Correction to "A cartesian presentation of weak n–categories"

CHARLES REZK

This paper gives a corrected proof for Proposition 6.6 of "A Cartesian presentation of weak n–categories" [1] by the author.

18D05; 55U40

1 Introduction

In this note, I provide a correction to the proof of Proposition 6.6 of my paper [1]. It forms a key part of the proof of the main Theorem 1.4 of that paper. The error lies in the use of Proposition 2.19, which is simply false as stated. In fact, it is easy to construct counterexamples to 2.19 when the poset $\mathcal{P}$ does not have contractible nerve.

The error the original paper was pointed out to me by Bill Dwyer, who also suggested the idea which forms the basis of the proof given here. I thank Bill Dwyer for his help.

The author was supported under NSF grant DMS-0505056.

I assume context and notation of [1].

2 Proof of Proposition 6.6

Let C be a small category, and let $A_1, \dots, A_m, B_1, \dots, B_n$ be objects of $s\mathrm{PSh}(C)$. Let $M \subseteq F[m]$ and $N \subseteq F[n]$ be covers, in the sense of [1, Section 6.2]. We need to prove the following.

Proposition 2.1 (Proposition 6.6 of [1]) *If $M \subseteq F[m]$ and $N \subseteq F[n]$ are covers in $s\mathrm{PSh}(\Delta)$, then*

$$V_M(A_1, \dots, A_m) \times V_N(B_1, \dots, B_n) \to V[m](A_1, \dots, A_m) \times V[n](B_1, \dots, B_m)$$

is an Se_C–equivalence.

Published: 29 October 2010 DOI: 10.2140/gt.2010.14.2301

As in [1, Proof of (6.6)], let $\mathcal{Q}_{m,n}$ denote the category whose objects are pairs of maps $\delta = (\delta_1\colon [p] \to [m], \delta_2\colon [p] \to [n])$ in Δ such that δ_1 and δ_2 are surjective, and $(F\delta_1, F\delta_2)\colon F[p] \to F[m] \times F[n]$ is a monomorphism in $s\mathrm{PSh}(\Delta)$.

Functorially associated to an object δ of $\mathcal{Q}_{m,n}$ is the following diagram.

$$\begin{array}{ccc} V_{(\delta_1,\delta_2)^{-1}M\times N}(D_1,\dots,D_p) & \xrightarrow{g} & V_M(A_1,\dots,A_m)\times V_N(B_1,\dots,B_n) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ V[p](D_1,\dots,D_p) & \xrightarrow[g']{} & V[m](A_1,\dots,A_m)\times V[n](B_1,\dots,B_n) \end{array} \tag{2.2}$$

The objects in the middle row are defined using the "intertwining functor" of [1, Section 4.4], and of the top row are defined using the construction of [1, Section 4.10]. The objects C_i are described by

$$D_i = \begin{cases} A_{\delta_1(i)} & \text{if } \delta_1(i) > \delta_1(i-1) \text{ and } \delta_2(i) = \delta_2(i-1), \\ B_{\delta_2(i)} & \text{if } \delta_1(i) = \delta_1(i-1) \text{ and } \delta_2(i) > \delta_2(i-1), \\ A_{\delta_1(i)} \times B_{\delta_2(i)} & \text{if } \delta_1(i) > \delta_1(i-1) \text{ and } \delta_2(i) > \delta_2(i-1). \end{cases}$$

The maps marked f and f' are induced by the inclusions $M \times N \to F[m] \times F[n]$ and $(\delta_1, \delta_2)^{-1}M \times N \to F[p]$. The maps marked g and g' are induced by the map $(F\delta_1, F\delta_2)$. (This is meant to be the same diagram described in the proof of Proposition 6.6 of [1]; however, in that paper the objects here called D_i are described incorrectly there.)

By [1, (6.4)], the map f is an Se_C–equivalence, and so Proposition 2.1 follows from the following Proposition 2.3, in which homotopy colimits are computed with respect to the level-wise weak equivalences in $s\mathrm{PSh}(\Theta C)$.

Proposition 2.3 *The induced maps*

$$\operatorname{hocolim}_{\mathcal{Q}_{m,n}} V_{(\delta,\delta')^{-1}M\times N} \to V_M(A_1,\dots,A_m)\times V_N(B_1,\dots,B_n)$$

and $$\operatorname{hocolim}_{\mathcal{Q}_{m,n}} V[p](D_1,\dots,D_p) \to V[m](A_1,\dots,A_m)\times V[n](B_1,\dots,B_n)$$

are levelwise weak equivalences.

Fix an object $\theta = [q](c_1,\dots,c_q)$ of ΘC. Given a pair of maps $\alpha = (\alpha_1\colon [q] \to [m], \alpha_2\colon [q] \to [n])$, let

$$D(\alpha) = \prod_{i=1}^{q} \left(\prod_{j=\alpha_1(i-1)+1}^{\alpha_1(i)} A_j \times \prod_{k=\alpha_2(i-1)+1}^{\alpha_2(i)} B_k \right),$$

an object of $s\mathrm{PSh}(\Theta C)$. For δ in $\mathcal{Q}_{m,n}$ define

$$G_\delta(\alpha) = \{\,\gamma\colon [q] \to [p] \mid \delta_1\gamma = \alpha_1, \delta_2\gamma = \alpha_2\,\}.$$

Since $(F\delta_1, F\delta_2)$ is a monomorphism, the set $G_\delta(\alpha)$ is either empty or is a singleton. By examining the definitions, we see that the map g' of (2.2), evaluated at the object θ, is the map

$$g'(\theta)\colon \coprod_\alpha D(\alpha) \times G_\delta(\alpha) \to \coprod_\alpha D(\alpha)$$

induced by the projections $G_\delta(\alpha) \to 1$. Both coproducts are taken over all pairs $\alpha = (\alpha_1, \alpha_2)$. Similarly, the map g of (2.2), evaluated at the object θ is either (i) isomorphic to $g'(\theta)$, if $(F\delta_1, F\delta_2)\colon F[p] \to F[m] \times F[n]$ factors through the subobject $M \times N$, or (ii) is an isomorphism between empty spaces, otherwise.

Thus we have reduced the proof of Proposition 2.3 to the following.

Lemma 2.4 *For each pair $\alpha = (\alpha_1\colon [q] \to [m], \alpha_2\colon [q] \to [n])$, the simplicial set* $\mathrm{hocolim}_{\mathcal{Q}_{m,n}} G_\delta(\alpha)$ *is weakly contractible.*

In what follows, it will be useful to view elements of the poset $\mathcal{Q}_{m,n}$ as paths $(x_0, y_0), \dots, (x_r, y_r)$ in the set $\mathbb{Z} \times \mathbb{Z}$, which (i) start at $(x_0, y_0) = (0, 0)$, (ii) end at $(x_r, y_r) = (m, n)$, and are such that (iii) each step in the path moves one unit left, one unit up or diagonally up and left by one unit (ie, $(x_k - x_{k-1}, y_k - y_{k-1})$ is one of $(1, 0)$, $(0, 1)$ or $(1, 1)$). The order relation is given by inclusion of paths.

Proof of Lemma 2.4 Observe that if $\beta\colon [q'] \to [q]$ is a surjection in Δ, then $G_\delta(\alpha) \approx G_\delta(\alpha\beta)$. Thus, we may assume without loss of generality that $(F\alpha_1, F\alpha_2)\colon F[q] \to F[m] \times F[n]$ is injective.

Let $\mathcal{Q}_{m,n,\alpha}$ denote the subposet of $\mathcal{Q}_{m,n}$ consisting of δ such that $G_\delta(\alpha)$ is nonempty. It is immediate that $\mathrm{hocolim}_{\mathcal{Q}_{m,n}} G_\delta(\alpha)$ is isomorphic to the nerve of $\mathcal{Q}_{m,n,\alpha}$.

Next, observe that the subposet $\mathcal{Q}_{m,n,\alpha}$ may be identified as the subposet of paths in $\mathcal{Q}_{m,n}$ which pass through the points $(\alpha_1(j), \alpha_2(j))$ for $j = 0, \dots, q$. Therefore, there is an isomorphism of posets $\mathcal{Q}_{m,n,\alpha} \approx \prod_{j=0}^{q+1} \mathcal{Q}_{m_j,n_j}$, where $(m_j, n_j) = (\alpha_1(j) - \alpha_1(j-1), \alpha_2(j) - \alpha_2(j-1))$ if $j = 1, \dots, q$, and $(m_0, n_0) = (\alpha_1(0), \alpha_2(0))$ and $(m_{q+1}, n_{q+1}) = (m - \alpha_1(q), n - \alpha_q(q))$. Thus, since nerve commutes with products, the result follows by Lemma 2.5. □

Lemma 2.5 *The simplicial nerve of $\mathcal{Q}_{m,n}$ is weakly contractible.*

Proof When $n = 0$, $\mathcal{Q}_{m,0}$ is the terminal category, and thus clearly has contractible nerve. Thus, it will suffice to produce, for each $n \geq 1$, a chain of inclusions

$$\mathcal{Q}_{m,n-1} \approx X_m \to Y_{m-1} \to X_{m-1} \to Y_{m-2} \to \cdots \to Y_0 \to X_0 = \mathcal{Q}_{m,n},$$

such that each inclusion induces a simplicial homotopy equivalence on nerves.

For $0 \leq k \leq m$ we let $X_k \subset \mathcal{Q}_{m,n}$ be the subposet of $\mathcal{Q}_{m,n}$ consisting of paths that do not contain the points (i,n) for $i < k$. We let $Y_k \subset X_k$ be the subposet consisting of those paths such that if (k,n) is in the path, then $(k,n-1)$ is also in the path; that is, the diagonal move from $(k-1,n-1)$ to (k,n) does not occur in the path. Thus $X_0 = \mathcal{Q}_{m,n}$, and X_m is isomorphic to $\mathcal{Q}_{m,n-1}$.

For each $k = 0, \dots, m-1$, there is a retraction $r\colon X_k \to Y_k$, which sends a path to a modified version of that path, in which the diagonal move $(k-1,n-1)$ to (k,n) (if it is present in the path) is replaced by two moves $(k-1,n-1)$ to $(k,n-1)$ to (k,n). Note that for any element $p \in X_k$, we have $r(p) \leq p$, and for any $p \in Y_k$, $r(p) = p$. This defines a simplicial homotopy retraction for the inclusion $NY_k \to NX_k$.

For each $k = 0, \dots, m-1$ there is a retraction $r'\colon Y_k \to X_{k+1}$, which sends a path to a modified version of that path, in which the two moves $(k,n-1)$ to (k,n) to $(k+1,n)$ (if present in the path), are replaced by the diagonal move $(k,n-1)$ to $(k+1,n)$. Note that for any element $p \in Y_k$, we have $r'(p) \geq p$, and for any $p \in X_k$, $r'(p) = p$. Thus r' defines a simplicial homotopy retraction for the inclusion $NX_{k+1} \to NY_k$. □

References

[1] **C Rezk**, *A Cartesian presentation of weak n–categories*, Geom. Topol. 14 (2010) 521–571 MR2578310

Department of Mathematics, University of Illinois at Urbana-Champaign
273 Altgeld Hall, MC-382, 1409 W Green Street, Urbana, IL 61801, USA

rezk@math.uiuc.edu

Received: 23 September 2010