Isosystolic genus three surfaces
critical for slow metric variations

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We show that the two piecewise flat surfaces with conical singularities conjectured by E Calabi as extremal surfaces for the isosystolic problem in genus 3 are critical with respect to some metric variations. The proof relies on a new approach to study isosystolic extremal surfaces.

1 Introduction

Let M be a non–simply connected closed surface. The systole of a Riemannian metric g on M, denoted by $\mathrm{sys}(g)$, is defined as the infimum of the lengths of the noncontractible loops of M. Define the systolic area of (M, g) as

$$\sigma(g) = \frac{\mathrm{area}(g)}{\mathrm{sys}(g)^2}.$$

The systolic area of a non–simply connected closed Riemannian surface (M, g) is bounded from below by $2/\pi$, where equality holds if and only if (M, g) is isometric to a round projective plane; see Pu [24] and Gromov [15].

Extremal metrics, ie metrics realizing the minimum of the systolic area for a surface of given topological type, are known in only three cases. Around 1949, C Loewner proved that extremal metrics on the torus are flat hexagonal; see Katz [19]. Then, P Pu [24] showed that extremal metrics on the projective plane are round. In comparison, C Bavard [4] showed that extremal metrics on the Klein bottle, formed of two round Möbius bands glued together, are not smooth and present a line of singularities.

M Gromov [15] established the existence of extremal metrics in the setting of "generalized metrics". A general study of extremal surfaces has been conducted by E Calabi [11] and pursued by R Bryant [8]. Loosely speaking they showed that extremal metrics tend to have flat regions. E Calabi [11] also described two piecewise flat metrics with conical singularities on the genus three surface extremal in their conformal classes. He

conjectured[1] that these two metrics represent local minima of the systolic area and that one of them corresponds to the global minimum.

The goal of this article is to show that these two piecewise flat metrics are critical for the systolic area with respect to some slow metric variations defined below.

Since the systolic area functional $g \mapsto \sigma(g)$ is not necessarily differentiable, we need to extend the notion of critical point. A similar extension was required by N Nadi-rashvili [23] and A El Soufi and S Ilias [14] to study Riemannian surfaces maximizing the product $\lambda_1(g) \cdot \mathrm{area}(g)$, where $\lambda_1(g)$ is the first nonzero eigenvalue of the Laplacian. By analogy with the notion introduced by these authors, we say that a metric g_0 on M is critical for the systolic area with respect to a metric variation $(g_t)_{t \geq 0}$ if

$$\sigma(g_t) \geq \sigma(g_0) + o(t).$$

We will also need the following.

Definition 1.1 A *slow metric variation* $(g_t)_{t \geq 0}$ is a one-parameter family of Riemannian metrics g_t with conical singularities on M (see Troyanov [27] or Section 12.1) such that the differential of the Busemann function $\beta_\xi^{g_t}$ in the direction ξ for the metric g_t satisfies

(1-1) $\|(d\beta_\xi^{g_t} - d\beta_{-\xi}^{g_t}) - (d\beta_\xi^{g_0} - d\beta_{-\xi}^{g_0})\|_{L^2(\widehat{B}_\xi)} = o(\sqrt{t})$

where $\widehat{B}_\xi$ is a fundamental domain of the systolic band B_ξ in the universal cover of (M, g_0).

The directions ξ involved in this definition correspond to the systolic bands B_ξ of the Calabi surface considered. We refer to Definitions 3.1 and 3.2 for precise definitions of the directions ξ and the systolic bands B_ξ (which rest on the description of the Calabi surfaces presented in Section 2). We also refer to Section 8 for a definition of Busemann functions.

The condition (1-1) is clearly satisfied if the L^2–norm of $d\beta_\xi^{g_t} - d\beta_\xi^{g_0}$ on $\widehat{B}_\xi$ is equal to $o(\sqrt{t})$. We could also use the L^1–norm in the condition (1-1).

Note that if the differentials of the Busemann functions vary smoothly, or even Lipschitz, with respect to t, then the metric variation $(g_t)_{t \geq 0}$ is slow. Actually, the first part of the article does not rely on metric variations, it only occurs in the last two sections.

Examples of slow metric variations are presented in the last section.

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[1]In a private discussion, he told me that he was less confident now than at the time of the writing of [11] that the metric with the higher systolic area represents a local minimum, though he still believes that the other metric corresponds to the global minimum.

Let (M_0, g_0) be one of the two piecewise flat surfaces of genus three with conical singularities conjectured extremal by E Calabi in [11]; see Section 2 for a description. Note that these two singular surfaces are nonpositively curved in the sense of Alexandrov.

We will deal with normalized metrics, that is, metrics with a systole equal to one.

Theorem 1.2 *The metric g_0 on M_0 is critical for the systolic area with respect to any normalized slow metric variation. That is,*

$$\sigma(g_t) \geq \sigma(g_0) + o(t)$$

for every normalized slow metric variation $(g_t)_{t \geq 0}$ of the piecewise flat metric g_0.

Of course, one can ask whether the assumption on the metric variation is always satisfied or can be dropped. To our knowledge, this is still an open question. Actually, it is possible that any metric variation is slow, and that the restriction on the type of metric variations can be relaxed. In any case, this result shows that the two Calabi surfaces have special isosystolic properties. We do not know any other surface of genus greater than one which is critical with respect to slow metric variations.

In spite of the restriction on the metric variations considered, we believe that the point of view developed in this article could be useful in the study of extremal isosystolic properties on surfaces. We present the general framework of the proof below.

The techniques used in this article fit in this contemporary approach of Riemannian geometry which consists of isometrically embedding a Riemannian manifold M into a space of large dimension, such as a Banach or Hilbert space, through the geometry of M. This point of view allows us to study the intrinsic geometry of M through the study of its isometric embedding. This approach was developed by Gromov [15], Besson, Courtois and Gallot [7], Ivanov [18] and Burago and Ivanov [9; 10], for instance, to obtain deep results on the geometry of Riemannian manifolds. We borrow several features from these pioneer works.

Thus, contrary to classical techniques available to study extremal isosystolic surfaces (cf Katz [19], Pu [24], Bavard [4; 5; 6], Katz and Sabourau [22] and Horowitz, Usadi Katz and Katz [17], our proof does not rest on the conformal length method nor does it make use of the uniformization theorem. Instead we embed the universal cover of a Riemannian surface M into an infinite dimensional space in an equivariant way using the geometry of the surface. This infinite dimensional space carries some special Euclidean metrics and two-forms along certain directions. These metrics and two-forms are modeled on a given Calabi surface M_0 so that its embedding into this

infinite dimensional space preserves some isometric properties. Here, the combinatorics of the systolic loops of M_0 plays an essential role. Patching the pullbacks of these metrics and two-forms by the embedding of M, we construct an auxiliary metric and a two-form on the competing surface M. A comparison of the area measures of the different metrics with the area measure of the two-form shows that it is enough to study the first variation of the integral of the two-form associated to the embedding of M. In some sense, the proof of the main theorem boils down to an infinitesimal calibration argument and the two-form considered above can be thought of as an "infinitesimally calibrating" form. The restriction on the metric variation occurs only in the final step to differentiate the integral of the infinitesimally calibrating form.

It seems that a local calibration argument could yield a local lower bound on the systolic area of the Calabi surfaces and not merely an infinitesimal one. However the presence of conical singularities on the Calabi surfaces makes the construction of a form which locally calibrates the embedding tricky. Let us emphasize that no form defined globally calibrates the embedding of the first Calabi surface since this surface is not a global minimum of the systolic area.

We also carried out a similar variational study directly on the auxiliary metric without using the infinitesimally calibrating form but the result turned out to be the same. We will not present this study. Instead we decided to develop the infinitesimally calibrating form approach as it lends itself to estimates for Finsler metrics too; cf Ivanov [18] and Burago and Ivanov [9]. In this case, we have to replace the Riemannian metric with an hexagonal metric as in Calabi [11].

In contrast with the genus three surface, we showed in [25] that extremal metrics on the genus two surface are not piecewise flat with conical singularities. In a joint work [22] with M Katz, we found the infimum of the systolic area on nonpositively curved genus two surfaces and described the extremal metrics. These extremal metrics are piecewise flat with conical singularities but are not extremal for the general isosystolic problem. This shows that extremal metrics for the general isosystolic problem on the genus two surface have regions with positive curvature and regions with negative curvature. To be complete with the genus two case, let us mention that the best available systolic inequality on this surface can be found in [21]. It is proved that every metric on the genus two surface has a systolic area greater or equal to the systolic area of the extremal metric on the torus. The same result holds for surfaces of genus at least 20 [20].

Locally extremal and critical metrics have recently been described on the two-sphere by F Balacheff [2; 3] and the author [26] and on compact rank-one symmetric spaces by J-C Álvarez Paiva and F Balacheff [1] for an extended notion of systolic volume.

We refer to the survey by C Croke and M Katz [12] and the monograph of M Katz [19] for an overview on other aspects of systolic geometry, especially on the notion of stable systole for which extremal metrics have been described in higher dimension.

2 Description of critical surfaces

In this section, we describe the two piecewise flat surfaces of genus three with conical singularities introduced by E. Calabi and arising in Theorem 1.2. This presentation follows the description of [11].

2.1 Piecewise flat surface modeled on Fermat's quartic

The first surface is conformal to the Fermat quartic, defined in the complex projective plane $\mathbb{C}P^2$ by the homogeneous polynomial equation

$$x^4 + y^4 + z^4 = 0.$$

It is modeled on the triangular hyperbolic surface representing the Fermat quartic. This triangular hyperbolic surface is tiled by 32 equilateral hyperbolic triangles with angles equal to $\pi/4$ and has 12 vertices with 8 equilateral triangles around each of them. These 12 vertices correspond to the points adjoined in the compactification of the quartic, that is to the points (x, y, z) where one of the coordinates is zero and the quotient of the other two is a primitive eighth root of unity.

Replacing the hyperbolic equilateral triangles by Euclidean equilateral triangles gives rise to a piecewise flat surface $M_0 = (M_0, g_0)$ with 12 conical singularities of angle $8\pi/3$, normalized to have a systole equal to one; see Figure 1. The orientation-preserving isometry group of M_0 is a group of order 96. Denote by $\mathcal{T}$ the collection of the equilateral flat triangles tiling M_0.

Every geodesic in the interior of one of the triangles of $\mathcal{T}$ and parallel to any side of this triangle extends as a geodesic and closes after passing through 6 triangles. These closed geodesics are systolic loops of M_0.

The systolic area of M_0 is equal to
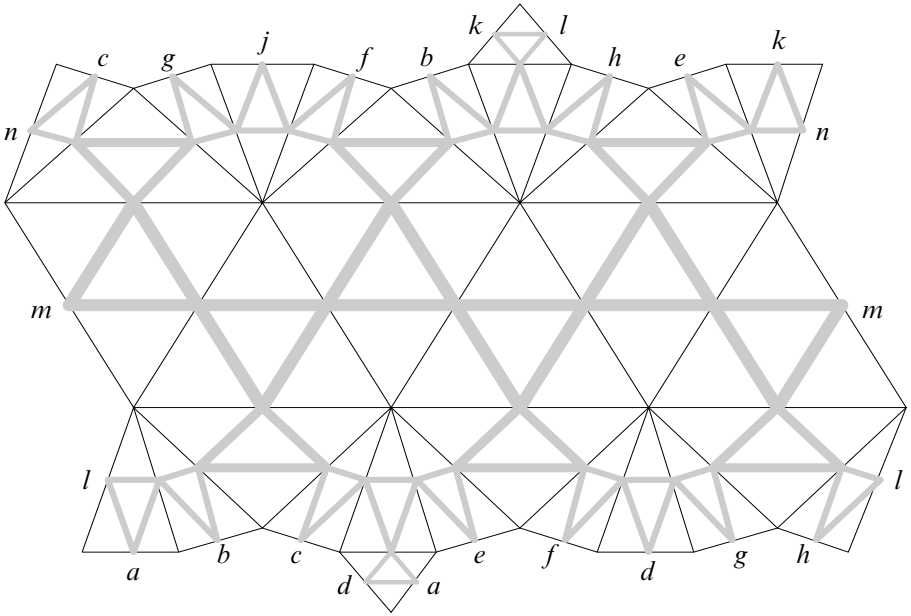
$$\sigma(M_0) = \frac{8\sqrt{3}}{9} \simeq 1.5396.$$

Figure 1

The piecewise flat surface M_0 can also be described as the polygonal domain of Figure 1 where the sides are identified (preserving the orientation) according to the letters. The 32 equilateral flat triangles of $\mathcal{T}$ are outlined in thin black lines, while the gray stripes, crossing each other pairwise through the midpoint of each edge represent the central portions of the bands foliated by the systolic loops on M_0. These systolic loops cover the surface M_0 and induce 32 free homotopy classes. Furthermore, exactly six of them (with the orientation taken into account) pass through every point lying in the interior of a triangle of $\mathcal{T}$.

Note that the conformal class of M_0 is given by a ramified 4–fold covering of the octahedron.

2.2 Piecewise flat surface modeled on Klein's quartic

The second surface is conformal to the Klein quartic, defined in the complex projective plane $\mathbb{C}P^2$ by the homogeneous polynomial equation

$$x^3 y + y^3 z + z^3 x = 0.$$

The hyperbolic surface representing the Klein quartic is tiled by 56 equilateral hyperbolic triangles with angles equal to $2\pi/7$ and has 24 vertices with 7 equilateral

triangles around each of them. These 24 vertices correspond to the points adjoined in the compactification of the quartic.

As before, replacing the hyperbolic equilateral triangles by Euclidean equilateral triangles gives rise to a piecewise flat surface $M_0 = (M_0, g_0)$ with 24 conical singularities of angle $7\pi/3$, normalized to have a systole equal to one; see Figure 2 for a representation. The orientation-preserving isometry group of M_0 is isometric to the group $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 168. Denote by $\mathcal{T}$ the collection of the equilateral flat triangles tiling M_0.
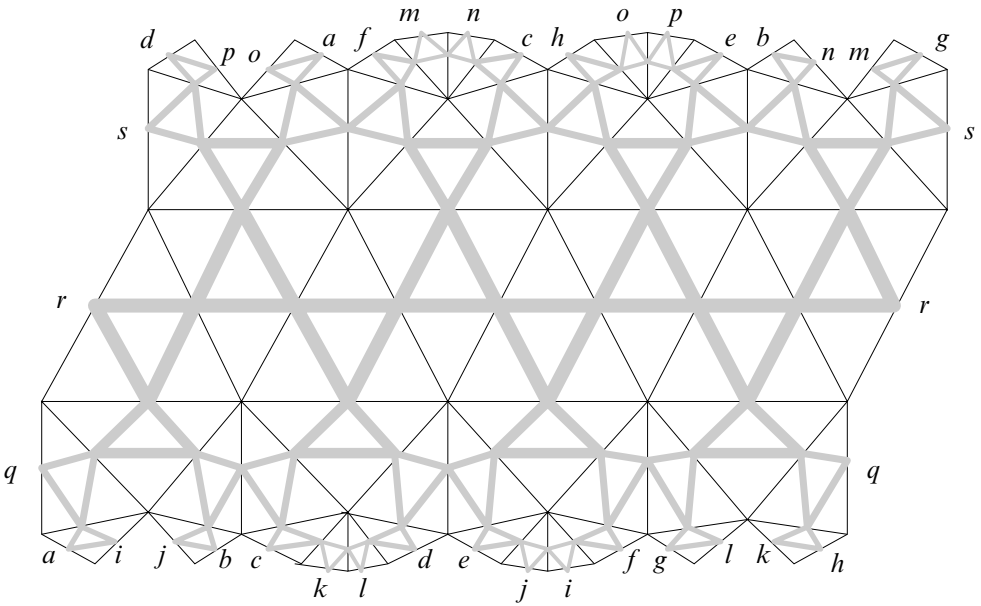
Figure 2

As previously, the systole is achieved by the length of the geodesics parallel to the edges of the triangles of $\mathcal{T}$, which close after passing through 8 triangles. The systolic loops on M_0 cover the surface and induce 56 free homotopy classes. As before, exactly six of them (with the orientation taken into account) pass through every point lying in the interior of a triangle of $\mathcal{T}$.

The systolic area of M_0 is equal to

$$\sigma(M_0) = \frac{7\sqrt{3}}{8} \simeq 1.5155.$$

2.3 Comparison with other surfaces

It might be instructive to compare the systolic areas of the two Calabi surfaces with the ones of surfaces of different topological types.

Given a non–simply connected closed surface Σ, define $\mathfrak{S}(\Sigma)$ as the infimum of $\sigma(g)$ over all the Riemannian metrics g on Σ.

The exact value of $\mathfrak{S}$ is known only for the projective plane $\mathbb{R}P^2$ [24], the torus $\mathbb{T}^2$ [19] and the Klein bottle $\mathbb{K}^2$ [4]:

$$\mathfrak{S}(\mathbb{R}P^2) = \frac{2}{\pi} \simeq 0.6366$$

$$\mathfrak{S}(\mathbb{T}^2) = \frac{\sqrt{3}}{2} \simeq 0.8660$$

$$\mathfrak{S}(\mathbb{K}^2) = \frac{2\sqrt{2}}{\pi} \simeq 0.9003.$$

In general, we only have

$$\frac{2}{\pi} \leq \mathfrak{S}(\Sigma)$$

while for the genus γ surfaces Σ_γ the following estimates hold:

$$\mathfrak{S}(\mathbb{T}^2) \leq \mathfrak{S}(\Sigma_2) \leq 3(\sqrt{2} - 1) \simeq 1.2426$$

$$\mathfrak{S}(\mathbb{T}^2) \leq \mathfrak{S}(\Sigma_\gamma) \text{ for every } \gamma \geq 20$$

(see [24; 15; 21; 22; 20]). Even for the genus three surface, we do not know whether $\mathfrak{S}(\mathbb{T}^2) \leq \mathfrak{S}(\Sigma_3)$.

3 Systolic directions and admissible maps

Let us introduce some notation which will be used throughout this article.

Given a genus three surface M, denote by $\widetilde{M}$ its universal covering. Denote also by Γ the deck transformation group of the universal covering $\widetilde{M}$ of M, that is $M \simeq \widetilde{M}/\Gamma$. The deck transformation group Γ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ isomorphic to the fundamental group $\pi_1(M)$ of M.

Let $M_0 = (M_0, g_0)$ be one of the two piecewise flat Calabi surfaces described in Section 2. Recall that $\mathrm{sys}(g_0) = 1$. The metric g_0 on M_0 lifts to a metric $\widetilde{g}_0$ on $\widetilde{M}_0$. Similarly, the natural triangulation $\mathcal{T}$ of M_0 lifts to a triangulation $\widetilde{\mathcal{T}}$ on $\widetilde{M}$. Fix an orientation on M_0. This orientation induces an orientation on the triangles $\widetilde{\Delta}$ of $\widetilde{\mathcal{T}}$ and on their boundaries $\partial\widetilde{\Delta}$.

Definition 3.1 A *systolic direction* of $\widetilde{M}_0$ is a $\widetilde{g}_0$–unit vector of $\widetilde{M}_0$ based at the midpoint of a height of an oriented triangle $\widetilde{\Delta}$ of $\mathcal{T}$, parallel to an edge of $\widetilde{\Delta}$ and pointing in the same direction as this edge. The set of systolic directions, denoted by $\mathcal{S}_0$, depends on the choice of the surface M_0 and its orientation. Set $\mathcal{S}_0^* = \{\xi \mid \xi \in \mathcal{S}_0\} \cup \{-\xi \mid \xi \in \mathcal{S}_0\}$.

A vector $\xi \in \mathcal{S}_0^*$ induces a geodesic line c_ξ on $\widetilde{M}_0$ such that $c_\xi'(0) = \xi$. It also defines a unique element α_ξ of $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, leaving c_ξ globally invariant, which takes $c_\xi'(0)$ to the vector $c_\xi'(1)$ (recall that $\mathrm{sys}(g_0) = 1$). Note that the line c_ξ projects onto a systolic loop of M_0 representing α_ξ.

Two vectors ξ and ξ' of $\mathcal{S}_0^*$ *point in the same direction* if they generate the same oriented geodesic line up to some time shift. That is,

$$c_{\xi'}(t) = c_\xi(t + \tau)$$

for some $\tau \in \mathbb{R}$ and every $t \in \mathbb{R}$.

Definition 3.2 The *systolic band* $\widetilde{B}_\xi$ of M_0 induced by a systolic direction ξ of $\widetilde{M}_0$ is the union of all triangles of $\mathcal{T}$ through which c_ξ passes. The projection B_ξ of $\widetilde{B}_\xi$ to M_0, which agrees with the quotient of $\widetilde{B}_\xi$ by $\langle \alpha_\xi \rangle$, is also called a *systolic band* of M_0.

Definition 3.3 Let us define the infinite product

$$\mathbb{R}^\infty = \prod_{\widetilde{\Delta} \in \mathcal{T}} \mathbb{R}^3_{\widetilde{\Delta}}$$

where $\mathbb{R}^3_{\widetilde{\Delta}}$ is a copy of $\mathbb{R}^3$. The space $\mathbb{R}^\infty$ is endowed with the product topology.

For every simplex $\widetilde{\Delta}$ of $\mathcal{T}$, number once and for all the systolic directions ξ_1, ξ_2, ξ_3 of $\widetilde{M}_0$ with basepoints in $\widetilde{\Delta}$ such that the angles between ξ_i and ξ_{i+1} are equal to $2\pi/3$ (all the indices are taken modulo 3).

Every systolic direction $\xi \in \mathcal{S}_0$ agrees with one of the three systolic directions ξ_i with basepoints in the same triangle $\widetilde{\Delta}_\xi$ of $\mathcal{T}$ containing the basepoint of ξ.

The ξ–*coordinate* of an element z of $\mathbb{R}^\infty$, denoted by z_ξ, is defined as the i–th coordinate of the canonical projection of z in $\mathbb{R}^3_{\widetilde{\Delta}_\xi}$.

The action of Γ on $\widetilde{M}_0$ induces an action on the (oriented) simplices $\widetilde{\Delta}$ of $\mathcal{T}$, on the systolic directions ξ of $\widetilde{M}_0$ and on the vectors of $\mathcal{S}_0^*$. It also defines an action on $\mathbb{R}^\infty$ given by

$$\gamma \cdot (z_\xi) = (z_{\gamma \cdot \xi})$$

for every $\gamma \in \Gamma$ and $z = (z_\xi) \in \mathbb{R}^\infty$. Clearly,

$$c_{\gamma.\xi} = \gamma \cdot c_\xi$$
$$\alpha_{\gamma.\xi} = \gamma \cdot \alpha_\xi \cdot \gamma^{-1}$$

for every $\xi \in \mathcal{S}_0^*$ and $\gamma \in \Gamma$.

Throughout this article, we will consider admissible maps defined as follows.

Definition 3.4 A continuous map $\Phi \colon \widetilde{M} \to \mathbb{R}^\infty$ is *admissible* if for every systolic direction $\xi \in \mathcal{S}_0$ the following conditions are satisfied:

(1) The map $\Phi_\xi \colon \widetilde{M} \to \mathbb{R}$ defined as the ξ–coordinate map of Φ, ie $\Phi_\xi = \Phi(\,.\,)_\xi$, is differentiable almost everywhere on $\widetilde{M}$.

(2) The one-forms $d\Phi_\xi$ and $d\Phi_{\xi'}$ defined almost everywhere on $\widetilde{M}$ agree whenever the systolic directions ξ and ξ' point in the same direction. That is,

$$d\Phi_{\xi'} = d\Phi_\xi.$$

(3) The one-form $d\Phi_\xi$ defined almost everywhere on $\widetilde{M}$ satisfies the following equivariance property

$$\gamma^*(d\Phi_{\gamma.\xi}) = d\Phi_\xi$$

for every $\gamma \in \Gamma$.

(4) The map Φ_ξ passes to the quotient and induces a map

$$\overline{\Phi_\xi} \colon \widetilde{M}/\langle \alpha_\xi \rangle \to \mathbb{R}/\mathbb{Z}$$

where $\langle \alpha_\xi \rangle$ is the subgroup of Γ generated by α_ξ; cf Definition 3.1.

4 Riemannian metrics induced by admissible maps

Let $\widetilde{\Delta}$ be a simplex of $\mathcal{T}$, and ξ_1, ξ_2 and ξ_3 be the three systolic directions with basepoints in $\widetilde{\Delta}$; cf Definition 3.3. Recall that the angles between ξ_i and ξ_{i+1} are equal to $2\pi/3$ (all the indices are taken modulo 3).

Define a linear projection $\pi_{\widetilde{\Delta}}$ as

$$\pi_{\widetilde{\Delta}} \colon \mathbb{R}^\infty \to \mathbb{R}^3_{\widetilde{\Delta}} \simeq \mathbb{R}^3$$
$$z \mapsto (z_{\xi_1}, z_{\xi_2}, z_{\xi_3}).$$

Let $\Phi \colon \widetilde{M} \to \mathbb{R}^\infty$ be an admissible map. Denote by Φ_i the ξ_i–coordinate map Φ_{ξ_i} of Φ. Even though Φ is not necessarily differentiable (no norm has even been set on $\mathbb{R}^\infty$), the composite $\pi_{\widetilde{\Delta}} \circ \Phi$, which sends x to $(\Phi_i(x))_{i=1}^3$, is differentiable almost everywhere on $\widetilde{M}$. This allows us to define a metric g_Φ on M as follows.

Definition 4.1 Let $\widetilde{\Delta} \in \mathcal{T}$. Denote by $\widetilde{g}_\Phi$ the pullback on $\widetilde{\Delta}$ of the standard Euclidean metric g_E of $\mathbb{R}^3$ by $\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}}$. That is,

$$\widetilde{g}_\Phi = (\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}})^*(g_E)$$

where $\Phi|_{\widetilde{\Delta}}$ is the restriction of Φ to $\widetilde{\Delta}$. Strictly speaking, the quadratic form $\widetilde{g}_\Phi$, which is only defined almost everywhere on $\widetilde{\Delta}$, can be degenerate (as the differential of $\pi_{\widetilde{\Delta}} \circ \Phi$). In spite of that, we will still refer to $\widetilde{g}_\Phi$ as a metric. Putting together the metrics $\widetilde{g}_\Phi$ on each simplex $\widetilde{\Delta}$ of $\mathcal{T}$ gives rise to a metric on $\widetilde{M}$ still denoted by $\widetilde{g}_\Phi$.

Note that the metric $\widetilde{g}_\Phi$ is not well defined on the one-skeleton of the triangulation $\mathcal{T}$ since the metrics defined on each simplex of $\mathcal{T}$ do not necessarily agree on their common edges.

Definition 4.2 Two maps $\varphi, \psi \colon X \to \mathbb{R}^3$ are said to *agree up to a coordinate permutation* of $\mathbb{R}^3$ if there exists a permutation σ of $\{1, 2, 3\}$ such that for every $x \in X$ and $i \in \{1, 2, 3\}$, the coordinates $\varphi(x)_i$ and $\psi(x)_i$ of $\varphi(x)$ and $\psi(x)$ satisfy $\varphi(x)_{\sigma(i)} = \psi(x)_i$. In this case, we write $\varphi \simeq \psi$. Note that the permutation σ does not depend on x.

This definition extends to vectors of $\mathbb{R}^3$ viewed as constant maps in $\mathbb{R}^3$.

Let us state a result that will be used to establish some invariance properties a couple of times in this article.

Lemma 4.3 Let $\widetilde{\Delta} \in \mathcal{T}$ and $\gamma \in \Gamma$. Then the two maps

$$\gamma^*[d(\pi_{\gamma.\widetilde{\Delta}} \circ \Phi|_{\gamma.\widetilde{\Delta}})] \quad \text{and} \quad d(\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}})$$

agree almost everywhere on Δ up to a coordinate permutation of $\mathbb{R}^3$. That is,

$$\gamma^*[d(\pi_{\gamma.\widetilde{\Delta}} \circ \Phi|_{\gamma.\widetilde{\Delta}})] \simeq d(\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}}).$$

Proof Let $(\xi_i)_{i=1}^3$ and $(\xi_i')_{i=1}^3$ be the three systolic directions of $\widetilde{M}_0$ with basepoints in $\widetilde{\Delta}$ and $\gamma.\widetilde{\Delta}$; cf Definition 3.3. Since γ is an orientation preserving isometry of $\widetilde{M}_0$, it preserves the systolic directions of $\widetilde{M}_0$. Hence, $\gamma.(\xi_i)_{i=1}^3 \simeq (\xi_i')_{i=1}^3$. Thus,

$$\pi_{\gamma.\widetilde{\Delta}} \circ \Phi|_{\gamma.\widetilde{\Delta}} = (\Phi_{\xi_i'}|_{\gamma.\widetilde{\Delta}})_{i=1}^3$$
$$\simeq (\Phi_{\gamma.\xi_i}|_{\gamma.\widetilde{\Delta}})_{i=1}^3.$$

Differentiating this relation yields

$$d(\pi_{\gamma.\widetilde{\Delta}} \circ \Phi|_{\gamma.\widetilde{\Delta}}) \simeq (d\Phi_{\gamma.\xi_i}|_{\gamma.\widetilde{\Delta}})_{i=1}^3.$$

From Definition 3.4 (3), we derive

$$\gamma^*(d\Phi_{\gamma.\xi_i}|_{\gamma.\widetilde{\Delta}})_{i=1}^3 = (d\Phi_{\xi_i}|_{\widetilde{\Delta}})_{i=1}^3$$
$$= d(\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}}).$$

Hence, $\gamma^*[d(\pi_{\gamma.\widetilde{\Delta}} \circ \Phi|_{\gamma.\widetilde{\Delta}})] \simeq d(\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}})$. □

The following result shows that the metric $\widetilde{g}_\Phi$ descends to M.

Lemma 4.4 *The metric $\widetilde{g}_\Phi$ is Γ–invariant on $\widetilde{M}$.*

Proof Let $\gamma \in \Gamma$ and $x \in \widetilde{M}$ such that $\widetilde{g}_\Phi$ and $\gamma^*\widetilde{g}_\Phi$ are defined at x. Denote by $\widetilde{\Delta}$ the simplex of $\mathcal{T}$ such that $x \in \widetilde{\Delta}$. We have

$$(\gamma^*\widetilde{g}_\Phi)|_x = [\gamma^*(\pi_{\gamma.\widetilde{\Delta}} \circ \Phi)^* g_E]|_x$$
$$= [\gamma^* d(\pi_{\gamma.\widetilde{\Delta}} \circ \Phi)]^*|_x \, g_E.$$

From Lemma 4.3 and since g_E is invariant by coordinate permutations, we derive

$$(\gamma^*\widetilde{g}_\Phi)|_x = d(\pi_{\widetilde{\Delta}} \circ \Phi)^*|_x \, g_E$$
$$= [(\pi_{\widetilde{\Delta}} \circ \Phi)^* g_E]|_x$$
$$= \widetilde{g}_\Phi|_x.$$

Hence, $\gamma^*\widetilde{g}_\Phi = \widetilde{g}_\Phi$. □

Definition 4.5 The metric induced by $\widetilde{g}_\Phi$ on the quotient M will be denoted by g_Φ.

5 Two-forms induced by admissible maps

Let $\Phi\colon \widetilde{M} \to \mathbb{R}^\infty$ be an admissible map. As in Section 4, we can define a two-form ω_Φ on M.

Definition 5.1 Assume $\widetilde{\Delta} \in \mathcal{T}$. Denote by $\widetilde{\omega}_\Phi$ the pullback on $\widetilde{\Delta}$ of the two-form $\sum_{i=1}^3 dx_i \wedge dx_{i+1}$ of $\mathbb{R}^3$ by $\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}}$ (all indices are taken modulo 3). That is,

$$\widetilde{\omega}_\Phi = (\pi_{\widetilde{\Delta}} \circ \Phi|_{\widetilde{\Delta}})^*\left(\sum_{i=1}^3 dx_i \wedge dx_{i+1}\right).$$

As previously, this two-form is only defined almost everywhere on $\widetilde{\Delta}$. Putting together the two-forms $\widetilde{\omega}_\Phi$ on each simplex $\widetilde{\Delta}$ gives rise to a two-form on $\widetilde{M}$ still denoted by $\widetilde{\omega}_\Phi$.

The following result shows that the two-form $\widetilde{\omega}_\Phi$ descends to M.

Lemma 5.2 *The one-form $\widetilde{\omega}_\Phi$ is Γ–invariant on $\widetilde{M}$.*

Proof The proof proceeds as in the proof of Lemma 4.4: simply replace $\widetilde{g}_\Phi$ by $\widetilde{\omega}_\Phi$ and g_E by $\sum_{i=1}^3 dx_i \wedge dx_{i+1}$. As previously, the result follows from Lemma 4.3 and the invariance of $\sum_{i=1}^3 dx_i \wedge dx_{i+1}$ by coordinate permutations. □

Definition 5.3 The two form induced by $\widetilde{\omega}_\Phi$ on the quotient M will be denoted by ω_Φ.

6 Comparison of the area forms dg_Φ and ω_Φ

Lemma 6.1 *Let $\Phi\colon \widetilde{M} \to \mathbb{R}^\infty$ be an admissible map. Then we have*

$$|\omega_\Phi| \le \sqrt{3}\, dg_\Phi$$

almost everywhere on M, where $|\omega_\Phi|$ is the area measure of ω_Φ on M and dg_Φ is the area measure of g_Φ on M.

Proof Let $\widetilde{\Delta} \in \mathcal{T}$. The two-form $\widetilde{\omega}_\Phi$ is represented on $\widetilde{\Delta}$ by the sum

$$\sum_{i=1}^3 d\Phi_i \wedge d\Phi_{i+1}$$

where $\Phi_i = \Phi_{\xi_i}$ and the ξ_i's are the three systolic directions with basepoints in $\widetilde{\Delta}$; cf Definition 3.3.

Let $x \in \widetilde{\Delta}$ such that the Φ_i's are differentiable at x for $i \in \{1, 2, 3\}$. Each bivector $d_x\Phi_i \wedge d_x\Phi_{i+1}$ of the cotangent plane $T_x^*\widetilde{M}$ can be represented by the oriented $\widetilde{g}_0$–area of the parallelogram $\#_i$ spanned by the $\widetilde{g}_0$–gradients $\nabla_x\Phi_i$ and $\nabla_x\Phi_{i+1}$ of Φ_i and Φ_{i+1} at x [18; 9] so that

$$(6\text{-}1) \qquad \widetilde{\omega}_\Phi = \sum_{i=1}^3 \operatorname{area}(\#_i)\, d\widetilde{g}_0.$$

Note that

$$(6\text{-}2) \qquad \operatorname{area}(\#_i) = \det_{\widetilde{g}_0}(\nabla_x\Phi_i, \nabla_x\Phi_{i+1}).$$

On the other hand, the differential of $\pi_{\widetilde{\Delta}} \circ \Phi$ at x is given by

$$d_x(\pi_{\widetilde{\Delta}} \circ \Phi)\colon T_x\widetilde{M}_0 \to \mathbb{R}^3$$

$$u \mapsto (\langle \nabla_x\Phi_i, u\rangle)_{i=1}^3.$$

Let (e_1, e_2) be a positively oriented $\widetilde{g}_0$–orthonormal basis of $T_x \widetilde{M}_0$. The cross-product of $d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_1)$ and $d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_2)$ is the vector of $\mathbb{R}^3$ given by

$$(6\text{-}3) \qquad d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_1) \times d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_2) = \left(\det\nolimits_{\widetilde{g}_0}(\nabla_x \Phi_{i+1}, \nabla_x \Phi_{i+2}) \right)_{i=1}^{3}$$

where the index i is taken modulo 3. Now, recall that $\widetilde{g}_\Phi = (\pi_{\widetilde{\Delta}} \circ \Phi)^* g_E$. Hence,

$$d\widetilde{g}_\Phi = \| d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_1) \times d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_2) \|_E \, d\widetilde{g}_0$$

$$= \sqrt{\sum_{i=1}^{3} \text{area}(\#_i)^2} \, d\widetilde{g}_0$$

from Equations (6-3) and (6-2). By the Cauchy–Schwarz inequality, we derive

$$\widetilde{\omega}_\Phi = \sum_{i=1}^{3} \text{area}(\#_i) \, d\widetilde{g}_0 \leq \sqrt{3} \sqrt{\sum_{i=1}^{3} \text{area}(\#_i)^2} \, d\widetilde{g}_0 = \sqrt{3} \, d\widetilde{g}_\Phi. \qquad \square$$

7 Area measures of 1–Lipschitz admissible maps

Let g be a metric on M and $\widetilde{g}$ its lift on $\widetilde{M}$. Even though the supremum norm $\|.\|_\infty$ is not always finite on $\mathbb{R}^\infty$, the following definition of Lipschitz maps for the supremum norm still makes sense.

Definition 7.1 A map $\Phi\colon \widetilde{M} \to \mathbb{R}^\infty$ is 1–Lipschitz if

$$\| \Phi(x) - \Phi(y) \|_\infty \leq d_{\widetilde{g}}(x, y)$$

for every $x, y \in \widetilde{M}$, or equivalently if Φ_ξ is 1–Lipschitz as a function with real values for every $\xi \in \mathcal{S}_0$.

With this notion, we can state the following result.

Lemma 7.2 Let $\Phi\colon \widetilde{M} \to \mathbb{R}^\infty$ be a 1–Lipschitz admissible map. Then we have

$$dg_\Phi \leq \frac{3}{2} \, dg$$

almost everywhere on M, where dg and dg_Φ represent the area measures of g and g_Φ on M.

Furthermore, equality holds if and only if $g_\Phi = \frac{3}{2} g$ almost everywhere on M.

Proof Let $x \in \widetilde{M}$ such that $\widetilde{g}_\Phi$ is defined at x. Denote by $\widetilde{\Delta}$ the simplex of $\mathcal{T}$ such that $x \in \widetilde{\Delta}$. Let $(\xi_i)_{i=1}^3$ be the three systolic directions with basepoints in $\widetilde{\Delta}$; cf Definition 3.3. We will write $\Phi_i = \Phi_{\xi_i}$.

Let (e_1, e_2) be a positively oriented $\widetilde{g}$–orthonormal basis of $T_x\widetilde{M}$. We have

(7-1) $$d\widetilde{g}_\Phi = \sqrt{\det_{\widetilde{g}} \widetilde{g}_\Phi}\, d\widetilde{g}$$

where $\det_{\widetilde{g}} \widetilde{g}_\Phi = \big(\det \widetilde{g}_\Phi(e_i, e_j)\big)_{i,j \in \{1,2\}}$ defines the determinant of $\widetilde{g}_\Phi$ with respect to $\widetilde{g}$. (This definition does not depend on the choice of the positively oriented $\widetilde{g}$–orthonormal basis (e_1, e_2).)

The trace of $\widetilde{g}_\Phi|_x$ with respect to $\widetilde{g}$ is given by

$$\mathrm{tr}_{\widetilde{g}}\,(\widetilde{g}_\Phi|_x) = \sum_{j=1}^2 (\pi_{\widetilde{\Delta}} \circ \Phi)^*(g_E)|_x(e_j, e_j)$$

$$= \sum_{j=1}^2 \|d_x(\pi_{\widetilde{\Delta}} \circ \Phi)(e_j)\|_E^2$$

$$= \sum_{j=1}^2 \sum_{i=1}^3 |d_x\Phi_i(e_j)|^2$$

$$= \sum_{i=1}^3 \|d_x\Phi_i\|_{\widetilde{g}}^2.$$

By assumption, the map Φ is 1–Lipschitz. Thus, the map Φ_ξ is 1–Lipschitz for every systolic direction ξ. Therefore, $\|d_x\Phi_i\|_{\widetilde{g}} \leq 1$. Hence,

$$\mathrm{tr}_{\widetilde{g}}\,(\widetilde{g}_\Phi) \leq 3.$$

From the geometric-arithmetic mean inequality, we have

$$\sqrt{\det_{\widetilde{g}} \widetilde{g}_\Phi} \leq \frac{3}{2}.$$

The result follows from (7-1). □

Corollary 7.3 *Under the same assumption as Lemma 7.2, we have*

$$\mathrm{area}(g_\Phi) \leq \frac{3}{2}\,\mathrm{area}(g)$$

with equality if and only if $g_\Phi = \frac{3}{2}g$ almost everywhere.

8 Admissible map induced by a metric on M

Fix a Calabi surface M_0; cf Section 2. Let g be a metric on M and $\widetilde{g}$ be the induced metric on the universal covering $\widetilde{M}$ of M.

Given $\xi \in \mathcal{S}_0^*$ (cf Definition 3.1), we define $\mathrm{sys}_g(\alpha_\xi)$ as the least length of a loop of M representing α_ξ. That is,

$$\text{(8-1)} \qquad \mathrm{sys}_g(\alpha_\xi) = \mathrm{sys}(\widetilde{M}/\langle\alpha_\xi\rangle)$$

where $\widetilde{M}/\langle\alpha_\xi\rangle$ is endowed with the metric induced by g.

Lemma 8.1 *There exists a constant $C = C(g)$ such that for every vector $\xi \in \mathcal{S}_0^*$ (cf Definition 3.1), we have*

$$\text{(8-2)} \qquad |d_{\widetilde{g}}(c_\xi(0), c_\xi(t)) - t\,\mathrm{sys}_g(\alpha_\xi)| \leq C.$$

Proof Let $m = E(t)$ be the integer part of t. The $\widetilde{g}$–distance between $c_{\xi(m)}$ and $c_\xi(t)$ is uniformly bounded from above.

Let γ be a g–length minimizing loop in M representing α_ξ. Denote by $\overline{c}_\xi(0)$ the projection of $c_\xi(0)$ to M and by c a segment joining $\overline{c}_\xi(0)$ to γ. The loop $c \cup \gamma^m \cup c^{-1}$ represents α_ξ^m and lifts to an arc of $\widetilde{M}$ with endpoints $c_\xi(0)$ and $c_\xi(m)$. Thus,

$$\text{(8-3)} \qquad d_{\widetilde{g}}(c_\xi(0), c_\xi(m)) \leq m\,\mathrm{sys}_g(\alpha_\xi) + 2\,\mathrm{diam}(g).$$

On the other hand, every arc with endpoints $c_\xi(0)$ and $c_\xi(m)$ projects to a loop of M representing α_ξ^m. From [16, Lemma 4.32], we obtain

$$\text{(8-4)} \qquad m\,\mathrm{sys}_g(\alpha_\xi) \leq d_{\widetilde{g}}(c_\xi(0), c_\xi(m)).$$

Combined with some triangular inequalities, the bounds (8-3) and (8-4) yield the desired estimate. $\qquad\square$

The bound (8-2) shows that the following limit-sup (8-5) is finite.

Definition 8.2 Every vector $\xi \in \mathcal{S}_0^*$ induces a Busemann function $\beta_\xi^g \colon \widetilde{M} \to \mathbb{R}$ with respect to g defined as

$$\text{(8-5)} \qquad \beta_\xi^g(x) = \limsup_{t\to\infty} d_{\widetilde{g}}(x, c_\xi(t)) - t\,\mathrm{sys}_g(\alpha_\xi)$$

where $x \in \widetilde{M}$. Note that $\beta_\xi^g(c_\xi(0)) = 0$.

The function β_ξ^g is 1–Lipschitz as a limit-sup of 1–Lipschitz functions. Thus, it is differentiable almost everywhere by the Rademacher theorem.

Suppose that β_ξ^g is differentiable at $x \in \widetilde{M}$. The g–geodesic ray γ arising from the g–gradient $\nabla_x \beta_\xi^g$ of β_ξ^g at x satisfies the relation

$$(8\text{-}6) \qquad \beta_\xi^g(\gamma(s)) - \beta_\xi^g(\gamma(0)) = s$$

for every $s \geq 0$. Furthermore, β_ξ^g is differentiable at every point of γ and

$$\nabla_x \beta_\xi^g = \gamma'(s).$$

The relation (8-6) also implies that the projection of γ to M is a geodesic curve without transverse self-intersecting points.

Definition 8.3 Let us define a map $\Phi^g \colon \widetilde{M} \to \mathbb{R}^\infty$ as follows.

As previously, given a vector $\xi \in \mathcal{S}_0^*$, we define $\phi_\xi^g \colon \widetilde{M} \to \mathbb{R}$ as

$$\phi_\xi^g(x) = \tfrac{1}{2}(\beta_\xi^g(x) - \beta_{-\xi}^g(x))$$

for every $x \in \widetilde{M}$. Note that $\phi_{-\xi}^g = -\phi_\xi^g$.

Given $\xi \in \mathcal{S}_0^*$, denote by $\xi(i)$ with $i \in \mathbb{Z}$ the vectors of $\mathcal{S}_0^*$ pointing in the same direction as ξ. More precisely, $\xi(i) = c_\xi'(i/n)$, where n is equal to 6 if M_0 is the first Calabi surface (cf Section 2.1), and to 8 if M_0 is the second Calabi surface; cf Section 2.2 (recall that $\mathrm{sys}(M_0) = 1$). Note that $\xi(0) = \xi$, $\xi(n) = \alpha_\xi . \xi$, $\xi(i)(j) = \xi(i+j)$ and $\alpha_{\xi(i)} = \alpha_\xi$.

We can now define a function $\Phi_\xi^g \colon \widetilde{M} \to \mathbb{R}$ as

$$(8\text{-}7) \qquad \Phi_\xi^g(x) = \left(\frac{1}{n} \sum_{i=0}^{n-1} \phi_{\xi(i)}^g(x) \right) \Big/ \mathrm{sys}_g(\alpha_\xi)$$

for every $x \in \widetilde{M}$ and every vector $\xi \in \mathcal{S}_0^*$. Note that $\Phi_{-\xi}^g = -\Phi_\xi^g$.

The functions Φ_ξ^g give rise to a map $\Phi^g \colon \widetilde{M} \to \mathbb{R}^\infty$.

Example 8.4 Let $g = g_0$ and $\xi \in \mathcal{S}_0$ be a systolic direction on M_0. Denote by $\widetilde{\Delta}$ the flat triangle of $\mathcal{T}$ where the basepoint of ξ lies in. The Busemann function β_ξ^g is differentiable on $\widetilde{\Delta}$ and its $\widetilde{g}_0$–gradient on $\widetilde{\Delta}$ is a unit vector parallel to ξ pointing in the same direction as ξ. Hence,

$$\nabla_x \Phi_\xi^{g_0} = \nabla_x \beta_\xi^{g_0} = \xi$$

for every $x \in \widetilde{\Delta}$.

Lemma 8.5 *Let g be a metric on M with $\mathrm{sys}(g) \geq 1$. Then the map $\Phi^g \colon \widetilde{M} \to \mathbb{R}^\infty$ induced by g is 1–Lipschitz and admissible.*

Proof Let $\xi \in \mathcal{S}_0$. Since $\mathrm{sys}_g(\alpha_\xi) \geq 1$, the ξ–coordinate maps Φ^g_ξ are 1–Lipschitz as an average of 1–Lipschitz functions. In particular, the condition (1) of Definition 3.4 is satisfied by the Rademacher theorem. Furthermore, the map $\Phi^g \colon \widetilde{M} \to \mathbb{R}^\infty$ is also 1–Lipschitz; cf Definition 7.1.

Let $x \in \widetilde{M}$ and $\gamma \in \Gamma$. Since $c_{\gamma.\xi} = \gamma . c_\xi$ and $d_{\widetilde{g}}$ is Γ–invariant, we have

$$d_{\widetilde{g}}(\gamma . x, c_{\gamma.\xi}(t)) = d_{\widetilde{g}}(x, c_\xi(t)).$$

We also have $\mathrm{sys}_g(\alpha_{\gamma.\xi}) = \mathrm{sys}_g(\alpha_\xi)$ since $\alpha_{\gamma.\xi}$ and α_ξ are conjugate in Γ. Thus,

$$\beta^g_{\gamma.\xi}(\gamma . x) = \beta^g_\xi(x),$$
(8-8)
$$\phi^g_{\gamma.\xi}(\gamma . x) = \phi^g_\xi(x).$$

Combined with the relations $(\gamma . \xi)(i) = c'_{\gamma.\xi}(i/n) = \gamma . c'_\xi(i/n) = \gamma . \xi(i)$ and the definition of Φ^g_ξ (cf (8-7)), we obtain

$$\Phi^g_{s\gamma.\xi}(\gamma . x) = \Phi^g_\xi(x)$$

Hence the equivariance of $d\Phi^g_\xi$; cf Definition 3.4 (3).

By definition of α_ξ (cf Definition 3.1), we have $\alpha_\xi . c_{\pm\xi}(t) = c_{\pm\xi}(t \pm 1)$. Recall also that $c_{-\xi}(t) = c_\xi(-t)$, $\alpha_{-\xi} = \alpha_\xi^{-1}$ and $\mathrm{sys}_g(\alpha_{-\xi}) = \mathrm{sys}_g(\alpha_\xi)$. Thus, the Busemann functions $\beta^g_{\pm\xi}$ satisfy

$$\begin{aligned}
\beta^g_{\pm\xi}(\alpha_\xi . x) &= \limsup_{t \to \infty} d_{\widetilde{g}}(\alpha_\xi . x, c_{\pm\xi}(t)) - t\,\mathrm{sys}_g(\alpha_{\pm\xi}) \\
&= \limsup_{t \to \infty} d_{\widetilde{g}}(x, c_{\pm\xi}(t \mp 1)) - (t \mp 1)\,\mathrm{sys}_g(\alpha_{\pm\xi}) \mp \mathrm{sys}_g(\alpha_\xi) \\
&= \beta^g_{\pm\xi}(x) - \mathrm{sys}_g(\alpha_\xi).
\end{aligned}$$

Hence

(8-9)
$$\phi^g_\xi(\alpha_\xi . x) = \phi^g_\xi(x) - \mathrm{sys}_g(\alpha_\xi).$$

Combined with the relation $\alpha_\xi = \alpha_{\xi(i)}$ and the definition of Φ_ξ (cf (8-7)), we obtain

$$\Phi^g_\xi(\alpha_\xi . x) = \Phi^g_\xi(x) - 1.$$

Hence the condition (4) of Definition 3.4.

Now, from Equations (8-8) and (8-9), we derive

(8-10)
$$\phi^g_{\alpha_\xi.\xi}(x) = \phi^g_\xi(x) + \mathrm{sys}_g(\alpha_\xi).$$

Let ξ and $\xi' = \xi(1)$ be two neighbor systolic directions pointing in the same direction. From the Equation (8-10) and the relations $\xi'(i) = \xi(i+1)$ and $\xi(n) = \alpha_\xi . \xi$, we deduce

$$\Phi^g_{\xi'}(x) = \left(\frac{1}{n}\sum_{i=0}^{n-1} \phi^g_{\xi(i+1)}(x)\right)\Big/ \mathrm{sys}_g(\alpha_\xi)$$

$$= \left(\frac{1}{n}\sum_{j=1}^{n} \phi^g_{\xi(j)}(x)\right)\Big/ \mathrm{sys}_g(\alpha_\xi)$$

$$= \left(\frac{1}{n}\sum_{j=0}^{n-1} \phi^g_{\xi(j)}(x)\right)\Big/ \mathrm{sys}_g(\alpha_\xi) + \frac{1}{n}$$

(8-11)
$$= \Phi^g_\xi(x) + \frac{1}{n}.$$

Repeatedly using this relation, we derive

$$\Phi^g_{\xi'}(x) = \Phi^g_\xi(x) + \frac{i}{n}$$

for every pair of systolic directions ξ and ξ' pointing in the same direction with $\xi' = \xi(i)$ and $i \in \mathbb{Z}$.

The condition (2) of Definition 3.4 immediately follows. $\square$

9   Induced metrics and two-forms of model admissible maps

Let Φ^0 be the 1–Lipschitz admissible map induced by the metric g_0 on M_0; see Definition 8.3.

Lemma 9.1   *We have*

$$g_{\Phi^0} = \frac{3}{2} g_0.$$

Proof   Let $x \in \widetilde{M}$ and $u \in T_x\widetilde{M}$. Denote by $\widetilde{\Delta}$ the triangle of $\mathcal{T}$ such that $x \in \widetilde{\Delta}$. Let $(\xi_i)_{i=1}^3$ be the three systolic directions with basepoints in $\widetilde{\Delta}$; cf Definition 3.3. We will write $\Phi^0_i = \Phi^{g_0}_{\xi_i}$.

$$\widetilde{g}_{\Phi^0}(u,u) = (\pi_{\widetilde{\Delta}} \circ \Phi^0)^*(g_E)(u,u)$$

$$= \sum_{i=1}^{3} |d_x\Phi^0_i(u)|^2.$$

Suppose u is the gradient of Φ_j^0 at x for the metric $\tilde{g}_0$, ie $u = \nabla_x \Phi_j^0$. Since the $\tilde{g}_0$–dot product between $\nabla_x \Phi_i^0$ and $\nabla_x \Phi_j^0$ is equal to $1 - \frac{3}{2}\delta_{i,j}$ (ie 1 when $i = j$ and $-\frac{1}{2}$ when $i \neq j$), we have

$$\tilde{g}_{\Phi^0}(u, u) = \sum_{i=1}^{3} g_0(\nabla_x \Phi_i^0, \nabla_x \Phi_j^0)^2$$
$$= \|(1, -\tfrac{1}{2}, -\tfrac{1}{2})\|_E^2$$
$$= \frac{3}{2}\tilde{g}_0(u, u).$$

since $u = \nabla \Phi_j^0$ is a $\tilde{g}_0$–unit vector. We derive that

$$\tilde{g}_{\Phi^0} = \frac{3}{2}\tilde{g}_0$$

on $\tilde{M}_0$ since the vectors $\nabla_x \Phi_j^0$ span the vector space $T_x\tilde{M}$. Hence the result. □

Lemma 9.2 *We have*

$$\omega_{\Phi^0} = \frac{3\sqrt{3}}{2}\, dg_0.$$

Proof We will use the same notation as in the proof of the previous lemma. By definition of $\tilde{\omega}_{\Phi^0}$, we have (cf (6-1))

$$\tilde{\omega}_{\Phi^0} = \sum_{i=1}^{3} \det_{\tilde{g}_0}(\nabla_x \Phi_i^0, \nabla_x \Phi_{i+1}^0)\, d\tilde{g}_0.$$

Now, the angle between the unit vectors $\nabla_x \Phi_i^0$ and $\nabla_x \Phi_{i+1}^0$ is equal to $2\pi/3$. Therefore,

$$\det_{\tilde{g}_0}(\nabla_x \Phi_i^0, \nabla_x \Phi_{i+1}^0) = \frac{\sqrt{3}}{2}.$$

Hence
$$\tilde{\omega}_{\Phi^0} = \frac{3\sqrt{3}}{2}\, d\tilde{g}_0.$$ □

10 The $\mathcal{A}$–functional of an admissible map

Definition 10.1 The $\mathcal{A}$–functional of an admissible map $\Phi\colon \tilde{M} \to \mathbb{R}^\infty$ is defined as

$$\mathcal{A}(\Phi) = \int_M \omega_\Phi.$$

Let g be a metric on M with $\mathrm{sys}(g) \geq 1$. Consider the admissible map Φ^g induced by g. Define $\widetilde{\varphi}^g \colon \widetilde{M} \to \mathbb{R}^\infty$ as the difference $\widetilde{\varphi}^g = \Phi^g - \Phi^0$, where Φ^0 is the admissible map induced by the metric g_0 on M_0.

Lemma 10.2 *Let $\xi \in \mathcal{S}_0$. The function $\widetilde{\varphi}_\xi^g \colon \widetilde{M} \to \mathbb{R}$ passes to the quotient by $\langle \alpha_\xi \rangle$ and induces a function denoted by $\varphi_\xi^g \colon \widetilde{M}/\langle \alpha_\xi \rangle \to \mathbb{R}$.*

Proof Passing to the quotient by $\langle \alpha_\xi \rangle$, the real-valued function Φ_ξ^g induces a map $\overline{\Phi_\xi^g} \colon \widetilde{M}/\langle \alpha_\xi \rangle \to \mathbb{R}/\mathbb{Z}$ which takes a (simple) systolic loop of $\widetilde{M}/\langle \alpha_\xi \rangle$ to a generator of the fundamental group of $\mathbb{R}/\mathbb{Z}$. This map $\overline{\Phi_\xi^g}$ induces an isomorphism in homotopy between $\pi_1(\widetilde{M}/\langle \alpha_\xi \rangle)$ and $\pi_1(\mathbb{R}/\mathbb{Z})$. Therefore, the map φ_ξ^g passes to the quotient and induces a map $\overline{\varphi_\xi^g} \colon \widetilde{M}/\langle \alpha_\xi \rangle \to \mathbb{R}/\mathbb{Z}$ which is homotopically trivial and lifts to the desired function $\varphi_\xi^g \colon \widetilde{M}/\langle \alpha_\xi \rangle \to \mathbb{R}$. $\qquad \square$

Definition 10.3 Let $\xi \in \mathcal{S}_0$ be a systolic direction of $\widetilde{M}_0$. Recall that the systolic band $\widetilde{B}_\xi$ of M induced by ξ is the union of all the triangles of $\mathcal{T}$ through which c_ξ passes. The projection B_ξ of $\widetilde{B}_\xi$ to M_0, which agrees with the quotient of $\widetilde{B}_\xi$ by $\langle \alpha_\xi \rangle$, is also called a systolic band of M_0.

We assign to every triangle $\Delta \in \mathcal{T}$ lying in a systolic band B the systolic direction $\xi_{\Delta,B}$ with basepoint in Δ pointing in the same direction as the edge of Δ lying in the boundary of B (the edges of a triangle inherit the orientation of the triangle).

Let $\mathcal{D}$ be a collection of triangles of $\mathcal{T}$ whose union forms a fundamental domain of M in $\widetilde{M}$ as in Figures 1 and 2. By definition of $\mathcal{A}$, we have

$$(10\text{-}1) \quad \mathcal{A}(\Phi^g) - \mathcal{A}(\Phi^0) = \sum_{\widetilde{\Delta} \in \mathcal{D}} \int_{\widetilde{\Delta}} \sum_{i=1}^{3} d\widetilde{\varphi}_i^g \wedge d\Phi_{i+1}^0 + d\Phi_i^0 \wedge d\widetilde{\varphi}_{i+1}^g + d\widetilde{\varphi}_i^g \wedge d\widetilde{\varphi}_{i+1}^g$$

$$= \underbrace{\sum_{\widetilde{\Delta} \in \mathcal{D}} \int_{\widetilde{\Delta}} \sum_{i=1}^{3} d\widetilde{\varphi}_i^g \wedge d\Phi_{i+1}^0 + d\Phi_i^0 \wedge d\widetilde{\varphi}_{i+1}^g}_{I_g}$$

$$+ \underbrace{\sum_{\widetilde{\Delta} \in \mathcal{D}} \sum_{i=1}^{3} \int_{\widetilde{\Delta}} d\widetilde{\varphi}_i^g \wedge d\widetilde{\varphi}_{i+1}^g}_{J_g}$$

where $\qquad \Phi_i^g = \Phi_{\xi_{\widetilde{\Delta},i}}^g, \quad \Phi_i^0 = \Phi_{\xi_{\widetilde{\Delta},i}}^0 \quad$ and $\quad \widetilde{\varphi}_i^g = \widetilde{\varphi}_{\xi_{\widetilde{\Delta},i}}^g.$

Here, the $\xi_{\widetilde{\Delta},i}$'s, with $i \in \{1, 2, 3\}$, are the three systolic directions arising from the triangle $\widetilde{\Delta}$ of $\mathcal{D}$.

The following proposition amounts to the criticality of the $\mathcal{A}$–functional at Φ^0.

Proposition 10.4 *We have*

$$I_g := \sum_{\widetilde{\Delta} \in \mathcal{D}} \int_{\widetilde{\Delta}} \sum_{i=1}^{3} d\widetilde{\varphi}_i^g \wedge d\Phi_{i+1}^0 + d\Phi_i^0 \wedge d\widetilde{\varphi}_{i+1}^g = 0.$$

Proof By rearranging the terms of the sum, we can write I_g as

$$I_g = \sum_{\widetilde{\Delta} \in \mathcal{D}} \sum_{i=1}^{3} \int_{\widetilde{\Delta}} d\widetilde{\varphi}_i^g \wedge (d\Phi_{i+1}^0 - d\Phi_{i-1}^0).$$

Now, note that $d\Phi_{i+1}^0 - d\Phi_{i-1}^0 = (\sqrt{3}/2)(d\Phi_i^0)^\star$, where $(d\Phi_i^0)^\star$ is the image of $d\Phi_i^0$ by the Hodge operator on the one-form space of $\widetilde{M}_0$. Thus,

$$(10\text{-}2) \qquad\qquad I_g = \frac{\sqrt{3}}{2} \sum_{\widetilde{\Delta} \in \mathcal{D}} \sum_{i=1}^{3} \int_{\widetilde{\Delta}} d\widetilde{\varphi}_i^g \wedge (d\Phi_i^0)^\star.$$

The one-forms $(d\Phi_i^0)^\star$ are constant on the systolic bands $\widetilde{B}_\xi$ of $\widetilde{M}_0$. We will still denote by $(d\Phi_i^0)^\star$ the one-form obtained by passing to the quotient on B_ξ. By rearranging the terms of the sum and passing to the quotient, we can write (10-2) as

$$I_g = \frac{\sqrt{3}}{2} \sum_{B} \sum_{\Delta \subset B} \int_{\Delta} d\varphi_{\Delta,B}^g \wedge (d\Phi_{\Delta,B}^0)^\star$$

where the first sum is taken over all the systolic bands B of M_0 and the second over all the triangles Δ of $\mathcal{T}$ lying in B. Here, $\varphi_{\Delta,B}^g = \varphi_{\xi_{\Delta,B}}^g$ and $\Phi_{\Delta,B}^g = \Phi_{\xi_{\Delta,B}}^g$ where $\xi_{\Delta,B}$ is defined in Definition 3.2.

Now, fix a triangle Δ_0 of $\mathcal{T}$ lying in B. Set $\varphi_B^g = \varphi_{\Delta_0,B}^g$ and $v_B^0 = (d\Phi_{\Delta_0,B}^0)^\star$. Let Δ be another triangle of $\mathcal{T}$ lying in B. If Δ and Δ_0 have an edge lying in the same connected component of ∂B, then $d\varphi_{\Delta,B}^g = d\varphi_B^g$ from Definition 3.4 (2), and $(d\Phi_{\Delta,B}^0)^\star = v_B^0$. Similarly, if Δ and Δ_0 do not have any edge lying in the same connected component of ∂B, then $d\varphi_{\Delta,B}^g = -d\varphi_B^g$ and $(d\Phi_{\Delta,B}^0)^\star = -v_B^0$. In both cases,

$$d\varphi_{\Delta,B}^g \wedge (d\Phi_{\Delta,B}^0)^\star = d\varphi_B^g \wedge v_B^0.$$

Therefore, $$I_g = \frac{\sqrt{3}}{2} \sum_{B} \int_{B} d\varphi_B^g \wedge v_B^0.$$

Since the one-form ν_B^0 is constant on B, and therefore closed on B, we deduce from Stokes' formula that

$$I_g = \frac{\sqrt{3}}{2} \sum_B \int_{\partial B} \varphi_B^g \nu_B^0.$$

But the one-form ν_B^0 vanishes on the vectors tangent to ∂B. Thus, I_g equals zero. $\quad\square$

11 Proof of the main result

Let $\Phi = \Phi^g$ be the 1–Lipschitz admissible map induced by a metric g on M. From Lemma 9.2, we derive that

$$\mathcal{A}(\Phi^0) = \frac{3\sqrt{3}}{2} \, \mathrm{area}(g_0).$$

Now, from Corollary 7.3 and Lemma 6.1, we have

$$\mathcal{A}(\Phi^g) \leq \sqrt{3} \, \mathrm{area}(g_\Phi) \leq \frac{3\sqrt{3}}{2} \, \mathrm{area}(g).$$

Let (g_t) be a slow metric variation of g_0 with $\mathrm{sys}(g_t) = 1$. By definition of a slow variation, we deduce from the Cauchy–Schwarz inequality that the term J_{g_t} in (10-1) is negligible with respect to t. Thus, Proposition 10.4 implies that

$$\mathcal{A}(\Phi^{g_t}) - \mathcal{A}(\Phi^0) = o(t).$$

Hence, $\qquad\qquad\qquad \mathrm{area}(g_t) \geq \mathrm{area}(g_0) + o(t).$

12 Examples of slow metric variations

The goal of this section is to present examples of slow metric variations and, more generally, to show how one can prove that a metric variation is slow.

12.1 A general bound for not-necessarily slow metric variations

Let $(g_t)_{t \geq 0}$ be a smooth deformation of g_0. More precisely, g_t is a Riemannian metric with conical singularities on M smoothly varying with t on $M_0 \backslash \{\text{conical singularities}\}$ (as a smooth section of the bundle of symmetric 2–forms on $M_0 \backslash \{\text{conical singularities}\}$). Recall from [27] that a point p of M is a conical singularity of order τ (or of angle $\theta = 2\pi(\tau + 1)$) of the metric g if there exists a nonsingular conformal map $z \colon U \to \mathbb{C}$ defined in a neighborhood U of p such that $z(p) = 0$ and $g = f(z)|z|^{2\tau}|dz|^2$ in U, for some continuous positive function f.

From [13], if p and q are two nonconjugate points on a complete simply connected manifold endowed with a Riemannian metric $\widetilde{g}$, there exists a neighborhood of $\widetilde{g}$ such that for every metric $\widetilde{h}$ in this neighborhood, the points p and q are connected by a unique $\widetilde{h}$–geodesic arc smoothly depending on $\widetilde{h}$. Thus, the g_t–minimizing arcs joining pairs of adjacent singularities smoothly vary with t, for t small enough. Therefore, by taking the pullback of g_t under a one-parameter family of diffeomorphisms converging to the identity map when t goes to zero, we can assume that the edges of the triangulation $\mathcal{T}$ are g_t–minimizing arcs for t small enough.

Recall that the systolic band $\widetilde{B}_\xi$ induced by a systolic direction $\xi \in \mathcal{S}_0$ (cf Definition 3.2) is a flat strip in $\widetilde{M}_0$ composed of equilateral flat triangles of $\mathcal{T}$. The trajectories (for positive time) of the dynamical system generated by $-\nabla_{g_0} \beta_\xi^{g_0}$ (where $\beta_\xi^{g_0}$ is the Busemann function defined by ξ; cf Definition 8.2) arising from the points of $\widetilde{B}_\xi$ are semilines parallel to the boundary of the flat strip $\widetilde{B}_\xi$ and pointing in the same direction as c_ξ; cf Definition 3.1. If the metric has nonpositive curvature (the case we will focus on), the Busemann function $\beta_\xi^{g_t}$ is differentiable and the trajectory γ_t^x of $-\nabla_{g_t} \beta_\xi^{g_t}$ arising from x is complete and leaves every compact set.

The following result shows that for t small enough the trajectories of $-\nabla_{g_t} \beta_\xi^{g_t}$ are still trapped in $\widetilde{B}_\xi$.

Lemma 12.1 *For t small enough, every complete minimizing $\widetilde{g}_t$–geodesic trajectory γ_t leaving $\widetilde{B}_\xi$ does not stay at bounded distance from c_ξ.*

In particular, for t small enough, the trajectories of $-\nabla_{g_t} \beta_\xi^{g_t}$ arising from the points of $\widetilde{B}_\xi$ lie in $\widetilde{B}_\xi$.

Proof We will assume that all the geodesics are parametrized by their arclength. Let $\widetilde{B}_\xi^+$ denote the $\widetilde{g}_0$–convex neighborhood of $\widetilde{B}_\xi$ formed of the triangles of $\mathcal{T}$ with a vertex in $\widetilde{B}_\xi$. For t small enough, the boundary of $\widetilde{B}_\xi$ is made of $\widetilde{g}_t$–minimizing segments and the angles of the singularities of $\widetilde{g}_t$ are greater than 2π. Therefore, for t small enough, the trajectories γ_t leaving $\widetilde{B}_\xi$ also leave $\widetilde{B}_\xi^+$. By translating the trajectories if necessary, we can extract a sequence γ_{t_n}, where $t_n \to 0$, leaving $\widetilde{B}_\xi^+$ with a speed vector u_n such that (u_n) converges to a $\widetilde{g}_0$–unit vector u_∞ pointing outside the interior of $\widetilde{B}_\xi^+$. Since the conical singularities of $\widetilde{g}_0$ are greater than 2π, the complete minimizing $\widetilde{g}_0$–geodesic trajectories γ_∞ generated by u_∞ do not stay at bounded distance from c_ξ. (Note that the trajectories γ_∞ are not necessarily uniquely determined by u_∞ as they may pass through conical singularities.) On the other hand, there exists a trajectory γ_∞ as above such that (γ_{t_n}) converges (up to a subsequence) to γ_∞ in restriction to any given bounded open set of $\widetilde{M}$. The conclusion follows. $\square$

We can now prove a general bound on the variation of the gradient of Busemann functions with respect to any (not-necessarily slow) metric deformation.

Proposition 12.2 *Let* $(g_t)_{t \geq 0}$ *be a smooth deformation of* g_0 *as above. Then for every systolic direction* $\xi \in S_0$ *and every triangle* $\widetilde{\Delta}$ *of* $\mathcal{T}$ *lying in* $\widetilde{B}_\xi$, *we have*

$$\|d\beta_\xi^{g_t} - d\beta_\xi^{g_0}\|_{L^\infty(\widetilde{\Delta})} = \mathcal{O}(\sqrt{t}).$$

Remark 12.3 We do not know whether this $\mathcal{O}(\sqrt{t})$–bound can be replaced with a $o(\sqrt{t})$–bound as in the definition of slow metric variations; cf Definition 1.1. A positive answer would show that any metric variation is slow.

Proof Fix $x \in \widetilde{\Delta}$ a point where $\beta_\xi^{g_t}$ is differentiable. Let $\mathfrak{h}$ be the $\widetilde{g}_0$–segment of $\widetilde{B}_\xi$ passing through x and $\widetilde{g}_0$–orthogonal to the boundary components of $\widetilde{B}_\xi$. Set $x_i = \alpha_\xi^i \cdot x$ and $\mathfrak{h}_i = \alpha_\xi^i \cdot \mathfrak{h}$; cf Definition 3.1. Denote by h the distance function to c_ξ with respect to $\widetilde{g}_0$. Suppose for simplicity that the angle $\theta_0 = \theta_0(x)$ between $-\nabla_{g_0}\beta_\xi^{g_0}(x_0)$ and $-\nabla_{g_t}\beta_\xi^{g_t}(x_0)$ is positive. Note that the function θ_0 tends to zero in $L^\infty(\widetilde{\Delta})$ when t goes to zero. Since the metric deformation is smooth and $\mathrm{sys}_{g_0}(\alpha_\xi) = 1$ (cf (8-1)), the trajectory γ_t^x of $-\nabla_{g_t}\beta_\xi^{g_t}$ arising from x_0, which is geodesic for $\widetilde{g}_t$, hits $\mathfrak{h}_1$ at a point x_1 such that $h(x_1) - h(x_0) \geq \sin\theta_0 - C\,t$ for t small enough, where, by compactness, C is an absolute constant (ie, a constant depending only on g_0 and the derivatives of the metric family (g_t) at $t = 0$, but not on t and x_0). Furthermore, the angle θ_1 between $-\nabla_{g_0}\beta_\xi^{g_0}(x_1)$ and $-\nabla_{g_t}\beta_\xi^{g_t}(x_1)$ is greater or equal to $\theta_0 - C'\,t$, where C' is another absolute constant. We repeat this process again and again. As a result, we obtain a sequence of points x_i along the trajectory γ_t^x and a sequence of angles θ_i between $-\nabla_{g_0}\beta_\xi^{g_0}(x_i)$ and $-\nabla_{g_t}\beta_\xi^{g_t}(x_i)$ is such that

$$(12\text{-}1) \qquad\qquad h(x_{i+1}) - h(x_i) \geq \sin\theta_i - C\,t,$$

$$(12\text{-}2) \qquad\qquad \theta_i \geq \theta_0 - i\,C'\,t$$

This last inequality implies that θ_i is positive for $i < i_0 := [\theta_0/C't]$. Note that for t small enough, θ_i is less than $\pi/4$ and so $\sin\theta_i \geq \frac{9}{10}\theta_i$. Thus, summing the inequality (12-1) from $i = 0$ to $i_0 - 1$ and using (12-2), we derive that for t small enough,

$$h(x_{i_0}) - h(x_0) \geq \sum_{i=0}^{i_0-1} \sin\theta_i - i_0\,C\,t \geq \frac{9}{10}\sum_{i=0}^{i_0-1}\theta_i - \frac{C}{C'}\theta_0$$

$$(12\text{-}3) \qquad\qquad\qquad\qquad \geq \frac{9}{10}\left(i_0\,\theta_0 - \frac{i_0(i_0-1)}{2}C't\right) - \frac{C}{C'}\theta_0$$

Now, since γ_t^x lies in $\widetilde{B}_\xi$ from Lemma 12.1, the sequence $h(x_i)$ is uniformly bounded, for instance by 1. Thus, from (12-3) and the expression of i_0, the angle θ_0 satisfies some inequality

$$\frac{a}{t}\theta_0^2 - b\,\theta_0 - 1 \le 0$$

where a and b are positive absolute constants. We immediately deduce that

$$\theta_0 \le \frac{b + \sqrt{b^2 + 4a/t}}{2a/t} = \mathcal{O}(\sqrt{t}).$$

Hence the result. □

12.2 Changing the edge lengths

A piecewise flat Calabi surface $M_0 = (M, g_0)$ is composed of Euclidean equilateral triangles with the same side length. We can deform the metric g_0 into another piecewise flat metric by changing the lengths of the sides of the Euclidean triangles of $\mathcal{T}$ tilling the surface; cf Section 2. This yields a 48–dimensional space of deformations for the Calabi surface modeled on Fermat's quartic and a 84–dimensional space of deformations for the one modeled on Klein's quartic. We consider a one-parameter family $(g_t)_{t \ge 0}$ of such deformations, where the lengths of the sides of the Euclidean triangles of $\mathcal{T}$ smoothly depend on t. We denote by $\mathcal{T}_t$ and $\mathcal{T}_t$ the deformations of the triangulations given by $\mathcal{T}$ and $\mathcal{T}$. For t small enough, the conical singularities of the piecewise flat metric g_t have nonpositive curvature in Alexandrov's sense and the Busemann functions of g_t are differentiable everywhere off the conical singularities.

Proposition 12.4 *Let $\widetilde{\Delta}$ be a triangle of $\mathcal{T}$ and $\xi \in \mathcal{S}_0$ be a systolic direction with basepoint in $\widetilde{\Delta}$. Then the metric family $(g_t)_{t \ge 0}$ satisfies*

$$(12\text{-}4) \qquad \|(d\beta_\xi^{g_t} - d\beta_{-\xi}^{g_t}) - (d\beta_\xi^{g_0} - d\beta_{-\xi}^{g_0})\|_{L^2(\widetilde{\Delta})} = \mathcal{O}(t).$$

In particular, the one-parameter family of metrics $(g_t)_{t \ge 0}$ is a slow metric variation.

Remark 12.5 The estimate (12-4) is stronger than the estimate (1-1) in the definition of a slow metric variation; cf Definition 1.1. Indeed, it yields a $\mathcal{O}(t)$–bound, which is stronger than a $o(\sqrt{t})$–bound. This still leaves room for improvement and for describing new slow metric variations.

It will be convenient to introduce the following definition.

Definition 12.6 We will write that $a(t) \lesssim b(t)$ if there exists a positive absolute constant C (ie, a constant depending only on g_0 and the derivatives of the metric family (g_t) at $t = 0$, but not on t) such that $a(t) \leq C\,b(t)$ for t small enough. Similarly, we define $a(t) \gtrsim b(t)$. If both $a(t) \leq b(t)$ and $a(t) \gtrsim b(t)$, we will write $a(t) \simeq b(t)$.

We will also use this definition for functions $a(x, t)$ depending on a variable x. In this case, we require the absolute constant C to be independent of x.

Proof of Proposition 12.4 As recalled in Section 12.1, the systolic band $\widetilde{B}_\xi$ of M_0 induced by ξ is a flat strip in $\widetilde{M}_0$ composed of equilateral flat triangles of $\mathcal{T}$. This band can also be seen as a horizontal strip in $\mathbb{R}^2$ endowed with the Euclidean metric. Moving along c_ξ in the positive direction, we denote by $\widetilde{\Delta}_1$ the triangle of $\mathcal{T}$ in $\widetilde{B}_\xi$ adjacent to $\widetilde{\Delta}_0 = \widetilde{\Delta}$, by $\widetilde{\Delta}_2$ the triangle of $\mathcal{T}$ in $\widetilde{B}_\xi$ adjacent to $\widetilde{\Delta}_1$, and so on. Similarly, we define $\widetilde{\Delta}_k$ for k negative by moving along c_ξ in the negative direction.

As t varies, the triangles of $\mathcal{T}_t$ and $\mathcal{T}_t$ are smoothly deformed. In order to compare the geometry of g_0 and g_t, it will be convenient to consider the triangles $\widetilde{\Delta}_k$ and their smooth deformations $\widetilde{\Delta}_k^t$ embedded in $\mathbb{R}^2$; see Figure 3. (This will implicitly be used in the sequel.) This can be done by keeping both the center of $\widetilde{\Delta}_0^t$ and the direction of a given edge of $\widetilde{\Delta}_0^t$ fixed in $\mathbb{R}^2$. With this identification, the restriction of g_t to the union $\widetilde{B}_\xi^t$ of the $\widetilde{\Delta}_k^t$ agrees with (more precisely, is the pullback of) the Euclidean metric of the plane. Note that even for small t, the center of $\widetilde{\Delta}_k^t$ can be far from the center of $\widetilde{\Delta}_k$ (of course, this occurs only for k large enough) and the band $\widetilde{B}_\xi^t$ may no longer be embedded in $\mathbb{R}^2$. However, we will later see that large portions of the band $\widetilde{B}_\xi^t$ do not overlap in $\mathbb{R}^2$.

A geodesic loop of a piecewise flat surface with conical singularities which does not pass through a singularity can be translated without increasing its length until it reaches a singularity. Thus, every homotopy class can be represented by a shortest representative passing through a conical singularity. In particular, the lift of some ξ–systolic loop of (M, g_t) passes through some vertex (conical singularity) x_0 of $\widetilde{B}_\xi^t$ at distance $\lesssim 1$ from $\widetilde{\Delta}$. Here, a ξ–systolic loop of M is the projection of a systolic loop of $\widetilde{M}/\langle \alpha_\xi \rangle$ on M. Consider now the piecewise straight line c of $\mathbb{R}^2$ composed of the segments c_i of $\mathbb{R}^2$ with endpoints $x_i = \alpha_\xi^i \cdot x_0$ and x_{i+1} in $\widetilde{B}_\xi^t$. Let ν be the angle between the directions of c_i and c_{i+1} (it does not depend on i since α_ξ acts by isometry on $\widetilde{B}_\xi^t$). Since the triangles $\widetilde{\Delta}_k^t$ smoothly vary in $\mathbb{R}^2$, we have $|\nu| \lesssim t$.

Suppose that ν is different from zero. Then c lies outside the disk of $\mathbb{R}^2$ bounded by the circle $\mathcal{C}_\rho$ of radius $\rho = \ell/(2\tan(\nu/2)) \simeq 1/t$ passing through the midpoints of the segments c_i, where ℓ is the length of c_i; see Figure 3. Changing the origin O of $\mathbb{R}^2$ if necessary, we can assume that the disk is centered at O. Since the boundary
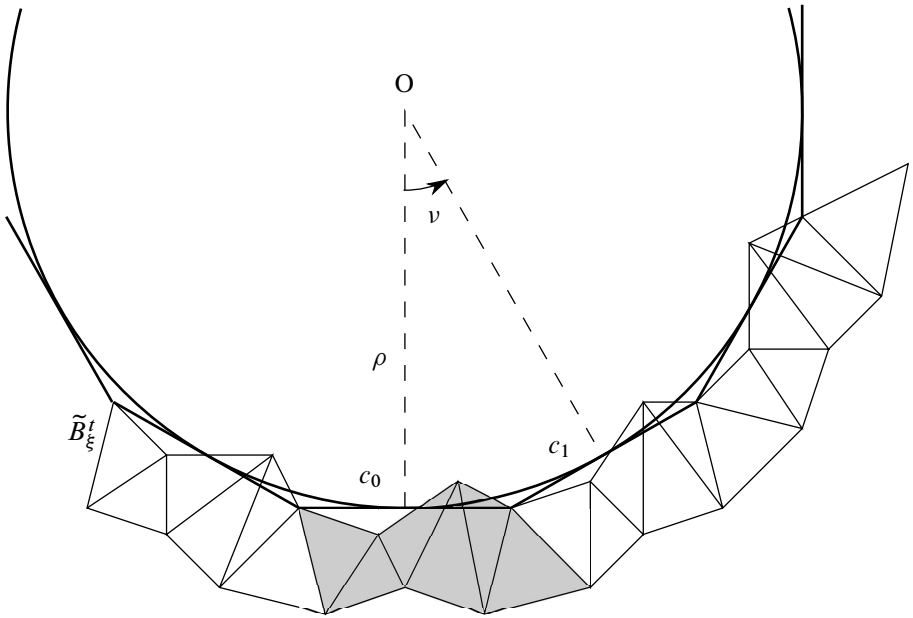
Figure 3: In gray, a fundamental domain for the projection of $\widetilde{B}^t_\xi$

component $\partial_0 \widetilde{B}^t_\xi$ of $\widetilde{B}^t_\xi$ passing through x_0 is at Hausdorff distance $\lesssim t$ from c in $\mathbb{R}^2$, it lies in a tubular neighborhood of $\mathcal{C}_\rho$ of width $\simeq t$, that is, an annulus $A(r, R)$ of $\mathbb{R}^2$ centered at O of inner radius r and outer radius R, where $\rho - r \simeq t$ and $R - \rho \simeq t$; see Figure 4. In particular, the band $\widetilde{B}^t_\xi$ is bounded in $\mathbb{R}^2$.

The area of $\widetilde{\Delta} \cap D(R+t)$ is $\lesssim t$, where $D(R+t)$ is the disk of radius $R+t$ centered at O. Thus, from Proposition 12.2,

$$\|d\beta^{g_t}_\xi - d\beta^{g_0}_\xi\|_{L^2(\widetilde{\Delta} \cap D(R+t))} \lesssim t\sqrt{t}.$$

Let us now bound the L^∞–variations of $d\beta^{g_s}_\xi - d\beta^{g_s}_{-\xi}$ between $s = 0$ and $s = t$ on $\widetilde{\Delta} \setminus D(R+t)$. As in Lemma 12.1, for t small enough, every geodesic of $(\widetilde{M}, \widetilde{g}_t)$ leaving $\widetilde{B}^t_\xi$ diverges so fast that it does not stay at bounded distance from c_ξ. (This is due to the angles of the singularities of g_t which are greater than 2π.) Thus, for t small enough, the trajectories of $-\nabla_{g_t}\beta^{g_t}_\xi$ arising from the points of $\widetilde{B}^t_\xi \setminus D(R+t)$ (which are geodesics possibly passing through conical singularities) lie in $\widetilde{B}^t_\xi$. Since they are length-minimizing geodesics, each of these trajectories lies in the convex cone bounded by two semilines arising from the same starting point as the trajectory and pointing in the same direction as ξ, where the first semiline is tangent to $\mathcal{C}_r$ and the second is tangent to $\mathcal{C}_R$; see Figure 4. Thus, for every x in $\widetilde{\Delta} \setminus D(R+t)$, the vector $-\nabla_{g_t}\beta^{g_t}_\xi(x)$ points into the convex cone previously described with vertex x.

Using a simple trigonometric argument, we deduce that

$$(12\text{-}5) \qquad \frac{r}{R+d} \leq \cos\theta \leq \frac{R}{R+d}$$

where $\theta = \theta(x)$ is the angle between $u(x)$ and $-\nabla_{g_t}\beta_\xi^{g_t}(x)$ (here, $u(x)$ is the vector based at x orthogonal to the line Ox and pointing in the same direction as ξ) and d is the distance from x to the disk bounded by $\mathcal{C}_R$. Strictly speaking, we need to estimate the angle in $\mathbb{R}^2$ between $-\nabla_{g_0}\beta_\xi^{g_0}(x)$ and $-\nabla_{g_t}\beta_\xi^{g_t}(x)$, but since this angle agrees with θ up to some constant $\lesssim t$, we can work with θ instead.
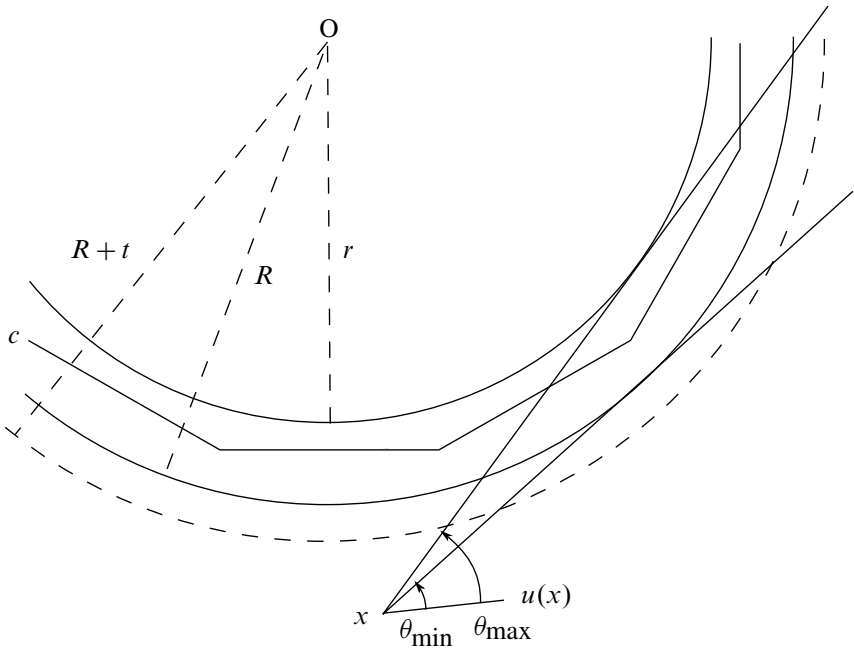

Figure 4

As x lies in $\tilde{\Delta} \setminus D(R+t)$, we have $t \leq d \lesssim 1$. Combined with the estimates on r and R, namely $r \simeq R \simeq 1/t$ and $r - R \simeq t$, we derive from (12-5) that $t \lesssim \theta \lesssim \sqrt{t}$. This does not yield the desired $o(\sqrt{t})$–bound on θ, only a $\mathcal{O}(\sqrt{t})$–bound. To obtain a $o(\sqrt{t})$–bound, we have to use a symmetry argument and deal with $-\xi$ as well.

Set $\theta_+ = \theta$. Replacing ξ with $-\xi$, we define θ_- as the angle between $-u(x)$ (which almost agrees with $-\nabla_{g_0}\beta_{-\xi}^{g_0}(x)$) and $-\nabla_{g_t}\beta_{-\xi}^{g_t}(x)$. The angle θ_- satisfies the same relation $t \lesssim \theta_- \lesssim \sqrt{t}$ and the same bound (12-5) as θ_+. Therefore, since $|R-r| \lesssim t$, we deduce that

$$|\cos\theta_+ - \cos\theta_-| \leq \frac{|R-r|}{R+d} \lesssim t^2.$$

From the mean value theorem, there exists θ_c between θ_+ and θ_- (and so $\theta_c \gtrsim t$) such that

$$|\cos\theta_+ - \cos\theta_-| = |\sin\theta_c| \cdot |\theta_+ - \theta_-|$$
$$\gtrsim t \cdot |\theta_+ - \theta_-|.$$

Hence, $|\theta_+ - \theta_-| \lesssim t$. Therefore,

$$|(d\beta_\xi^{g_t} - d\beta_{-\xi}^{g_t}) - (d\beta_\xi^{g_0} - d\beta_{-\xi}^{g_0})| \simeq |\cos\theta_+ + \cos\theta_- - 2 + i(\sin\theta_+ - \sin\theta_-)|$$
$$\lesssim t$$

on $\widetilde{\Delta} \setminus D(R+t)$.

Suppose now that ν is equal to zero. Then c is a straight line in $\mathbb{R}^2$. Since the boundary component $\partial_0 \widetilde{B}_\xi^t$ of $\widetilde{B}_\xi^t$ passing through x_0 is at Hausdorff distance $\lesssim t$ from c in $\mathbb{R}^2$, there exists a flat strip S_t, parallel to c, of width w_t with $w_0 - w_t \lesssim t$, contained in $\widetilde{B}_\xi^t$, where w_0 is the width of $\widetilde{B}_\xi$. The straight lines foliating S_t project to systolic loops of (M, g_t). Therefore, the trajectories of $-\nabla_{g_t}\beta_\xi^{g_t}$ arising from the points of S_t lie in these straight lines and so are parallel to c. As c_0 smoothly varies with t (see [13] and the beginning of Section 12.1), we derive that

$$\|d\beta_\xi^{g_t} - d\beta_\xi^{g_0}\|_{L^\infty(\widetilde{\Delta} \cap S_t)} \lesssim t.$$

On the other hand, the area of $\widetilde{\Delta} \setminus S_t$ is $\lesssim t$. Thus, from Proposition 12.2,

$$\|d\beta_\xi^{g_t} - d\beta_\xi^{g_0}\|_{L^2(\widetilde{\Delta} \setminus S_t)} \lesssim t\sqrt{t}.$$

The desired estimate follows in this case too. Hence the result. $\square$

Remark 12.7 For any oriented simple closed geodesic γ on M_0, we can define a one-parameter family of metrics (g_t) by twisting the metric g_0 along γ. More precisely, cut M_0 open along γ and glue the boundary components back together after making a twist of parameter t. The resulting metric on the surface, denoted by g_t, is piecewise flat with conical singularities of angles greater than 2π. (The conical singularities of M_0 may be split into two conical singularities in the process if γ passes through some of them.) The proof of Proposition 12.4 can easily be modified to show that (g_t) is a slow metric variation too.

Note that in the hyperbolic case, the metric variations obtained by twisting along simple closed geodesics induce a dense set of directions in the tangent space of the Teichmüller space at the initial metric.

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