in Q', since Q' is incompressible. Hence α' is isotopic in Q' to α and the new disk E'' can be viewed as a ∂–compressing disk for Q' isotopic to D. However, κ is eliminated and $|D \cap D'|$ is reduced after the isotopy while $|\text{int}(E'') \cap P| \leq |\text{int}(D) \cap P|$. This contradicts our minimality assumption on $|D \cap D'|$.

Case (8) If κ connects α to an arc α_i in D, then after pushing D across Δ, similar to Case (5), α and α_i merge into a pair of new arcs α_a and α_b which are boundary arcs of two new disks E_a and E_b respectively ($\partial\alpha \cup \partial\alpha_i = \partial\alpha_a \cup \partial\alpha_b$, E_a and E_b correspond to the two components of $D - D_i - \kappa$). Recall that the arc α_i bounds a ∂–compressing disk D_i for P and P^X. By our construction of Q', after the ∂–compressions on P^X that we performed to get Q, we may view α_i as a ∂–parallel arc in Q' that cuts off a disk in Q' corresponding to the ∂–compressing disk D_i. Since α is an essential arc in Q' and α_i can be viewed as a trivial arc in Q', at least one of α_a and α_b is an essential arc in Q' bounding a ∂–compressing (E_a or E_b) for Q'. After replacing D by a new disk E_a or E_b above, we get a contradiction to the minimality assumption of $|\text{int}(D) \cap P|$.

Therefore Q' must be ∂–incompressible on the minus side. In particular each component of P^X_∂ is ∂–incompressible on the minus side. Since P^X_∂ is incompressible by part (a), a component of P^X_∂ is either ∂–parallel in X or can be changed into an essential surface after some ∂–compressions on the plus side and deleting any resulting ∂–parallel components. So part (b) of the lemma holds. If the ∂–parallel components of P^X_∂ are nested in X (ie the two product regions in X bounded by two ∂–parallel components are nested), then this means that after some ∂–compressions on P^X_∂ on the plus side, the resulting surface becomes ∂–compressible on the minus side, a contradiction to the conclusion above. Thus part (c) holds. □

Lemma 3.5 *Let P and Q be properly embedded orientable surfaces in N with at least one boundary component in F. Suppose Q is essential and P is strongly irreducible and ∂–strongly irreducible. Note that by the definition of strongly irreducible surface, P is separating. Then either*

(1) $(\partial P \cap F) \cap (\partial Q \cap F) = \varnothing$ *after isotopy,*

(2) *after some compressions and ∂–compressions on the same side of P, one can obtain an essential surface with a boundary component in F, or*

(3) *after some isotopy, $P \cap Q$ is essential in both P and Q and $|\partial P \cap \partial Q|$ is minimal among curves isotopic to ∂P and ∂Q.*

Proof Suppose part (1) of the lemma is not true. So we may assume that $(\partial P \cap F) \cap (\partial Q \cap F) \neq \varnothing$ and $|\partial P \cap \partial Q|$ is minimal among curves isotopic to ∂P and ∂Q in ∂N.

Let X and Y be the closure of the two submanifolds of $N - P$ as in Definition 3.3. Let P^X and P^Y be the possibly disconnected surfaces obtained by maximally compressing P in X and Y respectively and removing all possible 2–sphere components. Let P^X_∂ and P^Y_∂ be the unions of the components of P^X and P^Y with boundary respectively. Since P has at a boundary component in F, $P^X_\partial \cap F \neq \varnothing$ and $P^Y_\partial \cap F \neq \varnothing$.

By Lemma 3.4, a component of P^X_∂ is either ∂–parallel in X or can be changed to an essential surface by some ∂–compressions on the X–side. Furthermore, by Lemma 3.4, the ∂–parallel components of P^X_∂ are non-nested. Thus either part (2) of Lemma 3.5 holds or the components of P^X_∂ and P^Y_∂ incident to F are ∂–parallel and non-nested in X and Y respectively. Let P^X_F and P^Y_F be the components of P^X_∂ and P^Y_∂ respectively whose boundary lie in F. Suppose part (2) of Lemma 3.5 is not true, then as above, P^X_F and P^Y_F are ∂–parallel and non-nested in X and Y respectively. Note that by our hypotheses and assumptions, $P^X_F \neq \varnothing$ and $P^Y_F \neq \varnothing$.

Let N_P be the submanifold of N between P^X and P^Y and we may assume P is properly embedded in N_P. By the construction of P^X and P^Y, there are graphs $G_X \subset X \cap N_P$ and $G_Y \subset Y \cap N_P$, which correspond to the compressions on P in X and Y respectively, such that $N_P - (P^X \cup G_X \cup P^Y \cup G_Y)$ is a product $P \times (0,1)$. Let $\Sigma_X = P^X \cup G_X$ and $\Sigma_Y = P^Y \cup G_Y$. We may view this as a sweepout $H\colon P \times (I, \partial I) \to (N_P, \Sigma_X \cup \Sigma_Y)$, where $I = [0,1]$ and $H|_{P \times (0,1)}$ is an embedding. We denote $H(P \times \{a\})$ by P_a for any $a \in I$. We may assume $P_0 = \Sigma_X$ and $P_1 = \Sigma_Y$ and each P_a ($a \neq 0,1$) is isotopic to P.

Since P^X_F and P^Y_F are ∂–parallel and non-nested, and by our hypothesis that Q has a boundary component in F, we may assume $Q \cap P^X_F$ and $Q \cap P^Y_F$ consist of non-nested ∂–parallel arcs in Q. Since we have assumed at the beginning that $|\partial P \cap \partial Q|$

is minimal among curves isotopic to ∂P and ∂Q, we may assume the arcs in $Q \cap P_F^X$ and $Q \cap P_F^Y$ are essential in P_F^X and P_F^Y respectively. Moreover, after isotopy, we may assume that for each $t \in (0,1)$, Q is transverse to P_t except for at most one center or saddle tangency. We call $t \in (0,1)$ a regular level if Q is transverse to P_t, otherwise, we call t a singular level. We may assume there are only finitely many singular levels.

Let X_t and Y_t ($t \in (0,1)$) be the closure of the two submanifolds of $N - P_t$ corresponding to X and Y respectively. For each regular level t, we label it X (resp. Y) if either a closed curve in $Q \cap P_t$ bounds a compressing disk for P_t in X_t (resp. Y_t), or an arc in $Q \cap P_t$ bounds a ∂–compressing disk for P_t in X_t (resp. Y_t).

Recall that we have assumed that $Q \cap P_F^X$ and $Q \cap P_F^Y$ consist of non-nested ∂–parallel arcs in Q (since P_F^X and P_F^Y are ∂–parallel in X and Y respectively) and the arcs in $Q \cap P_F^X$ and $Q \cap P_F^Y$ are essential in P_F^X and P_F^Y respectively. This means that, for any sufficiently small $\epsilon > 0$, ϵ is labelled X and $1-\epsilon$ is labelled Y.

Since P is strongly irreducible and ∂–strongly irreducible, no regular level t is labelled both X and Y.

Since Q is an essential surface, no arc or curve in $P_t \cap Q$ can be trivial in P_t but nontrivial in Q. For any regular level t, if a closed curve or an arc is trivial in both P_t and Q, since N is irreducible and ∂–irreducible, this curve or arc can be eliminated by an isotopy. So for any regular level t, if a closed curve (resp. an arc) in $P_t \cap Q$ is trivial in Q but essential in P_t, then we can find an innermost (resp. outermost) such curve (resp. arc) in Q that bounds a compressing disk (resp. ∂–compressing disk) for P_t and hence t is labelled X or Y. Thus if a level t has no label, after some isotopies removing curves and arcs trivial in both P_t and Q, $P_t \cap Q$ consists of curves and arcs essential in both P_t and Q and part (3) of the lemma holds. So to prove the lemma, it remains to consider the case that every regular level t is labelled.

Since ϵ is labelled X and $1-\epsilon$ is labelled Y for small $\epsilon > 0$, the conclusions above imply that there must be a singular level $s \in (0,1)$ such that $s-\epsilon$ is labelled X but $s+\epsilon$ is labelled Y for sufficiently small $\epsilon > 0$. Moreover, $P_s \cap Q$ contains a single saddle tangency.

Let Θ be the graph component of $P_s \cap Q$ containing the saddle tangency and let $N(\Theta)$ be the closure of a small regular neighborhood of Θ in Q. Since P_s, Q and N are all orientable and P_s is separating in N, every component of $P_{s\pm\epsilon} \cap Q$ is isotopic in Q to either a component of $\partial N(\Theta)$ or a component of $P_s \cap Q - \Theta$. Since $s-\epsilon$ is labelled X and $s+\epsilon$ is labelled Y, there are arcs or closed curves γ_X and γ_Y in $P_{s\pm\epsilon} \cap Q$ bounding compressing or ∂–compressing disks in $X_{s-\epsilon}$ and $Y_{s+\epsilon}$

respectively. As above, since P_s is separating and in particular P_s separates $P_{s-\epsilon}$ and $P_{s+\epsilon}$ in N, γ_X and γ_Y correspond to disjoint curves in $\partial N(\Theta) \cup (P_s \cap Q - \Theta)$. Let $Q \times J$ be a small product neighborhood of Q in N, where J is a closed interval. Let Q^+ and Q^- be the two components of $Q \times \partial J$. Note that by the configuration near a saddle tangency, the intersection patterns of $P_s \cap Q^{\pm}$ and $P_{s\pm\epsilon} \cap Q$ are the same. In particular, curves in $\partial N(\Theta) \cup (P_s \cap Q - \Theta) \subset Q$ are isotopic in $Q \times J$ to disjoint curves in $P_s \cap (Q \times \partial J)$. This means that there are two disjoint arcs or closed curves γ'_X and γ'_Y in $P_s \cap (Q \times \partial J)$ corresponding to γ_X and γ_Y above, such that γ'_X and γ'_Y bound compressing or ∂–compressing disks for P_s in X_s and Y_s respectively. This contradicts that P_s is strongly irreducible and ∂–strongly irreducible. □

For any integers g and b, let $C_{g,b}$ be the collection of orientable surfaces properly embedded in N, such that any $P \in C_{g,b}$ has at least one boundary component in F, ∂P is essential in ∂N, $g(P) \le g$ and $|\partial P - F| \le b$. Note that surfaces in $C_{g,b}$ need not to be essential and there is no restriction on the number of components of $\partial P \cap F$.

Lemma 3.6 *Let P be a surface in $C_{g,b}$. Let $P = P_0, P_1, \dots, P_k$ be surfaces in N such that each P_i is obtained by performing a ∂–compression on P_{i-1}. Suppose ∂P_i is essential in ∂N for each i. Then the distance $d(\partial P \cap F, \partial P_k \cap F) \le \max\{1,\ 4g+2b-2\}$ in $\mathcal{C}(F)$.*

Proof Since ∂N is incompressible in N and ∂P_i is essential in ∂N, $\chi(P_i) \le 0$ for each i. Let b_F be the number of components of $\partial P \cap F$. Since $P \in C_{g,b}$, the total number of boundary components of P is at most $b_F + b$. By our hypotheses, the total number of ∂–compressions is at most $-\chi(P)$, so $k \le -\chi(P) \le 2g - 2 + b + b_F$.

Let D_i be the ∂–compressing disk for P_i such that P_{i+1} is obtained by the ∂–compression along D_i. Let $\partial D_i = \alpha_i \cup \beta_i$ with $\alpha_i \subset P_i$ and $\beta_i \subset \partial N$. Since we are only concerned about how the curves change in $\mathcal{C}(F)$, we may assume $\beta_i \subset F$ for all i. Clearly, for any components γ_i and γ_{i+1} of $\partial P_i \cap F$ and $\partial P_{i+1} \cap F$ respectively, $d(\gamma_i, \gamma_{i+1}) \le 1$. Note that if $\partial \alpha_i \cap \gamma_i = \varnothing$, then the ∂–compression does not change γ_i and γ_i can be viewed as a component of ∂P_{i+1}.

We may view each α_i above as an arc properly embedded in $P = P_0$. As above, we may assume the endpoints of these α_i's all lie in $\partial P \cap F$. We have k such arcs α_i and $|\partial P \cap F| = b_F$. Hence there is a component γ of $\partial P \cap F$ that contains at most $2k/b_F$ endpoints of these arcs α_i's. Since $k \le 2g - 2 + b + b_F$, if $2g - 2 + b \ge 0$, then the number of those endpoints in γ is at most

$$\frac{2k}{b_F} \le \frac{4g-4+2b+2b_F}{b_F} = \frac{4g-4+2b}{b_F} + 2 \le (4g-4+2b) + 2 = 4g+2b-2.$$

If $2g-2+b<0$, we have

$$\frac{2k}{b_F} \leq \frac{4g-4+2b+2b_F}{b_F} = \frac{4g-4+2b}{b_F} + 2 < 2,$$

which means that the number of those endpoints in γ is at most 1.

This means that at most $\max\{1,\ 4g+2b-2\}$ ∂–compressions occur at the curve γ. Since each ∂–compression changes a curve by at most distance one in $\mathcal{C}(F)$, we have $d(\gamma, \partial P_k \cap F) \leq \max\{1,\ 4g+2b-2\}$ and hence the lemma holds. Note that this bound is not sharp and one can easily reduce the bound by a more delicate argument. □

Lemma 3.7 *Suppose N is not an I–bundle. Let P and Q be surfaces in $C_{g,b}$. Suppose Q is essential and suppose P is either essential or strongly irreducible and ∂–strongly irreducible. Then there exists a number K' that depends only on g and b, such that the distance $d(\partial P \cap F, \partial Q \cap F) \leq K'$ in $\mathcal{C}(F)$. Moreover, K' can be chosen to be an explicit quadratic function of g and b.*

Proof Before we proceed, we would like to mention a well-known result on the relation between the intersection number of two curves and their distance in the curve complex. Let α and β be two essential simple closed curves in F and let k be the minimal number of intersection points of $\alpha \cap \beta$ up to isotopy on α and β in F. If the genus of F is at least two, by [17, Lemma 2.1] (see [7, Lemma 2.1] for a better bound), the distance $d(\alpha, \beta) \leq 2k+1 \leq 2|\alpha \cap \beta| + 1$. The proof of [17, Lemma 2.1] also works in the case that F is a torus. For completeness, we include the proof. Suppose F is a torus and α and β realize the minimal intersection number k. If $k=1$, then by the definition of the curve complex for torus, $d(\alpha, \beta) = 1$. If $k \geq 2$, we fix two points x and y of $\alpha \cap \beta$ adjacent in α and let α' be the subarc of α between x and y with $\mathrm{int}(\alpha') \cap \beta = \varnothing$. We can do surgery at x and y, replacing a segment of β between x and y by α'. This operation produces a simple closed curve β_1 in F whose minimal intersection number with α is at most $k-1$. Moreover, since F is a torus and $\alpha \cap \beta$ realizes the minimal intersection number, the intersection points of $\alpha \cap \beta$ all have the same sign. This implies that (1) β_1 must be nontrivial in F and (2) the minimal intersection number of β and β_1 is one, which means $d(\beta_1, \beta) = 1$. Thus $d(\alpha, \beta) \leq d(\alpha, \beta_1) + d(\beta_1, \beta) = d(\alpha, \beta_1) + 1$. So by inductively applying the argument above, we have $d(\alpha, \beta) \leq k \leq |\alpha \cap \beta|$ if F is a torus. Therefore, for any essential simple closed curves α and β in F, we have $d(\alpha, \beta) \leq 2|\alpha \cap \beta| + 1$ if $g(F) \geq 2$ and $d(\alpha, \beta) \leq |\alpha \cap \beta|$ if F is a torus.

We have two cases to consider.

Case A N contains an essential annulus with at least one boundary component in F.

Let X be the set of essential annuli and Möbius bands in N such that for each A in X, (1) at least one boundary component of A lies in F and (2) after isotopy, $P \cap A$ is essential in both P and A. Note that if P is an essential surface, then every essential annulus or Möbius band satisfies property (2) above. We first consider the subcase that $X \neq \varnothing$.

Claim 1 *For any annulus or a Möbius band A_X in X, $d(\partial P \cap F, \partial A_X \cap F) \leq K_1$ for some constant K_1 which can be chosen to be a linear function of g and b.*

Proof of Claim 1 Let A be the annulus or Möbius band in X with the properties that $P \cap A$ consists of arcs essential in both P and A, and that $|A \cap P|$ is minimal among all such essential annuli and Möbius bands.

Note that for any essential Möbius band, the boundary of its small neighborhood in N gives an essential annulus disjoint from the Möbius band. By Lemma 3.2 the diameter of the annulus complex is at most 2. So if $d(\partial P \cap F, \partial A \cap F) \leq K_1$, then for any other annulus or Möbius band A_X in X, $d(\partial P \cap F, \partial A_X \cap F) \leq K_1 + 3$. Thus to prove the claim, it suffices to show that $d(\partial P \cap F, \partial A \cap F) \leq K_1$ for this particular annulus or Möbius band A.

If P is an annulus, then by Lemma 3.2, $d(\partial P \cap F, \partial A \cap F) \leq 2$. So we may assume $\chi(P) < 0$.

Suppose $d(\partial P \cap F, \partial A \cap F) \geq 2$, then every component of $\partial P \cap F$ intersects every component of $\partial A \cap F$. Let $\omega = \min\{|\alpha \cap \beta| : \alpha$ is a component of $\partial P \cap F$ and β is a component of $\partial A \cap F\}$. By the result we mentioned at the beginning of the proof, $d(\partial P \cap F, \partial A \cap F) \leq 2\omega + 1$ if $g(F) \geq 2$ [17, Lemma 2.1] and $d(\partial P \cap F, \partial A \cap F) \leq \omega$ if F is a torus. So to prove the claim, it suffices to show that ω is bounded from above by a linear function of g and b.

Let g_P be the genus of P and let b_F and b_P be the numbers of components of $\partial P \cap F$ and $\partial P - F$ respectively. By the definition of $C_{g,b}$, $g_P \leq g$ and $b_P \leq b$. Since each component of $\partial P \cap F$ intersects every component of $\partial A \cap F$ in at least ω points, the number of arcs of $P \cap A$ with an endpoint in F is at least $\omega b_F/2$. Since $\chi(P) < 0$, there are at most $6g_P + 3(b_F + b_P) - 6 \leq 3b_F + 6g + 3b - 6$ pairwise nonparallel arcs in P. Thus if $\omega > (2(3b_F + 6g + 3b - 6))/b_F$, at least two arcs of $P \cap A$ are parallel in P and each of the two arcs has at least one endpoint in F. Similar to the proof of Lemma 3.6, if $6g + 3b - 6 \geq 0$,

$$\frac{2(3b_F + 6g + 3b - 6)}{b_F} = 6 + \frac{12g + 6b - 12}{b_F} \leq 12g + 6b - 6,$$

and if $6g+3b-6<0$,

$$\frac{2(3b_F+6g+3b-6)}{b_F}<6.$$

Thus if $\omega>\max\{5,\ 12g+6b-6\}$ then there are 2 arcs in $P\cap A$, denoted by α and β, such that α and β are parallel in P, $\partial\alpha\cap F\neq\varnothing$ and $\partial\beta\cap F\neq\varnothing$.

Let R_P and R_A be rectangles bounded by α and β in P and A respectively. We may choose α and β so that $\mathrm{int}(R_P)\cap A=\varnothing$. Thus $A'=R_P\cup R_A$ is an embedded annulus or Möbius band. Next we show that A' is an essential annulus or Möbius band.

Since N is irreducible, if A' is a Möbius band, then $\partial A'$ must be essential in ∂N (otherwise the union of A' and a disk bounded by $\partial A'$ is an embedded projective plane in N). If A' is an annulus and a component of $\partial A'$ is trivial in ∂N, then since ∂N is incompressible, both components of $\partial A'$ must be trivial in ∂N and this means that $\Sigma_A=(A-R_A)\cup R_P$ is an annulus isotopic to A (Σ_A is embedded because $\mathrm{int}(R_P)\cap A=\varnothing$). Moreover, after a slight perturbation on Σ_A, Σ_A becomes transverse to P with $\Sigma_A\cap P$ essential in both Σ_A and P and $|\Sigma_A\cap P|<|A\cap P|$. This contradicts our choice of A. So if A' is an annulus, $\partial A'$ consists of essential curves in ∂N. Since ∂N is incompressible, the argument above says that no matter whether A' is an annulus or a Möbius band, A' is incompressible. Since A is essential and α is an essential arc in A, α must be an essential arc in N, which implies that A' is ∂–incompressible. Therefore A' must be an essential annulus or Möbius band in N. Moreover, after a small perturbation, $A'\cap P$ has fewer components than $A\cap P$, a contradiction to the minimality assumption on $|A\cap P|$. This means that $\omega\leq\max\{5,12g+6b-6\}$ and the claim holds. □

Claim 2 *For any essential annulus A_N in N with at least one boundary component in F, $d(\partial P\cap F,\partial A_N\cap F)\leq K_2$ for some constant K_2 which can be chosen to be a linear function of g and b.*

Proof of Claim 2 By Lemma 3.2 the diameter of the annulus complex is at most 2. So Claim 2 immediately follows from Claim 1 if $X\neq\varnothing$.

If P is essential in N, for any essential annulus A_N in the claim, after isotopy, $P\cap A_N$ is essential in both P and A_N. So $A_N\in X$ and $X\neq\varnothing$ in this subcase and Claim 2 follows from Claim 1.

If P is strongly irreducible and ∂–strongly irreducible, then by Lemma 3.5, either $(\partial A_N\cap F)\cap(\partial P\cap F)=\varnothing$, or one can obtain an essential surface P' (with $\partial P'\cap F\neq\varnothing$) by compressing and ∂–compressing P on the same side, or after isotopy $A_N\cap P$ consists of essential arcs in both A_N and P. If $(\partial A_N\cap F)\cap(\partial P\cap F)=\varnothing$, then

$d(\partial A_N \cap F, \partial P \cap F) \leq 1$ and the claim holds. If $A_N \cap P$ consists of essential arcs in both A_N and P, then $A_N \in X$ and Claim 2 follows from Claim 1. Thus it remains to consider the subcase that one can obtain an essential surface P' (with $\partial P' \cap F \neq \varnothing$) by compressing and ∂–compressing P as in Lemma 3.5. Since a compression does not change the boundary curve, the essential surface P' is obtained by ∂–compressing a surface whose boundary is the same as ∂P. By Lemma 3.6, $d(\partial P' \cap F, \partial P \cap F) \leq \max\{1,\ 4g + 2b - 2\}$. Since P' is an essential surface, we may assume $P' \cap A_N$ is essential in both P' and A_N. By Claim 1, $d(\partial P' \cap F, \partial A_N \cap F) \leq K_1$. As $d(\partial P' \cap F, \partial P \cap F) \leq \max\{1,\ 4g + 2b - 2\}$, we have

$$d(\partial P \cap F, \partial A_N \cap F) \leq d(\partial P \cap F, \partial P' \cap F) + 1 + d(\partial P' \cap F, \partial A_N \cap F) \leq K_2,$$

where $K_2 = \max\{1,\ 4g + 2b - 2\} + K_1 + 1$. □

The argument above also implies that $d(\partial Q \cap F, \partial A_N \cap F) \leq K_2$, since Q is an essential surface. Therefore, if N contains an essential annulus with at least one boundary component in F, $d(\partial P \cap F, \partial Q \cap F) \leq 2K_2 + 1$ and the lemma holds in Case A.

Case B N contains no essential annulus with a boundary component in F.

For simplicity, we assume P is strongly irreducible and ∂–strongly irreducible and the proof for the case that P is essential is the same. By Lemma 3.5, either $(\partial P \cap F) \cap (\partial Q \cap F) = \varnothing$, or one can obtain an essential surface P' (with $\partial P' \cap F \neq \varnothing$) by compressing and ∂–compressing P on the same side, or after isotopy $P \cap Q$ is essential in both P and Q and $|\partial P \cap \partial Q|$ is minimal up to isotopy on ∂P and ∂Q in ∂N.

If $(\partial P \cap F) \cap (\partial Q \cap F) = \varnothing$, the lemma holds trivially. So by Lemma 3.5, we have the following 2 subcases to consider.

Subcase 1 $P \cap Q$ is essential in both P and Q, and $|\partial P \cap \partial Q|$ is minimal up to isotopy on ∂P and ∂Q in ∂N.

Let $\omega = \min\{|\alpha \cap \beta| : \alpha$ is a component of $\partial P \cap F$ and β is a component of $\partial Q \cap F\}$. Let b_1 and b_2 be the numbers of components in $P \cap F$ and $Q \cap F$ respectively. Thus the number of arcs of $P \cap Q$ with an endpoint in F is at least $\omega b_1 b_2 / 2$.

Since N contains no essential annulus with a boundary component in F and since ∂N is incompressible, $\chi(P) < 0$ and $\chi(Q) < 0$. As in the argument in Claim 1 above, the maximal numbers of pairwise nonparallel arcs in P and Q are at most $6g + 3(b + b_1) - 6$ and $6g + 3(b + b_2) - 6$ respectively. Thus if ω is sufficiently large, there are a pair of arcs α and β in $P \cap Q$ such that $\partial\alpha \cap F \neq \varnothing$, $\partial\beta \cap F \neq \varnothing$, and α and β are parallel in both P and Q. Note that the bound for ω is an explicit quadratic function of g and b. Let R_P and R_Q be the rectangles bounded by $\alpha \cup \beta$ in P and Q respectively. We can choose α and β so that $\mathrm{int}(R_P) \cap \mathrm{int}(R_Q) = \varnothing$. So $A = R_P \cup R_Q$ is an embedded annulus or Möbius band in N. As $\alpha \cap F \neq \varnothing$, at least one component of ∂A lies in F. Similar to Claim 1 in Case A, since N is irreducible, if A is a Möbius band, ∂A must be essential in ∂N. Since $|\partial P \cap \partial Q|$ is minimal in their isotopy classes, if A is an annulus, ∂A is essential in ∂N. Hence A is incompressible. Since Q is an essential surface and α is an essential arc in Q, α must be an essential arc in N. Hence A is ∂–incompressible in N. So A is essential in N. If A is a Möbius band, a double cover of A is an essential annulus. This contradicts our hypothesis in Case B that no such essential annulus exists. Therefore, ω and $d(\partial P \cap F, \partial Q \cap F)$ must be bounded by a number K' that depends only on g and b. As above, K' can be chosen to be an explicit quadratic function of g and b and the lemma holds.

Subcase 2 One can obtain an essential surface P' (with $\partial P' \cap F \neq \varnothing$) by compressing and ∂–compressing P.

Since both P' and Q are essential, after isotopy, $P' \cap Q$ is essential in both P' and Q and $|\partial P' \cap \partial Q|$ is minimal up to isotopy on $\partial P'$ and ∂Q in ∂N. Then the argument in Subcase 1 above implies that $d(\partial P' \cap F, \partial Q \cap F) \leq K'$, where K' can be chosen to be an explicit quadratic function of g and b.

By Lemma 3.6, $d(\partial P' \cap F, \partial P \cap F) \leq \max\{1,\ 4g + 2b - 2\}$. Thus as in the proof of Claim 2 above, $d(\partial P \cap F, \partial Q \cap F) \leq d(\partial P \cap F, \partial P' \cap F) + 1 + d(\partial P' \cap F, \partial Q \cap F) \leq K' + \max\{1,\ 4g + 2b - 2\} + 1$. □

Lemma 3.8 *Suppose N is a twisted I–bundle over a closed nonorientable surface and $F = \partial N$. Let P be a properly embedded orientable genus–g surface with boundary and suppose P is either essential or strongly irreducible and ∂–strongly irreducible. Then there is a number K depending only on g, such that $d(\partial P, \mathcal{A}_N(F)) \leq K$, where $\mathcal{A}_N(F)$ is the annulus complex defined in Definition 3.1. Moreover, K can be chosen to be an explicit linear function of g.*

Proof Any orientable essential surface with boundary in the I–bundle N is an annulus. If P is an essential surface, P must be an annulus and $d(\partial P, \mathcal{A}_N(F)) = 0$. So we may assume that P is strongly irreducible and ∂–strongly irreducible.

By Lemma 3.5, for any vertical annulus Q, either $\partial P \cap \partial Q = \varnothing$, or one can obtain an essential annulus by compressing and ∂–compressing P on the same side, or after isotopy $P \cap Q$ is essential in both P and Q.

Suppose we can obtain an essential annulus P' after some compressions and ∂–compressions on the same side of P. By Lemma 3.6 and since $F=\partial N$, $P\in C_{g,0}$ and $d(\partial P,\mathcal{A}_N(F))\le d(\partial P,\partial P')\le \max\{1,\ 4g-2\}$. Thus by Lemma 3.5, it remains to consider the case that for any vertical annulus Q of N, we can isotope P so that $P\cap Q$ consists of arcs essential in both P and Q.

We may choose Q to be a vertical annulus or Möbius band so that $\partial P\cap\partial Q$ is minimal among all vertical annuli and Möbius bands with the property that $P\cap Q$ consists of arcs essential in both P and Q. If $\partial P\cap\partial Q=\varnothing$ then $d(\partial P,\mathcal{A}_N(F))\le 1$. So we may assume $\partial P\cap\partial Q\neq\varnothing$.

Let $\omega=\min\{|\alpha\cap\beta| : \alpha$ is a component of ∂P and β is a component of $\partial Q\}$. As in the proof of Lemma 3.7, if ω is large, then there are a pair of arcs α and β in $P\cap Q$ that are parallel in P and we can construct a new essential annulus or Möbius band with fewer intersection with P. As in the proof of Claim 1 in Lemma 3.7, this implies that $\omega\le\max\{5,\ 12g-6\}$. As before, by [17, Lemma 2.1] and the argument at the beginning of the proof of Lemma 3.7, $d(\partial P,\mathcal{A}_N(F))\le d(\partial P,\partial Q)\le 2\omega+1\le\max\{11,\ 24g-11\}$ if $g(F)\ge 2$, and $d(\partial P,\mathcal{A}_N(F))\le d(\partial P,\partial Q)\le\omega\le\max\{5,\ 12g-6\}$ if F is a torus. □

In the lemmas above, we proved some nice properties of strongly irreducible and ∂–strongly irreducible surfaces in N. Next we consider surfaces that are strongly irreducible but not ∂–strongly irreducible.

Lemma 3.9 *Let P be a strongly irreducible surface properly embedded in N. If P is not ∂–strongly irreducible, then there is a surface P' obtained by ∂–compressing P and deleting any resulting ∂–parallel components, such that P' is either*

(1) *strongly irreducible and ∂–strongly irreducible,*

(2) *essential in N, or*

(3) *$P'=\varnothing$, ie, after some ∂–compressions on P, every component of the resulting surface is ∂–parallel.*

Proof Let D be a ∂–compressing disk. We say D is disk-busting if every compressing disk on the other side of P intersects ∂D. If P has a ∂–compressing disk D that is not disk-busting, then we perform a ∂–compression along D. As D is not disk-busting, there is a compressing disk D' of P on the other side which remains a compressing disk after the ∂–compression. Moreover, since P is compressible on both sides, the surface obtained by ∂–compression along D remains compressible on both sides and strongly irreducible.

After some ∂–compressions as above, we may assume every ∂–compressing disk of P is disk-busting. If P is still not ∂–strongly irreducible, there must be a pair of ∂–compressing disks D_1 and D_2 on different sides of P with $\partial D_1 \cap \partial D_2 = \varnothing$. Now we perform ∂–compression on P along D_1 and D_2 simultaneously and obtain a surface P'. Since both D_1 and D_2 are disk-busting and D_1 and D_2 are on different sides of P, P' is incompressible in N. Therefore, after some more ∂–compressions on P', we obtain a surface of which every component is either essential or ∂–parallel. □

In the proof of Lemma 3.9, if a component of P is ∂–parallel and outermost, then we can simply eliminate this component. Next we discuss how the boundary curves of P change during the ∂–compressions in the proof of Lemma 3.9. This discussion will be used later. Since we are mainly interested in the curves in F, we suppose all the ∂–compressions occur at F. Each step in the proof of Lemma 3.9 is either a single ∂–compression or two simultaneous ∂–compressions on different sides of P. The resulting surface after each step is either strongly irreducible or incompressible. Note that a ∂–compression does not create any ∂–parallel disk, so the boundary of any resulting ∂–parallel component is essential in ∂N. As we pointed out in Definition 3.3, the boundary curves of a strongly irreducible surface are essential in ∂N, so we can view them as vertices in the curve complex $\mathcal{C}(F)$. Next we study how the distance of the boundary curves change after each step in the proof of Lemma 3.9.

Let D be a ∂–compressing disk for P in the proof of Lemma 3.9. Then a ∂–compression along D can be viewed as an isotopy pushing D into a product neighborhood of F. In fact, we can find a product neighborhood $F \times I$ of F in N such that every level $F \times \{t\}$ is transverse to P except for a singular level $s \in (0,1)$ where $F \times \{s\}$ is transverse to P except for a single saddle tangency. The saddle tangency corresponds to the ∂–compressing disk D. Suppose $F = F \times \{0\}$ and $N' = N - (F \times [0,1))$. Then $N' \cong N$ and $P \cap N'$ can be viewed as the surface obtained by ∂–compressing P along D. So $\chi(P \cap (F \times I)) = -1$. Since P is orientable, the component of $P \cap (F \times I)$ that contains the saddle tangency must be a pair of pants. Hence $P \cap (F \times I)$ consists of a pair of pants and a collection of vertical annuli. In the proof of Lemma 3.9, the surface after the ∂–compression along D remains either incompressible or strongly irreducible, so every component of $P \cap (F \times \partial I)$ is an essential curve in $F \times \partial I$. To simplify notation, we do not distinguish a curve γ in $F \times \{t\}$ from the vertex in $\mathcal{C}(F)$ represented by $\pi(\gamma)$, where $\pi\colon F \times I \to F$ is the projection. It is easy to see that for any curves γ_0 and γ_1 in $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, $d(\gamma_0, \gamma_1) \le 1 = -\chi(P \cap (F \times I))$ in $\mathcal{C}(F)$.

The situation is slightly more complicated when we simultaneously ∂–compressing P (on different sides) along two disjoint ∂–compressing disks D_1 and D_2 in the last part of the proof of Lemma 3.9. Similar to the argument above, we can find a product neighborhood $F \times I$ of F in N with $F \times \{0\} = F$ such that every level $F \times \{t\}$ is transverse to P except for a singular level $s \in (0,1)$ where $F \times \{s\}$ is transverse to P except for two saddle tangencies. The two saddle tangencies correspond to the ∂–compressing disks D_1 and D_2. Similar to the first case above, in the proof of Lemma 3.9, every component of $P \cap (F \times \partial I)$ is an essential curve in $F \times \partial I$. Let Θ be the possibly disconnected graph of $P \cap (F \times \{s\})$ containing the two saddle tangencies. So Θ has two vertices of valence 4. If the genus of F is at least 2, then there must be an essential simple closed curve α in $F \times \{s\}$ that is disjoint from Θ and $P \cap (F \times \{s\})$. This implies that if F is not a torus, for any components γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, $d(\gamma_0, \gamma_1) \le d(\gamma_0, \alpha) + d(\alpha, \gamma_1) \le 2 = -\chi(P \cap (F \times I))$ in $\mathcal{C}(F)$.

If F is a torus, since $P \cap (F \times \partial I)$ consists of essential curves, each $P \cap (F \times \{i\})$ ($i = 0, 1$) consists of parallel curves in the torus F. If $F \times \{s\} - \Theta$ is not a collection of disks, then there is an essential simple closed curve in $F \times \{s\}$ disjoint from P. This implies that any curves γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ represent the same vertex in $\mathcal{C}(F)$ and $d(\gamma_0, \gamma_1) = 0$. Next we suppose every component of $F \times \{s\} - \Theta$ is a disk. Then $P \cap (F \times I)$ contains no vertical annulus and $P \cap (F \times I)$ can be viewed as a small neighborhood of Θ. Since P is separating, $P \cap (F \times \{0\})$ contains at least two curves and the argument above implies that $P \cap (F \times \{0\})$ contains exactly two curves which cut the torus $F \times \{0\}$ into two annuli A_1 and A_2. Moreover, the two arcs $D_1 \cap (F \times \{0\})$ and $D_2 \cap (F \times \{0\})$ from the ∂–compressing disks are essential arcs in the two annuli A_1 and A_2 respectively. So it is easy to see that $P \cap (F \times \{1\})$ also consists of exactly two curves, and for any γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, the intersection number of γ_0 and γ_1 (after projecting to the torus F) is one and hence $d(\gamma_0, \gamma_1) = 1 < 2 = -\chi(P \cap (F \times I))$ in the case that F is a torus.

Therefore, in any case, for any curves γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, $d(\gamma_0, \gamma_1) \le -\chi(P \cap (F \times I))$ and $-\chi(P \cap (F \times I))$ equals to the number of saddle tangencies in $F \times I$.

4 Case I: The amalgamation surface F is incompressible

Let M_1, M_2, F and $M = M_1 \cup_F M_2$ be as in Theorem 1.5. We regard M_1 and M_2 as submanifolds of M with $F = \partial M_1 = \partial M_2$. In this section, we prove Theorem 1.5 in the case that both M_1 and M_2 have incompressible boundary, ie, F is incompressible in M.

Lemma 4.1 *Let M_1, M_2, M and F be as above. Then for any integer g, there is a number K_g which depends only on M_1, M_2 and g, such that, if $d(M) > K_g$, then any closed incompressible orientable surface of genus g in M can be isotoped disjoint from F.*

Proof Let S be a closed incompressible orientable surface of genus g in M. Suppose S cannot be isotoped disjoint from F. As both S and F are incompressible, we may assume $F \cap S$ is essential in both F and S.

Let $S_i = M_i \cap S$ $(i = 1, 2)$. So S_i has no disk component, each S_i is incompressible in M_i, $S = S_1 \cup S_2$ and $\chi(S) = \chi(S_1) + \chi(S_2)$. Since S cannot be isotoped disjoint from F, we obtain an essential surface S_i' in M_i after at most $-\chi(S_i)$ ∂–compressions on S_i. Each ∂–compression changes the boundary curves of the surface by at most distance one in $\mathcal{C}(F)$. Thus for any components γ_i and γ_i' of ∂S_i and $\partial S_i'$ respectively, the distance $d(\gamma_i, \gamma_i') \leq -\chi(S_i)$.

Suppose neither M_1 nor M_2 is a twisted I–bundle. Let Ω_i be the fixed essential surface with maximal Euler characteristic used in defining $d(M)$, ie, $d(M) = d(\partial\Omega_1 \cap F, \partial\Omega_2 \cap F)$. By Lemma 3.7, for any essential surface Q properly embedded in M_i with genus at most g and $\partial Q \subset F$, there is a number K_i such that $d(\partial\Omega_i \cap F, \partial Q) \leq K_i$. Thus there is a component γ_i' of $\partial S_i'$, such that $d(\partial\Omega_i \cap F, \gamma_i') \leq K_i$, $i = 1, 2$. Let γ be a component of $\partial S_1 = \partial S_2$. So we have $d(\partial\Omega_1 \cap F, \partial\Omega_2 \cap F) \leq d(\partial\Omega_1 \cap F, \gamma_1') + d(\gamma_1', \gamma) + d(\gamma, \gamma_2') + d(\gamma_2', \partial\Omega_2 \cap F) \leq K_1 - \chi(S_1) - \chi(S_2) + K_2 = K_1 - \chi(S) + K_2 = K_1 + K_2 + 2g - 2$. Thus Lemma 4.1 holds in the case that neither M_1 nor M_2 is a twisted I–bundle.

If M_i is a twisted I–bundle over a closed nonorientable surface, then S_i' must be a vertical annulus and each component of $\partial S_i'$ represents a vertex in the annulus complex of M_i. By the definition of $d(M)$ in the case that M_i is a twisted I–bundle, the argument above plus Lemma 3.8 also prove Lemma 4.1 in the case that some M_i is a twisted I–bundle. □

Let S be an unstabilized Heegaard surface of genus g. As in Theorem 1.4, the untelescoping of the Heegaard splitting [25] gives a decomposition $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M and $g(F_i) \leq g$. By Lemma 4.1, we may assume $d(M)$ is so large that $F_i \cap F = \varnothing$ for each i after isotopy. So we may suppose $F \subset \mathrm{int}(N_i)$ for some i. Without loss of generality, we may assume N_i is connected. By the untelescoping construction, N_i has a strongly irreducible Heegaard surface P_i and $g(P_i) \leq g$. Note that if S is strongly irreducible, then $N_i = M$. The following Lemma of Bachman, Schleimer and Sedgwick [2, Lemma 3.3] says that we can isotope P_i so that P_i intersects F nicely. If F is parallel to some F_j above, then Theorem 1.5 holds. Suppose Theorem 1.5 is not true, then F is not parallel to a component of ∂N_i.

Lemma 4.2 (Bachman–Schleimer–Sedgwick [2]) *Let N_i be a compact, irreducible, orientable 3–manifold with ∂N_i incompressible, if nonempty. Suppose P_i is a strongly irreducible Heegaard surface of N_i. Suppose further that N_i contains an incompressible, orientable, closed, non–boundary parallel surface F. Then either*

(1) *P_i may be isotoped to be transverse to F, with every component of $P_i - N(F)$ incompressible in the respective submanifold of $N_i - N(F)$, where $N(F)$ is a small neighborhood of F in N_i,*

(2) *P_i may be isotoped to be transverse to F, with every component of $P_i - N(F)$ incompressible in the respective submanifold of $N_i - N(F)$ except for exactly one strongly irreducible component, or*

(3) *P_i may be isotoped to be almost transverse to F (ie, P_i is transverse to F except for one saddle point), with every component of $P_i - N(F)$ incompressible in the respective submanifold of $N_i - N(F)$.*

Let $N(F) = F \times I$ be a product neighborhood of F in N_i and let X and Y be the two components of $N_i - \mathrm{int}(N(F))$. As P_i is a Heegaard surface of N_i and the incompressible surface F is not parallel to ∂N_i, $F \cap P_i \neq \varnothing$. Let $S_X = P_i \cap X$ and $S_Y = P_i \cap Y$. By Lemma 4.2, we may assume that each component of S_X and S_Y is either incompressible or strongly irreducible in X and Y respectively. Moreover, both S_X and S_Y are essential subsurfaces of P_i (ie ∂S_X and ∂S_Y are essential curves in P_i). Hence $\chi(S_X) + \chi(S_Y) \geq \chi(P_i)$. By projecting $F \times I$ to F, we may view ∂S_X and ∂S_Y as curves in F. By Lemma 4.2, P_i is transverse to every level surface $F \times \{t\}$ in $F \times I$ except for at most one saddle tangency which only occurs in Case (3) of Lemma 4.2. Thus, for any components γ_X and γ_Y of ∂S_X and ∂S_Y respectively, $d(\gamma_X, \gamma_Y) \leq 1$ in $\mathcal{C}(F)$.

Since $F \cap P_i \neq \varnothing$ after any isotopy, S_X or S_Y cannot be changed to a set of ∂–parallel surfaces by ∂–compressions on S_X or S_Y in X or Y respectively. Thus by Lemma 3.9, we can obtain a pair of surfaces S'_X and S'_Y by some ∂–compressions on S_X and S_Y respectively, such that S'_X and S'_Y are either essential or strongly irreducible and ∂–strongly irreducible in X and Y respectively. The numbers of ∂–compressions on S_X and S_Y are at most $-\chi(S_X)$ and $-\chi(S_Y)$ respectively. Since each ∂–compression changes a curve by distance at most one in the $\mathcal{C}(F)$, by the argument after Lemma 3.9, for any components γ_X and γ'_X of ∂S_X and $\partial S'_X$ respectively, $d(\gamma_X, \gamma'_X) \leq -\chi(S_X)$.

Since $-\chi(S_X) - \chi(S_Y) \leq -\chi(P_i) \leq 2g-2$ and since $d(\gamma_X, \gamma_Y) \leq 1$ for any components γ_X and γ_Y of ∂S_X and ∂S_Y respectively, for any components γ'_X and γ'_Y

in $\partial S'_X$ and $\partial S'_Y$, we have $d(\gamma'_X, \gamma'_Y) \le d(\gamma'_X, \gamma_X) + d(\gamma_X, \gamma_Y) + d(\gamma_Y, \gamma'_Y) \le -\chi(S_X) + 1 - \chi(S_Y) \le 1 - \chi(P_i) \le 2g - 1$.

Note that since N_i is a submanifold of $M = M_1 \cup_F M_2$ with $F \subset \mathrm{int}(N_i)$, to simplify notation, we will regard X and Y as submanifolds of M_1 and M_2 respectively with $F \subset \partial X$ and $F \subset \partial Y$. Since F is not parallel to a component of ∂N_i, X and Y are not I–bundles unless M_1 or M_2 is a twisted I–bundle.

We first suppose neither M_1 nor M_2 is a twisted I–bundle. Let Ω_j $(j = 1, 2)$ be the fixed essential surface in M_j used in defining $d(M)$, ie, $d(M) = d(\partial\Omega_1 \cap F, \partial\Omega_2 \cap F)$. Since ∂X and ∂Y are incompressible in M_1 and M_2 respectively, we may assume $\Omega_1 \cap X$ and $\Omega_2 \cap Y$ are essential surfaces in X and Y respectively. Moreover, we may assume $\Omega_1 \cap X$ and $\Omega_2 \cap Y$ are essential subsurfaces of Ω_1 and Ω_2 respectively, and in particular, $\chi(\Omega_1 \cap X) \ge \chi(\Omega_1)$ and $\chi(\Omega_2 \cap Y) \ge \chi(\Omega_2)$.

As $F \cap \partial(\Omega_1 \cap X) = \partial\Omega_1 \cap F$ and $F \cap \partial(\Omega_2 \cap Y) = \partial\Omega_2 \cap F$, by applying Lemma 3.7 to X and Y, we conclude that there is a number K depending only on g and $\max\{-\chi(\Omega_1), -\chi(\Omega_2)\}$, such that $d(\partial S'_X, \partial\Omega_1 \cap F) \le K$ and $d(\partial S'_Y, \partial\Omega_2 \cap F) \le K$. Let γ'_X and γ'_Y be any components of $\partial S'_X$ and $\partial S'_Y$ respectively. Recall that we have concluded earlier that $d(\gamma'_X, \gamma'_Y) \le 2g - 1$. So we have $d(M) = d(\partial\Omega_1 \cap F, \partial\Omega_2 \cap F) \le d(\partial\Omega_1 \cap F, \gamma'_X) + d(\gamma'_X, \gamma'_Y) + d(\gamma'_Y, \partial\Omega_2 \cap F) \le K + (2g - 1) + K = 2K + 2g - 1$.

If M_j is a twisted I–bundle, then it is possible that $X = M_1$ or $Y = M_2$ is a twisted I–bundle. In this case, we can replace $\partial\Omega_j \cap F$ by the annulus complex $\mathcal{A}_F(M_j)$ in the argument above. We can apply Lemma 3.8 instead of Lemma 3.7 and get the same inequalities on $d(M)$.

Therefore, if $d(M)$ is sufficiently large, we get a contradiction and this means that F must be parallel to some incompressible surface F_i in the untelescoping and Theorem 1.5 holds in the case that both M_1 and M_2 have incompressible boundary.

5 Case II: The amalgamation surface F is compressible on both sides

The case that both M_1 and M_2 have compressible boundary in Theorem 1.5 basically follows from a theorem of Scharlemann and Tomova [26] and a theorem of Hartshorn [5]; also see the author's paper [15].

Let $\mathcal{D}_i$ be the disk complex of M_i $(i = 1, 2)$. Recall that in this case $d(M)$ is defined to be $d(\mathcal{D}_1, \mathcal{D}_2)$. We may assume $d(\mathcal{D}_1, \mathcal{D}_2) \ge 2$ which implies that F is strongly irreducible in M. By Casson–Gordon [3] and Haken's lemma [4], this also implies that $M = M_1 \cup_F M_2$ is irreducible and ∂–irreducible and M is not S^3. Hence Theorem 1.2 holds in this case.

Next we suppose $d(M) > 2g$. Let S be an unstabilized Heegaard surface of genus g. As in Theorem 1.4, the untelescoping of the Heegaard splitting [25] gives a decomposition $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M and $g(F_i) \leq g$. By Hartshorn's theorem [5] (another proof is in [15]), either $F_i \cap F = \varnothing$ after isotopy or $d(M) = d(\mathcal{D}_1, \mathcal{D}_2) \leq 2g(F_i) \leq 2g$ for each i. Since $d(M) > 2g$, we may assume $F \subset \mathrm{int}(N_k)$ for some k. Without loss of generality, we suppose N_k is connected. As in Theorem 1.4, there is a strongly irreducible Heegaard surface P_k of the 3–manifold N_k and $g(P_k) \leq g$.

Let Q_j $(j = 1, 2)$ be the surface obtained by maximally compressing F in M_j and removing all resulting 2–sphere components. We may assume $Q_j \subset \mathrm{int}(M_j)$, $Q_1 \cup Q_2$ bounds a submanifold M_F in M, and F is a strongly irreducible Heegaard surface of M_F. Since $F \subset \mathrm{int}(N_k)$ and ∂N_k is incompressible in M, any compressing disk for F can be isotoped into N_k. So after isotopy, we may assume $M_F \subset N_k$.

Since P_k is strongly irreducible, a theorem of Scharlemann and Tomova [26] says that either $d(\mathcal{D}_1, \mathcal{D}_2) \leq 2g(P_k) \leq 2g$, or F and P_k are well-separated, or F and P_k are parallel.

Next we show that F and P_k are not well-separated. Suppose on the contrary that they are well-separated, ie, M_F can be isotoped disjoint from N_k. Let M'_F be a submanifold of M that is isotopic to M_F and disjoint from N_k. Since $M_F \subset N_k$ and $M'_F \cap N_k = \varnothing$, M'_F is disjoint from M_F. Recall that $M = M_1 \cup_F M_2$ is an amalgamation of M_1 and M_2 along F, so M'_F lies in either $\mathrm{int}(M_1)$ or $\mathrm{int}(M_2)$. Without loss of generality, suppose $M'_F \subset \mathrm{int}(M_1)$. By our construction of M_F, the surface $Q_2 \subset \partial M_F$ lies in M_2. Let Y be a component of Q_2 and let Y' be the component of $\partial M'_F$ isotopic to Y. Y and Y' are two-sided, incompressible, disjoint and isotopic surfaces in M, so $Y \cup Y'$ bounds a product region $Y \times I$ in M (one can see this easily after lifting Y and Y' to the covering space of M corresponding to $\pi_1(Y)$). Since $Y \subset \mathrm{int}(M_2)$ and $Y' \subset \partial M'_F \subset \mathrm{int}(M_1)$, the product region $Y \times I$ must contain the amalgamation surface F. Moreover, since Y and Y' are incompressible in M and $F \subset Y \times I$, a compressing disk for F can be isotoped into $Y \times I$. Hence after isotopy, we may assume M_F lies in the production region $Y \times I$. Each closed incompressible surface in the product $Y \times I$ is parallel to Y. This implies that M_F is isotopic to $Y \times I$ and F can be viewed as a Heegaard surface of $Y \times I$. By [24], Heegaard splittings of a product $Y \times I$ are standard and in particular the distance of the Heegaard splitting of $Y \times I$ along F is at most 2. This contradicts our assumption that $d(M) > 2g \geq 2$. So F and P_k are not well-separated.

So the theorem of Scharlemann and Tomova [26] implies that if $d(M) > 2g$, then F and P_k must be parallel. Therefore, Theorem 1.5 holds if both M_1 and M_2 have compressible boundary, and in this case we may choose the bound $K = 2g$.

6 Case III: The amalgamation surface F is compressible on one side

In the next two sections, we suppose F is compressible in M_1 but incompressible in M_2. We denote the disk complex of M_1 by $\mathcal{D}_1$.

Proposition 6.1 *Let γ be a nontrivial simple closed curve in F. Suppose γ bounds an embedded disk in $M = M_1 \cup_F M_2$. Then $d(\gamma, \mathcal{D}_1) \le 1$.*

Proof Let D be the embedded disk bounded by γ in M. We may assume that $|D \cap F|$ is minimal among all disks bounded by γ and transverse to F. Since F is incompressible in M_2, if $\mathrm{int}(D) \cap F = \varnothing$ then D must be a compressing disk of M_1 and $d(\gamma, \mathcal{D}_1) = 0$.

Let γ' be a component of $D \cap F$ that is innermost in D and let δ be the subdisk of D bounded by γ'. If γ' is a trivial curve in F, then a standard cutting and pasting yields a new disk bounded by γ with fewer intersection curves with F. So δ must be a compressing disk in M_1. Since D is embedded, γ and γ' are disjoint in F. Therefore, $d(\gamma, \mathcal{D}_1) \le d(\gamma, \gamma') \le 1$. □

Many parts of the proof of Theorem 1.5 in this case comes down to the situation that F lies in a submanifold M' of M with incompressible boundary, and we need to study how various surfaces intersect F. The following technical lemma deals with this situation and will be used in several places of the proof. The key point of Lemma 6.2 is that the bound on the distance in Lemma 6.2 depends on $-\chi(P)$ not on $-\chi(P_2)$ which can be large since F is compressible in M_1.

Lemma 6.2 *Let M' be a compact submanifold of $M = M_1 \cup_F M_2$ with $F \subset \mathrm{int}(M')$ and suppose $\partial M'$ is incompressible in M'. Let P be an orientable connected surface properly embedded in M'. Suppose P is either incompressible or strongly irreducible in M', $P \cap F \neq \varnothing$, and each component of $P \cap F$ is essential in F. Let $M_2' = M_2 \cap M'$ and $P_2 = P \cap M_2'$. Suppose P_2 is either incompressible or strongly irreducible in M_2' and P_2 does not lie in a product neighborhood of F in M_2'. Then there is a surface Q obtained by some ∂–compressions on P_2 in M_2' and removing all resulting ∂–parallel components, such that $d(Q \cap F, (P \cap F) \cup \mathcal{D}_1) \le \max\{3 - \chi(P), 2\}$ and Q is either an essential or a strongly irreducible and ∂–strongly irreducible surface properly embedded in M_2'.*

Proof If P_2 is incompressible in M_2', then after performing some ∂–compressions on P_2 in M_2', we get a surface Q such that each component of Q is either essential or

∂–parallel in M_2'. Similarly, if P_2 is strongly irreducible but not ∂–strongly irreducible, as in Lemma 3.9, we can obtain a surface Q after some ∂–compressions on P_2 in M_2' such that each component of Q is either essential, or ∂–parallel, or strongly irreducible and ∂–strongly irreducible in M_2'. Since P_2 does not lie in a product neighborhood of F, after discarding all the ∂–parallel components, we get a surface Q which is either essential or strongly irreducible and ∂–strongly irreducible in M_2'. As $\partial P_2 \cap F = P \cap F$, to prove the lemma, we need to study the distance between ∂P_2 and ∂Q in the curve complex $\mathcal{C}(F)$.

The surface F is a boundary component of M_2'. Since we are only interested in how the curves in $\partial P_2 \cap F$ change during ∂–compressions, to simplify notation, we will assume that all the ∂–compressions on P_2 in the construction above occur at F, ie, for any ∂–compressing disk D for P_2, we assume $D \cap \partial M_2' \subset F$.

A ∂–compression on P_2 is basically the same as an isotopy that pushes the ∂–compressing disk into a product neighborhood of F. Thus we can find a product neighborhood $F \times I$ of F in M_2' and assume $Q = P_2 \cap \overline{M_2' - (F \times I)}$. We denote $F \times \{t\}$ by F_t and suppose $F_0 = F \subset \partial M_2'$. By the discussion after the proof of Lemma 3.9, we may describe each ∂–compression using a saddle tangency in $F_t \cap P_2$. In the proof of Lemma 3.9, we have to simultaneously perform two ∂–compressions, so we allow two saddle tangencies at the same level surface F_t. Since Q is obtained by a sequence of ∂–compressions and pushing away the ∂–parallel components, we may assume that there are finitely many numbers $0 = s_0 < s_1 < \cdots < s_k = 1$, such that

(1) P_2 is transverse to each F_{s_i}, and each component of $P_2 \cap F_{s_i}$ is essential in F_{s_i},

(2) for each i, there is one special component of $P_2 \cap (F \times [s_i, s_{i+1}])$ that is transverse to every F_t except for a singular level $t_i \in (s_i, s_{i+1})$ where it is transverse to F_{t_i} except for one or two saddle tangencies, and

(3) every other component of $P_2 \cap (F \times [s_i, s_{i+1}])$ is either a vertical annulus or a ∂–parallel surface in $F \times [s_i, s_{i+1}]$ with boundary in F_{s_i}.

Each saddle tangency in the special component in (2) corresponds to a ∂–compression on P_2 and the ∂–parallel components in (3) are the possible ∂–parallel components after a ∂–compression. Note that it is possible to have two saddle tangencies at the same level F_{t_i} because in the proof of Lemma 3.9, we have to simultaneously ∂–compressing the surface on both sides in order to obtain an incompressible surface; see the discussion after the proof of Lemma 3.9. We regard $Q = P_2 \cap \overline{M_2' - (F \times I)}$, so $\partial Q \subset F_1 \cup (\partial M_2' - F_0)$.

To simplify notation, we do not distinguish a nontrivial curve γ in F_t from the vertex in $\mathcal{C}(F)$ representing $\pi(\gamma)$, where $\pi\colon F \times I \to F$ is the projection. Next we show that $d(Q\cap F_1,(P\cap F_0)\cup\mathcal{D}_1)\le\max\{3-\chi(P),2\}$. The argument is similar to [15, Claims 1 and 3 of Lemma 2.2].

We first point out a useful fact on curves in $P_2\cap F_{s_i}$. Let γ_i be any component of $P_2\cap F_{s_i}$ and let Q_i be the component of $P_2\cap(F\times[s_{i-1},s_i])$ that contains γ_i. By our assumption on $P_2\cap(F\times[s_{i-1},s_i])$ above, Q_i is either a vertical annulus or a special component in (2) above. Since the saddle tangencies in a special component correspond to ∂–compressions, $\partial Q_i\cap F_{s_{i-1}}\neq\varnothing$. Let γ_{i-1} be a component of $\partial Q_i\cap F_{s_{i-1}}$. By the discussion after the proof of Lemma 3.9, $d(\gamma_{i-1},\gamma_i)\le n_i$, where n_i is the number of saddle tangencies in the special component of $P_2\cap(F\times[s_{i-1},s_i])$ and n_i is either 1 or 2. Thus we can successively find a curve γ_i in each $P_2\cap F_{s_i}$ such that $d(\gamma_{i-1},\gamma_i)\le n_i$ for each i, where $n_i=1$ or 2 is the number of saddle tangencies in the special component of $P_2\cap(F\times[s_{i-1},s_i])$.

We say a component γ of $P_2\cap F_{s_i}$ is *good* if the component of $P_2\cap(F\times[s_i,1])$, denoted by Q_γ, that contains γ has a boundary component in F_1, ie $Q_\gamma\cap F_1\neq\varnothing$. Moreover, every component of $P_2\cap F_1=Q\cap F_1$ is regarded as a good component. Let C_i be the set of good components of $P_2\cap F_{s_i}$. As $s_k=1$, $C_k=Q\cap F_1$. Since P_2 does not lie in a product neighborhood of F in M_2', $C_i\neq\varnothing$ for all i.

Suppose the lemma is not true and $d(Q\cap F_1,(P\cap F_0)\cup\mathcal{D}_1)>2$. Since $s_k=1$ and $C_k=Q\cap F_1$, we have $d(C_k,(P\cap F_0)\cup\mathcal{D}_1)>2$. As $s_0=0$ and $P_2\cap F_{s_0}=P\cap F_0\supset C_0$, we have $d(C_0,P\cap F_0)\cup\mathcal{D}_1)=0$. Let m be the smallest number ($1\le m\le k$) such that $d(C_m,(P\cap F_0)\cup\mathcal{D}_1)\ge 2$. Since m is the smallest such number and $m\ge 1$, $d(C_{m-1},(P\cap F_0)\cup\mathcal{D}_1)\le 1$. By the discussion after the proof of Lemma 3.9 and as above, for any curves α and β in C_{m-1} and C_m respectively, $d(\alpha,\beta)$ is smaller than or equal to the number of saddle tangencies in the special component of $P_2\cap(F\times[s_{m-1},s_m])$ and $d(\alpha,\beta)\le 2$. We may choose α to be the curve in C_{m-1} realizing $d(\alpha,(P\cap F_0)\cup\mathcal{D}_1)=d(C_{m-1},(P\cap F_0)\cup\mathcal{D}_1)$. So $d(\alpha,(P\cap F_0)\cup\mathcal{D}_1)\le 1$. Hence for any curve β in C_m, $d(\beta,(P\cap F_0)\cup\mathcal{D}_1)\le d(\beta,\alpha)+d(\alpha,(P\cap F_0)\cup\mathcal{D}_1)\le 2+1=3$.

Let Q' be a component of $P_2\cap(F\times[s_m,1])$ that connects F_{s_m} and F_1, ie $\partial Q'$ contains curves in both F_{s_m} and F_1. By the definition of C_i, $\partial Q'\cap F_{s_m}\subset C_m$ and $\partial Q'\cap F_1\subset C_k$. Since $d(C_m,(P\cap F_0)\cup\mathcal{D}_1)\ge 2$, we have $d(C_m,\mathcal{D}_1)\ge 2$. Similarly, $d(C_k,\mathcal{D}_1)\ge 2$ by our assumption. Hence $d(\partial Q',\mathcal{D}_1)\ge 2$. This implies that $\partial Q'$ are essential curves in P, to see this, if a curve γ in $\partial Q'$ is trivial in P, then γ bounds a disk in P and by Proposition 6.1, $d(\partial Q',\mathcal{D}_1)\le d(\gamma,\mathcal{D}_1)\le 1$, a contradiction. Thus Q' must be an essential subsurface of P, and P cannot be a 2–sphere or disk. In particular, $\chi(P)\le\chi(Q')\le 0$.

Let Γ be the total number of saddle tangencies in those special components of $Q' \cap (F \times [s_i, s_{i+1}])$, $i = m, \dots, k-1$. Note that we are only counting the saddle tangencies in Q' not all saddle tangencies. As $\partial Q' \subset F_{s_m} \cup F_{s_k}$ $(s_k = 1)$, by our construction, $-\chi(Q') \geq \Gamma$ (note that this is an inequality because a component of $Q' \cap (F \times [s_i, s_{i+1}])$ may be ∂–parallel as in part (3) of our assumption above). Hence $-\chi(P) \geq -\chi(Q') \geq \Gamma$.

Let γ_k be a component of $\partial Q' \cap F_{s_k} \subset C_k = Q \cap F_1$. Recall that in the argument earlier, we showed that we can successively find a curve γ_i in each $Q' \cap F_{s_i}$ $(i = 1, \dots, k)$ such that $d(\gamma_{i-1}, \gamma_i) \leq n_i$ for all i, where n_i is the number of saddle tangencies in the special component of $Q' \cap (F \times [s_{i-1}, s_i])$. Thus $d(\gamma_m, \gamma_k) \leq \Gamma \leq -\chi(Q') \leq -\chi(P)$. Recall that we have concluded earlier that $d(\gamma_m, (P \cap F_0) \cup \mathcal{D}_1) \leq 3$. Since $s_k = 1$ and γ_k is a component of $P_2 \cap F_{s_k} = Q \cap F_1$, we have

$$d(Q \cap F_1, (P \cap F_0) \cup \mathcal{D}_1) \leq d(\gamma_k, \gamma_m) + d(\gamma_m, (P \cap F_0) \cup \mathcal{D}_1) \leq \Gamma + 3 \leq 3 - \chi(P). \quad \square$$

Corollary 6.3 *There is a number K depending on M_2 such that if $d(M) \geq K$ then $M = M_1 \cup_F M_2$ is irreducible and ∂–irreducible.*

Proof Suppose M is reducible or ∂–reducible and let P be either an essential 2–sphere or a compressing disk in M. Since M_i is irreducible and $\partial M_i - F$ is incompressible in M_i for both $i = 1, 2$, $P \cap F \neq \varnothing$ after any isotopy on P. If $P \cap M_2$ is compressible in M_2, then we can compress $P \cap M_2$ and obtain a new essential 2–sphere or compressing disk in M. So after finitely many such operations, we may assume $P \cap M_2$ is incompressible in M_2. Moreover, as in Lemma 6.2, after pushing parts of $P \cap M_2$ into M_1 via ∂–compressions, we may assume that $Q = P \cap M_2$ is an essential planar surface in M_2. Since each component of ∂Q bounds a disk in P, by Proposition 6.1, $d(\gamma, \mathcal{D}_1) \leq 1$ for each component γ of ∂Q.

If M_2 is not a twisted I–bundle, let Ω_2 be the fixed essential surface in M_2 used in defining $d(M) = d(\mathcal{D}_1, \partial\Omega_2 \cap F)$. Since Q is planar with all but at most one boundary component in F, by Lemma 3.7, there is a number K' depending on Ω_2 such that $d(\partial\Omega_2 \cap F, \gamma) \leq K'$, where γ is a component of ∂Q. Hence $d(M) = d(\partial\Omega_2 \cap F, \mathcal{D}_1) \leq d(\partial\Omega_2 \cap F, \gamma) + d(\gamma, \mathcal{D}_1) \leq K' + 1$.

If M_2 is a twisted I–bundle, then Q must be an essential annulus and hence $d(M) = d(\mathcal{A}_{M_2}, \mathcal{D}_1) \leq 1$, where $\mathcal{A}_{M_2}$ is the annulus complex of the twisted I–bundle.

Thus in any case, if $d(M) > K' + 1$, no essential 2–sphere or compressing disk P exists. $\square$

Corollary 6.4 *Let F' be the surface obtained by maximally compressing F in M_1 and removing all resulting 2–sphere components. Suppose $F' \neq \varnothing$. Then there is a number K depending on M_2 such that if $d(M) \geq K$, F' is incompressible in M.*

Proof We may assume that $F' \subset \mathrm{int}(M_1)$. By our construction, F' is incompressible in M_1. Suppose F' is compressible in M and let D be a compressing disk. So $D \cap F \neq \varnothing$. As in Corollary 6.3, we may assume a component Q of $D \cap M_2$ is essential in M_2. By Proposition 6.1, $d(\gamma, \mathcal{D}_1) \leq 1$ for each component γ of ∂Q. Now the proof is the same as the proof of Corollary 6.3. □

Part (2) of Theorem 1.2 also follows from the arguments above. However, we also need the following theorem from [16] which says that a graph complement in S^3 always contains a nice planar surface.

Lemma 6.5 [16] *Let Γ be any graph in S^3. Then there is a planar surface P properly embedded in $S^3 - N(\Gamma)$ such that all but at most one of the components of ∂P bound compressing disks in the handlebody $\overline{N(\Gamma)}$ and P is either*

(1) *strongly irreducible and ∂–strongly irreducible,*

(2) *essential (possibly an essential disk), or*

(3) *nonseparating and incompressible in $S^3 - N(\Gamma)$.*

Proof of Theorem 1.2 If F is incompressible in both M_1 and M_2, then Theorem 1.2 holds trivially. If F is compressible in both M_1 and M_2, then as in Section 5, Casson–Gordon [3] implies that if $d(M) \geq 2$, then M is irreducible and ∂–irreducible and $M \not\cong S^3$. Therefore we only need to consider the case that F is compressible on one side. Suppose F is compressible in M_1 but incompressible in M_2. Part (1) of Theorem 1.2 in this case is proved in Corollary 6.3. Suppose part (2) of Theorem 1.2 fails and $M = S^3$.

Since $M = S^3$ does not contain an incompressible surface, by Corollary 6.4, if $d(M)$ is sufficiently large, we may suppose $F' = \varnothing$ and M_1 must be a handlebody. We may view M_1 as a neighborhood of a graph in $M = S^3$. So there is a planar surface P properly embedded in M_2 as in Lemma 6.5. As F is incompressible in M_2, P is not a compressing disk in M_2 and hence a component of ∂P bounds a compressing disk in M_1 and $d(\mathcal{D}_1, \partial P) = 0$.

If P is nonseparating and incompressible as in part (3) of Lemma 6.5, then one can perform some ∂–compressions and obtains an essential planar surface Q. Moreover, by Lemma 3.6, $d(\partial P, \partial Q) \leq \max\{1,\ 4g + 2b - 2\} = 1$ since $g = 0$ and $b = 0$ in this case. Since $d(\mathcal{D}_1, \partial P) = 0$, this means that $d(\mathcal{D}_1, \partial Q) \leq 2$. Thus in any of the 3 possibilities of Lemma 6.5, we have a planar surface Q in M_2 that is either essential or strongly irreducible and ∂–strongly irreducible such that $d(\mathcal{D}_1, \partial Q) \leq 2$.

Since $M = S^3$, M_2 cannot be a twisted I–bundle. Let Ω_2 be the fixed essential surface in M_2 used in defining $d(M)$. Since Q is planar, by Lemma 3.7, there is a number K' depending on $g(\Omega_2)$ such that $d(\partial\Omega_2 \cap F, \gamma) \le K'$, where γ is a component of ∂Q. Hence $d(M) = d(\partial\Omega_2 \cap F, \mathcal{D}_1) \le d(\partial\Omega_2 \cap F, \gamma) + d(\gamma, \mathcal{D}_1) \le K' + 2$. Thus if $d(M) > K' + 2$, $M \ncong S^3$. □

In the remainder of the paper, we assume M is not S^3 and hence our Heegaard surfaces are not S^2.

Lemma 6.6 *For any $g \ge 1$, there is a number K depending only on M_2 and g, such that if $d(M) \ge K$ then any closed orientable incompressible surface in M of genus g can be isotoped disjoint from F.*

Proof By Corollary 6.3, we may assume $d(M)$ is so large that M is irreducible and ∂–irreducible. Let P be a closed orientable incompressible surface in M of genus g and suppose $F \cap P \neq \varnothing$ after any isotopy.

Let D be a compressing disk for F in M_1. If $P \cap D$ contains a closed curve, since P is incompressible, a standard isotopy on P can remove this intersection curve. Moreover, by shrinking D to be sufficiently small while fixing P, we can also isotope F to eliminate all the arcs in $P \cap D$. Thus, after isotopy, we may assume $P \cap D = \varnothing$. Since P is incompressible and M is irreducible, after isotopy, we may also assume every curve in $P \cap F$ is essential in F. Since $D \cap P = \varnothing$, for any component γ of $P \cap F$, $d(\gamma, \mathcal{D}_1) \le d(\gamma, \partial D) \le 1$.

Since P is incompressible in M and M is irreducible, after some isotopy, we may assume that $P \cap M_2$ is incompressible in M_2. Now we apply Lemma 6.2, setting M', P and P_2 in Lemma 6.2 to be M, P and $P \cap M_2$ above respectively. By Lemma 6.2 and since $F \cap P \neq \varnothing$ after any isotopy, there is an essential surface Q in M_2 obtained by ∂–compressing $P \cap M_2$ such that $d(\partial Q, (P \cap F) \cup \mathcal{D}_1) \le 3 - \chi(P) = 2g + 1$. For any component γ of $P \cap F$, by our earlier assumption, $d(\gamma, \mathcal{D}_1) \le 1$. This implies that $d(\partial Q, \mathcal{D}_1) \le 2g + 2$. Moreover, by our construction, the genus of Q is at most g.

If M_2 is a twisted I–bundle, then Q must be an essential annulus and hence $d(M) = d(\mathcal{A}_{M_2}, \mathcal{D}_1) \le d(\partial Q, \mathcal{D}_1) \le 2g + 2$, where $\mathcal{A}_{M_2}$ is the annulus complex of the twisted I–bundle.

If M_2 is not a twisted I–bundle, let Ω_2 be the fixed essential surface in M_2 used in defining $d(M) = d(\mathcal{D}_1, \partial\Omega_2 \cap F)$. Since $g(Q) \le g$, by Lemma 3.7, there is a number K' depending on Ω_2 and g, such that $d(\partial\Omega_2 \cap F, \partial Q) \le K'$. Hence we can find a component γ_Q of ∂Q such that $d(M) = d(\partial\Omega_2 \cap F, \mathcal{D}_1) \le d(\partial\Omega_2 \cap F, \gamma_Q) + d(\gamma_Q, \mathcal{D}_1) \le K' + 2g + 2$. Thus if $d(M) > K' + 2g + 2$, $P \cap F = \varnothing$ after isotopy. □

Let S be an unstabilized Heegaard surface of genus g. As in Theorem 1.4, the untelescoping of the Heegaard splitting [25] gives a decomposition $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M and $g(F_i) \leq g$. By Lemma 6.6, we may assume $d(M)$ is so large that each F_i is disjoint from F after some isotopy. Thus we may assume $F \subset N_j$ for some j. Without loss of generality, we may suppose N_j is connected. Now we consider the strongly irreducible Heegaard surface P_j of N_j in the untelescoping construction. Let X and Y be the two compression bodies in the Heegaard splitting of N_j along P_j. We have $P_j = \partial_+ X = \partial_+ Y$ and $g(P_j) \leq g$.

Let F' be the surface obtained by maximally compressing F in M_1 and removing all resulting 2–sphere components. We may assume $F' \subset \mathrm{int}(M_1)$. Let M_F be the compression body bounded by F and F' in M_1. If $F' = \varnothing$ then $M_F = M_1$ is a handlebody.

In the next two lemmas, we prove that if $d(M)$ is sufficiently large, then F cannot lie in a product neighborhood of any incompressible surface F_i in the untelescoping.

Lemma 6.7 *Let E be an orientable incompressible closed surface in M and $E \times I$ a product neighborhood of E. Suppose $M_2 \subset \mathrm{int}(E \times I)$, then $d(M) < K$ for some K depending only on M_2.*

Proof The hypothesis that $M_2 \subset \mathrm{int}(E \times I) \subset \mathrm{int}(M)$ implies that $F \subset \mathrm{int}(E \times I)$ and $\partial M_2 = F$ (ie M_2 has no other boundary component). Since E is incompressible in M and $F \subset \mathrm{int}(E \times I)$, every compressing disk for F can be isotoped into $E \times I$. Thus, after isotopy, we may assume the compression body M_F described above lies in $E \times I$. As $M_2 \subset E \times I$ and $E \times I \neq M$, $F' \neq \varnothing$.

By our construction of M_F and since $\partial M_2 = F$, the submanifold $M_2 \cup M_F$ of M is bounded by F'. The assumption above says that $M_2 \cup M_F \subset E \times I$. By Corollary 6.4, we may assume that $d(M)$ is so large that F' is incompressible in M. This means that we have a connected submanifold $M_2 \cup M_F$ of $E \times I$ bounded by the incompressible surface F'. As each component of a closed incompressible surface in $E \times I$ is parallel to E, the connected submanifold $M_2 \cup M_F$ must be isotopic to $E \times I$ in $E \times I$. Thus after isotopy, we may assume $M_2 \cup M_F = E \times I$ and $F' = E \times \partial I$.

The compression body M_F can be obtained by adding 1–handles to a small product neighborhood of $F' = E \times \partial I$. So there is a graph G properly embedded in $E \times I$ which corresponds to the 1–handles in M_F, such that after isotopy $M_F = \overline{N(G \cup (E \times \partial I))}$, where $N(G \cup (E \times \partial I))$ is a regular neighborhood of $G \cup (E \times \partial I)$ in $E \times I$. Hence we may view $M_2 = (E \times I) - N(G \cup (E \times \partial I))$. Since the compression body M_F is connected, the graph G connects the two components of $F' = E \times \partial I$.

Our goal is to use the intersection of M_2 with a vertical annulus in $E \times I$ to construct an essential surface in M_2, and then apply Lemma 3.7. We first show that a vertical annulus in $E \times I$ cannot be totally isotoped into $M_F = \overline{N(G \cup (E \times \partial I))}$.

We claim that $M_F = \overline{N(G \cup (E \times \partial I))}$ does not contain a properly embedded incompressible annulus A whose two boundary circles lie in different components of $F' = E \times \partial I$. As $M_F = \overline{N(G \cup (E \times \partial I))}$, after some handle slides if necessary, we may assume that there is a point x in the graph G that separates $E \times \partial I$ in the sense that no component of $G - x$ connects the two components of $F' = E \times \partial I$. This means that there is a compressing disk D for F in M_F such that D is separating in M_F and the two components of F' lie in different components of $M_F - D$. Suppose there is a properly embedded annulus A described above. As $\partial D \subset F$ and $\partial A \subset F'$, $A \cap \partial D = \varnothing$. This means that $A \cap D$ (if not empty) consists of simple closed curves. Since A is incompressible, any curve in $A \cap D$ must bound disks in both A and D. Hence after some isotopies removing closed curves in $A \cap D$ that are trivial in both A and D, we have $A \cap D = \varnothing$. However, this is impossible since A connects the two components of F' but the two components of F' lie in different components of $M_F - D$.

Let A be an essential vertical annulus in $E \times I$. We may assume either $A \cap G = \varnothing$ or $A \cap G$ consists of a finite number of points in $\operatorname{int}(A)$. Hence $P = \overline{A - M_F}$ is a planar surface properly embedded in M_2. After some standard cutting and pasting as in the proof of Corollary 6.3, we may assume P is incompressible in M_2.

The conclusion earlier says M_F does not contain a properly embedded incompressible annulus whose two boundary circles lie in different components of $F' = E \times \partial I$. So A cannot be isotoped totally into M_F and we cannot push P into M_F. This means that, after ∂–compressions on P, we obtain an essential planar surface Q ($Q \neq \varnothing$) properly embedded in M_2. Since we can view a ∂–compression on P as part of an isotopy on A pushing the ∂–compressing disk into M_F, we may view Q as a possibly disconnected subsurface of A and $Q = A \cap M_2$.

Next we show that there is a curve $\gamma_Q \subset \partial Q$ such that $d(\gamma_Q, \mathcal{D}_1) \leq 1$. Since F is incompressible in M_2, no component of Q is a disk. If a component of Q is not an essential subannulus of A, then there is a component γ_Q of ∂Q that bounds a disk in A. By Proposition 6.1, $d(\gamma_Q, \mathcal{D}_1) \leq 1$. If every component of Q is an essential subannulus of A, then every component of $A - \operatorname{int}(Q)$ is also an essential subannulus of A. Let A' be a component of $A - \operatorname{int}(Q)$ that contains a boundary circle of A. So A' is an annulus properly embedded in M_F with one component of $\partial A'$ in $F' = E \times \partial I$ and the other component of $\partial A'$, denoted by γ_Q, in $\partial Q \subset F$. Since ∂A is essential, A' is incompressible in M_F. If A' is ∂–compressible in M_F, then an essential arc of A' bounds a ∂–compressing disk and hence has both endpoints in the same boundary component of M_F, which implies that $\partial A'$ lies in the same boundary component of M_F. However, by our choice of A', one boundary circle of A' lies in $F' = E \times \partial I$ and the other boundary circle of A' lies in F. So A' must be ∂–incompressible. Hence A' is an essential annulus in M_F. After some standard cutting and pasting, one can always find a compressing disk for F in M_F disjoint from any essential annulus in M_F. Thus $d(\gamma_Q, \mathcal{D}_1) \le 1$. Hence, in any case, there is a curve $\gamma_Q \subset \partial Q$ such that $d(\gamma_Q, \mathcal{D}_1) \le 1$.

Note that M_2 cannot be a twisted I–bundle, since $E \times I$ does not contain any closed embedded nonorientable surface. Let Ω_2 be the essential surface used in defining $d(M)$. Since Q is a planar surface, by Lemma 3.7, there is a number K' depending on $g(\Omega_2)$ such that $d(\partial\Omega_2 \cap F, \gamma_Q) \le K'$. Therefore $d(M) = d(\partial\Omega_2 \cap F, \mathcal{D}_1) \le d(\partial\Omega_2 \cap F, \gamma_Q) + d(\gamma_Q, \mathcal{D}_1) \le K' + 1$. □

Lemma 6.8 *Let E be a closed orientable incompressible surface of genus g in M. Then there is a number K depending only on g and M_2 such that if F lies in a product neighborhood of E in M, then $d(M) < K$.*

Proof We may suppose $E \subset \text{int}(M)$ (E may be parallel to a boundary component of M). Let $E \times I$ be a product neighborhood of E in $\text{int}(M)$ and suppose $F \subset \text{int}(E \times I)$. By Lemma 6.7, we may assume $M_2 \not\subset E \times I$. So at least one component of $E \times \partial I$ lies in M_2.

Since both M_1 and M_2 are irreducible and $M \ncong S^3$, F does not lie in a 3–ball in $E \times I$ and hence we can find a vertical annulus A of $E \times I$ that cannot be isotoped disjoint from F. Let $N = M_2 \cap (E \times I)$ and $P_2 = A \cap N$. Since at least one component of $E \times \partial I$ lies in M_2, one or two components of ∂P_2 lie in $E \times \partial I$.

Let D be a compressing disk for F in M_1. Since E is incompressible in M and $F \subset \text{int}(E \times I)$, D can be isotoped into $E \times I$. By shrinking D to be sufficiently small, we may assume $D \cap A = \varnothing$ and hence $\partial P_2 \cap \partial D = \varnothing$. This means that $d(\gamma, \mathcal{D}_1) \le 1$ for any component γ of $\partial P_2 \cap F = A \cap F$. Moreover, since M is irreducible, after some standard cutting and pasting as in the proof of Corollary 6.3, we may assume that P_2 is incompressible in N.

Note that ∂N consists of F and one or both components of $E \times \partial I$. Moreover, a component of ∂P_2 lies in $E \times \partial I$ and this implies that P_2 is not totally in a product neighborhood of F in N. So we can apply Lemma 6.2, setting M', P and P_2 in Lemma 6.2 to be $E \times I$, A and P_2 above respectively. After performing some ∂–compressions on P_2 in N, we obtain an essential surface Q such that $Q \cap F \neq \varnothing$ and $d(Q \cap F, (A \cap F) \cup \mathcal{D}_1) \le 3 - \chi(A) = 3$. So there is a component δ of $Q \cap F$ such that $d(\delta, (A \cap F) \cup \mathcal{D}_1) \le 3$. Since $d(\gamma, \mathcal{D}_1) \le 1$ for any component γ of $\partial P_2 \cap F = A \cap F$, $d(\delta, \mathcal{D}_1) \le 4$ for some component δ of $Q \cap F$.

If M_2 is a twisted I–bundle over a closed nonorientable surface, since a component of $E \times \partial I$ lies in M_2 and is incompressible in M_2, E must be parallel to $\partial M_2 = F$. However, this contradicts that F is compressible in M_1 but E is incompressible in M. Thus M_2 cannot be a twisted I–bundle.

Let Ω_2 be the surface in M_2 used in defining $d(M) = d(\mathcal{D}_1, \partial\Omega_2 \cap F)$. Note that ∂N is incompressible in M_2, since ∂N consists of F and one or both components of $E \times \partial I$. So we may assume $\Omega_2 \cap N$ is an essential subsurface of Ω_2 and $-\chi(\Omega_2 \cap N) \leq -\chi(\Omega_2)$. N cannot be an I–bundle, since $F \subset \partial N$ is compressible in M but $\partial N - F \subset E \times \partial I$ is incompressible in M. By our construction of P_2 and Q, Q is a planar surface in N with all but one or two boundary components in F. Thus by Lemma 3.7, $d(\partial\Omega_2 \cap F, Q \cap F) \leq K'$ for some K' depending only on $\chi(\Omega_2)$. Since $d(\delta, \mathcal{D}_1) \leq 4$ for some component δ of $Q \cap F$, $d(M) = d(\partial\Omega_2 \cap F, \mathcal{D}_1) \leq d(\partial\Omega_2 \cap F, \delta) + d(\delta, \mathcal{D}_1) \leq K' + 5$. □

Let $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$ be the decomposition in Theorem 1.4 given by the untelescoping of an irreducible Heegaard splitting. Recall that by Lemma 6.6, we have assumed that $F \subset \mathrm{int}(N_j)$ for some j. In the next two lemmas, we discuss the case that F is also disjoint from the Heegaard surface P_j of N_j.

Lemma 6.9 *Let N_j be the submanifold of M between F_j and F_{j+1} in the untelescoping construction as above (we assume N_j is connected), and let P_j be the strongly irreducible Heegaard surface of N_j. Suppose $F \subset \mathrm{int}(N_j)$ and $F \cap P_j = \varnothing$. Then there is a number K such that if $d(M) > K$, $P_j \subset \mathrm{int}(M_2)$.*

Proof Since $P_j \cap F = \varnothing$, P_j lies in either $\mathrm{int}(M_1)$ or $\mathrm{int}(M_2)$. Suppose the lemma is not true and $P_j \subset \mathrm{int}(M_1)$. Let X and Y be the two compression bodies in the Heegaard splitting of N_j along P_j. We may suppose $F \subset \mathrm{int}(X)$ and let $Z = X \cap M_1$. Since $F \subset \mathrm{int}(X)$ and $P_j \subset \mathrm{int}(M_1)$, F and P_j are boundary components of Z. So we may view $Z \cup M_2$ as a submanifold of M. By our construction, $X \subset Z \cup M_2$. Since P_j is compressible on both sides, there is a compressing disk D for P_j in $X \subset Z \cup M_2$. We claim that D can be isotoped into Z if $d(M)$ is large.

Suppose on the contrary that D cannot be isotoped into Z. So $D \cap F \neq \varnothing$ after any isotopy on D and we may assume $|D \cap F|$ is minimal in the isotopy class of D. Let Q be a component of $D \cap M_2$. Since $D \cap F \neq \varnothing$ and $|D \cap F|$ is minimal, we may assume Q cannot be pushed into M_1 and Q is incompressible in M_2 after isotopy. As in the proofs of Corollary 6.3 and Corollary 6.4, we can perform some ∂–compressions on Q in M_2 and obtain an essential planar surface Q' in M_2. We may regard Q' as a subsurface of D. Since every component of $\partial Q'$ bounds a disk in D, by Proposition 6.1, for any component γ of $\partial Q'$, $d(\gamma, \mathcal{D}_1) \leq 1$. Now similar to the proofs of Corollary 6.3 and Corollary 6.4, this implies that $d(M) \leq K$ for some K depending only on M_2. Thus if $d(M)$ is sufficiently large, every compressing disk of P_j in $Z \cup M_2$ can be isotoped into Z.

Let W be the surface obtained by maximally compressing P_j in Z and removing all resulting 2–sphere components. Since a maximal compression on P_j in X yields $\partial_- X$, the conclusion above implies that W is parallel to $\partial_- X$. This means that F must lie in a product region bounded by W and $\partial_- X$. Now Lemma 6.9 follows from Lemma 6.8. □

Lemma 6.10 *Let N_j be the submanifold of M between F_j and F_{j+1} in the untelescoping construction as above (we assume N_j is connected), and let P_j be the strongly irreducible Heegaard surface of N_j. Suppose $F \subset \operatorname{int}(N_j)$, $P_j \subset \operatorname{int}(M_2)$ and P_j is compressible on both sides in M_2. Then there is a number K such that, if $d(M) > K$, F is isotopic to a middle surface of a compression body in the Heegaard splitting along P_j (see Definition 1.3).*

Proof Let X and Y be the two compression bodies in the Heegaard splitting of N_j along P_j. As $P_j \subset \operatorname{int}(M_2)$, $F \cap P_j = \varnothing$. Since $F \subset \operatorname{int}(N_j)$ and $F \cap P_j = \varnothing$, we may suppose $F \subset \operatorname{int}(X)$. Let $Z = X \cap M_2$. So P_j and F are boundary components of Z. Since P_j is compressible on both sides in M_2, P_j is compressible in Z. Let P' be the surface obtained by maximally compressing P_j in Z and removing all resulting 2–sphere components. So P' is incompressible in Z. Since P_j is strongly irreducible, Casson–Gordon [3] implies that P' is also incompressible on the other side. Hence P' is incompressible in M_2.

Let N be the submanifold of Z between F and P'. So ∂N is incompressible in M_2. If N is an I–bundle (ie, if F is parallel to a component of P'), then by our construction, F is a middle surface of the compression body X and the lemma holds. Next we suppose N is not an I–bundle.

If P' is parallel to $\partial_- X$, then by our construction, F lies in the product region bounded by P' and $\partial_- X$. By Lemma 6.8, we may suppose $d(M)$ is so large that F does not lie in a product neighborhood of $\partial_- X$. Thus we may assume P' is not parallel to $\partial_- X$, and this means that P' must be compressible in X but incompressible in Z. Let D be a compressing disk for P' in X. By the construction of P', $D \cap F \neq \varnothing$ after any isotopy on D. Let Q be the component of $D \cap N$ that contains ∂D. After isotopy, we may assume Q is an essential surface in N. Note that one component of ∂Q (ie, ∂D) lies in P' and all other components of ∂Q lie in F. Moreover, $Q \cap F = \partial Q - \partial D \neq \varnothing$ and every curve of $Q \cap F$ bounds a subdisk of D. By Proposition 6.1, $d(\mathcal{D}_1, \gamma) \leq 1$ for every component γ of $Q \cap F$.

Since P' is incompressible in M_2 and F is not parallel to P', M_2 cannot be a twisted I–bundle. Let Ω_2 be the fixed essential surface in M_2 used in defining $d(M) = d(\mathcal{D}_1, \partial\Omega_2 \cap F)$. Since P' is incompressible in M_2, we may assume $\Omega' = \Omega_2 \cap N$ is an essential subsurface of Ω_2 and Ω' is essential in N. Thus $-\chi(\Omega') \leq -\chi(\Omega_2)$.

By Lemma 3.7, there is a number K' depending on $\chi(\Omega_2)$ such that the distance $d(Q \cap F, \partial\Omega' \cap F) \leq K'$. Let γ be a component of $Q \cap F$ realizing $d(\gamma, \partial\Omega' \cap F) = d(Q \cap F, \partial\Omega' \cap F)$. Since $\partial\Omega' \cap F = \partial\Omega_2 \cap F$ and $d(\mathcal{D}_1, \gamma) \leq 1$ for every γ in $Q \cap F$, $d(M) = d(\mathcal{D}_1, \partial\Omega_2 \cap F) \leq d(\mathcal{D}_1, \gamma) + d(\gamma, \partial\Omega_2 \cap F) \leq 1 + K'$. □

Lemma 6.9 and Lemma 6.10 say that if $F \subset N_j$ and $F \cap P_j = \varnothing$, then either Theorem 1.5 holds, or (1) P_j lies in M_2 and (2) P_j cannot be compressible on both sides in M_2.

7 Intersection of F with sweepout surfaces

Let $M = M_1 \cup_F M_2$ be as in Section 6 and S an unstabilized genus g Heegaard surface of M. As in Theorem 1.4, let $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$ be the decomposition given by the untelescoping of S, where each F_i is incompressible in M. By Lemma 6.6, we may assume that $d(M)$ is so large that each F_i is disjoint from F. Suppose $F \subset \mathrm{int}(N_j)$. Without loss of generality, we may assume N_j is connected. Let $P_i = P$ be the strongly irreducible Heegaard surface of N_i in the untelescoping.

Let F' be the surface obtained by maximally compressing F in M_1 and deleting all the resulting 2–sphere components. We consider the compression body M_F bounded by F' and F. So $\partial_+ M_F = F$ and $\partial_- M_F = F'$. Since $F \subset \mathrm{int}(N_j)$ and ∂N_j is incompressible in M, every compressing disk of F in M_1 can be isotoped into N_j. Thus we may assume $M_F \subset \mathrm{int}(N_j)$.

Let X and Y be the two compression bodies in the Heegaard splitting of N_j along $P_j = P$. Let graphs G_X and G_Y be the cores of the compression bodies X and Y respectively, $\Sigma_X = G_X \cup \partial_- X$ and $\Sigma_Y = G_Y \cup \partial_- Y$, such that $N_j - (\Sigma_X \cup \Sigma_Y) \cong P \times (0,1)$. We consider the sweepout $f\colon P \times I \to N_j$ such that $f|_{P\times(0,1)}$ is an embedding, $f(P \times \{0\}) = \Sigma_X$ and $f(P \times \{1\}) = \Sigma_Y$. We denote $f(P \times \{t\})$ by P^t. Each P^t is isotopic to $P_j = P$ if $t \in (0,1)$.

Similarly, let graph G_F be the core of the compression body M_F and $\Sigma_F = G_F \cup F'$ such that $M_F - \Sigma_F \cong F \times (0,1]$. After isotopy, we may assume that the graphs G_X, G_Y and G_F are pairwise disjoint in N_j. We may suppose F' is transverse to the graphs G_X and G_Y and suppose F' is transverse to each P^t except for finitely many levels t where $F' \cap P^t$ contains exactly one center or saddle tangency. We may suppose $G_F \cap P^t$ consists of finitely many points for each $t \in I$. Moreover, we may suppose M_F is a small neighborhood of $\Sigma_F = G_F \cup F'$ in N_j and suppose F is transverse to each P^t except for finitely many levels t where $F \cap P^t$ contains exactly one center or saddle tangency. As M_F is a small neighborhood of Σ_F, F is transverse to the graphs G_X and G_Y.

We use Λ to denote the union of $\partial I = \{0, 1\}$ and the levels $t \in I$ where F is not transverse to P^t.

For any P^t, $t \in (0, 1)$, we use X_t (resp. Y_t) to denote the closure of the component of $N_j - P^t$ that contains Σ_X (resp. Σ_Y). Recall that $P = P_j$ is a Heegaard surface of N_j and P^t is parallel to P. So if $t \in (0, 1)$, X_t and Y_t are the two compression bodies corresponding to X and Y respectively.

Labelling A number $t \in I - \Lambda$ is labelled X (resp. Y) if, there is an essential curve γ in P^t such that

(1) γ bounds a compressing disk in X_t (resp. Y_t), and

(2) $\gamma \subset M_2$ and γ bounds an embedded disk D in M_2 that is transverse to P^t.

This is a labelling for all $t \in I - \Lambda$ and we do not assign any label for t if F is tangent to P^t.

Remark 7.1 The graph G_F can be viewed as the core of the 1–handles attached to (a product neighborhood of) $F' = \partial_- M_F$ in the compression body M_F. Recall that we have assumed M_F is a small neighborhood of $F' \cup G_F$ and F is a boundary component of this small neighborhood. For any point $t \notin \Lambda$, since we have assumed that P^t intersects the graph G_F in finitely many points and since M_F is a small neighborhood of $F' \cup G_F$, one can always find a compressing disk (which can be chosen to be a cocore of the 1–handles corresponding to some point in $G_F - P^t$) for F in M_F that is disjoint from P^t. Thus $d(\gamma, \mathcal{D}_1) \leq 1$ for any component γ of $F \cap P^t$ that is essential in F.

Theorem 1.5 follows from the following 4 claims.

Claim 1 *There is a number K_1 such that if $d(M) > K_1$, then for a sufficiently small $\epsilon > 0$, ϵ is labelled X and $1-\epsilon$ is labelled Y.*

Proof Suppose $d(M)$ is larger than the constant K in Lemma 6.9.

Since the compression body M_F lies in N_j, F is disjoint from $\partial N_j = \partial_- X \cup \partial_- Y$. The graph G_X cannot totally lie in M_1, because otherwise P^ϵ lies in M_1 for a sufficiently small ϵ, contradicting Lemma 6.9. Thus $G_X \cap \mathrm{int}(M_2) \neq \varnothing$ and X_ϵ must have a compressing disk D lying in M_2. Hence ϵ is labelled X.

We can apply the same argument to G_Y and conclude that $1-\epsilon$ is labelled Y for sufficiently small ϵ. □

Claim 2 *There is a number K_2 such that if $d(M) > K_2$, then every $t \in I - \Lambda$ has a label X or Y.*

Proof Suppose on the contrary that some $t \in I - \Lambda$ has no label. We assume $d(M)$ is larger than the constant K in Lemma 6.9.

Let $P_2 = P^t \cap M_2$. By Lemma 6.9, $P_2 \neq \varnothing$. Our goal is to use P_2 to construct an incompressible surface in M_2 and then apply the inequalities in Lemma 6.2 and Lemma 3.7 to get a bound on the distance $d(M)$.

We first suppose P_2 is compressible in M_2 and let D be a compressing disk for P_2 in M_2. Since t is not labelled, D cannot be a compressing disk for P^t and ∂D must be trivial in P^t but essential in P_2. We compress P^t along D and delete the resulting 2–sphere component. Let P' be the remaining surface after this operation. Since M is irreducible, P' is isotopic to P^t. Suppose $P' \cap M_2$ is still compressible in M_2 and let D' be a compressing disk of $P' \cap M_2$ in M_2. Suppose $\partial D'$ is essential in P'. Since $D' \cap P' = \partial D'$, by the operation above and after a slight perturbation on D' if necessary, we may view $\partial D'$ as an essential curve in P^t bounding an embedded disk $D' \subset M_2$. Since D' may intersect the 2–sphere component that we eliminated in the operation above, $\mathrm{int}(D') \cap P^t$ may not be empty. Nonetheless, since $\partial D'$ is essential in P^t, by Scharlemann's no-nesting lemma [21, Lemma 2.2], $\partial D'$ bounds a compressing disk for P^t in X_t or Y_t. This means that t is labelled X or Y, contradicting our hypothesis. Thus $\partial D'$ must also be trivial in P' and we can perform the same operation on P', ie compress P' along D' and remove the resulting 2–sphere component.

After finitely many such operations, we may assume that $P_2 = P^t \cap M_2$ is incompressible in M_2. If $P^t \cap F = \varnothing$, then by Lemma 6.9, $P^t \subset \mathrm{int}(M_2)$ and $P_2 = P^t$. However, since P^t is separating in N_j and compressible on both sides, $P_2 = P^t$ must be compressible in M_2, a contradiction to our assumption on P_2. Thus $P^t \cap F \neq \varnothing$.

By Lemma 6.9, P_2 does not lie in a product neighborhood of F (otherwise P_2 and hence P^t can be isotoped into M_1). So after some ∂–compressions on P_2, we get an essential surface Q properly embedded in M_2. Now we apply Lemma 6.2, setting M', P and P_2 in Lemma 6.2 to be N_j, P^t and P_2 above respectively, and get $d(\partial Q, (P^t \cap F) \cup \mathcal{D}_1) \leq 3 - \chi(P^t) \leq 2g+1$. By Remark 7.1, $d(\alpha, \mathcal{D}_1) \leq 1$ for every component α of $P^t \cap F$. Hence, $d(\partial Q, \mathcal{D}_1) \leq 2g+2$.

If M_2 is a twisted I–bundle, then Q must be a vertical annulus. Hence $d(M) \leq d(\partial Q, \mathcal{D}_1) \leq 2g+2$.

If M_2 is not a twisted I–bundle, let Ω_2 be the fixed essential surface used in defining $d(M)$. As the genus of Q is at most g, by Lemma 3.7, there is a K' depending on Ω_2 and g, such that $d(\partial\Omega_2 \cap F, \partial Q) \leq K'$. Since $d(\partial Q, \mathcal{D}_1) \leq 2g+2$, this means that $d(M) = d(\Omega_2 \cap F, \mathcal{D}_1) \leq d(\partial\Omega_2 \cap F, \partial Q) + 1 + d(\partial Q, \mathcal{D}_1) \leq K' + 2g + 3$.

Therefore if $d(M)$ is sufficiently large, every $t \in I - \Lambda$ has a label. □

Claim 3 *If some $t \in I - \Lambda$ is labelled both X and Y, then Theorem 1.5 holds.*

Proof In this claim, we assume $d(M)$ is larger than the constants K in Lemma 6.9 and Lemma 6.10.

Let D be an embedded disk in M_2 transverse to P^t with $\partial D \subset P^t \cap M_2$. We call D an *almost compressing disk* for X_t (resp. Y_t) if ∂D bounds a compressing disk in X_t (resp. Y_t).

Suppose $t \in I - \Lambda$ is labelled both X and Y. Then by definition, M_2 contains almost compressing disks D_X and D_Y for X_t and Y_t respectively. Since P^t is strongly irreducible, $\partial D_X \cap \partial D_Y \neq \varnothing$.

Let $P_2 = P^t \cap M_2$. Similar to the proof of Claim 2, our goal is to use P_2 to construct either an incompressible or a strongly irreducible surface in M_2, and then apply the inequalities in Lemma 6.2 and Lemma 3.7 to get a bound on the distance $d(M)$. Although P^t is strongly irreducible in M, P_2 may not be strongly irreducible in M_2 because the boundary curve of a compressing disk for P_2 in M_2 may be a trivial curve in P^t. So we need to perform some operations on P_2 first.

Let Δ be a compressing disk for P_2 in M_2. We say Δ is a *trivial compressing disk* if $\partial\Delta$ is essential in P_2 but trivial in P^t. Suppose a trivial compressing disk Δ lies in $X_t \cap M_2$ and there is an almost compressing disk D_Y for Y_t such that $\partial D_Y \cap \partial\Delta = \varnothing$. Then we can compress P^t along Δ and delete the resulting 2–sphere component. As in Claim 2, the remaining surface P' is isotopic to P^t. Since $\partial D_Y \cap \partial\Delta = \varnothing$, $\partial D_Y \subset P'$ and D_Y remains an almost compressing disk for P'.

For any almost compressing disk D_X for X_t, if a component γ of $\text{int}(D_X) \cap P^t$ is essential in P^t, then by Scharlemann's no-nesting lemma [21, Lemma 2.2], γ must bound a compressing disk for P^t. Since P^t is strongly irreducible and ∂D_X bounds a compressing disk in X_t, the subdisk of D_X bounded by γ must also be an almost compressing disk for X_t. Thus we may choose an almost compressing disk D_X for X_t so that every component of $\text{int}(D_X) \cap P^t$ is trivial in P^t. Since ∂D_X bounds a compressing disk in X_t, this implies that a small neighborhood of ∂D_X in D_X lies in X_t. Now we consider $D_X \cap \Delta$, where Δ is the trivial compressing disk in $X_t \cap M_2$ above. If $D_X \cap \Delta \neq \varnothing$, similar to the proof of Lemma 3.4, we can push the arcs in $D_X \cap \Delta$ across Δ. More specifically, we may suppose $D_X \cap \Delta$ does not contain any closed curve and let α be an arc in $D_X \cap \Delta$ that is outermost in Δ. Then α and a subarc of $\partial\Delta$ bound a subdisk E of Δ and $\text{int}(E) \cap D_X = \varnothing$. Since a small neighborhood of ∂D_X in D_X lies in X_t and $\Delta \subset X_t$, we have $E \cap D_X = \alpha$. So, similar to the proof of Lemma 3.4, we can perform an isotopy by pushing α and D_X across E to eliminate α. After the operation pushing D_X across E above, D_X becomes either one or two disks depending on whether or not both endpoints of α lie in ∂D_X. Since ∂D_X is essential in P^t and each component of $\text{int}(D_X) \cap P^t$ is trivial in P^t, after the operation, the boundary curve of at least one resulting disk is essential in P^t. Hence after the operation pushing D_X across E above, we obtain a new almost compressing disk for X_t with fewer intersection arcs with Δ. After finitely many these operations, we can construct an almost compressing disk D'_X (for X_t) that is disjoint from Δ.

The arguments above say that if there is a trivial compressing disk Δ in $X_t \cap M_2$ such that $\partial\Delta \cap \partial D_Y = \varnothing$ for some almost compressing disk D_Y for Y_t, then after compressing P^t along Δ and deleting the 2–sphere component, the resulting surface still has two almost compressing disks for X_t and Y_t respectively. Therefore, after finitely many such operations on trivial compressing disks as above, we may assume that for each trivial compressing disk Δ, if $\Delta \subset X_t$ then $\partial\Delta \cap \partial D_Y \neq \varnothing$ for every almost compressing disk D_Y for Y_t, and if $\Delta \subset Y_t$ then $\partial\Delta \cap \partial D_X \neq \varnothing$ for every almost compressing disk D_X for X_t. Note that this implies that every curve of $P^t \cap F$ must be essential in F, because otherwise the subdisk of F bounded by an innermost such curve is either a trivial compressing disk disjoint from all almost compressing disks, or a compressing disk of X_t (resp. Y_t) disjoint from an almost compressing disk D_Y (resp. D_X), which contradicts that P^t is strongly irreducible.

Next we show that $P_2 = P^t \cap M_2$ has compressing disks in both $X_t \cap M_2$ and $Y_t \cap M_2$. Suppose P_2 does not have any compressing disk lying in $X_t \cap M_2$. Let D_X be an almost compressing disk for X_t and we may assume $|\text{int}(D_X) \cap P^t|$ is minimal among all almost compressing disks for X_t. If $\text{int}(D_X) \cap P^t = \varnothing$, then D_X is a compressing disk for P_2 lying in $X_t \cap M_2$, contradicting our assumption. So we may suppose $\text{int}(D_X) \cap P^t \neq \varnothing$. Let γ be an innermost component of $\text{int}(D_X) \cap P^t$ and let d_γ be the subdisk of D_X bounded by γ. If γ is trivial in P_2, then we can perform a simple isotopy on D_X to remove γ and get a contradiction to the minimality assumption of $|\text{int}(D_X) \cap P^t|$. Thus γ is essential in P_2 and d_γ is a compressing disk for P_2. Since we have assumed that P_2 does not have any compressing disk lying in $X_t \cap M_2$, $d_\gamma \subset Y_t \cap M_2$. If γ is also essential in P^t, then d_γ is a compressing disk for P^t in Y_t.

However, since ∂D_X bounds a compressing disk for P^t in X_t and $\gamma \cap \partial D_X = \varnothing$, this contradicts that P^t is strongly irreducible. Hence γ must be trivial in P^t and d_γ is a trivial compressing disk for P_2 in Y_t, but this contradicts our earlier assumption that every trivial compressing disk in Y_t intersects every almost compressing disk for X_t because $\gamma \cap \partial D_X = \varnothing$. Therefore, $P_2 = P^t \cap M_2$ must have compressing disks in both $X_t \cap M_2$ and $Y_t \cap M_2$.

Suppose P_2 is not strongly irreducible in M_2. Then there are compressing disks Δ_X and Δ_Y for P_2 in M_2 such that $\Delta_X \subset X_t$ and $\Delta_Y \subset Y_t$ and $\partial\Delta_X \cap \partial\Delta_Y = \varnothing$. By our assumptions above, both Δ_X and Δ_Y must be trivial compressing disks. Now we compress P^t along Δ_X and Δ_Y simultaneously and delete the two resulting 2–sphere components. The remaining surface P' is isotopic to P^t. Suppose $P' \cap M_2$ has an almost compressing disk D'. As in Claim 2, after some perturbation, we may view $\partial D'$ as an essential curve in P^t and view D' as an almost compressing disk of P_2. However, since $D' \cap \Delta_X = \varnothing$ and $D' \cap \Delta_Y = \varnothing$ after isotopy, this contradicts our earlier assumption that every trivial compressing disk must intersect every almost compressing disk on the other side. So P' does not have any almost compressing disk in M_2, and this implies that every compressing disk of $P' \cap M_2$ in M_2 is a trivial compressing disk for P'. We can compress P' along each trivial compressing disk of $P' \cap M_2$ in M_2 and delete the resulting 2–sphere component. By the argument above, the resulting surface does not have any almost compressing disk in M_2 neither. Therefore, after finitely many such operations, we obtain a surface P'' isotopic to P^t and $P'' \cap M_2$ is incompressible in M_2.

The arguments above imply that, after some isotopies/operations on P^t described above, we may assume that $P_2 = P^t \cap M_2$ is either strongly irreducible or incompressible in M_2. If $P^t \cap F = \varnothing$ after the operations above, then by Lemma 6.9 $P^t \subset \mathrm{int}(M_2)$ and hence $P_2 = P^t$. Since P^t is separating in N_j and compressible on both sides, $P^t \cap F = \varnothing$ implies that $P_2 = P^t$ cannot be incompressible in M_2. Hence $P_2 = P^t$ is strongly irreducible and in particular P_2 is compressible on both sides in M_2. In this case, by Lemma 6.10, F is isotopic to a middle surface of the compression body X^t or Y^t and Theorem 1.5 holds.

Suppose Theorem 1.5 fails, then the argument above implies that $P^t \cap F \neq \varnothing$.

By Lemma 6.9, P_2 does not lie in a product neighborhood of F (otherwise P_2 and hence P^t can be isotoped into M_1). As P_2 is either strongly irreducible or incompressible in M_2, Claim 3 basically follows from Lemma 6.2. By Lemma 6.2 (setting M', P and P_2 in Lemma 6.2 to be N_j, P^t and P_2 above respectively), we can perform some ∂–compressions on P_2 in M_2 and obtain a surface Q which is either essential or strongly irreducible and ∂–strongly irreducible, such that $d(\partial Q, (P^t \cap F) \cup \mathcal{D}_1) \leq 3 - \chi(P^t) \leq 2g + 1$. By Remark 7.1, $d(\alpha, \mathcal{D}_1) \leq 1$ for every component α of $P^t \cap F$. Hence, $d(\partial Q, \mathcal{D}_1) \leq 2g + 2$.

Suppose M_2 is not a twisted I–bundle and let Ω_2 be the essential surface used in defining $d(M)$. As the genus $g(Q) \leq g$, by Lemma 3.7, there is a number K' depending on Ω_2 and g such that $d(\partial\Omega_2 \cap F, \partial Q) \leq K'$. Thus $d(M) = d(\partial\Omega_2 \cap F, \mathcal{D}_1) \leq d(\partial\Omega_2 \cap F, \partial Q) + 1 + d(\partial Q, \mathcal{D}_1) \leq K' + 2g + 3$.

If M_2 is a twisted I–bundle, then we can apply Lemma 3.8 instead of Lemma 3.7 in the argument above and get the same inequality.

Therefore, if $d(M)$ is sufficiently large and some $t \in I - \Lambda$ is labelled both X and Y, F must be isotopic to a middle surface in X or Y as in Lemma 6.10 and Theorem 1.5 holds. □

Claim 4 *Suppose every $t \in I - \Lambda$ is labelled, then Theorem 1.5 holds.*

Proof By Claim 3, we may assume that no t is labelled both X and Y. By Claim 1, as t increases from ϵ to $1-\epsilon$, its label changes from X to Y. As $t \in I - \Lambda$ is labelled, then there is a number $t_0 \in \Lambda$ such that $t_0 - \epsilon$ is labelled X and $t_0 + \epsilon$ is labelled Y for sufficiently small $\epsilon > 0$. Since $t_0 \in \Lambda$, $F \cap P^{t_0}$ contains a single tangency. Since $t_0 - \epsilon$ and $t_0 + \epsilon$ have different labels, the tangency in $F \cap P^{t_0}$ must be a saddle tangency.

Let $F \times J$ be a small product neighborhood of F in M, where J is a closed interval, and let F^+ and F^- be the two components of $F \times \partial J$. F^+ and F^- are parallel and close to F but lie on different sides of F. By considering how the intersection curves change near a saddle tangency, it is easy to see that, we may choose F^+ and F^- so that, for a sufficiently small ϵ, the intersection patterns of $F^\pm \cap P^{t_0}$ and $F \cap P^{t_0 \pm \epsilon}$ are the same. In fact, we may assume $F^+ \cup P^{t_0}$ and $F^- \cup P^{t_0}$ are isotopic to $F \cup P^{t_0+\epsilon}$ and $F \cup P^{t_0-\epsilon}$ in M respectively.

Let $M_1^\pm$ and $M_2^\pm$ be components of the closure of $M - F^\pm$ corresponding to M_1 and M_2 respectively. There are two subcases depending on whether $F^\pm$ lies in M_1 or M_2.

The first subcase is that $F^- \subset \mathrm{int}(M_1)$ and $F^+ \subset \mathrm{int}(M_2)$. In this subcase $M_1^- \subset \mathrm{int}(M_1^+)$ and $M_2^+ \subset \mathrm{int}(M_2^-)$. Since $t_0 - \epsilon$ is labelled X and since intersection pattern of $F^- \cap P^{t_0}$ is the same as $F \cap P^{t_0-\epsilon}$, there is a curve γ_X in $P^{t_0} \cap M_2^-$ such that (1) γ_X bounds a compressing disk in X_{t_0}, and (2) γ_X bounds an almost compressing disk D_X in M_2^-. Similarly, since $t_0+\epsilon$ is labelled Y, there is a curve γ_Y in $P^{t_0} \cap M_2^+$ such that (1) γ_Y bounds a compressing disk in Y_{t_0}, and (2) γ_Y bounds an almost compressing disk D_Y in M_2^+. Since $M_2^+ \subset \mathrm{int}(M_2^-)$, $D_Y \subset M_2^+ \subset M_2^-$. So both D_X and D_Y lie in M_2^-. As $F^- \cup P^{t_0}$ and $F \cup P^{t_0-\epsilon}$ are isotopic in M, there are a pair of almost compressing disks D'_X and D'_Y for $P^{t_0-\epsilon}$ in M_2 corresponding to D_X and D_Y. This means that $t_0 - \epsilon$ is labelled both X and Y. Now Theorem 1.5 follows from Claim 3.

The second subcase is that $F^+ \subset \mathrm{int}(M_1)$ and $F^- \subset \mathrm{int}(M_2)$. This subcase is basically the same as the first one. One can simply interchange all the plus and minus signs and interchange all the labels X and Y in the proof above for the first subcase to conclude that $t_0 + \epsilon$ is labelled both X and Y. By Claim 3, Theorem 1.5 holds. □

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Department of Mathematics, Boston College
Chestnut Hill, MA 02467

`taoli@bc.edu`

`http://www2.bc.edu/~taoli/`

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