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Describing the universal cover of a noncompact limit

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Suppose that X is the Gromov–Hausdorff limit of a sequence of Riemannian manifolds M_i^n with a uniform lower bound on Ricci curvature. In a previous paper the authors showed that when X is compact the universal cover $\widetilde{X}$ is a quotient of the Gromov–Hausdorff limit of the universal covers $\widetilde{M}_i^n$. This is not true when X is noncompact. In this paper we introduce the notion of pseudo-nullhomotopic loops and give a description of the universal cover of a noncompact limit space in terms of the covering spaces of balls of increasing size in the sequence.

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1 Introduction

Gromov–Hausdorff convergence is very useful in many subjects and understanding the structure of the limit spaces often helps one understand the structure of the spaces in the sequence. For manifolds with a lower bound on Ricci curvature, one also has the benefit of Gromov's precompactness theorem, which says that any sequence of complete Riemannian n–manifolds with a uniform lower bound on Ricci curvature has convergent subsequence. In addition one can show that the limit space of any convergent sequence of such manifolds is a complete length space (see eg Burago, Burago and Ivanov [1]).

Cheeger and Colding have made significant progress in understanding the regularity and geometric structure of the limit spaces of manifolds with uniform lower Ricci curvature bound [3; 4; 5]; see also Cheeger [2] and Wei [11]. In particular Cheeger and Colding have demonstrated that such limits spaces are in many ways well behaved but unfortunately some of this behavior is not local and hence cannot be automatically lifted to covering spaces. For example such limit spaces have a renormalized limit measure that satisfies the relative volume comparison [3]. On the other hand Menguy constructed examples showing that the limit space could have infinite topology in an arbitrarily small neighborhood [7]. Sormani and Wei [8; 9] showed that the limit space has a universal cover (not assuming it is simply connected; see Definition 2.1), and when the limit space is compact it was shown that the universal cover is the limit of

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some covers of the sequence. Therefore the universal cover of the limit space carries the same regularity and geometric structure as the limit space. More recently the authors of this paper showed that in the case of a compact limit space the universal cover can also be described as a quotient of the limit of the universal covers of the manifolds in the sequence [6]. Unfortunately, if the limit space is not compact none of these characterizations of the universal cover of the limit space are true. In particular we have the following well-known example showing that the universal cover of a Gromov–Hausdorff limit space may not be the limit of any sequence of covering spaces of the manifolds.

Example 1.1 Let M be the half-cylinder capped off by a hemisphere and smoothed to have nonnegative sectional curvature, and let p_i be a sequence of points going to infinity. Then (M, p_i) converges to the cylinder X. But the universal cover $\mathbb{R}^2$ of X is the not the Gromov–Hausdorff limit of any sequence of covers of M since the only cover of M is itself.

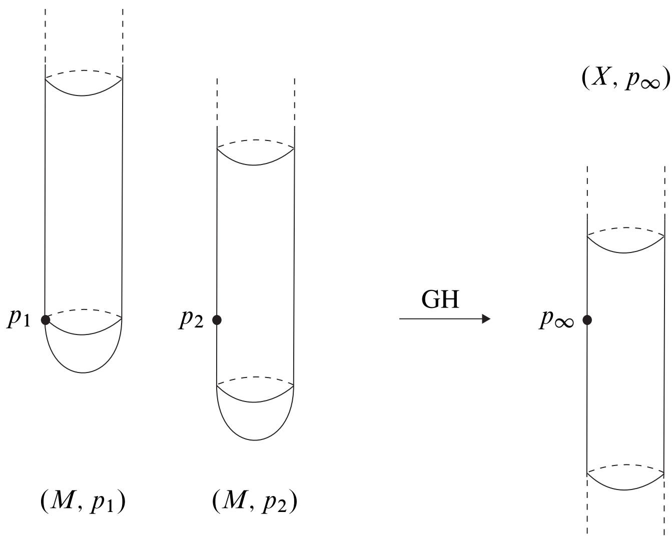

Figure 1: Example 1.1

In this paper we study the relationship between the universal cover of the limit and covers of the sequence when the limit is noncompact. Motivated by Example 1.1 together with the fact that pointed Gromov–Hausdorff convergence is defined with respect to balls, we first relate the universal cover of a noncompact space to the universal cover of larger and larger balls. Namely we have:

Proposition 1.2 *If X is a complete locally compact and semilocally simply connected length space, and there is a point $x \in X$, $R_i \to \infty$ such that the universal covers $\widetilde{B}(x, R_i)$ of the closed balls $B(x, R_i)$ exist, then the universal cover of X is given by*

$$\widetilde{X} = \mathrm{GH} \lim_{i\to\infty} (\widetilde{B}(x, R_i), \tilde{x}).$$

Remark 1.3 The assumptions are clearly satisfied when X is a Riemannian manifold. On the other hand Proposition 1.2 is not true for general metric spaces; see for instance Example 3.7.

When $(X, x) = \mathrm{GH} \lim_{i\to\infty}(X_i, x_i)$, then for any $R > 0$,

$$(B(x, R), x) = \mathrm{GH} \lim_{i\to\infty} (B(x_i, R), x_i).$$

By [8; 6] the universal cover of a closed compact length space is the limit of some covering of the sequence. From these one naturally expects a relation between the universal cover of the limit space and some covering of balls of increasing size in the sequence.

Nevertheless, there are still some difficulties. On the one hand there is the subtlety of dealing with the balls which have boundaries instead of closed ones treated in [8; 6]. On the other hand even though one knows [8; 9] that the Gromov–Hausdorff limit of manifolds with Ricci curvature bounded from below has a universal cover, one does not know if the limit is semilocally simply connected or whether the universal cover of its balls exist. Overcoming these, in particular we prove the following result.

Theorem 1.4 *Suppose*

$$(X, x) = \mathrm{GH} \lim_{i\to\infty} (M_i, p_i),$$

where M_i^n have $\mathrm{Ric}_{M_i} \geq (n-1)H$. If the universal cover $\widetilde{X}$ is simply connected then after passing to a subsequence there are $R_i \to \infty$ and covering spaces $\widehat{B}(p_i, R_i)$ of the closed balls $B(p_i, R_i)$ such that

$$(\widetilde{X}, \tilde{x}) = \mathrm{GH} \lim_{i\to\infty} (\widehat{B}(p_i, R_i), \hat{p}_i).$$

For the general version see Theorem 3.8, where the simply connected condition is replaced by a weaker assumption and a precise description of the covers of the balls $B(p_i, R_i)$ is given.

As a consequence of Theorem 1.4 the universal cover of a noncompact limit space carries the same regularity and geometric structure as the limit space itself. Now in [9, Corollary 4.3], Sormani and Wei showed that each ball in $\widetilde{X}$ is the Gromov–Hausdorff limit of some covering spaces of balls in the sequence, but these covering spaces depend on the structure of the sequence spaces at infinity while the covering spaces in our result only depend on the structure of the sequence spaces within finite balls.

The paper is organized as follows. In Section 2 we review several different types of covering spaces and their properties. In Section 3 we study conditions under which covering maps lift and we introduce the concept of pseudo-nullhomotopy. Throughout the paper we have to consider both the intrinsic and restricted metrics on balls in complete locally compact length spaces and in Proposition 3.3 we show that the topologies on the balls are the same with respect to these two metrics. Finally we state and prove the main theorem.

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2 Coverings of length spaces

2.1 δ–Covers

An effective way to study the coverings of length spaces is using δ–covers, which were introduced by Sormani and Wei to study the existence of universal covers [8]. Here we do not assume the universal cover is simply connected. Namely:

Definition 2.1 (Universal cover [10, pages 62,83]) We say $\tilde{X}$ is a universal cover of X if $\tilde{X}$ is a cover of X such that for any other cover $\bar{X}$ of X, there is a commutative triangle formed by a continuous map $f\colon \tilde{X} \to \bar{X}$ and the two covering projections.

Recall that a length space X is a metric space whose metric is defined as the infimum of the lengths of paths in the space (see [1] for more detail). For any subset $A \subseteq X$, there are two naturally associated metrics: the restricted metric, d_X, which is simply the restriction of the metric of X to the subset and the intrinsic metric, d_A, which is computed using the lengths of paths contained entirely inside the subspace. In what follows we use the intrinsic metric unless otherwise specified.

Definition 2.2 (δ–Cover) Suppose X is a complete length space. For $x \in X$ and $\delta > 0$, let $\pi_1(X,x,\delta)$ be the subgroup of $\pi_1(X,x)$ generated by elements of the form $= [\alpha * \beta * \alpha^{-1}]$, where α is a path from x to some $y \in X$ and β is a loop contained in some open δ–ball in X (see Figure 2).

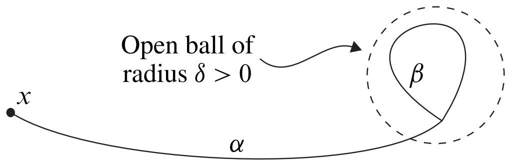

Figure 2: A typical generator for $\pi_1(X, x, \delta)$

The δ–cover of X is the covering space

$$\pi^\delta\colon \widetilde{X}^\delta \to X$$

with

$$(\pi^\delta)_*(\pi_1(\widetilde{X}^\delta, \tilde{x})) = \pi_1(X, x, \delta).$$

Intuitively, a δ–cover is the result of unwrapping all but the smallest loops in X.

Remarks 2.3 (1) $\widetilde{X}^{\delta'}$ covers $\widetilde{X}^\delta$ for $\delta' \le \delta$.

(2) δ–covers exist for connected, locally path connected metric spaces. See [10] for more details.

Returning to the case of Riemannian manifolds M_i converging to a noncompact limit space X, we note that although the δ–covers of balls of increasing size in the manifolds will converge to a cover of the limit space, it may not be possible to realize the universal cover of the limit space in this way.

Example 2.4 Let Y_i be the result of rotating the graph of $y = e^{-x}$, where $x \le i$, about the x–axis. Then let X_i be Y_i with its rightmost end capped off with a hemisphere. Then X_i converges to X, which is the result of rotating the graph of $y = e^{-x}$ about the x–axis.

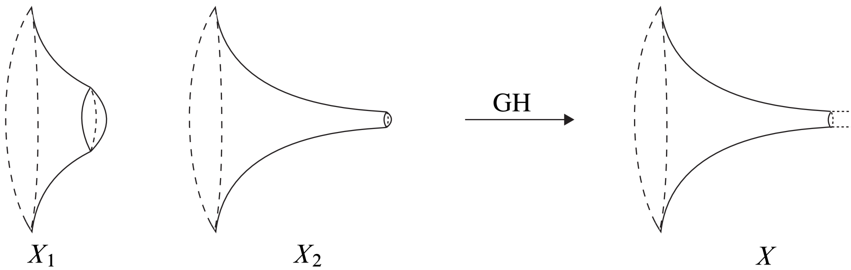

Figure 3: Example 2.4

For any fixed $\delta > 0$ in this example, the δ–covers of balls of increasing size are eventually the balls themselves, which do not converge to the universal cover of the limit space.

2.2 Relative δ–covers

Another issue that arises when balls are used to study the topology of a larger space is that the fundamental groups of balls could be much bigger than the total space.

Example 2.5 Let M be the torus obtained from the unit square. Then any ball with radius greater than 1 and less than $\sqrt{2}$ is homotopic to the wedge of two circles. Thus the fundamental group of such a ball has exponential growth.

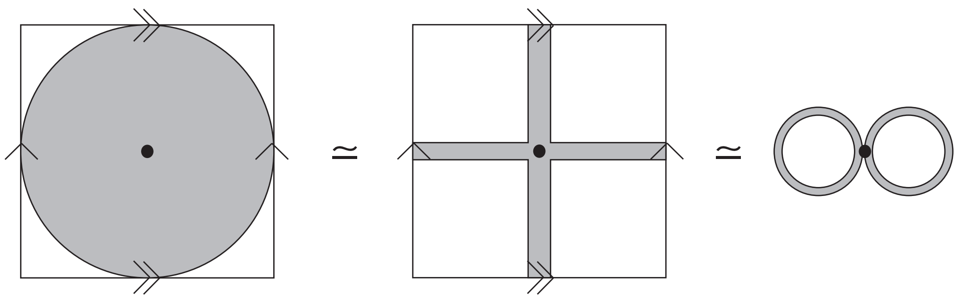

Figure 4: Example 2.5

To stay away from the boundaries of balls, Sormani and Wei introduced relative δ–covers [9].

Definition 2.6 (Relative δ–cover) Suppose X is a length space, $x \in X$ and $0 < r < R$. Let

$$\pi^\delta\colon \widetilde{B}_R(x)^\delta \to B_R(x)$$

be the δ–cover of the open ball $B_R(x)$. A connected component of $(\pi^\delta)^{-1}(B(x,r))$, where $B(x,r)$ is a closed ball, is called a relative δ–cover of $B(x,r)$ and is denoted $\widetilde{B}(x,r,R)^\delta$.

Relative δ–covers are covering spaces [9]. Although Sormani and Wei's original definition of relative δ–covers in [9] only used closed balls we find that this modification by Wylie [12], in which the larger ball is open, is easier to work with.

Remark 2.7 We will use the following notation:

(1) Open balls are denoted $B_R(x)$ while closed balls are denoted $B(x,R)$.

(2) The covering group $\pi_1(x,\delta)$ of $\widetilde{X}^\delta$ is the set of all equivalence classes of loops γ in X based at x that are homotopic in X to products of loops of the form $\alpha * \beta * \alpha^{-1}$ where each β lies in some open δ–ball in X.

(3) The covering group $\pi_1(x, R, \delta)$ of $\widetilde{B}_R(x)^\delta$ is the set of all equivalence classes of loops γ in $B_R(x)$ based at x that are homotopic in $B_R(x)$ to products of loops of the form $\alpha * \beta * \alpha^{-1}$ where each β lies in some open δ–ball in $B_R(x)$.

(4) The covering group $\pi_1(x, r, R, \delta)$ of $\widetilde{B}(x, r, R)^\delta$ is the set of all equivalence classes of loops γ in $B(x, r)$ based at x that are homotopic in $B_R(x)$ to products of loops of the form $\alpha * \beta * \alpha^{-1}$ where each β lies in some open δ–ball in $B_R(x)$.

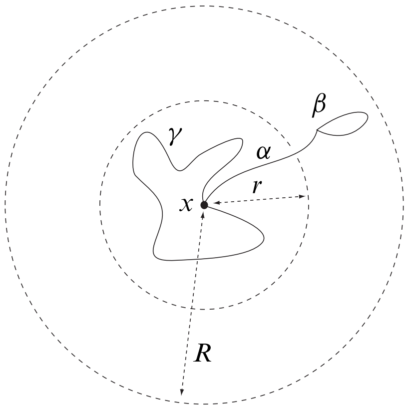

Figure 5: A typical generator for $\pi_1(x, r, R, \delta)$

To relate the δ–cover $\widetilde{B}(x, r)^\delta$ to the relative δ–cover $\widetilde{B}(x, r, R)^\delta$, we note that each δ–ball in $B(x, r)$ defined according to the metric $d_{B(x,r)}$ is contained in the corresponding δ–ball in $B(x, r)$ defined according to the metric $d_{B_R(x)}$. Working through the definitions, we see that the δ–cover $\widetilde{B}(x, r)^\delta$ covers the relative δ–cover $\widetilde{B}(x, r, R)^\delta$:

$$\begin{array}{c} \widetilde{B}(x, r)^\delta \\ \downarrow \\ \widetilde{B}(x, r, R)^\delta \\ \downarrow \\ B(x, r) \end{array}$$

Definition 2.8 (Stable relative δ–cover) Let X be a length space, $x \in X$ and let $0 < r < R$ be fixed. Suppose there is $\delta_0 > 0$ such that

$$\widetilde{B}(x, r, R)^\delta = \widetilde{B}(x, r, R)^{\delta_0}$$

for all $0 < \delta < \delta_0$. Then the relative δ–covers $\widetilde{B}(x, r, R)^\delta$ are said to stabilize, and $\widetilde{B}(x, r, R)^{\delta_0}$ is called stable.

Sormani and Wei have proven a number of results regarding relative δ–covers in [9], including the following lemma.

Lemma 2.9 (Sormani–Wei) *Suppose X is a length space and there is a point $x \in X$ such that for all $r > 0$ there exists $R \geq r$ such that $\widetilde{B}(x, r, R)^{\delta(r,R)}$ is stable for some $\delta(r, R) > 0$. Then the universal cover $\widetilde{X}$ exists.*

Lemma 2.9 is a general result for length spaces. Returning to sequences of manifolds with a uniform Ricci curvature lower bound, Sormani and Wei have proved the following theorem.

Theorem 2.10 (Sormani–Wei) *Suppose M_i^n have $\mathrm{Ric}_{M_i} \geq (n-1)H$ and that*

$$(M_i, p_i) \xrightarrow{\mathrm{GH}} (X, x).$$

Then for all $0 < r < R$ there is $\delta(r, R)$ such that $\widetilde{B}(x, r, R)^{\delta(r,R)}$ is stable. Thus the universal cover $\widetilde{X}$ of X exists. In addition, each stable relative δ–cover of X is a limit of relative δ–covers of the sequence:

$$(\widetilde{B}(x, r, R)^{\delta(r,R)}, \tilde{x}) = \mathrm{GH}\lim_{i\to\infty} (\widetilde{B}(p_i, r, R)^{\delta(r,R)}, \tilde{p}_i).$$

Thus the relative δ–covers in the limit space stabilize, and these stable relative δ–covers are themselves the limit of the relative δ–covers from the sequence.

We conclude this subsection on relative δ–covers with a lemma that will be important in the proof of the main theorem.

Lemma 2.11 *Let X be a complete length space and suppose $\pi\colon \bar{X} \to X$ is a covering with $\pi(\bar{x}) = x$. If the relative δ–covers $\widetilde{B}(x, r, R)^{\delta}$ stabilize then the relative δ–covers $\widetilde{B}(\bar{x}, r, R)^{\delta}$ also stabilize.*

Proof First note that π restricts to a map from $B(\bar{x}, R)$ to $B(x, R)$. Moreover, the compactness of $B(\bar{x}, R)$ implies that there is $\delta_2 > 0$ such that $\pi\colon B(\bar{x}, R) \to B(x, R)$ is an isometry on balls of radius δ_2. Next we may choose $0 < \delta_1 < \delta_2$ such that $\widetilde{B}(x, r, R)^{\delta_1}$ is stable. We will show that $\widetilde{B}(\bar{x}, r, R)^{\delta_1}$ is also stable by showing that $\pi_1(\bar{x}, r, R, \delta) = \pi_1(\bar{x}, r, R, \delta_1)$ for all $0 < \delta < \delta_1$.

Let $0 < \delta < \delta_1$ be given. Clearly $\pi_1(\bar{x}, r, R, \delta) \leq \pi_1(\bar{x}, r, R, \delta_1)$. Conversely suppose $\bar{\gamma}$ is a loop in $B(\bar{x}, r)$ that is homotopic in $B_R(\bar{x})$ to a product of paths of the form $\bar{\alpha} * \bar{\beta} * \bar{\alpha}^{-1}$ where each $\bar{\beta}$ is a loop that lies in some δ_1–ball in $B_R(\bar{x})$. Since π is continuous and distance nonincreasing it follows that $\pi(\bar{\gamma})$ is a loop that lies in

$B(x,r)$ and is homotopic in $B_R(x)$ to a product of paths $\alpha * \beta * \alpha^{-1}$ where each β is a loop that lies in some δ_1–ball in $B_R(x)$.

Now $\widetilde{B}(x,r,R)^{\delta_1}$ is stable so there is a homotopy H in $B_R(x)$ that carries $\pi(\overline{\gamma})$ to a product of paths $\alpha * \beta * \alpha^{-1}$ where each β is a loop that lies in some δ–ball in $B_R(x)$. Since $\pi\colon B(\overline{x},R) \to B(x,R)$ is an isometry on all balls of radius $\delta \le \delta_2$ we can lift H to see that $\overline{\gamma}$ is homotopic in $B_R(\overline{x})$ to a product of paths $\overline{\alpha} * \overline{\beta} * \overline{\alpha}^{-1}$ where each $\overline{\beta}$ is a loop that lies in some δ–ball in $B_R(\overline{x})$. □

3 Description of universal cover

In this section we will state and prove the main theorem of this paper. Our goal is to relate the universal cover of a Gromov–Hausdorff limit space with covering spaces of balls in the sequence. To this end we will first show that balls in $\widetilde{X}$ are eventually isometric to balls in the relative δ–covers $\widetilde{B}(x,r,R)^\delta$. Note that we use $\pi_1(\widetilde{x},r)$ in place of $\pi_1(B(\widetilde{x},r),\widetilde{x})$ and that when we say that a loop lifts we mean that it lifts to a loop.

3.1 Lifting covering maps

We begin by specifying conditions on generators of

$$G(\widetilde{x},r,R,\delta) = \frac{\pi_1(\widetilde{x},r)}{\pi_1(\widetilde{x},r,R,\delta)}$$

that allow us to lift the covering map $\pi\colon \widetilde{X} \to X$ from the ball $B(x,r)$ to $\widetilde{B}_R(x)^\delta$.

Lemma 3.1 *Suppose X is a complete length space with universal cover $\widetilde{X}$ and $\{[\![\gamma_j]\!]\}_{j\in S}$ is a collection of generators for $G(\widetilde{x},r,R,\delta)$. If each loop $\pi(\gamma_j)$ lifts to $\widetilde{B}_R(x)^\delta$ then*

$$\pi\colon B(\widetilde{x},r) \to B_R(x)$$

lifts to

$$\widetilde{\pi}\colon B(\widetilde{x},r) \to \widetilde{B}_R(x)^\delta.$$

Proof We have

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi"'] & B(\widetilde{x},r) \arrow[l, hook'] \arrow[d, "\pi"'] & \widetilde{B}_R(x)^\delta \arrow[d, "\pi_R"] \\
X & B(x,r) \arrow[l, hook'] \arrow[r, hook] & B_R(x)
\end{tikzcd}$$

and we will show that

$$\pi_*(\pi_1(\tilde{x}, r)) \leq (\pi_R)_*(\pi_1(\widetilde{B}_R(x)^\delta)).$$

In other words we will show that every loop in $B(\tilde{x}, r)$, when pushed into $B(x, r)$, lifts to $\widetilde{B}_R(x)^\delta$. We start by showing that this is true for loops representing elements in $\pi_1(\tilde{x}, r, R, \delta)$.

Suppose $[\sigma] \in \pi_1(\tilde{x}, r, R, \delta)$. Then σ is homotopic in $B_R(\tilde{x})$ to a product of loops of the form $\tilde{\alpha} * \tilde{\beta} * \tilde{\alpha}^{-1}$, where each $\tilde{\beta}$ lies in some open δ–ball in $B_R(\tilde{x})$.

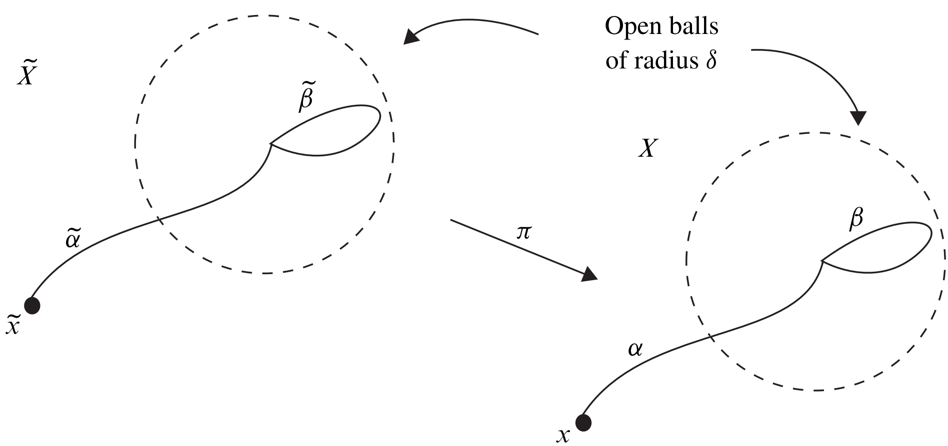

Figure 6: Lemma 3.1

Applying π to this homotopy, we see that $\pi(\sigma)$ is homotopic in $B_R(x)$ to a product of loops of the form $\alpha * \beta * \alpha^{-1}$, where each β is a loop in some δ–ball in * $B_R(x)$. Thus $\pi_*[\sigma] = [\pi(\sigma)] \in \pi_1(x, r, R, \delta)$.

Next suppose $[\![\gamma_j]\!]$ is a generator for $\pi_1(\tilde{x}, r)/\pi_1(\tilde{x}, r, R, \delta)$. By assumption $\pi(\gamma_j)$ is a loop in $B(x, r)$ that lifts to $\widetilde{B}_R(x)^\delta$. Thus $\pi_*[\gamma_j] = [\pi(\gamma_j)] \in \pi_1(x, r, R, \delta)$.

Next suppose γ is a loop in $B(\tilde{x}, r)$. Then $[\![\gamma]\!] \in \pi_1(\tilde{x}, r)/\pi_1(\tilde{x}, r, R, \delta)$ can be expressed as a product

$$\begin{aligned} [\![\gamma]\!] &= [\![\gamma_{j_1}]\!] * \cdots * [\![\gamma_{j_{N(\gamma)}}]\!] \\ &= [\![\gamma_{j_1} * \cdots * \gamma_{j_{N(\gamma)}}]\!]. \end{aligned}$$

Hence $[\gamma] = [\gamma_{j_1} * \cdots * \gamma_{j_{N(\gamma)}} * \sigma] \in \pi_1(\tilde{x}, r)$ for some $[\sigma] \in \pi_1(\tilde{x}, r, R, \delta)$. But then

$$\begin{aligned} \pi_*[\gamma] &= \pi_*[\gamma_{j_1} * \cdots * \gamma_{j_{N(\gamma)}} * \sigma] \\ &= \pi_*[\gamma_{j_1}] \cdots \pi_*[\gamma_{j_{N(\gamma)}}] \cdot \pi_*[\sigma] \in \pi_1(x, r, R, \delta). \end{aligned}$$

Thus
$$\pi_*(\pi_1(\tilde{x}, r)) \le \pi_1(x, R, \delta),$$
so π lifts
$$\begin{tikzcd} B(\tilde{x}, r) \arrow[r, dashed, "\tilde{\pi}"] \arrow[d, "\pi"'] & \widetilde{B}_R(x)^\delta \arrow[d, "\pi_R"] \\ B(x, r) \arrow[r, hook] & B_R(x) \end{tikzcd}$$
which completes the proof. □

To apply the previous lemma, it will be helpful to know that $G(\tilde{x}, r, R, \delta)$ is finitely generated. First we observe the following:

Lemma 3.2 *If X is a compact length space then for any $x \in X$ and any $\delta > 0$ the covering group*
$$\frac{\pi_1(X, x)}{\pi_1(X, x, \delta)}$$
is finitely generated.

We would like to apply this to $B(\tilde{x}, r)$ but as noted previously we must be careful what metric the ball carries. When $\widetilde{X}$ is a complete, locally compact length space and $B(\tilde{x}, r)$ is given the restricted metric $d_{\widetilde{X}}$ it is compact by the Hopf–Rinow theorem [1], but we want to use the intrinsic metric $d_{B(\tilde{x},r)}$ for the purposes of defining δ–covers.

In general, a closed and bounded subset of a length space may no longer be compact when given an intrinsic metric. One can see this by considering for example a cone over a Hawaiian earring. Fortunately, in a complete locally compact length space, closed balls are still compact when given the intrinsic metric. In fact, changing the metric does not change the topology in this case.

Proposition 3.3 *Suppose X is a complete locally compact length space, $x \in X$ and let $r > 0$ be given. Then the identity map*
$$i\colon (B(x, r), d_{B(x,r)}) \to (B(x, r), d_X)$$
is a homeomorphism.

Proof First observe that since $d_X \le d_{B(x,r)}$, i is continuous. To show that i^{-1} is continuous, it is enough to show that if a sequence y_i converges with respect to d_X

then y_i converges with respect to $d_{B(x,r)}$. We start by showing that this fact holds on the open ball

$$B_r(x) = \{y \in X \mid d(x,y) < r\}.$$

So suppose y and y_i are in $B_r(x)$ and that

$$d_X(y, y_i) \to 0.$$

Connect x and y by a path γ that minimizes distance with respect to d_X, and connect y to y_i by paths σ_i that minimize distance with respect to d_X. Then

$$\ell(\gamma) = d_X(x,y) = r_0 < r.$$

Since we also know that when i is large,

$$\ell(\sigma_i) = d_X(y, y_i) < r - r_0,$$

we have $\mathrm{Im}(\sigma_i) \subset B(x,r)$ for large i by the triangle inequality. Thus

$$d_{B(x,r)}(y, y_i) \le \ell(\sigma_i),$$

which goes to zero as i goes to infinity.

Now suppose y and y_i are in $B(x,r)$ and that

$$d_X(y, y_i) \to 0$$

as $i \to \infty$. By the above, we may assume that $d(x,y) = r$. Connect x to y_i by paths γ_i, parameterized by arc length, that minimize distance in X. Since $(B(x,r), d_X)$ is a compact metric space, the Arzela–Ascoli lemma implies that some subsequence γ_j of γ_i converges uniformly to a path γ connecting x to y. Since X is a complete locally compact length space, γ minimizes distance in X. We first show that the subsequence y_j of y_i converges to y with respect to $d_{B(x,r)}$.

Let $\epsilon > 0$ be given and set $r_j = \ell(\gamma_j)$. For j large, $r_j > r - \epsilon$. By the uniform convergence of γ_j to γ in $(B(x,r), d_X)$,

$$d_X(\gamma(r-\epsilon), \gamma_j(r-\epsilon)) \to 0.$$

Since $\gamma(r-\epsilon)$ and $\gamma_j(r-\epsilon)$ are in $B_r(x)$, the above argument shows that

$$d_{B(x,r)}(\gamma(r-\epsilon), \gamma_j(r-\epsilon)) \to 0.$$

In particular, for j large,

$$d_{B(x,r)}(\gamma(r-\epsilon), \gamma_j(r-\epsilon)) < \epsilon.$$

But then, for j large

$$\begin{aligned} d_{B(x,r)}(y,y_j) &\le d_{B(x,r)}(y,\gamma(r-\epsilon)) + d_{B(x,r)}(\gamma(r-\epsilon),\gamma_j(r-\epsilon)) \\ &\qquad + d_{B(x,r)}(\gamma_j(r-\epsilon),y_j) \\ &< 3\epsilon. \end{aligned}$$

We have shown that if y_i converges in $B(x,r)$ to y with respect to d_X, then some subsequence y_j converges to y with respect to $d_{B(x,r)}$. To complete the proof, suppose that y_i does not converge to y with respect to $d_{B(x,r)}$. Then there is a neighborhood U of y in $(B(x,r), d_{B(x,r)})$ and a subsequence y_k of y_i with $y_k \notin U$ for all k. But y_k converges to y with respect to d_X, so some subsequence of y_k converges to y with respect to $d_{B(x,r)}$. This is a contradiction, as desired. □

Combining Lemma 3.2 and Proposition 3.3 we have:

Lemma 3.4 *If $\widetilde{X}$ is a complete locally compact length space then $\pi_1(\tilde{x},r)/\pi_1(\tilde{x},r,\delta)$ is finitely generated. Thus its quotient*

$$G(\tilde{x},r,R,\delta) = \frac{\pi_1(\tilde{x},r)}{\pi_1(\tilde{x},r,R,\delta)}$$

is finitely generated.

Our next step is to state a condition that will ensure that each generator of $G(\tilde{x},r,R,\delta)$ lifts. Although this condition will be trivially satisfied in the case where $\widetilde{X}$ is simply connected, our goal is to prove our main theorem in as much generality as possible.

3.2 Pseudo-nullhomotopic loops

Definition 3.5 Suppose X is a metric space, $x \in X$ and Y is a subset of X. A loop γ based at x is called pseudo-nullhomotopic in Y if, for all $\delta > 0$, γ is homotopic in Y to a product of loops of the form $\alpha * \beta * \alpha^{-1}$, where each β lies in some open δ–ball in Y.

In practice the subset Y will either be a closed ball or an open ball. In those cases a loop is pseudo-nullhomotopic in a ball if it can be freely homotoped inside that ball to a product of arbitrarily small loops.

Lemma 3.6 *Suppose X is a complete length space, $x \in X$ and that the universal cover $\widetilde{X}$ exists. If a loop γ is pseudo-nullhomotopic in $B(x,R)$ for some $R > 0$ then γ lifts to $\widetilde{X}$.*

Proof Consider $\widetilde{B}(x,R,\infty)$, the connected lift of $B(x,R)$ to $\widetilde{X}$. Since $B(x,R)$ is compact there is $\delta_0>0$ such that $\pi\colon \widetilde{B}(x,R,\infty)\to B(x,R)$ is an isometry on balls of radius δ_0. Then $\widetilde{B}(x,R)^{\delta_0}$ covers $\widetilde{B}(x,R,\infty)$:

$$\begin{tikzcd}
\widetilde{B}(x,R)^{\delta_0} \arrow[d] \arrow[dd, bend right=60] \\
\widetilde{B}(x,R,\infty) \arrow[d, "\pi"'] \\
B(x,R)
\end{tikzcd}$$

If γ is pseudo-nullhomotopic in $B(x,R)$ then γ lifts to $\widetilde{B}(x,R)^{\delta}$ for all $\delta>0$. In particular γ lifts to $\widetilde{B}(x,R)^{\delta_0}$. But this means that γ lifts to $\widetilde{B}(x,R,\infty)\subset\widetilde{X}$. $\square$

The converse of Lemma 3.6 is not true for general metric spaces.

Example 3.7 For each $x\ge 0$ let C_x be the circle in $\mathbb{R}^3$ with radius e^{-x} centered at $(x,0,e^{-x})$. Let X_0 be the union of the circles C_x, and for $i=1,2,\dots$ let X_i be the union of X_{i-1} with the cylinder created by rotating the segment $z=2e^{-x}$, $x\le i$, about the line $z=e^{-i}$ in the xz plane. Then let

$$X=\mathrm{GH}\lim_{n\to\infty}X_n.$$

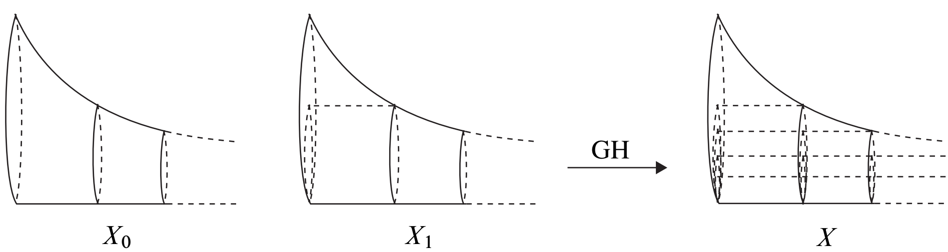

Figure 7: Example 3.7

In this case X is its own universal cover but the homotopies that carry the nontrivial loop C_0 to smaller and smaller loops lie inside bigger and bigger balls.

3.3 Main Theorem

We now state and prove the main theorem of this paper.

Theorem 3.8 (Description of universal cover) *Suppose*

$$(X, x) = \mathrm{GH} \lim_{i\to\infty} (M_i, p_i),$$

where M_i^n have $\mathrm{Ric}_{M_i} \geq (n-1)H$. Suppose further that for every loop γ that lifts to a loop in $\widetilde{X}$ there is $R(\gamma)$ such that γ is pseudo-nullhomotopic in $B(x, R(\gamma))$. Then if $\pi\colon \widetilde{X} \to X$ is the universal cover with $\pi(\tilde{x}) = x$ there is an increasing sequence $R_i \to \infty$ and numbers $\delta_i > 0$ such that

$$(\widetilde{X}, \tilde{x}) = \mathrm{GH} \lim_{i\to\infty} (\widetilde{B}(p_i, R_i, R_{i+1})^{\delta_i}, \tilde{p}_i),$$

where we pass to a subsequence of the manifolds M_i as well.

Proof We start by constructing an increasing sequence of numbers R_i so that $B(\tilde{x}, R_i) \subset \widetilde{X}$ and $B(\tilde{x}_i, R_i) \subset \widetilde{B}(x, R_i, R_{i+1})^{\delta_i}$ are isometric. Note that by Theorem 3.15 of [9] and by Lemma 2.11 of this paper the relative δ–covers $\widetilde{B}(\tilde{x}, S_k, S_{k+1})^{\delta}$ stabilize for any increasing sequence of $S_k \to \infty$. Using this fact we will construct our sequence R_i inductively. Set $R_1 = 1$ and suppose R_i has been determined. Let $R_{i+1,0} = R_i + 1$ and pick $\epsilon_i > 0$ so that $\widetilde{B}(\tilde{x}, R_i, R_{i+1,0})^{\epsilon_i}$ is stable.

The stability of $\widetilde{B}(\tilde{x}, R_i, R_{i+1,0})^{\epsilon_i}$ together with Lemma 3.4 implies that the group of covering transformations

$$G(\tilde{x}, R_i, R_{i+1,0}, \delta) = \frac{\pi_1(\tilde{x}, R_i)}{\pi_1(\tilde{x}, R_i, R_{i+1,0}, \delta)}$$

is finitely generated and identical for all nonzero $\delta \leq \epsilon_i$. Let $[\tilde{\gamma}_{ij}]_{j=1}^{N(i)}$ be generators for $G(\tilde{x}, R_i, R_{i+1,0}, \epsilon_i)$, where $\tilde{\gamma}_{ij}$ are loops in $B(\tilde{x}, R_i)$. Set $\gamma_{ij} = \pi(\tilde{\gamma}_{ij})$. Then γ_{ij} lifts to $\widetilde{X}$. By assumption there is $R_{i+1,j} = R(\gamma_{ij})$ so that γ_{ij} is pseudo-nullhomotopic in $B(x, R_{i+1,j})$.

Set
$$R_{i+1} = \max_{j=0,\dots,N(i)} \{R_{i+1,j}\} + 1.$$

Then each γ_{ij} is pseudo-nullhomotopic in $B_{R_{i+1}}(x)$. Working through the definitions we see this means that each γ_{ij} lifts to $\widetilde{B}_{R_{i+1}}(x)^{\delta}$ for each $\delta > 0$. Thus, for all $\delta > 0$ we may use Lemma 3.1 to lift π to

$$\tilde{\pi}^{\delta}\colon B(\tilde{x}, R) \to \widetilde{B}_{R_{i+1}}(x)^{\delta}.$$

On the other hand, Theorem 2.10 implies that we may pick a nonzero $\delta_i \le \epsilon_i$ so that $\widetilde{B}(x, R_i, R_{i+1})^{\delta_i}$ is stable. So far we have:

$$\begin{tikzcd}
\widetilde{X} & B(\tilde{x}, R_i) \arrow[l, hook'] \arrow[rr, dashed, bend left, "\tilde{\pi}^{\delta_i}"] \arrow[dr, "\pi"] & & \widetilde{B}_{R_{i+1}}(x)^{\delta_i} \arrow[d, "\pi_i"] \\
X \arrow[from=u, "\pi"'] & & B(x, R_i) \arrow[ll, hook'] \arrow[r, hook] & B_{R_{i+1}}(x)
\end{tikzcd}$$

The next step is to note that $\tilde{\pi}^{\delta_i}$ is a surjective local isometry

$$\tilde{\pi}^{\delta_i} \colon B(\tilde{x}, R_i) \to B(\tilde{x}_i, R_i) \subset \widetilde{B}(x, R_i, R_{i+1})^{\delta_i},$$

where $\pi_i(\tilde{x}_i) = x$. This gives us:

$$\begin{tikzcd}
\widetilde{X} & B(\tilde{x}, R_i) \arrow[l, hook'] \arrow[rr, dashed, bend left, "\tilde{\pi}^{\delta_i}"] \arrow[dr, "\pi"] & & B(\tilde{x}_i, R_i) \arrow[r, hook] & \widetilde{B}(x, R_i, R_{i+1})^{\delta_i} \arrow[r, hook] & \widetilde{B}_{R_{i+1}}(x)^{\delta_i} \arrow[d, "\pi_i"] \\
X \arrow[from=u, "\pi"'] & & B(x, R_i) \arrow[ll, hook'] \arrow[rrr, hook] & & & B_{R_{i+1}}(x)
\end{tikzcd}$$

To show $\tilde{\pi}^{\delta_i}$ is an isometry $B(\tilde{x}, R_i) \to B(\tilde{x}_i, R_i)$ we need only show that $\tilde{\pi}^{\delta_i}$ is injective. We will prove this by contradiction.

Suppose $\tilde{q} \ne \tilde{q}'$ have

$$\tilde{\pi}^{\delta_i}(\tilde{q}) = \tilde{\pi}^{\delta_i}(\tilde{q}') = \tilde{q}_i.$$

Connect $\tilde{x}$ to $\tilde{q}$ by distance minimizing path $\tilde{\sigma}$ and connect $\tilde{x}$ to $\tilde{q}'$ by a distance minimizing path $\tilde{\sigma}'$. Let $\tilde{\sigma}_i = \tilde{\pi}^{\delta_i}(\tilde{\sigma})$ and $\tilde{\sigma}_i' = \tilde{\pi}^{\delta_i}(\tilde{\sigma}')$. (See Figure 8.)

Set $\tilde{\gamma}_i = \tilde{\sigma}_i^{-1} * \tilde{\sigma}_i'$. Then

$$\gamma = \pi_i(\tilde{\gamma}_i) = \pi_i(\tilde{\sigma}_i^{-1}) * \pi_i(\tilde{\sigma}_i')$$

is a loop. Moreover, γ lifts to a loop $\tilde{\gamma}_i$ in $\widetilde{B}(x, R_{i+1})^{\delta}$ for every $\delta \le \delta_i$. Thus γ acts trivially on every $\widetilde{B}(x, R_{i+1})^{\delta}$. By Lemma 3.6, γ acts trivially on $\widetilde{X}$, so the lift of γ to $\widetilde{X}$ based at $\tilde{q}$ is a loop. But the lift of γ to $\widetilde{X}$ based at $\tilde{q}$ is the nonloop $\tilde{\sigma}^{-1} * \tilde{\sigma}'$, which is a contradiction.

Thus $\tilde{\pi}^{\delta_i}$ is an isometry $B(\tilde{x}, R_i) \to B(\tilde{x}_i, R_i)$. In particular,

$$\tilde{\pi}^{\delta_i} \colon B(\tilde{x}, R) \to B(\tilde{x}_i, R)$$

is an isometry for all $R \le R_i$.

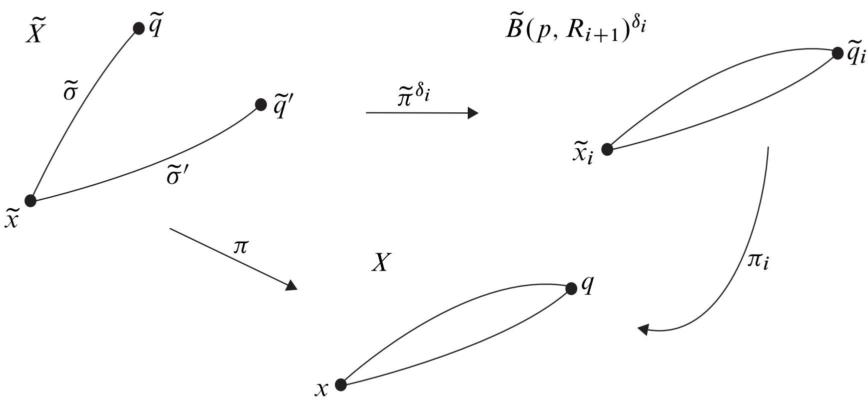

Figure 8: $\widetilde{\pi}^{\delta_i}$ is injective.

Thus the balls in $\widetilde{X}$ are isometric to the balls in $\widetilde{B}(x, R_i, R_{i+1})^{\delta_i}$ for i large, which is to say

$$(\widetilde{B}(x, R_i, R_{i+1})^{\delta_i}, \widetilde{x}_i) \xrightarrow{\text{GH}} (\widetilde{X}, \widetilde{x}).$$

Since each stable relative δ–cover $\widetilde{B}(x, R_i, R_{i+1})^{\delta_i}$ is itself the Gromov–Hausdorff limit of the relative δ–covers $\widetilde{B}(p_j, R_i, R_{i+1})^{\delta_i}$ of the balls in the sequence (Theorem 2.10) we use a diagonalization procedure to pass to a subsequence of the manifolds and obtain the result. □

Note that when $\widetilde{X}$ is simply connected, any loop that lifts to a simply connected universal cover is nullhomotopic. This means that the pseudo-nullhomotopy condition of the Theorem 3.8 is trivially satisfied, giving us Theorem 1.4.

In the proof we showed that for a complete locally compact length space X, if there is $x \in X$ and an increasing sequence $R_i \to \infty$ such that the relative δ–covers $\widetilde{B}(x, R_i, R_{i+1})^{\delta}$ stabilize and we assume that every loop γ that lifts to a loop in $\widetilde{X}$ is pseudo-nullhomotopic in some ball, then $\widetilde{X} = \mathrm{GH}\lim_{i\to\infty} \widetilde{B}(x, R_i, R_{i+1})^{\delta_i}$ for some subsequence of R_i. In addition the relative δ–covers $\widetilde{B}(x, R_i, R_{i+1})^{\delta_i}$ are stable. When the universal covers of the balls $B(x, R_i)$ exist then the δ–covers $\widetilde{B}(x, R_i)^{\delta}$ are also stable which means we can use these instead of the relative δ–covers $\widetilde{B}(x, R_i, R_{i+1})^{\delta}$. Therefore, if we also know that X is semilocally simply connected (which implies that the pseudo-nullhomotopy condition is satisfied), we have that $\widetilde{X} = \mathrm{GH}\lim_{i\to\infty} \widetilde{B}(x, R_i)$. This proves Proposition 1.2.

Note that all known examples of limits of Riemannian manifolds with a uniform lower bound on Ricci curvature have simply connected universal covers. It may well be that this is true in general.

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