Directed immersions of closed manifolds

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Given any finite subset $X$ of the sphere $S^n$, $n \geq 2$, which includes no pairs of antipodal points, we explicitly construct smoothly immersed closed orientable hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ whose Gauss map misses $X$. In particular, this answers a question of M Gromov.

53A07, 53C42; 57R42, 58K15

1 Introduction

To every $C^1$ immersion $f: M^n \to \mathbb{R}^{n+1}$ of a closed oriented $n$–manifold $M$, there corresponds a unit normal vector field or Gauss map $G_f: M \to S^n$, which generates a set $G_f(M) \subset S^n$ known as the spherical image of $f$. Conversely, one may ask (see Gromov [8, page 3]): for which sets $A \subset S^n$ is there an immersion $f: M \to \mathbb{R}^{n+1}$ such that $G_f(M) \subset A$? Such a mapping would be called an $A$–directed immersion of $M$; see Eliashberg and Mishachev [1], Gromov [7], Rourke and Sanderson [13] and Spring [14]. It is well-known that when $A \neq S^n$, $f$ must have double points (Section 4.1), and $M$ must be parallelizable, eg, $M$ can only be the torus $T^2$ when $n = 2$ (Section 4.2). Furthermore, the only known necessary condition on $A$ is the elementary observation that $A \cup -A = S^2$, while there is also a sufficient condition due to Gromov [7, Theorem $(D')$, page 186]:

**Condition 1.1** $A \subset S^n$ is open, and there is a point $p \in S^n$ such that the intersection of $A$ with each great circle passing through $p$ includes a (closed) semicircle.

A great circle is the intersection of $S^n$ with a 2–dimensional subspace of $\mathbb{R}^{n+1}$. Note that, when $n \geq 2$, examples of sets $A \subset S^n$ satisfying the above condition include those which are the complement of a finite set of points without antipodal pairs. Thus the spherical image of a closed hypersurface can be remarkably flexible. Like most h-principle or convex integration type arguments, however, the proof does not yield specific examples. It is therefore natural to ask, for instance:

**Question 1.2** [7, page 186] Is there a “simple” immersion $T^2 \to \mathbb{R}^3$ whose spherical image misses the four vertices of a regular tetrahedron in $S^2$?

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Here we give an affirmative answer to this question (Section 2), and more generally present a short constructive proof of the sufficiency of a slightly stronger version of Condition 1.1 for the existence of $A$–directed immersions of parallelizable manifolds $M^{n-1} \times S^1$, where $M^{n-1}$ is closed and orientable. Any such manifold admits an immersion $f: M^{n-1} \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ (Section 4.3). We then extend $f$ to $M^{n-1} \times S^1$ by using the figure eight curve

$$E_\delta(t) := (\cos(t), \delta \sin(2t))$$

(1)

to put a copy of $S^1 \simeq \mathbb{R}/2\pi$ in each normal plane of $f$, as described below. Note that the midpoint of $G_{E_\delta}(S^1)$ is assumed to be at $(1,0)$; see Figure 1 which shows $E_{1/2}$ and its spherical image. Further, the unit normal bundle of $f$ may be naturally identified with the pencil of great circles of $S^n$ passing through $(0, \ldots, 0, 1)$.

![Figure 1](image_url)

Theorem 1.3  Let $A \subset S^n$ satisfy Condition 1.1 with respect to $p = (0, \ldots, 0, 1)$. Further, if $n \geq 3$, suppose that the semicircle in Condition 1.1 contains $p$, or that no great circle through $p$ is contained in $A$. Let $f: M^{n-1} \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ be a smooth ($C^\infty$) immersion of a closed orientable $(n-1)$–manifold, and, for every $q \in M$, let $C_q \subset S^n$ be the unit normal space of $f$ at $q$. Then there is a smooth orthogonal frame \{$N_i: M \to S^n$\}, $i = 1, 2$, for the normal bundle of $f$ such that the semicircle in $C_q$ centered at $N_1(q)$ lies in $A$. For any such frame, and sufficiently small $\varepsilon$, $\delta > 0$,

$$F(q, t) := f(q) + \varepsilon \sum_{i=1}^{2} E^i_\delta(t) N_i(q)$$

(2)

yields a smooth $A$–directed immersion $M \times S^1 \to \mathbb{R}^{n+1}$, where $E^i_\delta$ are the components of the figure eight curve $E_\delta$ given by (1).

2 Example

If \( A = S^2 \setminus X \) for a finite set \( X \) without antipodal pairs, we may always find a point \( p \in S^2 \) with respect to which \( A \) satisfies the hypothesis of Theorem 1.3 (eg, let \( p \notin X \) be in the complement of all great circles which pass through at least two points of \( X \) other than \(-p\)). After a rigid motion (which may be arbitrarily small) we may assume that \( p = (0, 0, 1) \) or \((0, 0, -1)\), and let \( f(\theta) := (\cos(\theta), \sin(\theta), 0) \) be the standard immersion of \( S^1 \cong \mathbb{R}/2\pi \) in \( \mathbb{R}^3 \). Then the desired framing for the normal bundle of \( f \) may always take the form

\[
N_1(\theta) := f'(\theta) \times N_2(\theta), \quad N_2(\theta) := \frac{(\cos(\theta), \sin(\theta), z(\theta))}{\sqrt{1 + z^2(\theta)}},
\]

where \( z: \mathbb{R}/2\pi \to \mathbb{R} \) is a smooth function with \( z(\theta) = -z(\theta + \pi) \) and such that \( X \) is contained entirely in one of the components of \( S^2 - N_2(S^1) \). For instance, when \( X \) is the vertices of a regular tetrahedron, we may set \( z(\theta) := \cos(3\theta) \) in (3). Then, for \( \varepsilon, \delta \leq 1/8 \), the mapping \( F(\theta, t) \) given by (2) yields an immersion \( T^2 \cong \mathbb{R}/2\pi \times \mathbb{R}/2\pi \to \mathbb{R}^3 \) which answers Question 1.2. The resulting surface, for \( \varepsilon = \delta = 1/8 \), is depicted in Figure 2 together with its spherical image (note that here \( p = (0, 0, -1) \)). To find \( z(\theta) \)

in general, we may order the points in \( X' \cup -X' \), where \( X' := X \setminus \{-p\} \), according to their “longitude” \( \theta \), and connect them by geodesic segments to obtain a simple closed symmetric curve \( \gamma(\theta) \). A perturbation of \( \gamma \) then yields a smooth symmetric curve \( \tilde{\gamma} \) such that \( X \) is contained in one of the components of \( S^2 - \tilde{\gamma}(S^1) \). The third coordinate of \( \tilde{\gamma} \) gives our desired height function \( z \).

3 Proof of Theorem 1.3

3.1 First we construct the frame \( \{N_i\} \). For every \( q \in M \), \( C_q \) is a great circle passing through \( p \). So it contains a semicircle in \( A \) by assumption (Condition 1.1). Let
$m_q \subset C_q$ be the set of midpoints of all such semicircles. We need to find a smooth map $N_1: M \to S^n$ such that $N_1(q) \in m_q$ for all $q \in M$. To this end note that $m_q$ is open and connected. Further, if $m_q$ contains any pairs of antipodal points, then $m_q = C_q$; otherwise, $m_q$ lies in the interior a semicircle of $C_q$. Consequently,

$$\text{Cone}(m_q) := \{ \lambda x \mid x \in m_q \text{ and } \lambda \geq 0 \},$$

is a convex set in $\mathbb{R}^{n+1}$. In particular, for any finite set of points $x_i \in \text{Cone}(m_q)$ and numbers $\lambda_i \geq 0$, $\sum_i \lambda_i x_i \in \text{Cone}(m_q)$. Now let $B$ be the set of all points $q \in M$ such that $m_q \neq C_q$. Then $B$ is closed (and therefore compact) since $M \setminus B$ is open; indeed the set of great circles contained in $A$ is open, since $A$ is open. Further, note that for any point $q \in M$, normal vector $x \in m_q$, and continuous local extension $v$ of $x$ to a normal vector field of $M$, we have $v(q') \in m_q$ for all $q'$ in an open neighborhood $U$ of $q$ (because the set of semicircles contained in $A$ is open). Let $\{v_i: U_i \to S^n\}$, $i = 1, \ldots, k$, be a finite collection of such local vector fields so that $\bigcup_i U_i$ covers $B$ and $v_i$ are smooth. Also let $\{\phi_i: M \to \mathbb{R}\}$ be a smooth partition of unity subordinate to $\{U_i\}$, and, for $q \in \bigcup_i U_i$, set

$$N_1(q) := \frac{\sum_{i=1}^k \phi_i(q)v_i(q)}{\|\sum_{i=1}^k \phi_i(q)v_i(q)\|}.$$

If $q \in B$, then $v_i(q) \in m_q$ which lies in the interior of a semicircle $S \subset C_q$, and so $\|\sum_{i=1}^k \phi_i(q)v_i(q)\| \neq 0$. Indeed, if $x$ is the midpoint of $S$, then

$$\langle \sum_{i=1}^k \phi_i(q)v_i(q), x \rangle = \sum_{i=1}^k \phi_i(q)\langle v_i(q), x \rangle > 0.$$

Thus $N_1$ is well defined (and smooth) on an open neighborhood $V$ of $B$. Further, $N_1(q) \in m_q$ for all $q \in V$, since $\text{Cone}(m_q)$ is convex. In particular we are done if $B = M$; otherwise, note that we may write

$$N_1(q) = \cos(\theta(q)) p + \sin(\theta(q)) G_f(q),$$

for some function $\theta: V \to \mathbb{R}$, since $G_f$ is well defined due to the orientability of $M$, and thus $\{p, G_f(q)\}$ forms an orthonormal basis for the normal plane $df(T_qM)^\perp$. Further, it is easy to see that we may choose $\theta$ continuously (and therefore smoothly) if $n = 2$. This also holds for $n > 2$ if each $C_q$ contains a semicircle passing through $p$; for then $\theta$ is uniquely determined within the range $[-\pi/2, \pi/2]$. Indeed, we may choose the vectors $v_i$ above so that $\langle v_i(q), p \rangle \geq 0$ which would in turn yield that $\langle N_1(q), p \rangle \geq 0$. Now let $V'$ be an open neighborhood of $B$ with closure $\overline{V'} \subset V$. Using Tietze’s theorem, followed by a perturbation and a gluing, we may extend $\theta|_{V'}$ smoothly to all of $M$. Then (4) yields the desired vector field on $M$, since for any
Also, for fixed $q \in M \setminus B$, $N_1(q) \in C_q = m_q$. Finally, set

$$N_2(q) := \sin(\theta(q)) p - \cos(\theta(q)) G_f(q).$$

### 3.2 It remains to show that $G_F(M \times S^1) \subset A$, for small $\varepsilon, \delta > 0$. For all $q \in M$, $C_q \cap A$ contains an arc of length $\geq \pi + \alpha$ with midpoint $N_1(q)$ for some uniform constant $\alpha > 0$. Indeed, if we let $g(q)$ be the supremum of lengths of all arcs in $C_q \cap A$ with midpoint $N_1(q)$, then $g: M \to \mathbb{R}$ is lower semicontinuous, ie, $\lim_{q \to q_0} g(q) \geq g(q_0)$, since $A$ is open. Thus, since $g > \pi$ and $M$ is compact, $g \geq \pi + \alpha$. Now choose $\delta > 0$ so small that the length $\ell$ of the spherical image of $E_\delta$ is $\leq \pi + \alpha$ (this is possible since $\ell \to \pi$ as $\delta \to 0$). Next, for $(q, t) \in M \times S^1$, let $\widetilde{G}_F(q, t)$ be the normalized projection of $G_F(q, t)$ into $df(T_q M)^\perp$, ie,

$$\widetilde{G}_F(q, t) := \frac{\sum_{i=1}^{2} \langle G_F(q, t), N_i(q) \rangle N_i(q)}{\sqrt{\sum_{i=1}^{2} \langle G_F(q, t), N_i(q) \rangle^2}}.$$

Also, for fixed $t \in S^1$, let $F_t(q) := F(q, t)$. Then, by the tubular neighborhood theorem, $F_t: M \to \mathbb{R}^{n+1}$ is a smooth immersion for small $\varepsilon$. Further, as $\varepsilon \to 0$, $F_t$ converges to $f$ with respect to the $C^1$--topology. Thus, for each $q \in M$, the normal plane $dF_t(T_q M)^\perp$ (which contains $G_F(q, t)$) converges to $df(T_q M)^\perp$. Consequently $G_F$ is well-defined for small $\varepsilon$, and converges to $\widetilde{G}_F$ as $\varepsilon \to 0$. So it suffices to check that $\widetilde{G}_F(M \times S^1) \subset A$, which follows from our choice of $\delta$. Indeed for each fixed $q \in M$, $\widetilde{G}_F(q \times S^1)$ is the spherical image of the figure eight curve $\sum_{i=1}^{2} E_\delta^i(t) N_i(q)$ in $df(T_q M)^\perp$, which is an arc of $C_q$ with midpoint $N_1(q)$ and length $\leq \pi + \alpha$. \qed

### 4 Notes

#### 4.1 It is well-known that $G_f(M) = S^n$ for any embedding $f: M^n \to \mathbb{R}^{n+1}$ of a closed oriented $n$--manifold [7, page 187]. More generally, this also holds for “Alexandrov embeddings”, ie, immersions $f: M \to \mathbb{R}^{n+1}$ which may be extended to an immersion $\tilde{f}: \tilde{M} \to \mathbb{R}^{n+1}$ of a compact $(n + 1)$-manifold $\tilde{M}$ with $\partial \tilde{M} = M$. Indeed if $v$ is any vector field along $M$ which points “outward” with respect to $\tilde{M}$, then for $p \in M$, the normalized projection of $df(v(p))$ into the line $df(T_p M)^\perp$ defines a normal vector field $M \to S^n$ which coincides with $G_f$ (after a reflection of $G_f$ if necessary). Then, for any $u \in S^n$, if $p$ is a point which maximizes the height function $\langle \cdot, u \rangle$ on $M$, we have $G_f(p) = u$. On the other hand, being only regularly homotopic to an embedding, is not enough to ensure that $G_f(M) = S^n$. Indeed the example in Figure 2 is regularly homotopic to an embedded torus of revolution by Pinkall [12].

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4.2 If $G_f(M) \neq S^n$ for an immersion $f: M^n \to \mathbb{R}^{n+1}$ of an oriented $n$–manifold, then, as is well-known (see Milnor [11]), $M$ must be parallelizable. Here we include a brief geometric argument for this fact. If $\{0, \ldots, 0, 1\} \not\subset G_f(M)$, we may define a continuous map $F: TM \to \mathbb{R}^n \simeq \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ as follows; cf [5, Lemma 2.2]. There is a continuous map $\rho: S^n \sim \{0, \ldots, 0, 1\} \to SO(n+1), u \mapsto \rho_u$ such that $\rho_u(u) = (0, \ldots, 0, -1)$. Let $\pi: TM \to M$ be the canonical projection, and for $X \in TM$ set $F(X) := \rho_{G_f(\pi(X))}(df(X))$. Also let $F_p := F|_{T_p M}$. Then $\{F_p^{-1}(e_i)\}$, where $\{e_i\}$ is a fixed basis of $\mathbb{R}^n$, gives a framing for $TM$ as desired. So in particular, when $M$ is closed and $n = 2$, we have $M = T^2$. The last observation also follows from Gauss–Bonnet theorem via degree theory when $f$ is $C^2$; since if $G_f(M) \neq S^2$, then

$$0 = \deg(G_f) = \frac{1}{4\pi} \int_M \det(df_G) = \frac{1}{4\pi} \int_M K = \frac{1}{2} \chi(M),$$

where $K$ is the Gaussian curvature and $\chi$ is the Euler characteristic.

4.3 To generate some concrete examples of the immersions $f: M^{n-1} \to \mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ in Theorem 1.3, note that if $f_0: M_0^{n-k-1} \to \mathbb{R}^{n-k} \times \{0\}$ is any immersion such that $f_0(M_0)$ is disjoint from the subspace $L := \mathbb{R}^{n-k-1} \times \{(0, 0)\}$, then spinning $f_0$ about $L$ yields an immersion $f_1: M_0 \times S^1 \to \mathbb{R}^{n-k+1}$ given by

$$f_1(q,t) := \begin{bmatrix} 1 & 0 \\ 0 & \cos(t) & \sin(t) \\ & -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} f_0^1(q) \\ \vdots \\ f_0^{n-k}(q) \end{bmatrix},$$

where $f_0^j$ are the components of $f_0$. Thus, for instance, one may inductively construct immersions of $S^{n-k-1} \times T^k$ in $\mathbb{R}^n$, for $k = 1, \ldots, n-1$. More generally, if $M^{n-1} \times S^1$ is parallelizable, then so is the open manifold $M^{n-1} \times (0, 1)$, which may be immersed in $\mathbb{R}^n$ [10] by the h-principle [7], or more specifically, the “holonomic approximation theorem" of Eliashberg and Mishachev [1; 5].

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