

## Adams operations in smooth $K$ -theory

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We show that the Adams operation  $\Psi^k$ ,  $k \in \{-1, 0, 1, 2, \dots\}$ , in complex  $K$ -theory lifts to an operation  $\hat{\Psi}^k$  in smooth  $K$ -theory. If  $V \rightarrow X$  is a  $K$ -oriented vector bundle with Thom isomorphism  $\text{Thom}_V$ , then there is a characteristic class  $\rho^k(V) \in K[1/k]^0(X)$  such that  $\Psi^k(\text{Thom}_V(x)) = \text{Thom}_V(\rho^k(V) \cup \Psi^k(x))$  in  $K[1/k](X)$  for all  $x \in K(X)$ . We lift this class to a  $\hat{K}^0(\dots)[1/k]$ -valued characteristic class for real vector bundles with geometric  $\text{Spin}^c$ -structures.

If  $\pi: E \rightarrow B$  is a  $K$ -oriented proper submersion, then for all  $x \in K(X)$  we have  $\Psi^k(\pi_!(x)) = \pi_!(\rho^k(N) \cup \Psi^k(x))$  in  $K[1/k](B)$ , where  $N \rightarrow E$  is the stable  $K$ -oriented normal bundle of  $\pi$ . To a smooth  $K$ -orientation  $o_\pi$  of  $\pi$  we associate a class  $\hat{\rho}^k(o_\pi) \in \hat{K}^0(E)[1/k]$  refining  $\rho^k(N)$ . Our main theorem states that if  $B$  is compact, then  $\hat{\Psi}^k(\hat{\pi}_!(\hat{x})) = \hat{\pi}_!(\hat{\rho}^k(o_\pi) \cup \hat{\Psi}^k(\hat{x}))$  in  $\hat{K}(B)[1/k]$  for all  $\hat{x} \in \hat{K}(E)$ . We apply this result to the  $e$ -invariant of bundles of framed manifolds and  $\rho$ -invariants of flat vector bundles.

### 1 Introduction

The formalism of smooth extensions of generalized cohomology theories is designed to capture secondary invariants in topology, global analysis and geometry. The first example was the smooth extension of ordinary cohomology introduced by Cheeger and Simons [9]. Among other applications it was used to construct secondary characteristic classes for flat vector bundles.

Motivated by applications in mathematical physics, in particular string theory, smooth extensions of other generalised cohomology theories, in particular of  $K$ -theory, have been considered for example by Moore and Witten [16], Freed [11; 10] and Szabo and Valentino [20]. The existence of smooth extensions of generalised cohomology theories has been shown by Hopkins and Singer [14]. Axioms and uniqueness results have been discussed by Simons and Sullivan [18] and the author and Schick [7]. In [7] we have shown that there is, up to unique isomorphism, a unique smooth extension of complex  $K$ -theory.

An important tool for the construction of primary and secondary invariants is the integration or pushforward map for suitably oriented maps. The integration for smooth extensions of generalised cohomology theories has been discussed by Hopkins and Singer [14]. The notion of a smooth orientation of a submersion has been formalised for bordism theories by the author, Schick, Schröder and Wiethaup [8] and by the author and Schick [6] for complex  $K$ -theory.

In general, Riemann–Roch type index theorems are assertions about the compatibility of natural operations between cohomology theories and the pushforward. In the prototypical example its states the compatibility of the Chern character

$$\text{ch}: K \rightarrow HP\mathbb{Q}$$

from  $K$ -theory to periodic rational cohomology with the pushforward along a  $K$ -oriented proper submersion  $\pi: E \rightarrow B$  between smooth manifolds:

$$(1) \quad \begin{array}{ccc} K^*(E) & \xrightarrow{\text{ch}} & HP\mathbb{Q}^*(E) \\ \downarrow \pi_!^K & & \downarrow \pi_!^{HP\mathbb{Q}}(\widehat{\mathbf{A}}^c(T^v\pi)\cup\dots) \\ K^{*-n}(B) & \xrightarrow{\text{ch}} & HP\mathbb{Q}^{*-n}(B) \end{array}$$

Here  $n := \dim(E) - \dim(B)$  is the dimension of the fibres of  $\pi$  and  $\widehat{\mathbf{A}}^c(T^v\pi) \in HP\mathbb{Q}^0(E)$  is the  $\text{Spin}^c$ -generalisation of the  $\widehat{\mathbf{A}}$ -genus (see [6, Definition 3.3]) of the vertical bundle  $T^v\pi := \ker(d\pi)$  which has a  $\text{Spin}^c$ -structure by the  $K$ -orientation. The maps  $\pi_!^K$  and  $\pi_!^{HP\mathbb{Q}}$  are the integration maps in the corresponding cohomology theories<sup>1</sup>.

The prototypical result for the smooth extensions shown in [6, Theorem 6.19] states, that if  $\pi$  is smoothly  $K$ -oriented, then the smooth refinement of the diagram (1)

$$(2) \quad \begin{array}{ccc} \widehat{K}^*(E) & \xrightarrow{\widehat{\text{ch}}} & \widehat{HP\mathbb{Q}}^*(E) \\ \downarrow \widehat{\pi}_!^K & & \downarrow \widehat{\pi}_!^{HP\mathbb{Q}}(\widehat{\mathbf{A}}^c(o_\pi)\cup\dots) \\ \widehat{K}^{*-n}(B) & \xrightarrow{\widehat{\text{ch}}} & \widehat{HP\mathbb{Q}}^{*-n}(B) \end{array}$$

commutes, too. Here  $\widehat{K}$  and  $\widehat{HP\mathbb{Q}}$  denote the smooth extensions of complex  $K$ -theory and periodic rational cohomology theory,  $\widehat{\text{ch}}: \widehat{K} \rightarrow \widehat{HP\mathbb{Q}}$  is the smooth lift of the Chern character, and  $\widehat{\mathbf{A}}^c(o_\pi) \in \widehat{HP\mathbb{Q}}^0(E)$  is the smooth refinement  $\widehat{\mathbf{A}}^c(o_\pi)$  determined by the smooth  $K$ -orientation  $o_\pi$  of  $\pi$ .

<sup>1</sup>In the main body of the paper we will omit the superscripts since we only consider integration in  $K$ -theory.

In the present paper instead of the Chern character we consider the Adams operation

$$\Psi^k: K[1/k]^*(X) \rightarrow K[1/k]^*(X)$$

for  $k \in \{-1\} \cup \mathbb{N}$ . In this case the Riemann–Roch type theorem states that

$$(3) \quad \begin{array}{ccc} K[1/k]^*(E) & \xrightarrow{\Psi^k} & K[1/k]^*(E) \\ \downarrow \pi_!^K & & \downarrow \pi_!^K(\rho^k(T^v\pi)^{-1} \cup \dots) \\ K[1/k]^{*-n}(B) & \xrightarrow{\Psi^k} & K[1/k]^{*-n}(B) \end{array}$$

commutes, where  $\rho^k(T^v\pi) \in K[1/k]^0(E)$  is an invertible  $K$ -theoretic characteristic class of the  $\text{Spin}^c$ -bundle  $T^v\pi$  (see Section 2) below.

The main results of the present paper are the following three theorems:

**Theorem 1.1** (Theorem 3.1) *There exists a natural lift of the Adams operation to a natural transformation  $\widehat{\Psi}^k: \widehat{K}(\dots)[1/k] \rightarrow \widehat{K}(\dots)[1/k]$  of functors on the category of compact manifolds.*

**Theorem 1.2** (Definition 4.6 and Theorem 4.1) *If  $\pi: E \rightarrow B$  is a submersion with compact  $E$  which is smoothly  $K$ -oriented by  $o_\pi$ , then there exists a natural smooth refinement  $\widehat{\rho}^k(o_\pi) \in \widehat{K}^0(E)[1/k]$  of the class  $\rho^k(T^v\pi)^{-1}$ .*

For details, in particular for the meaning of the word *natural*, we refer to the main body of the present paper. The analog of (2) is given by the third theorem.

**Theorem 1.3** (Theorem 5.1) *If  $\pi: E \rightarrow B$  is a smoothly  $K$ -oriented proper submersion over a compact base, then the smooth refinement of (3) commutes:*

$$(4) \quad \begin{array}{ccc} \widehat{K}^*(E)[1/k] & \xrightarrow{\widehat{\Psi}^k} & \widehat{K}^*(E)[1/k] \\ \downarrow \widehat{\pi}_!^K & & \downarrow \widehat{\pi}_!^K(\widehat{\rho}^k(o_\pi) \cup \dots) \\ \widehat{K}^{*-n}(B)[1/k] & \xrightarrow{\widehat{\Psi}^k} & \widehat{K}^{*-n}(B)[1/k] \end{array}$$

The Dirac operator model of smooth  $K$ -theory [6] provides the link between the pushforward in smooth  $K$ -theory and spectral geometric invariants of families of Dirac operators. So in principle, the diagram (4) can be interpreted as a relation between these invariants for different families of Dirac operators. We discuss this aspect in greater detail in Section 6.

In [6], we have constructed a version of Adams  $e$ -invariant  $e(\pi) \in K\mathbb{R}/\mathbb{Z}^{-n-1}(B)$  for families of  $n$ -dimensional framed manifolds  $\pi: E \rightarrow B$  using only elements of the formalism of smooth  $K$ -theory. As an immediate consequence of (4) we show in Theorem 6.1 that

$$k^L(\widehat{\Psi}^k - 1)e(\pi) = 0$$

for sufficiently large  $L$ . In the case that  $B$  is a point these relations for all  $k \in \mathbb{N} \cup \{-1\}$  together imply the well-known (in fact optimal) upper bound of the range of the  $e$ -invariant; see Adams [1] and Ravenel [17, Theorem 1.1.13 and comments below].

## 2 Adams operations

Let  $\mathbb{Z}[1/k]$  denote the abelian group obtained from the integers by inverting  $k \in \mathbb{Z} \setminus \{0\}$ . If  $A$  is any abelian group, then we write  $A[1/k] := A \otimes \mathbb{Z}[1/k]$ . Let us consider a cohomology theory  $E$  represented by a spectrum  $E$  (which we will denote by the same symbol). Using the Moore space  $M\mathbb{Z}[1/k]$  we can define the spectrum  $E[1/k] := E \wedge M\mathbb{Z}[1/k]$  which represents the cohomology theory  $E[1/k]$ . Since  $\mathbb{Z}[1/k]$  is flat the coefficients of the latter are given by  $E[1/k]^* \cong E^* \otimes \mathbb{Z}[1/k]$ . Note that we can define another contravariant functor from spaces to  $\mathbb{Z}$ -graded abelian groups by  $E^*(X)[1/k] := E^*(X)[1/k]$ . Since  $\mathbb{Z}[1/k]$  is flat this functor admits Mayer-Vietoris sequences. But it does not satisfy the wedge axiom. The map  $M\mathbb{Z} \rightarrow M\mathbb{Z}[1/k]$  induces a natural transformation  $E \rightarrow E[1/k]$  which extends to a natural transformation  $E^*(\dots)[1/k] \rightarrow E[1/k]^*(\dots)$  preserving Mayer-Vietoris sequences. If  $X$  has the homotopy type of a finite CW-complex, then this map induces an isomorphism  $E^*(X)[1/k] \cong E[1/k]^*(X)$ .

In the present paper we want to apply this construction to the cohomology theory  $K$  called complex  $K$ -theory. In order to avoid to provide the details of the construction of a smooth extension of  $K[1/k]$  we simply tensorize the smooth extension of  $K$  by  $\mathbb{Z}[1/k]$ . This dictates the same choice on the topological side and forces us to restrict to manifolds which have the homotopy type of finite CW-complexes, eg compact manifolds.

So let  $k \in \mathbb{Z} \setminus \mathbb{N} \cup \{-1\}$  and let  $K$  denote the generalized cohomology theory complex  $K$ -theory. By the discussion above, for a finite CW-complex  $X$  we have

$$(5) \quad K[1/k]^*(X) \cong K^*(X)[1/k].$$

By the Landweber formalism [15] complex  $K$ -theory is associated to the multiplicative formal group law

$$(x, y) \mapsto x + y + bxy$$

over the ring  $K^* \cong \mathbb{Z}[b, b^{-1}]$  generated by the Bott element  $b$  with  $\deg(b) = 2$ . The cohomology theory  $K[1/k]$  is then given by the same law considered over  $K[1/k]^* = \mathbb{Z}[1/k][b, b^{-1}]$ . The diagram

$$\begin{array}{ccc}
 \mathbb{Z}[1/k][b, b^{-1}][[x]] & \xrightarrow[\Psi_{\text{FGL}}^k]{x \mapsto kx, b \mapsto k^{-1}b} & \mathbb{Z}[1/k][b, b^{-1}][[x]] \\
 \uparrow & & \uparrow \\
 \mathbb{Z}[1/k][b, b^{-1}] & \xrightarrow[\Psi^k]{b \mapsto k^{-1}b} & \mathbb{Z}[1/k][b, b^{-1}]
 \end{array}$$

gives a morphism  $\Psi_{\text{FGL}}^k$  of formal group laws over the morphism of rings  $\Psi^k$ . It induces the Adams operation  $\Psi^k$  which is a multiplicative cohomology operation of the generalised cohomology theory  $K[1/k]$ .

It is the stable version of the classical Adams operation

$$\Psi^k: K^0 \rightarrow K^0$$

which is already defined before inverting  $k$ . If  $L \rightarrow X$  is a one-dimensional complex vector bundle over a finite CW-complex  $X$  and  $[L] \in K^0(X)$  denotes the corresponding  $K$ -theory class, then we have

$$(6) \quad \Psi^k([L]) = [L^k]$$

in  $K^0(X)$ .

The Bott periodicity isomorphism  $\text{Bott}: K[1/k]^*(X) \xrightarrow{\sim} K[1/k]^{*+2}(X)$  is given by multiplication with the Bott element  $b \in K^2$  so that the following diagram commutes for all  $n \in \mathbb{Z}$ :

$$(7) \quad \begin{array}{ccc}
 K[1/k]^*(X) & \xrightarrow{k^{-n}\Psi^k} & K[1/k]^*(X) \\
 \downarrow \text{Bott}^n & & \downarrow \text{Bott}^n \\
 K[1/k]^{*+2n}(X) & \xrightarrow{\Psi^k} & K[1/k]^{*+2n}(X)
 \end{array}$$

The Adams operations satisfy

$$(8) \quad \Psi^k \circ \Psi^l = \Psi^{kl}$$

(here we invert  $k$  and  $l$ ).

We define the multiplicative cohomology operation  $\Psi_H^k: HP\mathbb{Q}^* \rightarrow HP\mathbb{Q}^*$  on the periodic rational cohomology  $HP\mathbb{Q}^*(X) := H^*(X; K_{\mathbb{Q}}^*)$  such that  $\Psi_H^k(x) = x$  for

$x \in H^*(X; \mathbb{Q})$  and  $\Psi_H^k(b) = k^{-1}b$ . Periodic rational cohomology is the natural target of the Chern character  $\text{ch}: K^* \rightarrow HP\mathbb{Q}^*$ , and we have

$$(9) \quad \text{ch} \circ \Psi^k = \Psi_H^k \circ \text{ch} .$$

A real  $n$ -dimensional vector bundle  $V \rightarrow X$  with a  $\text{Spin}^c$ -structure is  $K$ -oriented. We have a Thom isomorphism

$$\text{Thom}_V: K[1/k]^*(X) \rightarrow \tilde{K}[1/k]^{*+n}(\text{Thom}(V)) ,$$

where  $\tilde{K}[1/k](\text{Thom}(V))$  denotes the reduced  $K[1/k]$ -theory of the Thom space of  $V$ . There exists a unique invertible characteristic class

$$\rho^k(V) \in K[1/k]^0(X)$$

(called the cannibalistic class in [1]) such that

$$(10) \quad \Psi^k(\text{Thom}(x)) = \text{Thom}(\rho^k(V) \cup \Psi^k(x)) , \quad \forall x \in K[1/k]^*(X) .$$

A  $K$ -orientation of a proper submersion  $\pi: E \rightarrow B$  is determined by a  $\text{Spin}^c$ -structure of the vertical bundle  $T^v\pi = \ker(d\pi)$ . If  $\pi$  is  $K$ -oriented, then we have an integration map

$$\pi_!: K[1/k]^*(E) \rightarrow K[1/k]^{*-n}(B) ,$$

where  $n = \dim(E) - \dim(B)$  is the dimension of the fibres. The compatibility of the Adams operations and the integration is expressed by the identity

$$(11) \quad \Psi^k(\pi_!(x)) = \pi_!(\rho^k(T^v\pi)^{-1} \cup \Psi^k(x)) , \quad \forall x \in K[1/k]^*(E) .$$

It is an immediate consequence of the usual construction of  $\pi_!$  and (10).

### 3 The lift of the Adams operations

We consider the smooth extension  $(\hat{K}, R, I, a, f)$  of complex  $K$ -theory [6; 7] on the category of compact manifolds. We restrict to compact manifolds since we will frequently use the isomorphism (5). In order to avoid this restriction one could alternatively start with a smooth extension of  $K[1/k]$ .

In the present paper it is useful to keep track of degrees properly. So for the domain of  $a$  and the target of  $R$  we will take the periodic differential forms  $\Omega P^*(M) := \Omega^*(M, K_{\mathbb{R}}^*)$  with  $K_{\mathbb{R}}^* := K^* \otimes \mathbb{R}$ , and the corresponding cohomology is the periodic de Rham cohomology  $HP_{\text{dR}}(M, K_{\mathbb{R}}^*) \cong HP^*(M)$ . We define the natural transformation  $\Psi_{\Omega}^k: \Omega P^*(M) \rightarrow \Omega P^*(M)$  of ring-valued functors such that  $\Psi_{\Omega}^k(\omega) = \omega$  for  $\omega \in \Omega^*(M)$ , and  $\Psi_{\Omega}^k(b) = k^{-1}b$ . It induces a corresponding transformation  $\Psi_H^k$  on the periodic cohomology.

**Theorem 3.1** *There exists a unique natural transformation*

$$\widehat{\Psi}^k: \widehat{K}(\dots)[1/k] \rightarrow \widehat{K}(\dots)[1/k]$$

*of set-valued functors on the category of compact manifolds such that*

$$(12) \quad I \circ \widehat{\Psi}^k = \Psi^k \circ I, \quad R \circ \widehat{\Psi}^k = \Psi_{\Omega}^k \circ R,$$

*and the following diagram commutes:*

$$(13) \quad \begin{array}{ccc} \widehat{K}^0(S^1 \times M)[1/k] & \xrightarrow{\widehat{\Psi}^k} & \widehat{K}^0(S^1 \times M)[1/k] \\ \downarrow f & & \downarrow f \\ \widehat{K}^{-1}(M)[1/k] & \xrightarrow{\widehat{\Psi}^k} & \widehat{K}^{-1}(M)[1/k] \end{array}$$

*The transformation  $\widehat{\Psi}^k$  preserves the ring structure and satisfies*

$$(14) \quad \widehat{\Psi}^k \circ \widehat{\Psi}^l = \widehat{\Psi}^{kl}.$$

**Proof** We first show that there is a unique natural transformation of set-valued functors

$$\widehat{\Psi}^k: \widehat{K}^0 \rightarrow \widehat{K}^0$$

which satisfies (12). We then show that this transformation preserves the ring structure and satisfies (14). Finally we extend  $\widehat{\Psi}^k$  to all degrees using Bott periodicity and (13).

The space  $\mathbf{K}_0 := \mathbb{Z} \times BU$  represents the homotopy type of the classifying space of the functor  $K^0$ . We choose by [7, Proposition 2.1] a sequence of compact manifolds  $(\mathcal{K}_i)_{i \geq 0}$  together with maps

$$x_i: \mathcal{K}_i \rightarrow \mathbf{K}_0, \quad \kappa_i: \mathcal{K}_i \rightarrow \mathcal{K}_{i+1}$$

such that

- (1)  $\mathcal{K}_i$  is homotopy equivalent to an  $i$ -dimensional CW-complex,
- (2)  $\kappa_i: \mathcal{K}_i \rightarrow \mathcal{K}_{i+1}$  is an embedding of a closed submanifold,
- (3)  $x_i: \mathcal{K}_i \rightarrow \mathbf{K}_0$  is  $i$ -connected,
- (4)  $x_{i+1} \circ \kappa_i = x_i$ .

Let  $u \in K^0(\mathbf{K}_0)$  be the universal class represented by the identity map  $\mathbf{K}_0 \rightarrow \mathbf{K}_0$ . By [7, Proposition 2.6] we can further choose a sequence  $\widehat{u}_i \in \widehat{K}^0(\mathcal{K}_i)$  such that  $I(\widehat{u}_i) = x_i^*u$  and  $\kappa_i^*\widehat{u}_{i+1} = \widehat{u}_i$  for all  $i \geq 0$ . By [7, Lemma 3.8] for  $k \geq 2i + 2$  we have  $H^{2j+1}(\mathcal{K}_k, \mathbb{R}) = 0$  for all  $j \leq i$ .

The requirements  $I(\widehat{\psi}^k(u_i)) = \Psi^k(I(\widehat{u}_i)) \in K^0(\mathcal{K}_i)$  and  $R(\widehat{\psi}^k(\widehat{u}_i)) = \Psi_\Omega^k(R(\widehat{u}_i))$  fix a class  $\widehat{\psi}^k(\widehat{u}_i) \in \widehat{K}^0(\mathcal{K}_i)$  uniquely up to elements of the form  $a(\alpha)$  for  $\alpha \in F^{\geq i}HP^{-1}(\mathcal{K}_i)$ , where

$$(15) \quad F^{\geq l}HP^{-1}(X) := \bigoplus_{2j+1 \geq l} b^{-j-1}H^{2j+1}(X; \mathbb{R}) \subseteq HP^{-1}(X).$$

Let  $M$  be a compact manifold. In the following we construct a map  $\widehat{\Psi}^k: \widehat{K}^0(M) \rightarrow \widehat{K}^0(M)$ . Let  $\widehat{y} \in \widehat{K}^0(M)$  be given. Then we choose  $i > \dim(M)$  and  $f: M \rightarrow \mathcal{K}_i$  such that  $I(\widehat{y}) = f^*x_i^*(u)$ . We further choose  $\rho \in \Omega P^{-1}(M)$  such that

$$(16) \quad \widehat{y} = f^*\widehat{u}_i + a(\rho).$$

We define

$$(17) \quad \widehat{\Psi}^k(\widehat{y}) := f^*\psi^k(\widehat{u}_i) + a(\Psi_\Omega^k(\rho)).$$

By a direct calculation we verify that (12) holds true:

$$I(\widehat{\Psi}^k(\widehat{y})) = \Psi^k(I(y)), \quad R(\widehat{\Psi}^k(\widehat{y})) = \Psi_\Omega^k(R(\widehat{y})).$$

**Lemma 3.2** *The right-hand side of (17) does not depend on the choices.*<sup>2</sup>

**Proof** If  $\rho'$  is a second choice for  $\rho$  in (16), then  $\rho' - \rho = \text{ch}(x)$  in  $\Omega P^{-1}(M)/\text{im}(d)$  for some  $x \in K^{-1}(M)$ . But then  $\Psi_\Omega^k(\rho) - \Psi_\Omega^k(\rho') = \Psi_H^k(\text{ch}(x)) = \text{ch}(\Psi^k(x))$ . This implies  $a(\Psi_\Omega^k \rho) = a(\Psi_\Omega^k \rho')$ .

Since we take  $i$  sufficiently large it follows that  $f^*\psi^k(\widehat{u}_i)$  is independent of the choice of the actual element  $\psi^k(\widehat{u}_i)$ .

We further can increase  $i$  to  $i + 1$  by replacing  $f$  by  $\kappa_i \circ f$  without changing the right-hand side of (17).

Finally, any two choices of  $f$  become homotopic after increasing  $i$  sufficiently many times. Therefore let  $f'$  be another choice for  $f$  in (16) and  $H: [0, 1] \times M \rightarrow \mathcal{K}_i$  be a homotopy from  $f$  to  $f'$ . We choose  $\tilde{\rho} \in \Omega P^{-1}([0, 1] \times M)$  such that  $\text{pr}_M^* \widehat{y} = H^* \widehat{u}_i + a(\tilde{\rho})$ . In (16) we can then take  $\rho = \tilde{\rho}|_{\{0\} \times M}$  and  $\rho' := \tilde{\rho}|_{\{1\} \times M}$ , and by the

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<sup>2</sup>At the moment we consider the existence of a transformation  $\widehat{\Psi}^k$ . We must show that our construction of  $\widehat{\Psi}^k$  is well-defined. It follows from the next Lemma 3.3 that  $\widehat{\Psi}^k$  is independent of the choice of the approximation  $(\mathcal{K}_i)_i$  of  $\mathbf{K}_0$ .



homotopy formula (see [7, (1)]) we have

$$a\left(\rho' - \rho + \int_{[0,1] \times M/M} H^* R(\hat{u}_k)\right) = 0,$$

that is,

$$\rho' - \rho + \int_{[0,1] \times M/M} H^* R(\hat{u}_k) = \text{ch}(x)$$

for some  $x \in K^{-1}(M)$ .

We evaluate the difference of the right-hand sides of (17) for the two choices and get

$$\begin{aligned} f'^* \psi^k(\hat{u}_i) + a(\Psi_\Omega^k(\rho')) - f^* \psi^k(\hat{u}_i) - a(\Psi_\Omega^k(\rho)) \\ &= a\left(\int_{[0,1] \times M/M} R(H^* \psi^k(\hat{u}_k)) + \Psi_\Omega^k(\rho') - \Psi_\Omega^k(\rho)\right) \\ &= a\left(\Psi_\Omega^k\left(\int_{[0,1] \times M/M} H^* R(\hat{u}_k) + \rho' - \rho\right)\right) \\ &= a(\text{ch}(\Psi^k(x))) \\ &= 0, \end{aligned}$$

where we use the homotopy formula in the first equality. □

We now have constructed for each manifold  $M$  a map of sets  $\hat{\Psi}^k: \hat{K}^0(M) \rightarrow \hat{K}^0(M)$  satisfying (12).

**Lemma 3.3**  $\hat{\Psi}^k$  is a natural transformation of set-valued functors on the category of compact manifolds. It is the unique natural transformation satisfying (12).

**Proof** Let  $g: M' \rightarrow M$  be a smooth map of compact manifolds. If we take in addition  $i > \dim(M')$ , then we can start the construction with  $g^* \hat{y} = f' \hat{u}_i + a(g^* \rho)$ , the analog of (16), where  $f' := f \circ g$ . With this choice we have

$$\hat{\Psi}^k(g^* \hat{y}) = (f \circ g)^* \psi^k(\hat{u}_i) + a(\Psi_\Omega^k g^* \rho) = g^*(f^* \psi^k(\hat{u}_i) + a(\Psi_\Omega^k(\rho))) = g^* \hat{\Psi}^k(\hat{y}).$$

We now show uniqueness. If  $\Psi'^{k}: \hat{K}^0 \rightarrow \hat{K}^0$  is another natural transformation satisfying (12), then for  $i > \dim(M)$  and  $f: M \rightarrow \mathcal{K}_i$  we have

$$(18) \quad \hat{\Psi}'^{k}(f^* \hat{u}_i) = f^* \hat{\Psi}'^{k}(\hat{u}_i) = f^* \psi^k(\hat{u}_i) = \hat{\Psi}^k(f^* \hat{u}_i).$$

For  $\rho \in \Omega P^{-1}(M)$  we consider the class  $\hat{y} := f^*\hat{u}_i + a(\rho)$  and  $\hat{x} := \text{pr}_M^* f^*\hat{u}_i + a(t\text{pr}_M^*\rho) \in \hat{K}^0([0, 1] \times M)$ , where  $t$  is the coordinate of  $[0, 1]$ . The homotopy formula gives

$$\begin{aligned} \hat{\Psi}^{k, \prime}(\hat{y}) - \hat{\Psi}^{k, \prime}(f^*\hat{u}_i) &= a \left( \int_{[0, 1] \times M/M} R(\hat{\Psi}^{\prime, k}(\hat{x})) \right) \\ &= a \left( \int_{[0, 1] \times M/M} \Psi_{\Omega}^k(R(\hat{x})) \right) \\ &= \hat{\Psi}^k(\hat{y}) - \hat{\Psi}^k(f^*\hat{u}_i) . \end{aligned}$$

In view of (18) we get  $\hat{\Psi}^{k, \prime}(\hat{y}) = \hat{\Psi}^k(\hat{y})$ . □

**Lemma 3.4**  $\hat{\Psi}^k: \hat{K}^0 \rightarrow \hat{K}^0$  is a natural transformation of ring-valued functors and satisfies (14).

**Proof** We first consider the additive structure. Let

$$\hat{B}(\hat{x}, \hat{y}) := \hat{\Psi}^k(\hat{x} + \hat{y}) - \hat{\Psi}^k(\hat{x}) - \hat{\Psi}^k(\hat{y}) .$$

Since  $\hat{\Psi}^k$  is compatible with  $I$  and  $R$  we immediately see that  $\hat{B}$  takes values the subfunctor  $HP^{-1}/\text{im}(\text{ch}) \subset \hat{K}^0$ . Furthermore, since by the explicit formula (17) we have  $\hat{\Psi}^k(\hat{y} + a(\rho)) = \hat{\Psi}^k(\hat{y}) + \Psi^k(a(\rho))$ , it follows that  $\hat{B}$  factorises over a natural transformation

$$B: K^0 \times K^0 \rightarrow HP^{-1}/\text{im}(\text{ch}) .$$

The same argument as for [7, Theorem 3.6] shows that such a transformation vanishes. This shows that  $\hat{\Psi}^k$  preserves the additive structure.

In order to show that  $\hat{\Psi}^k$  is multiplicative we argue similarly. We consider

$$\hat{E}(\hat{x}, \hat{y}) := \hat{\Psi}^k(\hat{x} \cup \hat{y}) - \Psi^k(\hat{x}) \cup \Psi^k(\hat{y}) .$$

We again see that  $\hat{E}$  factors over a transformation

$$E: K^0 \times K^0 \rightarrow HP^{-1}/\text{im}(\text{ch})$$

which necessarily vanishes.

For the relation  $\hat{\Psi}^l \circ \hat{\Psi}^k = \hat{\Psi}^{lk}$  we argue similarly using

$$\hat{C}(\hat{x}) := \Psi^l \circ \hat{\Psi}^k(\hat{x}) - \hat{\Psi}^{lk}(\hat{x}) .$$

We again see that  $\hat{C}$  factors over a natural transformation

$$C: K^0 \rightarrow HP^{-1}/\text{im}(\text{ch})$$

which necessarily vanishes. □

**Lemma 3.5** *There exists a unique natural transformation  $\widehat{\Psi}^k: \widehat{K}^{-1} \rightarrow \widehat{K}^{-1}$  such that (13) commutes. It further satisfies (12) and (14), and for  $\widehat{z} \in \widehat{K}^0(M)$  and  $\widehat{y} \in \widehat{K}^{-1}(M)$  we have*

$$(19) \quad \widehat{\Psi}^k(\widehat{z} \cup \widehat{y}) = \widehat{\Psi}^k(\widehat{z}) \cup \widehat{\Psi}^k(\widehat{y}) .$$

**Proof** We choose  $\widehat{e} \in \widehat{K}^1(S^1)$  such that  $\int \widehat{e} = 1$  and  $R(\widehat{e})$  is rotation invariant. Note that  $\widehat{e}$  is unique up to addition of elements  $a(c)$ , where  $c \in HP^0(S^1)$ . For  $\widehat{x} \in \widehat{K}^{-1}(M)$  by (13) we are forced to define

$$\widehat{\Psi}^k(\widehat{x}) := \int \widehat{\Psi}^k(\widehat{e} \times \widehat{x}) .$$

This gives a natural transformation such that (13) commutes. In particular, it is independent of the choice of  $\widehat{e}$ . The relations (12), (14) and (19) follow by direct calculations. □

By the relations (12) and (7) we are forced to extend the transformation  $\widehat{\Psi}^k$  to all degrees by Bott periodicity, ie such that

$$(20) \quad \begin{array}{ccc} \widehat{K}^*(M)[1/k] & \xrightarrow{k^{-n}\widehat{\Psi}^k} & \widehat{K}^*(M)[1/k] \\ \downarrow \text{Bott}^n & & \downarrow \text{Bott}^n \\ \widehat{K}^{*+2n}(M)[1/k] & \xrightarrow{\widehat{\Psi}^k} & \widehat{K}^{*+2n}(M)[1/k] \end{array}$$

commutes. The relation (14) holds true automatically.

In order to finish the proof we must show that  $\widehat{\Psi}^k$  is multiplicative. Let  $\widehat{x}, \widehat{y} \in \widehat{K}^*(M)$ . Then we can write  $\widehat{x} = b^l \widehat{x}_1$ ,  $\widehat{y} = b^n \widehat{y}_1$ , where  $\widehat{x}_1, \widehat{y}_1$  have degrees in  $\{0, -1\}$ . In this way by (20) we reduce the problem to the multiplicativity of  $\widehat{\Psi}^k$  in degree zero and (19). This finishes the proof of Theorem 3.1. □

An alternative way to construct the lift of the Adams operations would be to use the model [8, Theorem 2.5]

$$\widehat{K}(M) := \widehat{MU}(M) \otimes_{MU^*} K^* ,$$

together with [8, Corollary 2.8].

The following example shows that the lift of the Adams operations to smooth  $K$ -theory act on the classes of geometric line bundles in the expected way lifting (6). Let  $\mathbf{L} := (L, h^L, \nabla^L)$  be a hermitian line bundle with connection over  $M$ . It gives rise to a geometric family  $\mathcal{L}$  (see [8, 2.1.4]) and a smooth  $K$ -theory class  $[\mathbf{L}] := [\mathcal{L}, 0] \in \widehat{K}^0(M)$  in the model of smooth  $K$ -theory (compare [8, Lemma 2.16]).

**Proposition 3.6** We have  $\widehat{\Psi}^k([\mathbf{L}]) = [\mathbf{L}^k]$ .

**Proof** We first consider the canonical bundle  $U \rightarrow \mathbb{C}\mathbb{P}^n$ . Equipped with geometry we get the geometric bundle  $\mathbf{U} = (U, h^U, \nabla^U)$  and the class  $[\mathbf{U}] \in \widehat{K}^0(\mathbb{C}\mathbb{P}^n)$ . Note that by a direct calculation

$$I(\widehat{\Psi}^k([\mathbf{U}])) = I([\mathbf{U}^k]), \quad R(\widehat{\Psi}^k[\mathbf{U}]) = R([\mathbf{U}^k]).$$

Since  $HP^{-1}(\mathbb{C}\mathbb{P}^n) = 0$  the class  $\widehat{\Psi}^k([\mathbf{U}])$  is uniquely determined by its curvature and its underlying topological  $K$ -theory class. This implies

$$(21) \quad \widehat{\Psi}^k([\mathbf{U}]) = [\mathbf{U}^k].$$

In the general case there exists  $n \geq 0$  and  $f: M \rightarrow \mathbb{C}\mathbb{P}^n$  such that  $L \cong f^*U$ . We consider the bundle  $K := \text{pr}_M^*L \rightarrow [0, 1] \times M$  with a geometry  $\mathbf{K}$ , which coincides with  $f^*\mathbf{U}$  on  $\{0\} \times M$  and with  $\mathbf{L}$  on  $\{1\} \times M$ . From the homotopy formula [7, (1)] we get

$$\begin{aligned} \widehat{\Psi}^k([\mathbf{L}]) - f^*\widehat{\Psi}^k([\mathbf{U}]) &= a \left( \int_{[0,1] \times M/M} R(\widehat{\Psi}^k([\mathbf{K}])) \right) \\ &= a \left( \int_{[0,1] \times M/M} R([\mathbf{K}^k]) \right) \\ &= [\mathbf{L}^k] - f^*[\mathbf{U}^k]. \end{aligned}$$

In view of (21) this implies the assertion. □

Let  $(\widehat{HP}\mathbb{Q}, R, I, a, f)$  denote the smooth extension of the periodic rational cohomology theory. In [6] we have constructed a lift of the Chern character to a natural transformation

$$\widehat{\text{ch}}: \widehat{K} \rightarrow \widehat{HP}\mathbb{Q}$$

of ring-valued functors. We let  $\widehat{\Psi}_H^k: \widehat{HP}\mathbb{Q}^* \rightarrow \widehat{HP}\mathbb{Q}^*$  denote the obvious lift of  $\Psi_H^k$  which multiplies  $b^n$  by  $k^{-n}$ .

**Proposition 3.7** We have  $\widehat{\text{ch}} \circ \widehat{\Psi}^k = \widehat{\Psi}_H^k \circ \widehat{\text{ch}}$ .

**Proof** We first consider the even case. The difference

$$\widehat{D} := \widehat{\text{ch}} \circ \widehat{\Psi}^k - \widehat{\Psi}_H^k \circ \widehat{\text{ch}}$$

factors over a natural transformation

$$D: K^0 \rightarrow HP^{-1}/HP\mathbb{Q}^{-1}.$$

Using that  $\mathbf{K}_0$  is an even space similar arguments as in the proof of Lemma 3.4 show that  $D = 0$ . The odd case follows from the compatibility of the Adams operations and the Chern character with integration.  $\square$

Let  $M$  be a compact connected manifold with base point  $* \in M$ . We consider the multiplicative subgroups

$$U := \{x \in K^0(M)[1/k] \mid x|_* = 1\}, \quad \hat{U} := \{\hat{x} \in \hat{K}^0(M)[1/k] \mid \hat{x}|_* = 1\}$$

of  $K^0(M)[1/k]$  and  $\hat{K}^0(M)[1/k]$ .

**Lemma 3.8** *The maps*

$$\Psi^k: U \rightarrow U, \quad \hat{\Psi}^k: \hat{U} \rightarrow \hat{U}$$

*are surjective.*

**Proof** We first consider the topological case. Let  $F^{2n}K^0(M)[1/k] \subseteq K^0(M)[1/k]$  denote the  $2n$ -th step of the Atiyah–Hirzebruch filtration which is finite. The Atiyah–Hirzebruch filtration is compatible with the multiplication in the sense that

$$F^{2n}K(M)[1/k] \cup F^{2m}K(M)[1/k] \subseteq F^{2n+2m}K(M)[1/k].$$

Let  $x \in U$ . Then we find inductively approximations  $z \in U$  such that  $\Psi^k(z) - x \in F^{2n}K^0[1/k](M)$  as follows. The first approximation is  $z = 1$ . Then  $\Psi^k(1) - x \in F^2K^0(M)[1/k]$ . Assume that  $\Psi^k(z) - x =: d \in F^{2n-2}K^0(M)[1/k]$ . Then we take

$$z' := z - (1/k^n)d.$$

Then  $\Psi^k((1/k^n)d) - d \in F^{2n}K^0[1/k](M)$  and therefore  $\Psi^k(z') - x = \Psi^k(z) - (1/k^n)\Psi^k(d) - x \in F^{2n}K^0(M)[1/k]$ .

We now consider the smooth case. Let  $\hat{x} \in \hat{U}$ . Then  $I(\hat{x}) \in U$ . We thus can choose  $z \in U$  such that  $\Psi^k(z) = I(\hat{x})$ . Let  $\hat{z}$  be a smooth lift. Then  $\hat{\Psi}^k(\hat{z}) - \hat{x} = a(\omega)$  for some  $\omega \in \Omega P^{-1}(M)$ . We define  $\hat{z}' := \hat{z} - a((\Psi^k_\Omega)^{-1}(\omega))$ . Then  $\hat{\Psi}^k(\hat{z}') = \hat{x}$ .  $\square$

#### 4 The characteristic class $\hat{\rho}^k$

A  $\text{Spin}^c$ -structure  $(P, \phi)$  on a real  $n$ -dimensional vector bundle  $V \rightarrow M$  is a pair of a  $\text{Spin}^c$ -principal bundle  $P \rightarrow M$  together with an isomorphism  $\phi: P \times_{\text{Spin}^c(n)} \mathbb{R}^n \xrightarrow{\sim} V$ ,

where  $\text{Spin}^c(n)$  acts on  $\mathbb{R}^n$  via the natural projection  $\text{Spin}^c(n) \rightarrow \text{SO}(n)$ . A  $\text{Spin}^c$ -vector bundle is  $K$ -oriented. Let  $\text{Thom}_V: K[1/k]^*(M) \rightarrow \tilde{K}[1/k]^{*+n}(\text{Thom}(V))$  denote the Thom isomorphism. Then there is a characteristic class  $\rho^k(V) \in K[1/k]^0(M)$  of  $\text{Spin}^c$ -vector bundles uniquely characterised by the relation

$$\Psi^k(\text{Thom}_V(x)) = \text{Thom}_V(\rho^k(V) \cup \Psi^k(x)), \quad \forall x \in K[1/k]^*(M)$$

(see [1]). We consider the characteristic class  $\hat{\mathbf{A}}^c(V) \in HP^0(M)$  of  $\text{Spin}^c$ -bundles. A definition is given in [6, Definition 3.3]. In the present paper we modify this definition by inserting suitable powers of the Bott element  $b \in K^*$  in order to shift all homogenous components to degree zero. The  $K$ -theoretic characteristic class  $\rho^k$  has the following properties:

- (1)  $\rho^k(V \oplus W) = \rho^k(V) \cup \rho^k(W)$ .
- (2)  $\rho^k(M \times \mathbb{R}^n) = 1$ .
- (3)  $\rho^k(V)$  is a unit in  $K[1/k]^0(M)$ .
- (4)  $\text{ch}(\rho^k(V)) = \Psi_H^k(\hat{\mathbf{A}}^c(V))/\hat{\mathbf{A}}^c(V)$ .

A geometric  $\text{Spin}^c$ -structure on a vector bundle  $V \rightarrow M$  is a triple  $\mathbf{V} = (P, \phi, \tilde{\nabla})$ , where  $(P, \phi)$  is a  $\text{Spin}^c$ -structure on  $V$  and  $\tilde{\nabla}$  is a connection on  $P$ . By Chern–Weil theory we can define a closed differential form  $\hat{\mathbf{A}}^c(\mathbf{V}) \in \Omega P^0(M)$  which represents the cohomology class  $\hat{\mathbf{A}}^c(V)$ ; see [6, Definition 3.3].

There is a natural definition of the sum  $\mathbf{V} \oplus \mathbf{W}$  of two geometric  $\text{Spin}^c$ -vector bundles. We consider the contravariant functor

$$\text{Vect}^{\text{Spin}^c}: \text{smooth compact manifolds} \longrightarrow \text{semigroups}$$

which associates to a compact smooth manifold  $M$  the semigroup  $\text{Vect}^{\text{Spin}^c}(M)$  of geometric  $\text{Spin}^c$  vector bundles.

**Theorem 4.1** *There exists a unique natural transformation of set-valued functors*

$$\hat{\rho}^k: \text{Vect}^{\text{Spin}^c} \rightarrow \hat{K}^0(\dots)[1/k]$$

such that

$$(22) \quad I \circ \hat{\rho}^k = \rho^k,$$

$$(23) \quad R(\hat{\rho}^k(\mathbf{W})) = \frac{\Psi_\Omega^k(\hat{\mathbf{A}}^c(\mathbf{W}))}{\hat{\mathbf{A}}^c(\mathbf{W})}, \quad \mathbf{W} \in \text{Vect}^{\text{Spin}^c}(M).$$

It in addition satisfies

$$(24) \quad \widehat{\rho}^k(\mathbf{W} \oplus \mathbf{W}') = \widehat{\rho}^k(\mathbf{W}) \cup \widehat{\rho}^k(\mathbf{W}'), \quad \mathbf{W}, \mathbf{W}' \in \text{Vect}^{\text{Spin}^c}(M),$$

$$(25) \quad \widehat{\rho}^k(\mathbf{W}) \text{ is a unit in } \widehat{K}^0(M)[1/k], \quad \mathbf{W} \in \text{Vect}^{\text{Spin}^c}(M).$$

**Proof** Let us fix  $n \in \mathbb{N}_0$ . We first construct  $\widehat{\rho}^k$  on the subfunctor  $\text{Vect}^{\text{Spin}^c}_n$  of  $n$ -dimensional geometric  $\text{Spin}^c$ -vector bundles. Note that the classifying space  $B\text{Spin}^c(n)$  of  $\text{Spin}^c(n)$  is a simply connected rationally even space. Using [7, Proposition 2.1] we choose a sequence of compact manifolds  $(\mathcal{B}_i)_{i \geq 0}$  together with maps

$$x_i: \mathcal{B}_i \rightarrow B\text{Spin}^c(n), \quad \kappa_i: \mathcal{B}_i \rightarrow \mathcal{B}_{i+1}$$

such that

- (1)  $\mathcal{B}_i$  is homotopy equivalent to an  $i$ -dimensional CW-complex,
- (2)  $\kappa_i: \mathcal{B}_i \rightarrow \mathcal{B}_{i+1}$  is an embedding of a closed submanifold,
- (3)  $x_i: \mathcal{B}_i \rightarrow B\text{Spin}^c(n)$  is  $i$ -connected,
- (4)  $x_{i+1} \circ \kappa_i = x_i$ .

Let  $\xi_n \rightarrow B\text{Spin}^c(n)$  be the universal  $n$ -dimensional  $\text{Spin}^c$  vector bundle. Let  $E_i := x_i^* \xi_n$ . Note that  $\kappa_i^* E_{i+1} \cong E_i$ . Since  $\kappa_i: \mathcal{B}_i \rightarrow \mathcal{B}_{i+1}$  is an embedding of a closed submanifold we can inductively choose geometric refinements  $\mathbf{E}_i \in \text{Vect}^{\text{Spin}^c}_n(\mathcal{B}_i)$  such that  $\kappa_i^* \mathbf{E}_{i+1} \cong \mathbf{E}_i$ . For each  $i$  we choose a class  $\widehat{r}^k(\mathbf{E}_i) \in \widehat{K}^0(\mathcal{B}_i)[1/k]$  such that

$$I(\widehat{r}^k(\mathbf{E}_i)) = \rho^k(E_i), \quad R(\widehat{r}^k(\mathbf{E}_i)) = \frac{\Psi_{\Omega}^k(\widehat{\mathbf{A}}^c(\mathbf{E}_i))}{\widehat{\mathbf{A}}^c(\mathbf{E}_i)}.$$

This element is uniquely determined up to elements in  $a(F^{\geq i}HP^{-1}(\mathcal{B}_i))$ ; see (15).

Let now  $M$  be a compact manifold and  $\mathbf{W} \in \text{Vect}^{\text{Spin}^c}_n(M)$ . Then we choose  $i \in \mathbb{N}$  such that  $i > \dim(M)$  and there exists  $f: M \rightarrow \mathcal{B}_i$  and an isomorphism  $f^* E_i \cong \mathbf{W}$ . We are forced to define  $\widehat{\rho}^k(f^* \mathbf{E}_i) = f^* \widehat{r}^k(\mathbf{E}_i)$ . This class is independent of the choice of  $\widehat{r}^k(\mathbf{E}_i)$ .

We now consider the bundle  $V := \text{pr}_M^* \mathbf{W} \rightarrow [0, 1] \times M$ . We choose a geometry  $\mathbf{V} \in \text{Vect}^{\text{Spin}^c}_n([0, 1] \times M)$  which coincides on  $\{0\} \times M$  with  $f^* \mathbf{E}_i$ , and on  $\{1\} \times M$  with  $\mathbf{W}$ . By the homotopy formula we are forced to define

$$(26) \quad \begin{aligned} \widehat{\rho}^k(\mathbf{W}) &:= \widehat{\rho}^k(f^* \mathbf{E}_i) + a \left( \int_{[0,1] \times M/M} R(\widehat{\rho}^k(\mathbf{V})) \right) \\ &= f^* \widehat{r}^k(\mathbf{E}_i) + a \left( \int_{[0,1] \times M/M} \frac{\Psi_{\Omega}^k(\widehat{\mathbf{A}}^c(\mathbf{V}))}{\widehat{\mathbf{A}}^c(\mathbf{V})} \right). \end{aligned}$$

By a straightforward calculation we verify that (22) and (23) hold true.

**Lemma 4.2**  $\hat{\rho}^k: \text{Vect}^{\text{Spin}^c}_n \rightarrow \widehat{K}^0(\dots)[1/k]$  is a well-defined natural transformation.

**Proof** If we choose a second geometry  $\mathbf{V}' \in \text{Vect}^{\text{Spin}^c}_n([0, 1] \times M)$  interpolating between  $\mathbf{W}$  and  $f^*\mathbf{E}_i$ , then the difference

$$\int_{[0,1] \times M/M} \frac{\Psi^k_{\Omega}(\widehat{\mathbf{A}}^c(\mathbf{V}))}{\widehat{\mathbf{A}}^c(\mathbf{V})} - \int_{[0,1] \times M/M} \frac{\Psi^k_{\Omega}(\widehat{\mathbf{A}}^c(\mathbf{V}'))}{\widehat{\mathbf{A}}^c(\mathbf{V}')}$$

is exact. This follows by Stokes' theorem from the fact, that the geometries  $\mathbf{V}$  and  $\mathbf{V}'$  can again be connected by geometric bundle over  $[0, 1]^2 \times M$ . We conclude that  $\hat{\rho}^k(\mathbf{W})$  does not depend on the choice of  $\mathbf{V}$ .

If we increase  $i$  by one and set  $f' := \kappa_i \circ f$ , then we get  $f'^*\mathbf{E}_{i+1} \cong f^*\mathbf{E}_i$ ,  $f^*\hat{\rho}^k(\mathbf{E}_i) = f'^*\hat{\rho}^k(\mathbf{E}_{i+1})$ , and therefore the same result for  $\hat{\rho}^k(\mathbf{W})$ .

Any two choices of maps  $f: M \rightarrow \mathcal{B}_i$  become homotopic after increasing  $i$  sufficiently many times. If  $f$  and  $f'$  are homotopic by a homotopy  $H: [0, 1] \times M \rightarrow \mathcal{B}_i$ , then we apply the construction for  $\text{pr}_M^*\mathbf{W}$  and  $H$ . In this way we get a class  $\hat{\rho}^k(\text{pr}_M^*\mathbf{W})$ . Note that

$$R(\hat{\rho}^k(\text{pr}_M^*\mathbf{W})) = \text{pr}_M^* \frac{\Psi^k_{\Omega}(\widehat{\mathbf{A}}^c(\mathbf{W}))}{\widehat{\mathbf{A}}^c(\mathbf{W})}$$

has no  $dt$ -component, where  $t$  is the coordinate of  $[0, 1]$ . It thus follows from the homotopy formula that

$$\hat{\rho}^{k, \prime}(\mathbf{W}) = \hat{\rho}^k(\text{pr}_M^*\mathbf{W})|_{\{1\} \times M} = \hat{\rho}^k(\text{pr}_M^*\mathbf{W})|_{\{0\} \times M} = \hat{\rho}^k(\mathbf{W}) .$$

Finally we verify that  $\hat{\rho}^k: \text{Vect}^{\text{Spin}^c}_n \rightarrow \widehat{K}^0(\dots)[1/k]$  is a natural transformation. Let  $g: M' \rightarrow M$  be a smooth map. Then in the definition of  $\hat{\rho}^k(g^*\mathbf{W})$  we can take  $f' := f \circ g$  and  $\mathbf{V}' := (\text{id}_{[0,1]} \times g)^*\mathbf{V}$ . With these choices we have

$$g^*\hat{\rho}^k(\mathbf{W}) = \hat{\rho}^k(g^*\mathbf{W}) . \quad \square$$

**Lemma 4.3** The relation (24) holds true.

**Proof** We consider the transformation

$$B: \text{Vect}^{\text{Spin}^c}_n \times \text{Vect}^{\text{Spin}^c}_m \rightarrow \widehat{K}^0(\dots)[1/k]$$

given by  $B(\mathbf{W}, \mathbf{W}') := \hat{\rho}^k(\mathbf{W} \oplus \mathbf{W}') - \hat{\rho}^k(\mathbf{W}) \cup \hat{\rho}^k(\mathbf{W}')$ .

We must show that  $B = 0$ . By construction we have

$$R \circ B = 0 , \quad I \circ B = 0 .$$



Therefore  $B$  takes values in the homotopy invariant subfunctor

$$HP^{-1}(\dots)/\text{im}(\text{ch})[1/k] \subset \widehat{K}^0(\dots)[1/k].$$

Since it is homotopy invariant it is clear that  $B(\mathbf{W}, \mathbf{W}') \in HP^{-1}(M)/\text{im}(\text{ch})[1/k]$  only depends on the underlying topological  $\text{Spin}^c$ -bundles  $W$  and  $W'$  over  $M$ . There exists  $j > \dim(M)$  such that can find maps  $f, f': M \rightarrow \mathcal{B}_j$  such that  $W \cong f^*E_j$  and  $W' \cong f'^*E_j$ . We set  $F := f \times f', V := \text{pr}_1^*E_j$ , and  $V' := \text{pr}_2^*E_j$ , where  $\text{pr}_k: \mathcal{B}_j \times \mathcal{B}_j \rightarrow \mathcal{B}_j, k = 1, 2$  are the projections. Then we get

$$B(W, W') = B(F^*V, F^*V') = F^*B(V, V').$$

Now  $B(V, V') \in F^{\geq j}HP^{-1}(\mathcal{B}_j \times \mathcal{B}_j)/\text{im}(\text{ch})[1/k]$ , and therefore  $F^*B(V, V') = 0$ . This shows that  $B(W, W') = 0$ . □

**Lemma 4.4** For  $\mathbf{W} \in \text{Vect}^{\text{Spin}^c}(M)$  the class  $\widehat{\rho}^k(\mathbf{W}) \in \widehat{K}^0(M)[1/k]$  is a unit.

**Proof** We write  $\widehat{\rho}^k(\mathbf{W}) = 1 + (\widehat{\rho}^k(\mathbf{W}) - 1)$ . It suffices to show that  $\widehat{\rho}^k(\mathbf{W}) - 1$  is nilpotent. First of all, since

$$R(\widehat{\rho}^k(\mathbf{W})) = 1 + \text{higher order forms}$$

we see that  $R(\widehat{\rho}^k(\mathbf{W}) - 1)$  is nilpotent. Similarly, the restriction of

$$I(\widehat{\rho}^k(\mathbf{W}) - 1) = \rho^k(W) - 1$$

to a point vanishes. Therefore  $\rho^k(W) - 1$  belongs to a lower step of the Atiyah–Hirzebruch filtration of  $K^0(M)[1/k]$  and is therefore nilpotent.

We conclude that for some large  $l \in \mathbb{N}$

$$(\widehat{\rho}^k(\mathbf{W}) - 1)^l \in HP^{-1}(M)/\text{im}(\text{ch})[1/k].$$

But then  $(\widehat{\rho}^k(\mathbf{W}) - 1)^{2l} = 0$ . □

This finishes the proof of Theorem 4.1. □

In view of the homomorphism  $\text{Spin}(n) \rightarrow \text{Spin}^c(n)$  a  $\text{Spin}$ -structure on  $V$  naturally induces a  $\text{Spin}^c$ -structure. We define the notion of a geometric  $\text{Spin}$ -bundle in a similar manner as the notion of a geometric  $\text{Spin}^c$ -bundle. Notice that a geometric  $\text{Spin}$ -bundle  $\widetilde{\mathbf{W}}$  gives rise to a geometric  $\text{Spin}^c$ -bundle  $\mathbf{W}$ . We have  $\widehat{\mathbf{A}}^c(\mathbf{W}) = \widehat{\mathbf{A}}(\widetilde{\mathbf{W}})$ , and this form is invariant under  $\Psi_\Omega^{-1}$ .

**Lemma 4.5** If  $\widetilde{\mathbf{W}}$  is a geometric  $\text{Spin}$ -bundle, then  $\widehat{\rho}^{-1}(\mathbf{W}) = 1$ .

**Proof** If  $\tilde{W}$  is an  $n$ -dimensional geometric Spin-bundle on a compact manifold, then there exist another geometric Spin-bundle  $\tilde{E}$  on a manifold  $B$  which has no real cohomology in odd degree below  $\dim(M) + 1$ , and a map  $f: M \rightarrow B$  such that  $f^*E \cong W$ . In fact, for  $B$  we can take some approximation of a finite skeleton of  $BSpin(n)$ . We let  $\mathbf{E}$  denote the geometric  $Spin^c$ -bundle induced by  $\tilde{E}$ .

A spin bundle  $W$  has a  $KO$ -orientation<sup>3</sup>. Its  $KO$ -theory Thom class induces the  $K$ -theory Thom class via the natural transformation (complexification)  $KO \rightarrow K$ . Therefore the  $K$ -theory Thom class of  $W$  is fixed by complex conjugation  $\Psi^{-1}$ . Hence, if the  $Spin^c$ -structure of  $W$  comes from a Spin-structure, then we have  $\rho^{-1}(W) = 0$ .

We conclude that

$$I(\hat{\rho}^{-1}(\mathbf{E}) - 1) = \rho^{-1}(E) - 1 = 0 .$$

Furthermore, as argued above,  $\hat{\mathbf{A}}^c(\mathbf{E})$  is fixed by  $\Psi^{-1}$  so we have  $R(\hat{\rho}^{-1}(\mathbf{E}) - 1) = 0$ . It follows that  $\hat{\rho}^{-1}(\mathbf{E}) - 1 \in F^{\geq \dim(M)+1}HP^{-1}(B)/\text{im}(\text{ch})$  by the assumption on the odd-degree cohomology of  $B$ . Therefore  $\hat{\rho}^{-1}(f^*\mathbf{E}) = 1$ . We now consider a geometric Spin-bundle  $\tilde{V}$  over  $[0, 1] \times M$  which connects  $\tilde{W}$  and  $f^*\mathbf{E}$ . Since  $\hat{\mathbf{A}}(\tilde{V}) = \hat{\mathbf{A}}^c(\mathbf{V})$  is invariant under  $\Psi_{\Omega}^{-1}$  we have  $R(\hat{\rho}^{-1}(\mathbf{V})) = 1$ . If we use these observations in (26) we get  $\hat{\rho}^{-1}(\mathbf{W}) = 1$ . □

Let us now consider a submersion  $\pi: E \rightarrow B$  from a compact manifold  $E$ . We assume that the vertical bundle  $T^v\pi := \ker(d\pi)$  has a  $Spin^c$ -structure. It induces a  $K$ -orientation of  $\pi$ . Recall that a smooth  $K$ -orientation  $o$  [6, Definition 3.5] of  $f$  is represented by a tuple  $(g^{T^v\pi}, T^h\pi, \tilde{\nabla}, \sigma)$ , where  $g^{T^v\pi}$  is a vertical metric,  $T^h\pi$  is a horizontal distribution,  $\tilde{\nabla}$  is a  $Spin^c$ -extension of the Levi-Civita connection  $\nabla^{T^v\pi}$  on  $T^v\pi$  (induced by  $g^{T^v\pi}$  and  $T^h\pi$ ), and  $\sigma \in \Omega P^{-1}(E)/\text{im}(d)$  (here we again use the modified definition based on the insertion powers of the Bott element in order to shift all forms to degree  $-1$ ). The curvature of the  $K$ -orientation is defined by

$$R(o) := \hat{\mathbf{A}}^c(\tilde{\nabla}) - d\sigma ,$$

where we write  $\hat{\mathbf{A}}^c(\tilde{\nabla})$  instead of  $\hat{\mathbf{A}}^c(\mathbf{T}^v\mathbf{B})$ . The smooth  $K$ -orientation in particular induces a geometric  $Spin^c$ -structure  $\mathbf{T}^v\mathbf{B}$  on  $T^v\pi$ .

We consider the bundle  $\bar{\pi} = \text{id} \times \pi: [0, 1] \times E \rightarrow [0, 1] \times B$  and choose a representative of a smooth  $K$ -orientation  $\bar{o}$  which interpolates from  $(g^{T^v\pi}, T^h\pi, \tilde{\nabla}, 0)$  to  $(g^{T^v\pi}, T^h\pi, \tilde{\nabla}, \sigma)$ .

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<sup>3</sup>  $KO$  denotes real  $K$ -theory.

**Definition 4.6** We define

$$\hat{\rho}^k(o) := \hat{\rho}^k(\mathbf{T}^v \mathbf{B})^{-1} + a \left( \int_{[0,1] \times E/E} \frac{\Psi_{\Omega}^k(R(\bar{o}))}{R(\bar{o})} \right) \in \hat{K}^0(E)[1/k].$$

Note that two tuples  $(g^{T^v \pi}, T^h \pi, \tilde{\nabla}, \sigma)$  and  $(g^{T^v \pi'}, T^{h'} \pi', \tilde{\nabla}', \sigma')$  represent the same smooth  $K$ -orientation if the underlying  $\text{Spin}^c$ -structures are isomorphic and

$$\sigma' - \sigma = \tilde{\mathbf{A}}^c(\tilde{\nabla}', \tilde{\nabla}).$$

Here  $\tilde{\mathbf{A}}^c(\tilde{\nabla}', \tilde{\nabla})$  is the transgression form defined in [6, Definition 3.4] (again shifted to degree  $-1$ ).

**Proposition 4.7** The class  $\hat{\rho}^k(o)$  is independent of the choice of the representative of  $o$ .

**Proof** We choose a representative of a smooth  $K$ -orientation  $\bar{\sigma}$  of  $[0, 1] \times E \rightarrow [0, 1] \times B$  which interpolates from  $(g^{T^v \pi'}, T^{h'} \pi', \tilde{\nabla}', \sigma')$  to  $(g^{T^v \pi}, T^h \pi, \tilde{\nabla}, \sigma)$ . Furthermore we let  $\bar{o}'$  be the smooth  $K$ -orientation of  $[0, 1] \times E \rightarrow [0, 1] \times B$  obtained by concatenating  $\bar{\sigma}$  with  $\bar{o}$ . With these choices we get

$$\hat{\rho}^{k'}(o) - \hat{\rho}^k(o) = a \left( \int_{[0,1] \times E/E} \frac{\Psi_{\Omega}^k(R(\bar{o}'))}{R(\bar{o}')} \right).$$

In order to go further we adopt a very special choice for  $\bar{\sigma}$ :

$$\bar{\sigma}(t) := \text{pr}_E^* \sigma' + \int_{[0,t] \times E/E} \hat{\mathbf{A}}^c(\bar{\nabla}).$$

Indeed,  $\bar{\sigma}(1) = \text{pr}_E^* \sigma' + \hat{\mathbf{A}}^c(\tilde{\nabla}, \tilde{\nabla}') = \text{pr}_E^* \sigma$ .

Then we have  $d\bar{\sigma} = dt \wedge i_{\partial_t} \hat{\mathbf{A}}^c(\bar{\nabla}) + \theta$  with  $i_{\partial_t} \theta = 0$ . It follows that  $i_{\partial_t} R(\bar{o}) = 0$  and therefore

$$\int_{[0,1] \times E/E} \frac{\Psi_{\Omega}^k(R(\bar{o}))}{R(\bar{o})} = 0. \quad \square$$

We now consider an iterated bundle

$$\begin{array}{ccc} & & r \\ & \curvearrowright & \\ W & \xrightarrow{p} & E \xrightarrow{q} B \end{array}$$

of compact manifolds. We assume that  $T^v p$  and  $T^v q$  are equipped with  $\text{Spin}^c$ -structures. We choose smooth  $K$ -orientations  $o_p$  and  $o_q$  lifting these  $\text{Spin}^c$ -structures.

Then there is an induced  $\text{Spin}^c$ -structure on  $T^v r \cong T^v p \oplus p^*T^v q$  and a smooth orientation  $o_r = o_q \circ o_p$  (see [6, Definition 3.21]).

**Proposition 4.8** We have  $\widehat{\rho}^k(o_p \circ o_q) = \widehat{\rho}^k(o_p) \cup p^*\widehat{\rho}^k(o_q)$ .

**Proof** We consider the difference

$$\Delta(o_p, o_q) := \widehat{\rho}^k(o_q \circ o_p) - \widehat{\rho}^k(o_p) \cup p^*\widehat{\rho}^k(o_q) .$$

We first check by a direct calculation that

$$I(\Delta(o_p, o_q)) = 0 , \quad R(\Delta(o_p, o_q)) = 0 .$$

It follows that  $\Delta(o_p, o_q) \in HP^{-1}(W)/\text{im}(\text{ch})$ . Since two choices of smooth  $K$ -orientations refining a fixed underlying topological  $K$ -orientation can be connected by a path it follows by homotopy invariance that  $\Delta(o_p, o_q)$  only depends on the topological  $\text{Spin}^c$ -structures of  $T^v p$  and  $T^v q$ . Let us now recall the construction of  $o_r$ . We take  $o_p := (g^{T^v p}, T^h p, \widetilde{\nabla}^{T^v p}, 0)$  and  $o_q := (g^{T^v q}, T^h q, \widetilde{\nabla}^{T^v q}, 0)$ . Then for  $\lambda > 0$  we get an induced metric  $g_\lambda^{T^v r} = \lambda^2 g^{T^v p} \oplus p^*g^{T^v q}$  and splitting  $T^h r$ . As explained in [6, 3.3.1] we also get an induced connection  $\widetilde{\nabla}_\lambda^{T^v r}$  which has a limit  $\widetilde{\nabla}^{\text{adia}} := \lim_{\lambda \rightarrow 0} \widetilde{\nabla}_\lambda^{T^v r}$ . The composition of the smooth  $K$ -orientations is represented by (see [8, Definition 3.21])

$$o_q \circ o_p = (g_\lambda^{T^v r}, T^h r, \widetilde{\nabla}_\lambda^{T^v r}, -\widetilde{\mathbf{A}}^c(\widetilde{\nabla}^{\text{adia}}, \widetilde{\nabla}_\lambda^{T^v r})) .$$

In [8, (20)] we have observed that  $\widetilde{\nabla}^{\text{adia}} = \widetilde{\nabla}^{T^v p} \oplus \widetilde{p}^*\nabla^{T^v q}$  under the decomposition  $\mathbf{T}^v \mathbf{r} \cong \mathbf{T}^v \mathbf{p} \oplus p^*\mathbf{T}^v \mathbf{q}$  of vector bundles with geometric  $\text{Spin}^c$ -structures. We have by Definition 4.6 and Proposition 4.7 that

$$\begin{aligned} \widehat{\rho}^k(o_q \circ o_p) &= \widehat{\rho}^k(\mathbf{T}^v \mathbf{r})^{-1} + a(T(\lambda)) \\ &= \widehat{\rho}^k(\mathbf{T}^v \mathbf{p} \oplus p^*\mathbf{T}^v \mathbf{q})^{-1} + a(T'(\lambda)) \\ &= \widehat{\rho}^k(\mathbf{T}^v \mathbf{p})^{-1} \cup p^*\widehat{\rho}^k(\mathbf{T}^v \mathbf{q})^{-1} + a(T'(\lambda)) \\ &= \widehat{\rho}^k(o_p) \cup p^*\widehat{\rho}^k(o_q) + a(T'(\lambda)) , \end{aligned}$$

where the forms  $T(\lambda)$  and  $T'(\lambda)$  depend on the difference of  $\widetilde{\nabla}^{\text{adia}}$  and  $\widetilde{\nabla}_\lambda^{T^v r}$  and vanish as  $\lambda \rightarrow 0$ . We now take the limit  $\lambda \rightarrow 0$  and get

$$\widehat{\rho}^k(o_q \circ o_p) = \widehat{\rho}^k(o_p) \cup p^*\widehat{\rho}^k(o_q) . \quad \square$$

We now consider a cartesian diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & E \\ \downarrow q & & \downarrow p \\ A & \xrightarrow{f} & B . \end{array}$$

A smooth  $K$ -orientation  $o_p$  of  $p$  induces a smooth  $K$ -orientation  $o_q$  of  $q$ .

**Lemma 4.9** *In this situation we have*

$$\hat{\rho}^k(o_q) = g^* \hat{\rho}^k(o_p) .$$

**Proof** The geometric  $\text{Spin}^c$ -structure on  $T^v q$  induced by  $o_q$  is  $\mathbf{T}^v \mathbf{q} \cong g^* \mathbf{T}^v \mathbf{p}$ . If we choose  $\bar{o}_q := (\text{id}_{[0,1]} \times g)^* \bar{o}_p$  in the construction of  $\hat{\rho}^k(o_q)$ , then we immediately get  $\hat{\rho}^k(o_q) = g^* \hat{\rho}^k(o_p)$  from Definition 4.6.  $\square$

Consider a submersion  $\pi : E \rightarrow B$  from a compact manifold  $E$ . In [6, 5.11] we have observed that a stable framing of  $T^v \pi$  provides a canonical  $K$ -orientation. Let us assume that  $T^v \pi \oplus \underline{\mathbb{R}}^N_E$  is framed, where  $\underline{\mathbb{R}}^N_E := E \times \mathbb{R}^N$  denotes the trivial  $N$ -dimensional real vector bundle over  $E$ . The associated smooth  $K$ -orientation is

$$o_\pi := (g^{T^v \pi}, T^h \pi, \tilde{\nabla}^{T^v \pi}, \tilde{\mathbf{A}}^c(\tilde{\nabla}^{T^v \pi} \oplus \nabla^{\underline{\mathbb{R}}^N_E}, \tilde{\nabla}^{T^v \pi \oplus \underline{\mathbb{R}}^N_{E, \text{frame}}})) ,$$

where  $\tilde{\nabla}^{T^v \pi \oplus \underline{\mathbb{R}}^N_{E, \text{frame}}}$  is the connection induced by the framing.

**Proposition 4.10** *If  $o_\pi$  is induced by a stable framing of  $T^v \pi$ , then  $\hat{\rho}^k(o_\pi) = 1$ .*

**Proof** We have by the homotopy formula

$$(27) \quad \hat{\rho}^k(\mathbf{T}^v \mathbf{B})^{-1} = 1 + a \left( \int_{[0,1] \times E/E} \frac{\Psi_\Omega^k(\hat{\mathbf{A}}^c(\bar{\nabla}))}{\hat{\mathbf{A}}^c(\bar{\nabla})} \right) ,$$

where  $\bar{\nabla}$  is a family of  $\text{Spin}^c$ -connections interpolating from

$$\tilde{\nabla}^{T^v \pi \oplus \underline{\mathbb{R}}^N_{E, \text{frame}}} \quad \text{to} \quad \tilde{\nabla}^{T^v \pi} \oplus \tilde{\nabla}^{\underline{\mathbb{R}}^N_E} .$$

Let  $\bar{o}$  be a form on  $[0, 1] \times E$  interpolating from

$$\tilde{\mathbf{A}}^c(\tilde{\nabla}^{T^v \pi} \oplus \tilde{\nabla}^{\underline{\mathbb{R}}^N_E}, \tilde{\nabla}^{T^v \pi \oplus \underline{\mathbb{R}}^N_{E, \text{frame}}})$$

to zero. This family induces a smooth  $K$ -orientation  $\bar{o}$  on  $[0, 1] \times E \rightarrow [0, 1] \times B$  which interpolates from  $o_\pi$  to  $(g^{T^v \pi}, T^h \pi, \tilde{\nabla}^{T^v \pi}, 0)$ . Then we have

$$(28) \quad \hat{\rho}^k(o_\pi) = \hat{\rho}^k(\mathbf{T}^v \mathbf{B})^{-1} - a \left( \int_{[0,1] \times E/E} \frac{\Psi_\Omega^k(R(\bar{o}))}{R(\bar{o})} \right) .$$

Furthermore, we have

$$\tilde{\widehat{\mathbf{A}}}^c(\tilde{\nabla}^{T^v\pi} \oplus \tilde{\nabla}^{\mathbb{R}^N_E}, \tilde{\nabla}^{T^v\pi} \oplus \mathbb{R}^N_{E, \text{frame}}) = \int_{[0,1] \times E/E} \widehat{\mathbf{A}}^c(\bar{\nabla}) .$$

As in the proof of Proposition 4.7 we can choose

$$\bar{\sigma}(t) := \int_{[t,1] \times E/E} \widehat{\mathbf{A}}^c(\bar{\nabla})$$

so that

$$\begin{aligned} d\bar{\sigma}(t) &= -dt \wedge i_{\partial_t} \widehat{\mathbf{A}}^c(\bar{\nabla}) + d\tilde{\widehat{\mathbf{A}}}^c(\tilde{\nabla}^{T^v\pi} \oplus \tilde{\nabla}^{\mathbb{R}^N_E}, \bar{\nabla}_{\{t\} \times E}) \\ &= -dt \wedge i_{\partial_t} \widehat{\mathbf{A}}^c(\bar{\nabla}) - \widehat{\mathbf{A}}^c(\bar{\nabla}_{\{t\} \times E}) + \widehat{\mathbf{A}}^c(\tilde{\nabla}^{T^v\pi}) . \end{aligned}$$

We get

$$R(\bar{\sigma}) = \text{pr}_E^* \widehat{\mathbf{A}}^c(\tilde{\nabla}^{T^v\pi}) - d\bar{\sigma} = \widehat{\mathbf{A}}^c(\bar{\nabla}) .$$

In combination with (27) and (28) this implies  $\hat{\rho}^k(o_\pi) = 1$ . □

### 5 The index theorem

Let  $\pi: E \rightarrow B$  be a proper submersion over a compact base  $B$  with fibre dimension  $n := \dim(E) - \dim(B)$ . We assume that  $\pi$  is topologically  $K$ -oriented by the datum of a  $\text{Spin}^c$ -structure on  $T^v\pi$ . Let  $o$  be a smooth  $K$ -orientation of  $\pi$  which refines this topological  $K$ -orientation. Then we have the pushforward

$$\hat{\pi}_!: \widehat{K}^*(E) \rightarrow \widehat{K}^{*-n}(B)$$

(see [6, Definition 3.18]). The following theorem refines the identity (11) to the smooth case.

**Theorem 5.1** *In  $\widehat{K}^{*-n}(B)[1/k]$  we have the identity*

$$(29) \quad \widehat{\Psi}^k(\hat{\pi}_!(\hat{x})) = \hat{\pi}_!(\hat{\rho}^k(o) \cup \widehat{\Psi}^k(\hat{x})) , \quad \forall x \in \widehat{K}^*(E)[1/k] .$$

**Proof** We begin with a lemma.

**Lemma 5.2** *The equality (29) holds true after applying  $I$  or  $R$ . Moreover it holds true if  $\hat{x} = a(\alpha)$  for  $\alpha \in \Omega P^{*-1}(E)$ .*

**Proof** The proof goes by straightforward calculations. □

We now consider the difference of the left- and right-hand sides of (29):

$$\widehat{\Delta}_\pi(\hat{x}) := \widehat{\Psi}^k(\hat{\pi}_!(\hat{x})) - \hat{\pi}_!(\hat{\rho}^k(o) \cup \widehat{\Psi}^k(\hat{x})) \in \widehat{K}^{*-n}(B)[1/k] .$$

By Lemma 5.2 we know that  $R \circ \widehat{\Delta}_\pi = 0$  and  $I \circ \widehat{\Delta}_\pi = 0$ . It follows that

$$\widehat{\Delta}_\pi(\widehat{x}) \in HP^{*-n-1}(B)/\text{im}(\text{ch})[1/k].$$

Moreover, since it vanishes on classes of the form  $\widehat{x} = a(\alpha)$ , it factors over a homomorphism

$$\Delta_\pi: K^*(E) \rightarrow HP^{*-n-1}(B)/\text{im}(\text{ch})[1/k].$$

**Lemma 5.3** *Assume that  $p: F \rightarrow B$  is a smoothly  $K$ -oriented zero bordism of the smoothly  $K$ -oriented bundle  $\pi: E \rightarrow B$ , and that  $y \in K^l(F)[1/k]$ . Then we have  $\Delta_\pi(y|_E) = 0$ .*

**Proof** We let  $o_p$  denote a smooth  $K$ -orientation of  $p$  with a product structure near the boundary which restricts to a smooth  $K$ -orientation  $o$  of  $\pi$ . We further choose a smooth lift  $\widehat{y} \in \widehat{K}^l(F)$ . We let  $\widehat{x} := \widehat{y}|_E$  and  $x := I(\widehat{x}) = y|_E$ . Then we calculate using the bordism formula [6, Proposition 5.18]

$$\begin{aligned} \Delta_\pi(x) &= \widehat{\Psi}^k(\widehat{\pi}_!(\widehat{x})) - \widehat{\pi}_!(\widehat{\rho}^k(o) \cup \widehat{\Psi}^k(\widehat{x})) \\ &= \widehat{\Psi}^k\left(a\left(\int_{F/B} R(o_p) \wedge R(\widehat{y})\right)\right) - a\left(\int_{F/B} R(o_p) \wedge R(\widehat{\rho}^k(o_p)) \wedge R(\widehat{\Psi}^k(\widehat{y}))\right) \\ &= a\left(\Psi_\Omega^k\left(\int_{F/B} R(o_p) \wedge R(\widehat{y})\right)\right) - \int_{F/B} \Psi_\Omega^k(R(o_p)) \wedge \Psi_\Omega^k(R(\widehat{y})) \\ &= 0. \end{aligned} \quad \square$$

**Lemma 5.4** *The homomorphism  $\Delta_\pi$  only depends on the underlying topological  $K$ -orientation of the bundle  $\pi: E \rightarrow B$ .*

**Proof** Let  $o_0$  and  $o_1$  be two smooth  $K$ -orientations with the same underlying topological  $K$ -orientations which gives rise to  $\Delta_{\pi,0}$  and  $\Delta_{\pi,1}$ . Then we choose a smooth  $K$ -orientation  $o_p$  on  $p := \pi \circ \text{pr}_E: F := [0, 1] \times E \rightarrow B$  which restricts to  $o_i$  on the endpoints of the interval. We apply Lemma 5.3 to the class  $y = \text{pr}_E^*x$  in order to see that  $\Delta_{\pi,0}(x) = \Delta_{\pi,1}(x)$ . □

The homomorphism  $\Delta_\pi$  has the following naturality property. Let

$$\begin{array}{ccc} E' & \xrightarrow{G} & E \\ \downarrow \pi' & & \downarrow \pi \\ B' & \xrightarrow{g} & B \end{array}$$

be cartesian with a compact manifold  $B'$  and the topological  $K$ -orientation of  $\pi'$  be induced by that of  $\pi$ .

**Lemma 5.5** We have  $\Delta_{\pi'} \circ G^* = g^* \circ \Delta_{\pi}$ .

**Proof** This follows from the naturality of  $\widehat{\Psi}^k$ ,  $G^* \widehat{\rho}^k(o_{\pi}) = \widehat{\rho}^k(o_{\pi'})$  (Lemma 4.9) and  $\widehat{\pi}_1 \circ G^* = g^* \circ \widehat{\pi}_1$  (see [6, Lemma 3.20]). □

The following proposition is the nontrivial heart of the proof of Theorem 5.1.

**Proposition 5.6** If  $\dim(E) = n = 2m - 1$  and  $B = *$ , then  $\Delta_{\pi} = 0$ .

**Proof** Let  $E$  be a closed  $\text{Spin}^c$ -manifold of dimension  $2m - 1$  together with a class  $x \in K^0(E)$  which we view as a homotopy class of maps  $E \rightarrow \mathbb{Z} \times BU$ . The pair  $(E, x)$  represents a  $\text{Spin}^c$ -bordism class  $[E, x] \in \Omega_{2m-1}^{\text{Spin}^c}(\mathbb{Z} \times BU)$ . The integral cohomology of  $\mathbb{Z} \times BU$  is concentrated in even degrees, and the odd part of  $\Omega_{*}^{\text{Spin}^c}$  is a torsion 2-group. Using the Atiyah–Hirzebruch spectral sequence we see that  $\Omega_{2m-1}^{\text{Spin}^c}(\mathbb{Z} \times BU)$  is a torsion 2-group. Hence there exists  $l \in \mathbb{N}$  of the form  $l = 2^{l'}$  such that  $l[E, x] = 0$ . Thus there exists a  $2m$ -dimensional  $\text{Spin}^c$ -manifold  $W$  with boundary  $\partial W \cong lE$  together with an extension  $y: W \rightarrow \mathbb{Z} \times BU$  of the map  $\partial W \rightarrow \mathbb{Z} \times BU$  induced by  $x$ . More precisely, the boundary of  $\partial W$  decomposes as  $\partial W = \bigsqcup_{i=1}^l \partial_i W$ , and we can choose identifications of  $\text{Spin}^c$ -manifolds  $w_i: E \xrightarrow{\sim} \partial_i W$  such that  $x = y \circ w_i$  for all  $i = 1, \dots, l$ . We choose a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle  $Y \rightarrow W$  such that its  $K$ -theory class satisfies  $[Y] = y$ . We further define  $X := w_1^* Y$  such that  $[X] = x$ . After stabilisation of  $Y$ , if necessary, we can assume that there are isomorphisms  $w_i^* Y \cong X$  for all  $i = 1, \dots, l$ . These choices constitute a  $\text{Spin}^c$ - $\mathbb{Z}/l\mathbb{Z}$ -manifold  $\check{W}$  in the sense of [12] and [13] together with a  $\mathbb{Z}/l\mathbb{Z}$ -bundle  $\check{Y}$  over  $\check{W}$ .

We consider  $\mathbb{R}^2 \cong \mathbb{C}$ . Let  $\xi$  be a primitive  $l$ -th root of unity. We fix  $r > 0$  so small that the discs  $B(\xi^i, r)$  are pairwise disjoint. We let  $\check{\mathbb{R}}^2 := \mathbb{R}^2 \setminus \bigcup_{i=0}^{l-1} \text{int } B(\xi^i, r)$ . In order to define the structure of a  $\mathbb{Z}/l\mathbb{Z}$ -manifold  $\check{\mathbb{R}}^2$  we fix the identifications  $v_i: S^1 \rightarrow \partial_i \check{\mathbb{R}}^2 = \partial B(\xi^i, r)$  as  $v_i(u) = \xi^i + ur\xi^i$ . Let  $\bar{\mathbb{R}}^2$  be the quotient of  $\check{\mathbb{R}}^2$  obtained by identifying the boundary components with  $S^1$  using the maps  $v_i$ .

As shown in [12] we have  $\check{K}^0(\bar{\mathbb{R}}^2) \cong \mathbb{Z}/l\mathbb{Z}$ . Let us describe this isomorphism explicitly. We choose  $R > 4$  and consider the decomposition  $\bar{\mathbb{R}}^2 \cong (\mathbb{R}^2 \setminus \text{int } B(0, R)) \cup_{S(0,R)} B(0, R) \cap \bar{\mathbb{R}}^2$ . We define line bundles  $L_m$  on  $\bar{\mathbb{R}}^2$  using a clutching function  $S(0, R) \cong S^1 \rightarrow U(1)$  of degree  $m$ . In greater detail, we define  $L_m := ((\mathbb{R}^2 \setminus \text{int } B(0, R)) \times \mathbb{C} \sqcup B(0, R) \cap \bar{\mathbb{R}}^2 \times \mathbb{C}) / \sim$ , where the gluing is given by  $(u', z) \sim (u, (u/R)^m z)$  if  $u' \in S(0, R) \cap (\mathbb{R}^2 \setminus \text{int } B(0, R))$ ,  $u \in S(0, R) \cap (B(0, R) \cap \bar{\mathbb{R}}^2)$ ,  $u = u'$  as points in  $\mathbb{R}^2 = \mathbb{C}$ .

If  $m$  is divisible by  $l$ , then the clutching function extends to  $B(0, R) \cap \bar{\mathbb{R}}^2$  and therefore defines the trivial line bundle. The element  $[m] \in \mathbb{Z}/l\mathbb{Z} \cong \check{K}^0(\bar{\mathbb{R}}^2)$  is represented by



$L_m - L_0$ . With this description it is easy to see how the Adams operation  $\Psi^k$  acts on  $K^0(\mathbb{R}^2)$ . Indeed,  $\Psi^k([L_m - L_0]) = [L_m^k - L_0^k] = [L_{km} - L_0]$ . Note that by Bott periodicity  $\tilde{K}^{2-2g}(\mathbb{R}^2) \cong \mathbb{Z}/l\mathbb{Z}$ , too. Therefore on  $\tilde{K}^{2-2g}(\mathbb{R}^2) \cong \mathbb{Z}/l\mathbb{Z}$  the action of the Adams operation  $\Psi^k$  is given by multiplication by  $k^g$ .

We now recall the definition of the topological  $\mathbb{Z}/l\mathbb{Z}$ -index given by [13].

We choose collars  $c_i: [0, 1) \times E \rightarrow W$ . Then we define the map of  $\mathbb{Z}/l\mathbb{Z}$ -manifolds  $\check{\rho}: \check{W} \rightarrow \check{\mathbb{R}}^2$  as follows. On the  $i$ -th collar we require that

$$\begin{CD} [0, 1] \times E @>{c_i}>> \check{W} \\ @V{\text{pr}_1}VV @VV{\check{\rho}}V \\ [0, 1] @>{\gamma_i}>> \check{\mathbb{R}}^2 \end{CD}$$

commutes, where  $\gamma_i: [0, 1] \rightarrow \check{\mathbb{R}}^2$  is given by  $\gamma_i(t) := (1-t)(1-r)\xi^i$ . The complement of the union of collars is mapped to the origin  $0 \in \check{\mathbb{R}}^2$ .

Let furthermore  $i: W \rightarrow V$  be an embedding of the manifold  $W$  into a real vector space  $V$ . Then we consider the embedding of  $\mathbb{Z}/l\mathbb{Z}$ -manifolds  $i \times \check{\rho}: \check{W} \rightarrow V \times \check{\mathbb{R}}^2$ . We can choose a  $\mathbb{Z}/l\mathbb{Z}$ -normal bundle  $\check{N} \rightarrow \check{W}$  and extend the embedding to an open embedding  $\check{I}: \check{N} \rightarrow V \times \check{\mathbb{R}}^2$ . We let  $\bar{N} \rightarrow \bar{W}$  be the quotient obtained by identifying the boundary components to one. We get an induced map  $\bar{I}: \bar{N} \rightarrow V \times \bar{\mathbb{R}}^2$ . The bundle  $\bar{N} \rightarrow \bar{W}$  has an induced  $\text{Spin}^c$ -structure and therefore has a Thom isomorphism  $\text{Thom}_{\bar{N}}$ . The topological index  $\text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}: K^0(\bar{W}) \rightarrow \mathbb{Z}/l\mathbb{Z}$  is given by

$$(30) \quad \text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}: K^*(\bar{W}) \xrightarrow{\text{Thom}_{\bar{N}}} K_c^{*+\dim(\bar{N})}(\bar{N}) \xrightarrow{\text{excision}} K_c^{*+\dim(\bar{N})}(V \times \bar{\mathbb{R}}^2) \xrightarrow{\text{Thom}_V^{-1}} \tilde{K}^{*-\dim(W)}(\bar{\mathbb{R}}^2).$$

We now choose a smooth  $K$ -orientation  $o_E$  of the  $K$ -oriented map  $\pi: E \rightarrow *$  and a geometry  $\mathbf{X}$  on  $X$ . We extend this orientation to an orientation  $o_W$  of the  $W \rightarrow *$ . Similarly we extend the geometry  $\mathbf{X}$  to a geometry  $\mathbf{Y}$  of  $Y$ . In this way we get geometric manifolds  $\mathcal{W}$  and  $\mathcal{E}$  such that  $\partial\mathcal{W} = l\mathcal{E}$  (see [5, Definition 2.1.30]). We let  $\mathcal{E} \otimes \mathbf{X}$  denote the geometric manifold obtained from  $\mathcal{E}$  by twisting the Dirac bundle of  $\mathcal{E}$  with  $\mathbf{X}$ . Since  $\text{ind}(\mathcal{E} \otimes \mathbf{X}) = 0$  (eg since  $l(\mathcal{E} \otimes \mathbf{X})$  is zero bordant) we can choose a taming  $(\mathcal{E} \otimes \mathbf{X})_t$ . It gives a boundary taming  $(\mathcal{W} \otimes \mathbf{Y})_{bt}$ . The class

$$\text{ind}_a^{\mathbb{Z}/l\mathbb{Z}}(D(\mathcal{W} \otimes \mathbf{Y})) := [\text{ind}(\mathcal{W} \otimes \mathbf{Y})_{bt}] \in \mathbb{Z}/l\mathbb{Z}$$

is the analytic  $\mathbb{Z}/l\mathbb{Z}$ -index. We refer to [5, Definitions 2.1.44, 2.1.47] for the definition of a taming or boundary taming and the corresponding index theory.

The index theorem of [12; 13] states that

$$(31) \quad \text{ind}_a^{\mathbb{Z}/l\mathbb{Z}}(D(\mathcal{W} \otimes \mathbf{Y})) = \text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\bar{y}) ,$$

where  $\bar{y} \in K^0(\bar{W})$  is represented by the map  $\bar{y}: \bar{W} \rightarrow \mathbb{Z} \times BU$  induced by  $y$ , and on the right-hand side we use the identification  $\tilde{K}^{*-2m}(\mathbb{R}^2) \cong \mathbb{Z}/l\mathbb{Z}$  given above. We apply the APS-index theorem for boundary tamed manifolds [5, Theorem 2.2.18] and get

$$(32) \quad \text{ind}(\mathcal{W} \otimes \mathbf{Y})_{bt} = \Omega(\mathcal{W} \otimes \mathbf{Y}) - l\eta((\mathcal{E} \otimes \mathbf{X})_t) ,$$

where the first term is the usual local contribution and the second term is the boundary correction.

We now choose a finite formal sum  $\mathbf{Q} := \sum_{\alpha} a_{\alpha} \mathbf{Q}_{\alpha}$  of geometric bundles  $\mathbf{Q}_{\alpha} \rightarrow W$  with coefficients  $a_{\alpha} \in \mathbb{Z}[1/k]$  which represents  $\hat{\rho}^k(o_E) \in \hat{K}^0(W)[1/k]$ . This is possible; see eg [19]). More precisely, if  $\mathcal{Q}_{\alpha}$  denotes the geometric family induced by  $\mathbf{Q}_{\alpha}$ , then we assume that  $\sum_{\alpha} a_{\alpha}[\mathcal{Q}_{\alpha}, 0] = \hat{\rho}^k(o_E)$ . In the following we will suppress this sum decomposition. The pullback  $\mathbf{R} := c_1^* \mathbf{Q}$  represents  $\hat{\rho}^k(o_E) \in \hat{K}^0(E)$ . Indeed we can choose the geometry of  $\mathbf{Q}$  such that we have isomorphisms  $c_i^* \mathbf{Q} \cong c_1^* \mathbf{Q}$ . In this way we get a  $\mathbb{Z}/l\mathbb{Z}$ -bundle  $\check{\mathbf{Q}}$  over  $\check{W}$ . Its underlying topological  $\mathbb{Z}/l\mathbb{Z}$ -bundle represents  $\rho^k(\bar{N})$ . From the construction of the topological index (30) and the calculation of the action of the Adams operation on  $\tilde{K}^{2-2m}(\mathbb{R}^2)$  given above (note that  $\text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\bar{y}) \in \tilde{K}^{2-2m}(\mathbb{R}^2)$ ) we have the following identities:

$$\begin{aligned} \Psi^k \text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\bar{y}) &= \text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\rho^k(\bar{N}) \cup \bar{y}), \\ \Psi^k \text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\bar{y}) &= k^m \text{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\bar{y}) . \end{aligned}$$

We let  $[\mathbf{X}] \in \hat{K}^0(E)$  denote the smooth  $K$ -theory class induced by the geometric bundle  $\mathbf{X}$ . The following calculation uses the explicit cycle level description of the pushforward in smooth  $K$ -theory [6, (17)] and the relations [6, Definition 2.10]. We get

$$\begin{aligned} \Delta_{\pi}(x) &= \hat{\Psi}^k(\hat{\pi}_!([\mathbf{X}])) - \hat{\pi}_!(\hat{\rho}^k(o_E) \cup [\mathbf{X}]) \\ &= \hat{\Psi}^k([\mathcal{E} \otimes \mathbf{X}, 0]) - [\mathcal{E} \otimes \mathbf{X} \otimes \mathbf{R}, 0] \\ &= \Psi^k([\emptyset, \eta((\mathcal{E} \otimes \mathbf{X})_t)]) - [\emptyset, \eta((\mathcal{E} \otimes \mathbf{X} \otimes \mathbf{R})_t)] \\ &= a(k^m \eta((\mathcal{E} \otimes \mathbf{X})_t) - \eta((\mathbf{E} \otimes \mathbf{X} \otimes \mathbf{R})_t)) . \end{aligned}$$

In  $\mathbb{R}/\mathbb{Z}$  we have by (32) the identity

$$\begin{aligned} [k^m \eta((\mathcal{E} \otimes \mathbf{X})_t) - \eta((\mathbf{E} \otimes \mathbf{X} \otimes \mathbf{R})_t)]_{\mathbb{R}/\mathbb{Z}} \\ = [k^m l^{-1} \Omega(\mathcal{W} \otimes \mathbf{Y}) - l^{-1} \Omega(\mathcal{W} \otimes \mathbf{Y} \otimes \mathbf{Q})]_{\mathbb{R}/\mathbb{Z}} \\ + l^{-1} \text{ind}_a^{\mathbb{Z}/l\mathbb{Z}}(D(\mathcal{W} \otimes \mathbf{Y} \otimes \mathbf{Q})) - l^{-1} k^m \text{ind}_a^{\mathbb{Z}/l\mathbb{Z}}(D(\mathcal{W} \otimes \mathbf{Y})) , \end{aligned}$$

where we interpret  $\mathbb{Z}/l\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$  via multiplication by  $l^{-1}$ . We now observe that

$$k^m \Omega(\mathcal{W} \otimes \mathbf{Y}) = \Omega(\mathcal{W} \otimes \mathbf{Y} \otimes \mathbf{Q})$$

and that in  $\mathbb{R}/\mathbb{Z}[1/k]$  we have by (31) the identity

$$\begin{aligned} & [k^m l^{-1} \operatorname{ind}_a^{\mathbb{Z}/l\mathbb{Z}}(D(\mathcal{W} \otimes \mathbf{Y})) - l^{-1} \operatorname{ind}_a^{\mathbb{Z}/l\mathbb{Z}}(D(\mathcal{W} \otimes \mathbf{Y} \otimes \mathbf{Q}))]_{\mathbb{R}/\mathbb{Z}[1/k]} \\ &= [l^{-1} k^m \operatorname{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\bar{y}) - l^{-1} \operatorname{ind}_t^{\mathbb{Z}/l\mathbb{Z}}(\rho^k(\bar{N}) \cup \bar{y})]_{\mathbb{R}/\mathbb{Z}[1/k]} \\ &= 0. \end{aligned}$$

This implies that in  $\mathbb{R}/\mathbb{Z}[1/k]$  we have

$$[k^m \eta((\mathcal{E} \otimes \mathbf{X})_t) - \eta((\mathbf{E} \otimes \mathbf{X} \otimes \mathbf{R})_t)]_{\mathbb{R}/\mathbb{Z}[1/k]} = 0$$

and hence  $\Delta_\pi(x) = 0$ . □

**Lemma 5.7** *Let  $n := \dim(E) - \dim(B)$  be even and  $x \in K^0(E)$ . Then*

$$\Delta_\pi(x) = 0 \in HP^{-n-1}(B)/\operatorname{im}(\operatorname{ch})[1/k].$$

**Proof** For  $x \in K^0(E)$  we have

$$\Delta_\pi(x) \in HP^{-n-1}(B)/\operatorname{im}(\operatorname{ch})[1/k] \subseteq \widehat{K}_{\text{flat}}^{-n}(B)[1/k] \cong K\mathbb{R}/\mathbb{Z}^{-n-1}(B)[1/k]$$

(see [7, Theorem 5.5] for the last isomorphism). We use the universal coefficient formula [21]

$$K\mathbb{R}/\mathbb{Z}^{-n-1}(B)[1/k] \cong \operatorname{Hom}(K_{-n-1}(B), \mathbb{R}/\mathbb{Z})[1/k].$$

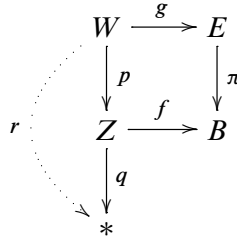
In order to show that  $\Delta_\pi(x) = 0$  it therefore suffices to show that

$$\langle u, \Delta_\pi(x) \rangle = 0 \in \mathbb{R}/\mathbb{Z}[1/k]$$

for all  $K$ -homology classes  $u \in K_{-n-1}(B)$ . We now use the geometric picture of  $K$ -homology [3; 4]. Given  $u \in K_{-n-1}(B)$  there exists a  $k$ -dimensional  $\operatorname{Spin}^c$ -manifold  $Z$  (where  $k$  is odd) together with a map  $f: Z \rightarrow B$  such that  $u = b^j f_*([Z])$ . Here  $[Z] \in K_k(Z)$  is the  $K$ -homology orientation of  $Z$  given by the  $\operatorname{Spin}^c$ -structure and  $j := (-n - 1 - k)/2$ . Let  $q: M \rightarrow *$  be the projection. Then we have for  $z \in K\mathbb{R}/\mathbb{Z}^{-n-1}(B)$

$$\langle u, z \rangle = q_!(b^{-j} f^* z) \in K\mathbb{R}/\mathbb{Z}^0 \cong \mathbb{R}/\mathbb{Z}.$$

We consider the following diagram:



We choose smooth  $K$ -orientations  $o_q$  and  $o_\pi$  on  $q$  and  $\pi$  lifting the topological ones. Furthermore we choose a smooth lift  $\hat{x}$  of  $x$ . We equip  $p$  with the induced smooth  $K$ -orientation  $o_p$ . By Lemma 3.8 there exists a class  $\hat{z} \in \widehat{K}^0(Z)[1/k]$  such that  $\widehat{\Psi}^k(\hat{z}) = \widehat{\rho}^k(o_q)^{-1}$ . Then we calculate using Proposition 5.6 at the marked equalities, omitting the powers of the Bott element  $b^j$  in order to increase readability, and using the projection formula [6, Proposition 4.5]

$$\begin{aligned}
 \langle u, \Delta_\pi(x) \rangle &= q_!(f^* \Delta_\pi(x)) \\
 &= \hat{q}_!(f^* \widehat{\Delta}_\pi(\hat{x})) \\
 &= q_!(\widehat{\Delta}_p(g^* \hat{x})) \quad (\text{by Lemma 5.5}) \\
 &= \hat{q}_!(\widehat{\Psi}^k(\hat{p}_!(\hat{x})) - \hat{p}_!(\widehat{\rho}^k(o_p) \cup \widehat{\Psi}^k(\hat{x}))) \\
 &= \hat{q}_!(\widehat{\rho}^k(o_q) \cup \widehat{\rho}^k(o_q)^{-1} \cup \widehat{\Psi}^k(\hat{p}_!(\hat{x}))) \\
 &\quad - \hat{p}_!(p^* \widehat{\rho}^k(o_q) \cup p^* \widehat{\rho}^k(o_q)^{-1} \cup \widehat{\rho}^k(o_p) \cup \widehat{\Psi}^k(\hat{x})) \\
 &= \hat{q}_!(\widehat{\rho}^k(o_q) \cup \widehat{\Psi}^k(\hat{z}) \cup \widehat{\Psi}^k(\hat{p}_!(\hat{x})) - \hat{p}_!(p^* \widehat{\rho}^k(o_q) \cup p^* \widehat{\Psi}^k(\hat{z}) \cup \widehat{\rho}^k(o_p) \cup \widehat{\Psi}^k(\hat{x}))) \\
 &= \hat{q}_!(\widehat{\rho}^k(o_q) \cup \widehat{\Psi}^k(\hat{z} \cup \hat{p}_!(\hat{x}))) - \hat{q}_! \circ \hat{p}_!(p^* \widehat{\rho}^k(o_q) \cup \widehat{\rho}^k(o_p) \cup \widehat{\Psi}^k(p^* \hat{z} \cup \hat{x})) \\
 &\stackrel{!}{=} \widehat{\Psi}^k(\hat{q}_!(\hat{z} \cup \hat{p}_!(\hat{x}))) - \hat{q}_! \circ \hat{p}_!(\widehat{\rho}^k(o_r) \cup \widehat{\Psi}^k(p^* \hat{z} \cup \hat{x})) \\
 &= \widehat{\Psi}^k(\hat{r}_!(p^* \hat{z} \cup \hat{x})) - \hat{r}_!(\widehat{\rho}^k(o_r) \cup \widehat{\Psi}^k(p^* \hat{z} \cup \hat{x})) \\
 &\stackrel{!}{=} 0. \quad \square
 \end{aligned}$$

**Lemma 5.8** *Let  $n := \dim(E) - \dim(B)$  be odd and  $x \in K^{-1}(E)$ . Then*

$$\Delta_\pi(x) = 0 \in HP^{-n-1}(B)/\text{im}(\text{ch})[1/k].$$

**Proof** We consider the bundle  $q = \pi \circ \text{pr}_E: S^1 \times E \rightarrow B$  with even-dimensional fibres and the class  $y = e \times x \in K^0(S^1 \times E)$ , where  $e \in K^1(S^1) \cong \mathbb{Z}$  is the generator.

We choose smooth lifts  $\hat{e}$  and  $\hat{x}$  of  $e$  and  $x$ . Furthermore we choose a smooth  $K$ -orientation  $o_p$  lifting the underlying topological  $K$ -orientation, and we let  $o_{\text{pr}_E}$  be such that  $f = \text{pr}_{E,1}$ . Then we have  $\text{pr}_{E,1}(\hat{e} \times \hat{x}) = \hat{x}$ ,  $\hat{\Psi}^k(\hat{e}) = \hat{e}$ , and  $\text{pr}_E^* \hat{\rho}^k(o_\pi) = \hat{\rho}^k(o_q)$ . Applying Lemma 5.7 to  $\Delta_q(e \times x)$  we get

$$\begin{aligned} \Delta_\pi(x) &= \Psi^k(\hat{\pi}_!(\hat{x})) - \hat{\pi}_!(\hat{\rho}^k(o_\pi) \cup \hat{\Psi}^k(\hat{x})) \\ &= \hat{\Psi}^k(\hat{q}_!(\hat{e} \times \hat{x})) - \hat{q}_!(\text{pr}_E^* \hat{\rho}^k(o_\pi) \cup \hat{e} \cup \text{pr}_E^* \hat{\Psi}^k(\hat{x})) \\ &= \Delta_q(e \times x) \\ &= 0. \end{aligned}$$

□

**Lemma 5.9** *Let  $n := \dim(E) - \dim(B)$  be even and  $x \in K^{-1}(E)$ . Then*

$$\Delta_\pi(x) = 0 \in HP^{-n-1}(B)/\text{im}(\text{ch})[1/k].$$

**Proof** We consider the diagram

$$\begin{array}{ccc} S^1 \times E & \xrightarrow{\text{pr}_E} & E \\ q := \text{id} \times \pi \downarrow & & \downarrow \pi \\ S^1 \times B & \xrightarrow{\text{pr}_B} & B. \end{array}$$

We claim that  $e \times \text{pr}_B^* \Delta_\pi(x) = e \times \Delta_q(\text{pr}_E^* x)$ .

Indeed, after choosing smooth orientations and smooth lifts we calculate using  $\hat{\rho}^k(o_q) = \text{pr}_E^* \hat{\rho}^k(o_p)$  (Lemma 4.9),  $\hat{\Psi}^k(\hat{e}) = \hat{e}$ , the equality  $\hat{e} \times \hat{\Psi}(\hat{y}) = \hat{\Psi}^k(\hat{e} \times \hat{y})$ , and  $\hat{q}_!(\hat{e} \times \hat{y}) = \hat{e} \times \hat{\pi}_!(\hat{y})$  (a special case of the projection formula [6, Proposition 4.5])

$$\begin{aligned} e \times \Delta_q(\text{pr}_E^* x) &= \hat{e} \times (\hat{\Psi}^k(\hat{\pi}_!(\hat{x})) - \hat{\pi}_!(\hat{\rho}^k(o_\pi) \cup \hat{\Psi}^k(\hat{x}))) \\ &= \hat{\Psi}^k(\hat{e} \times \hat{\pi}_!(\hat{x})) - \hat{q}_!(\hat{e} \times (\hat{\rho}^k(o_\pi) \cup \hat{\Psi}^k(\hat{x}))) \\ &= \hat{\Psi}^k(\hat{q}_!(\hat{e} \times \hat{x})) - \hat{q}_!(\hat{\rho}^k(o_q) \cup \hat{\Psi}^k(\hat{e} \times \hat{x})) \\ &= \Delta_q(e \times x) \\ &= 0, \end{aligned}$$

where the last equality holds by Lemma 5.7.

□

**Lemma 5.10** *Let  $n := \dim(E) - \dim(B)$  be odd and  $x \in K^0(E)$ . Then*

$$\Delta_\pi(x) = 0 \in HP^{-n-1}(B)/\text{im}(\text{ch})[1/k].$$

**Proof** Let  $q := \pi \circ \text{pr}_E: S^1 \times E \rightarrow B$ . This bundle has even-dimensional fibres. We calculate (again after choosing smooth lifts  $\hat{x}$  and  $\hat{e}$  of  $x$  and  $e$  and smooth  $K$ -orientations  $o_\pi, o_{\text{pr}_E}$  refining the underlying topological ones)

$$\begin{aligned} \Delta_\pi(x) &= \widehat{\Psi}^k(\widehat{\pi}_!(\hat{x})) - \widehat{\pi}_!(\widehat{\rho}^k(o_\pi) \cup \widehat{\Psi}^k(\hat{x})) \\ &= \widehat{\Psi}^k(\widehat{q}_!(\hat{e} \times \hat{x})) - \widehat{q}_!(\hat{e} \times \widehat{\rho}^k(o_\pi) \cup \widehat{\Psi}^k(\hat{x})) \\ &= \widehat{\Psi}^k(\widehat{q}_!(\hat{e} \times \hat{x})) - \widehat{q}_!(\widehat{\rho}^k(o_q) \cup \widehat{\Psi}^k(\hat{e} \times \hat{x})) \\ &= \Delta_q(e \times x) = 0. \end{aligned} \quad \square$$

The collection of the Lemmas 5.7, 5.8, 5.9 and 5.10 gives Theorem 5.1. □

## 6 Applications

Let  $\pi: E \rightarrow B$  be a proper submersion over a compact base with fibre-dimension  $n$  together with a stable framing of  $T^v\pi$ . In this situation we have the canonical smooth  $K$ -orientation  $o_\pi$  and the class  $e(\pi) := \widehat{\pi}_!(1) \in \widehat{K}^{-n}(B)$ ; see [6, 5.11]. This class is actually flat and therefore belongs to  $\widehat{K}_{\text{flat}}^{-n}(B) \cong K\mathbb{R}/\mathbb{Z}^{-n-1}(B)$ . It is an invariant of the bordism class of bundles with stably framed vertical bundles.

**Theorem 6.1** *The  $e$ -invariant satisfies*

$$(\widehat{\Psi}^k - 1)e(\pi) = 0 \in K\mathbb{R}/\mathbb{Z}^{-n-1}(B)[1/k].$$

**Proof** Since  $\widehat{\rho}^k(o_\pi) = 1$  by Proposition 4.10 using Theorem 5.1 we get

$$\widehat{\Psi}^k e(\pi) = \widehat{\Psi}^k(\widehat{\pi}_!(1)) = \widehat{\pi}_!(\widehat{\rho}^k(o_\pi)) = \widehat{\pi}_!(1) = e(\pi). \quad \square$$

In the special case that  $B = *$  and  $n = 2m - 1$  is odd,  $e(\pi) \in K\mathbb{R}/\mathbb{Z}^{-n-1} \cong \mathbb{R}/\mathbb{Z}$  is the  $e$ -invariant of Adams of the framed bordism class  $[E] \in \Omega_n^{\text{fr}}$  represented by  $E$ , or equivalently of the element in the stable stem  $\pi_n^S \cong \Omega_n^{\text{fr}}$  corresponding to  $[E]$  via the Pontrjagin–Thom construction. Originally, the  $e$ -invariant has been introduced in order to detect elements in the image of the  $j$ -homomorphism  $j: \pi_{*+1}(BO) \rightarrow \pi_*^S$ . The order of the image of  $j$ -homomorphism is known (see [1] and the other papers in that series) and conforms with the following special case of Theorem 6.1.

**Corollary 6.2** *If  $\pi: E \rightarrow *$  is the projection from a compact stably framed manifold to the point and  $\dim(E) = 2m - 1$ , then for every  $k \in \{-1\} \cup \mathbb{N}$  there exists  $L \in \mathbb{N}$  such that*

$$k^L(k^m - 1)e(\pi) = 0.$$

**Proof** Indeed  $0 = (\widehat{\Psi}^k - 1)e(\pi) = (k^m - 1)e(\pi) \in \mathbb{R}/\mathbb{Z}[1/k]$ . □

The determination of the order of the image of the  $j$ -homomorphism in the work of Adams also uses Adams operations, namely in order to characterise the kernel of  $j$ . The proof of the upper bound of the order of the  $e$ -invariant Corollary 6.2 employs the Adams operations in different manner.

Let us now discuss some application to higher  $\rho$ -invariants. Let  $\pi: E \rightarrow B$  be a proper submersion over a compact base  $B$  which is  $K$ -oriented by a  $\text{Spin}^c$ -structure on the vertical bundle  $T^v\pi$ . We fix a base point of  $E$ , choose a character  $\chi: \pi_1(E, *) \rightarrow \mathbb{Z}/k\mathbb{Z}$ , and we let  $\mathbf{H} := (H, h^H, \nabla^H)$  be the corresponding geometric line bundle. It represents a class  $[\mathbf{H}] \in \widehat{K}^0(E)$ . Since  $\mathbf{H}^k$  is trivial we have the relation  $[\mathbf{H}]^k = 1$  in  $\widehat{K}^0(E)$ . We choose a smooth  $K$ -orientation  $o_\pi$  which refines the topological one. The higher  $\rho$ -invariant is then defined by

$$\rho(\chi) := \widehat{\pi}_!([\mathbf{H}]) - \widehat{\pi}_!(1) .$$

Note that  $R(\rho(\chi)) = 0$  so that  $\rho(\chi) \in K_{\text{flat}}^{-n}(B) \cong K\mathbb{R}/\mathbb{Z}^{-n-1}(B)$ , and it is independent of the choice of  $o_\pi$ . Therefore  $\rho(\chi)$  is a differential-topological invariant of the  $K$ -oriented bundle  $\pi$ .

**Proposition 6.3** *We have the relation  $\widehat{\Psi}^k(\rho(\chi)) = 0 \in K\mathbb{R}/\mathbb{Z}^{-n-1}(B)[1/k]$ .*

**Proof** We calculate

$$\begin{aligned} \widehat{\Psi}^k(\widehat{\pi}_!(1)) - \widehat{\Psi}^k(\widehat{\pi}_!([\mathbf{H}])) &= \widehat{\pi}_!(\widehat{\rho}^k(o_\pi)) - \widehat{\pi}_!(\widehat{\rho}^k(o_\pi) \cup \widehat{\Psi}^k([\mathbf{H}])) \\ &= 0 \in K\mathbb{R}/\mathbb{Z}^{-n-1}(B)[1/k] \end{aligned}$$

since  $\widehat{\Psi}^k([\mathbf{H}]) = [\mathbf{H}]^k = 1$  by Proposition 3.6. □

If  $M$  is a point and  $\dim(E) = 2m-1$ , then  $\rho(\chi) \in K\mathbb{R}/\mathbb{Z}^{-2m}(\ast) \cong \mathbb{R}/\mathbb{Z}$ . Furthermore  $\widehat{\Psi}^k(\rho(\chi)) = k^m \rho(\chi)$ .

**Corollary 6.4** *There exists  $L \in \mathbb{N}$  such that*

$$k^L \rho(\chi) = 0 \in \mathbb{R}/\mathbb{Z} .$$

This can also be verified independently from the present formalism as we will indicate now. We consider the  $\text{Spin}^c$ -bordism group  $\Omega_*^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z})$  of the classifying space  $B\mathbb{Z}/k\mathbb{Z}$  of the group  $\mathbb{Z}/k\mathbb{Z}$ . The character  $\chi$  determines a map  $b\chi: E \rightarrow B\mathbb{Z}/k\mathbb{Z}$ . The pair  $(E, b\chi)$  of a closed  $\text{Spin}^c$ -manifold and a map to  $B\mathbb{Z}/k\mathbb{Z}$  represents a class  $[E, b\chi] \in \Omega_{2m-1}^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z})$ . The element  $\rho([E, b\chi]) := \rho(\chi) \in \mathbb{R}/\mathbb{Z}$  only depends

on the class  $[E, b\chi]$  and vanishes if  $b\chi$  is homotopic to the constant map. In this way we define a homomorphism

$$\rho: \tilde{\Omega}_{2m-1}^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z},$$

where  $\tilde{\Omega}_*^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z}) \subset \Omega_*^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z})$  denotes the reduced  $\text{Spin}^c$ -bordism group of  $B\mathbb{Z}/k\mathbb{Z}$  (see [2] for a related construction). Note that  $H_*(B\mathbb{Z}/k\mathbb{Z}; \mathbb{Z}[1/k]) \cong H_*(*; \mathbb{Z}[1/k])$ . This implies that

$$\tilde{\Omega}_*^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z})[1/k] = 0$$

via the Atiyah–Hirzebruch spectral sequence for  $\Omega_*^{\text{Spin}^c}(B\mathbb{Z}/k\mathbb{Z})$ , and therefore gives Corollary 6.4.

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