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A short proof of the Göttsche conjecture

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We prove that for a sufficiently ample line bundle L on a surface S, the number of δ–nodal curves in a general δ–dimensional linear system is given by a universal polynomial of degree δ in the four numbers L^2, $L\,.\,K_S$, K_S^2 and $c_2(S)$.

The technique is a study of Hilbert schemes of points on curves on a surface, using the BPS calculus of Pandharipande and the third author [22] and the computation of tautological integrals on Hilbert schemes by Ellingsrud, Göttsche and Lehn [8].

We are also able to weaken the ampleness required, from Göttsche's $(5\delta-1)$–very ample to δ–very ample.

14C05, 14N10; 14C20, 14N35

1 Introduction

Throughout this paper we fix a compact complex surface S with very ample line bundle L with no higher cohomology. Curves C in the linear system $\mathbb{P} := \mathbb{P}(H^0(L))$ have arithmetic genus $g = g(C)$, where $2g-2 = L\,.\,(L+K_S)$.

Call a (possibly reducible) curve δ–nodal if it is smooth away from δ points at which it looks analytically like $\{xy=0\} \subset \mathbb{C}^2$. For sufficiently ample L, δ–nodal curves occur in codimension δ, and it is a classical question how many such curves appear in a general δ–dimensional linear subsystem $\mathbb{P}^\delta \subset \mathbb{P}$: see Kleiman and Piene [14] for a history going back to 1848 and Steiner, Cayley and Salmon. For $S = \mathbb{P}^2$, Ran [23], and later Caporaso and Harris [4], found recursions that determine these Severi degrees. For general S it was conjectured (see Di Francesco and Itzykson [6], Göttsche [10], Kleiman and Piene [13] and Vainsencher [27]) that they should be topological – in fact a universal polynomial of degree δ in the numbers L^2, $L\,.\,K_S$, K_S^2 and $c_2(S)$. This is often referred to as the Göttsche conjecture; in fact Göttsche [10] gave a partial conjectural form for the generating series (motivated by the Yau–Zaslow formula [28] and involving quasimodular forms) about which we have nothing to say in this paper. It is plausible that our recursive formulae could be used to identify these generating

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series, but it seems unlikely: the same integrals are evaluated for $S = K3$ by Kawai and Yoshioka [11] and Maulik, Pandharipande and Thomas [20] but by an indirect method.

There are now proofs for $\mathbb{P}^2$ (see Choi [5] and Fomin and Mikhalkin [9]) and other surfaces (see Beauville [1], Bryan and Leung [3; 2], Kazarian [12] and Liu [18; 17]) using wildly different techniques. A completely general algebro-geometric proof using degeneration has recently been found by Tzeng [26].

Our method of counting is rather simpler; it is a generalisation of the following elementary standard technique for $\delta = 1$. Suppose we have a pencil of curves, a finite number of which have one node (and the rest are smooth). The smooth curves have Euler characteristic $2-2g$, while the nodal curves have Euler characteristic $(2-2g)+1$. Thus we can determine the number of nodal curves from the topological Euler characteristic of the universal curve $\mathcal{C} \to \mathbb{P}^1$:

$$e(\mathcal{C}) = (2-2g)e(\mathbb{P}^1) + \#(1\text{–nodal curves}). \tag{1.1}$$

Since $\mathcal{C}$ is the blow up of S at L^2 points this gives the classical formula

$$\#(1\text{–nodal curves}) = c_2(S) + L^2 + 2L\,.\,(L + K_S).$$

For a curve C, define $e_i := e\big(C^{[i]}\big)$, where $C^{[i]}$ is the Hilbert scheme of i points on C. Then the point is that

$$e_1 - (2-2g)e_0 \tag{1.2}$$

is 1 when C is 1–nodal, and 0 if C is smooth. Summing over all the curves in the pencil gives (1.1).

Similarly for arbitrary δ there is a linear formula in the e_i, $0 \le i \le \delta$, which gives 1 for δ–nodal curves and 0 for all curves of geometric genus $\overline{g} > g-\delta$ (those which are less singular in some sense). The result – Theorem 3.4 below – is taken from Pandharipande and Thomas [22], where it was proved in the context of stable pairs (these are in some sense dual to points of the Hilbert scheme).

Summing over $\mathbb{P}^\delta$ gives a formula for the number of δ–nodal curves in terms of the Euler characteristics of the *relative Hilbert schemes* of the universal curve. In turn we compute these Euler characteristics of relative Hilbert schemes in terms of certain tautological integrals over the Hilbert schemes of points on the surface. These can be handled by a recursion due to Ellingsrud, Göttsche and Lehn [8]. (Göttsche also expressed the counts of nodal curves in terms of integrals on the Hilbert scheme, but to which [8] does not apply. They are evaluated by degeneration techniques by Tzeng in [26].)

Motivation This paper arose from our project [15] to define the invariants counting nodal curves in terms of virtual classes, thus extending them from the very ample case to more general curve classes. There are two obvious ways to do this, using Gromov–Witten theory or stable pairs (using equivariant *reduced* 3–fold virtual classes as in Maulik, Pandharipande and Thomas [20]). These should be related by the famous MNOP conjecture of Maulik, Nekrasov, Okounkov and Pandharipande [19] adapted to stable pair as in Pandharipande and Thomas [21]. To see the δ–nodal curves in GW theory we change the genus and look at stable maps from curves of genus $g-\delta$. Via the MNOP conjecture, the analogue in stable pairs theory is to allow up to δ free points to roam around the curves and use the BPS calculus of [22] to identify the lower genus curves. This is essentially what we do in this paper. But we forget about the motivation, working from the start with Hilbert schemes of points on curves. We also confine ourselves to the classical case of very ample curves, so that no virtual cycles enter.

Notation For a reduced curve C of arithmetic genus $g=g(C)$, we let $\overline{g}:=g(\overline{C})$ be its geometric genus: the genus of its normalisation $\overline{C}\to C$. If $\overline{C}$ is disconnected with connected components C_i then its genus is $1+\sum_i(g(C_i)-1)$, which can be negative. The total Chern class of a bundle E is denoted by $c_\bullet(E)$. A line bundle L on S is said to be n–very ample if $H^0(L)\to H^0(L|_Z)$ is onto for all length–$(n+1)$ subschemes $Z\subset S$. We use $X^{[k]}$ for the Hilbert scheme of k points on any variety X, but $\mathrm{Hilb}^k(X/B)$ for the *relative* Hilbert scheme of points on the fibres of a family $X\to B$. Using the obvious projections and the universal subscheme $\mathcal{Z}_k\subset X\times X^{[k]}$ we get the rank k tautological bundle

$$L^{[k]}:=\pi_{2*}\big((\pi_1^*L)|_{\mathcal{Z}_k}\big)\quad\text{on } X^{[k]}.$$

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2 Sufficiently ample linear systems

We begin with a description of the curves in a general sufficiently ample linear system, strengthening a result of Göttsche [10, Proposition 5.2]. The bound on the geometric genus is key to our results.

Proposition 2.1 *If L is δ–very ample then the general δ–dimensional linear system $\mathbb{P}^\delta\subset\mathbb{P}(H^0(L))$ contains a finite number of δ–nodal curves appearing with multiplicity 1. All other curves are reduced with geometric genus $\overline{g}>g-\delta$.*

Proof We recall some deformation theory of singularities of curves in surfaces; an excellent reference is Diaz and Harris [7]. Since everything is local we consider the germ of a reduced plane curve $C = \{f = 0\} \subset \mathbb{C}^2$ about its singular points. This has a miniversal deformation space $\mathrm{Def} = H^0(\mathcal{O}_C/J)$, where $J = (\partial f/\partial x, \partial f/\partial y)$ is the Jacobian ideal. (Denoting the composition $\mathbb{C}[x,y] \to H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C/J)$ by $\epsilon \mapsto [\epsilon]$, to first order the construction associates $[\epsilon] \in \mathrm{Def}$ to the deformation $\{f+\epsilon = 0\}$ of $\{f = 0\}$.) Inside Def is the *equigeneric locus* of deformations with the same geometric genus; it has tangent cone $H^0(A/J) \subset H^0(\mathcal{O}_C/J)$ where $A \subset \mathcal{O}_C$ is the *conductor ideal* (see Teissier [25]) of colength $g - \overline{g}$. Inside that is the equisingular locus, which is smooth with tangent space $H^0(\mathrm{ES}/J) \subset H^0(\mathcal{O}_C/J)$ where $\mathrm{ES} \subset \mathcal{O}_C$ is the *equisingular ideal*. The inclusion $\mathrm{ES} \subset A$ is *strict* unless the singularities are nodal.

First we show that the locus of δ–nodal curves in $\mathbb{P} := \mathbb{P}(H^0(L))$ is smooth of codimension δ, from which the first result of the Proposition follows. Fix $[s] \in \mathbb{P}$ corresponding to a δ–nodal curve $C \subset S$. The germ of $\mathbb{P}$ about $[s]$ maps to the miniversal deformation space Def of the singularities of C. Its tangent map $T_{[s]}\mathbb{P} = H^0(L)/\langle s\rangle \to H^0(L \otimes \mathcal{O}_C/J)$ is onto because $\mathrm{length}(\mathcal{O}_C/J) = \delta$ and L is $(\delta-1)$–very ample. Therefore this is a smooth map. Inside Def the locus of δ–nodal curve germs is smooth of codimension δ, so the locus of δ–nodal curves in $\mathbb{P}$ is also smooth of codimension δ.

Now we show in turn that curves of the following kind

- nonreduced,
- reduced with geometric genus $< g - \delta$, and
- reduced with geometric genus $g - \delta$ with singularities other than nodes,

sit in subschemes of $\mathbb{P}$ of codimension $> \delta$.

• Fix a nonreduced curve C cut out by $s \in H^0(L)$, with underlying reduced curve C^{red}. Nearby curves in $\mathbb{P}$ are of the form $s + \epsilon = 0$ for small ϵ in a fixed complement to $\langle s\rangle \subset H^0(L)$. A local computation shows that where $\epsilon|_{C^{\mathrm{red}}} \neq 0$, the resulting curve is reduced. So the tangent cone to the nonreduced curves lies in the kernel of $T_{[s]}\mathbb{P} = H^0(L)/\langle s\rangle \to H^0(L|_{C^{\mathrm{red}}})$. This map need not be onto, but its composition with the projection to any length–$(\delta+1)$ subscheme of C^{red} is, since L is δ–very ample. Therefore its kernel has codimension at least $\delta + 1$, so the nonreduced curves have codimension $> \delta$ inside $\mathbb{P}$, as required.

• Given a curve C of geometric genus $\overline{g} < g - \delta$ defined by $s \in H^0(L)$, the tangent cone to the equigeneric locus of curves of the same geometric genus in $\mathbb{P}(H^0(L))$ is given by the kernel of $H^0(L)/\langle s\rangle \to H^0(L\otimes(\mathcal{O}_C/A))$. This map need not be onto, but its composition with the projection to any length–$(\delta+1)$ subscheme of $\mathcal{O}_C/A$ is, since L is δ–very ample. Therefore the kernel has codimension at least $\delta+1$, and the equigeneric locus has codimension $>\delta$.

- Finally we deal with reduced curves of geometric genus $g-\delta$ which are *not* δ–nodal. Let $S_m \subset \mathbb{P}$ be the locus of curves of geometric genus $g-\delta$ whose total Milnor number (summed over the singular points of C) is m. For plane curve singularities, it is a classical fact that $m \geq \delta$ with equality only for the δ–nodal curves. Thus S_δ is the locus of nodal curves, which we have observed is smooth of codimension δ.

For $m > \delta$ we will show that S_m is in the closure of $S_\delta \cup S_{\delta+1} \cup \cdots \cup S_{m-1}$ and so has codimension $\geq \delta+1$ by induction, as required. In fact it will be enough to prove that for each $C \in S_m$, some equigeneric deformation of C is not equisingular, and so, by a classical result (see Lê and Ramanujam [16]), has smaller Milnor number.

Since L is $(\delta-1)$–very ample, $H^0(L)/\langle s\rangle$ surjects onto $H^0(L\otimes\mathcal{O}_C/A)$. Therefore the image of the germ of $\mathbb{P}$ about $[s]$ is *transverse* to the equigeneric locus of curves in Def. Since C has non-nodal singularities, A/ES has length ≥ 1, so there is a nonzero map $A/\mathrm{ES} \to \mathcal{O}_p$, where $p \in C$ is one of the singular points. Its kernel defines an ideal I with $J \subset \mathrm{ES} \subset I \subset A \subset \mathcal{O}_C$ and A/I one dimensional. By the δ–very ampleness of L, $H^0(L)/\langle s\rangle$ surjects onto $H^0(L\otimes\mathcal{O}_C/I)$. Therefore $\mathbb{P}$ maps onto an equigeneric 1–dimensional family of deformations of C whose tangent cone in $H^0(C, L\otimes A/J)$ is transverse to $H^0(C, L\otimes I/J)$ (which by construction contains the tangent space to the smooth locus of equisingular deformations). So we get our equigeneric family in which the Milnor number drops, as required. □

3 BPS calculus

Proposition 3.1 [22, Appendix B.1] *For C a reduced curve embedded in a smooth surface, the generating series of Euler characteristics*

$$q^{1-g}\sum_{i=0}^{\infty} e(C^{[i]})q^i$$

can be written uniquely in the form

$$\sum_{r=g(\bar{C})}^{g(C)} n_{r,C}\, q^{1-r}(1-q)^{2r-2}$$

for integers $n_{r,C}$, $r=\bar{g},\dots,g$.

Moreover, the $n_{r,C}$ are determined by only the numbers $e(C^{[i]})$, $i=1,\dots,g-\overline{g}$ by a linear relation. Explicitly, $n_{r,C}=0$ for $r>g$ and $n_{g,C}=1$, while for $r<g$ the $n_{r,C}$ are determined inductively by

$$n_{g-r,C}=e(C^{[r]})-\sum_{i=g-r+1}^{g} n_{i,C}\, e(\mathrm{Sym}^{r-(g-i)}\,\Sigma_i),$$

where Σ_i is any smooth proper curve of genus i. □

Proposition 3.1 is a consequence of Serre duality for the fibres of the Abel–Jacobi map taking a subscheme $Z\subset C$ to the sheaf I_Z^*. Its force is that these formulae give $n_r=0$ for all $r<\overline{g}$, *regardless of the singularities of C*.

The result says that at the level of Euler characteristics of Hilbert schemes, the singular curve C looks like a disjoint union of smooth curves Σ_i of genera i between $\overline{g}$ and g; the number of genus i being $n_{i,C}$. (Note that the generating series for Σ_i is precisely $q^{1-i}(1-q)^{2i-2}$. The example of nodal curves in Proposition 3.2 below is illustrative.) It is most natural and comprehensible in the context of stable pairs [22] (which are dual to the ideal sheaves parametrised by the Hilbert scheme), but a self contained account using only Hilbert schemes is given by Shende [24, Proposition 2]. Moreover, it is shown there that the $n_{r,C}$ are *positive*.

Proposition 3.2 [22, Proposition 3.23] *For C a δ–nodal curve,*

$$n_{r,C}=\binom{\delta}{g-r}$$

is the number of partial normalisations of C at $g-r$ of its δ nodes – ie the number of curves of arithmetic genus r mapping to C by a finite map which is generically an isomorphism. In particular, $n_{g-1,C}=\delta$ is the number of nodes, while $n_{g,C}=1=n_{g-\delta,C}$ and $n_{i,C}=0$ for $i\notin\{g-\delta,\dots,g\}$. □

To give some feeling for these two results, we explain how the resulting formulae for a δ–nodal curve C,

$$e(C^{[k]})=\sum_{j=0}^{k}\binom{\delta}{j}e(\mathrm{Sym}^{k-j}\,\Sigma_{g-j}), \tag{3.3}$$

arise from a decomposition of $C^{[k]}$ into Zariski locally closed subsets. The first piece is $\mathrm{Sym}^k\, C_0\subset C^{[k]}$, where $C_0=C\backslash\{\text{nodes}\}$ is the smooth locus. Since

$$e(C_0)=e(\Sigma_g)\quad\text{and so}\quad e(\mathrm{Sym}^k\,C_0)=e(\mathrm{Sym}^k\,\Sigma_g),$$

this gives the first term of (3.3). To describe the j–th piece of the decomposition, fix j distinct nodes $p_{i_1}, \dots, p_{i_j} \in C$ at which we normalise to produce an arithmetic genus $g-j$ curve $\overline{C}^{i_1 \cdots i_j}$ with $\delta - j$ nodes and smooth locus $(\overline{C}^{i_1 \cdots i_j})_0$. Then we have

$$\mathrm{Sym}^{k-j}(\overline{C}^{i_1 \cdots i_j})_0 \hookrightarrow C^{[k]}$$

by projecting the $(k-j)$–cycle in $(\overline{C}^{i_1 \cdots i_j})_0$ down to C and adding in the nodes $p_{i_1}, \dots, p_{i_j}$. Around the normalised nodes p_{i_l} we have to explain what we mean by this. Describing the node p_{i_l} locally as $\{xy = 0\} \subset \mathbb{C}^2$, the partial normalisation has two local branches corresponding to the x– and y–axes. If the $(k-j)$–cycle has multiplicities a and b at the two preimages of the node on these branches, then we push this down to the length–$(a+b+1)$ subscheme with local ideal (x^{a+1}, y^{b+1}). That is, we thicken p_{i_l} further by a units down the x–axis and b units down the y–axis.

Taking the disjoint union over all $n_{g-j,C}$ choices of the nodes $p_{i_1}, \dots, p_{i_j}$ gives the j–th piece. Since

$$e((\overline{C}^{i_1 \cdots i_j})_0) = e(\Sigma_{g-j}) \quad \text{and so} \quad e(\mathrm{Sym}^k(\overline{C}^{i_1 \cdots i_j})_0) = e(\mathrm{Sym}^k \Sigma_{g-j}),$$

its Euler characteristic gives the j–th term in (3.3). What we are missing of $C^{[k]}$ in this decomposition is those subschemes which are Cartier divisors with nonempty intersection with the set $\{p_1, \dots, p_\delta\}$ of nodes. The set of such subschemes has a $\mathbb{C}^*$–action (with weights $(1,0)$ on $\mathbb{C}^2 \supset \{xy = 0\}$) with no fixed points, and so zero Euler characteristic.

A simple consequence of Proposition 3.1 and Proposition 3.2 is the following.

Theorem 3.4 *Fix a general linear system $\mathbb{P}^\delta$ as in Proposition 2.1, with universal curve $\mathcal{C} \to \mathbb{P}^\delta$. Then the number of δ–nodal curves in $\mathbb{P}^\delta$ is a linear combination of the numbers $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$, $i = 1, \dots, \delta$.*

It is the coefficient $n_{g-\delta}$ of $q^{1-g+\delta}(1-q)^{2g-2\delta-2}$ in the generating series

$$q^{1-g} \sum_{i=0}^{\infty} e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)) q^i = \sum_{r=g-\delta}^{g} n_r \, q^{1-r}(1-q)^{2r-2}.$$

Proof Compute the Euler characteristics fibrewise, stratifying $\mathbb{P}^\delta$ by the topological type of the curve. By Proposition 3.1, $n_{g-\delta,C}$ vanishes for all but the curves C in our linear system with geometric genus $g-\delta$. By Proposition 2.1 these are the δ–nodal curves, for which $n_{g-\delta,C} = 1$ by Proposition 3.2. Therefore the only contributions to the coefficient $n_{g-\delta}$ of the above generating series are precisely 1 for each δ–nodal curve. □

4 Tautological integrals

These relative Hilbert schemes are relatively easy to compute with. Let $\mathbb{P} := \mathbb{P}(H^0(L))$ denote the complete linear system. The sections $H^0(L) \otimes H^0(L)^*$ of $L \boxtimes \mathcal{O}(1)$ on $S \times \mathbb{P}$ include the canonical section $\mathrm{id}_{H^0(L)}$. Pull it back to $S \times S^{[i]} \times \mathbb{P}$, restrict to the universal subscheme $\mathcal{Z}_i \times \mathbb{P}$, and push down to $S^{[i]} \times \mathbb{P}$ to give the tautological section of $L^{[i]} \boxtimes \mathcal{O}(1) =: L^{[i]}(1)$. (Thinking of it as a section of $\mathcal{H}om(\mathcal{O}(-1), L^{[i]})$ on $S^{[i]} \times \mathbb{P}$, at a point $(Z, [s])$ it maps $s \in \mathcal{O}(-1)$ to $s|_Z \in H^0(L|_Z) = L^{[i]}|_Z$.)

This tautological section has scheme theoretic zero locus

$$\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}) \subset S^{[i]} \times \mathbb{P}.$$

This is smooth, fibring over $S^{[i]}$ with fibres $\mathbb{P}\big(\ker(H^0(L) \to H^0(L|_Z)\big)$ of constant rank, because L is $(i-1)$–very ample. Since its codimension equals the rank i of $L^{[i]}(1)$, the tautological section was in fact transverse and $[\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P})]$ is Poincaré dual to $c_i(L^{[i]}(1))$. Intersecting general hyperplanes in $\mathbb{P}$, by Bertini we retain smoothness when $\mathbb{P}$ is replaced by a general $\mathbb{P}^\delta$.

It follows that the Euler characteristic of the relative Hilbert scheme – the integral of $c_\bullet(T\,\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$ over $\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)$ – can be computed as

$$\int_{\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)} \frac{c_\bullet(T(S^{[i]} \times \mathbb{P}^\delta))}{c_\bullet(L^{[i]}(1))} = \int_{S^{[i]} \times \mathbb{P}^\delta} c_i(L^{[i]}(1)) \frac{c_\bullet(T(S^{[i]} \times \mathbb{P}^\delta))}{c_\bullet(L^{[i]}(1))}.$$

Theorem 4.1 *Suppose that L is δ–very ample. Then in a $\mathbb{P}^\delta$ linear system, general in the sense of Proposition 2.1, the number of δ–nodal curves is a polynomial of degree δ in L^2, $L\,.\,K_S$, K_S^2 and $c_2(S)$.*

Proof By Proposition 2.1, we want to compute the intersection of $\mathbb{P}^\delta$ with the quasiprojective variety of δ–nodal curves in $\mathbb{P}$. Thus we may perturb $\mathbb{P}^\delta$ without changing this number to make it general in the sense of both Proposition 2.1 and the Bertini theorem above. So we can assume that $\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P})$ is smooth.

By the above computation, $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$ is then the integral over $S^{[i]}$ of a polynomial in the Chern classes of $L^{[i]}$ and $S^{[i]}$: the coefficient of ω^δ in

$$\frac{c_\bullet(T(S^{[i]}))(1+\omega)^{\delta+1}\sum_{j=0}^{i}\omega^j c_{i-j}(L^{[i]})}{\sum_{j=0}^{i}(1+\omega)^j c_{i-j}(L^{[i]})}. \tag{4.2}$$

The recursion of [8] applied i times turns this into an integral over S^i of a polynomial in $c_1(L)$, $c_1(S)$, $c_2(S)$ (pulled back from different S factors) and Δ_*1, $\Delta_* c_1(S)$,

$\Delta_* c_1(S)^2$, $\Delta_* c_2(S)$ (pulled back from different $S \times S$ factors), where $\Delta\colon S \hookrightarrow S \times S$ is the diagonal. The result is a degree $\le i$ polynomial in L^2, $L\,.\,K_S$, K_S^2 and $c_2(S)$.

By Theorem 3.4 the number $n_{g-\delta}$ of δ–nodal curves is a linear combination of these degree $\le i$ polynomials, for $0 \le i \le \delta$. Their coefficients are polynomials of degree $\le \delta - i$ in $g = 1 + (L^2 + L\,.\,K_S)/2$. Thus it has total degree $\le \delta$. In fact one easily sees from [8] that our integral over $S^{[i]}$ produces a single term in $c_2(S)^i$ (the coefficient being $(\delta - i + 1)/i!$) arising from the $(\delta - i + 1)c_{2i}(T(S^{[i]}))$ term in (4.2). Therefore $n_{g-\delta}$ contains a term $c_2(S)^\delta/\delta!$ which does not cancel with any other terms, so the polynomial has degree precisely δ. □

The recursion is constructive, though not terribly efficient. Our computer programme only calculates up to 4 nodes without trouble, finding agreement with [13; 27] (after correcting a small sign error in [8]).

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