

## Strongly contracting geodesics in Outer Space

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We study the Lipschitz metric on Outer Space and prove that fully irreducible elements of  $\text{Out}(F_n)$  act by hyperbolic isometries with axes which are strongly contracting. As a corollary, we prove that the axes of fully irreducible automorphisms in the Cayley graph of  $\text{Out}(F_n)$  are Morse, meaning that a quasi-geodesic with endpoints on the axis stays within a bounded distance from the axis.

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### Introduction

There exists a striking analogy between the mapping class groups of surfaces, and the outer automorphism group  $\text{Out}(F_n)$  of a rank  $n$  free group. At the core of this analogy lies Culler and Vogtmann’s Outer Space  $\mathcal{X}_n$  [16], a contractible finite dimensional cell complex on which  $\text{Out}(F_n)$  has a properly discontinuous action. Like Teichmüller space, Outer Space has an invariant spine on which the action is cocompact, making it a good topological model for the study of  $\text{Out}(F_n)$ . Indeed, Outer Space has played a key role in proving theorems for  $\text{Out}(F_n)$ , which were classically known for the mapping class group. For example, Bestvina and Feighn [6] show that  $\text{Out}(F_n)$  is a virtual duality group by showing that the Borel–Serre bordification of Outer Space is  $2n - 5$  connected at infinity. There have been three well studied metrics on Teichmüller space: the Teichmüller metric, the Weil–Petersson metric, and the Lipschitz metric. When the present work was conducted, the study of the geometry of Outer Space was still in its infancy (see Handel and Mosher [23] and Francaviglia and Martino [20]). Since then, Bestvina [4] has found a new proof of the classification theorem of outer automorphisms using the geometry of Outer Space.

One would like to define a metric on Outer Space so that fully irreducible elements of  $\text{Out}(F_n)$  (which are analogous to pseudo-Anosov elements in  $\mathcal{MCG}(S)$ ) act by hyperbolic isometries with meaningful translation lengths. But immediately one encounters a problem: it isn’t clear whether to require the metric to be symmetric. To clarify, we follow the discussion in Handel and Mosher [24]. Consider the situation of a pseudo-Anosov map  $\psi$  acting on Teichmüller space  $\mathcal{T}(S_{g,p})$  with the Teichmüller metric  $d_{\mathcal{T}}$ . Associated to  $\psi$  is an expansion factor  $\lambda_{\psi}$  and two measured foliations

$\mathcal{F}^s$  and  $\mathcal{F}^u$  so that  $\psi$  expands the transverse measure of  $\mathcal{F}^u$  by  $\lambda_\psi$  and contracts the transverse measure of  $\mathcal{F}^s$  by  $\lambda_\psi^{-1}$ . Incidentally,  $\lambda_\psi = \lambda_{\psi^{-1}}$ . Furthermore, by Teichmüller's theorem, the translation length of  $\psi$  in the Teichmüller metric is  $\log(\lambda_\psi)$ . Going back to  $\text{Out}(F_n)$ , one can associate to a fully irreducible outer automorphism  $\Phi$  a Perron–Frobenius eigenvalue  $\lambda_\Phi$  which plays much the same roll as the expansion factor in the pseudo-Anosov case. If we did have an honest metric on Outer Space where  $\Phi$  was a hyperbolic isometry then the axis for  $\Phi$  would also be an axis for  $\Phi^{-1}$ . Thus for a point  $x$  on the axis of  $\Phi$ ,  $d(x, \Phi(x)) = \log(\lambda_\Phi)$  and  $d(\Phi(x), x) = \log(\lambda_{\Phi^{-1}})$ . However, it is not always the case that  $\lambda_\Phi = \lambda_{\Phi^{-1}}$ . Therefore one would have to abandon either the symmetry of the metric or the relationship between the translation length  $\Phi$  and its Perron–Frobenius eigenvalue. We choose to do the former in order to preserve the ties between the action of  $\Phi$  on  $\mathcal{X}_n$  and its action on the conjugacy classes of  $F_n$ .

The (non-symmetric) metric that we carry over from  $\mathcal{T}(S_{g,p})$  to Outer Space is the Lipschitz metric introduced by Thurston [32]. Given two marked hyperbolic structures  $(X, f), (Y, g)$  on a surface  $S$  define

$$d_L((X, f), (Y, g)) = \inf \{ \text{Lip}(h) \mid h \text{ is Lipschitz, and homotopic to } g \circ f^{-1} \}$$

In [12] Rafi and Choi proved that this metric is Lipschitz equivalent to  $d_T$  in the thick part of  $\mathcal{T}(S_{g,p})$ .

While  $\mathcal{T}(S_{g,p})$  with  $d_T$  is not CAT(0) (see Masur [28]) or Gromov hyperbolic (see Masur and Wolf [29]) it does exhibit some features of negative curvature in the thick part. A geodesic is *strongly contracting* if its nearest point projection takes balls disjoint from the geodesic to sets of bounded diameter, where the bound is independent of the radius of the ball. That is, the “shadow” that a ball casts on the geodesic is bounded. For example, geodesics in a Gromov hyperbolic space are strongly contracting. In [30] Minsky proved that geodesics contained in the  $\epsilon$ -thick part of  $\mathcal{T}(S_{g,p})$  are uniformly strongly contracting, with the bound only depending on  $\epsilon$  and the topology of  $S$ . Note that any axis of a pseudo-Anosov map is contained in the  $\epsilon$ -thick part of  $\mathcal{T}(S_{g,p})$  for a sufficiently small  $\epsilon$ . We prove

**Theorem** *An axis of a fully irreducible outer automorphism is strongly contracting.*

A geodesic  $L$  in a metric space is *Morse* if every quasi-geodesic segment with endpoints on  $L$  stays within a bounded neighborhood of  $L$  which only depends on the quasi-geodesic constants. As an application of the theorem above we prove:

**Corollary** *In the Cayley graph of  $\text{Out}(F_n)$ , the axis of a fully irreducible automorphism is Morse.*

Lenzhen, Rafi, and Tao [25] have since shown that geodesics with bounded combinatorics in Teichmüller space with the Lipschitz metric have a strongly contracting projection. Hamenstaedt [22] has used different methods to show that lines of minima are contracting in Outer Space.

This paper is organized as follows:

- In Section 1 we go over some definitions and background on Outer Space. The well informed reader could skip this part.
- In Section 2 we define the Lipschitz metric on Outer Space, and deduce a formula which expresses the relationship between the metric and the lengths of loops in  $X$  and  $Y$  (proof due to Tad White and first written in Francaviglia and Martino [20]).
- In Section 3 we describe axes of fully irreducible automorphisms. Given such an axis, we define a coarse projection of  $\mathcal{X}_n$  onto this axis. Note that the axis for  $\Phi$  will not necessarily be an axis for  $\Phi^{-1}$ , however the projections of a point to both axes are uniformly close.
- In Section 4 we define the Whitehead graph  $Wh_X(\Lambda^\pm)$  of the attracting and repelling laminations of  $\Phi$  for a rose  $X \in \mathcal{X}_n$ . We prove that there exists a point  $F \in \mathcal{X}_n$  for which  $Wh_F(\Lambda^+) \cup Wh_F(\Lambda^-)$  is connected and does not contain a cut vertex.
- In Section 5 we use the previous result to show that any loop  $\alpha$  which represents a basis element cannot contain long pieces of both laminations. Next we prove our main “negative curvature” property. If the projections of  $x$  and  $y$  are sufficiently far apart then  $d(x, y)$  is coarsely larger than  $d(x, p(x)) + d(p(x), p(y))$ . We show that this is enough to prove that  $L$  is a strongly contracting geodesic. We end the Chapter by proving that in the Cayley graph of  $\text{Out}(F_n)$ , axes of fully irreducible automorphisms are Morse.
- In Section 6 we have collected some applications: the asymptotic cone of the Cayley graph of  $\text{Out}(F_n)$  contains many cut points and is in fact tree graded, the divergence function in  $\mathcal{X}_n$  is at least quadratic. Finally, we show that projections onto two axes  $A, B$  of independent irreducible automorphisms satisfy a dichotomy similar to the one shown by Behrstock [3] for subsurface projections.

**A note on notation** Many of the theorems and propositions in this article contain several constants which we usually denote  $s$  or  $c$  within the proposition. When referring to a constant from a previous proposition, we add its number as a subscript.

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## 1 Preliminary notions

### The outer automorphism group of the free group

**Definition 1.1** The group of outer automorphisms of the free group of rank  $n$  is

$$\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inner}(F_n)$$

If  $\phi$  is an automorphism we denote its class by  $\Phi$ .

**Definition 1.2** Let  $\{x_1, \dots, x_n\}$  be a basis for  $F_n$ . Let  $A \subset \{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$  and  $a \in A$  so that  $a^{-1} \notin A$ . Then the *Whitehead automorphism*  $\phi_{(A,a)}$  associated with  $(A, a)$  is defined as follows.  $\phi_{(A,a)}(a) = a$  and for  $x \neq a, a^{-1}$ :

$$\begin{aligned} x &\rightarrow axa^{-1} && \text{if } x, x^{-1} \in A \\ x &\rightarrow xa^{-1} && \text{if } x \in A \text{ and } x^{-1} \notin A \\ x &\rightarrow ax && \text{if } x \notin A \text{ and } x^{-1} \in A \\ x &\rightarrow x && \text{if } x, x^{-1} \notin A \end{aligned}$$

**Theorem** (Whitehead generators) *The following set generates  $\text{Out}(F_n)$*

$$\{[\phi_{(A,a)}] \mid \text{all possible } a, A\}$$

### Bases of $\text{Out}(F_n)$ and Whitehead’s Theorem

**Definition 1.3** (The Whitehead Graph) Let  $\mathcal{B} = \{y_1, \dots, y_n\}$  be a basis of  $F_n$ , let  $\alpha$  be the conjugacy class, in  $F_n$  and  $w \in \alpha$  a cyclically reduced word written in the basis  $\mathcal{B}$ . Then the *Whitehead graph* of  $\alpha$  with respect to  $\mathcal{B}$  is denoted  $Wh_{\mathcal{B}}(\alpha)$  and constructed as follows: The vertex set of this graph is the set  $\mathcal{B} \cup \mathcal{B}^{-1}$ .  $z_i$  and  $z_j$  are connected by an edge if  $z_i^{-1}z_j$  or  $z_j^{-1}z_i$  appears in the cyclic word  $w$ , that is, if  $w = \dots z_i^{-1}z_j \dots$  or  $w = \dots z_j^{-1}z_i \dots$  or  $w = z_j \dots z_i^{-1}$  or  $w = z_i \dots z_j^{-1}$ .

The Whitehead graph  $Wh_{\mathcal{B}}([w_1], \dots, [w_k])$  of the set  $\{[w_1], \dots, [w_k]\}$  is the superposition of the individual Whitehead graphs  $Wh_{\mathcal{B}}([w_i])$ .

**Definition 1.4** We say that the set of conjugacy classes  $\{\alpha_1, \dots, \alpha_k\}$  can be completed to a basis if there are  $w_i \in \alpha_i$  and  $w_{k+1}, \dots, w_n$  such that  $\{w_1, \dots, w_n\}$  is a basis for  $F_n$ .

Let  $v$  be a vertex in the graph  $G$ . Let  $C_v$  be the connected component of  $G$  containing  $v$ .  $v$  is a cut vertex if  $C_v \setminus \{v\}$  is disconnected. We will discuss cut vertices only when  $G$  is connected, that is, when  $C_v = G$ .

**Theorem 1.5** (Whitehead [33]) *If  $\alpha_1, \dots, \alpha_k$  can be completed to a basis and  $Wh_{\mathcal{B}}(\alpha_1, \dots, \alpha_k)$  is connected, then  $Wh_{\mathcal{B}}(\alpha_1, \dots, \alpha_k)$  has a cut vertex.*

A related notion is the following

**Definition 1.6** (Free Factor) Let  $A$  be a subgroup of  $F_n$ ,  $A$  is a free factor of  $F_n$  if there exists a subgroup  $B$  such that  $F_n = A * B$ .

Let  $\mathcal{B} = \{x_1, \dots, x_n\}$  be a basis for  $F_n$ , and suppose we want to determine whether  $[y_1], \dots, [y_k]$  can be completed to a basis, where  $y_i$  are cyclically reduced words. We construct  $W = Wh_{\mathcal{B}}([y_1], \dots, [y_k])$ .

- If  $W$  is connected and does not contain a cut vertex then  $[y_1], \dots, [y_k]$  cannot be completed to a basis by Theorem 1.5.
- If  $W$  is connected and contains a cut vertex  $a$  we will construct a new basis  $\mathcal{B}'$  such that

$$(1) \quad \sum_{i=1}^k |y_i|_{\mathcal{B}} > \sum_{i=1}^k |y_i|_{\mathcal{B}'}$$

Let  $W'$  be the induced subgraph of  $W$  on all of the vertices except  $a$ . Let  $W''$  be a connected component of  $W'$ , which does not contain  $a^{-1}$ . Take  $A$  to be the elements of  $\mathcal{B}$  whose vertices are in  $W''$  and  $a$ . Then  $\mathcal{B}' = \phi_{(A,a)}(\mathcal{B})$ . It is straightforward to check that (1) is satisfied.

- If  $W$  is not connected, one can continue carrying out the algorithm on a subset of the generators, until either there is some subset of the  $y_i$ s that do not form a free factor (since their Whitehead graph is connected with no cut vertex) or there is a basis  $\mathcal{B}''$  where  $|y_i|_{\mathcal{B}''} = 1$  for all  $i$ , which means  $\mathcal{B}''$  contains  $y_1, \dots, y_k$ .

Whitehaed's Theorem 1.5 may be reformulated in the following way.

**Definition 1.7**  $[y_1], [y_2], \dots, [y_k]$  are compatible with a free decomposition of  $F_n$ , if there exists a free splitting  $A * B$  so that for all  $i$ , either  $[y_i] \in [A]$  or  $[y_i] \in [B]$ .

**Theorem 1.8** (Martin [27]) *The following are equivalent:*

- (1)  $[y_1], \dots, [y_k]$  are compatible with a free decomposition of  $F_n$ .
- (2) If  $\mathcal{B}$  is a basis such that  $Wh_{\mathcal{B}}([y_1], \dots, [y_k])$  contains no cut vertex then it is disconnected.

## Outer Space

A *graph* will always be a finite cell complex of dimension 1 with all vertices of valence  $> 2$ . A *metric* on a graph  $G$  is a function  $\ell: E(G) \rightarrow [0, 1]$  defined on the set of edges of  $G$  such that

- $\sum_{e \in E(G)} \ell(e) = 1$ . We shall denote the total sum of lengths of edges in the metric graph  $G$  by  $\text{vol}(G)$ .
- $\bigcup_{\ell(e)=0} e$  is a forest, that is, it contains no circles.

The space  $\Sigma_G$  of all such metrics  $\ell$  on  $G$  is a “simplex with missing faces”; the missing faces correspond to degenerate metrics that vanish on a subgraph which is not a forest. If  $G'$  is obtained from  $G$  by collapsing a forest, then we have a natural inclusion  $\Sigma_{G'} \subset \Sigma_G$ .

The rose  $R_0$  is the wedge of  $n$  circles. A *marking* is a homotopy equivalence  $f: R_0 \rightarrow G$  from the rose to a graph. A *marked graph* is a pair  $(G, f)$  where  $f: R_0 \rightarrow G$  is a marking.

**Definition 1.9** (Outer Space: graph definition) Outer Space  $\mathcal{X}_n$  consists of the set of equivalence classes of triples  $(G, f, \ell)$  where  $G$  is a graph,  $\ell$  is a metric, and  $f$  is a marking, and so that  $(G, f, \ell) \sim (G', f', \ell')$  if there is an isometry  $\phi: (G, \ell) \rightarrow (G', \ell')$  so that  $\phi \circ f$  is homotopic to  $f'$ .

**Definition 1.10** Throughout the paper we will abuse notation by referring to a point in  $\mathcal{X}_n$  as  $G$ .

An equivalent definition is the following:

**Definition 1.11** (Outer Space: tree definition) Outer Space  $\mathcal{X}_n$  is the space of equivalence classes of free, simplicial, minimal  $F_n$ -trees, with the equivalence:  $T \sim T'$  if there exists an  $F_n$ -equivariant homothety  $\rho: T \rightarrow T'$ .

There is a natural right action of  $\text{Aut}(F_n)$ , the group of automorphisms of  $F_n$ , on  $\mathcal{X}_n$ . Let  $\phi$  be an automorphism and let  $g: R_0 \rightarrow R_0$  be a map such that  $g_*: \pi_1(R_0, \text{ver}) \rightarrow \pi_1(R_0, \text{ver})$  equals  $\phi$ . Then

$$\begin{aligned} \phi: \quad \mathcal{X}_n &\rightarrow \mathcal{X}_n \\ (X, f, \ell) &\rightarrow (X, f \circ g, \ell) \end{aligned}$$

Notice that this action does not depend on the choice of  $g$ , and that inner automorphisms act trivially. Thus we get an action of  $\text{Out}(F_n)$  on  $\mathcal{X}_n$ . When we define the Lipschitz metric it will be evident that this is an isometric action.

### The axes topology

Consider the set of non-trivial conjugacy classes  $\mathcal{C}$  in  $F_n$ . Each  $F_n$ -tree  $T$  induces a length function  $\ell_T: F_n \rightarrow \mathbb{R}$  by  $\ell_T(x) = \text{tr}(x)$  the translation length of  $x$  as an isometry of  $T$ . Since the translation length is a class function,  $\ell_T$  descends to a map  $\ell_T: \mathcal{C} \rightarrow \mathbb{R}$ . Therefore we can define a map

$$\begin{aligned} \ell: \mathcal{X}_n &\rightarrow \mathbb{RP}^{\mathcal{C}} \\ [T] &\rightarrow [\ell_T] \end{aligned}$$

In [15] Culler and Morgan proved that this map is injective. Thus  $\mathcal{X}_n$  inherits a topology from  $\mathbb{RP}^{\mathcal{C}}$  known as the axes topology. We remark (although we will not need this) that there are other ways to define a topology on  $\mathcal{X}_n$ : using the cellular structure of  $\mathcal{X}_n$ , and using the Gromov topology on the space of metric  $F_n$ -trees. Paulin [31] proved that all three topologies are equivalent.

### The boundary of Outer Space

In [15] Culler and Morgan showed that  $\overline{\mathcal{X}_n}$  is compact. It was later shown by Cohen and Lustig [13] and Bestvina and Feighn [5] that  $\overline{\mathcal{X}_n}$  is the space of homothety classes of very small, minimal  $F_n$ -trees.

### Train-track structures and maps

**Definition 1.12** (Turns and train-track structures) Let  $G$  be a graph. An unordered pair of oriented edges  $\{e_1, e_2\}$  is a *turn* if  $e_1, e_2$  have the same initial endpoint. Let  $\bar{e}$

denote the edge  $e$  with the opposite orientation. If an edge path  $\alpha = \dots \bar{e}_1 e_2 \dots$  or  $\alpha = \dots \bar{e}_2 e_1 \dots$  then we say that  $\alpha$  *crosses* or *contains* the turn  $\{e_1, e_2\}$ .

A *train track structure* on  $G$  is an equivalence relation on the set of oriented edges  $E(G)$  with the property that if  $e_1 \sim e_2$  then  $\{e_1, e_2\}$  is a turn.

**Definition 1.13** (Legal turns and gates) A turn  $\{e_1, e_2\}$  is *legal* with respect to a fixed train-track structure on  $G$  if  $e_1 \not\sim e_2$ . An edge path is legal if every turn it crosses is legal. The equivalence classes of the edges are called *gates*.

**Definition 1.14** (A t-t structure induced by a self-map) Let  $g: G \rightarrow G$  be map which restricts on each edge to either an immersion or a constant map. The train-track structure induced by  $g$  is the following equivalence relation:  $e_1 \sim e_2$  if they have the same initial endpoint and there is some  $m \geq 1$  such that there are small enough initial subsegments of  $g^m(e_1)$  and  $g^m(e_2)$  which coincide.

**Definition 1.15** (Train-track maps) Let  $g: G \rightarrow G$  be map which restricts on each edge to either an immersion or a constant map.  $g$  is a (*weak*) *train-track map* if for all  $e \in E(G)$ , the path  $g(e)$  is legal (with respect to the t-t structure in 1.14). A weak train-track map is called a *train-track map* if in addition, vertices are mapped to vertices.

For us, the distinction between a train-track map and a weak train-track map will not be important so we subsequently drop the adjective “weak”.

## Irreducible outer automorphisms

**Definition 1.16** Let  $\Phi$  be an outer automorphism.  $\Phi$  is *reducible* if there exists a free product decomposition

$$F_n = H_1 * \dots * H_m * U$$

where all  $H_i$  are nontrivial,  $m \geq 1$  and where  $\Phi$  permutes the conjugacy classes of  $H_1, \dots, H_m \subseteq F_n$ . If  $m = 1$  we also require that  $U$  is non-trivial. An outer automorphism  $\Phi$  is said to be *irreducible* if it is not reducible.

**Definition 1.17**  $\Phi$  is called *fully irreducible* if all of its powers are irreducible.

The main content of the classification theorem of  $\text{Out}(F_n)$  is that irreducible outer automorphisms have “nice” representatives that are called train-track maps.

**Definition 1.18** Let  $\Phi \in \text{Out}(F_n)$  a *topological representative* of  $\Phi$  is a marked graph  $(G, h)$  and a self map  $f: G \rightarrow G$  such that



- (1) the restriction of  $f$  to each edge to either an immersion or a constant map.
- (2) if  $k: G \rightarrow R_0$  is a homotopy inverse of  $h$  then  $(h \circ f \circ k)_* \in \Phi$ .

**Definition 1.19** (irreducible maps) A *core* graph  $H$  is a graph, all of whose vertices have valence  $\geq 2$ . Let  $g: G \rightarrow G$  be map that restricts on each edge to either an immersion or a constant map.  $g$  is *reducible* if there is a proper, nonempty core subgraph  $H$  of  $G$  which is invariant under  $g$ . If  $g$  is not reducible it is called *irreducible*.

**Theorem 1.20** (Bestvina and Handel [10]) *If  $\Phi \in \text{Out}(F_n)$  is irreducible then  $\Phi$  has an irreducible train-track representative  $(G, h, f)$ .*

**Definition 1.21** If  $\Phi$  is an irreducible outer automorphism and  $f: G \rightarrow G$  is a train-track representative of  $\Phi$  one can endow  $G$  with a metric  $\ell: E(G) \rightarrow (0, 1)$  so that  $f$  stretches each edge of  $G$  by the same amount  $c_{\text{PF}}(f) > 1$ .  $c_{\text{PF}}(f)$  is called the *expansion factor* of  $f$ , or the Perron–Frobenius eigenvalue of  $f$ . Even though the automorphism  $\Phi$  has many different train-track representatives, their stretch factors are equal and thus we may write  $c_{\text{PF}}(\Phi)$ .

### Laminations of fully irreducible automorphisms

Let  $f: G \rightarrow G$  be a train-track representative of an irreducible outer automorphism  $\phi$ , let  $c$  be the expansion factor of  $f$ . By replacing  $f$  with a power if necessary, we may assume that  $f$  has a fixed point  $p$  in the interior of an edge. Let  $I$  be an  $\epsilon$  neighborhood of  $p$  so that  $f(I) \supset I$ . Choose an isometry  $\lambda: (-\epsilon, \epsilon) \rightarrow I$  and extend uniquely to a local isometric immersion  $\lambda: \mathbb{R} \rightarrow G$  so that  $\lambda(c^m t) = f^m(t)$  for all  $t \in \mathbb{R}$ .  $\lambda$  is a *periodic leaf in the lamination  $\Lambda_f^+(G)$*  (for a definition of  $\Lambda_f^+$  see Bestvina, Feighn and Handel [7]). A *stable leaf subsegment* is the restriction of  $\lambda$  to a subinterval of  $\mathbb{R}$ . Given a different metric graph  $H \in \mathcal{X}_n$  and a homotopy equivalence  $g: G \rightarrow H$ ,  $\Lambda_f^+(H)$  the attracting lamination in the  $H$  coordinate is the collection of immersions  $[g\lambda]$ . This definition does not depend on the train-track representative, so we can denote it by  $\Lambda_\phi^+$ . An important feature of the leaves of  $\Lambda_\phi^+$  is given by the following proposition.

**Proposition 1.22** (Bestvina, Feighn and Handel [7, Proposition 1.8]) *Every periodic leaf of  $\Lambda_\phi^+$  is quasi-periodic.*

This means that for every length  $L$  there is a length  $L'$  such that if  $\alpha, \beta \subseteq \lambda$  are subleaf segments with  $\text{length}(\alpha) = L$  and  $\text{length}(\beta) > L'$  then  $\beta$  contains an occurrence of  $\alpha$ . One can think of  $\lambda$  as a necklace made of beads. The segments of length  $L$  that appear in  $\lambda$  are beads of different colors. The proposition tells us that in any subchain of  $\frac{L'}{L}$  consecutive beads we can find beads of all possible colors.

## Geometric and nongeometric automorphisms

**Definition 1.23** An outer automorphism  $\Phi$  of  $F_n$  is *geometric* if there is a surface automorphism  $F: S_{g,p} \rightarrow S_{g,p}$  with  $\pi_1(S_{g,p}) = F_n$  so that  $F_* \in \Phi$ .

When  $n = 2$  all elements of  $\text{Out}(F_n)$  are geometric. This is false for  $n > 2$ . In this section we describe when an irreducible automorphism is geometric.

**Notation 1.24** Consider a graph  $G$  if  $\alpha$  is a loop we denote by  $[\alpha]$  the immersed loop which is freely homotopic to  $\alpha$ . If  $\alpha$  is a path then  $[\alpha]$  is the immersed path homotopic to  $\alpha$  relative to its endpoints.

**Definition 1.25** (Nielsen paths) Let  $f: G \rightarrow G$  be a map, a *Nielsen path* of  $f$  is a path  $\beta$  such that there is some integer  $i$  such that  $[f^i(\beta)] = \beta$ . Note that if  $\alpha, \beta$  are Nielsen paths then so is  $\alpha\beta$ . A Nielsen path is *indivisible* if it cannot be expressed as a concatenation of other Nielsen paths.

**Definition 1.26** Let  $f: G \rightarrow G$  be a train-track map.  $f$  is *stable* if it has no more than one invariant indivisible Nielsen path.

**Theorem 1.27** (Bestvina and Handel [10]) *Every irreducible  $\Phi \in \text{Out}(F_n)$  has a stable irreducible train track representative.*

**Theorem 1.28** (Bestvina and Handel [10]) *Let  $\Phi$  be an irreducible outer automorphism and  $f: G \rightarrow G$  a stable train-track representative for  $\Phi$ .  $\Phi$  is geometric if and only if  $f$  has an invariant Nielsen loop  $\beta$ , that is,  $[f(\beta)] = \beta$ . Moreover,  $\beta$  has the property that it crosses every edge twice.*

If  $F: S_{g,p} \rightarrow S_{g,p}$  represents  $\Phi$  then one of the boundary components will be a Nielsen loop of the type described in the theorem. If  $f: G \rightarrow G$  is a train-track map with a Nielsen loop  $\beta$  as described in the theorem then one can attach an annulus along one of its boundary components to  $G$  along  $\beta$  and get a surface  $S_{g,1}$  and an induced map  $f$  on it which represents  $\Phi$ .  $\beta$  will correspond to the boundary on this surface.

## 2 The Lipschitz metric on $\mathcal{X}_n$

Let  $(G, f, \ell)$  and  $(G', f', \ell')$  represent two points  $x, y$  in  $\mathcal{X}_n$ . A *difference of markings* is a map  $h: G \rightarrow G'$  with  $h \circ f$  homotopic to  $f'$ . We will assume that  $h$  is Lipschitz.

By  $\text{Lip}(h)$  denote the Lipschitz constant of  $\phi$ , that is, the smallest number such that  $d_y(\phi(p), \phi(p')) \leq \text{Lip}(\phi) \cdot d_x(p, p')$  for all  $p, p' \in G$ . Define the distance

$$d(x, y) = \min_h \log \text{Lip}(h)$$

where  $\min$  is taken over all differences of markings (it is attained by Arzela–Ascoli).

We claim that we may restrict our attention to maps  $h$  that are linear on edges, since the minimum is realized by a linear map: For any map  $h$  one can construct a map  $k \sim h$  which is linear on edges. Define  $k(v) = h(v)$  on every vertex  $v$  of  $G$  and let the image of the edge  $(v, w)$  under  $k$  be the immersed path  $[h(v), h(w)]$ , which is homotopic to  $\text{im}(h|_{[v,w]})$  rel endpoints and parameterized at a constant speed. It is clear that  $\text{Lip}(k) \leq \text{Lip}(h)$ . Therefore, we can usually restrict our attention to linear maps.

**Lemma 2.1** *The Lipschitz distance satisfies the following:*

- (1)  $d(x, y) \geq 0$  with equality only if  $x = y$ .
- (2)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in \mathcal{X}_n$ .
- (3)  $d$  is a geodesic metric; for any  $x, y$  there is a path from  $x$  to  $y$  whose length is  $d(x, y)$ .

**Proof** (1) Let  $h$  be the linear map realizing  $d(x, y)$ . Since  $h$  is a difference in marking, it is a homotopy equivalence. If  $h$  were not onto, then  $h_*$  would not be an isomorphism (since there are no edges in  $y$  that can be homotoped away from every loop, recall that there are no free edges). Thus,  $\text{vol}(y) \leq \text{vol}(\text{Im}(h)) \leq \text{Lip}(h) \text{vol}(x)$  (because  $h$  stretches all edges by at most  $\text{Lip}(h)$ ). Since  $\text{vol}(y) = \text{vol}(x) = 1$  we get  $\text{Lip}(h) \geq 1$  hence  $d(x, y) \geq 0$ . We get equality if only if  $\text{Lip}(h) = 1$  and  $h$  is a local isometry, hence if and only if  $h$  is an isometry. This implies  $x = y$ .

(2) Suppose  $h: x \rightarrow y, k: y \rightarrow z$  realize the distance, we shall also call them *optimal maps*, then  $j = k \circ h$  is a difference in marking from  $x$  to  $z$ , thus by the chain rule  $\text{Lip}(h)\text{Lip}(k) \geq \text{Lip}(j) \geq \exp(d(x, z))$ . Taking log we get  $d(x, y) + d(y, z) \geq d(x, z)$ .

(3) See, for example, Francaviglia and Martino [20]. □

## 2.1 Candidates and computing distances

**Definition 2.2** Suppose  $(G, f, \ell)$  represents  $x$  in  $\mathcal{X}_n$ . Let  $\alpha$  be a loop in  $G$ .  $\ell(\alpha)$  is the length of an immersed loop  $[\alpha]$  freely homotopic to  $\alpha$  in  $x$ .

We say that a loop  $\alpha$  in  $G$  is a *candidate* if either

- it is embedded, or
- (figure eight) there are two embedded circles  $u, v$  in  $G$  that intersect in one point and  $\alpha$  crosses  $u, v$  once and does not cross any edges outside of  $u$  and  $v$ , or
- (barbell) there are two disjoint embedded circles  $u, v$  in  $G$  and an arc  $w$  that connects them, and whose interior is disjoint from  $u$  and  $v$ , and  $\alpha$  crosses  $u, v$  once,  $w$  twice, and no edges outside  $u \cup v \cup w$ .

The following is due to Tad White, and first published in the paper by Francaviglia and Martino [20]. We give a shorter proof here.

**Proposition 2.3** *Let  $x, y \in \mathcal{X}_n$ ,  $x = (G, f, \ell)$ ,  $y = (G', f', \ell')$  and let  $g: G \rightarrow G'$  be a difference of markings. Then*

$$d(x, y) = \log \sup_{\alpha} \frac{\ell'(g(\alpha))}{\ell(\alpha)}$$

where the supremum is taken over all loops  $\alpha$  in  $x$ . Moreover, there is a candidate loop  $\alpha$  in  $G$  which realizes the supremum.

**Definition 2.4** (a t-t structure induced by a map) Let  $g: G \rightarrow H$  be a map whose restriction to each edge is either an immersion or constant. The train-track structure induced by  $g$  is the following equivalence relation:  $e_1 \sim e_2$  if they have the same initial vertex and if there are small enough initial segments of  $g(e_1)$  and  $g(e_2)$ , which coincide (they define the same germ).

**Remark 2.5** Notice that the train-track structure defined in 2.4 is smaller (has less illegal turns) than the one in Definition 1.14. When we refer to a t-t structure defined by  $g$  and  $g$  is a self-map we will always mean the structure defined in 1.14.

**Proof of Proposition 2.3** Let  $\alpha$  be an immersed loop in  $G$ . The loop  $g(\alpha)$  might not be immersed.

$$(2) \quad l'(g(\alpha)) \leq \text{Lip}(g)l(\alpha)$$

This is a strict inequality if one of the edges that  $\alpha$  crosses is stretched less than  $\text{Lip}(g)$  or if the loop  $g(\alpha)$  is not immersed. We have

$$(3) \quad d(x, y) \geq \log \sup_{\alpha} \frac{\ell'(g(\alpha))}{\ell(\alpha)}$$

We define the *green graph* of  $G$  with respect to  $g$  as the set of edges on which the slope of  $g$  is  $\text{Lip}(g)$ . We denote it by  $G_g$ . There is a train-track structure on  $G_g$  induced by  $g$  as in Definition 2.4.

Now suppose  $g$  is an optimal map, that is, it satisfies  $d(x, y) = \log \text{Lip}(g)$ , and suppose that  $G_g$  is the smallest among all optimal maps. To show equality in (3) we will find a candidate loop  $\alpha$  that is contained in  $G_g$  and is legal with respect to the train-track structure of  $g$  thus obtaining equality in (2).

First, we claim there are at least two gates at each vertex. See Figure 1 for an example. Suppose by way of contradiction that there is a vertex  $v$  which has only one gate, we will construct a new map  $h$  homotopic to  $g$  so that either  $\text{Lip}(h) < \text{Lip}(g)$  or  $\text{Lip}(h) = \text{Lip}(g)$  and  $G_h \subsetneq G_g$  which gives a contradiction. Define a map  $h$  by  $h(u) = g(u)$  for all vertices  $u \neq v$ . To define  $h(v)$ , recall that all images of edges in  $G_g$  adjacent to  $v$  start with the same initial subsegment in  $G'$  (there is one gate). Take the subsegment to be small enough as to not contain a vertex. Let  $h(v)$  be the point on this subsegment, a distance  $\varepsilon$  (to be chosen later) away from  $g(v)$ . Define  $h$  to be homotopic to  $g$  and linear on edges.

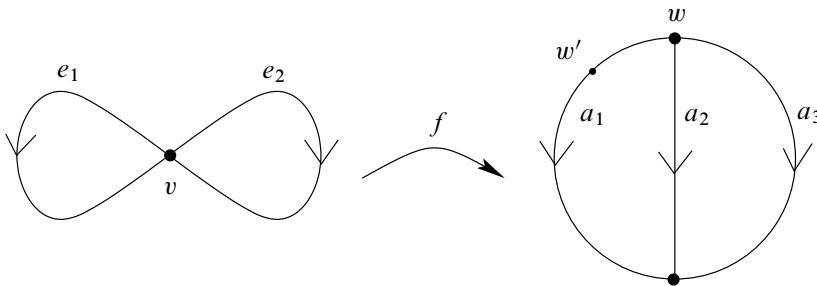


Figure 1: The following is an example of a map  $f$  where  $X_f$  has a vertex with one gate. We show that the map is not optimal. In the graph on the left both edges have length  $\frac{1}{2}$  and in the graph on the right all three edges have length  $\frac{1}{3}$ . Suppose the map  $f$  takes  $e_1 \rightarrow a_2\bar{a}_3$  and  $e_2 \rightarrow a_1\bar{a}_2a_1\bar{a}_3a_2\bar{a}_1$ . The stretch of  $f$  on  $e_1$  is  $\frac{2/3}{1/2} = \frac{4}{3}$  and the stretch of  $f$  on  $e_2$  is  $\frac{6/3}{1/2} = 4$ , so  $\text{Lip}(f) = 4$  and the green graph of  $f$  is  $X_f$  which is just  $e_2$ . Both  $e_2, \bar{e}_2$  begin with  $a_1$  so  $X_f$  contains only one gate at  $v$ . Let  $w'$  be a point on  $a_1$  which is  $\varepsilon$  away from  $v$  where  $8\varepsilon < 4 - \frac{4}{3}$ .  $w'$  divides  $a_1$  into two edges  $b_1, b_2$ . Consider the map  $f_1$  which takes  $e_1 \rightarrow \bar{b}_1a_2\bar{a}_3b_1$  and  $e_2 \rightarrow b_2\bar{a}_2b_1b_2\bar{a}_3a_2\bar{b}_2$ .  $f_1$  is homotopic to  $f$ .  $f_1$  stretches  $e_1$  by  $\frac{2/3+2\varepsilon}{1/2} = \frac{4}{3} + 4\varepsilon$  and  $e_2$  by  $\frac{2-2\varepsilon}{1/2} = 4 - 4\varepsilon$ . Since  $\varepsilon$  is small enough  $\text{Lip}(f_1) = 4 - 4\varepsilon$  which is smaller than  $\text{Lip}(f)$ .

For  $e \in G$  which is not adjacent to  $v$  the images of  $e$  under  $g$  and  $h$  coincide so the slope of  $h$  on  $e$  is still  $\leq \text{Lip}(g)$ . If  $e \in G_g$  is adjacent to  $v$ , we have made  $g(e)$  shorter by  $\varepsilon$  so the slope of  $h$  on  $e$  is strictly smaller than  $\text{Lip}(g)$ . Let  $S$  be the second largest slope of  $g$  in  $G$ . If  $e \in G \setminus G_g$  then  $\ell'(h(e)) \leq S \cdot \ell(e) + 2\varepsilon$ . So if we take

$$\varepsilon < \frac{1}{2}(\text{Lip}(g) - S)(\text{length of shortest edge in } G)$$

then the slope on  $e$  will still be strictly smaller than  $\text{Lip}(g)$ . This proves the claim.

Construct a legal loop in  $\alpha \in G_g$  as follows. Start constructing an embedded legal path  $\alpha$  until it intersects itself. That is, until  $\alpha(t_1) = \alpha(t_2)$  for some  $0 \leq t_1 < t_2$  (we now consider  $\alpha$  as a map from  $[0, L]$  to  $G$ ). If the turn  $\{D_+\alpha(t_1), D_-\alpha(t_2)\}$  is legal, then  $\alpha|_{[t_1, t_2]}$  is a legal loop. If it is illegal, let  $\alpha'(t) = \alpha(t + t_1)$  and rename  $\alpha, t_1$  to get  $\alpha(0) = \alpha(t_1)$  and  $\{D_+\alpha(0), D_-\alpha(t_1)\}$  is illegal. By the previous paragraph there is another gate at  $\alpha(0)$  in  $G_g$ . Extend  $\alpha$  to cross this gate and continue until there are  $t_2 < t_3$  so that  $\alpha(t_2) = \alpha(t_3)$ . If  $0 < t_2 \leq t_1$  then either  $\alpha|_{[t_2, t_3]}$  is a legal loop or  $\alpha|_{[0, t_2]} \cup \alpha^{-1}|_{[t_3, t_1]}$  is a legal loop. If  $t_1 < t_2 < t_3$  then either  $\alpha|_{[t_2, t_3]}$  is a legal loop or  $\alpha|_{[0, t_4]} \cup \alpha^{-1}|_{[t_1, t_2]}$  is a legal loop (barbell or figure 8 loops).  $\square$

Note that  $d(x, y) \geq \log \frac{\ell'(g(\alpha))}{\ell(\alpha)}$  for any loop  $\alpha$  and any difference in marking  $g$ . The right hand side does not depend on a particular choice of  $g$  (among the different choices of differences in markings), so one can effectively compute the distance by maximizing the ratio over the finitely many candidate loops.

For  $x \in \mathcal{X}_n$  represented by  $(G, f, \ell)$ , any conjugacy class  $\alpha$  of  $F_n$  may be identified with an immersed loop  $\alpha_G$  in  $G$ . We will use the same notation for both the conjugacy class and the loop representative. If we want to emphasize that the loop  $\alpha$  is in the graph  $G$  we will denote it by  $\alpha_G$

**Definition 2.6** We say that the conjugacy class  $\alpha$  is a basis element if  $\{\alpha\}$  can be completed to a basis of  $F_n$ .

**Proposition 2.7** Let  $\alpha_X, \beta_X$  be different candidates in  $(X, f, \ell)$ . Then there is a third candidate  $\gamma_X$  so that  $\{\alpha, \gamma\}$  and  $\{\beta, \gamma\}$  can each (separately) be completed to a basis of  $F_n$ .

**Proof** Suppose  $\alpha_X$  is a candidate and  $\gamma_X$  is an embedded loop such that  $\gamma_X \setminus \alpha_X \supseteq \{e_i\}$ . Let  $J$  be a maximal forest in  $X$  which doesn't contain  $e_i$ . Collapse  $J$  to get  $R_X$  a wedge of circles. Since  $e_i$  was not collapsed,  $\gamma_R \setminus \alpha_R \supseteq \{e_i\}$ . Let  $e_j$  be any edge that  $\alpha$  crosses exactly once then  $\{\alpha_R, \gamma_R, e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n\}$  represents a basis for  $F_n$  because  $\{e_j, e_i, e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n\}$  is a basis for  $F_n$ .

Suppose  $\alpha$  is a figure 8 or barbell candidate and  $\gamma$  is an embedded circle so that  $\gamma \subset \alpha$ . Choose an edge  $e_i$  in  $\alpha \setminus \gamma$  which  $\alpha$  crosses only once. Collapse a maximal forest that doesn't contain  $e_i$ . Now choose  $e_j$  in  $\gamma_R$ . Then  $\{\alpha_R, \gamma_R, e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n\}$  is a basis for  $F_n$ .

Let  $\alpha_X, \beta_X$  be any two candidates. If one of them is an embedded loop whose image is not contained in the other then  $\alpha, \beta$  can be completed to a basis. If  $\alpha, \beta$  have the same image then find an embedded loop  $\gamma$  as in the previous paragraph so that  $\alpha, \gamma$  and  $\beta, \gamma$  can be completed to a basis. If they have different images and are not embedded, let  $\gamma$  be an embedded loop so that  $\text{im } \gamma \subset \text{im } \alpha$ . Then  $\alpha, \gamma$  can be completed to a basis. If  $\text{im } \gamma \subset \text{im } \beta$  then  $\beta, \gamma$  can be completed to a basis. If  $\gamma$  is not contained in  $\beta$  then again by the previous paragraph  $\beta, \gamma$  can be completed to a basis.  $\square$

### 2.2 Asymmetry of $d(x, y)$

In general, the lipschitz distance is not symmetric as is shown in Figure 2.

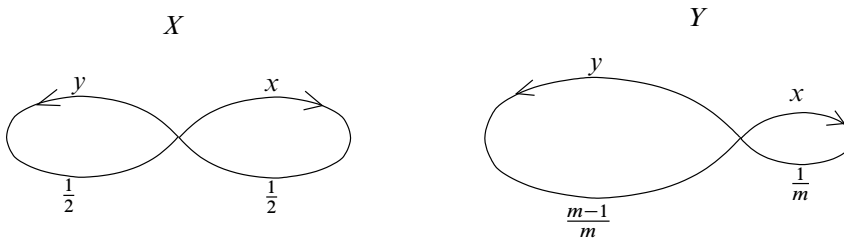


Figure 2: An example where  $d(X, Y) = \log \frac{2m-2}{m} \sim \log 2$  and  $d(Y, X) = \log \frac{m}{2}$

We will make use of the following result:

**Definition 2.8** For  $\theta > 0$  the  $\theta$ -thick part of  $\mathcal{X}_n$  is

$$\mathcal{X}_n(\theta) = \{(X, f, l) \in \mathcal{X}_n \mid l(\alpha) \geq \theta \text{ for all } \alpha\}$$

**Theorem 2.9** (Algom-Kfir and Bestvina [2]) For any  $\theta > 0$  there is constant  $c = c(\theta, n)$  such that:

$$d(x, y) \leq c \cdot d(y, x)$$

for any  $x, y \in \mathcal{X}_n(\theta)$ .

### 3 The axis of an irreducible automorphism and its projection

It is straightforward to check that the right action of  $\text{Out}(F_n)$  on  $\mathcal{X}_n$  is an isometric action.

**Notation 3.1** Let  $x = (G, g, \ell)$  be a point in  $\mathcal{X}_n$ . In this section and in later chapters, we will repress  $g$  and  $\ell$  and denote  $x$  with  $G$ .

Let  $\phi$  be an irreducible outer automorphism. Observe that if  $f: G \rightarrow G$  is a train-track representative for  $\phi$  then so is  $f: G \cdot \phi \rightarrow G \cdot \phi$ . We get a sequence of points  $G \cdot \phi^m$  so that  $f: G \cdot \phi^m \rightarrow G \cdot \phi^m$  is a train-track representative for  $\phi$ . We would like to define a line  $\mathcal{L}_f = \{G_t\}_{t \in \mathbb{R}} = \{G(t)\}_{t \in \mathbb{R}}$  so that:

- (1)  $G_t$  is a directed geodesic parameterized according to arc-length in the Lipschitz metric. That is, for  $t < t'$  we have  $d(G_t, G_{t'}) = t' - t$ . In particular, for  $t < t' < t''$ :  $d(G_t, G_{t''}) = d(G_t, G_{t'}) + d(G_{t'}, G_{t''})$
- (2)  $G_0 = G$
- (3)  $\mathcal{L}_f$  is invariant under  $\phi$ .
- (4) For each  $t$ , there are maps  $f_t: G_t \rightarrow G_t$  that are irreducible train-track representatives of  $\phi$ .

The way to achieve this is to start “folding”  $G$  onto itself by identifying appropriate segments in edges which form an illegal turn until we reach  $G \cdot \phi$ . This defines a path  $[G, G \cdot \phi] := \{G_t\}_{0 \leq t \leq \log \lambda}$  where  $\log \lambda = d(G, G \cdot \phi)$ . Then we translate this path using  $\phi$  to construct a line  $\mathcal{L}_f = \bigcup_{m=-\infty}^{\infty} [G, G \cdot \phi] \cdot \phi^m$ . This line is automatically invariant under  $\phi$ . It is not hard to see that it is a directed geodesic. For more details see the author’s thesis [1], Handel and Mosher [23] or Francaviglia and Martino [20].

#### 3.1 The projection to an axis

**Notation 3.2** In this section we denote the point  $x = (G, h, \ell) \in \mathcal{X}_n$  by  $G$ . The length of the (immersed) loop  $\alpha$  in  $G$  will now be denoted by  $l(\alpha, G)$ .

Let  $\phi$  be an outer automorphism, and suppose  $f: G \rightarrow G$  is a train-track representative for  $\phi$  and  $g: H \rightarrow H$  is a train-track representative for  $\phi^{-1}$ . We may suppose  $f$  and  $g$  are stable. Let  $\mathcal{L}_f = \{G(t)\}_{t \in \mathbb{R}}$  be an axis for  $\phi$ , and  $\mathcal{L}_g = \{H(t)\}_{t \in \mathbb{R}}$  an axis for  $\phi^{-1}$ . Let  $\lambda, \nu$  be the expansion factors of  $\phi, \phi^{-1}$ . We will show that for a conjugacy class of basis elements  $\alpha$  there is a  $k_0$  such that  $l(\phi^{k-k_0}(\alpha), G) \sim \lambda^{k-k_0} l_0 + \nu^{k_0-k} l_0$ . This  $k_0$  will allow us to define the projection to  $\mathcal{L}_f$ .



**Notation 3.3** Fix a graph  $G$ . If  $\alpha$  is a loop we denote by  $[\alpha]$  the immersed loop which is freely homotopic to  $\alpha$ . If  $\alpha$  is a path then  $[\alpha]$  is the immersed path homotopic to  $\alpha$  relative to its endpoints.

**Definition 3.4** (The bounded cancellation constant) Let  $f: T \rightarrow T'$  be an equivariant map of free simplicial metric  $\mathbb{R}$ -trees. There is a constant denoted  $BCC(f)$  so that for all  $p, q, r \in T$  such that  $p \in [q, r]$  we have  $d(f(p), [f(q), f(r)]) < BCC(f)$ . This was first observed by Cooper [14].  $BCC(f)$  is called *the bounded cancellation constant* of  $f$ . When  $T, T'$  are very small the existence of a bounded cancellation constant was proved by Gaboriau, Jaeger, Levitt, and Lustig in [21].

We need the following lemmas and notions due to Bestvina, Feighn, and Handel [7]. We include some details for completeness.

**Definition 3.5** We call the constant  $\kappa = \frac{2BCC(f)}{\lambda-1}$  the legality threshold (note the slight difference from the “critical constant” of [7]). The legality of  $\delta$  with respect to the train-track structure of  $f$  is

$$LEG_f(\delta, G) = \frac{\text{Total length of all legal pieces of length } > \kappa}{l(\delta, G)}$$

If  $LEG_f(\delta, G) > \epsilon$  we say that  $\delta$  is  $\epsilon$ -legal.

**Lemma 3.6** For every  $\epsilon > 0$  there is a constant  $C = C(\epsilon)$  so that if  $\delta$  is  $\epsilon$ -legal then  $l(f^n(\delta), G) > C\lambda^n l(\delta, G)$ .

**Proof** Let  $\beta_1, \dots, \beta_k$  be legal subsegments of  $\delta$  of length  $> \kappa$  and let

$$\delta = \sigma_1\beta_1\sigma_2\beta_2 \dots \sigma_k\beta_k\sigma_{k+1}$$

Let  $e_1$  be the initial edge of  $\delta$  and  $e_2$  be the terminal edge of  $\delta$  and suppose that the turn  $\{\bar{e}_2, e_1\}$  is legal. Thus when we reduce  $f(\delta)$  it is automatically cyclically reduced.

$$[f(\delta)] \sim [f(\sigma_1)] \cdot f(\beta_1) \cdot [f(\sigma_2)] \cdot f(\beta_2) \cdots [f(\sigma_k)] \cdot f(\beta_k) \cdot [f(\sigma_{k+1})]$$

where the cancellation occurs only at the dots. Note that  $l(f(\beta_i), G) > \kappa > BCC(f)$ . If there was a subsegment  $t$  of  $f(\beta_1)$  that cancelled with a subsegment  $s$  of  $[f(\sigma_1)]$  and  $l(t, G) = l(s, G) > BCC(f)/2$  then this would contradict the bounded cancellation lemma. Thus there is a middle segment  $\beta_1^1$ , obtained from  $f(\beta_1)$  by truncating a segment of length  $BCC(f)/2$  from both ends which survives the cancellation. Similarly,

there are middle segments  $\beta_i^1$  of  $f(\beta_i)$  for  $1 \leq i \leq k$  which survive after the cancellation in  $[f(\delta)]$ . Thus

$$l(f(\delta), G) \geq \sum_{i=1}^k l(\beta_i^1) = \sum_{i=1}^k \lambda l(\beta_i, G) - BCC(f).$$

We continue to apply  $f$  and get subsegments  $\beta_i^m$  of  $f(\beta_i^{m-1})$  obtained by truncating subsegments of length  $BCC(f)/2$  from both ends of  $f(\beta_i^{m-1})$ , which survive in  $[f^m(\delta)]$  after the cancellation. The length of  $\beta_i^m$  is no less than  $\lambda l(\beta_i^{m-1}, G) - BCC(f) > l(\beta_i^{m-1}, G) > \kappa$  hence

$$\begin{aligned} l(\beta_i^m, G) &\geq \lambda^m l(\beta_i, G) - BCC(f)(1 + \lambda + \dots + \lambda^{m-1}) \\ &= \lambda^m l(\beta_i, G) - \frac{BCC(f)(\lambda^m - 1)}{\lambda - 1} \\ &= \lambda^m l(\beta_i, G) \left(1 - \frac{BCC(f)}{l(\beta_i, G)(\lambda - 1)}\right) + \frac{BCC(f)}{\lambda - 1} \\ &\geq \lambda^m l(\beta_i, G) \left(1 - \frac{BCC(f)}{l(\beta_i, G)(\lambda - 1)}\right) \end{aligned}$$

Since  $l(\beta_i, G) \geq \kappa = \frac{2BCC(f)}{\lambda - 1}$  the term in the parenthesis is no smaller than  $\frac{1}{2}$  thus  $l(\beta_i^m, G) \geq \frac{1}{2} \lambda^m l(\beta_i, G)$ . Therefore,

$$\begin{aligned} l(f^m(\delta), G) &> \sum_{i=1}^k l(\beta_i^m, G) > \frac{1}{2} \lambda^m \sum_{i=1}^k l(\beta_i, G) \\ &= \frac{1}{2} \lambda^m \epsilon l(\delta, G) \end{aligned}$$

Therefore  $C(\epsilon) = \frac{\epsilon}{2}$ . □

**Lemma 3.7** (Bestvina, Feighn and Handel [7, Lemma 5.6]) *If  $\phi$  is non-geometric there is a constant  $\epsilon_0 > 0$  and an integer  $N$  such that for any conjugacy  $\alpha$ :*

$$LEG_f(\phi^N(\alpha), G) > \epsilon_0 \text{ or } LEG_g(\phi^{-N}(\alpha), H) > \epsilon_0$$

We want some version of this lemma for geometric automorphisms. In the proof of 3.7 the assumption that  $\phi$  is nongeometric is used only to bound the number of consecutive Nielsen paths appearing in  $\alpha_G$  (the immersed loop representing the conjugacy class  $\alpha$  in  $G$ ).

Recall the definitions of Nielsen paths from Definition 1.25. Recall that if  $\phi$  is geometric and fully irreducible and  $f$  is stable, then the unique invariant indivisible Nielsen path

$\beta$  is a loop. Therefore, if  $\alpha = \beta^m$  there is no bound for the number of consecutive Nielsen paths that appear in  $\alpha_G$ . We shall restrict our attention henceforth to the case where  $\alpha$  is the conjugacy class of a basis element.

**Proposition 3.8** *There is a bound  $K$  that depends only on  $\phi$ , such that if  $\alpha$  is a conjugacy class of a basis element, then  $\alpha_G$  cannot cross more than  $K$  consecutive Nielsen paths.*

**Proof** Let  $f: G \rightarrow G$  be a train-track representative of  $\phi$ , by a lemma of Bestvina, Feighn and Handel [8, Lemma 4.2.5] there is a finite set of periodic Nielsen paths  $\gamma_1, \gamma_2, \dots, \gamma_m$  of  $f$ . There is a constant  $K$  so that it is either impossible to concatenate more than  $K$  consecutive Nielsen paths or if we concatenate  $K$  of the  $\gamma_i$ s then we will get a path containing a loop which is a concatenation of the Nielsen paths. This loop  $\beta'$  is fixed by some power  $f^k$ . Since  $\phi$  is fully irreducible this implies that  $\phi^k$  is geometric and  $\beta'$  represents a conjugacy class in the fixed subgroup of  $\phi^k$ . The fixed subgroup of  $\phi^k$  is cyclic, let  $\beta$  be its generator. Let  $h: H \rightarrow H$  be a stable train-track representative of  $\phi^k$ . Let  $\rho: G \rightarrow H$  be a Lipschitz homotopy equivalence respecting the markings of  $G, H$ . By making  $K$  even larger we can insure that if  $\alpha$  contains a concatenation of  $K$  indivisible Nielsen paths then  $\alpha$  contains  $\beta^2$  with a padding of segments of length  $BCC(\rho)$  at each end. Thus a path  $\alpha$  that contains  $K$  consecutive Nielsen paths would map under  $\rho$  to a path containing 2 consecutive appearances of  $\beta$ .

$h: H \rightarrow H$  may be extended to an automorphism of a surface of genus greater than 0 and one boundary component  $\hat{h}: S_{g,1} \rightarrow S_{g,1}$  where  $H$  is a spine for  $S_{g,1}$  (see Bestvina and Handel [10]). In this case, the indivisible Nielsen path  $\beta$  can be represented by the boundary circle. Collapse  $H$  to a rose  $R$  with vertex  $*$  (also embedded in the surface) and notice the Whitehead graph of  $\beta$  with respect to the basis represented by the edges of  $R$  is equal to  $\text{Link}(*, S_g)$  which is a circle. Thus,  $Wh_R(\beta)$  is connected with no cut vertex. Now if  $\alpha_H$  contains 2 consecutive appearances of  $\beta$  then  $Wh_R(\alpha) \supseteq Wh_R(\beta)$ . Hence  $Wh_R(\alpha)$  is connected with no cut-vertex. By Whitehead's theorem  $\alpha$  is not a basis element, which is a contradiction.  $\square$

**Lemma 3.9** *For any irreducible outer automorphism  $\phi$  there is a constant  $\epsilon_0 > 0$  and an integer  $N$  such that for any basis element  $\alpha$ ,*

$$LEG_f(\phi^N(\alpha), G) > \epsilon_0 \quad \text{or} \quad LEG_g(\phi^{-N}(\alpha), H) > \epsilon_0$$

*if  $\phi$  is nongeometric this holds for all  $\alpha$ . In particular, the above holds for a conjugacy class  $\alpha$  that is represented by a candidate loop in some  $x \in \mathcal{X}_n$ .*

**Proof** There is an integer  $K$  so that for any basis element  $\alpha$ ,  $\alpha$  does not contain  $K$  consecutive Nielsen paths. Apply the proof of Lemma 3.7 as it appears in the paper [7] by Bestvina, Feighn and Handel to get the first part of the statement. By Proposition 2.7 all candidates are basis elements so the statement applies to them.  $\square$

Fix a conjugacy class of a basis element  $\alpha$ . Notice that if  $LEG_f(\phi^N(\alpha), G) \geq \epsilon$  then  $LEG_f(\phi^m(\alpha), G) \geq \epsilon$  for all  $m > N$  (once a legal segment has length greater than the threshold, the exponential expansion dominates over the bounded cancelation and makes the surviving subsegments longer than their predecessors). Define

$$k_0 = k_0(\alpha) = \max\{k \mid LEG_f(\phi^k(\alpha), G) \leq \epsilon_0\}$$

$$k'_0 = k'_0(\alpha) = \min\{k \mid LEG_g(\phi^k(\alpha), H) \leq \epsilon_0\}$$

To see the existence of some  $k, j$  such that

$$LEG_f(\phi^k(\alpha), G) \leq \epsilon_0 \quad \text{and} \quad LEG_g(\phi^j(\alpha), H) \leq \epsilon_0$$

first suppose that  $LEG_f(\alpha, G) > \epsilon_0$  we show that  $LEG_g(\phi^m(\alpha), H) \leq \epsilon_0$  for some  $m \geq 0$ . By Lemma 3.6, for all  $m \geq 0$

$$l(f^m(\alpha), G) > C_1 \lambda^m \epsilon_0 l(\alpha, G)$$

Let  $\tau: G \rightarrow H$  be an optimal difference in marking, if  $LEG_g(\phi^m(\alpha), H) > \epsilon_0$  then by applying Lemma 3.6 to  $g$  we get

$$l(g^m(\tau(f^m(\alpha))), H) \geq C_2 \nu^m \epsilon_0 l(\tau(f^m(\alpha)), H) \geq C_1 C_2 \lambda^m \nu^m \epsilon_0^2 l(\alpha, G)$$

But  $g^m \circ \tau \circ f^m$  is homotopic to  $\tau$ , thus

$$l(g^m(\tau(f^m(\alpha))), H) = l(\tau(\alpha), H) \leq \text{Lip}(\tau) l(\alpha, G).$$

This gives a bound on  $m$ . Thus there is some  $k$  such that  $LEG_g(\phi^k(\alpha), H) < \epsilon_0$ . The sequence  $\alpha, g(\alpha), \dots, g^k(\alpha), \dots$  weakly converges to the lamination  $\Lambda_{\phi}^-(H)$  (see Bestvina, Feighn and Handel [7]). This means that every leaf segment  $\beta$  of a leaf  $\ell$  of the lamination  $\Lambda_{\phi}^-(H)$  appears in all but finitely many elements of the sequence  $\{g^k(\alpha)\}$ . Since every leaf of this lamination is  $g$  legal we get that for some  $k$ ,  $LEG_g(\phi^k(\alpha), H) \geq \epsilon_0$ . This gives us the existence of  $k'_0$  and a symmetric argument shows the existence of  $k_0$ .

**Lemma 3.10** *There is an  $N$  so that for all  $\alpha$ ,  $|k_0(\alpha) - k'_0(\alpha)| < N$*

**Proof** At  $k_0$ ,  $LEG_f(\phi^{k_0}(\alpha), G) \not\geq \epsilon_0$ , thus by Lemma 3.9,  $LEG_g(\phi^{k_0-2N}(\alpha), H) > \epsilon_0$ . This implies that  $k'_0 > k_0 - 2N$ . By symmetry we get the result.  $\square$

For each conjugacy class  $\alpha$  and  $t \in \mathbb{R}$  define

$$l_\alpha(t) = l(\alpha, G(t))$$

We now show that if  $\alpha$  is a basis element, then there is a bounded set on which  $l_\alpha(t)$  achieves its minimum, the bound is uniform over all conjugacy classes  $\alpha$ .

**Definition 3.11** Let  $t_0(\alpha) = k_0 \log \lambda$  and  $t'_0(\alpha) = k'_0 \log \lambda$ .

**Lemma 3.12** There exists a  $C$  such that for every basis element  $\alpha$  let  $n(t) = \lfloor \frac{|t|}{\log \lambda} \rfloor$  then for  $t > 0$

$$(4) \quad \frac{1}{C} \cdot \lambda^{n(t)} \cdot l_\alpha(t_0) < l_\alpha(t_0 + t) \leq C \cdot \lambda^{n(t)} \cdot l_\alpha(t_0)$$

and for  $t < 0$

$$(5) \quad \frac{1}{C} \cdot \nu^{n(t)} \cdot l_\alpha(t_0) < l_\alpha(t_0 + t) \leq C \cdot \nu^{n(t)} \cdot l_\alpha(t_0)$$

**Proof** For any integer  $m$ :

$$C_{3.6} \lambda^m l(\alpha, G(t_0)) \leq l(\phi^m(\alpha), G(t_0)) \leq \lambda^m l(\alpha, G(t_0))$$

Now

$$l_\alpha(t_0 + t) = l(\alpha, G(t)) \leq \lambda l(\alpha, G(t_0 + n(t) \log \lambda)) \leq \lambda \lambda^{n(t)} l(\alpha, G(t_0))$$

which is the right hand side of (4). To get the left hand side,

$$l_\alpha(t_0 + t) \geq \frac{l(\alpha, G(t_0 + (n(t) + 1) \log \lambda))}{\lambda} \geq C_{3.6} \lambda^{n(t)} l(\alpha, G(t_0)).$$

We now prove the right hand side of (5). Let

$$d = \max\{t_0 - t'_0, d(G(t_0), H(t_0)), d(H(t_0), G(t_0))\} \quad \text{and} \quad D = e^d.$$

Notice that  $D$  doesn't depend on  $\alpha$ . When  $t < 0$ ,  $-(n(t) + 1) \leq t < -n(t)$  hence

$$l_\alpha(t_0 + t) = l(\alpha, G(t_0 + t)) \leq \lambda l(\alpha, G(t_0 - (n(t) + 1) \log \lambda)) = \lambda l(\phi^{-(n(t)+1)}(\alpha), G(t_0)).$$

Now

$$l(\phi^{-(n(t)+1)}(\alpha), G(t_0)) \leq D l(\phi^{-(n(t)+1)}(\alpha), H(t_0)) \leq D^2 l((\phi^{-1})^{n(t)+1}(\alpha), H(t'_0))$$

and

$$\begin{aligned} l((\phi^{-1})^{n(t)+1}(\alpha), H(t'_0)) &\leq \nu^{n(t)+1} l(\alpha, H(t'_0)) \\ &\leq D \nu^{n(t)+1} l(\alpha, H(t_0)) \leq D^2 \nu^{n(t)+1} l(\alpha, G(t_0)). \end{aligned}$$

Thus we get

$$l(\alpha, G(t_0 + t)) \leq \lambda D^4 v \cdot v^{n(t)} l(\alpha, G(t_0))$$

The left hand side of (5) is proved similarly. □

Since the length function is coarsely  $e^t + e^{-t}$  it attains a minimum.

**Definition 3.13** (min set) For a conjugacy class  $\alpha$  of a basis element: let  $L = \min\{l_\alpha(t) \mid t \in \mathbb{R}\}$  and denote by  $T_\alpha$  the set of  $t_\alpha$  such that  $l_\alpha(t_\alpha) = L$ . The min set of  $\alpha$  is  $\pi_f(\alpha) = \{G(t_\alpha) \mid t_\alpha \in T_\alpha\} \neq \emptyset$ .

It follows from Lemma 3.12 that

**Corollary 3.14** *There exists an  $s > 0$  so that for any basis element  $\alpha$  and for all  $t_\alpha \in T_\alpha$ ,  $|t_\alpha - t_0| < s$ .*

Which implies the following.

**Corollary 3.15** *There is an  $s > 0$  such that for every basis element  $\alpha$ ,  $\text{diam}\{T_\alpha\} < s$  hence  $\text{diam}\{\pi_f(\alpha)\}$  is bounded independently of  $\alpha$ .*

From now on  $t_\alpha$  denotes any choice of element in  $T_\alpha$ , for example the smallest one. The following proposition follows from Corollary 3.14 and Lemma 3.10.

**Corollary 3.16** *There is an  $s > 0$  such that for every primitive  $\alpha$ :*

$$d(G', H') < s \text{ for all } G' \in \pi_f(\alpha), H' \in \pi_g(\alpha).$$

*That is, the min sets of  $\alpha$  with respect to  $\mathcal{L}_f$  and  $\mathcal{L}_g$  are uniformly close.*

**Corollary 3.17** *There is an  $s > 0$  such that for every basis element  $\alpha$ , if  $t > t_\alpha + s$  then  $LEG(\alpha, G(t)) > \epsilon_0$  (the legality is computed with respect to the train track structure induced by  $f_t: G(t) \rightarrow G(t)$  (see Definition 1.14) from item (4) in the list of properties of  $\mathcal{L}_f$ ).*

The following observation states that if  $\alpha$  is almost legal in  $G(t)$ , then it almost realizes the distance  $d(G(t), G(t + t'))$ . Denote  $\text{St}_\alpha(X, Y) = \frac{l(\alpha, Y)}{l(\alpha, X)}$ .

**Proposition 3.18** *There is a  $C$  so that if  $\alpha$  is  $\epsilon_0$ -legal in  $G(t)$  with respect to  $g_t$  then for all  $t' > 0$ ,*

$$\log \text{St}_\alpha(G(t), G(t + t')) - C \geq d(G(t), G(t + t')) = t'$$

**Proof** Since  $\alpha$  is  $\epsilon$ -legal  $t \geq t_0$ . Let  $K = C_{3.12}$  from Lemma 3.12.

$$l_\alpha(t + t') \geq \frac{1}{K} \lambda^{n(t+t')} l_\alpha(t_0)$$

$$l_\alpha(t) \leq K \lambda^{n(t)} l_\alpha(t_0)$$

Then

$$\text{St}_\alpha(G(t), G(t + t')) = \frac{l_\alpha(t + t')}{l_\alpha(t)} \geq \frac{1}{K^2} \lambda^{n(t+t')-n(t)}. \quad \square$$

Now we are in a position to define a coarse projection  $\pi_f: \mathcal{X}_n \rightarrow \mathcal{L}_f$ . Let  $X \in \mathcal{X}_n$  and  $T_X = \{t \mid d(X, G(t)) = d(X, \mathcal{L}_f)\}$ . Define the projection of  $X$  to  $\mathcal{L}_f$  by  $\pi_f(X) = \{G(t) \mid t \in T_X\}$ .

**Proposition 3.19** *There is an  $s > 0$  such that, for every point  $X \in \mathcal{X}_n$ ,*

$$\text{diam}(\pi(X)) < s.$$

**Proof** Let  $u(t)$  be a coarsely exponential function, that is, a function that satisfies Lemma 3.12. Let  $C = C_{3.12}$ ,  $s(u) = \frac{2 \log(C)}{\log \lambda} + 1$ . Let  $t_0 < t$  and  $t' > t + s(u)$  then  $\lambda^{n(t')-n(t)} > C^2$ . Thus by Lemma 3.12 applied to  $u$  we have:

$$u(t') - u(t) > \frac{1}{C} \lambda^{n(t')} u(t_0) - C \lambda^{n(t)} u(t_0) > 0$$

Similarly for  $t' < t - s(u)$  and  $t < t_0$  for some appropriate  $s(u)$ . To sum up:

$$\text{If } t_0 < t, t + s(u) < t' \text{ then } u(t) < u(t')$$

$$\text{If } t_0 > t, t - s(u) > t' \text{ then } u(t) < u(t')$$

Let  $\alpha, \beta$  be two candidates in  $X$ . The function  $u(t) = \text{St}(\alpha_t) = \frac{l_\alpha(t)}{l(\alpha, X)}$  differs from  $l_\alpha(t)$  by the multiplicative constant  $\frac{1}{l(\alpha, X)}$ . So  $u(t)$  and  $v(t) = \text{St}(\beta_t)$  are also coarsely exponential.

We claim that  $h(t) = \max\{u(t), v(t)\}$  also has a coarse minimum. Let  $t_u = t_0(u) = t_0(\alpha)$  and  $t_v = t_0(\beta)$  and for concreteness, assume  $t_u < t_v$ . Note that  $\lim_{t \rightarrow \pm\infty} h(t) = \infty$  so  $h$  obtains a minimum at some  $t_h$ .

If  $h(t_u) = u(t_u)$  let  $t > t_u + s(u)$  then we have

$$h(t_u - t) \geq u(t_u - t) > u(t_u) = h(t_u)$$

$$h(t_u + t) \geq u(t_u + t) > u(t_u) = h(t_u)$$

hence the diameter of the min-set of  $h$  is at most  $2s(u)$ . Similarly, if  $h(t_v) = v(t_v)$  we are done. So we can assume  $h(t_u) = v(t_u) > u(t_u)$  and  $h(t_v) = u(t_v) > v(t_v)$ . By

continuity, there is a point  $t_1$  so that  $u(t_1) = v(t_1) = h(t_1)$ . Then  $h(t_1 + t) \geq u(t_1 + t) > u(t_1)$  for  $t > s(u)$  (since  $u$  is increasing in this domain) and  $h(t_1 - t) \geq v(t_1 - t) > v(t_1)$  for  $t > s(v)$ . Therefore the min-set of  $h$  is bounded by  $2(s(u) + s(v))$  see Figure 3.

Since there is only a finite number of candidates (depending only on  $n$ ) then the diameter of  $\pi(X)$  is uniformly bounded. □

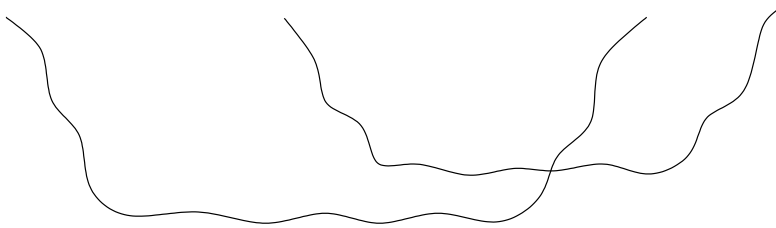


Figure 3: If two functions have a coarse minimum then their max has a coarse minimum.

Using the fact that the length map  $l_\alpha(t)$  is coarsely exponential one could show that  $\pi_f: \mathcal{X}_n \rightarrow \mathcal{L}_f$  is “coarsely Lipschitz”. However we will get a better result in Corollary 5.12.

## 4 The Whitehead graph of the attracting and repelling laminations

### 4.1 The length of a lamination

Let  $\phi$  be a fully irreducible automorphism, and  $f: G_0 \rightarrow G_0$  a stable train-track representative of  $\phi$ . Let  $T_0$  be the universal cover of  $G_0$ , and  $\tilde{f}: T_0 \rightarrow T_0$  a lift of  $f$ . Suppose  $\Lambda_\phi^+(G_0)$  is the attracting lamination of  $\phi$  realized as bi-infinite lines in  $G_0$ .

**Definition 4.1** (length of  $\Lambda_\phi^+$  in  $T$  scaled with respect to  $T_0$ ) Given a very small minimal  $\mathbb{R}$ -tree  $T$  (that is, where we do not identify homothetic trees) one can define the length of the lamination  $\Lambda^+$  in  $T$ , scaled with respect to  $T_0$ , as follows. Let  $h: T_0 \rightarrow T$  be an equivariant Lipschitz map and let  $\sigma$  be a subsegment of the leaf  $\lambda \in \Lambda_\phi^+(G_0)$  and  $\tilde{\sigma}$  be a lift of  $\sigma$  to  $T_0$ . Let  $[h(\tilde{\sigma})]$  the tightened image of  $\tilde{\sigma}$  in  $T$  then

$$(6) \quad l_{T_0}(\Lambda^+, T) = \lim_{\sigma \rightarrow \lambda} \frac{l([h(\tilde{\sigma})], T)}{l(\tilde{\sigma}, T_0)}$$



Note that Definition 4.1 does not depend on the choice of leaf  $\lambda$  of  $\Lambda_\phi^+$  since all leaves of  $\Lambda_\phi^+$  have the same leaf segments. In Lemma 4.3 we will prove that this definition is mathematically sound. But first, let us observe that the limit in equation (6) is not invariant when resealing the metric of  $T$ . Therefore, we modify the definition for  $[T]$

**Definition 4.2** (length of  $\Lambda_\phi^+$  in the projective class of  $T$  scaled with respect to  $T_0$ ) Let  $[w_1], \dots, [w_J]$  be a set of conjugacy classes in  $F_n$  that cannot be simultaneously elliptic in any very small  $F_n$ -tree (for example the set of words of length at most 2 in some basis). Let  $\text{tr}(w_1, T), \dots, \text{tr}(w_J, T)$  be their translation lengths in  $T$  and  $d(T) = \sum_{i=1}^J \text{tr}(w_i, T)$  then

$$(7) \quad pl_{T_0}(\Lambda^+, [T]) = \frac{l_{T_0}(\Lambda^+, T)}{d(T)}$$

**Lemma 4.3** *The limit in equation (6) exists, and it is independent of the choice of  $h$ . Moreover, the map  $pl_{T_0}(\Lambda^+, \cdot): \bar{\mathcal{X}}_n \rightarrow \mathbb{R}$  is continuous.*

**Proof** We begin by showing that the limit exists. This boils down to the fact that  $\lambda$  is quasi-periodic. We first give the idea of the proof: If  $\sigma \subseteq \lambda$  is long enough then  $\sigma$  is a concatenation of a list of words  $\tau_1^k, \dots, \tau_m^k$  called  $k$ -tiles, which are  $f^k$  images of edges. The  $k$ -tiles appear with fixed frequencies  $r_1, \dots, r_m$ . We can choose the tiles long enough so that the cancellation in  $h_\#(\tilde{\tau}_i^k)h_\#(\tilde{\tau}_j^k)$  is negligible with respect to the length of  $\tau_i^k$ . Thus  $h_\#(\tilde{\sigma})$  (up to small cancellation) is a concatenation of the tiles  $h_\#(\tilde{\tau}_i^k)$ , which appear with frequency  $r_i$ . Thus

$$\frac{l_{T_0}(h(\tilde{\sigma}), T)}{l_{T_0}(\tilde{\sigma}, T_0)} \sim \frac{\sum_{i=1}^m r_i l(h(\tilde{\tau}_i^k), T)}{\sum_{i=1}^m r_i l(\tilde{\tau}_i^k, T_0)}.$$

This expression can be shown to converge as  $k \rightarrow \infty$ .

Let  $L = \text{Lip}(h)$  and  $C = \text{BCC}(h)$ . Denote the edges of  $G_0$  by  $e_1, \dots, e_m$ . For each  $k$  the  $i$ th  $k$ -tile is  $\tau_i^k = f^k(e_i)$  where  $1 \leq i \leq m$ . We use  $l_i^k(T_0) = l(\tilde{\tau}_i^k, T_0)$ , and  $l_i^k(T) = l([h(\tilde{\tau}_i^k)], T)$  for shorthand. Let  $A = \max\{l_i^k(T_0) \mid 1 \leq i \leq m\}$  and  $B = \min\{l_i^k(T_0) \mid 1 \leq i \leq m\}$ . Suppose  $k$  is large enough so that  $\frac{2C}{B} < \epsilon$ .

Each leaf  $\lambda$  of  $\Lambda_0^+$  has a natural 1-tiling by edges in  $G_0$ . The standard  $j$ -tiling of  $\lambda$  is the  $f^j$  image of the 1-tiling of  $f^{-j}(\lambda)$ . Let  $\sigma$  be a subsegment of  $\lambda$ ,  $\sigma$  itself might not be “nicely” tiled because it might begin and end in the middle of a tile, but we can sandwich it  $\sigma_1 \subseteq \sigma \subseteq \sigma_2$  with leaf segments which are tiled. Let  $\sigma_1$  be the longest subsegment of  $\sigma$  which is tiled by  $\{\tau_i^k\}_{i=1}^m$  and  $\sigma_2$  the shortest subsegment of  $\lambda$  which contains  $\sigma$  and is tiled by  $\{\tau_i^k\}_{i=1}^m$ .

$$l(\tilde{\sigma}_1, T_0) \leq l(\tilde{\sigma}, T_0) \leq l(\tilde{\sigma}_2, T_0) \leq l(\tilde{\sigma}_1, T_0) + 2A$$

Let  $N_i^k = \#\text{occurrences of the tile } \tau_i^k \text{ in the tiling of } \sigma_1$ , and  $N^k = \sum_{i=1}^m N_i^k$ . By Perron–Frobenius theory there are  $r_1, \dots, r_m$  (independent of  $k$ ) such that  $N_i^k/N^k \rightarrow r_i$  as  $\sigma_1 \rightarrow \lambda$ . Let

$$a_k = \frac{\sum_{i=1}^m r_i l_i^k(T)}{\sum_{i=1}^m r_i l_i^k(T_0)}.$$

We show that for large enough  $\sigma$ ,  $\frac{l(h[\tilde{\sigma}], T)}{l(\tilde{\sigma}, T_0)}$  is in  $[a_k - \epsilon, a_k + \epsilon]$ .

We have

$$\frac{l([h(\tilde{\sigma}_1)], T)}{l(\tilde{\sigma}_2, T_0)} \leq \frac{l([h(\tilde{\sigma})], T)}{l(\tilde{\sigma}, T_0)} \leq \frac{l([h(\tilde{\sigma}_2)], T)}{l(\tilde{\sigma}_1, T_0)}.$$

The right hand side is

$$\begin{aligned} \frac{l([h(\tilde{\sigma}_2)], T)}{l(\tilde{\sigma}_1, T_0)} &\leq \frac{\sum_{i=1}^m N_i^k l_i^k(T) + 2AL}{\sum_{i=1}^m N_i^k l_i^k(T_0)} \\ &= \frac{\sum_{i=1}^m \frac{N_i^k}{N^k} l_i^k(T) + \frac{2AL}{N^k}}{\sum_{i=1}^m \frac{N_i^k}{N^k} l_i^k(T_0)} \xrightarrow{N^k \rightarrow \infty} \frac{\sum_{i=1}^m r_i l_i^k(T)}{\sum_{i=1}^m r_i l_i^k(T_0)} = a_k. \end{aligned}$$

The left hand side limits to

$$\begin{aligned} \frac{l(h[\tilde{\sigma}_1], T)}{l(\tilde{\sigma}_2, T_0)} &\geq \frac{\sum_{i=1}^m N_i^k [l_i^k(T) - 2C]}{\sum_{i=1}^m N_i^k l_i^k(T_0) + 2A} \\ &= \frac{\sum_{i=1}^m N_i^k l_i^k(T)}{\sum_{i=1}^m N_i^k l_i^k(T_0) + 2A} - \frac{\sum_{i=1}^m N_i^k 2C}{\sum_{i=1}^m N_i^k l_i^k(T_0) + 2A} \\ &\geq \frac{\sum_{i=1}^m \frac{N_i^k}{N^k} l_i^k(T)}{\sum_{i=1}^m \frac{N_i^k}{N^k} l_i^k(T_0) + 2A} - \frac{N^k \cdot 2C}{N^k B + 2A} \xrightarrow{N^k \rightarrow \infty} \frac{\sum_{i=1}^m r_i l_i^k(T)}{\sum_{i=1}^m r_i l_i^k(T_0)} - \frac{2C}{B} \\ &= \frac{\sum_{i=1}^m r_i l_i^k(T)}{\sum_{i=1}^m r_i l_i^k(T_0)} - \epsilon = a_k - \epsilon. \end{aligned}$$

Thus, for all  $\epsilon$ , and for large enough  $\sigma$ ,

$$(8) \quad a_k - \epsilon \leq \frac{l([h(\tilde{\sigma})], T)}{l(\tilde{\sigma}, T_0)} \leq a_k + \epsilon.$$

Let  $J(L)$  be the smallest closed interval containing  $\left\{ \frac{l([h(\tilde{\sigma})], T)}{l(\tilde{\sigma}, T_0)} \mid \sigma \subseteq \lambda, l(\tilde{\sigma}, T) \geq L \right\}$  then we have  $J(L + 1) \subseteq J(L)$ . By equation (8) the diameter of  $J(L)$  is bounded by  $2\epsilon$ . By Cantor’s nested intervals lemma  $a_k$  converges to a limit  $c$ . Thus

$$(9) \quad \lim_{\sigma \rightarrow \lambda} \frac{l(h[\tilde{\sigma}], T)}{l(\tilde{\sigma}, T_0)} = c.$$

Next, we show that this limit does not depend on the choice of  $h$ . We claim that if  $h': T_0 \rightarrow T$  is another equivariant Lipschitz map, then

$$|l([h(\sigma)], T) - l([h'(\sigma)], T)| < 2D$$

for some  $D$ . Thus the limit in equation (9) is the same for both  $h$  and  $h'$ . Indeed let  $p$  be some point in  $T_0$ . Then for all  $x \in T_0$  there is a  $g \in F_n$  such that  $d_{T_0}(x, g \cdot p) \leq 1$ . Hence

$$\begin{aligned} d(h(x), h'(x)) &\leq d(h(x), h(gp)) + d(h(gp), h'(gp)) + d(h'(gp), h'(x)) \\ &\leq \text{Lip}(h) + d(h(p), h'(p)) + \text{Lip}(h'). \end{aligned}$$

Denote this constant by  $D$ . Thus, for any path  $\sigma \subseteq T_0$  the initial and terminal endpoints of  $h(\sigma), h'(\sigma)$  are  $D$ -close, so  $|l([h(\sigma)], T) - l([h'(\sigma)], T)| < 2D$ .

Finally we want to show that  $pl_{[T_0]}(\Lambda^+, [T])$  depends continuously on  $[T]$ .

$$pl_{[T_0]}(\Lambda^+, [T]) = \lim_{k \rightarrow \infty} \frac{1}{\sum_{i=1}^m r_i l_i^k(T_0)} \frac{\sum_{i=1}^m r_i l_i^k(T)}{d(T)}$$

Without loss of generality suppose  $\text{tr}(w_1, T) \neq 0$ . If  $[T_j] \xrightarrow{j \rightarrow \infty} [T]$  then

$$\frac{l_i^k(T_j)}{\text{tr}(w_1, T_j)} \rightarrow \frac{l_i^k(T)}{\text{tr}(w_1, T)}$$

for all  $1 \leq i \leq m$  so

$$\begin{aligned} \frac{\sum_{i=1}^m r_i l_i^k(T_j)}{d(T_j)} &= \frac{\sum_{i=1}^m r_i l_i^k(T_j) / \text{tr}(w_1, T_j)}{d(T_j) / \text{tr}(w_1, T_j)} \xrightarrow{j \rightarrow \infty} \\ &= \frac{\sum_{i=1}^m r_i l_i^k(T) / \text{tr}(w_1, T)}{d(T) / \text{tr}(w_1, T)} = \frac{\sum_{i=1}^m r_i l_i^k(T)}{d(T)}. \end{aligned}$$

This completes the proof. □

### 4.2 The Whitehead graph of the attracting and repelling laminations

We've defined the Whitehead graph of a conjugacy class  $\alpha$  in the basis  $\mathcal{B}$ . If  $R \in \mathcal{X}_n$  is a rose, that is, a wedge of  $n$  circles, then a marking inverse identifies it's edges with a basis  $\mathcal{B}(R)$  of  $F_n$ . The Whitehead graph of  $\alpha$  in  $R$  is  $Wh_R(\alpha) = Wh_{\mathcal{B}(R)}(\alpha)$ .

Let  $\eta$  be a quasi-periodic bi-infinite edge path in  $R$ . Since it is quasi-periodic, there is some  $M$  so that  $\{\text{the turns taken by } \eta\} \subseteq \{\text{the turns taken by } \eta'\}$  where  $\eta'$  is any subpath of  $\eta$  whose length is at least  $M$ . We may assume that  $\eta'$  is a closed path. Let

$Wh_R^*(\eta')$  be the Whitehead graph of  $\eta'$  taking into account all turns except the one connecting the end of  $\eta'$  and the beginning of  $\eta'$ . Define  $Wh_R(\eta) = Wh_R^*(\eta')$  (We exclude the “last” turn of  $\eta'$  since it need not be taken in  $\eta$ ).  $Wh_R(\eta)$  does not depend on our choice of  $\eta'$ .

**Proposition 4.4** *Let  $X \in \mathcal{X}_n$ , every leaf of  $\Lambda^+(X)$  is quasi-periodic.*

**Proof** Bestvina, Feighn and Handel [7] prove this for a graph  $G$  which admits a train-track representative  $f: G \rightarrow G$  of  $\phi$ . For a general  $X \in \mathcal{X}_n$  let  $\rho: X \rightarrow G$  and  $\rho': G \rightarrow X$  be optimal differences in markings and  $C = \max\{\text{Lip}(\rho), \text{Lip}(\rho')\}$ . Let  $\lambda_X \in \Lambda^+(X)$  and  $\gamma \subseteq \lambda_X$  be a subsegment of length  $s$ . Let  $B = BCC(\rho')$  and extend  $\gamma$  to  $\gamma' \subset \lambda_X$  by adding segments of length  $B$  at both ends of  $\gamma$ . Let  $\delta \subset \lambda_G$  be a segment such that  $[\rho'(\delta)] = \gamma'$ .  $\text{len}(\delta) \leq C(s + 2B)$  and there is an  $L'$  such that any leaf segment  $\alpha \subseteq \lambda_G$  of length greater than  $L'$  contains every leaf segment of length  $C(s + 2B)$ , in particular,  $\alpha \supseteq \delta$ . Thus  $[\rho'(\alpha)] \supseteq \gamma$ . Let  $S' = CL'$  then for any leaf segment  $\beta \subset \lambda_X$  of length  $CL'$  we have  $\beta = [\rho'(\alpha)]$  for a leaf segment  $\alpha \subset \lambda_G$  of length at least  $L'$  hence  $\alpha$  contains  $\delta$  and  $\beta$  contains an appearance of  $\gamma$ .  $\square$

Let  $X$  be a rose, every leaf of  $\Lambda^+(X)$  is quasi-periodic, and we may define  $Wh_X(\lambda)$  for  $\lambda \in \Lambda^+$ . If  $\lambda_1, \lambda_2 \in \Lambda^+(X)$  are leaves of the attracting lamination then they share the same leaf segments so,  $Wh_X(\lambda_1) = Wh_X(\lambda_2)$ . Since the choice of the leaves does not affect the whitehead graph, fix leaves  $\lambda \in \Lambda^+$  and  $\nu \in \Lambda^-$  once and for all.

Let  $Wh_R(\lambda, \nu)$  be the superposition of the Whitehead graphs  $Wh_R(\lambda), Wh_R(\nu)$ .

**Lemma 4.5** *There is a point  $F \in \mathcal{X}_n$  such that for any leaves  $\lambda \in \Lambda_\phi^+(F)$  and  $\nu \in \Lambda_\phi^-(F)$ , the whitehead graph  $Wh_F(\lambda, \nu)$  is connected and contains no cut vertex.*

To prove this lemma we will need the following proposition which boils down to a result by Levitt and Lustig [26].

**Proposition 4.6** *If  $pl_{T_0}(\Lambda^+, [T]) = 0$  then  $pl_{T_0}(\Lambda^-, [T]) \neq 0$*

**Proof** Let  $\lambda \in \Lambda^+(T_0/F_n)$  be any leaf and represent it as a bi-infinite geodesic in  $T_0$ . Let  $h: T_0 \rightarrow T$  be an equivariant Lipschitz map. We map  $\lambda$  to  $T$  by  $h$ , and denote by  $\lambda_T$  the tightened image of  $\lambda$  in  $T$ . We claim that if  $\lambda_T$  is unbounded then  $l_{T_0}(\Lambda^+, T) > 0$ . Indeed, if  $\lambda_T$  is unbounded then there exists a lift of a tile  $\sigma = f^k(e_i)$  such that  $l([h(\tilde{\sigma})], T) > 2BCC(h)$  denote their positive difference by  $b$ . If  $\rho$  is a leaf segment of  $\lambda$  tiled by  $k$ -tiles then  $[h(\rho)]$  is no shorter than  $N_i b$  where

$N_\sigma$  is the number of times  $\sigma$  appears in  $\rho$ . Let  $A$  be the length of the longest  $k$ -tile and let  $N$  be the number of  $k$ -tiles in  $\rho$ . Thus, as  $\rho$  gets larger,

$$l_{T_0}(\Lambda^+, T) \sim \frac{l([h(\rho)], T)}{l(\rho, T_0)} > \frac{N_\sigma b}{N \cdot A} \sim r_i \frac{b}{A} > 0.$$

In [26, Section 6] it is shown that if  $\lambda \in \Lambda^+$  is bounded then  $T$  has dense orbits. Thus if  $pl_{T_0}(\Lambda^+, [T]) = 0$  then  $l_{T_0}(\lambda, T) = 0$  and  $\lambda$  is bounded which implies that  $T$  has dense orbits. [26, Proposition 5.1] shows that if  $T$  has dense orbits and  $\Lambda^+$  is bounded then  $\Lambda^-$  is unbounded. Therefore  $l_{T_0}(\Lambda^-, T) > 0$  hence  $pl_{T_0}(\Lambda^-, [T]) > 0$ .  $\square$

**Proof of Lemma 4.5** Pick a point  $X_0 \in \mathcal{X}_n$  whose underlying graph is a rose where all edges have length  $\frac{1}{n}$ . It was proven by Bestvina, Feighn and Handel [7] that  $Wh_{X_0}(v), Wh_{X_0}(\lambda)$  are both connected. If  $Wh_{X_0}(v, \lambda)$  contains a cut vertex  $a$  then let  $X_1 = X_0 \cdot \phi_{(A,a)}$  the automorphism described in Whitehead’s algorithm (see description in the paragraph following Definition 1.6). Continue this way to get a sequence  $X_0, X_1, X_2, \dots$ . We will show that this process terminates in a finite number of steps with a graph  $F = X_N$  such that  $Wh_F(v, \lambda)$  does not contain a cut vertex.

A priori, two other cases are possible:  $X_k = X_j$  for some  $j > k$ , and the process never terminates producing an infinite sequence  $\{X_i\}_{i=1}^\infty$ .

**Observation 4.7** For all  $i$  we have

$$l_{T_0}(\Lambda^+, \tilde{X}_i) > l_{T_0}(\Lambda^+, \tilde{X}_{i+1}) \quad \text{and} \quad l_{T_0}(\Lambda^-, \tilde{X}_i) > l_{T_0}(\Lambda^-, \tilde{X}_{i+1}).$$

We delay the proof of this observation to finish the proof of Lemma 4.5.  $X_k = X_j$  for  $k < j$  is impossible since the lengths get strictly smaller. If the process doesn’t terminate then we get an infinite sequence  $\{X_i\}_{i=1}^\infty$  which has a subsequence converging projectively to a limit  $[T] \in \partial\mathcal{X}_n$ . We will argue that  $pl_{T_0}(\Lambda^+, [T]) = pl_{T_0}(\Lambda^-, [T]) = 0$  and get a contradiction to Proposition 4.6.

Let  $L = l_{T_0}(\Lambda^+, \tilde{X}_0)$  then  $l_{T_0}(\Lambda^+, \tilde{X}_i) < L$ , and together with  $d(\tilde{X}_i) \geq 1$  we get  $pl_{T_0}(\Lambda^+, [\tilde{X}_i]) < L$ . Therefore,  $pl_{T_0}(\Lambda^+, [T]) < L$ . Now assume by way of contradiction that  $pl_{T_0}(\Lambda^+, [T]) = L' > 0$ . There exists some conjugacy class  $[w]$  such that  $\text{tr}(w, T) < \frac{L'd(T)}{2nL}$  (if  $T$  is simplicial then there is a conjugacy class  $[w]$  which is elliptic and if  $T$  is not simplicial, it has a quotient tree with dense orbits. In either case we can find conjugacy classes with arbitrarily small translation length). Since the smallest loop in  $X_k$  has length  $\frac{1}{n}$ ,

$$\frac{l_{T_0}(\Lambda^+, \tilde{X}_k)}{\text{tr}(w, \tilde{X}_k)} < \frac{L}{1/n} = nL$$

Since  $\tilde{X}_k$  converges projectively to  $[T]$ ,

$$\frac{l_{T_0}(\Lambda^+, \tilde{X}_k)}{\text{tr}(w, \tilde{X}_k)} = \frac{l_{T_0}(\Lambda^+, \tilde{X}_k)}{d(\tilde{X}_k)} \cdot \frac{d(\tilde{X}_k)}{\text{tr}(w, \tilde{X}_k)} > \frac{L'}{2} \cdot \frac{d(T)}{L'd(T)/2nL} = nL$$

So we get a contradiction to  $pl_{T_0}(\Lambda^+, [T]) \neq 0$ . A similar argument shows that  $pl_{[T_0]}(\Lambda^-, [T]) = 0$  and we get a contradiction. Therefore, the process must end in a finite number of steps with a graph  $F$  such that  $Wh_F(\lambda, \nu)$  is connected without a cut vertex. □

**Proof of Observation 4.7** Let

$$a_k(T) = \frac{\sum_{i=1}^m r_i l_i(T)}{\sum_{i=1}^m r_i l_i(T_0)}$$

where the notation is established in the proof of Lemma 4.3. We must estimate  $\lim_{k \rightarrow \infty} a_k(T)$  for  $T = \tilde{X}_j$  and  $T = \tilde{X}_{j+1}$ . We will show that  $l_i(\tilde{X}_j) > l_i(\tilde{X}_{j+1})$ .

$\lambda$  in  $X_j$  is denoted  $\lambda(X_j)$ , let  $a$  be a cut vertex in  $Wh_{X_j}(\lambda)$  and recall that  $X_{j+1} = X_j \cdot \phi_{(A,a)}$ . This implies that any leaf segment  $\rho \subset \lambda(X_j)$ , the length of  $[\phi_{(A,a)}(\rho)]$  in  $X_j$  is not longer than  $\rho$  in  $X_j$ . Moreover, the length shrinks by a definite amount. Let  $\bar{x} \in A$  then subwords of  $\rho$  of the form  $\bar{x}a$  and  $\bar{a}x$  reduce to  $x$ , so their length is decreased by  $\frac{1}{n}$ . So by taking  $\rho$  long enough we can make sure it contains  $M$  appearances of such turns and thus in  $X_{j+1}$  the length would be decreased by  $\frac{M}{n}$ .

Now let  $\sigma = \tau_i^k$  be  $k$ -tiles in  $G_0$ ,  $k$  is large (to be specified later). Let  $h: G_0 \rightarrow X_j$  be a difference in marking. Then  $[h(\sigma)] = \alpha \cdot \rho \cdot \beta$  where  $\rho \subset \lambda(X_j)$  and  $l(\alpha, X_j), l(\beta, X_j) < BCC(h)$ .  $\alpha, \beta$  contain at most  $nBCC(h)$  edges. Each edge in  $\alpha, \beta$  may transform to at most 3 edges  $x \rightarrow ax\bar{a}$ , hence they may add to the total length of  $[\phi_{(A,a)}(\rho)]$  at most  $2 \cdot nBCC(h) \cdot \frac{2}{n}$ . Thus

$$l([h(\sigma)], X_{j+1}) \geq l(\rho, X_j) + 4BCC(h) - \frac{M}{n}$$

Choose  $M = 5nBCC(h)$  and let  $N$  be the number of  $k$  tiles in  $\sigma$ ,

$$\begin{aligned} a_k(\tilde{X}_{j+1}) &= \frac{\sum r_i l_i(X_{j+1})}{\sum r_i l_i(G_0)} \leq \frac{\sum r_i (l_i(X_j) - 1)}{\sum r_i l_i(G_0)} \\ &= a_k(\tilde{X}_j) - \frac{1}{\sum r_i l_i(G_0)} = a_k(\tilde{X}_j) - \frac{N}{l(\sigma, G_0)} \end{aligned}$$

And  $\frac{N}{l(\sigma, G_0)} \geq 1$ . Thus  $k$  must be chosen long enough so that  $\rho$  contains  $M$  appearances of  $\bar{x}$  for some  $x \in A$ . □

**Remark 4.8** If  $Wh_F(\lambda, \nu)$  is connected and does not contain a cut vertex, then  $Wh_{F \cdot \phi}(\lambda, \nu)$  and  $Wh_{F \cdot \phi^{-1}}(\lambda, \nu)$  satisfy the same property. In fact  $Wh_F(\lambda, \nu) = Wh_{F \cdot \phi}(\lambda, \nu) = Wh_{F \cdot \phi^{-1}}(\lambda, \nu)$ . Indeed let  $k: F \rightarrow F$  and  $k': F \rightarrow F$  be topological representatives of  $\phi, \phi^{-1}$  (see Definition 1.18). Then  $Wh_F(\lambda) = Wh_{F \cdot \phi}(k_{\#}(\lambda))$  this is because  $\lambda \in \Lambda^+(F)$  implies  $k_{\#}(\lambda) \in \Lambda^+(F \cdot \phi)$ .  $Wh_{F \cdot \phi}(k_{\#}(\lambda)) = Wh_{F \cdot \phi}(\lambda)$  because  $\Lambda^+(F \cdot \phi)$  is  $\phi, \phi^{-1}$ -invariant. Thus  $Wh_F(\lambda) = Wh_{F \cdot \phi}(\lambda)$ . Similarly,  $Wh_F(\nu) = Wh_{F \cdot \phi}(k'_{\#}(\nu)) = Wh_{F \cdot \phi}(\nu)$ . Thus,  $Wh_F(\lambda, \nu) = Wh_{F \cdot \phi}(\lambda, \nu)$ . The argument for  $Wh_{F \cdot \phi^{-1}}(\lambda, \nu)$  is identical.

## 5 Axes are strongly contracting

### 5.1 Projections of horoballs are finite

**Definition 5.1** Let  $\eta$  a leaf of  $\Lambda^+$  or  $\Lambda^-$  in  $X \in \mathcal{X}_n$ . Let  $\gamma$  be an edge path contained in  $\eta$ . We say that  $\gamma$  is an  $r$ -piece of  $\eta$  if the  $l(\gamma, X) \geq r$ .

The next proposition states that basis elements cannot contain long pieces of both  $\Lambda^+$  and  $\Lambda^-$ .

**Proposition 5.2** *There exists a constant  $j > 0$  so that for all  $G_t \in \mathcal{L}_f$ :*

- (1) *Let  $\beta$  conjugacy class of  $F_n$ .  $\beta$  is represented by an immersed loop which we shall also denote by  $\beta$  in  $G_t$ . Suppose there exist leaves  $\lambda \in \Lambda_f^+(G_t)$  and  $\nu \in \Lambda_f^-(G_t)$  such that  $\beta$  contains a  $j$ -piece of  $\lambda$  or the inverse of a  $j$ -piece of  $\lambda$  and a  $j$ -piece of  $\nu$ . Then  $\beta$  is not a basis element.*
- (2) *Let  $\alpha, \beta$  be tight loops in  $G_t$  (also thought of as conjugacy classes). Suppose  $\alpha, \beta$  are compatible with a free decomposition of  $F_n$ . If  $\alpha$  contains a  $j$ -piece of  $\lambda$  or the inverse of a  $j$ -piece of  $\lambda$  (a  $j$ -piece of  $\nu$  or the inverse of a  $j$ -piece of  $\nu$ ) then  $\beta$  doesn't contain a  $j$ -piece of  $\nu$  (a  $j$ -piece of  $\lambda$  or the inverse of a  $j$ -piece of  $\lambda$ ).*
- (3) *If  $\alpha$  is a primitive conjugacy class (a conjugacy class of a basis element) then  $\alpha^m$  cannot simultaneously contain both a  $j$ -piece of  $\lambda$  or a  $j$  piece of  $\lambda^{-1}$  and a  $j$  piece of  $\nu$ .*

#### Proof

- (1) We first prove this for  $G_0$ . By Lemma 4.5, there is an  $F \in CV_n$  such that  $Wh_F(\lambda, \nu)$  is connected and contains no cut point. Suppose  $d = d(F, G_0)$  and  $k = \exp(d)$  so for all loops  $\alpha: \frac{l(\alpha, G_0)}{l(\alpha, F)} \leq k$ . Hence  $l(\alpha, F) \geq \frac{1}{k}l(\alpha, G_0)$ .

Let  $h_0: G_0 \rightarrow F$  be an optimal Lipschitz difference in marking, and let  $B = BCC(h_0)$ . Since  $\lambda_F, \nu_F$  are quasi-periodic there is a length  $r$  such that if  $\gamma_F$  is an  $r$ -piece of  $\lambda_F$  then  $\gamma_F$  contains all of the 2-edge leaf segments in  $\lambda_F$  hence  $Wh_F(\lambda_F) = Wh_F^*(\gamma_F)$ . Similarly, if  $\delta_F$  is an  $r$ -piece of  $\nu_F$  then  $\delta_F$  contains all of the 2-edge leaf segments in  $\nu_F$  hence  $Wh_F(\nu_F) = Wh_F^*(\delta_F)$ . Let  $p$  be large enough so that if  $\beta$  is a  $p$ -piece of  $\lambda_{G_0}, \nu_{G_0}$  then all edges of  $G_0$  appear in  $\beta$ .

Define  $J = k(r + 2B) + p$ . If  $\beta_{G_0}$  contains  $\beta'_1$  or  $\beta'^{-1}_1$  where  $\beta'_1$  is a  $J$ -piece of  $\lambda_0 \in \Lambda^+(G_0)$ . By truncating a piece of length at most  $p$  from  $\beta'_1$  we can find  $\beta_1 \subseteq \beta'_1$  a loop such that  $l(\beta_1, G_0) > k(r + 2C)$ . Thus  $l([h_0(\beta_1)], F) > r + 2B$  and  $[h_0(\beta_1)]$  is contained in  $\lambda_F$  apart from some initial and terminal segments of length at most  $B$ . Hence  $[h_0(\beta_{G_0})]$  contains an  $r$ -piece of  $\lambda_F$ . Similarly, if  $\beta_{G_0}$  contains an  $J$ -piece of  $\nu_0 \in \Lambda^-(G_0)$  then there is a loop  $\beta_2 \in \nu$  so that  $l(\beta_2, G_0) > k(r + 2B)$ . Hence  $l([h_0(\beta_2)], F) > r + 2B$  and  $[h_0(\beta_{G_0})]$  contains an  $r$ -piece of  $\nu_F$ . Therefore, if  $\beta_{G_0}$  contains such  $\beta'_1, \beta'_2$  then  $Wh_F(\beta) \supseteq Wh_F(\lambda, \nu)$ . By the definition of the Whitehead graph  $Wh_F(\gamma_F) = Wh_F(\gamma_F^{-1})$  so if  $\beta$  contains  $\beta'^{-1}_1, \beta'_2$  then again,  $Wh_F(\beta) \supseteq Wh_F(\lambda, \nu)$ . Thus  $Wh_F(\beta)$  is connected and does not contain a cut vertex. By Whitehead's Theorem 1.5  $\beta$  is not a basis element.

We can do the same for all graphs  $G_t \in \mathcal{L}_f$  and  $J$  depends on  $d(F, G_t)$ , which varies continuously with  $t$ . Therefore if we vary  $t$  across a fundamental domain of the  $\phi$  action on  $\mathcal{L}_f$ , then there is an upper bound for  $J$  (which we still denote  $J$ ). Now by Remark 4.8 the same is true (with the same  $J$ ) for any translate of the fundamental domain (we translate  $F$  as well so the distance and the optimal map remain the same).

- (2) The proof of the second claim is similar to (1). If  $\alpha_G$  contains a long enough piece of  $\lambda_G$  and  $\beta_G$  contains a long enough piece of  $\nu_G$  then  $Wh_F(\alpha, \beta) \supseteq Wh_F(\lambda_F, \nu_F)$  but by Theorem 1.8  $\alpha, \beta$  are then not compatible with a free decomposition.
- (3) This follows from the fact that in any basis the Whitehead graph of  $\alpha$  and  $\alpha^m$  are one and the same. If  $\alpha^m$  contains such pieces then  $Wh_F(\alpha^m)$  would not contain a cut vertex contradicting Whitehead's Theorem.  $\square$

We now turn to prove some applications:

**Lemma 5.3** *There is an  $s > 0$  such that: if  $\alpha, \beta$  are conjugacy classes which are compatible with a free decomposition of  $F_n$  then  $|t_\alpha - t_\beta| < s$*



**Proof** Suppose for concreteness  $t_\beta > t_\alpha$ . Let  $\alpha_t$  represent  $\alpha$  in  $G_t$ , and  $\beta_t$  represent  $\beta$  in  $G_t$ . We claim that there is a  $t_0$  such that if  $t < t_\beta - t_0$  then  $\beta_t$  contains a  $J$ -piece of  $\nu_{G(t)}$ , and if  $t > t_\alpha + t_0$  then  $\alpha_t$  contains a  $J$ -piece of  $\lambda_{G(t)}$ . Thus, if  $|t_\beta - t_\alpha| > 2t_0$  let  $r = t_\alpha + t_0$  then  $\alpha_r$  contains a  $J$ -piece of  $\lambda_{G(r)}$  and  $\beta_r$  contains a  $J$  piece of  $\nu_{G(r)}$  which contradicts Proposition 5.2.

To find  $t_0$ : by Corollary 3.17, there is an  $s_1 = s_{3,17}$  such that if  $t > t_\alpha + s_1$  then  $LEG_f(\alpha_t, G(t)) > \epsilon_0$ . Let  $\alpha'_t \subseteq \alpha_t$  be a legal segment of length  $> \kappa$  (the legality threshold). There is an  $N$  such that  $[f_t^N(\alpha_t)]$  will contain a  $J$ -piece of the lamination contributed from  $f_t^N(\alpha'_t)$ . Let  $s_2 = s_1 + N \log(\text{PF}(f))$  then at  $t_0 = t_\alpha + s_2$ ,  $\alpha$  contains a  $J$ -piece of  $\lambda$ , contributed by  $\alpha'_t$ .

We apply this argument for  $\beta$  in  $H(t'_\beta)$ . Thus there is an  $s_3$  so that if  $t < t'_\beta - s_3$  then  $\beta$  in  $H(t)$  contains a  $DJ$  piece of  $\nu$  where

$$D = \max \{e^{d(G_u, H_u)}, e^{d(H_u, G_u)} \mid u \in \mathbb{R}\}.$$

Now  $|t_\beta - t'_\beta| < c_{5.6}$  thus if  $s_4 = s_3 + c_{5.6}$  and  $t < t_\beta - s_4$  then  $\beta$  in  $H(t)$  contains a  $DJ$  piece of  $\nu$ . Thus  $\beta_t$  in  $G(t)$  contains a  $J$  piece of  $\nu$ . □

**Corollary 5.4** *There exists a constant  $s > 0$  such that if  $\alpha$  and  $\beta$  are candidates in  $X$  then  $|t_\alpha - t_\beta| < s$*

**Proof** By Proposition 2.7 there is a candidate  $\gamma$  so that  $\alpha, \gamma$  and  $\gamma, \beta$  can be completed to a basis of  $F_n$ . Therefore by Lemma 5.3 there is an  $s = s_{5,3}$  such that  $|t_\alpha - t_\gamma| < s$  and  $|t_\gamma - t_\beta| < s$ . Thus  $|t_\alpha - t_\beta| < 2s$  □

**Corollary 5.5** *There exists a constant  $s > 0$  so that if  $\alpha$  is a candidate in  $X$  then  $|t_X - t_\alpha| < s$*

**Proof** Let  $\alpha_1, \dots, \alpha_N$  be the candidates of  $X$ , for each  $i$  denote by  $t_i$  the parameter realizing  $\min_{t \in \mathbb{R}} \text{St}_{\alpha_i}(X, G(t))$ , that is,  $t_i = t_{\alpha_i}$ .  $t_X$  is the parameter realizing  $\min_{t \in \mathbb{R}} \max_{1 \leq i \leq N} \text{St}_{\alpha_i}(X, G(t))$ . In the proof of Proposition 2.7, we show that

$$t_X \in \left[ \min_{1 \leq i \leq N} \{t_i\}, \max_{1 \leq i \leq N} \{t_i\} \right]$$

By Corollary 5.4 the length of this interval is  $s = s_{5,4}$ . Thus  $|t_X - t_\alpha| < s$ . □

**Corollary 5.6** *There exists an  $s > 0$  such that if the translation length of  $\alpha \in F_n$  in both  $X$  and  $Y$  is smaller than 1 then  $|\pi(X) - \pi(Y)| < s$ .*

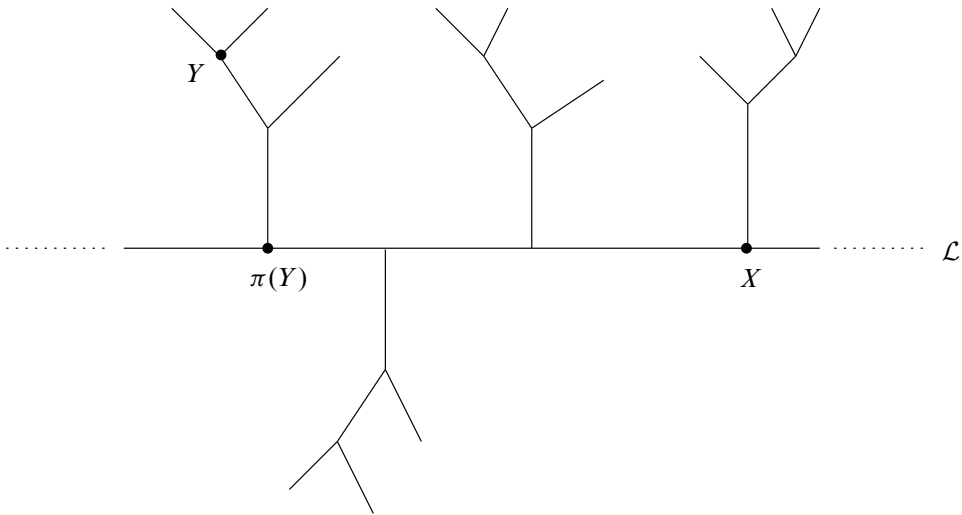


Figure 4: In a tree, the geodesic from  $Y$  to a point on a geodesic visits  $\pi(Y)$

**Proof** Since  $\text{vol}(X) = 1$ ,  $\alpha$  does not cross all edges of  $X$ . Let  $e_i$  be the edge that  $\alpha$  misses. If  $e_i$  is non-separating collapse a maximal tree  $J$  of  $X$  which avoids  $e_i$  to obtain  $R_X$  and choose a short basis of  $\pi_1(R_X)$  so that the image of  $e_i$  is the basis element  $x_n$ . Then  $\alpha$  is carried by the free factor  $\langle x_1, \dots, x_{n-1} \rangle$ . If  $\alpha$  misses  $e_i$  which is separating then again it is carried by a free factor which we denote  $\langle x_1, \dots, x_m \rangle$ . Similarly,  $\alpha$  is carried by  $\langle y_1, \dots, y_k \rangle$ . Therefore,  $|t_\alpha - t_{[x_n]}| < s_{5.3}$ . Similarly, for  $Y, |t_\alpha - t_{[y_n]}| < s_{5.3}$ . So  $t_{[x_n]}$  and  $t_{[y_n]}$  are uniformly close. By Corollary 5.5, we have that  $t_X$  and  $t_Y$  are uniformly close.  $\square$

A horoball based at the conjugacy class  $\alpha$  is the (unbounded) subset  $H(\alpha, r) = \{x \in \mathcal{X}_n \mid l(\alpha, x) < r\}$  of  $\mathcal{X}_n$ . Corollary 5.6 shows that the  $\pi_f(H(\alpha, 1))$  is a bounded interval of  $\mathcal{L}_f$ .

### 5.2 Projections to axes are like projections in trees

Consider a geodesic  $\mathcal{L}$  in a tree  $T$ , and let  $\pi: T \rightarrow \mathcal{L}$  be the closest point projection. The next lemma is motivated by the following observation (see Figure 4): If  $X$  is a point on  $\mathcal{L}$  then  $d(Y, X) = d(Y, \pi(Y)) + d(\pi(Y), X)$ .

**Lemma 5.7** *There exist constants  $s, c > 0$  such that for any  $Y$ , if  $|t - t_Y| > s$  then  $d(Y, G(t)) \geq d(Y, \pi(Y)) + d(\pi(Y), G(t)) - c$*

**Proof** Denote  $X = G(t)$ . Let us first prove it for  $t > t_Y$ . There is an  $s_1 = s_{5,5}$  such that for all candidates  $\alpha$  of  $Y$ :  $|t_\alpha - t_Y| < s_1$ . There is an  $s_2 = s_{3,17}$  such that if  $t > t_\alpha + s_2$  then  $LEG_f(\alpha_t, G(t)) > \epsilon_0$ . Let  $Z = G(t_Y + s_1 + s_2)$  then for any candidate  $\beta$  of  $Y$ ,  $LEG_f(\beta, Z) > \epsilon_0$ . Now suppose  $\beta_Y$  in  $Y$  is the loop that realizes  $d(Y, Z)$ , that is,  $St_\beta(Y, Z) = e^{d(Y,Z)}$ . Then, since  $\beta$  is  $\epsilon_0$ -legal in  $Z$  then by Proposition 3.18 there is a  $C = C_{3,18}$  so that  $St_\beta(Z, X) \geq Ce^{d(X,Z)}$  so  $St_\beta(Y, X) = St_\beta(Y, Z)St_\beta(Z, X) \geq Ce^{d(Y,Z)}e^{d(Z,X)} = Ce^{d(Y,Z)+d(Z,X)}$ . We have  $St(Y, X) \geq St_\beta(Y, X) \geq Ce^{d(Y,Z)+d(Z,X)}$ . Thus  $d(Y, X) \geq \log(C) + d(Y, Z) + d(Z, X)$ . Now recall that  $Z = G(t_Y + s_1 + s_2)$  so  $d(\pi(Y), Z) = s_1 + s_2$ . We have

$$d(Y, Z) > d(Y, \pi(Y))$$

$$d(Z, X) > d(\pi(Y), X) - (s_1 + s_2)$$

thus  $d(Y, X) \geq d(Y, \pi(Y)) + d(\pi(Y), X) - (s_1 + s_2) + \log(C)$  let  $c = s_1 + s_2 - \log(C)$  and we get  $d(Y, X) \geq d(Y, \pi(Y)) + d(\pi(Y), X) - c$ .

If  $t < t_Y$ : there is an  $s'$  such that the above holds for  $g$ . The claim now follows from the fact that  $\pi_f, \pi_g$  are uniformly close (see Corollary 3.16). □

Getting back to the tree  $T$ , if  $X, Y$  are any two points such that  $\pi(Y) \neq \pi(X)$  then the geodesic from  $Y$  to  $X$  passes through  $\pi(X)$ , see Figure 5. In particular  $d(Y, X) > d(Y, \pi(X))$ . In  $\mathcal{X}_n$ :

**Lemma 5.8** *There exist constants  $s, c > 0$  such that for  $X, Y \in \mathcal{X}_n$  if  $|t_Y - t_X| > s$ , then  $d(Y, X) \geq d(Y, \pi(X)) - c$*

To prove this we recall from Proposition 5.2, that if  $\alpha$  and  $\beta$  are loops in  $G(t)$  representing candidates of  $X$  then they cannot contain long pieces of both laminations  $\Lambda^+, \Lambda^-$ . We will need a slightly souped up version of this.

**Definition 5.9** We call a point  $X$  in  $\mathcal{X}_n$  minimal if the underlying topological graph of  $X$  is either a bouquet of circles or a graph with two vertices, one edge between them which we will refer to as a bar, and all other edges are loops.

**Proposition 5.10** *Suppose  $X$  is minimal. Let  $v$  be one of its vertices and the basepoint for  $\pi_1(X, v)$  and let  $e$  denote the bar of  $X$  initiating from  $v$  (if  $X$  is a rose then  $e$  is empty). Let  $\alpha_X, \beta_X$  be either one edge loops based at  $v$  or loops of the form  $e\gamma\bar{e}$  where  $\gamma$  is a one edge loop based at the other vertex. Fix  $Z \in \mathcal{L}_f$  and let  $h: X \rightarrow Z$  be a map homotopic to the difference in marking so that  $h(\alpha_X)$  is an immersed loop and  $h(\beta_X)$  is a loop which is immersed as a path. If  $h(\alpha_X), [h(\beta_X)]$  both contain a  $J$ -piece of  $v \in \Lambda^-$  then  $h(\beta_X)$  does not contain a  $3J$ -piece of  $\lambda \in \Lambda^+$ .*

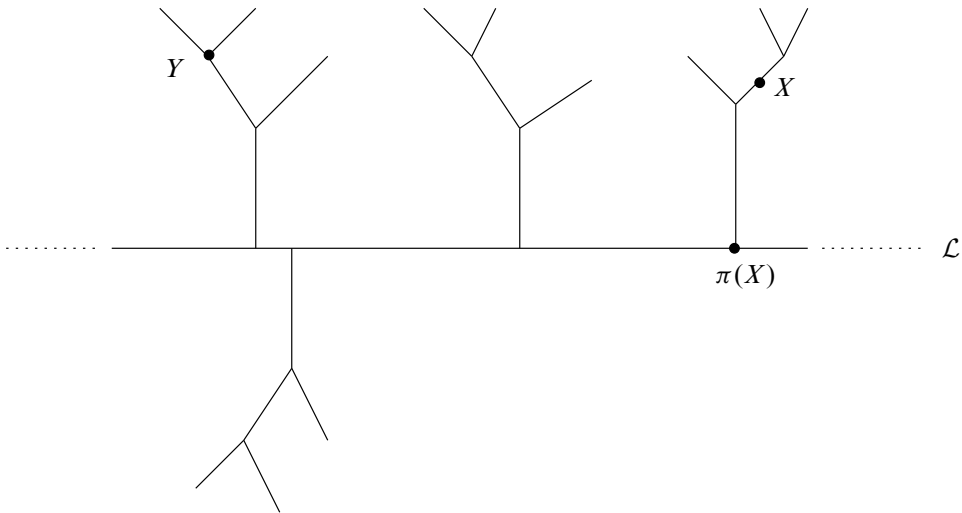


Figure 5: In a tree, if  $X, Y$  project to different points then the geodesic between them visits both of the projections.

**Proof** We emphasize that by Proposition 5.2,  $[h(\beta)]$  does not cross a  $J$ -piece of  $\lambda$  but we want it not to contain any such pieces in the part that gets cancelled when we tighten the loop.

We represent  $h(\alpha)$  by the cyclically reduced edge path  $x$  in  $G(t)$  and  $\beta$  by the reduced  $u = wyw^{-1}$ , with  $y$  cyclically reduced. We proceed to prove the proposition by way of contradiction. We'll show that if  $w$  contains a  $2J$  piece of  $\lambda$  then there is an  $m$  such that the immersed loop homotopic to  $x^m u$  contains both a  $J$  piece of  $\nu$  and a  $J$  piece of  $\lambda$ . Since  $x^m u$  is a basis element this is a contradiction to Proposition 5.2.

First notice that  $w \not\subseteq x^m$  and  $w \not\subseteq x^{-m}$  for all  $m$ . For otherwise  $x^m$  or  $x^{-m}$  would contain both a  $J$  piece of  $\nu$  and a  $J$  piece of  $\lambda$  contradicting Proposition 5.2. Moreover, there is an  $m$  such that  $x^m \not\subseteq w$  and  $x^{-m} \not\subseteq w$ . Let  $u = \text{gcd}\{x^{-m}, w\}$  and  $v = \text{gcd}\{x^m, w^{-1}\}$  then  $x^m = vx_1 = x_2u^{-1}$  with  $x_1, x_2$  non-empty,  $w = uw_1 = w_2v^{-1}$  with  $w_1, w_2$  non empty and

$$z = x^m x^m \cdot wyw^{-1} \cdot x^m = x^m x_2 w_1 y w_2 x_1$$

the word on the right hand side is cyclically reduced.

Now the assumption was that  $w = uw_1$  contains a  $2J$  piece of  $\lambda$ . If  $w_1$  contains a  $J$ -piece of  $\lambda$  then  $[z]$  represents a basis element and contains a  $J$ -piece of  $\lambda$  and a  $J$ -piece of  $\nu$  thereby contradicting Proposition 5.2. If  $u$  contains a  $J$ -piece of  $\lambda$  then

$x^m$  contains the inverse of a  $J$ -piece of  $\lambda$  and also a  $J$ -piece of  $\nu$  again contradicting Proposition 5.2. □

**Proof of Lemma 5.8** We prove the claim for  $X, Y$  such that  $t_Y < t_X$ , the case where  $t_Y > t_X$  follows by applying the same argument to  $g$ . We also make the assumption that  $X$  is minimal.

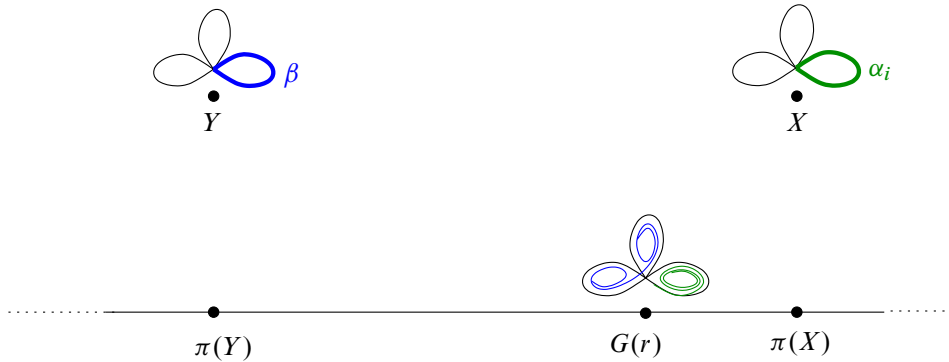


Figure 6: In  $G(r)$ ,  $\beta$  contains many  $l_1$ -pieces of  $\lambda$  and  $\alpha_i$  contain  $l_1$ -pieces of  $\nu$

Let  $\lambda$  be a periodic leaf of  $\Lambda_f^+$  and  $\nu$  a periodic leaf of  $\Lambda_f^-$ . Let  $J = J_{5.10}$ . The idea of the proof is as follows. If  $t_Y \ll t_X$ , then for  $r$  in the middle of  $[t_Y, t_X]$ , any loop which is short in  $Y$ , would contain many  $J$ -pieces of  $\lambda$  in  $G(r)$ . And any loop which is short in  $X$  would contain many  $J$ -pieces of  $\nu$  in  $G(r)$ , see Figure 6. If a candidate in  $Y$  was short in  $X$ , then it would contain long pieces of both  $\lambda$  and  $\nu$  in  $G(r)$  contradicting the fact that it is a basis element. To make the argument precise we need to argue that for a candidate  $\beta$  in  $Y$ ,  $l(\beta, X)$  is longer than a definite fraction of  $l(\beta, \pi(X))$ . This is done by bounding  $l(\beta, \pi(X))$  above by the number of disjoint  $J$ -pieces of  $\lambda$  that appear in  $\beta_{\pi(X)}$ .

Let  $s_1 = s_{5.5}$  that is, for any candidate  $\beta$  in  $Y$ ,  $|t_Y - t_\beta| < s_1$ . Let  $s_2 = s_{3.17}$ , that is, for any primitive conjugacy class  $\beta$  if  $t > t_\beta + s_2$  then  $LEG_f(\beta, G(t)) > \epsilon_0$ . Let  $s_3$  be such that if  $t > t_\beta + s_2 + s_3$  then  $\beta$  crosses a  $J$ -piece of  $\lambda$  in  $G(t)$  (contributed by one of the  $\kappa$  long legal segments). Let  $s_4$  be such that for any primitive conjugacy class  $\beta$  if  $t < t_\beta - s_4$  then  $\beta$  contains a  $J$ -piece of  $\nu$  in  $G(t)$ . Let  $s = 2s_1 + s_2 + s_3 + s_4$  and suppose that  $t_X - t_Y > s$  we will show that there exists a  $c$  as in the statement of the Lemma.

Let  $\beta$  be a loop in  $Y$  such that  $d(Y, \pi(X)) = \log(\text{St}_\beta(Y, \pi(X)))$ . Then by Corollary 5.4  $t_\beta < t_Y + s_1$ . Let  $r = t_X - s_1 - s_4$  then  $r > t_Y + s_1 + s_2 + s_3$  (see Figure 6). Let  $k(r)$

be the number of  $J$ -pieces of  $\lambda$  in  $\beta_r \subseteq G(r)$  with disjoint interiors, then

$$(10) \quad k(r) \cdot J > \epsilon_0 \cdot l(\beta, G(r))$$

Recall that  $X$  is minimal. Let  $\alpha_1, \dots, \alpha_n$  denote loops representing a short basis in  $X$  where  $\alpha_i$  is either a one edge loop or is  $e\alpha'\bar{e}$  where  $\alpha'$  is a one edge loop based at the other vertex and  $e$  is the bar of  $X$ . Let  $\alpha_1$  be the longest one-edge-loop.

Choose a map  $h: X \rightarrow G(r)$ , homotopic to the difference in marking, so that  $h(\alpha_1)$  is an immersed loop and  $h(\alpha_i)$  are immersed as paths. Each  $h(\alpha_i)$  in  $G(r)$  contains a  $J$ -piece of  $v$ . By Proposition 5.10 for  $1 \leq i \leq n$ ,  $h(\alpha_i)$  does not contain any  $J$ -pieces of  $\lambda$ .

**Claim** *Let  $\gamma$  be a conjugacy class in  $F_n$  and write it as a cyclically reduced word in  $\alpha_1, \dots, \alpha_n$  the basis of  $\pi_1(X, v)$ . If  $[h(\gamma_X)]$  contains  $k$  occurrences of  $J$ -pieces of  $\Lambda^+$  in  $G(r)$  (with disjoint interiors) then  $\gamma_X$  traverses each  $\alpha_q$  at least  $k$  times.*

*Moreover, for  $i \neq j$ ,  $\gamma$  contains at least  $\lfloor \frac{k}{2} \rfloor$  disjoint appearances of subpaths of the type  $\alpha_i \dots \alpha_j$ .*

**Proof of Claim** First note that if  $\gamma_X$  is a loop that does not traverse  $\alpha_q$  at all then it is carried by the free factor  $\langle \alpha_1, \dots, \widehat{\alpha_q}, \dots, \alpha_n \rangle$ . Using Proposition 5.2 applied to  $h(\alpha_q), [h(\gamma)]$  in  $G(r)$ , we get that  $[h(\gamma)]$  does not contain any  $l$ -pieces of  $\Lambda^+$  in  $G(r)$ .

Now suppose that  $\gamma_X = \alpha_{i_1} \dots \alpha_{i_N}$  so that  $\alpha_{i_j} = \alpha_q$  for at most  $k - 1$  choices of  $j$ s.  $[h(\gamma_X)]$  is the result of reducing  $h(\alpha_{i_1}) \cdot h(\alpha_{i_2}) \dots h(\alpha_{i_N})$  to get  $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_N}$  where  $\sigma_{i_j}$  are the subpaths of  $h(\alpha_{i_j})$  that survive after the cancellation (some  $\sigma_{i_j}$  might be trivial).  $J$ -pieces of  $\lambda$  can appear only if they are split between different  $\sigma_{i_j}$ s. If there are  $k$  disjoint  $J$ -pieces of  $\lambda$  in  $\gamma_{G(r)}$  then there is a  $J$ -piece of  $\lambda$  appearing in  $\sigma_{i_m} \dots \sigma_{i_l} \subseteq [h(\alpha_{i_m}) \dots h(\alpha_{i_l})]$  where none of the  $\alpha_{i_j}$  are equal to  $\alpha_q$ . This is a contradiction to the first paragraph. Since every  $J$ -piece in  $\lambda$  must stem from  $[h(\alpha_{i_m}) \dots h(\alpha_{i_l})]$  with all  $\alpha_i$  appearing in the product then  $\gamma_X$  contains at least  $\lfloor \frac{k}{2} \rfloor$  disjoint appearances of subpaths of the type  $\alpha_i \dots \alpha_j$ .  $\square$

By the claim above  $\beta_X$  in  $X$  must traverse  $\alpha_1$  at least  $k = k(r)$  times. If  $l(\alpha_1, X) > \frac{1}{n+1}$  then  $l(\beta, X) > \frac{k}{n+1}$ . Otherwise,  $X$  has a separating edge  $e$  and  $l(e, X) > \frac{1}{n+1}$ . Let  $\delta$  be a one-edge-loop so that  $\alpha_1$  and  $\delta$  are loops on opposite sides of  $e$ . By the claim above  $\beta_X$  traverses  $\alpha_1$  and  $\delta$  alternately at least  $\frac{k}{2}$  times therefore it must cross  $e$  at least  $\frac{k}{2}$  times. Again we get  $l(\beta, X) > \frac{k}{2(n+1)}$ . Combining with equation (10)

$$(11) \quad l(\beta, X) > \frac{\epsilon_0}{2(n+1)J} l(\beta, G(r))$$

$\mathcal{L}_f$  is contained in the  $\theta$ -thick part of  $\mathcal{X}_n$  for some  $\theta$ . Let  $\mu = \exp(D_{2.9}(\theta)(s_1 + s_4))$  then  $l(\beta, G(r)) \geq \mu l(\beta, G(t_X))$ . By equation (11) we get  $l(\beta, X) > \frac{\epsilon_0 \mu}{2(n+1)^J} l(\beta, \pi(X))$ . Thus, we get  $\frac{l(\beta, X)}{l(\beta, Y)} > \frac{\epsilon_0 \mu}{2(n+1)^J} \frac{l(\beta, \pi(X))}{l(\beta, Y)}$ , that is,

$$d(Y, X) > d(Y, \pi(X)) - \log \left( \frac{2(n+1)^J}{\epsilon_0 \mu} \right)$$

Which proves the statement in the case that  $X$  is minimal. Now we deal with the case that  $X$  is not minimal. We claim that there is a constant  $b$  such that for any  $X \in \mathcal{X}_n$  there is a minimal  $K$  so that  $d(X, K) < b$ . Moreover, there exists a short loop in  $X$  (whose length is smaller than 1) that is still short in  $K$ . Therefore, by Corollary 5.6  $d(\pi(K), \pi(X)) < s_{5.6}$ . So

$$\begin{aligned} d(Y, X) &\geq d(Y, K) - d(X, K) \geq d(Y, K) - b > d(Y, \pi(K)) - c - b \geq \\ &d(Y, \pi(K)) - d(\pi(K), \pi(X)) - c - b > d(Y, \pi(X)) - c - b - s_{5.6}. \end{aligned}$$

To prove that each point in  $\mathcal{X}_n$  lies a uniform distance away from a minimal  $K$ : Let  $e$  be the longest edge in  $X$ . Note that  $l(e, X) \geq \frac{1}{3n-3}$ . If  $e$  is nonseparating let  $J$  be a maximal tree in  $X$  that does not contain  $e$ , otherwise let  $J$  be the forest obtained from this maximal tree by deleting  $e$ . Collapse  $J$  to get a new unnormalized graph  $X'$  with volume  $> \frac{1}{3n-3}$ . Notice that  $X'$  is a minimal graph. Normalize  $X'$  to get  $K$ . Then  $d(X, K) \leq \log \left( \frac{1}{1/(3n-3)} \right) = \log(3n-3)$ . If  $\alpha$  is an embedded path in  $X$  its image is embedded in  $K$  thus the claim is proved.  $\square$

**Corollary 5.11** *There are constants  $s, c > 0$  such that if  $d(\pi(Y), \pi(X)) > s$  then*

$$d(Y, X) > d(Y, \pi(Y)) + d(\pi(Y), \pi(X)) - c.$$

**Proof** By Lemma 5.7 if  $d(\pi(Y), \pi(X)) > s_{5.7}$  then

$$d(Y, \pi(X)) > d(Y, \pi(Y)) + d(\pi(Y), \pi(X)) - c_{5.7}.$$

By Lemma 5.8 if  $d(\pi(Y), \pi(X)) > s_{5.8}$  then

$$d(Y, X) > d(Y, \pi(X)) - c_{5.8}.$$

So let  $s = \max\{s_{5.7}, s_{5.8}\}$  and  $c = c_{5.7} + c_{5.8}$  then  $d(\pi(Y), \pi(X)) > s$  implies

$$\begin{aligned} d(Y, X) &> d(Y, \pi(X)) - c_{5.8} > d(Y, \pi(Y)) + d(\pi(Y), \pi(X)) - c_{5.7} - c_{5.8} \\ &= d(Y, \pi(Y)) + d(\pi(Y), \pi(X)) - c. \end{aligned}$$

This completes the proof.  $\square$

As a corollary we get that the projection is coarsely Lipschitz.

**Corollary 5.12** *There is a constant  $c > 0$  such that for all  $X, Y \in \mathcal{X}_n$ ,*

$$d(X, Y) \geq d(\pi(X), \pi(Y)) - c.$$

### 5.3 Strongly contracting geodesics

**Definition 5.13** Let  $A$  be a subset in  $\mathcal{U}$  an asymmetric metric space. The outgoing neighborhood and the incoming neighborhood of  $A$  are respectively:

$$\begin{aligned} N_\delta(A_{\rightarrow}) &= \{x \in \mathcal{U} \mid d(a, x) < \delta \text{ for some } a \in A\} \\ N_\delta(A_{\leftarrow}) &= \{x \in \mathcal{U} \mid d(x, a) < \delta \text{ for some } a \in A\} \end{aligned}$$

**Definition 5.14** Let  $r > 0$ . The ball of outward radius  $r$  centered at  $x$  is

$$B_r(x_{\rightarrow}) = \{y \in \mathcal{X}_n \mid d(x, y) < r\}$$

The ball of inward radius  $r$  centered at  $x$  is

$$B_r(x_{\leftarrow}) = \{y \in \mathcal{X}_n \mid d(y, x) < r\}$$

We will use the ball of outward radius to define the notion of a strongly contracting geodesic in this case.

**Definition 5.15** (Strongly contracting geodesics in an asymmetric space) Let  $L$  be a directed geodesic in  $\mathcal{U}$ , and let  $\pi_L: \mathcal{U} \rightarrow L$  be the closest point projection.  $L$  is  $D$ -strongly contracting if, for any ball  $B_r(x_{\rightarrow}) \subseteq \mathcal{U}$  disjoint from  $L$ ,

$$\text{diam}(\pi_L(B_r(x_{\rightarrow}))) < D.$$

**Theorem 5.16** *If  $f: G \rightarrow G$  is a train-track representative of a fully irreducible outer automorphism  $\phi$ , then  $\mathcal{L}_f$  is  $D$ -strongly contracting.*

**Proof** It is enough to show that there exists a  $D > 0$  such that  $\text{diam}\{\pi(B_r(Y_{\rightarrow}))\} < D$  for  $r = d(Y, \pi(Y))$ . We'll show that if  $X \in B_r(Y_{\rightarrow})$  then  $d(\pi(Y), \pi(X)) < D$  where  $D = \max\{s_{5.11}, c_{5.11}\}$ . If  $X \in B_r(Y_{\rightarrow})$  then  $d(Y, X) < r$  and by Corollary 5.11 either  $d(\pi(Y), \pi(X)) \leq s_{5.11}$  or  $d(Y, X) > d(Y, \pi(Y)) + d(\pi(Y), \pi(X)) - c_{5.11}$ . If the latter occurs then

$$r > r + d(\pi(Y), \pi(X)) - c_{5.11}$$

Thus  $d(\pi(Y), \pi(X)) < c_{5.11}$ . □



### 5.4 The Morse Lemma

**Definition 5.17** The map  $\alpha: [0, l] \rightarrow \mathcal{U}$  is a directed  $(k, c)$ -quasi-geodesic if for all  $0 \leq t < t' \leq l$  we have

$$\frac{1}{k}(t' - t) - c \leq d(\alpha(t), \alpha(t')) \leq k(t' - t) + c$$

**Definition 5.18** Let  $\alpha: [0, l] \rightarrow \mathcal{U}$  be a path. As in a symmetric metric space  $\alpha$  is rectifiable if the following limit exists. Let  $P$  be a partition of  $[0, l]$ ,  $P: t_0 = 0 < t_1 < \dots < t_m = l$ . Let  $\Delta(P) = \max_i \{t_{i+1} - t_i\}$

$$\lim_{\Delta(P) \rightarrow 0} \sum_{i=0}^{m-1} d(\alpha(t_i), \alpha(t_{i+1}))$$

This limit is the length of the path  $\alpha$ .

**Definition 5.19** A quasi-geodesic  $\alpha: [0, l] \rightarrow \mathcal{U}$  is  $(m, p)$ -tame if for all  $0 \leq t < t' \leq l$  we have

$$\text{len}(\alpha|_{[t, t']}) \leq m(t' - t) + p$$

**Lemma 5.20** For every  $(k, c)$ -quasi-geodesic  $\alpha: [0, l] \rightarrow \mathcal{U}$  there is an  $(m, p)$ -tame  $(k', c')$ -quasi-geodesic  $\beta: [0, l] \rightarrow \mathcal{U}$  with

- (1)  $\beta(0) = \alpha(0), \beta(l) = \alpha(l)$
- (2)  $k' = k, c' = 2(k + c)$
- (3)  $m = k(k + c), \text{ and } p = (k + c)(2k^2 + 2kc + 3)$
- (4)  $N_{k+c}(\text{im } \alpha) \supseteq \text{im } \beta \text{ and } N_{k+c}(\text{im } \beta) \supseteq \text{im } \alpha$

The proof of Lemma 5.20 for a symmetric metric space can be found in the book by Bridson and Haefliger [11]. The proof for a nonsymmetric space is the same hence we omit it.

**Definition 5.21** A point  $x \in \mathcal{U}$  is high if there exists a constant  $A$  such that  $d(x, y) \leq \text{Ad}(y, x)$  for all  $y \in \mathcal{U}$ . A set  $S \in \mathcal{U}$  is high if there are constants  $A$  so that for all  $x \in S$  and  $y \in \mathcal{U}$  such that  $d(x, y) \leq \text{Ad}(y, x)$

We recall the definition of Hausdorff distance

**Definition 5.22** Let  $S, T \subset \mathcal{U}$  be closed. Define the Hausdorff distance

$$d_{\text{Haus}}(S, T) = \inf\{\epsilon \mid S \subseteq N_\epsilon(T_{\rightarrow}) \text{ and } T \subseteq N_\epsilon(S_{\rightarrow})\}$$

**The Morse Lemma** *If  $L$  is a directed,  $A$ -high,  $D$ -strongly contracting geodesic in  $\mathcal{U}$  and  $\alpha$  is an  $(a, b)$ -quasi-geodesic with endpoints on  $L$  then there exists a constant  $C$ , depending only on  $A, D, a$  and  $b$ , such that  $d_{\text{Haus}}(\text{im } L, \text{im } \alpha) < C$ .*

**Remark 5.23** In fact, we only need  $\odot$  to satisfy:  $\text{len}(\alpha|_{[t',t]}) < a[d(\alpha(t'), \alpha(t))] + b$  for the corollary above to hold true.

**Proof of the Morse Lemma** We may assume that  $\alpha$  is an  $(a, b)$ -tame quasi-geodesic. We fix the following notation. Let  $c = \max\{a, b, 1\}$ ,  $R = \max\{d(\alpha(t), \text{im } L) | t \in [0, l]\}$  and suppose  $R > cD$ . Let  $[s_1, s_2]$  be a maximum subinterval such that for every  $s \in [s_1, s_2]$ :  $d(\alpha(s), \text{im } L) \geq cD$ . Subdivide  $[s_1, s_2]$  into:  $s_1 = r_1, \dots, r_m, r_{m+1} = s_2$  where  $d(\alpha(r_i), \alpha(r_{i+1})) = 2cD$  for  $i \leq m$  and  $d(\alpha(r_m), \alpha(r_{m+1})) \leq 2cD$ . Thus:

$$(12) \quad \text{len}(\alpha|_{[s_1, s_2]}) \geq \sum_{i=1}^{m+1} d(\alpha(r_i), \alpha(r_{i+1})) \geq 2cDm$$

On the other hand, let  $p_i = \pi_L(\alpha(r_i))$ . Since  $d(\alpha(r_i), p_i) \geq cD$ , and since  $L$  is  $D$ -strongly contracting we get

$$d(p_1, p_{m+1}) \leq D(m + 1)$$

So  $d(\alpha(r_1), \alpha(r_{m+1})) \leq cD + (m + 1)D + AcD$  where  $L$  is  $A$ -high. Therefore,

$$(13) \quad \text{len}(\alpha|_{[s_1, s_2]}) \leq cd(\alpha(s_1), \alpha(s_2)) + c \leq c(cD + (m + 1)D + AcD) + c$$

Combining the inequalities (12) and (13) we get:

$$2mcD \leq c^2D + (m + 1)cD + Ac^2D + c$$

After some manipulation we get:  $m \leq c + c \cdot A + \frac{1}{D} + 1 = K$ .

Hence  $\text{len}(\alpha|_{[s_1, s_2]}) \leq m2cD < 2KcD$ . Thus for each  $s \in [s_1, s_2]$ :

$$d(\alpha(s), \text{im } L) < d(\alpha(s), \alpha(s_2)) + d(\alpha(s_2), \text{im } L) \leq \text{len}(\alpha|_{[s_1, s_2]}) + cD < 2KcD + cD$$

Since  $\text{im } L$  is  $A$ -high there is a constant  $C$  so that  $d_{\text{Haus}}(\text{im } L, \text{im } \alpha) < C$ . □

Since  $\mathcal{L}_f$  is periodic there is an  $\epsilon$  so that  $\text{im } \mathcal{L}_f \subseteq \mathcal{X}_n^{\geq \epsilon}$ . By Theorem 2.9 the set  $\mathcal{L}_f$  is  $A$ -high. Thus applying the Morse Lemma we get the following.

**Theorem 5.24**  *$\mathcal{L}_f$  is a Morse geodesic: For any  $(a, b)$ -quasi-geodesic  $\mathcal{Q}$  with endpoints on  $\mathcal{L}_f$  there is a  $C$  that depends only on the constants  $a, b, \epsilon$  and  $D_{5.16}$  so that  $d_{\text{Haus}}(\text{im } \mathcal{L}_f, \text{im } \mathcal{Q}) < C$ .*

## 6 Applications

### 6.1 Morse quasi-geodesics in the Cayley graph of $\text{Out}(F_n)$

Let  $\mathcal{C}$  be the Cayley graph of  $\text{Out}(F_n)$  with the generating set of Whitehead automorphisms  $\{\phi_i\}_{i=1}^N$ , that is, its vertices  $V(\mathcal{C})$  are the elements of  $\text{Out}(F_n)$  and  $\psi_1, \psi_2 \in \text{Out}(F_n)$  are connected by an edge if there is a Whitehead generator  $\phi_i$  so that  $\psi_1 = \phi_i \circ \psi_2$  (we want  $\text{Out}(F_n)$  to act on the right). Let  $\phi$  be a fully irreducible outer automorphism. Let  $f: G \rightarrow G$  be a stable train-track representative for  $\phi$ . Define the equivariant map  $\iota: \mathcal{C} \hookrightarrow \mathcal{X}_n$  by taking  $\text{id} \in \mathcal{C}$  to  $G$ , and mapping the edges of  $\mathcal{C}$  to geodesics between images of vertices.

Let  $M = \max\{d_{\mathcal{X}_n}(\iota(\text{id}), \iota(\phi_i)) \mid \phi_i \text{ is a generator}\}$  then for the vertices of  $\mathcal{C}$  we have:

$$d_{\mathcal{X}_n}(\iota(\psi_1), \iota(\psi_2)) \leq M \cdot d_{\mathcal{C}}(\psi_1, \psi_2)$$

For other points in  $\mathcal{C}$  a similar inequality holds (by adding  $2M$ ). The reverse inequality is false.

**Example 6.1** Let

$$\psi_1 = \text{id} \quad \text{and} \quad \psi_2 = \begin{cases} x \rightarrow x \\ y \rightarrow xy^m \end{cases}$$

Suppose  $\iota(\psi_1) = R$  is a bouquet of 2 circles each of length  $\frac{1}{2}$  with the identity marking. Then

$$d_{\mathcal{X}_n}(\iota(\psi_1), \iota(\psi_2)) = d_{\mathcal{X}_n}(R, R \cdot \psi_2) = \log\left(\frac{(m+1)/2}{1/2}\right) = \log(m+1),$$

while  $d_{\mathcal{C}}(\psi_1, \psi_2) = m$ .

**Lemma 6.2** ( $\iota$  is proper) *For all  $a > 0$  there is a  $b > 0$  such that: If  $d_{\mathcal{X}_n}(\iota(\psi), \iota(\chi)) < a$  then  $d_{\mathcal{C}}(\psi, \chi) < b$ .*

**Proof** Consider the set  $\mathcal{A} = \{\psi \mid d_{\mathcal{X}_n}(\iota(\text{id}), \iota(\psi)) < a\}$ . Note that  $V(\mathcal{C})$  is mapped to the orbit of  $G$  which is in the  $\theta$ -thick part of  $\text{Out}(F_n)$  hence  $\mathcal{A} \subseteq B_a(G_{\rightarrow}) \cap \mathcal{X}_n^{\geq \theta}$ . Hence  $\mathcal{A} \subseteq B_r(G_{\leftarrow})$  where  $r = D_{2,9}a$ .  $B_r(G_{\leftarrow})$  is compact thus  $\mathcal{A}$  is finite. Let  $b = \max\{d_{\mathcal{C}}(\text{id}, \psi) \mid \psi \in \mathcal{A}\}$ . Suppose  $d_{\mathcal{X}_n}(\iota(\psi), \iota(\chi)) < a$  then  $d_{\mathcal{X}_n}(\iota(\text{id}), \iota(\chi\psi^{-1})) < a$ , so  $d_{\mathcal{C}}(\text{id}, \chi\psi^{-1}) < b$  and  $d_{\mathcal{C}}(\psi, \chi) < b$   $\square$

Let  $p = [\text{id}, \phi]$  be a geodesic path from  $\text{id}$  to  $\phi$  in  $\mathcal{C}$  and define  $L: \mathbb{Z} \rightarrow \mathcal{C}$  by  $k \rightarrow \phi^k$ .  $L$  can be extended to a map  $\mathbb{R} \rightarrow \bigcup_{i=-\infty}^{\infty} p \cdot \phi^i$ .  $L$  is a quasi-geodesic: for  $i < j \in \mathbb{Z}$  we have

$$d(L(i), L(j)) \geq \frac{1}{M}d(\iota \circ L(i), \iota \circ L(j)) = \frac{1}{M}d(G\phi^i, G\phi^j) = \frac{1}{M}(j-i) \log \lambda$$

and  $d(L(i), L(j)) \leq (j - i) \text{len}(p)$ .

**Theorem 6.3**  $L$  is a Morse quasi-geodesic in  $\mathcal{C}$ .

**Proof** Let  $\alpha$  be an  $(a, b)$ -quasi geodesic in  $\mathcal{C}$  with endpoints on  $\text{im } L$ . We may assume that  $\alpha$  is tame. Consider  $\mathcal{Q} = \iota \circ \alpha$ , then  $\text{len}_{\mathcal{X}_n} \mathcal{Q}|_{[t, t']} \leq M \cdot \text{len}_{\mathcal{C}} \alpha|_{[t, t']} \leq Ma(t' - t) + Mb$ . By Remark 5.23, there exists a  $d$ , depending only on  $a, b, M, D$  and  $\epsilon$  (where  $\mathcal{L}_f$  is in the  $\epsilon$  thick part of  $\mathcal{X}_n$ ) such that  $d_{\text{Haus}}(\text{im } \mathcal{Q}, \text{im } \mathcal{L}_f) < d$ . Since  $\iota$  is proper (Lemma 6.2) we have  $d_{\text{Haus}}(\text{im } L, \text{im } \alpha) < D$  for some  $D$  depending only on  $d$ . □

### 6.2 The asymptotic cone of $\mathcal{C}$

**Definition 6.4** A nonprincipal maximal ultrafilter  $\omega$  on the integers is a nonempty collection of subsets of  $\mathbb{Z}$  so that:

- $\omega$  is closed under inclusion
- $\omega$  is closed under finite intersection
- $\omega$  does not contain any finite sets
- $A \subset \mathbb{Z}$ , if  $A \notin \omega$  then  $\mathbb{Z} \setminus A \in \omega$

**Definition 6.5** Let  $\omega$  be a nonprinciple maximal ultrafilter on the integers. Let  $(X_i, x_i, d_i)$  be a sequence of based metric spaces. Define the following pseudo-distance on  $\prod_{i \in \mathbb{N}} X_i$ :

$$d_\omega(\{a_i\}, \{b_i\}) = \lim_\omega d_{X_i}(a_i, b_i)$$

The ultralimit of  $(X_i, x_i)$  is then

$$\lim_\omega (X_i, x_i, d_i) = \left\{ y \in \prod_{i \in \mathbb{N}} X_i \mid d_\omega(y, \{x_i\}) < \infty \right\} / \sim$$

where  $y \sim y'$  if  $d_\omega(y, y') = 0$ .

Consider a space  $X$ , a point  $x \in X$  and a sequence of integers  $k_i$  such that  $\lim_{i \rightarrow \infty} k_i = \infty$ .

**Definition 6.6** The asymptotic cone of  $(X, x, \{k_i\})$  relative to the ultrafilter  $\omega$  is:

$$\text{Cone}_\omega(X, x, k_i) = \lim_\omega \left( X, x, \frac{1}{k_i} d_X(\cdot, \cdot) \right)$$

The asymptotic cone of a geodesic metric space is a geodesic metric space.

We recall the following definition from Druţu and Sapir [18].

**Definition 6.7** Let  $W$  be a complete metric space and let  $\mathcal{P}$  be a collection of closed geodesic subsets (called pieces). The space  $W$  is said to be tree-graded with respect to  $\mathcal{P}$  if the following properties are satisfied:

- (1) The intersection of two pieces is either empty or a single point.
- (2) Every simple geodesic triangle in  $X$  is contained in one piece.

The arcs starting in a given point  $w \in W$  intersecting each piece in at most one point compose a real tree called a *transversal tree*.

In particular, if  $p$  is in a transversal tree then  $p$  is a cut point of  $W$ .

**Theorem 6.8** (Druţu, Mozes and Sapir [17, Proposition 3.24]) *Let  $X$  be a metric space and let  $\mathfrak{q}$  be a quasi-geodesic. The following are equivalent:*

- *The image of  $\mathfrak{q}$  in every asymptotic cone of  $X$  is either empty or contained in a transversal tree of  $X$  for some tree graded structure.*
- *$\mathfrak{q}$  is a Morse quasi-geodesic.*

We get the following corollary from Theorem 6.3.

**Corollary 6.9** *The image of an axis of an irreducible automorphism in  $\text{Cone}_\omega \mathcal{C}$  is either empty or is contained in a transversal tree for some tree graded structure of  $\text{Cone}_\omega \mathcal{C}$ .*

**Remark 6.10** It is tempting to try to define the asymptotic cone of Outer Space itself. One would like to conclude that the cone is itself an asymmetric metric space. We choose a basepoint  $x_0 \in \mathcal{X}_n$  and define:

$$\lim_{\omega}(\mathcal{X}_n, \{x_0\}, d_{\text{Lip}}) = \left\{ \{y_i\} \in \prod_{i \in \mathbb{N}} \mathcal{X}_n \mid d_{\omega}(\{y_i\}, \{x_0\}) < \infty \right\}$$

Note that by the asymmetry theorem  $d_{\omega}(\{x_0\}, \{y_i\}) < \infty$  but now it does not make sense to mod out by the equivalence relation  $y \sim z$  if  $d(y, z) = 0$  because  $d(z, y)$  might be positive. For example, if we choose  $y_i$  so that  $d_{\text{Lip}}(y_i, x_0) = i$  and  $d_{\text{Lip}}(x_0, y_i)$  is bounded then  $d(\{y_i\}, \{x_0\}) = 1$  and  $d(\{x_0\}, \{y_i\}) = 0$ . We could always restrict our attention to the thick part of Outer Space where the distances are almost symmetric. However, this space is quasi-isometric to  $\mathcal{C}$  so we will not get anything new.

### 6.3 Divergence in Outer Space

**Definition 6.11** Let  $\gamma_1, \gamma_2$  be two geodesic rays in  $\mathcal{X}_n$ , with  $\gamma_1(0) = \gamma_2(0) = x$ . The divergence function from  $\gamma_1$  to  $\gamma_2$  is:

$$\text{div}(\gamma_1, \gamma_2, t) = \inf \left\{ \text{length}(\gamma) \mid \begin{array}{l} \gamma: [0, 1] \rightarrow \mathcal{X}_n \setminus B_t(x_{\leftarrow}) \\ \gamma(0) = \gamma_1(t), \gamma(1) = \gamma_2(t) \end{array} \right\}$$

If  $f(t)$  is a function such that:

- (1) for every  $\gamma_1, \gamma_2$ :  $\text{div}(\gamma_1, \gamma_2, t) \prec f(t)$  (we use  $g(t) \prec f(t)$  to denote the relationship  $f(t) \leq c \cdot g(t) + c'$  for all  $t$ )
- (2) there exist geodesics  $\gamma_1, \gamma_2$  such that  $\text{div}(\gamma_1, \gamma_2, t) \asymp f(t)$ .

then we say that the divergence function of  $\mathcal{X}_n$  is on the order of  $f(t)$ . If only (1) holds we say that  $f$  is an upper bound for the divergence of  $\mathcal{X}_n$ , if only (2) holds we say that  $f$  is a lower bound for the divergence of  $\mathcal{X}_n$ .

Behrstock [3] proved that the divergence in  $\mathcal{MCG}(S)$  is quadratic. Duchin and Rafi [19] prove that the divergence in Teichmüller space is quadratic. The proof that the divergence is at least quadratic in the Outer Space setting needs very little modification, but we include it for the reader’s convenience.

**Proposition 6.12** Let  $\gamma$  be a path in  $\mathcal{X}_n$ , from  $x$  to  $y$  with  $d(\pi(x), \pi(y)) = 2R$ . Let  $z$  the point on  $\mathcal{L}_f$  in the middle of the segment  $[\pi(x), \pi(y)] \subseteq \mathcal{L}$ . Further assume that the image of  $\gamma$  lies outside the ball  $B_R(z_{\leftarrow})$ . If  $R > 2D_{5.16}$  then there is a constant  $c$  such that  $\text{Len}(\gamma) \geq cR^2$  where  $c$  only depends on the constants  $D_{5.16}$  and  $c_{5.12}$ .

**Proof** Subdivide  $\gamma$  into  $n > 1$  subsegments  $I_1, I_2, \dots, I_n$ , each of which has length  $\frac{R}{2}$  except for possibly  $\text{Len}(I_n) \leq \frac{R}{2}$ . Therefore  $\text{Len}(\gamma) \geq (n - 1)\frac{R}{2}$ . Let  $\mathcal{L}_0$  be the subsegment of  $\mathcal{L}_f$  centered at  $z$  of length  $R$ . Since  $\mathcal{L}$  is  $b$ -contracting for  $b = D_{5.16}$  then  $\mathcal{L}_0$  is  $b'$ -contracting for  $b' = b + 4c_{5.12} + 3$  (see Bestvina and Fujiwara [9, Lemma 3.2]). Each segment  $I_j$  is contained in a ball  $B_{R/2}(x'_{\leftarrow})$  disjoint from  $\mathcal{L}_0$ . Thus the length of each  $\pi(I_j) \leq b'$ , since these segments cover  $\mathcal{L}_0$  we get  $R \leq nb'$ . Therefore  $\text{Len}(\gamma) \geq (n - 1)\frac{R}{2} > \left(\frac{R}{b'} - 1\right)\frac{R}{2} = \frac{1}{2b'}R^2 - \frac{1}{2}R$ . □

The exact behavior of the divergence function of  $\mathcal{X}_n$  remains open.

### 6.4 The Behrstock inequality

In this section let  $\phi, \psi \in \text{Out}(F_n)$  be two irreducible outer automorphisms and  $f, g$  their respective train-track representatives. Denote  $A = \mathcal{L}_f, B = \mathcal{L}_g$  and  $p_A = \pi_f$  and  $p_B = \pi_g$ . Our first goal is to show that either  $A, B$  are parallel or the diameter of  $p_A(B)$  is bounded, and we would like to understand what the bound depends on. We introduce the following notation for the next lemma: if  $x, y \in A$  denote by  $[x, y]_A$  the subinterval of  $A$  whose endpoints are  $x$  and  $y$ .

**Lemma 6.13** *There exist constants  $c, d$  such that if  $x, y \in B$  with  $d(p_A(x), p_A(y)) > c$ , then*

$$[p_A(x), p_A(y)]_A \subset \mathcal{N}_d(B),$$

where  $c, d$  depend only on the constants  $s_{5.8}, c_{5.8}$  applied to  $A$  and  $B$  and on  $D_{2.9}(\theta)$  where  $\theta$  is small enough so that  $A, B \subset \mathcal{X}_n(\theta)$ .

**Proof** Denote  $c_1 = D_{2.9}(\theta)$ . For all  $z, w \in \mathcal{X}_n(\theta)$ :  $d(w, z) < c_1 \cdot d(z, w)$ . Let  $s_A, c_A$  be the constants from Lemma 5.8 applied to  $A$ , thus if  $z, w$  are points such that  $d(p_A(z), p_A(w)) > s_A$  then  $d(z, w) > d(z, p_A(z)) + d(p_A(z), p_A(w)) - c_A$ . Let  $a = 1 + (c_1)^2, b = c_A(1 + c_1)$  and  $d = c_{5.24}(a, b)$  from Theorem 5.24 applied to  $B$ , that is, for every  $(a, b)$ -quasi-geodesic  $\mathcal{Q}$  with endpoints on  $B, N_d(B) \supset \text{im } \mathcal{Q}$ . We prove that  $[x, p_A(x)] \cup [p_A(x), p_A(y)]_A \cup [p_A(y), y]$  is an  $(a, b)$ -quasi-geodesic.

First note

$$(14) \quad d(x, y) > d(x, p_A(x)) + d(p_A(x), p_A(y)) - c_A.$$

Similarly,

$$\begin{aligned} d(y, x) &> d(y, p_A(y)) + d(p_A(y), p_A(x)) - c_A \\ d(y, x) &> d(y, p_A(y)) - c_A > \frac{1}{c_1} d(p_A(y), y) - c_A \end{aligned}$$

So

$$(15) \quad (c_1)^2 \cdot d(x, y) > c_1 d(y, x) > d(p_A(y), y) - c_1 c_A$$

Adding equations (14) and (15) we get

$$(1 + (c_1)^2) d(x, y) > d(x, p_A(x)) + d(p_A(x), p_A(y)) + d(p_A(y), y) - c_A(1 + c_1).$$

Therefore  $[x, p_A(x)] \cup [p_A(x), p_A(y)]_A \cup [p_A(y), y]$  is a  $(1 + (c_1)^2, c_A(1 + c_1))$ -quasi-geodesic. Hence  $[p_A(x), p_A(y)]_A \subset N_d(B)$ . □

The next Lemma is motivated by the following observation. Let  $X$  is a proper, metric space with a properly discontinuous isometric  $G$ -action. Let  $g, h \in G$  be hyperbolic isometries of  $X$  and let  $A_g, A_h$  denote their axes. Then for every  $d$  there is a constant  $k$  which depends only on  $d, \text{tr}(g), \text{tr}(h)$  such that either  $A_g, A_h$  are parallel, or the length of  $A_g \cap N_d(A_h)$  is shorter than  $k$ . In our case, Outer Space is not proper. The closure of a ball  $B_r(x_{\rightarrow}) = \{y \in \mathcal{X}_n \mid d(x, y) < r\}$  need not be compact. However

**Claim** *The closure of the ball  $B_r(x_{\leftarrow}) = \{y \in \mathcal{X}_n \mid d(y, x) < r\}$  in  $\overline{\mathcal{X}_n}$  is contained in  $\mathcal{X}_n$ , hence the closure of the ball  $B_r(x_{\leftarrow})$  in  $\mathcal{X}_n$  is compact.*

**Proof** For each  $y \in B_r(x_{\leftarrow})$ , and for all conjugacy classes  $\alpha$ ,  $l(\alpha, y) \geq \frac{l(\alpha, x)}{e^r}$ . Thus the length functions are bounded away from 0 by some  $c = \frac{\text{shortest loop in } \overline{\mathcal{X}}}{e^r}$ . Hence,  $B_r(x_{\leftarrow})$  is contained in the  $c$ -thick part of  $\mathcal{X}_n$ , which is quasi-symmetric. Thus, there is some  $h$  such that for all  $y \in B_r(x_{\leftarrow})$ ,  $d(x, y) < h$ . Let  $y_n \in B_r(x_{\leftarrow})$  converge in  $\overline{\mathcal{X}_n}$ . For each  $\alpha$ , the sequence  $l(\alpha, y_n)$  is bounded so there is a convergent subsequence. Since there are countably many conjugacy classes, by a diagonal argument we can find a subsequence of  $y_n$  so that all length functions  $l(\alpha, y_n)$  converge without rescaling. They must converge to a free simplicial tree since the length functions are bounded away from zero. Thus the limit point in  $\overline{\mathcal{X}_n}$  is contained in  $\mathcal{X}_n$ .  $\square$

Recall that the  $\text{Out}(F_n)$  action is properly discontinuous. Thus for every  $r$  there is a number  $N_r$  such that  $B_r(x_{\leftarrow})$  contains no more than  $N_r$  points of any orbit.

**Definition 6.14** Let  $A, B$  be two axes in  $\mathcal{X}_n(\theta)$  and  $r > 0$ . Define the closed set

$$A_B(r) = \{x \in A \mid d(x, p_B(x)) \leq r \text{ and } d(p_B(x), x) \leq r\}$$

**Lemma 6.15** *For every  $r$ , there exists a constant  $c = c(\text{tr}(g), \text{previous constants})$  such that either  $f, g$  have common powers, or the diameter of  $A_B(r)$  is smaller than  $c \text{tr}(f)$ .*

**Proof** Let  $\ell$  denote the diameter of  $A_B(d)$ . Let  $a$  be the leftmost point on  $A_B(r)$  assuming that  $f$  translates points to the right. Without loss of generality, assume  $f$  and  $g$  translate points in the same direction. Denote  $b = p_B(a)$  and  $k = \lfloor \frac{\ell}{\text{tr}(f)} \rfloor$ . We will show that there is an  $R$  such that for each  $1 \leq i \leq k$  there is a  $j$  such that  $d(a f^i g^{-j}, a) < R$ . There are at most  $N_R$  points in  $\{a \cdot \text{Out}(F_n)\} \cap B_R(a_{\leftarrow})$  thus there are two possibilities:

- $k \leq N_R$ .



- There are  $i, j, s, t$  such that  $af^i g^{-j} = af^s g^{-t}$  hence  $f^i g^{t-j} f^{-s}$  is in the stabilizer of  $a$ . If  $k > N_R |\text{Stab}(a)|$  then there are integers  $i, j, q, w, s, t$  such that  $f^i g^j f^q = f^w g^s f^t$  hence  $f, g$  have common powers.

Therefore if  $f, g$  don't have common powers then  $k < C$  and  $\ell < (C + 1)\text{tr}(f)$ . We now prove the existence of  $R$ :

For  $1 \leq i \leq k$  there is a unique  $j$  such that

$$d(p_B(a), p_B(af^i)g^{-j}) < \text{tr}(g)$$

Next we want to show that  $d(p_B(af^i), af^i)$  is bounded.  $y = af^i$  is in the interval  $[a, a']_A$  where  $a, a'$  are the endpoints of  $A_B$ .  $[p_b(a), a] \cup [a, a'] \cup [a', p_B(a')]$  is a  $(1, 2d)$  quasi-geodesic thus it is contained in the  $r_1$  neighborhood of  $[p_B(a), p_B(a')]_B$  by 5.24. Thus there is some point  $b' \in B$  such that  $d(y, b') < r_1$  hence  $d(y, p_B(y)) < r_1$ . By making  $r_1$  a bit larger we can also satisfy  $d(p_B(y), y) < r_1$  (we are in the thick part).

Since  $d(p_B(af^i), af^i) < r_1$  then  $d(p_B(af^i)g^{-j}, af^i g^{-j}) < r_1$  hence

$$\begin{aligned} d(a, af^i g^{-j}) &\leq d(a, p_B(a)) + d(p_B(a), p_B(af^i)g^{-j}) + d(p_B(af^i)g^{-j}, af^i g^{-j}) \\ &\leq r_1 + \text{tr}(g) + r_1 \\ &= \text{tr}(g) + 2r_1 \end{aligned}$$

Therefore,  $d(af^i g^{-j}, a) < c_1(\text{tr}(g) + 2r_1) = R$  where  $c_1 = D_{2.9}(\theta)$ . □

**Corollary 6.16** *There exists a constant  $k$ , depending only on the constants from Lemma 6.15 and Lemma 6.13, such that either  $f, g$  have common powers or*

$$\text{diam}\{p_A(B)\} < k$$

**Proof** Let  $\{x_i\}, \{y_i\}$  be sequences on  $B$  so that  $x_i$  converges to one end of  $B$  and  $y_i$  to the other. If  $d(p_A(x_i), p_A(y_i)) > c_{6.13}$  then  $[p_A(x_i), p_A(y_i)]_A \subseteq N_d(B)$ . Thus  $[p_A(x_i), p_A(y_i)]_A$  is contained in  $A_B(d)$ . Therefore by Lemma 6.15 there is a  $c = c_{6.15}\text{tr}(f)$  such that either  $f, g$  have common powers or the diameter of  $A_B(d) < c$  and hence the diameter of  $[p_A(x_i), p_A(y_i)]_A$  is smaller than  $c$ . □

Let us go back for a moment to the surface case. We denote by  $\mathcal{M}(S)$  the marking complex of  $S$ . Let  $Y, Z$  be subsurfaces of  $S$ , denote by  $\mathcal{C}(Z), \mathcal{C}(Y)$  the curve complexes of  $Z, Y$ . For definitions of the curve complex and the marking complex consult Behrstock [3]. Define the projections (slightly abusing notation)  $p_Y: \mathcal{C}(S) \rightarrow \mathcal{C}(Y)$ ,  $p_Y: \mathcal{M}(S) \rightarrow \mathcal{C}(Y)$  and  $p_Z: \mathcal{C}(S) \rightarrow \mathcal{C}(Z)$ ,  $p_Z: \mathcal{M}(S) \rightarrow \mathcal{C}(Z)$ . In [3,

Theorem 4.3], Behrstock proved that if  $Y, Z$  are overlapping subsurfaces of  $S$ , neither of which is an annulus, then for any marking  $\mu$  of  $S$ :

$$d_{C(Y)}(p_Y(\partial Z), p_Y(\mu)) > M \implies d_{C(Z)}(p_Z(\partial Y), p_Z(\mu)) < M$$

And the constant  $M$  depends only on the topological type of  $S$ . We prove an analogous estimate for our projections.

Suppose  $f, g, h$  are stable train-track maps representing fully irreducible automorphisms and  $A, B, C$  are their axes. Suppose that no two of these automorphisms have common powers. We define the coarse distance from  $B$  to  $C$  with respect to  $A$  as

$$d_A(B, C) = \text{diam}\{p_A(C) \cup p_A(B)\}$$

**Lemma 6.17** *There exists a constant  $M > 0$  depending only on the constants from Lemma 5.8 and Corollary 6.16 such that at most one of  $d_A(B, C), d_B(A, C)$  and  $d_C(A, B)$  is greater than  $M$ .*

**Proof** Let  $s_A, c_A, s_B, c_B, s_C, c_C$  be the constants from Lemma 5.8 applied to any of the geodesics  $A, B, C$  respectively. Let  $b \geq k_{6.16}$  the constant from Corollary 6.16 applied to any two of the three geodesics. Let  $M > \max\{s_A, c_A, s_B, c_B, s_C, c_C\} + 2b$ . We claim that if  $d_B(A, C) > M$  then  $d_C(A, B) < M$ . Assume by way of contradiction that both are greater than  $M$ . Let  $y \in A$  and  $q \in B$  such that  $d(y, q) = d_{\text{Haus}}(A, B)$ . Let  $z = p_C(y) \sim p_C(A)$ ,  $p = p_B(z) \sim p_B(C)$  and  $x = p_C(q) \sim p_C(B)$  (see Figure 7).

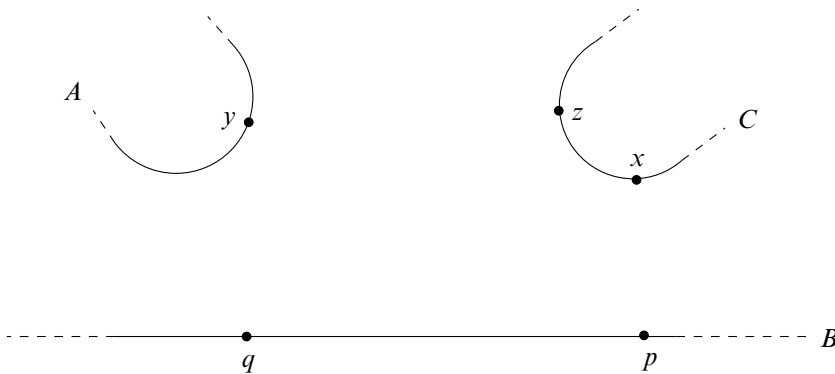


Figure 7: If  $d(p, q) > M$  then  $d(x, z) < M$ .

Because  $d(p_C(y), p_C(q)) = d(z, x) > M - 2b > s_C$ ,

$$d(y, q) > d(y, z) + d(z, x) - c_C > d(y, z) + M - 2b - c_C.$$

Since  $d(p_B(y), p_B(z)) = d(q, p) > M - 2b > s_B$  we have

$$d(y, z) > d(y, q) + d(q, p) - c_B > d(y, q) + M - 2b - c_B.$$

Therefore

$$d(y, q) > d(y, q) + 2M - c_C - c_B - 4b,$$

which implies  $2M < c_C + c_B + 4b$  which is a contradiction.  $\square$

**Theorem 6.18** *Let  $\phi_1, \dots, \phi_k$  be fully irreducible outer automorphisms such that no two have common powers and  $f_1, \dots, f_k$  their respective train track representatives with axes  $A_1, \dots, A_k$ . Let  $\mathcal{F}$  be the set of translates of  $A_1, \dots, A_k$  under the action of  $\text{Out}(F_n)$ . Then there exists a constant  $M > 0$  such that for any  $A, B, C \in \mathcal{F}$  then*

$$d_A(B, C) > M \implies d_B(A, C) < M.$$

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