

Quantum traces for representations of surface groups in $SL_2(\mathbb{C})$

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We relate two different quantizations of the character variety consisting of all representations of surface groups in SL_2 . One is the Kauffman skein algebra considered by Bullock, Frohman and Kania-Bartoszyńska, Przytycki and Sikora, and Turaev. The other is the quantum Teichmüller space introduced by Chekhov and Fock and by Kashaev. We construct a homomorphism from the skein algebra to the quantum Teichmüller space which, when restricted to the classical case, corresponds to the equivalence between these two algebras through trace functions.

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Let S be an oriented surface of finite topological type. The goal of this paper is to establish a connection between two quantizations of the character variety

$$\mathcal{R}_{SL_2(\mathbb{C})}(S) = \{r: \pi_1(S) \rightarrow SL_2(\mathbb{C})\} // SL_2(\mathbb{C}),$$

consisting of all group homomorphisms r from the fundamental group $\pi_1(S)$ to the Lie group $SL_2(\mathbb{C})$, considered up to conjugation by elements of $SL_2(\mathbb{C})$. The double bar indicates here that the quotient is taken in the algebraic geometric sense of geometric invariant theory.

The first quantization was introduced by D Bullock, C Frohman, J Kania-Bartoszyńska [12], J Przytycki and A Sikora [32] and V Turaev [35] and uses the *Kauffman skein algebra* $\mathcal{S}^A(S)$. This algebra is obtained by considering the vector space freely generated by all isotopy classes of framed links in $S \times [0, 1]$, and then taking the quotient of this space under the Kauffman skein relation; see Section 3.1. What makes $\mathcal{S}^A(S)$ a quantization of $\mathcal{R}_{SL_2(\mathbb{C})}(S)$ is that, when $A = -1$, the skein algebra $\mathcal{S}^{-1}(S)$ has a natural identification with the commutative algebra of regular functions on $\mathcal{R}_{SL_2(\mathbb{C})}(S)$ and that, as A tends to -1 , the lack of commutativity of $\mathcal{S}^A(S)$ is infinitesimally measured by the Goldman–Weil–Petersson Poisson structure [19; 20; 31; 36] on $\mathcal{R}_{SL_2(\mathbb{C})}(S)$; see [35]. There is a similar situation when $A = +1$, in which case $\mathcal{S}^{+1}(S)$ has a natural identification with the algebra of functions on a twisted version of $\mathcal{R}_{SL_2(\mathbb{C})}(S)$; see Section 3.2.

The second quantization of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, with respect to the same Goldman–Weil–Petersson Poisson structure, is the quantum Teichmüller space $\widehat{\mathcal{T}}_S^q$ introduced by V Fock and L Chekhov [18; 14; 15] or, in a slightly different form, by R Kashaev [25]; see also Bonahon and Liu [5], Liu [28] and Guo and Liu [21]. This quantization takes advantage of the fact that, if one restricts to matrices with real coefficients, a large subset of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{R})}(S)$ with nonempty interior has a natural identification with the Teichmüller (or Fricke–Klein) space $\mathcal{T}(S)$, consisting of isotopy classes of all complete hyperbolic metrics on S . Starting with an ideal triangulation of the surface, Thurston [33; 34] introduced for the Teichmüller space $\mathcal{T}(S)$ a set of coordinates, called shear coordinates, in which the Goldman–Weil–Petersson form is expressed in a particularly simple way. The *quantum Teichmüller space* is a quantization of $\mathcal{T}(S)$ that is based on these shear coordinates. As this construction requires the existence of an ideal triangulation, the surface must have at least one puncture.

A natural conjecture is that these two quantizations are “essentially equivalent”.

In the classical cases where $q = 1$ and $A = \pm 1$, the correspondence is relatively clear because of the identifications of $\mathcal{S}^{\pm 1}(S)$ and $\widehat{\mathcal{T}}_S^1$ with algebras of functions on $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and $\mathcal{T}(S)$. The only minor problem is that the functions considered in each case are not quite the same; the first point of view involves Laurent polynomials in the square roots of the shear coordinates, whereas the second approach is based on rational functions in the same coordinates.

More precisely, the correspondence between the skein algebra $\mathcal{S}^{\pm 1}(S)$ and the algebra of regular functions on $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ uses the trace functions $\mathrm{Tr}_K: \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \rightarrow \mathbb{R}$, associated to all closed curves K immersed in S , which associates to a homomorphism $r: \pi_1(S) \rightarrow \mathrm{SL}_2(\mathbb{C})$ the trace of $r(K) \in \mathrm{SL}_2(\mathbb{C})$ (see Sections 1.3 and 3.2 for technical details).

Shear coordinates depend on the choice of some topological information, namely on the choice of an ideal triangulation λ for the surface S . More importantly for us, for a real representation $r_m \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{R})}(S)$ corresponding to a hyperbolic metric $m \in \mathcal{T}(S)$, the trace of $r_m(K)$ can then be explicitly computed in terms of the shear coordinates (with sign issues which are completely resolved in Section 3.2). This trace is actually expressed as a Laurent polynomial in the *square roots* of the shear coordinates of m . This means that we must consider an algebra $\widehat{\mathcal{Z}}_\lambda^1$ consisting of rational fractions in the square roots of the shear coordinates. We now have an algebra homomorphism

$$\mathrm{Tr}_\lambda^1: \mathcal{S}^1(S) \rightarrow \widehat{\mathcal{Z}}_\lambda^1$$

which to $[K] \in \mathcal{S}^1(S)$ associates the Laurent polynomial expressing the trace of $r_m(K)$ in terms of the shear coordinates of $m \in \mathcal{T}(S)$.

In the quantum case, one similarly introduces a noncommutative algebra $\widehat{\mathcal{Z}}_\lambda^\omega$ consisting of rational fractions in certain skew-commuting variables associated to the square roots of the shear coordinates. When $q = \omega^4$, the quantum Teichmüller space $\widehat{\mathcal{T}}_S^q$ consists of those rational fractions in $\widehat{\mathcal{Z}}_\lambda^\omega$ that involve only even powers of the variables. See Section 2.3 for details.

Theorem 1 For $A = \omega^{-2}$, there is an algebra homomorphism

$$\text{Tr}_\lambda^\omega: \mathcal{S}^A(S) \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega,$$

depending continuously on ω in an appropriate sense, which corresponds to the above homomorphism $\text{Tr}_\lambda^1: \mathcal{S}^1(S) \rightarrow \mathcal{Z}_\lambda^1$ when $\omega = 1$. In addition, the image $\text{Tr}_\lambda^\omega([K]) \in \widehat{\mathcal{Z}}_\lambda^\omega$ of every $[K] \in \mathcal{S}^A(S)$ is a Laurent polynomial in the variables generating $\widehat{\mathcal{Z}}_\lambda^\omega$.

The homomorphism Tr_λ^ω is shown to be injective in Proposition 29.

A major step in the construction of the quantum Teichmüller space $\widehat{\mathcal{T}}_S^q$ is to make it independent of a choice of ideal triangulation. The homomorphism Tr_λ^ω of Theorem 1 is similarly independent of choices. Making sense of this statement uses work of C Hiatt in [24] that extends to the square root setup the original coordinate changes of Chekhov and Fock. More precisely, for any two ideal triangulations λ and λ' of the surface S , Hiatt constructs a coordinate change isomorphism $\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$ that restricts to the identity on $\widehat{\mathcal{T}}_S^q$, considering the quantum Teichmüller space $\widehat{\mathcal{T}}_S^q$ as a subalgebra of both $\widehat{\mathcal{Z}}_\lambda^\omega$ and $\widehat{\mathcal{Z}}_{\lambda'}^\omega$.

Theorem 2 Given two ideal triangulations λ and λ' of the surface S and an element $[K] \in \mathcal{S}^A(S)$ of the skein algebra of S , the coordinate change map

$$\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$$

sends the Laurent polynomial $\text{Tr}_{\lambda'}^\omega(K)$ to the Laurent polynomial $\text{Tr}_\lambda^\omega(K)$.

While the proof of Theorem 1 is rather elaborate, the proof of Theorem 2 results from an easy application of the technology developed by Hiatt in [24].

Theorems 1 and 2 were conjectured by Chekhov and Fock [18; 15] and proved for certain small surfaces by Chekhov and Penner [16] and Hiatt [24]. Our proof is much more 3-dimensional than these earlier attempts. The technical challenge is to figure out a “good” way to order the noncommuting variables in each monomial of the Laurent polynomials considered; this is a classical problem in mathematical physics, where it is known as the search for a quantum ordering. Our solution is based on a careful control of the elevations of the strands of a link K in $S \times [0, 1]$ with respect to the $[0, 1]$ factor. The exposition that we give here is very computational and involves a few miraculous identities that

the reader may find somewhat frustrating (see in particular the proof of Proposition 26). Recent conversations with C Kassel seem to provide a more conceptual explanation for these identities, based on the fundamental representation of the dual $\mathrm{SL}_2(q)$ of the quantum group $U_q(\mathfrak{sl}_2)$; in particular, it might be possible to place our construction within the framework of Bullock, Frohman and Kania-Bartoszyńska [11; 13].

This work was motivated by a comparison of the respective advantages and drawbacks of the two points of view on the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, and by the consideration of the impact of these on each corresponding quantization. The algebraic geometric approach to $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ through trace functions is very natural. For instance, its coordinate functions are polynomials. However, it is hard to extract much information from this point of view. Conversely, the shear coordinates for the Teichmüller space are very concrete and have deep geometric meaning, but they also suffer from some practical drawbacks. For instance, they are not defined for closed surfaces, they do not behave well under restriction to subsurfaces and, generally, they are less intrinsic. This last property is particularly true for hyperbolic surfaces with infinite area, for which additional data is needed. The same features can be found at the quantum level. The skein algebra is very natural and occurs in many different contexts. However, its algebraic structure is quite difficult to handle at this point, except for small surfaces (see for instance our article [6, Section 3] for a discussion). Conversely, the quantum Teichmüller space has a very simple algebraic structure (it is a quantum torus), but it suffers from the lack of canonicity inherited from the classical shear coordinates.

One great advantage of the quantum Teichmüller space is that it has a finite-dimensional representation theory which is very simple and completely understood. In particular, an irreducible representation of the quantum Teichmüller space is essentially determined by a point in the character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ [5; 2]. By composition with the trace homomorphism $\mathrm{Tr}_\lambda^\omega: \mathcal{S}^A(S) \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$ provided by Theorem 1, one obtains a wide family of finite-dimensional representations of the skein algebra $\mathcal{S}^A(S)$. These representations behave well with respect to the action of the mapping class group, and a particularly useful feature of the corresponding machinery is that it works even for closed surfaces as seen in our papers [6; 7; 8]. Thus the results of the current paper represent a key technical step in a long-term program to study the representation theory of the skein algebra $\mathcal{S}^A(S)$; see [6] for a discussion.

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1 The classical case

1.1 Ideal triangulations

The introduction was restricted to surfaces with no boundary, but it is convenient to allow boundary as well.

Let S be an oriented punctured surface with boundary, obtained by removing finitely many points v_1, v_2, \dots, v_p from a compact connected oriented surface \bar{S} with (possibly empty) boundary $\partial\bar{S}$. We require that each component of $\partial\bar{S}$ contains at least one puncture v_i , that there is at least one puncture, and that $\chi(S) < \frac{d}{2}$, where $\chi(S)$ is the Euler characteristic of S and d is the number of components of ∂S . These topological restrictions are equivalent to the existence of an *ideal triangulation* for S , namely a triangulation of the closed surface \bar{S} whose vertex set is exactly $\{v_1, \dots, v_p\}$. In particular, an ideal triangulation λ has $n = -3\chi(S) + 2d$ edges and $m = -2\chi(S) + d$ faces. Its edges provide n infinite arcs $\lambda_1, \dots, \lambda_n$ in S , going from puncture to puncture, which decompose the surface S into m infinite triangles T_1, T_2, \dots, T_m whose vertices sit “at infinity” at the punctures. Note that d of these λ_i are just the boundary components of S .

1.2 The shear parameters

Suppose that we are given a positive weight $X_i \in \mathbb{R}_+$ for each interior edge λ_i of the ideal triangulation λ . We can associate to this data a group homomorphism $r: \pi_1(S) \rightarrow PSL_2(\mathbb{R})$ as follows.

Lift the ideal triangulation λ to an ideal triangulation $\tilde{\lambda}$ of the universal cover \tilde{S} . We can then construct an orientation-preserving immersion $\tilde{f}: \tilde{S} \rightarrow \mathbb{H}^2$ from \tilde{S} to the hyperbolic plane \mathbb{H}^2 such that

- (1) \tilde{f} sends each face \tilde{T} of $\tilde{\lambda}$ to an ideal triangle of \mathbb{H}^2 , delimited by three disjoint geodesics and touching the circle at infinity $\partial_\infty\mathbb{H}^2$ in 3 points;
- (2) when two faces \tilde{T} and \tilde{T}' meet along an edge $\tilde{\lambda}_i$ that projects to the edge λ_i of λ , then $\tilde{f}(\tilde{T}')$ is obtained from $\tilde{f}(\tilde{T})$ by performing a hyperbolic reflection across the geodesic $\tilde{f}(\tilde{\lambda}_i)$ followed by a hyperbolic translation of $\log X_i$ along the same geodesic $\tilde{f}(\tilde{\lambda}_i)$, if we orient $\tilde{f}(\tilde{\lambda}_i)$ by the boundary orientation of \tilde{T} .

The immersion \tilde{f} is easily constructed stepwise, and uniquely determined up to isotopy of \tilde{S} respecting $\tilde{\lambda}$, once we have chosen the image of a single face of \tilde{S} . In particular, the family of the ideal triangles $\tilde{f}(\tilde{T}) \subset \mathbb{H}^2$ is unique up to an orientation-preserving isometry of \mathbb{H}^2 , namely up to composition by an element of $PSL_2(\mathbb{R})$.

From the construction, it is immediate that there is a unique group homomorphism $r: \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ such that $\tilde{f}(\gamma\tilde{T}) = r(\gamma)(\tilde{T})$ for every face \tilde{T} of $\tilde{\lambda}$. Since the family of the ideal triangles $\tilde{f}(\tilde{T})$ is unique up to composition by an element of $\mathrm{PSL}_2(\mathbb{R})$, r is unique up to conjugation by an element of $\mathrm{PSL}_2(\mathbb{R})$.

We say that $r: \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is associated to the *shear parameters* $X_i \in \mathbb{R}_+$.

1.3 The classical trace function

For a group homomorphism $r: \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ and immersion $\tilde{f}: \tilde{S} \rightarrow \mathbb{H}^2$ as above, consider a closed curve K immersed in S .

The fact that K is immersed provides a natural lift $\hat{r}(K) \in \mathrm{SL}_2(\mathbb{R})$ of $r(K) \in \mathrm{PSL}_2(\mathbb{R})$. Indeed, lift K to an immersed path $\tilde{K}: [0, 1] \rightarrow \tilde{S}$. Then $r(K) \in \mathrm{PSL}_2(\mathbb{R})$ is the unique orientation-preserving isometry of \mathbb{H}^2 sending the point $\tilde{f} \circ \tilde{K}(0) \in \mathbb{H}^2$ to $\tilde{f} \circ \tilde{K}(1)$, and sending the vector $(\tilde{f} \circ \tilde{K})'(0)$ to $(\tilde{f} \circ \tilde{K})'(1)$. Now, for every $t \in [0, 1]$, we can consider the isometry $r(K)_t \in \mathrm{PSL}_2(\mathbb{R})$ that sends the point $\tilde{f} \circ \tilde{K}(0) \in \mathbb{H}^2$ to $\tilde{f} \circ \tilde{K}(t)$, and the vector $(\tilde{f} \circ \tilde{K})'(0)$ to a positive real multiple of $(\tilde{f} \circ \tilde{K})'(t)$. We now have a constructed a path $t \mapsto r(K)_t \in \mathrm{PSL}_2(\mathbb{R})$ that joins $r(K)_0 = \mathrm{Id}_{\mathbb{H}^2}$ to $r(K)_1 = r(K)$. This path defines an element of the universal cover of $\mathrm{PSL}_2(\mathbb{R})$, which projects to an element $\hat{r}(K)$ of the 2-fold cover $\mathrm{SL}_2(\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{R})$.

We are particularly interested in the trace $\mathrm{Tr} \hat{r}(K)$ of $\hat{r}(K) \in \mathrm{SL}_2(\mathbb{R})$. Note that, when K is just a small circle bounding a disk embedded in S , our designated lift $\hat{r}(K)$ is minus the identity matrix of $\mathrm{SL}_2(\mathbb{R})$, and $\mathrm{Tr} \hat{r}(K) = -2$.

If the homomorphism $r: \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is associated to shear parameters $X_i \in \mathbb{R}_+$ assigned to the edges of the ideal triangulation λ , then the construction of the map $\tilde{f}: \tilde{S} \rightarrow \mathbb{H}^2$ and of the homomorphism r is sufficiently explicit that $\hat{r}(K) \in \mathrm{SL}_2(\mathbb{R})$ can be computed in practice.

More precisely, suppose K transversely meets the edges $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_k}, \lambda_{i_{k+1}} = \lambda_{i_1}$, in this order. After crossing the edge λ_{i_j} , the curve K enters a face T of λ , which it exits through the edge $\lambda_{i_{j+1}}$. There are three possible choices for $\lambda_{i_{j+1}}$: it can be the edge immediately to the left as one enters T through λ_{i_j} , the one immediately to the right, or it can be λ_{i_j} again if γ makes a U-turn in T . In addition, because K is immersed, we can measure the amount by which the tangent to K turns between λ_{i_j} and $\lambda_{i_{j+1}}$. We then define a matrix M_j according to the various possible configurations.

If $\lambda_{i_{j+1}}$ is the edge immediately to the left as one enters T through λ_{i_j} , let $t_j \in \mathbb{Z}$ denote the number of full turns to the left that the tangent to K makes between λ_{i_j} and $\lambda_{i_{j+1}}$, and let $\varepsilon_j = (-1)^{t_j} = \pm 1$. Here the topological number of turns $t_j \in \mathbb{Z}$

is measured so that $t_j = 0$ when K has no self-intersection between λ_{i_j} and $\lambda_{i_{j+1}}$; in fact, t_j has the same parity as the number of double points of K between λ_{i_j} and $\lambda_{i_{j+1}}$. In this case, define

$$M_j = \begin{pmatrix} \varepsilon_j & \varepsilon_j \\ 0 & \varepsilon_j \end{pmatrix}.$$

For the analogous case where $\lambda_{i_{j+1}}$ is the edge immediately to the right as one enters T through λ_{i_j} , let again $t_j \in \mathbb{Z}$ denote the number of full turns to the left that the tangent to K makes between λ_{i_j} and $\lambda_{i_{j+1}}$, and set $\varepsilon_j = (-1)^{t_j} = \pm 1$. Then define

$$M_j = \begin{pmatrix} \varepsilon_j & 0 \\ \varepsilon_j & \varepsilon_j \end{pmatrix}.$$

In the case of a U-turn, where $\lambda_{i_{j+1}} = \lambda_{i_j}$, let $t_j \in \mathbb{Z}$ be defined so that the tangent to K makes $2t_j + 1$ half-turns to the left between λ_{i_j} and $\lambda_{i_{j+1}}$, and set again $\varepsilon_j = (-1)^{t_j} = \pm 1$. Then define

$$M_j = \begin{pmatrix} 0 & \varepsilon_j \\ -\varepsilon_j & 0 \end{pmatrix}.$$

Finally, having defined M_j in every case, consider for $X > 0$ the matrix

$$S(X) = \begin{pmatrix} X^{1/2} & 0 \\ 0 & X^{-1/2} \end{pmatrix}.$$

Lemma 3 *Up to conjugation by an element of $SL_2(\mathbb{R})$,*

$$\hat{r}(K) = S(X_{i_1})M_1S(X_{i_2})M_2 \cdots S(X_{i_k})M_k,$$

where the matrices M_j and $S(X_{i_j})$ are associated as above to the way the immersed curve K crosses the edges of the ideal triangulation λ , and where $X_i \in \mathbb{R}_+$ are the shear parameters defining the homomorphism $r: \pi_1(S) \rightarrow PSL_2(\mathbb{R})$.

Proof This is an easy exercise in hyperbolic geometry. See, for instance, Exercises 8.5–8.7 and 10.14 in the first author’s book [4]. □

1.4 State sums

As preparation for the quantum extension, we now give a state sum formula for the trace $\text{Tr } \hat{r}(K)$, where $\hat{r}(K) \in SL_2(\mathbb{R})$.

Let a *state* assign a sign $s_1, s_2, \dots, s_k, s_{k+1} = s_1 \in \{+, -\}$ to each point where K crosses an edge λ_{i_j} of λ , in this order. For $j = 1, 2, \dots, k$, write the matrix M_j

defined above as

$$M_j = \begin{pmatrix} m_j^{++} & m_j^{+-} \\ m_j^{-+} & m_j^{--} \end{pmatrix}$$

with $m_j^{\pm\pm} = 0, +1$ or -1 . Then, the following formula immediately follows from Lemma 3 combined with elementary linear algebra.

Lemma 4 *The trace $\text{Tr } \hat{r}(K)$ satisfies the state sum formula*

$$\text{Tr } \hat{r}(K) = \sum_s m_1^{s_1 s_2} m_2^{s_2 s_3} \dots m_k^{s_k s_1} X_{i_1}^{s_1/2} X_{i_2}^{s_2/2} \dots X_{i_k}^{s_k/2},$$

where the sum is over all possible states s for K and λ , and where in the exponents we identify the sign $s_j = \pm$ to the number $s_j = \pm 1$. □

2 The quantum Teichmüller space

2.1 The Chekhov–Fock algebra of an ideal triangulation

Let T_1, T_2, \dots, T_m be the faces of the ideal triangulation λ . Index the sides of each face T_j as $\lambda_{j1}, \lambda_{j2}, \lambda_{j3}$, in such a way that they occur in this order clockwise around T_j . We then associate to T_j a copy $\mathcal{T}_{T_j}^q$ of the *triangle algebra*, generated by three elements X_{j1}, X_{j2}, X_{j3} and their inverses $X_{j1}^{-1}, X_{j2}^{-1}, X_{j3}^{-1}$, and defined by the relations that $X_{j1} X_{j2} = q^2 X_{j2} X_{j1}$, $X_{j2} X_{j3} = q^2 X_{j3} X_{j2}$ and $X_{j3} X_{j1} = q^2 X_{j1} X_{j3}$. We here think of each generator X_{ja} as being associated to the side λ_{ja} of T_j .

In the tensor product algebra $\mathcal{T}_{T_1}^q \otimes \dots \otimes \mathcal{T}_{T_m}^q = \bigotimes_{j=1}^m \mathcal{T}_{T_j}^q$, we now associate to the edge λ_i of λ an element X_i , defined by:

- (1) $X_i = X_{ja} \otimes X_{kb}$ if λ_i separates two distinct faces T_j and T_k , and if $X_{ja} \in \mathcal{T}_{T_j}^q$ and $X_{kb} \in \mathcal{T}_{T_k}^q$ are the generators associated to the sides of T_j and T_k corresponding to λ_i ;
- (2) $X_i = q^{-1} X_{ja} X_{jb} = q X_{jb} X_{ja}$ if λ_i corresponds to two sides of the same face T_j , if $X_{ja}, X_{jb} \in \mathcal{T}_{T_j}^q$ are the generators associated to these two sides, and if X_{ja} is associated to the side that comes first when going counterclockwise around their common vertex.

By convention, when describing an element $Z_1 \otimes \dots \otimes Z_m$ of $\bigotimes_{j=1}^m \mathcal{T}_{T_j}^q$, we omit in the tensor product those Z_j that are equal to the identity element 1 of $\mathcal{T}_{T_j}^q$.

The *Chekhov–Fock algebra* of the ideal triangulation λ is the subalgebra \mathcal{T}_λ^q of $\bigotimes_{j=1}^m \mathcal{T}_{T_j}^q$ generated by the elements X_i associated as above to the edges of λ , and by their inverses X_i^{-1} .

Note that $X_i X_j = q^{2\sigma_{ij}} X_j X_i$, where the integers $\sigma_{ij} \in \{0, \pm 1, \pm 2\}$ are defined as follows: Let a_{ij} be the number of angular sectors delimited by λ_i and λ_j in the faces of λ , and with λ_i coming first counterclockwise; then $\sigma_{ij} = a_{ij} - a_{ji}$.

2.2 Coordinate changes between Chekhov–Fock algebras

As one switches from one ideal triangulation λ to another ideal triangulation λ' , the geometry of the Teichmüller space provides coordinate changes between the shear coordinates associated to λ and those associated to λ' . Because shear coordinates can be expressed as cross-ratios, one easily sees that these coordinate changes are given by rational maps.

In the quantum case, there is no underlying geometry to provide us with similar coordinate changes, and one has to find algebraic isomorphisms that have the required properties.

As in the classical case, these will involve rational fractions, and we consequently have to introduce the *fraction division algebra* $\widehat{\mathcal{T}}_\lambda^q$ of the Chekhov–Fock algebra \mathcal{T}_λ^q . Such a fraction division algebra exists because \mathcal{T}_λ^q satisfies the so-called Ore Condition; see for instance Cohn [17] and Kassel [26]. In practice, $\widehat{\mathcal{T}}_\lambda^q = \mathbb{C}(X_1, \dots, X_n)_\lambda^q$ consists of noncommutative rational fractions in the variables X_1, \dots, X_n which are manipulated according to the q -commutativity relations $X_i X_j = q^{2\sigma_{ij}} X_j X_i$.

Chekhov and Fock [18; 14; 15] (and Kashaev [25] in the context of length coordinates) construct such coordinate isomorphisms; see also Bonahon and Liu [5; 28].

Theorem 5 (Chekhov–Fock, Kashaev) *There exists a family of algebra isomorphisms*

$$\Phi_{\lambda\lambda'}^q: \widehat{\mathcal{T}}_{\lambda'}^q \rightarrow \widehat{\mathcal{T}}_\lambda^q,$$

defined for any two ideal triangulations λ, λ' , such that

$$\Phi_{\lambda\lambda''}^q = \Phi_{\lambda\lambda'}^q \circ \Phi_{\lambda'\lambda''}^q$$

for any three ideal triangulations λ, λ' and λ'' . □

This enables us to define the *quantum Teichmüller space* $\widehat{\mathcal{T}}_S^q$ of the punctured surface S as the quotient

$$\widehat{\mathcal{T}}_S^q = \coprod_{\lambda} \widehat{\mathcal{T}}_\lambda^q / \sim$$

of the disjoint union of the $\widehat{\mathcal{T}}_\lambda^q$ of all ideal triangulations λ of S , where the equivalence relation \sim identifies $\widehat{\mathcal{T}}_\lambda^q$ with $\widehat{\mathcal{T}}_{\lambda'}^q$ by the coordinate change isomorphism $\Phi_{\lambda\lambda'}^q$. Note that the property that $\Phi_{\lambda\lambda''}^q = \Phi_{\lambda\lambda'}^q \circ \Phi_{\lambda'\lambda''}^q$ is crucial to guarantee that \sim is an

equivalence relation. This property is much stronger than one could have thought at first glance, as indicated by the uniqueness result of Bai [1].

Because the $\Phi_{\lambda\lambda'}^q$ are algebra isomorphisms, the quantum Teichmüller space $\widehat{\mathcal{T}}_S^q$ inherits an algebra structure from the $\widehat{\mathcal{T}}_\lambda^q$.

2.3 The Chekhov–Fock square root algebra

The formulas of Lemma 4 involve *square roots* of shear coordinates. This lead us to consider formal square roots $Z_i = X_i^{1/2}$ of the generators of the Chekhov–Fock algebra \mathcal{T}_λ^q .

In practice, one just considers the Chekhov–Fock algebra $\mathcal{T}_\lambda^\omega$ associated to a fourth root $\omega = q^{1/4}$ of q . To avoid confusion, we denote by Z_i the generator of $\mathcal{T}_\lambda^\omega$ associated to the edge λ_i of λ while, as before, X_i is the generator of \mathcal{T}_λ^q associated to the same λ_i . Then, there exists an injective algebra homomorphism $\mathcal{T}_\lambda^q \rightarrow \mathcal{T}_\lambda^\omega$ associating the element Z_i^2 to the generator X_i , so that we can consider \mathcal{T}_λ^q as a subalgebra of $\mathcal{T}_\lambda^\omega$. This also induces a similar inclusion $\widehat{\mathcal{T}}_\lambda^q \subset \widehat{\mathcal{T}}_\lambda^\omega$ between the corresponding fraction division algebras.

In the classical case, the coordinate changes between square roots of shear coordinates are not as nice as those between shear coordinates, because they are not rational anymore. The same consequently holds in the quantum setup. However, there is a subalgebra of the algebra $\mathcal{T}_\lambda^\omega$ which is better behaved with respect to coordinate changes.

A monomial $Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}$ in the generators Z_i of $\mathcal{T}_\lambda^\omega$ is said to be *balanced* if, for every triangle face T_j of λ , the exponents k_i of the generators Z_i associated to the three sides of T_j add up to an even number. (When the same edge λ_i corresponds to two distinct sides of T_j , the exponent k_i is counted twice in the sum.) This is equivalent to the property that there exists a homology class $\alpha \in H_1(S; \mathbb{Z}_2)$ such that the class of the exponent k_i in \mathbb{Z}_2 is equal to the algebraic intersection number of α with the edge λ_i . In this case, we will say that the monomial $Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}$ is *α -balanced*.

In the Chekhov–Fock algebra $\mathcal{T}_\lambda^\omega$, let $\mathcal{Z}_\lambda^\omega$ denote the linear subspace generated by all balanced monomials. Note that it splits as a direct sum

$$\mathcal{Z}_\lambda^\omega = \bigoplus_{\alpha \in H_1(S, \mathbb{Z}_2)} \mathcal{Z}_\lambda^\omega(\alpha)$$

where $\mathcal{Z}_\lambda^\omega(\alpha)$ denotes the linear subspace generated by all α -balanced monomials. The product of an element of $\mathcal{Z}_\lambda^\omega(\alpha)$ with an element of $\mathcal{Z}_\lambda^\omega(\beta)$ belongs to $\mathcal{Z}_\lambda^\omega(\alpha + \beta)$,

so that $\mathcal{Z}_\lambda^\omega$ is a subalgebra of $\mathcal{T}_\lambda^\omega$. We will refer to $\mathcal{Z}_\lambda^\omega$ as the *Chekhov–Fock square root algebra* of the ideal triangulation λ .

Note that $\mathcal{Z}_\lambda^\omega(0)$ coincides with the subalgebra $\mathcal{T}_\lambda^q \subset \mathcal{T}_\lambda^\omega$ generated by the $Z_i^2 = X_i$.

In the fraction algebra $\widehat{\mathcal{T}}_\lambda^\omega$, we now consider the subset $\widehat{\mathcal{Z}}_\lambda^\omega$ consisting of all fractions P/Q where $P \in \mathcal{Z}_\lambda^\omega$ and $Q \in \mathcal{Z}_\lambda^\omega(0) = \mathcal{T}_\lambda^q$. One easily checks that $\widehat{\mathcal{Z}}_\lambda^\omega$ is a subalgebra of $\widehat{\mathcal{T}}_\lambda^\omega$. It also contains the Chekhov–Fock fraction algebra $\widehat{\mathcal{T}}_\lambda^q$.

Hiatt [24] extends the Chekhov–Fock coordinate change isomorphism $\Phi_{\lambda\lambda'}: \widehat{\mathcal{T}}_{\lambda'}^q \rightarrow \widehat{\mathcal{T}}_\lambda^q$ of Theorem 5 to the Chekhov–Fock square root algebra $\mathcal{Z}_\lambda^\omega$.

Theorem 6 (Hiatt) *When $q = \omega^4$, there exists for any two ideal triangulations λ, λ' an algebra isomorphism*

$$\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$$

extending the Chekhov–Fock coordinate change isomorphism $\Phi_{\lambda\lambda'}: \widehat{\mathcal{T}}_{\lambda'}^q \rightarrow \widehat{\mathcal{T}}_\lambda^q$. In addition, $\Theta_{\lambda\lambda''}^\omega = \Theta_{\lambda\lambda'}^\omega \circ \Theta_{\lambda'\lambda''}^\omega$ for any three ideal triangulations λ, λ' and λ'' .

Proof Hiatt does not quite prove the result in this form, so we need to explain how to obtain it from [24, Sections 6–7].

For every $\alpha \in H_1(S; \mathbb{Z}_2)$, let $\widehat{\mathcal{Z}}_\lambda^\omega(\alpha)$ consist of all fractions P/Q where $P \in \mathcal{Z}_\lambda^\omega(\alpha)$ and $Q \in \mathcal{Z}_\lambda^\omega(0) = \mathcal{T}_\lambda^q$, so that

$$\widehat{\mathcal{Z}}_\lambda^\omega = \bigoplus_{\alpha \in H_1(S, \mathbb{Z}_2)} \widehat{\mathcal{Z}}_\lambda^\omega(\alpha).$$

Hiatt constructs in [24, Section 6] a linear map $\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega(\alpha) \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega(\alpha)$. The construction of this map in [24] *a priori* depends on the choice of a 1–dimensional submanifold K immersed in S and representing the homology class $\alpha \in H_1(S; \mathbb{Z}_2)$. However, it easily follows from [24, Lemma 17] that this map depends only on α .

Linearly extend these maps $\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega(\alpha) \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega(\alpha)$ to a linear map $\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$.

To show that this is an algebra homomorphism we need to check that, for every $A \in \widehat{\mathcal{Z}}_{\lambda'}^\omega(\alpha)$ and $B \in \widehat{\mathcal{Z}}_{\lambda'}^\omega(\beta)$, $\Theta_{\lambda\lambda'}^\omega(A)\Theta_{\lambda\lambda'}^\omega(B) = \Theta_{\lambda\lambda'}^\omega(AB)$ in $\widehat{\mathcal{Z}}_\lambda^\omega(\alpha + \beta)$. This is an immediate consequence of [24, Sublemma 19] and of the construction of $\Theta_{\lambda\lambda'}^\omega$.

The fact that the restriction of $\Theta_{\lambda\lambda'}^\omega$ to $\widehat{\mathcal{Z}}_{\lambda'}^\omega(0) = \widehat{\mathcal{T}}_{\lambda'}^q$ coincides with $\Phi_{\lambda\lambda'}^q$ immediately follows from its construction in [24]. The property that $\Theta_{\lambda\lambda''}^\omega = \Theta_{\lambda\lambda'}^\omega \circ \Theta_{\lambda'\lambda''}^\omega$ is proved in [24, Theorem 25]. □

Remark 7 For $A \in \widehat{\mathcal{Z}}_\lambda^\omega$, the operator point of view of [18; 14; 15] much more easily provides a natural square root $\Theta_{\lambda\lambda'}^\omega(A)$ of $\Phi_{\lambda\lambda'}^q(A^2)$. The real content of Theorem 6 is that this square root can be expressed as a rational fraction in the generators $Z_i = X_i^{1/2}$. The restriction to $\widehat{\mathcal{Z}}_{\lambda'}^\omega$ is here crucial.

3 The skein algebra

3.1 Links and skeins

We begin with the *framed link algebra* $\mathcal{K}(S)$. This is the vector space (over \mathbb{C} , say) freely generated by the isotopy classes of (unoriented) 1–dimensional framed submanifolds $K \subset S \times [0, 1]$ such that

- (1) $\partial K = K \cap \partial(S \times [0, 1])$ consists of finitely many points in $(\partial S) \times [0, 1]$;
- (2) at every point of ∂K , the framing is vertical, namely parallel to the $[0, 1]$ factor, and points in the direction of 1;
- (3) for every component k of ∂S , the points of ∂K that are in $k \times [0, 1]$ sit at different elevations, namely have different $[0, 1]$ –coordinates.

An isotopy of such framed submanifolds of course is required to respect all three conditions. The third condition will turn out to be crucial for our analysis.

Perhaps we should have begun by specifying what we mean by a *framing* for K . For us here, a framing is a continuous choice of a vector transverse to K at each point of K .

The vector space $\mathcal{K}(S)$ can be endowed with a multiplication, where the product of K_1 and K_2 is defined by the framed link $K \subset S \times [0, 1]$ that is the union of K_1 rescaled in $S \times [0, \frac{1}{2}]$ and K_2 rescaled in $S \times [\frac{1}{2}, 1]$. In other words, the product $K_1 K_2$ is defined by superposition of the framed links K_1 and K_2 . Note that this *superposition operation* is compatible with isotopies, and therefore provides a well-defined algebra structure on $\mathcal{K}(S)$.

Three links K_1 , K_0 and K_∞ in $S \times [0, 1]$ form a *Kauffman triple* if the only place where they differ is above a small disk in S , where they are as represented in Figure 1 (as seen from above) and where the framing is vertical and pointing upwards (namely the framing is parallel to the $[0, 1]$ factor and points towards 1).

The *Kauffman skein algebra* $\mathcal{S}^A(S)$ is the quotient of the framed link algebra $\mathcal{K}(K)$ by the two-sided ideal generated by all elements $K_1 - A^{-1}K_0 - AK_\infty$ as (K_1, K_0, K_∞) ranges over all Kauffman triples. The superposition operation descends to a multiplication in $\mathcal{S}^A(S)$, endowing $\mathcal{S}^A(S)$ with the structure of an algebra. The class $[\emptyset]$ of the empty link is an identity element in $\mathcal{S}^A(S)$, and is usually denoted by 1.

An element $[K] \in \mathcal{S}^A(S)$, represented by a framed link $K \in \mathcal{K}(S)$, is a *skein* in S . The construction is defined to ensure that the *skein relation*

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

holds in $\mathcal{S}^A(S)$ for every Kauffman triple (K_1, K_0, K_∞) .

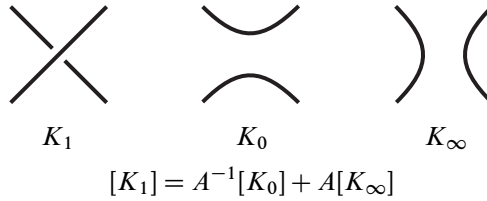


Figure 1: The skein relation

3.2 The classical cases $A = \pm 1$

Bullock [9; 10], Bullock, Frohman and Kania-Bartoszyńska [12] and Przytycki and Sikora [32] observe that there is a strong connection between the skein algebra with $A = -1$ and the character variety

$$\mathcal{R}_{SL_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r: \pi_1(S) \rightarrow SL_2(\mathbb{C})\} // SL_2(\mathbb{C}).$$

The quotient is under the action by conjugation, and should be understood in the sense of geometric invariant theory (see Mumford, Fogarty and Kirwan [29]) to avoid pathologies near the reducible homomorphisms.

Note that, for every A , there is a unique algebra homomorphism $\mathcal{S}^A(S) \rightarrow \mathbb{C}$ that sends each nonempty skein $[K] \in \mathcal{S}^A(S)$ to 0 and sends the empty skein $[\emptyset] = 1$ to 1. This homomorphism is the *trivial homomorphism* $\mathcal{S}^A(S) \rightarrow \mathbb{C}$.

Theorem 8 [23; 9; 10; 12; 32] *Assume that the surface S has no boundary (but is still allowed to have punctures), and consider the skein algebra $\mathcal{S}^{-1}(S)$ corresponding to $A = -1$. Every group homomorphism $r: \pi_1(S) \rightarrow SL_2(\mathbb{C})$ defines a unique nontrivial algebra homomorphism $T_r: \mathcal{S}^{-1}(S) \rightarrow \mathbb{C}$ by the property that*

$$T_r([K]) = -\text{Tr } r(K)$$

for every connected skein $[K] \in \mathcal{S}^{-1}(S)$.

Conversely, every nontrivial algebra homomorphism $T: \mathcal{S}^{-1}(S) \rightarrow \mathbb{C}$ is associated to a unique $r \in \mathcal{R}_{SL_2(\mathbb{C})}(S)$ in this way. □

Note that the definition of $r(K) \in SL_2(\mathbb{C})$ implicitly supposes the choice of an orientation for the closed curve K . However, reversing this orientation replaces $r(K)$ by its inverse, and leaves the trace $\text{Tr } r(K)$ unchanged.

There is a similar result for the other case where the skein algebra $\mathcal{S}^A(S)$ is commutative, corresponding to $A = 1$. This statement uses the correspondence $\mathcal{S}^A(S) \cong \mathcal{S}^{-A}(S)$ established by J Barrett [3], and requires the use of spin structures.

Let $\text{Spin}(S)$ be the set of isotopy classes of spin structures on S or, equivalently, the set of isotopy classes of spin structures on $S \times [0, 1]$. Any two elements of $\text{Spin}(S)$ differ by an obstruction in $H^1(S; \mathbb{Z}_2)$, so that there is an action of $H^1(S; \mathbb{Z}_2)$ on $\text{Spin}(S)$.

Similarly, the cohomology group $H^1(S; \mathbb{Z}_2)$ acts on $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ by the property that, if $\alpha \in H^1(S; \mathbb{Z}_2)$ and $r: \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$, then $\alpha r \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ associates $(-1)^{\alpha(\gamma)} r(\gamma) \in \text{SL}_2(\mathbb{C})$ to $\gamma \in \pi_1(S)$. Note that the quotient of $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ under this action of $H^1(S; \mathbb{Z}_2)$ is just the character variety

$$\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})\} // \text{PSL}_2(\mathbb{C})$$

of homomorphisms valued in $\text{PSL}_2(\mathbb{C})$ instead of $\text{SL}_2(\mathbb{C})$. (It is here important that S is noncompact so that, because the fundamental group $\pi_1(S)$ is free, every homomorphism $\pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ lifts to $\text{SL}_2(\mathbb{C})$.)

We can then combine these actions of $H^1(S; \mathbb{Z}_2)$ on $\text{Spin}(S)$ and $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$, and consider the twisted product

$$\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S) = \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) \tilde{\times} \text{Spin}(S) = (\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \times \text{Spin}(S)) / H^1(S; \mathbb{Z}_2).$$

Note that, just like $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$, this twisted product $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S)$ is a finite cover of $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ with fiber $H^1(S; \mathbb{Z}_2) \cong \text{Spin}(S)$.

If $\sigma \in \text{Spin}(S)$ is a spin structure and K is a framed knot in $S \times [0, 1]$, the monodromy of the framing of K with respect to σ defines an element $\sigma(K) \in \mathbb{Z}_2$. If, in addition, we are given a group homomorphism $r: \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$, we can consider the element $T_{(r,\sigma)}(K) = (-1)^{\sigma(K)} \text{Tr } r(K)$. Note that $T_{(r,\sigma)}(K)$ is invariant under the action of $H^1(S; \mathbb{Z}_2)$ on the pair (r, σ) , and therefore depends only on the class of (r, σ) in $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S)$.

Theorem 9 *Assume that the surface S has no boundary (but is still allowed to have punctures), and consider the skein algebra $S^1(S)$ corresponding to $A = +1$. Every group homomorphism $r: \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$ and spin structure $\sigma \in \text{Spin}(S)$ define a unique algebra homomorphism $T_{(r,\sigma)}: S^1(S) \rightarrow \mathbb{C}$ by the property that*

$$T_{(r,\sigma)}([K]) = (-1)^{\sigma(K)} \text{Tr } r(K)$$

for every connected skein $[K] \in S^1(S)$. This homomorphism $T_{(r,\sigma)}$ is nontrivial and depends only on the class of (r, σ) in $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S)$.

Conversely, every nontrivial algebra homomorphism $T: S^1(S) \rightarrow \mathbb{C}$ is associated to a unique element $(r, \sigma) \in \mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S)$ in this way.

Proof Fix a spin structure $\sigma \in \text{Spin}(S)$. Then Barrett [3] defines an algebra isomorphism $\mathcal{S}^1(S) \rightarrow \mathcal{S}^{-1}(S)$ associating $(-1)^{k+\sigma(K)}[K] \in \mathcal{S}^{-1}(S)$ to every skein $[K] \in \mathcal{S}^1(S)$ represented by a link K with k components (see [32, Section 2] for a proof that this is an algebra homomorphism). The result then immediately follows by combining Theorem 8 with this correspondence. \square

To connect the setup of Section 1.3 to Theorem 9, consider a hyperbolic metric $m \in \mathcal{T}(S)$. It is convenient to move to a 3-dimensional framework, by extending $m \in \mathcal{T}(S)$ to a 3-dimensional hyperbolic metric on a small thickening $S \times (0, 1)$ of $S = S \times \{\frac{1}{2}\}$. We can even consider the more general case of a hyperbolic metric m on $S \times (0, 1)$, not necessarily complete. Classically, this hyperbolic metric m on $S \times (0, 1)$ has a well-defined monodromy homomorphism $r \in \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$. What seems less well-known is that m provides additional spin information, and uniquely determines an element of the twisted product $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S) = \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) \tilde{\times} \text{Spin}(S)$.

A spin structure $\sigma \in \text{Spin}(S)$ specifies a way to lift the monodromy homomorphism $r: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ to a homomorphism $r_\sigma: \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$ as follows. For this, first extend σ to a spin structure on the thickened surface $S \times (0, 1)$. Then consider a *developing map* for the metric m , namely an isometric immersion $\tilde{f}: \tilde{S} \times (0, 1) \rightarrow \mathbb{H}^3$ from the universal cover $\tilde{S} \times (0, 1)$ to the hyperbolic space \mathbb{H}^3 that is equivariant with respect to the monodromy $r: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$.

Represent $[\gamma] \in \pi_1(S) = \pi_1(S; x_0)$ by a path $\gamma: [0, 1] \rightarrow S \times (0, 1)$ satisfying $\gamma(0) = \gamma(1) = x_0$. Pick an arbitrary orthonormal frame $F(t)$ at each $\gamma(t)$, depending continuously on t and such that $F(1) = F(0)$ at $\gamma(0) = \gamma(1) = x_0$. Lift γ to $\tilde{\gamma}: [0, 1] \rightarrow \tilde{S} \times (0, 1)$, and $F(t)$ to an orthonormal frame $\tilde{F}(t)$ at $\tilde{\gamma}(t)$. For every $t \in [0, 1]$, we can now consider the unique isometry $r(\gamma)_t \in \text{PSL}_2(\mathbb{C})$ of \mathbb{H}^3 that sends $\tilde{f}(\tilde{\gamma}(0))$ to $\tilde{f}(\tilde{\gamma}(t))$ and $\tilde{f}(\tilde{F}(0))$ to $\tilde{f}(\tilde{F}(t))$. By construction, $r(\gamma)_0 = \text{Id}_{\mathbb{H}^3}$ and $r(\gamma)_1 = r(\gamma)$. The path $t \mapsto r(\gamma)_t$ then defines a lift $\hat{r}(\gamma) \in \text{SL}_2(\mathbb{C})$ of $r(\gamma) = r(\gamma)_1$ to the universal cover $\text{SL}_2(\mathbb{C})$ of $\text{PSL}_2(\mathbb{C})$. This $\hat{r}(\gamma) \in \text{SL}_2(\mathbb{C})$ clearly depends on the framing F but, if $\sigma(F) \in \mathbb{Z}_2$ denotes the monodromy of the framing F around γ with respect to the spin structure σ ,

$$r_\sigma(\gamma) = (-1)^{\sigma(F)} \hat{r}(\gamma) \in \text{SL}_2(\mathbb{C})$$

does not. One easily checks this defines a group homomorphism $r_\sigma: \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$.

By definition of the action of $H^1(S; \mathbb{Z}_2)$ on $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ and $\text{Spin}(S)$, a different choice of spin structure $\sigma \in \text{Spin}(S)$ does not change the class of (r_σ, σ) in $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S) = (\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \times \text{Spin}(S))/H^1(S; \mathbb{Z}_2)$.

Proposition 10 *The element (r_σ, σ) in $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S) = \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) \tilde{\times} \text{Spin}(S)$ depends only on the hyperbolic metric m on $S \times (0, 1)$. \square*

For a connected skein $[K] \in \mathcal{S}^1(S)$, note that the element $T_{(r_\sigma, \sigma)}([K])$ associated to $(r_\sigma, \sigma) \in \mathcal{R}_{\text{PSL}_2(\mathbb{C})}^{\text{Spin}}(S)$ by Theorem 9 is just the trace of $\hat{r}(K)$ defined above. As a consequence, in the situation of Section 1.3 where m comes from a 2-dimensional hyperbolic metric on S , where K is obtained by perturbing to remove double points a curve immersed in S , and where the framing is chosen everywhere vertical, $T_{(r_\sigma, \sigma)}([K])$ is exactly the trace $\text{Tr } \hat{r}(K)$ considered in Section 1.3.

3.3 Gluing skeins

In addition to the multiplication by superposition, there is another operation which can be performed on framed links and skeins.

Given two surfaces S_1 and S_2 and two boundary components $k_1 \subset \partial S_1$ and $k_2 \subset \partial S_2$, we can glue S_1 and S_2 by identifying k_1 and k_2 to obtain a new oriented surface S . There is a unique way to perform this gluing so that the orientations of S_1 and S_2 match to give an orientation of S . We allow the “self-gluing” case, where the surfaces S_1 and S_2 are equal as long as the boundary components k_1 and k_2 are distinct. If we are given an ideal triangulation λ_1 of S_1 and an ideal triangulation λ_2 of S_2 , these two triangulations fit together to give an ideal triangulation λ of the glued surface S .

Now, suppose in addition that we are given skeins $[K_1] \in \mathcal{S}^A(S_1)$ and $[K_2] \in \mathcal{S}^A(S_2)$ such that $K_1 \cap (k_1 \times [0, 1])$ and $K_2 \cap (k_2 \times [0, 1])$ have the same number of points. We can then arrange by an isotopy of framed links that K_1 and K_2 fit together to give a framed link $K \subset S \times [0, 1]$; note that it is here important that the framings be vertical pointing upwards on the boundary, so that they fit together to give a framing of K . By our hypothesis that the points of $K_1 \cap (k_1 \times [0, 1])$ (and of $K_2 \cap (k_2 \times [0, 1])$) sit at different elevations, the framed link K is now uniquely determined up to isotopy. Also, this operation is well behaved with respect to the skein relations, so that K represents a well-defined element $[K] \in \mathcal{S}^A(S)$. We will say that $[K] \in \mathcal{S}^A(S)$ is *obtained by gluing* the two skeins $[K_1] \in \mathcal{S}^A(S_1)$ and $[K_2] \in \mathcal{S}^A(S_2)$.

3.4 The main theorem

Let a *state* for a skein $[K] \in \mathcal{S}^A(S)$ be the assignment $s: \partial K \rightarrow \{+, -\}$ of a sign \pm to each point of ∂K . Let $\mathcal{S}_s^A(S)$ be the algebra consisting of linear combinations of *stated skeins*, namely of skeins endowed with states.

In the case where $K \in \mathcal{S}^A(S)$ is obtained by gluing the two skeins $K_1 \in \mathcal{S}^A(S_1)$ and $K_2 \in \mathcal{S}^A(S_2)$, the states $s: \partial K \rightarrow \{+, -\}$, $s_1: \partial K_1 \rightarrow \{+, -\}$, $s_2: \partial K_2 \rightarrow \{+, -\}$ are *compatible* if s_1 and s_2 coincide on $\partial K_1 \cap (k_1 \times [0, 1]) = \partial K_2 \cap (k_2 \times [0, 1])$ for the

identification given by the gluing, and if s coincides with the restrictions of s_1 and s_2 on $\partial K \subset \partial K_1 \cup \partial K_2$.

The main result of the paper is the following. Recall that, for an ideal triangulation of the surface S , $\mathcal{Z}_\lambda^\omega$ is the square-root Chekhov–Fock algebra defined in Section 2.3.

Theorem 11 *For $A = \omega^{-2}$, there is a unique family of algebra homomorphisms*

$$\text{Tr}_S: \mathcal{S}_s^A(S) \rightarrow \mathcal{Z}_\lambda^\omega,$$

defined for each surface S and each ideal triangulation λ of S , such that:

- (1) (State Sum Property) *If the surface S is obtained by gluing S_1 to S_2 , if the ideal triangulation λ of S is obtained by combining the ideal triangulations λ_1 of S_1 and λ_2 of S_2 , and if the skeins $[K_1] \in \mathcal{S}^A(S_1)$ and $[K_2] \in \mathcal{S}^A(S_2)$ are glued together to give $[K] \in \mathcal{S}^A(S)$, then*

$$\text{Tr}_S([K, s]) = \sum_{\text{compatible } s_1, s_2} \text{Tr}_{S_1}([K_1, s_1]) \otimes \text{Tr}_{S_2}([K_2, s_2])$$

where the sum is over all states $s_1: \partial K_1 \rightarrow \{+, -\}$ and $s_2: \partial K_2 \rightarrow \{+, -\}$ that are compatible with $s: \partial K \rightarrow \{+, -\}$ and with each other. Similarly if the surface S , the ideal triangulation λ of S , and the skein $[K] \in \mathcal{S}^A(S)$ are obtained by gluing the surface S_1 , the ideal triangulation λ_1 of S_1 , and the skein $[K_1] \in \mathcal{S}^A(S_1)$, respectively, to themselves, then

$$\text{Tr}_S([K, s]) = \sum_{\text{compatible } s_1} \text{Tr}_{S_1}([K_1, s_1]).$$

- (2) (Elementary Cases) *When S is a triangle and K projects to a single arc embedded in S , with vertical framing, then*
 - (a) *in the case of Figure 2(a), where $\varepsilon_1, \varepsilon_2 = \pm$ are the signs associated by the state s to the end points of K ,*

$$\text{Tr}_S([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = - \text{ and } \varepsilon_2 = +, \\ [Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] & \text{if } \varepsilon_1 \neq - \text{ or } \varepsilon_2 \neq +, \end{cases}$$

where Z_1 and Z_2 are the generators of $\mathcal{Z}_\lambda^\omega$ associated to the sides respectively carrying weights ε_1 and ε_2 , and where $[Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] = \omega^{-\varepsilon_1 \varepsilon_2} Z_1^{\varepsilon_1} Z_2^{\varepsilon_2} = \omega^{\varepsilon_1 \varepsilon_2} Z_2^{\varepsilon_2} Z_1^{\varepsilon_1}$ (identifying the sign $\varepsilon = \pm$ to the exponent $\varepsilon = \pm 1$);

- (b) in the case of Figure 2(b), where the end point of K marked by ε_1 is higher in $\partial S \times [0, 1]$ than the point marked by ε_2 ,

$$\text{Tr}_S([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = \varepsilon_2, \\ -\omega^{-5} & \text{if } \varepsilon_1 = + \text{ and } \varepsilon_2 = -, \\ \omega^{-1} & \text{if } \varepsilon_1 = - \text{ and } \varepsilon_2 = +. \end{cases}$$

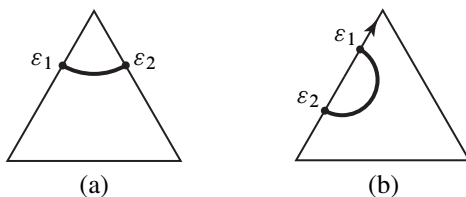


Figure 2: Elementary skeins in the triangle

In particular, Theorem 11 immediately gives Theorem 1 of the introduction by restriction to surfaces without boundary.

In the State Sum Condition (1), note that Z_λ^ω is contained in $Z_{\lambda_1}^\omega \otimes Z_{\lambda_2}^\omega$ when the surfaces S_1 and S_2 are distinct, and in $Z_{\lambda_1}^\omega$ in the case of a self-gluing.

In Condition (2)(a), the formula $[Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] = \omega^{-\varepsilon_1 \varepsilon_2} Z_1^{\varepsilon_1} Z_2^{\varepsilon_2} = \omega^{\varepsilon_1 \varepsilon_2} Z_2^{\varepsilon_2} Z_1^{\varepsilon_1}$ is more natural than one might think at first glance, as it corresponds to the classical Weyl quantum ordering for the monomial $Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}$. More generally, if Y_1, Y_2, \dots, Y_k are elements of an algebra such that $Y_i Y_j = \omega^{2a_{ij}} Y_j Y_i$, the Weyl quantum ordering of the monomial $Y_1 Y_2 \cdots Y_k$ is the monomial

$$[Y_1 Y_2 \cdots Y_k] = \omega^{-\sum_{i < j} a_{ij}} Y_1 Y_2 \cdots Y_k.$$

The formula is specially designed to be invariant under all permutations of the Y_i .

3.5 Picture conventions

To work more efficiently with framed links and skeins, we need a convenient way to describe and manipulate them.

In practice, we will represent a link $K \subset S \times [0, 1]$ by its projection to S , namely by a 1-dimensional manifold K' immersed in S with $K' \cap \partial S = \partial K'$, and whose only singularities are transverse double points in the interior of S ; in addition, these double points are endowed with over- or under-crossing information, describing which strand of K lies above the other in $S \times [0, 1]$ (with the convention that, when oriented from 0 to 1, the $[0, 1]$ factor points towards the eye of the reader).

By adding kinks if necessary, we can always arrange that the framing is vertical at every point of K , with the framing vector parallel to the $[0, 1]$ factor and pointing towards 1.

A crucial piece of information encoded in a framed link $K \subset S \times [0, 1]$ is that, for a component k of ∂S , the points of $(\partial K) \cap (k \times [0, 1])$ are ordered by their elevation. This ordering is not altogether easy to describe on a 2-dimensional picture, and we will resort to the following method to specify this orientation. We choose an arbitrary orientation of k . We now have two orderings on $(\partial K) \cap (k \times [0, 1])$: one is by order of increasing elevations; the other one is given by the orientation of k if we identify each point of $(\partial K) \cap (k \times [0, 1])$ to its projection in k . After an isotopy of K (which is elevation-preserving near the boundary), we can always arrange that these two orderings coincide, and we will require this condition to hold in all pictures.

Note that reversing the orientation of k will then oblige us to modify the projection of K by a half-twist near k , as in Figure 3.

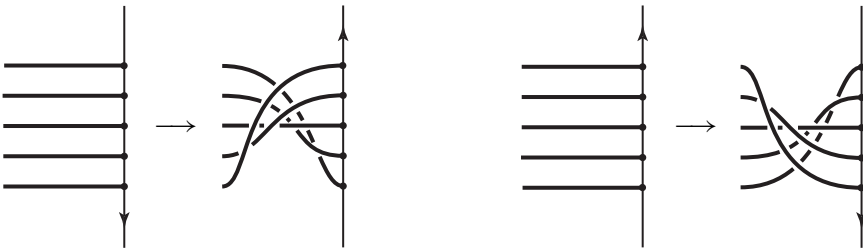


Figure 3: Reversing a boundary orientation

With these conventions, the isotopy class of the framed link $K \subset S \times [0, 1]$ is then immediately recovered from its projection K' to S .

3.6 Unknots and kinks

For future reference, we note the following classical facts.

Lemma 12 *If the framed link K' is obtained from K by adding a positive kink as in Figure 4(a), then $[K'] = -A^{-3}[K]$ in $\mathcal{S}^A(S)$. If K' is obtained from K by adding a negative kink as in Figure 4(b), then $[K'] = -A^3[K]$.*

If K' is obtained from K by adding a small unknotted circle as in Figure 4(c), then $[K'] = -(A^2 + A^{-2})[K]$ in $\mathcal{S}^A(S)$.

In the above statement, we of course assume that those skeins drawn in Figure 4 follow the picture conventions that we just introduced. Adding a positive or negative kink does not change the isotopy class of the link but modifies the framing.

Proof This is an immediate consequence of the skein relations and of the invariance of skeins under the isotopy of Figure 4(d). See, for instance, [27, Lemmas 3.2 and 3.3]. \square

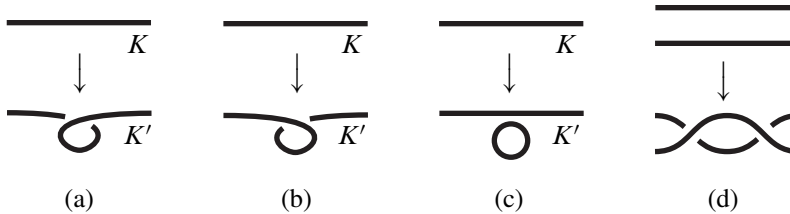


Figure 4: Adding kinks and unknotted components

4 The case of the biangle

Our proof of Theorem 11 will make use of *ideal biangles* in addition to ideal triangles. An ideal biangle is the surface B obtained from a closed disk by removing two points from its boundary. In particular, it has two (infinite) edges, and it is also diffeomorphic to the strip delimited by two parallel lines in the plane.

There is a skein algebra $\mathcal{S}^A(B)$ of the biangle B defined as before. States for skeins are similarly defined.

In this context, we have the following simpler analog of Theorem 11.

Proposition 13 *Let two numbers $\alpha, \beta \in \mathbb{C}$ be given, with $\alpha^2 + \beta^2 = A^5 + A$ and $\alpha\beta = -A^3$. Then, there is a unique family of algebra homomorphisms*

$$\text{Tr}_B: \mathcal{S}_s^A(B) \rightarrow \mathbb{C},$$

defined for all oriented biangles B , such that:

- (1) (State Sum Property) *If the biangle B is obtained by gluing together two distinct biangles B_1 and B_2 , and if $[K_1] \in \mathcal{S}^A(B_1)$ and $[K_2] \in \mathcal{S}^A(B_2)$ are glued together to give $[K] \in \mathcal{S}^A(B)$, then*

$$\text{Tr}_B([K, s]) = \sum_{\text{compatible } s_1, s_2} \text{Tr}_{B_1}([K_1, s_1]) \text{Tr}_{B_2}([K_2, s_2]),$$

where the sum is over all states $s_1: \partial K_1 \rightarrow \{+, -\}$ and $s_2: \partial K_2 \rightarrow \{+, -\}$ that are compatible with $s: \partial K \rightarrow \{+, -\}$ and with each other.

- (2) (Elementary Cases) *If, using the picture conventions of Section 3.5, the biangle B is represented by a vertical strip in the plane as in Figure 5 and if K projects to a single arc embedded in B , then*

(a) in the case of Figure 5 (a), where $\varepsilon_1, \varepsilon_2 = \pm$ are the signs associated by the state s to the end points of K ,

$$\text{Tr}_B([K, s]) = \begin{cases} 1 & \text{if } \varepsilon_1 = \varepsilon_2, \\ 0 & \text{if } \varepsilon_1 \neq \varepsilon_2; \end{cases}$$

(b) in the case of Figure 5 (b),

$$\text{Tr}_B([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = \varepsilon_2, \\ \alpha & \text{if } \varepsilon_1 = + \text{ and } \varepsilon_2 = -, \\ \beta & \text{if } \varepsilon_1 = - \text{ and } \varepsilon_2 = +. \end{cases}$$

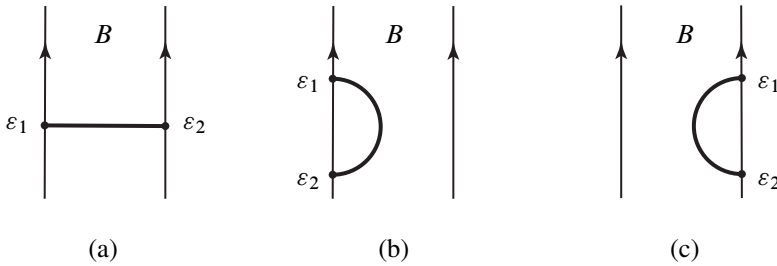


Figure 5: Elementary skeins in the biangle

Note that the equations $\alpha^2 + \beta^2 = A^5 + A$ and $\alpha\beta = -A^3$ only admit four solutions, namely $(\alpha, \beta) = \pm(A^{5/2}, -A^{1/2})$ and $\pm(A^{1/2}, -A^{5/2})$.

Proof In the case considered, the homomorphism Tr_B is essentially a version of the Kauffman bracket for tangles. In particular, everything here is fairly classical. However, it is useful to go through the details of the construction to see where the hypotheses on α and β come up.

We will split the proof of Proposition 13 into several steps. We begin with a lemma.

Lemma 14 For a family of homomorphisms Tr_B that satisfies the properties of Proposition 13, then necessarily

$$\text{Tr}_B([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = \varepsilon_2, \\ -A^{-3}\alpha & \text{if } \varepsilon_1 = + \text{ and } \varepsilon_2 = -, \\ -A^{-3}\beta & \text{if } \varepsilon_1 = - \text{ and } \varepsilon_2 = +, \end{cases}$$

when K is as in Figure 5 (c).

Proof The proof is provided by Figure 6. The equivalence of Figures 6(a) and 6(b) is just obtained by rotating B by 180 degrees. Reversing the boundary orientations then introduces a half-twist as in Figure 3, which gives the skein of Figure 6(c). Removing the kink, Lemma 12 then shows that this skein is equal to $-A^{-3}$ times the skein of Figure 5(b). The result then follows from Property (2)(b) of Proposition 13. \square

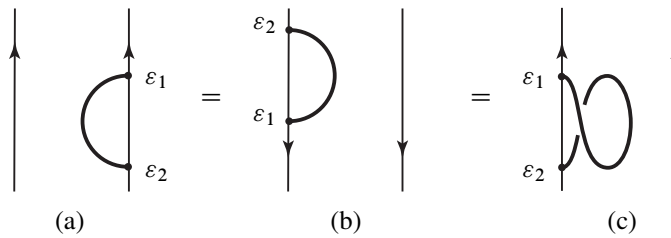


Figure 6: The proof of Lemma 14

From now on, when representing a skein in a biangle B , we will use the conventions of Section 3.5 where the two boundary components of B are oriented in a parallel way, as in Figure 5.

We now prove the uniqueness of the homomorphisms Tr_B .

Lemma 15 *If there exists a family of homomorphisms Tr_B satisfying the properties of Proposition 13, then it is unique.*

Proof We first restrict attention to a skein $[K] \in \mathcal{S}^A(B)$ that is represented by a family of arcs and curves *without crossings* in B . By general position, isotop K so that it is in bridge position namely so that, as we sweep B from one boundary component to the other, the local maxima and minima are generic and occur at distinct positions. We can then subdivide B into a union of biangles B_1, \dots, B_n so that each $K_i = K \cap (B_i \times [0, 1])$ contains at most one maximum or minimum. Each K_i then is of one of the three types pictured in Figure 7.

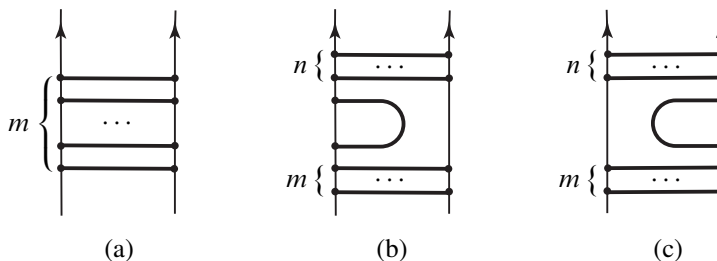


Figure 7: The subdivision skeins K_i

In particular, if a, b, c are the three skeins respectively represented in Figures 5(a), 5(b) and 5(c), $[K_i] = [a]^m, [a]^m[b][a]^n$ or $[a]^m[c][a]^n$ in the algebra $\mathcal{S}^A(B_i)$. Thus, for any state s_i , the image of $[K_i, s_i] \in \mathcal{S}_s^A(B_i)$ under the algebra homomorphism Tr_{B_i} is uniquely determined by Properties (2)(a)–(b) of Proposition 13 and by Lemma 14.

By the State Sum Property,

$$\text{Tr}_B([K, s]) = \sum_{\text{compatible } s_i} \text{Tr}_{B_1}([K_1, s_1]) \cdots \text{Tr}_{B_n}([K_n, s_n]),$$

so that $\text{Tr}_B([K, s])$ is uniquely determined.

In the case of a skein with crossings, the Kauffman skein relations allow $[K] \in \mathcal{S}^A(B)$ to be expressed as a linear combination of skeins without crossings. By linearity of Tr_B and by uniqueness in the case without crossings, $\text{Tr}_B[K, s]$ is uniquely determined in this general case as well.

This proves Lemma 15, namely the uniqueness of the homomorphisms Tr_B . □

We now demonstrate the existence of the homomorphisms Tr_B .

First consider the case of a link $K \subset B \times [0, 1]$ whose projection to B has no crossing. As in the proof of Lemma 15, put K in bridge position, and decompose B as a union of biangles B_i such that $K_i = K \cap (B_i \times [0, 1])$ has at most one local maximum or one local minimum for the sweep.

In this case with no crossing, define

$$(1) \quad \text{Tr}_B([K, s]) = \sum_{\text{compatible } s_i} \text{Tr}_{B_1}([K_1, s_1]) \cdots \text{Tr}_{B_n}([K_n, s_n]),$$

where each $\text{Tr}_{B_i}([K_i, s_i])$ is defined by Conditions (2)(a)–(b) of Proposition 13 or by Lemma 14.

Note that, in the above sum, there are few states s_i for which $\text{Tr}_{B_i}([K_i, s_i])$ is nonzero.

Lemma 16 *For a skein $[K] \in \mathcal{S}^A(B)$ without crossing, the number $\text{Tr}_B([K, s])$ defined above is independent of the subdivision of B into biangles B_i .*

Proof For a given bridge position of K , the only freedom in the choice of the biangles B_i is that we can successively add or delete biangles B_i where K_i has no local maximum or minimum. Because of the definition of the corresponding $\text{Tr}_B([K_i, s_i])$ by Condition (2)(a) of Proposition 13, the state sum (1) providing $\text{Tr}_B([K, s])$ remains unchanged if we add or delete such a B_i .

It therefore suffices to prove independence under the bridge position. By general position, any two bridge positions are related to each other by a sequence of the following moves:

- (1) the “Snake Move” of Figure 8, where a local maximum and a local minimum collide and cancel out; this Snake Move actually comes in two types, related to each other by a reflection, according to whether the local maximum sits above or below the local minimum just before the collision;
- (2) the inverse of the snake move, which creates a pair of a local maximum and a local minimum;
- (3) the “Time Switch Move”, where the times at which two different local extrema occur during the sweep of B are switched; there are 4 types of such Time Switch Moves and their inverses (according to whether they involve local maxima or minima), two of which are represented in Figures 9 and 10.

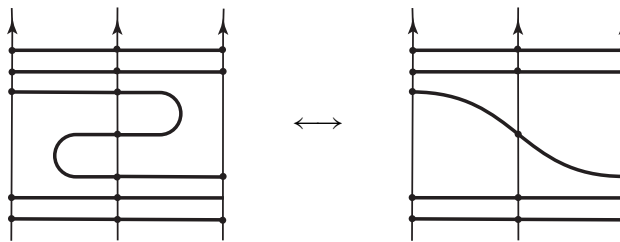


Figure 8: The Snake Move and its inverse

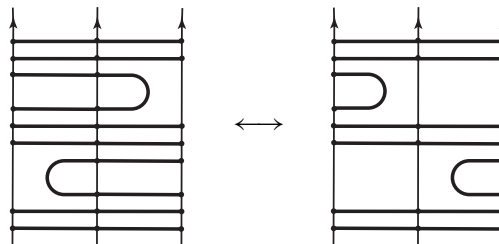


Figure 9: A Time Switch Move

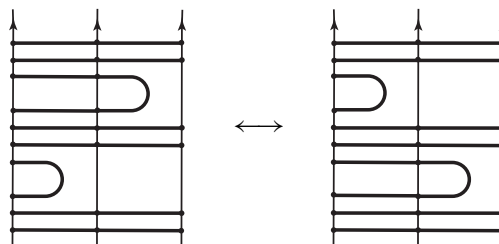


Figure 10: Another Time Switch Move

The invariance of the state sum formula (1) for $\text{Tr}_B([K, s])$ under these Snake and Time Switch Moves follows from an easy computation using the definition of the $\text{Tr}_B([K_i, s_i])$ by Lemma 14 and Condition (2) of Proposition 13. In particular, the hypothesis that $\alpha\beta = -A^3$ is critical for the Snake Move.

This proves Lemma 16, and uniquely defines $\text{Tr}_B([K, s])$ for every skein with no crossing. □

In particular:

Lemma 17 *If K projects to a small circle embedded in the biangle B ,*

$$\text{Tr}_B([K, \emptyset]) = -A^2 - A^{-2}.$$

Proof Putting K in bridge position with only one local maximum and one local minimum, the state sum formula (1) involves only four compatible states, two of which contribute 0 to the sum. Using the hypothesis that $\alpha^2 + \beta^2 = A + A^5$, this gives

$$\text{Tr}_B([K, \emptyset]) = 0 + 0 + \alpha(-A^{-3}\alpha) + \beta(-A^{-3}\beta) = -A^2 - A^{-2}. \quad \square$$

We now define $\text{Tr}_B([K, s])$ for an arbitrary link K with a state s . By resolving all the crossings of K and applying the skein relation, write $[K] \in \mathcal{S}^A(B)$ as a linear combination

$$[K] = \sum_{i=1}^k A^{n_i} [K_i]$$

of skeins $[K_i]$ where the link K_i has no crossing. Then, define

$$\text{Tr}_B([K, s]) = \sum_{i=1}^k A^{n_i} \text{Tr}_B([K_i, s_i]).$$

Lemma 18 *The number $\text{Tr}_B([K, s])$ defined above is independent of the framed isotopy class of K .*

Proof It suffices to show invariance under the second and third Reidemeister Moves. This is a classical consequence of Lemma 17 (see for instance [27, Lemma 3.3]). □

By construction, it is immediate that the $\text{Tr}_B([K, s])$ satisfy the skein relation. Therefore, the construction provides a linear map

$$\text{Tr}_B: \mathcal{S}_s^A(B) \rightarrow \mathbb{C}.$$

It is also immediate that this linear map also satisfies the State Sum Property (1) of Proposition 13. It remains to show that it is an algebra homomorphism.

Lemma 19 For any two stated skeins $[K, s], [K', s'] \in \mathcal{S}_s^A(B)$,

$$\text{Tr}_B([K, s][K', s']) = \text{Tr}_B([K, s]) \text{Tr}_B([K', s']).$$

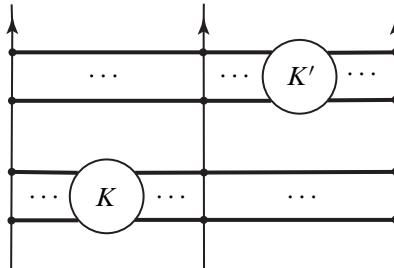


Figure 11: Superposition of skeins K and K'

Proof Our convention of using parallel orientations for the two boundary components of B turns out to be convenient here. The product $[K, s][K', s']$ is equal to $[KK', s \cup s']$, where KK' denotes the superposition of the links K and K' . Because of the orientation convention, the superposition KK' of K and K' can be isotoped so that K and K' sit side by side in B , with K' above K on the sheet of paper as in Figure 11. If we use this configuration in our construction of $\text{Tr}_B([KK', s \cup s'])$, it is then immediate that $\text{Tr}_B([KK', s \cup s']) = \text{Tr}_B([K, s]) \text{Tr}_B([K', s'])$. □

This completes the proof of Proposition 13. □

Although the definition of Tr_B may seem complicated, its computation is much simpler in practice. Indeed, if K is a link whose projection to B has no crossing, each of its components is a closed curve, or an arc of one of the three types of Figure 5. If, in addition, K is endowed with a state s and if $\varepsilon_1 = \pm$ and $\varepsilon_2 = \pm$, let $a_{\varepsilon_2}^{\varepsilon_1}$ be the number of components of the type of Figure 5(a) where the state s assigns signs ε_1 and ε_2 as in that figure; let $b_{\varepsilon_2}^{\varepsilon_1}$ be the number of components of the type of Figure 5(b); let $c_{\varepsilon_2}^{\varepsilon_1}$ be the number of components of the type of Figure 5(c); and let d be the number of closed components of K .

Lemma 20 For a stated skein $[K, s] \in \mathcal{S}_s^A(B)$ with no crossing, let $a_{\varepsilon_2}^{\varepsilon_1}, b_{\varepsilon_2}^{\varepsilon_1}, c_{\varepsilon_2}^{\varepsilon_1}$ and d be defined as above. If one of $a_{\pm}^{\pm}, a_{\pm}^{\mp}, b_{\pm}^{\pm}, b_{\pm}^{\mp}, c_{\pm}^{\pm}, c_{\pm}^{\mp}$ is nonzero, then $\text{Tr}_B([K, s]) = 0$. Otherwise,

$$\text{Tr}_B([K, s]) = \alpha^{b_{\pm}^{\pm}} \beta^{b_{\mp}^{\mp}} (-A^{-3}\alpha)^{c_{\pm}^{\pm}} (-A^{-3}\beta)^{c_{\mp}^{\mp}} (-A^2 - A^{-2})^d.$$

Proof Isotop K so that it is in bridge position and so that: each arc component of K has only one local maximum or minimum; the projection of each closed component of K to B bounds a disk whose interior is disjoint from the projection of K . The formula then follows from the definition of $\text{Tr}_B([K, s])$. \square

In particular, for a skein with no crossings, $\text{Tr}_B([K, s])$ is independent of the relative nesting of the components of the projection of K to B .

The following two observations will be useful later on.

Lemma 21 *If $\text{Tr}_B([K, s])$ is nonzero, the stated skein $[K, s] \in S_s^A(B)$ is balanced in the sense that the sum of the signs assigned by s to the components of ∂K in one component of ∂B is equal to the sum of the signs in the other component of ∂B .*

Proof Lemma 20 proves this for skeins with no crossings. The general case follows from this one by resolving all the crossings and applying the skein relations. \square

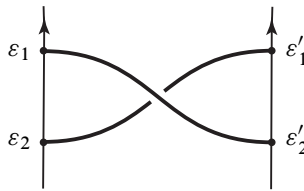


Figure 12: A right-handed half-twist

The following is an immediate computation, based on the definitions.

Lemma 22 *For the right half-twist $[K, s]$ represented in Figure 12,*

$$\text{Tr}_B[K, s] = \begin{cases} A & \text{if } \varepsilon_1 = \varepsilon'_1 = \varepsilon_2 = \varepsilon'_2, \\ A - A^{-4}\alpha^2 & \text{if } \varepsilon_1 = \varepsilon'_1 = + \text{ and } \varepsilon_2 = \varepsilon'_2 = -, \\ A - A^{-4}\beta^2 & \text{if } \varepsilon_1 = \varepsilon'_1 = - \text{ and } \varepsilon_2 = \varepsilon'_2 = +, \\ -A^{-4}\alpha\beta & \text{if } \varepsilon_1 = \varepsilon'_2 \neq \varepsilon_2 = \varepsilon'_1, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

5 Split ideal triangulations

A split ideal triangulation $\hat{\lambda}$ is obtained from an ideal triangulation λ by replacing each edge of λ by two parallel copies of it, separated by a biangle. In particular, $\hat{\lambda}$ is a cell

decomposition of S whose faces consists of finitely many triangles T_1, T_2, \dots, T_m (each corresponding to a face of λ) and finitely many biangles B_1, B_2, \dots, B_n (each corresponding to an edge of λ).

Lemma 23 *Let K be a framed link in $S \times [0, 1]$ and let $\hat{\lambda}$ be a split ideal triangulation of S . Then K can be isotoped so that*

- (1) *for every edge $\hat{\lambda}_i$ of $\hat{\lambda}$, K is transverse to $\hat{\lambda}_i \times [0, 1]$;*
- (2) *for every triangle face T_j of $\hat{\lambda}$, $K \cap (T_j \times [0, 1])$ consists of finitely many disjoint arcs, each of which is contained in a constant elevation surface $S \times *$ and joins two distinct components of $\partial T_j \times [0, 1]$;*
- (3) *for every triangle face T_j of $\hat{\lambda}$, the components of $K \cap (T_j \times [0, 1])$ lie at different elevations, and their framings are vertical pointing upwards.*

The effect of Lemma 23 is to push all the complexities of K into the part of $S \times [0, 1]$ that lies above the biangles of $\hat{\lambda}$.

Proof Select a spine Y_j for each ideal triangle T_j , namely an infinite Y-shaped subset such that T_j properly collapses on Y_j , as in Figure 13.

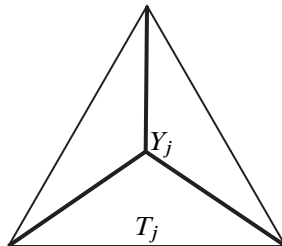


Figure 13: The spine of an ideal triangle

By generic position we can arrange that, for each spine Y_j , the link K is disjoint from the singular locus $\{v_j\} \times [0, 1]$ of each $Y_j \times [0, 1]$, corresponding to the 3-valent vertex $v_j \in Y_j$, and transverse to the rest of $Y_j \times [0, 1]$. With a further isotopy we can assume that on a small neighborhood U_j of $Y_j \times [0, 1]$ each component of $K \cap U_j$ has constant elevation, and that distinct components have distinct elevations. Finally, the framing can be modified so that it is vertical and pointing upwards on $K \cap U_j$.

By definition of the spines Y_j , the union of the $T_j \times [0, 1]$ can be isotoped inside the union of the U_j , and this by an isotopy which respects all level surfaces $S \times *$ and which sends vertical arc $* \times [0, 1]$ to vertical arc. Modifying K by the inverse of this isotopy puts it in the required position. \square

When K satisfies the conclusions of Lemma 23, we will say that it is in *good position* with respect to the split ideal triangulation $\hat{\lambda}$.

When K is in good position, we can always isotop it so that, over each triangle face T_j , $K \cap (T_j \times [0, 1])$ projects to finitely many disjoint arcs embedded in T_j . However, the projection is usually much more complicated when we use the conventions of Section 3.5 to represent the ordering of the components of $K \cap (T_j \times [0, 1])$. The two pictures of Figure 14 illustrate the same example of a link in good position over a triangle T_j , drawn with the conventions of Section 3.5 but with two different choices of orientations for the sides of T_j .

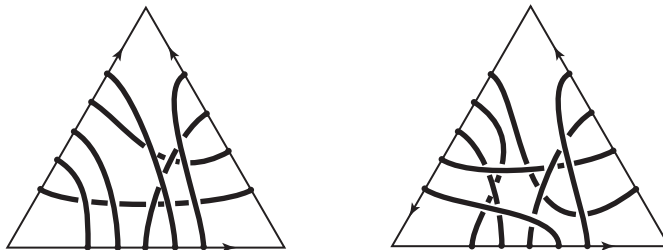


Figure 14: A link in good position over a triangle

Figures 15–19 describe five moves, occurring in a neighborhood of a triangle T_j , which isotop K from one good position to another. It is understood there that K is in good position with respect to the split ideal triangulation, and in particular that each component of $K \cap (T_j \times [0, 1])$ has constant elevation. The intersection $K \cap (T_j \times [0, 1])$ may include more components than the zero, one or two arcs shown. However, when two arcs are shown, it is understood that they are adjacent to each other for the ordering of the components of $K \cap (T_j \times [0, 1])$ given by their elevation; their ordering with respect to each other is determined by the ordering of their end points, indicated by the arrows with the conventions of Section 3.5.

Moves (I) and (II) of Figures 15–16 eliminate a U-turn in biangles adjacent to T_j . Moves (III) and (IV) transpose the elevations of two components of $K \cap (T_j \times [0, 1])$. The kinks added by Move (V) change the framing in two biangles adjacent to T_j (if these two biangles are distinct), but not the isotopy class of the global framing of K since the two kinks have opposite signs.

Lemma 24 *Let K and L be two framed links in $S \times [0, 1]$ satisfying the conclusions of Lemma 23, namely in good position with respect to $\hat{\lambda}$. If K and L are isotopic, then they can be connected by a sequence of framed links $K = K_0, K_1, K_2, \dots, K_l, K_{l+1} = L$ such that each K_i is in good position with respect to $\hat{\lambda}$, and such that one goes from K_i to K_{i+1} by either an isotopy keeping the link in good position with respect to $\hat{\lambda}$, or one of the moves (I)–(V) of Figures 15–19 and their inverses.*

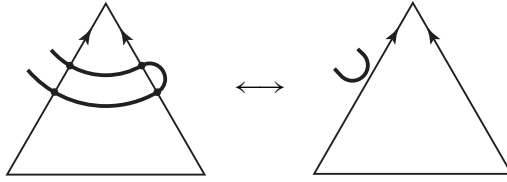


Figure 15: Move (I)

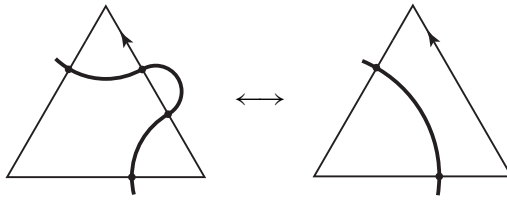


Figure 16: Move (II)

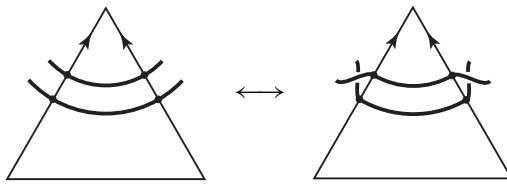


Figure 17: Move (III)

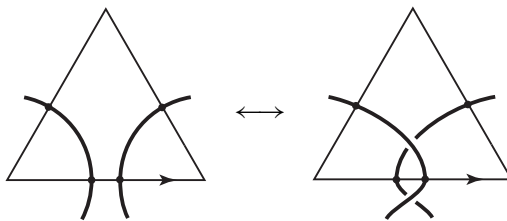


Figure 18: Move (IV)

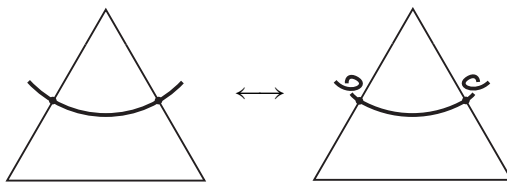


Figure 19: Move (V)

Proof Choose a spine Y_j in each triangle face T_j of the split ideal triangulation $\hat{\lambda}$, as in the proof of Lemma 23. For a framed link K in $S \times [0, 1]$, this proof of Lemma 23 shows that being in good position with respect to the split triangulation $\hat{\lambda}$ is essentially equivalent to the property that, for each spine Y_j ,

- (1) the link K is transverse to $Y_j \times [0, 1]$, and in particular is disjoint from the singular locus $\{v_j\} \times [0, 1]$, where v_j is the trivalent vertex of Y_j ;
- (2) the elevations of the points of $K \cap (Y_j \times [0, 1])$ are distinct;
- (3) the framing of K is vertical, pointing upwards, at each point of $K \cap (Y_j \times [0, 1])$.

When these properties hold, we say that K is in good position with respect to the spine Y_j .

Indeed, by adjusting K in a neighborhood of the Y_j as in the proof of Lemma 23 and by collapsing the triangles T_j in this neighborhood by isotopies, one easily goes back and forth between links that are in good position with respect to the split triangulation $\hat{\lambda}$ and links that are in good position with respect to the spines Y_j .

Let us first neglect the elevations of the intersection points and the framings.

If the two framed links K and L are in good position with respect to the spines Y_j , and if they are isotopic, let us choose the isotopy to be generic with respect to the $Y_j \times [0, 1]$. Then, during the isotopy, the link remains transverse to the $Y_j \times [0, 1]$ except at finitely many times where, either the link crosses one of the singular loci $\{v_j\} \times [0, 1]$, or it is tangent to one of the smooth parts $(Y_j - \{v_j\}) \times [0, 1]$.

When the link crosses the singular locus $\{v_j\} \times [0, 1]$, an intersection point of the link with a component of $(Y_j - \{v_j\}) \times [0, 1]$ gets replaced with two intersection points, one in each of the two other components of $(Y_j - \{v_j\}) \times [0, 1]$, or the converse holds. Translating this in terms of links in good position with respect to the split ideal triangulation $\hat{\lambda}$ gives Move (II) and its inverse.

When the link becomes tangent to $(Y_j - \{v_j\}) \times [0, 1]$ then, generically, either two intersection points of the link with $(Y_j - \{v_j\}) \times [0, 1]$ collide and cancel out, or a pair of intersection points gets created in the inverse process. This is described by Move (I) and its inverse.

So far, we had neglected the elevation of the intersection points. At finitely many times during the generic isotopy, the elevations of two points of some $K \cap (Y_j \times [0, 1])$ will cross each other. This is described by Moves (III) and (IV), according to whether the points are in the same component of $(Y_j - \{v_j\}) \times [0, 1]$ or not.

In particular, when we care about elevations, Moves (III) and (IV) enable us to avoid having to consider two versions of Moves (I) and (II), one for each ordering of the two points of $K \cap (Y_j \times [0, 1])$.

Finally, we have to worry about framings. At some time in the isotopy, we will need to move the framing at a point of $K \cap (Y_j \times [0, 1])$ from vertical position to vertical position by rotating it by a certain number of full turns. This is accomplished by several applications of Move (V) or its inverse. □

6 The quantum trace as a state sum

We now begin the proof of Theorem 11.

Let K be a framed link in $S \times [0, 1]$, with a state $s: \partial K \rightarrow \{+, -\}$. Let λ be an ideal triangulation for S .

Let $\hat{\lambda}$ be a split ideal triangulation associated to λ . By an isotopy, put K in good position with respect to $\hat{\lambda}$ as in Lemma 23. The conclusions of this lemma guarantee that, for every triangle face T_j or biangle face B_i of $\hat{\lambda}$, the intersections $K_j = K \cap (T_j \times [0, 1])$ and $L_i = K \cap (B_i \times [0, 1])$ are framed links in $T_j \times [0, 1]$ or $B_i \times [0, 1]$.

Suppose that, in addition to the state $s: \partial K \rightarrow \{+, -\}$, we are given a state $s_j: \partial K_j \rightarrow \{+, -\}$ for each K_j with $j = 1, 2, \dots, m$, and a state $t_i: \partial L_i \rightarrow \{+, -\}$ for each L_i with $i = 1, 2, \dots, n$. Note that exactly two of these states are defined at every point of $\partial K \cup \bigcup_{j=1}^m \partial K_j \cup \bigcup_{i=1}^n \partial L_i$. We say that all these states s, s_j and t_i are *compatible* if they coincide whenever they are defined at the same point.

For a triangle T_j , let k_1, k_2, \dots, k_l be the components of K_j , in order of increasing elevation (remember that the elevation is constant on each k_i , and that distinct k_i have distinct elevations). Then $K_j = k_1 k_2 \dots k_l$ in the link algebra $\mathcal{K}(T_j)$; note that the order of the terms in this product is important. Let $\text{Tr}_{T_j}(k_i, s_j) \in \mathcal{Z}_{T_j}^\omega$ be defined as in Condition (2)(a) of Theorem 11. Then, define

$$\text{Tr}_{T_j}(K_j, s_j) = \text{Tr}_{T_j}(k_1, s_j) \text{Tr}_{T_j}(k_2, s_j) \dots \text{Tr}_{T_j}(k_l, s_j) \in \mathcal{Z}_{T_j}^\omega.$$

For a biangle B_i of $\hat{\lambda}$, let $\text{Tr}_{B_i}(L_i, t_i) \in \mathbb{C}$ be the scalar provided by Proposition 13.

We can then consider the tensor product

$$\left(\prod_{i=1}^n \text{Tr}_{B_i}(L_i, t_i) \right) \left(\bigotimes_{j=1}^m \text{Tr}_{T_j}(K_j, s_j) \right) \in \bigotimes_{j=1}^m \mathcal{Z}_{T_j}^\omega$$

Recall that the triangles T_j are identified to the faces of the ideal triangulation λ , and that the Chekhov–Fock square root algebra $\mathcal{Z}_\lambda^\omega$ is also contained in $\bigotimes_{j=1}^m \mathcal{Z}_{T_j}^\omega$.

Lemma 25 *If the states s, s_j and t_i are compatible, the element*

$$\left(\prod_{i=1}^n \text{Tr}_{B_i}(L_i, t_i)\right) \left(\bigotimes_{j=1}^m \text{Tr}_{T_j}(K_j, s_j)\right) \in \bigotimes_{j=1}^m \mathcal{Z}_{T_j}^\omega$$

is contained in the square root Chekhov–Fock algebra $\mathcal{Z}_\lambda^\omega$ of Section 2.3.

Proof We first have to check that, when the monomial

$$\left(\prod_{i=1}^n \text{Tr}_{B_i}(L_i, t_i)\right) \left(\bigotimes_{j=1}^m \text{Tr}_{T_j}(K_j, s_j)\right)$$

is different from 0, the generators Z_{ja} and Z_{ka} associated to the two sides of an edge λ_i of λ appear with the same exponent in the monomial. This is an immediate consequence of Lemma 21 and of the definition of the terms $\text{Tr}_{T_j}(K_j, s_j)$.

The fact that this monomial satisfies the parity condition defining the square root algebra $\mathcal{Z}_\lambda^\omega$ automatically follows from the definitions. □

Define

$$\text{Tr}_S(K, s) = \sum_{\text{compatible } s_j, t_i} \left(\prod_{i=1}^n \text{Tr}_{B_i}(L_i, t_i)\right) \left(\bigotimes_{j=1}^m \text{Tr}_{T_j}(K_j, s_j)\right) \in \mathcal{Z}_\lambda^\omega(K),$$

where the sum is over all choices of states $s_j: \partial K_j \rightarrow \{+, -\}$, $j = 1, 2, \dots, m$, and $t_i: \partial L_i \rightarrow \{+, -\}$, $i = 1, 2, \dots, n$, that are compatible with $s: \partial K \rightarrow \{+, -\}$ and with each other.

The key step in the proof of Theorem 11 is the following.

Recall that our definition of Tr_B for biangles B in Section 4 depended on two parameters α and β such that $\alpha^2 + \beta^2 = A + A^5$ and $\alpha\beta = -A^3$. This allowed four possibilities $(\alpha, \beta) = \pm(A^{1/2}, -A^{5/2})$ and $\pm(A^{5/2}, -A^{1/2})$ for these parameters. They now need to be even more restricted. Going over the proof of Proposition 26, the reader will readily check that these restrictions on A, α and β are necessary for the statement to hold.

Proposition 26 *If $A = \omega^{-2}$, $\alpha = -\omega^{-5}$ and $\beta = \omega^{-1}$, the above element*

$$\text{Tr}_S(K, s) \in \mathcal{Z}_\lambda^\omega(K)$$

depends only on the isotopy class of K and on the state s .

Proof By Proposition 13, $\text{Tr}_S(K, s)$ is invariant under isotopy respecting good position with respect to the split ideal triangulation $\hat{\lambda}$. Therefore, we only need to check that it remains unchanged under the Moves (I)–(V) of Lemma 24.

These moves involve a triangle T_j , adjacent to three biangles $B_{i_1}, B_{i_2}, B_{i_3}$. We will restrict attention to the case where these three biangles are distinct. Since it involves only minor modifications in notation and no new arguments, we leave as an exercise to the reader the task of adapting our proof to the case where two of the biangles coincide.

To alleviate the notation, we can assume that the triangle involved is the triangle T_1 , while the adjacent biangles are B_1, B_2, B_3 . In addition, in each of Figures 15–19, we will assume that the B_i are indexed as in Figure 20. In particular, the square root algebra $\mathcal{Z}_{T_1}^\omega$ is defined by generators Z_{11}, Z_{12} and Z_{13} , respectively associated to the edges $T_1 \cap B_1, T_1 \cap B_2$ and $T_1 \cap B_3$, such that $Z_{1i}Z_{1(i+1)} = \omega^2 Z_{1(i+1)}Z_{1i}$ if we count indices modulo 3.

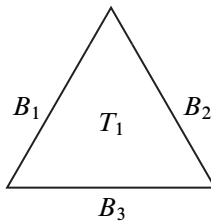


Figure 20: Labeling conventions

Consider a move of type (I)–(V), going from a framed link K (on the left of each of Figures 15–19) to a framed link K' (on the right of Figures 15–19). In the above state sums for $\text{Tr}_S(K, s)$ and $\text{Tr}_S(K', s)$ we group terms so that, in each group, the families of compatible states s_j, t_i for K and s'_j, t'_i for K' coincide outside of the parts of K and K' shown. We then have to show that, in each group, the sum of the contributions to $\text{Tr}_S(K, s)$ of the compatible states s_j, t_i for K considered is equal to the sum of the contributions to $\text{Tr}_S(K', s)$ of the corresponding compatible states s'_j, t'_i for K' . We will group terms even further according to the powers of the generators Z_{11}, Z_{12}, Z_{13} of $\mathcal{Z}_{T_1}^\omega$ (associated to the sides of the triangle T_1) contributed by these states.

In the case of Move (I), for a given family of compatible states s'_j, t'_i for K' , there are 2^4 families of compatible states s_j, t_i for K that coincide with s'_j, t'_i outside of the area shown, but only four of these give a nonzero contribution to $\text{Tr}_S(K, s)$. These are indicated in Figure 21. The equality sign in this figure means that we have to show that the contributions of the terms on one side of the equation add up to the contributions of the other side.



Figure 21: States for Move (I)

For the labeling conventions of Figure 20 and remembering that L_i denotes the portion of K that is above the biangle B_i , let $t_1^{++}, t_1^{-+}, t_1^{+-}$ be the states for L_1 described by the first, second and fourth triangles in Figure 21, in this order. (To explain the notation, note the signs assigned by these states to the two points of $K \cap B_1 \cap T_1$ shown, for the orientation of the edge $B_1 \cap T_1$ specified by the arrow.) Similarly, let t_2^{+-} and t_2^{--} be the states for L_2 represented in the first and third triangles, respectively.

By the State Sum Property of Proposition 13,

$$\begin{aligned} \text{Tr}_{B_1}(L'_1, t'_1) &= \alpha \text{Tr}_{B_1}(L_1, t_1^{-+}) + \beta \text{Tr}_{B_1}(L_1, t_1^{+-}) \\ &= -\omega^{-5} \text{Tr}_{B_1}(L_1, t_1^{-+}) + \omega^{-1} \text{Tr}_{B_1}(L_1, t_1^{+-}) \end{aligned}$$

while

$$\begin{aligned} \text{Tr}_{B_2}(L_2, t_2^{+-}) &= \beta \text{Tr}_{B_2}(L'_2, t'_2) = \omega^{-1} \text{Tr}_{B_2}(L'_2, t'_2), \\ \text{Tr}_{B_2}(L_2, t_2^{--}) &= \alpha \text{Tr}_{B_2}(L'_2, t'_2) = -\omega^{-5} \text{Tr}_{B_2}(L'_2, t'_2). \end{aligned}$$

For each of these states s_1 for K_1 , those components of K_1 that are not represented on Figure 21 and sit below the two arcs shown have the same contribution $X \in \mathcal{Z}_{T_1}^\omega$ to $\text{Tr}_{T_1}(K_1, s_1)$, while the components of K_1 sitting above the two arcs shown contribute $Y \in \mathcal{Z}_{T_1}^\omega$.

Therefore, the contributions to $\text{Tr}_S(K, s)$ of the four families of states t_1, s_1, t_2 on the left of Figure 21 add up to

$$\begin{aligned} &\text{Tr}_{B_1}(L_1, t_1^{++})X[Z_{11}Z_{12}^{-1}][Z_{11}Z_{12}]Y \text{Tr}_{B_2}(L_2, t_2^{-+}) \\ &\quad + \text{Tr}_{B_1}(L_1, t_1^{-+})X[Z_{11}^{-1}Z_{12}^{-1}][Z_{11}Z_{12}]Y \text{Tr}_{B_2}(L_2, t_2^{-+}) \\ &\quad + \text{Tr}_{B_1}(L_1, t_1^{+-})X[Z_{11}Z_{12}][Z_{11}Z_{12}^{-1}]Y \text{Tr}_{B_2}(L_2, t_2^{+-}) \\ &\quad + \text{Tr}_{B_1}(L_1, t_1^{--})X[Z_{11}Z_{12}][Z_{11}^{-1}Z_{12}^{-1}]Y \text{Tr}_{B_2}(L_2, t_2^{+-}) \\ &= \text{Tr}_{B_1}(L_1, t_1^{++})X(\omega Z_{11}Z_{12}^{-1})(\omega Z_{12}Z_{11})Y(-\omega^{-5}) \text{Tr}_{B_2}(L'_2, t'_2) \\ &\quad + \text{Tr}_{B_1}(L_1, t_1^{-+})X(\omega^{-1}Z_{11}^{-1}Z_{12}^{-1})(\omega Z_{12}Z_{11})Y(-\omega^{-5}) \text{Tr}_{B_2}(L'_2, t'_2) \\ &\quad + \text{Tr}_{B_1}(L_1, t_1^{+-})X(\omega^{-1}Z_{11}Z_{12})(\omega^{-1}Z_{12}^{-1}Z_{11})Y\omega^{-1} \text{Tr}_{B_2}(L'_2, t'_2) \\ &\quad + \text{Tr}_{B_1}(L_1, t_1^{--})X(\omega^{-1}Z_{11}Z_{12})(\omega Z_{12}^{-1}Z_{11}^{-1})Y\omega^{-1} \text{Tr}_{B_2}(L'_2, t'_2) \end{aligned}$$

$$\begin{aligned}
 &= -\omega^{-3} \operatorname{Tr}_{B_1}(L_1, t_1^{++}) X Z_{11}^2 Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &\quad - \omega^{-5} \operatorname{Tr}_{B_1}(L_1, t_1^{-+}) X Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &\quad + \omega^{-3} \operatorname{Tr}_{B_1}(L_1, t_1^{++}) X Z_{11}^2 Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &\quad + \omega^{-1} \operatorname{Tr}_{B_1}(L_1, t_1^{+-}) X Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &= (-\omega^{-5} \operatorname{Tr}_{B_1}(L_1, t_1^{-+}) + \omega^{-1} \operatorname{Tr}_{B_1}(L_1, t_1^{+-})) X Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &= \operatorname{Tr}_{B_1}(L'_1, t'_1) X Y \operatorname{Tr}_{B_2}(L'_2, t'_2),
 \end{aligned}$$

which, as required, is the contribution of the right hand side of Figure 21 to $\operatorname{Tr}_S(K', s)$. Recall that square brackets [] denote here the Weyl quantum ordering (see the end of Section 3.4). Also, note that the order in which we multiply the contributions of the various components of K_1 is really crucial in the above computations.

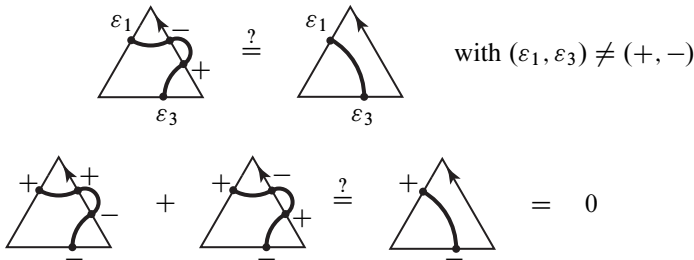


Figure 22: States for Move (II)

We now consider Move (II). If we again group terms according to compatible states s'_j, t'_i for K' , Figure 22 lists the possible restrictions to the part of K involved in Move (II) of all compatible states s_j, t_i for K that make nontrivial contributions to $\operatorname{Tr}_S(K, s)$.

In the case of the first line of Figure 22, note that $\operatorname{Tr}_{B_2}(L_2, t_2) = \beta \operatorname{Tr}_{B_2}(L'_2, t'_2)$ by the State Sum Property of Proposition 13. As before, let $X \in \mathcal{Z}_{T_1}^\omega$ be the contribution of those components of K_1 that are not represented on the figure and sit below the two arcs shown, while Y represents the contribution of the components that sit above these two arcs. Then if, as usual, we identify the sign $\varepsilon = \pm$ to the number $\varepsilon = \pm 1$,

$$\begin{aligned}
 \operatorname{Tr}_{T_1}(K_1, s_1) \operatorname{Tr}_{B_2}(L_2, t_2) &= X[Z_{12} Z_{13}^{\varepsilon_3}][Z_{11}^{\varepsilon_1} Z_{12}^{-1}] Y \beta \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &= X(\omega^{\varepsilon_3} Z_{13}^{\varepsilon_3} Z_{12})(\omega^{-\varepsilon_1} Z_{12}^{-1} Z_{11}^{\varepsilon_1}) Y \omega^{-1} \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &= X(\omega^{\varepsilon_3 - \varepsilon_1 - 1} Z_{13}^{\varepsilon_3} Z_{11}^{\varepsilon_1}) Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &= X[Z_{13}^{\varepsilon_3} Z_{11}^{\varepsilon_1}] Y \operatorname{Tr}_{B_2}(L'_2, t'_2) \\
 &= \operatorname{Tr}_{T_1}(K'_1, s'_1) \operatorname{Tr}_{B_2}(L'_2, t'_2)
 \end{aligned}$$

because $\varepsilon_3 - \varepsilon_1 - 1 = -\varepsilon_1 \varepsilon_3$ when $(\varepsilon_1, \varepsilon_3) \neq (+1, -1)$.

For the second line of Figure 22, there are two families of compatible states s_1^{-+}, t_2^{-+} and s_1^{+-}, t_2^{+-} for K that correspond to compatible states s'_1, t'_2 for K' and give nontrivial contributions to $\text{Tr}_S(K, s)$. Note that s'_1, t'_2 contribute 0 to $\text{Tr}_S(K', s)$. Then,

$$\begin{aligned} & \text{Tr}_{T_1}(K_1, s_1^{-+}) \text{Tr}_{B_2}(L_2, t_2^{-+}) + \text{Tr}_{T_1}(K_1, s_1^{+-}) \text{Tr}_{B_2}(L_2, t_2^{+-}) \\ &= X[Z_{12}^{-1} Z_{13}^{-1}][Z_{11} Z_{12}]Y\alpha \text{Tr}_{B_2}(L'_2, t'_2) \\ & \quad + X[Z_{12} Z_{13}^{-1}][Z_{11} Z_{12}^{-1}]Y\beta \text{Tr}_{B_2}(L'_2, t'_2) \\ &= X(\omega Z_{13}^{-1} Z_{12}^{-1})(\omega Z_{12} Z_{11})Y(-\omega^{-5}) \text{Tr}_{B_2}(L'_2, t'_2) \\ & \quad + X(\omega^{-1} Z_{13}^{-1} Z_{12})(\omega^{-1} Z_{12}^{-1} Z_{11})Y\omega^{-1} \text{Tr}_{B_2}(L'_2, t'_2) \\ &= -\omega^{-3} X Z_{13}^{-1} Z_{11} Y \text{Tr}_{B_2}(L'_2, t'_2) + \omega^{-3} X Z_{13}^{-1} Z_{11} Y \text{Tr}_{B_2}(L'_2, t'_2) \\ &= 0 = \text{Tr}_{T_1}(K'_1, s'_1) \text{Tr}_{B_2}(L'_2, t'_2). \end{aligned}$$

This concludes the proof of invariance under Move (II).

For Move (III), instead of state sums, it is more convenient to use the compatibility of the $\text{Tr}_{B_i}(L_i, t_i)$ with the skein relations, as proved by Proposition 13.

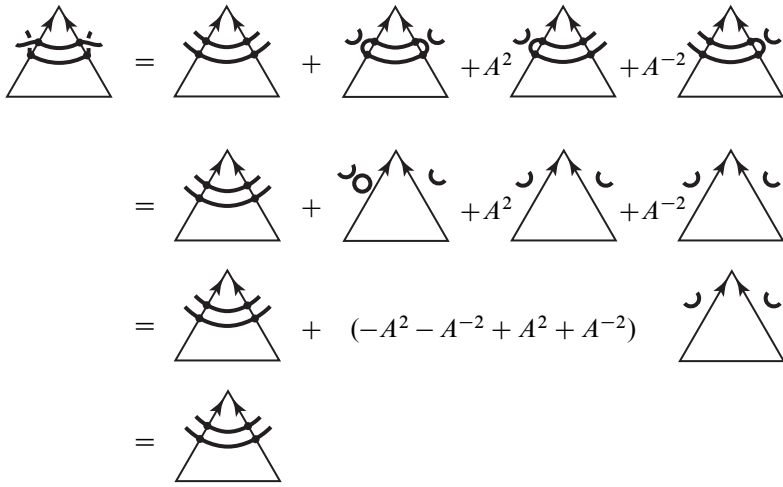


Figure 23: Skein manipulations for Move (III)

The proof of the invariance under Move (III) is described by Figure 23, where the equalities between linear combinations are understood to apply to the images of the corresponding links under Tr_S . The first equality comes from the fact that Tr_{B_1} and Tr_{B_2} are compatible with the skein relations, as proved by Proposition 13. The second equality is a consequence of the invariance of Tr_S under Move (I), which we just

proved. The third equality is a consequence of the fact that, in the skein algebra $S^A(B_1)$, adding a small unknotted unlinked loop to a skein $[L]$ just multiplies $[L]$ by the scalar $-(A^2 + A^{-2})$ (see Lemma 12).

We now turn to Move (IV). All the states that have nontrivial contributions are listed in Figure 24. Grouping the contributions of compatible states according to their restrictions outside of the pictures, and according to powers of the generators Z_{11}, Z_{12}, Z_{13} of $\mathcal{Z}_{T_1}^\omega$, we have to show the equalities of contributions indicated.

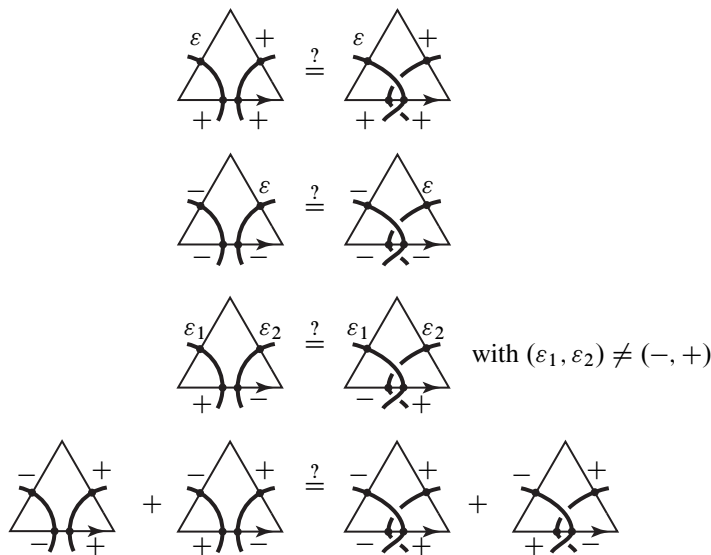


Figure 24: State sums for Move (IV)

As usual, let $t_3^{\varepsilon\varepsilon'}$ be the state for L_3 or L'_3 where the two boundary points represented are respectively labelled by $\varepsilon, \varepsilon' \in \{+, -\}$, in this order for the orientation of the edge $T_1 \cap B_3$ specified by the arrow (while the value of $t_3^{\varepsilon\varepsilon'}$ on the other points of ∂L_3 and $\partial L'_3$ is determined by the group of compatible states that we are considering). By combining Proposition 13 and Lemma 22,

$$\begin{aligned} \text{Tr}_{B_3}(L'_3, t_3^{++}) &= \omega^{-2} \text{Tr}_{B_3}(L_3, t_3^{++}), \\ \text{Tr}_{B_3}(L'_3, t_3^{--}) &= \omega^{-2} \text{Tr}_{B_3}(L_3, t_3^{--}), \\ \text{Tr}_{B_3}(L'_3, t_3^{+-}) &= \omega^2 \text{Tr}_{B_3}(L_3, t_3^{+-}) + (\omega^{-2} - \omega^6) \text{Tr}_{B_3}(L_3, t_3^{+-}), \\ \text{Tr}_{B_3}(L'_3, t_3^{-+}) &= \omega^2 \text{Tr}_{B_3}(L_3, t_3^{-+}). \end{aligned}$$

Then, for the first line of Figure 24,

$$\begin{aligned} \text{Tr}_{T_1}(K'_1, s'_1) \text{Tr}_{B_3}(L'_3, t'_3) &= X[Z_{12}Z_{13}][Z_{11}^\varepsilon Z_{13}]Y \text{Tr}_{B_3}(L'_3, t_3^{+++}) \\ &= X\omega^2[Z_{11}^\varepsilon Z_{13}][Z_{12}Z_{13}]Y\omega^{-2} \text{Tr}_{B_3}(L_3, t_3^{+++}) \\ &= X[Z_{11}^\varepsilon Z_{13}][Z_{12}Z_{13}]Y \text{Tr}_{B_3}(L_3, t_3^{+++}) \\ &= \text{Tr}_{T_1}(K_1, s_1) \text{Tr}_{B_3}(L_3, t_3), \end{aligned}$$

where, as usual, X and Y denote the contributions of the components of K_1 and K'_1 that respectively sit below and above the two arcs represented.

The case of the second line is almost identical:

$$\begin{aligned} \text{Tr}_{T_1}(K'_1, s'_1) \text{Tr}_{B_3}(L'_3, t'_3) &= X[Z_{12}^\varepsilon Z_{13}^{-1}][Z_{11}^{-1} Z_{13}^{-1}]Y \text{Tr}_{B_3}(L'_3, t_3^{---}) \\ &= X\omega^2[Z_{11}^{-1} Z_{13}^{-1}][Z_{12}^\varepsilon Z_{13}^{-1}]Y\omega^{-2} \text{Tr}_{B_3}(L_3, t_3^{---}) \\ &= X[Z_{11}^{-1} Z_{13}^{-1}][Z_{12}^\varepsilon Z_{13}^{-1}]Y \text{Tr}_{B_3}(L_3, t_3^{---}) \\ &= \text{Tr}_{T_1}(K_1, s_1) \text{Tr}_{B_3}(L_3, t_3). \end{aligned}$$

For the third line of Figure 24,

$$\begin{aligned} \text{Tr}_{T_1}(K'_1, s'_1) \text{Tr}_{B_3}(L'_3, t'_3) &= X[Z_{12}^{\varepsilon_2} Z_{13}^{-1}][Z_{11}^{\varepsilon_1} Z_{13}]Y \text{Tr}_{B_3}(L'_3, t_3^{+-}) \\ &= X\omega^{2(-\varepsilon_1\varepsilon_2 + \varepsilon_2 - \varepsilon_1)}[Z_{11}^{\varepsilon_1} Z_{13}][Z_{12}^{\varepsilon_2} Z_{13}^{-1}]Y\omega^2 \text{Tr}_{B_3}(L_3, t_3^{+-}) \\ &= X[Z_{11}^{\varepsilon_1} Z_{13}][Z_{12}^{\varepsilon_2} Z_{13}^{-1}]Y \text{Tr}_{B_3}(L_3, t_3^{+-}) \\ &= \text{Tr}_{T_1}(K_1, s_1) \text{Tr}_{B_3}(L_3, t_3) \end{aligned}$$

as required. Note that $-\varepsilon_1\varepsilon_2 + \varepsilon_2 - \varepsilon_1 = -1$ exactly when $(\varepsilon_1, \varepsilon_2) \neq (-1, +1)$.

The case of the fourth line of Figure 24 is more elaborate.

$$\begin{aligned} \text{Tr}_{T_1}(K'_1, s_1^{-+}) \text{Tr}_{B_3}(L'_3, t_3^{-+}) + \text{Tr}_{T_1}(K'_1, s_1^{+-}) \text{Tr}_{B_3}(L'_3, t_3^{+-}) &= X[Z_{12}Z_{13}^{-1}][Z_{11}^{-1} Z_{13}]Y \text{Tr}_{B_3}(L'_3, t_3^{-+}) \\ &\quad + X[Z_{12}Z_{13}][Z_{11}^{-1} Z_{13}^{-1}]Y \text{Tr}_{B_3}(L'_3, t_3^{+-}) \\ &= X(\omega Z_{12}Z_{13}^{-1})(\omega Z_{13}Z_{11}^{-1})Y\omega^2 \text{Tr}_{B_3}(L_3, t_3^{-+}) \\ &\quad + X(\omega^{-1}Z_{12}Z_{13})(\omega^{-1}Z_{13}^{-1}Z_{11}^{-1})Y\omega^2 \text{Tr}_{B_3}(L_3, t_3^{+-}) \\ &\quad + X(\omega^{-1}Z_{12}Z_{13})(\omega^{-1}Z_{13}^{-1}Z_{11}^{-1})Y(\omega^{-2} - \omega^6) \text{Tr}_{B_3}(L_3, t_3^{+-}) \\ &= \omega^4 XZ_{12}Z_{11}^{-1}Y \text{Tr}_{B_3}(L_3, t_3^{-+}) + XZ_{12}Z_{11}^{-1}Y \text{Tr}_{B_3}(L_3, t_3^{+-}) \\ &\quad + \omega^{-4} XZ_{12}Z_{11}^{-1}Y \text{Tr}_{B_3}(L_3, t_3^{+-}) - \omega^4 XZ_{12}Z_{11}^{-1}Y \text{Tr}_{B_3}(L_3, t_3^{+-}) \end{aligned}$$

$$\begin{aligned}
&= \omega^2 X Z_{11}^{-1} Z_{12} Y \operatorname{Tr}_{B_3}(L_3, t_3^{-+}) + \omega^{-2} X Z_{11}^{-1} Z_{12} Y \operatorname{Tr}_{B_3}(L_3, t_3^{+-}) \\
&= X(\omega Z_{11}^{-1} Z_{13}^{-1})(\omega Z_{13} Z_{12}) Y \operatorname{Tr}_{B_3}(L_3, t_3^{-+}) \\
&\quad + X(\omega^{-1} Z_{11}^{-1} Z_{13})(\omega^{-1} Z_{13}^{-1} Z_{12}) Y \operatorname{Tr}_{B_3}(L_3, t_3^{+-}) \\
&= X[Z_{11}^{-1} Z_{13}^{-1}][Z_{12} Z_{13}] Y \operatorname{Tr}_{B_3}(L_3, t_3^{-+}) \\
&\quad + X[Z_{11}^{-1} Z_{13}][Z_{12} Z_{13}^{-1}] Y \operatorname{Tr}_{B_3}(L_3, t_3^{+-}).
\end{aligned}$$

This concludes our proof that $\operatorname{Tr}_S(K, s)$ remains invariant under Move (IV).

The case of Move (V) is much simpler. Indeed, by Lemma 12,

$$\begin{aligned}
\operatorname{Tr}_{B_1}(L'_1, t_1) &= -A^{-3} \operatorname{Tr}_{B_1}(L_1, t_1), \\
\operatorname{Tr}_{B_2}(L'_2, t_2) &= -A^3 \operatorname{Tr}_{B_2}(L_2, t_2).
\end{aligned}$$

Therefore, when computing $\operatorname{Tr}_S(K', s)$, the two scalars $-A^{-3}$ and $-A^3$ cancel out, and $\operatorname{Tr}_S(K', s) = \operatorname{Tr}_S(K, s)$.

This concludes our proof that $\operatorname{Tr}_S(K, s)$ is invariant under the moves (I)–(V), at least under our original assumption that the biangles B_1, B_2, B_3 touching the triangle T_1 where each move takes place are distinct. As indicated at the beginning, we are leaving as an exercise to the reader the task of adapting our arguments to the case where two of these three biangles are equal.

By Lemma 24, this concludes the proof of Proposition 26. \square

Lemma 27 *The above element*

$$\operatorname{Tr}_S(K, s) \in \mathcal{Z}_\lambda^\omega(K)$$

depends only on the class $[K, s] \in \mathcal{S}_s^A(S)$ of the framed link K and its state s in the skein algebra.

Proof We have to show that Tr_S is compatible with the skein relations, namely that

$$\operatorname{Tr}_S(K_1, s) = A^{-1} \operatorname{Tr}_S(K_0, s) + A \operatorname{Tr}_S(K_\infty, s)$$

when the framed links K_1, K_0 and K_∞ form a Kauffman triple, namely are related as in Figure 1.

When we put K_1 in good position with respect to the split ideal triangulation $\hat{\lambda}$ as in Lemma 23, we can always arrange that the little ball where K_1, K_0 and K_∞ differ is located above a biangle B_j . For that biangle, Proposition 13 asserts that Tr_{B_j} is

compatible with the skein relations. In particular, if L_j^0, L_j^1, L_j^∞ are the respective intersections of K_1, K_0 and K_∞ with $B_j \times [0, 1]$, it follows from Proposition 13 that

$$\text{Tr}_{B_j}(L_j^1, s_j) = A^{-1} \text{Tr}_{B_j}(L_j^0, s_j) + A \text{Tr}_{B_j}(L_j^\infty, s_j)$$

for every state s_j . By definition of Tr_S as a state sum, this immediately proves the desired result. \square

We are now ready to prove Theorem 11. Indeed, the combination of Proposition 26 and Lemma 27 provides a linear map

$$\text{Tr}_S: \mathcal{S}_s^A(S) \rightarrow \mathcal{Z}_\lambda^\omega$$

defined by $\text{Tr}_S([K, s]) = \text{Tr}_S(K, s)$. This linear map is well-behaved under the superposition operation, so that it is actually an algebra homomorphism.

Because of its construction as a state sum, it is also immediate that the family of homomorphisms Tr_S satisfy the State Sum Condition (1) of Theorem 11.

This State Sum Condition also shows that the homomorphisms Tr_S are uniquely determined by their restriction to the case where S is a triangle. When S is a triangle, the skein algebra $\mathcal{S}^A(S)$ is generated by simple arcs of the type appearing in Condition (2) of Theorem 11 (use the skein relations to eliminate all crossings, and apply Lemma 12 to remove all simple closed curves). The uniqueness part of Theorem 11 immediately follows.

This concludes the proof of Theorem 11. \square

7 Invariance under changes of ideal triangulations

The homomorphism $\text{Tr}_S: \mathcal{S}_s^A(S) \rightarrow \mathcal{Z}_\lambda^\omega$ provided by Theorem 11 depends of course on the ideal triangulation λ of S considered. We now show that it is well behaved under change of ideal triangulation.

Since we now have to worry about different ideal triangulations, we will write $\text{Tr}_S^\lambda([K, s])$ for the element that we have so far denoted $\text{Tr}_S([K, s]) \in \mathcal{Z}_\lambda^\omega$.

Given two ideal triangulations λ and λ' of S , let $\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$ be the coordinate change map provided by Theorem 6.

Theorem 28 *Given two ideal triangulations λ and λ' of S , and a stated skein $[K, s] \in \mathcal{S}_s^A(S)$, the coordinate change map*

$$\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$$

sends the polynomial $\text{Tr}_S^{\lambda'}([K, s])$ to the polynomial $\text{Tr}_S^\lambda([K, s])$.

Note that, in general, the coordinate change map $\Theta_{\lambda\lambda'}^\omega$ sends a polynomial $P \in \mathcal{Z}_{\lambda'}^\omega$ to a rational fraction in $\widehat{\mathcal{Z}}_\lambda^\omega$. It is therefore surprising that the trace polynomials $\text{Tr}_S^\lambda([K, s])$ remain polynomial under coordinate change.

Proof of Theorem 28 By [22; 30], any two ideal triangulations can be connected to each other by a sequence of diagonal exchanges, as in Figure 25. Since it is proved in [24, Theorem 25] that $\Theta_{\lambda\lambda''}^\omega = \Theta_{\lambda\lambda'}^\omega \circ \Theta_{\lambda'\lambda''}^\omega$ for every three ideal triangulations λ , λ' and λ'' , it will be sufficient to restrict attention to the case where λ and λ' differ only by a diagonal exchange.

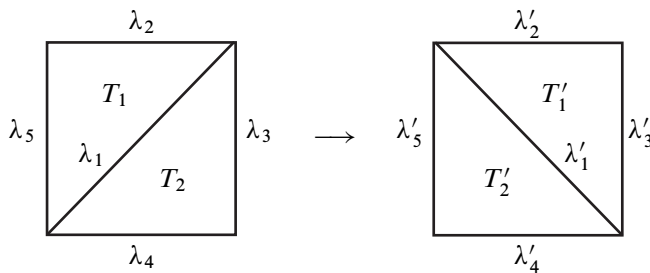


Figure 25: A diagonal exchange

We will assume that the indexing of the edges and faces of λ and λ' is as in Figure 25. Beware that it is quite possible that there exists identifications between the sides of the square represented, for instance that $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_3$; however, this will have no impact on our arguments.

For the split ideal triangulation $\widehat{\lambda}$ associated to λ let, as usual, T_j denote the triangle face associated to the j -th face of λ , and let B_i be the biangle face corresponding to the edge λ_i of λ . We use similar conventions for the split ideal triangulation $\widehat{\lambda}'$ associated to λ' .

Put the framed link K in good position with respect to the split ideal triangulation $\widehat{\lambda}$, as in Lemma 23. When doing so we can always arrange that, above the square $T_1 \cup B_1 \cup T_2$ formed by the triangles T_1 , T_2 and the biangle B_1 , the components of $K \cap ((T_1 \cup B_1 \cup T_2) \times [0, 1])$ are all horizontal arcs. Indeed, we can always push any complication of the picture away from the square $T_1 \cup B_1 \cup T_2$ and into one of the bingles B_i with $i > 1$.

The same property will then hold in $\widehat{\lambda}'$ since we can always arrange that $T'_1 \cup B'_1 \cup T'_2 = T_1 \cup B_1 \cup T_2$. In particular, K is now in good position with respect to both $\widehat{\lambda}$ and $\widehat{\lambda}'$.

In the state sum expression of $\text{Tr}_S^\lambda([K, s])$, we can then group the contributions of the components of $K \cap ((T_1 \cup B_1 \cup T_2) \times [0, 1])$ into blocks in $\mathcal{Z}_{T_1}^\omega \otimes \mathcal{Z}_{T_2}^\omega$ of one of the following types.

- (1) For components going from $\lambda_2 \times [0, 1]$ to $\lambda_3 \times [0, 1]$:
 - (a) $[Z_{12}Z_{11}] \otimes [Z_{21}Z_{23}]$
 - (b) $[Z_{12}Z_{11}] \otimes [Z_{21}Z_{23}^{-1}] + [Z_{12}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{23}^{-1}]$
 - (c) $[Z_{12}^{-1}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{23}^{-1}]$
- (2) For components going from $\lambda_2 \times [0, 1]$ to $\lambda_4 \times [0, 1]$:
 - (a) $[Z_{12}Z_{11}] \otimes [Z_{21}Z_{24}] + [Z_{12}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{24}]$
 - (b) $[Z_{12}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{24}^{-1}]$
 - (c) $[Z_{12}^{-1}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{24}^{-1}]$
 - (d) $[Z_{12}^{-1}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{24}^{-1}]$
- (3) For components going from $\lambda_2 \times [0, 1]$ to $\lambda_5 \times [0, 1]$:
 - (a) $[Z_{12}Z_{15}] \otimes 1$
 - (b) $[Z_{12}^{-1}Z_{15}] \otimes 1$
 - (c) $[Z_{12}^{-1}Z_{15}^{-1}] \otimes 1$
- (4) For components going from $\lambda_3 \times [0, 1]$ to $\lambda_4 \times [0, 1]$:
 - (a) $1 \otimes [Z_{23}Z_{24}]$
 - (b) $1 \otimes [Z_{23}Z_{24}^{-1}]$
 - (c) $1 \otimes [Z_{23}^{-1}Z_{24}^{-1}]$
- (5) For components going from $\lambda_3 \times [0, 1]$ to $\lambda_5 \times [0, 1]$:
 - (a) $[Z_{15}Z_{11}] \otimes [Z_{21}Z_{23}]$
 - (b) $[Z_{15}Z_{11}] \otimes [Z_{21}Z_{23}^{-1}]$
 - (c) $[Z_{15}^{-1}Z_{11}] \otimes [Z_{21}Z_{23}]$
 - (d) $[Z_{15}^{-1}Z_{11}] \otimes [Z_{21}Z_{23}^{-1}] + [Z_{15}^{-1}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{23}^{-1}]$
- (6) For components going from $\lambda_4 \times [0, 1]$ to $\lambda_5 \times [0, 1]$:
 - (a) $[Z_{15}Z_{11}] \otimes [Z_{21}Z_{24}]$
 - (b) $[Z_{15}^{-1}Z_{11}] \otimes [Z_{21}Z_{24}] + [Z_{15}^{-1}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{24}]$
 - (c) $[Z_{15}^{-1}Z_{11}^{-1}] \otimes [Z_{21}^{-1}Z_{24}^{-1}]$

Then $\text{Tr}_S^{\lambda'}([K, s])$ is obtained from $\text{Tr}_S^\lambda([K, s])$ by replacing each of the above blocks by the corresponding block in the list below, while the remaining Z_{ji} with $j > 2$ are replaced with the corresponding Z'_{ji} .

- (1) For components going from $\lambda'_2 \times [0, 1]$ to $\lambda'_3 \times [0, 1]$:
 - (a) $[Z'_{12}Z'_{13}] \otimes 1$
 - (b) $[Z'_{12}Z'_{13}^{-1}] \otimes 1$
 - (c) $[Z'_{12}^{-1}Z'_{13}^{-1}] \otimes 1$

- (2) For components going from $\lambda'_2 \times [0, 1]$ to $\lambda'_4 \times [0, 1]$:
- (a) $[Z'_{12} Z'_{11}] \otimes [Z'_{21} Z'_{24}]$
 - (b) $[Z'_{12} Z'_{11}] \otimes [Z'_{21} Z'_{24}{}^{-1}]$
 - (c) $[Z'_{12}{}^{-1} Z'_{11}] \otimes [Z'_{21} Z'_{24}]$
 - (d) $[Z'_{12}{}^{-1} Z'_{11}] \otimes [Z'_{21} Z'_{24}{}^{-1}] + [Z'_{12}{}^{-1} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{24}{}^{-1}]$
- (3) For components going from $\lambda'_2 \times [0, 1]$ to $\lambda'_5 \times [0, 1]$:
- (a) $[Z'_{12} Z'_{11}] \otimes [Z'_{21} Z'_{25}]$
 - (b) $[Z'_{12}{}^{-1} Z'_{11}] \otimes [Z'_{21} Z'_{25}] + [Z'_{12}{}^{-1} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{25}]$
 - (c) $[Z'_{12}{}^{-1} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{25}{}^{-1}]$
- (4) For components going from $\lambda'_3 \times [0, 1]$ to $\lambda'_4 \times [0, 1]$:
- (a) $[Z'_{12} Z'_{11}] \otimes [Z'_{21} Z'_{24}]$
 - (b) $[Z'_{12} Z'_{11}] \otimes [Z'_{21} Z'_{24}{}^{-1}] + [Z'_{12} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{24}{}^{-1}]$
 - (c) $[Z'_{12}{}^{-1} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{24}{}^{-1}]$
- (5) For components going from $\lambda'_3 \times [0, 1]$ to $\lambda'_5 \times [0, 1]$:
- (a) $[Z'_{13} Z'_{11}] \otimes [Z'_{21} Z'_{25}] + [Z'_{13} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{25}]$
 - (b) $[Z'_{13} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{25}]$
 - (c) $[Z'_{13} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{25}{}^{-1}]$
 - (d) $[Z'_{13}{}^{-1} Z'_{11}{}^{-1}] \otimes [Z'_{21}{}^{-1} Z'_{25}{}^{-1}]$
- (6) For components going from $\lambda'_4 \times [0, 1]$ to $\lambda'_5 \times [0, 1]$:
- (a) $1 \otimes [Z'_{24} Z'_{25}]$
 - (b) $1 \otimes [Z'_{24} Z'_{25}{}^{-1}]$
 - (c) $1 \otimes [Z'_{24}{}^{-1} Z'_{25}{}^{-1}]$

The coordinate change map $\Theta_{\lambda\lambda'}^\omega: \widehat{\mathcal{Z}}_{\lambda'}^\omega \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$ is defined in [24] by a similar block-by-block analysis. It turns out that it is very well-behaved with respect to the blocks in the above two lists. Indeed, Hiatt proves in [24, Lemma 21] that $\Theta_{\lambda\lambda'}^\omega$ sends each block of $\mathcal{Z}_{T'_1}^\omega \otimes \mathcal{Z}_{T'_2}^\omega$ in the list above to the corresponding block in $\mathcal{Z}_{T_1}^\omega \otimes \mathcal{Z}_{T_2}^\omega$ in the first list, while sending each element of $\mathcal{Z}_{T'_j}^\omega$ with $j > 2$ to the element of $\mathcal{Z}_{T_j}^\omega$ obtained by removing the primes $'$. When combined with our original observations, this proves that

$$\Theta_{\lambda\lambda'}^\omega(\mathrm{Tr}_S^{\lambda'}([K, s])) = \mathrm{Tr}_S^\lambda([K, s])$$

in the case considered, namely when λ and λ' differ by a diagonal exchange.

As noted at the beginning of our discussion, this completes the proof of Theorem 28. \square

8 Injectivity of the quantum trace homomorphism

We conclude with a simple observation.

Proposition 29 *The quantum trace homomorphism*

$$\mathrm{Tr}_\lambda^\omega: \mathcal{S}^A(S) \rightarrow \widehat{\mathcal{Z}}_\lambda^\omega$$

of Theorem 1 is injective.

Proof As a vector space, the skein algebra is clearly generated by the family of all skeins $[K] \in \mathcal{S}^A(S)$ that are *simple*, in the sense that they are represented by 1-submanifolds of S (with no crossings, and with vertical framing) whose components are not homotopic to 0.

For such a simple skein $[K]$, our state sum construction of the Laurent polynomial $\mathrm{Tr}_\lambda^\omega([K]) \in \widehat{\mathcal{Z}}_\lambda^\omega$ shows that its highest degree term is a nonzero scalar multiple of $Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}$, where $k_i \geq 0$ is the geometric intersection number of K with the i -th edge λ_i of the ideal triangulation λ .

The key observation is now that a simple skein $[K]$ can be completely recovered from the collection (k_1, k_2, \dots, k_n) of its geometric intersection numbers. It easily follows that the image under $\mathrm{Tr}_\lambda^\omega$ of a nontrivial linear combination of simple skeins cannot be 0 (focus attention on a term for which (k_1, k_2, \dots, k_n) is maximal), which proves that the kernel of $\mathrm{Tr}_\lambda^\omega$ is trivial. \square

Incidentally, the above argument also provides another proof that simple skeins are linearly independent in $\mathcal{S}^A(S)$.

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