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Directed immersions of closed manifolds

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Given any finite subset X of the sphere $\mathbf{S}^n$, $n \geq 2$, which includes no pairs of antipodal points, we explicitly construct smoothly immersed closed orientable hypersurfaces in Euclidean space $\mathbf{R}^{n+1}$ whose Gauss map misses X. In particular, this answers a question of M Gromov.

53A07, 53C42; 57R42, 58K15

1 Introduction

To every $(\mathcal{C}^1)$ immersion $f\colon M^n \to \mathbf{R}^{n+1}$ of a closed oriented n–manifold M, there corresponds a unit normal vector field or *Gauss map* $G_f\colon M \to \mathbf{S}^n$, which generates a set $G_f(M) \subset \mathbf{S}^n$ known as the *spherical image* of f. Conversely, one may ask (see Gromov [8, page 3]): *for which sets $A \subset \mathbf{S}^n$ is there an immersion $f\colon M \to \mathbf{R}^{n+1}$ such that $G_f(M) \subset A$?* Such a mapping would be called an *A–directed immersion* of M; see Eliashberg and Mishachev [1], Gromov [7], Rourke and Sanderson [13] and Spring [14]. It is well-known that when $A \neq \mathbf{S}^n$, f must have double points (Section 4.1), and M must be parallelizable, eg, M can only be the torus $\mathbf{T}^2$ when $n = 2$ (Section 4.2). Furthermore, the only known necessary condition on A is the elementary observation that $A \cup -A = \mathbf{S}^2$, while there is also a sufficient condition due to Gromov [7, Theorem (D'), page 186]:

Condition 1.1 $A \subset \mathbf{S}^n$ is open, and there is a point $p \in \mathbf{S}^n$ such that the intersection of A with each great circle passing through p includes a (closed) semicircle.

A *great circle* is the intersection of $\mathbf{S}^n$ with a 2–dimensional subspace of $\mathbf{R}^{n+1}$. Note that, when $n \geq 2$, examples of sets $A \subset \mathbf{S}^n$ satisfying the above condition include those which are the complement of a finite set of points without antipodal pairs. Thus the spherical image of a closed hypersurface can be remarkably flexible. Like most h-principle or convex integration type arguments, however, the proof does not yield specific examples. It is therefore natural to ask, for instance:

Question 1.2 [7, page 186] Is there a “simple” immersion $\mathbf{T}^2 \to \mathbf{R}^3$ whose spherical image misses the four vertices of a regular tetrahedron in $\mathbf{S}^2$?

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Here we give an affirmative answer to this question (Section 2), and more generally present a short constructive proof of the sufficiency of a slightly stronger version of Condition 1.1 for the existence of A–directed immersions of parallelizable manifolds $M^{n-1} \times \mathbf{S}^1$, where M^{n-1} is closed and orientable. Any such manifold admits an immersion $f\colon M^{n-1} \to \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ (Section 4.3). We then extend f to $M^{n-1} \times \mathbf{S}^1$ by using the *figure eight curve*

$$E_\delta(t) := \big(\cos(t),\ \delta \sin(2t)\big) \tag{1}$$

to put a copy of $\mathbf{S}^1 \simeq \mathbf{R}/2\pi$ in each normal plane of f, as described below. Note that the midpoint of $G_{E_\delta}(\mathbf{S}^1)$ is assumed to be at $(1,0)$; see Figure 1 which shows $E_{1/2}$ and its spherical image. Further, the unit normal bundle of f may be naturally identified with the pencil of great circles of $\mathbf{S}^n$ passing through $(0,\dots,0,1)$.

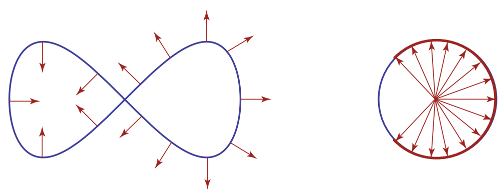

Figure 1

Theorem 1.3 *Let $A \subset \mathbf{S}^n$ satisfy Condition 1.1 with respect to $p = (0,\dots,0,1)$. Further, if $n \geq 3$, suppose that the semicircle in Condition 1.1 contains p, or that no great circle through p is contained in A. Let $f\colon M^{n-1} \to \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ be a smooth ($\mathcal{C}^\infty$) immersion of a closed orientable $(n-1)$–manifold, and, for every $q \in M$, let $C_q \subset \mathbf{S}^n$ be the unit normal space of f at q. Then there is a smooth orthogonal frame $\{N_i\colon M \to \mathbf{S}^n\}$, $i = 1, 2$, for the normal bundle of f such that the semicircle in C_q centered at $N_1(q)$ lies in A. For any such frame, and sufficiently small ε, $\delta > 0$,*

$$F(q,t) := f(q) + \varepsilon \sum_{i=1}^{2} E^i_\delta(t) N_i(q) \tag{2}$$

yields a smooth A–directed immersion $M \times \mathbf{S}^1 \to \mathbf{R}^{n+1}$, where E^i_δ are the components of the figure eight curve E_δ given by (1).

It is not known if Condition 1.1 is necessary for the existence of A–directed closed hypersurfaces, and the question posed in the first paragraph is open, even for $n = 2$. See Ghomi [3; 4] and Ghomi and Tabachnikov [6] for some other recent results on Gauss maps of closed submanifolds, Ghomi [2], Hartman and Nirenberg [9], Milnor [11] and Wu [16] for still more studies of spherical images, and Spring [15] for historical background.

2 Example

If $A = \mathbf{S}^2 \smallsetminus X$ for a finite set X without antipodal pairs, we may always find a point $p \in \mathbf{S}^2$ with respect to which A satisfies the hypothesis of Theorem 1.3 (eg, let $p \notin X$ be in the complement of all great circles which pass through at least two points of X other than $-p$). After a rigid motion (which may be arbitrarily small) we may assume that $p = (0,0,1)$ or $(0,0,-1)$, and let $f(\theta) := (\cos(\theta), \sin(\theta), 0)$ be the standard immersion of $\mathbf{S}^1 \simeq \mathbf{R}/2\pi$ in $\mathbf{R}^3$. Then the desired framing for the normal bundle of f may always take the form

$$N_1(\theta) := f'(\theta) \times N_2(\theta), \quad N_2(\theta) := \frac{\big(\cos(\theta),\ \sin(\theta),\ z(\theta)\big)}{\sqrt{1+z^2(\theta)}}, \tag{3}$$

where $z\colon \mathbf{R}/2\pi \to \mathbf{R}$ is a smooth function with $z(\theta) = -z(\theta+\pi)$ and such that X is contained entirely in one of the components of $\mathbf{S}^2 - N_2(\mathbf{S}^1)$. For instance, when X is the vertices of a regular tetrahedron, we may set $z(\theta) := \cos(3\theta)$ in (3). Then, for $\varepsilon, \delta \leq 1/8$, the mapping $F(\theta, t)$ given by (2) yields an immersion $\mathbf{T}^2 \simeq \mathbf{R}/2\pi \times \mathbf{R}/2\pi \to \mathbf{R}^3$ which answers Question 1.2. The resulting surface, for $\varepsilon = \delta = 1/8$, is depicted in Figure 2 together with its spherical image (note that here $p = (0,0,-1)$). To find $z(\theta)$

Figure 2

in general, we may order the points in $X' \cup -X'$, where $X' := X \smallsetminus \{-p\}$, according to their "longitude" θ, and connect them by geodesic segments to obtain a simple closed symmetric curve $\gamma(\theta)$. A perturbation of γ then yields a smooth symmetric curve $\widetilde{\gamma}$ such that X is contained in one of the components of $\mathbf{S}^2 - \widetilde{\gamma}(\mathbf{S}^1)$. The third coordinate of $\widetilde{\gamma}$ gives our desired height function z.

3 Proof of Theorem 1.3

3.1 First we construct the frame $\{N_i\}$. For every $q \in M$, C_q is a great circle passing through p. So it contains a semicircle in A by assumption (Condition 1.1). Let

$m_q \subset C_q$ be the set of midpoints of all such semicircles. We need to find a smooth map $N_1\colon M \to \mathbf{S}^n$ such that $N_1(q) \in m_q$ for all $q \in M$. To this end note that m_q is open and connected. Further, if m_q contains any pairs of antipodal points, then $m_q = C_q$; otherwise, m_q lies in the interior a semicircle of C_q. Consequently,

$$\mathrm{Cone}(m_q) := \{\, \lambda x \mid x \in m_q \text{ and } \lambda \geq 0 \,\},$$

is a convex set in $\mathbf{R}^{n+1}$. In particular, for any finite set of points $x_i \in \mathrm{Cone}(m_q)$ and numbers $\lambda_i \geq 0$, $\sum_i \lambda_i x_i \in \mathrm{Cone}(m_q)$. Now let B be the set of all points $q \in M$ such that $m_q \neq C_q$. Then B is closed (and therefore compact) since $M \smallsetminus B$ is open; indeed the set of great circles contained in A is open, since A is open. Further note that for any point $q \in M$, normal vector $x \in m_q$, and continuous local extension v of x to a normal vector field of M, we have $v(q') \in m_{q'}$ for all q' in an open neighborhood U of q (because the set of semicircles contained in A is open). Let $\{v_i\colon U_i \to \mathbf{S}^n\}$, $i = 1, \dots, k$, be a finite collection of such local vector fields so that $\bigcup_i U_i$ covers B and v_i are smooth. Also let $\{\phi_i\colon M \to \mathbf{R}\}$ be a smooth partition of unity subordinate to $\{U_i\}$, and, for $q \in \bigcup_i U_i$, set

$$N_1(q) := \frac{\sum_{i=1}^k \phi_i(q) v_i(q)}{\| \sum_{i=1}^k \phi_i(q) v_i(q) \|}.$$

If $q \in B$, then $v_i(q) \in m_q$ which lies in the interior of a semicircle $S \subset C_q$, and so $\| \sum_{i=1}^k \phi_i(q) v_i(q) \| \neq 0$. Indeed, if x is the midpoint of S, then

$$\left\langle \sum_{i=1}^k \phi_i(q) v_i(q), x \right\rangle = \sum_{i=1}^k \phi_i(q) \langle v_i(q), x \rangle > 0.$$

Thus N_1 is well defined (and smooth) on an open neighborhood V of B. Further, $N_1(q) \in m_q$, for all $q \in V$, since $\mathrm{Cone}(m_q)$ is convex. In particular we are done if $B = M$; otherwise, note that we may write

$$N_1(q) = \cos(\theta(q))\, p + \sin(\theta(q))\, G_f(q), \tag{4}$$

for some function $\theta\colon V \to \mathbf{R}$, since G_f is well defined due to the orientability of M, and thus $\{p, G_f(q)\}$ forms an orthonormal basis for the normal plane $df(T_q M)^\perp$. Further, it is easy to see that we may choose θ continuously (and therefore smoothly) if $n = 2$. This also holds for $n > 2$ if each C_q contains a semicircle passing through p; for then θ is uniquely determined within the range $[-\pi/2, \pi/2]$. Indeed, we may choose the vectors v_i above so that $\langle v_i(q), p \rangle \geq 0$ which would in turn yield that $\langle N_1(q), p \rangle \geq 0$. Now let V' be an open neighborhood of B with closure $\overline{V'} \subset V$. Using Tietze's theorem, followed by a perturbation and a gluing, we may extend $\theta|_{V'}$ smoothly to all of M. Then (4) yields the desired vector field on M, since for any

$q \in M \setminus B$, $N_1(q) \in C_q = m_q$. Finally, set

$$N_2(q) := \sin(\theta(q))\, p - \cos(\theta(q))\, G_f(q).$$

3.2 It remains to show that $G_F(M \times \mathbf{S}^1) \subset A$, for small $\varepsilon, \delta > 0$. For all $q \in M$, $C_q \cap A$ contains an arc of length $\geq \pi + \alpha$ with midpoint $N_1(q)$ for some uniform constant $\alpha > 0$. Indeed, if we let $g(q)$ be the supremum of lengths of all arcs in $C_q \cap A$ with midpoint $N_1(q)$, then $g\colon M \to \mathbf{R}$ is lower semicontinuous, ie, $\lim_{q \to q_0} g(q) \geq g(q_0)$, since A is open. Thus, since $g > \pi$ and M is compact, $g \geq \pi + \alpha$. Now choose $\delta > 0$ so small that the length ℓ of the spherical image of E_δ is $\leq \pi + \alpha$ (this is possible since $\ell \to \pi$ as $\delta \to 0$). Next, for $(q,t) \in M \times \mathbf{S}^1$, let $\widetilde{G}_F(q,t)$ be the normalized projection of $G_F(q,t)$ into $df(T_qM)^\perp$, ie,

$$\widetilde{G}_F(q,t) := \frac{\sum_{i=1}^2 \langle G_F(q,t), N_i(q)\rangle N_i(q)}{\sqrt{\sum_{i=1}^2 \langle G_F(q,t), N_i(q)\rangle^2}}.$$

Also, for fixed $t \in \mathbf{S}^1$, let $F_t(q) := F(q,t)$. Then, by the tubular neighborhood theorem, $F_t\colon M \to \mathbf{R}^{n+1}$ is a smooth immersion for small ε. Further, as $\varepsilon \to 0$, F_t converges to f with respect to the $\mathcal{C}^1$–topology. Thus, for each $q \in M$, the normal plane $dF_t(T_qM)^\perp$ (which contains $G_F(q,t)$) converges to $df(T_qM)^\perp$. Consequently G_F is well-defined for small ε, and converges to $\widetilde{G}_F$ as $\varepsilon \to 0$. So it suffices to check that $\widetilde{G}_F(M \times \mathbf{S}^1) \subset A$, which follows from our choice of δ. Indeed for each fixed $q \in M$, $\widetilde{G}_F(\{q\} \times \mathbf{S}^1)$ is the spherical image of the figure eight curve $\sum_{i=1}^2 E_\delta^i(t) N_i(q)$ in $df(T_qM)^\perp$, which is an arc of C_q with midpoint $N_1(q)$ and length $\leq \pi + \alpha$. □

4 Notes

4.1 It is well-known that $G_f(M) = \mathbf{S}^n$ for any embedding $f\colon M^n \to \mathbf{R}^{n+1}$ of a closed oriented n–manifold [7, page 187]. More generally, this also holds for "Alexandrov embeddings", ie, immersions $f\colon M \to \mathbf{R}^{n+1}$ which may be extended to an immersion $\overline{f}\colon \overline{M} \to \mathbf{R}^{n+1}$ of a compact $(n+1)$-manifold $\overline{M}$ with $\partial \overline{M} = M$. Indeed if v is any vector field along M which points "outward" with respect to $\overline{M}$, then for $p \in M$, the normalized projection of $df(v(p))$ into the line $df(T_pM)^\perp$ defines a normal vector field $M \to \mathbf{S}^n$ which coincides with G_f (after a reflection of G_f if necessary). Then, for any $u \in \mathbf{S}^n$, if p is a point which maximizes the height function $\langle \cdot, u \rangle$ on M, we have $G_f(p) = u$. On the other hand, being only regularly homotopic to an embedding, is not enough to ensure that $G_f(M) = \mathbf{S}^n$. Indeed the example in Figure 2 is regularly homotopic to an embedded torus of revolution by Pinkall [12].

4.2 If $G_f(M) \neq \mathbf{S}^n$ for an immersion $f\colon M^n \to \mathbf{R}^{n+1}$ of an oriented n–manifold, then, as is well-known (see Milnor [11]), M must be parallelizable. Here we include a brief geometric argument for this fact. If $(0,\dots,0,1) \notin G_f(M)$, we may define a continuous map $F\colon TM \to \mathbf{R}^n \simeq \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ as follows; cf [5, Lemma 2.2]. There is a continuous map $\rho\colon \mathbf{S}^n \setminus \{(0,\dots,0,1)\} \to \mathrm{SO}(n+1)$, $u \mapsto \rho_u$ such that $\rho_u(u) = (0,\dots,0,-1)$. Let $\pi\colon TM \to M$ be the canonical projection, and for $X \in TM$ set $F(X) := \rho_{G_f(\pi(X))}(df(X))$. Also let $F_p := F|_{T_pM}$. Then $\{F_p^{-1}(e_i)\}$, where $\{e_i\}$ is a fixed basis of $\mathbf{R}^n$, gives a framing for TM as desired. So in particular, when M is closed and $n=2$, we have $M = \mathbf{T}^2$. The last observation also follows from Gauss–Bonnet theorem via degree theory when f is $\mathcal{C}^2$; since if $G_f(M) \neq \mathbf{S}^2$, then

$$0 = \deg(G_f) = \frac{1}{4\pi}\int_M \det(dG_f) = \frac{1}{4\pi}\int_M K = \frac{1}{2}\chi(M),$$

where K is the Gaussian curvature and χ is the Euler characteristic.

4.3 To generate some concrete examples of the immersions $f\colon M^{n-1} \to \mathbf{R}^n \simeq \mathbf{R}^n \times \{0\}$ in Theorem 1.3, note that if $f_0\colon M_0^{n-k-1} \to \mathbf{R}^{n-k} \times \{0\}$ is any immersion such that $f_0(M_0)$ is disjoint from the subspace $L := \mathbf{R}^{n-k-1} \times \{(0,0)\}$, then spinning f_0 about L yields an immersion $f_1\colon M_0 \times \mathbf{S}^1 \to \mathbf{R}^{n-k+1}$ given by

$$f_1(q,t) := \left[\begin{array}{c|cc} \mathrm{I} & \multicolumn{2}{c}{0} \\ \hline 0 & \cos(t) & \sin(t) \\ & -\sin(t) & \cos(t) \end{array}\right] \begin{bmatrix} f_0^1(q) \\ \vdots \\ f_0^{n-k}(q) \\ 0 \end{bmatrix},$$

where f_0^i are the components of f_0. Thus, for instance, one may inductively construct immersions of $\mathbf{S}^{n-k-1} \times \mathbf{T}^k$ in $\mathbf{R}^n$, for $k = 1,\dots,n-1$. More generally, if $M^{n-1} \times \mathbf{S}^1$ is parallelizable, then so is the open manifold $M^{n-1} \times (0,1)$, which may be immersed in $\mathbf{R}^n$ [10] by the h-principle [7], or more specifically, the "holonomic approximation theorem" of Eliashberg and Mishachev [1; 5].

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