

## Small generating sets for the Torelli group

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Proving a conjecture of Dennis Johnson, we show that the Torelli subgroup  $\mathcal{I}_g$  of the genus  $g$  mapping class group has a finite generating set whose size grows cubically with respect to  $g$ . Our main tool is a new space called the handle graph on which  $\mathcal{I}_g$  acts cocompactly.

20F05; 20F38, 57M07, 57N05

### 1 Introduction

Let  $\Sigma_{g,n}$  be a compact connected oriented genus  $g$  surface with  $n$  boundary components. The *mapping class group* of  $\Sigma_{g,n}$ , denoted  $\text{Mod}_{g,n}$ , is the group of orientation-preserving homeomorphisms of  $\Sigma_{g,n}$  that fix the boundary pointwise modulo isotopies that fix the boundary pointwise. We will often omit the  $n$  if it vanishes. For  $n \leq 1$ , the *Torelli group*, denoted  $\mathcal{I}_{g,n}$ , is the kernel of the action of  $\text{Mod}_{g,n}$  on  $H_1(\Sigma_{g,n}; \mathbb{Z})$ . The Torelli group has been the object of intensive study ever since the seminal work of Dennis Johnson in the early '80's. See [10] for a survey of Johnson's work.

**Finite generation of Torelli** One of Johnson's most celebrated theorems says that  $\mathcal{I}_{g,n}$  is finitely generated for  $g \geq 3$  and  $n \leq 1$  (see [9]). This is a surprising result – though  $\text{Mod}_{g,n}$  is finitely presentable,  $\mathcal{I}_{g,n}$  is an infinite-index normal subgroup of  $\text{Mod}_{g,n}$ , so there is no reason to hope that  $\mathcal{I}_{g,n}$  has any finiteness properties. Moreover, McCullough and Miller [13] proved that  $\mathcal{I}_{2,n}$  is *not* finitely generated for  $n \leq 1$ , and later Mess [14] proved that  $\mathcal{I}_2$  is an infinite rank free group.

**Johnson's generating set** Johnson's generating set for  $\mathcal{I}_{g,n}$  when  $g \geq 3$  and  $n \leq 1$  is enormous. Indeed, for  $\mathcal{I}_g$  (resp.  $\mathcal{I}_{g,1}$ ), it contains  $9 \cdot 2^{2g-3} - 4g^2 + 2g - 6$  (resp.  $9 \cdot 2^{2g-3} - 4g^2 + 4g - 5$ ) elements. In [11], Johnson proved that the abelianization of  $\mathcal{I}_g$  (resp.  $\mathcal{I}_{g,1}$ ) has rank  $\frac{1}{3}(4g^3 + 5g + 3)$  (resp.  $\frac{1}{3}(4g^3 - g)$ ). These give large lower bounds on the size of generating sets for  $\mathcal{I}_{g,n}$ ; however, there is a huge gap between this cubic lower bound and Johnson's exponentially growing generating set. At the end of [9] and in [10, page 168], Johnson conjectures that there should be a generating set for  $\mathcal{I}_{g,n}$  whose size grows cubically with respect to the genus. Later, in

[4, Problem 5.7] Farb asked whether there at least exists a generating set whose size grows polynomially.

**Main theorem** In this paper, we prove Johnson’s conjecture. Our main theorem is as follows.

**Theorem A** For  $g \geq 3$ , the group  $\mathcal{I}_g$  has a generating set of size at most  $57\binom{g}{3}$  and the group  $\mathcal{I}_{g,1}$  has a generating set of size at most  $57\binom{g}{3} + 2g + 1$ .

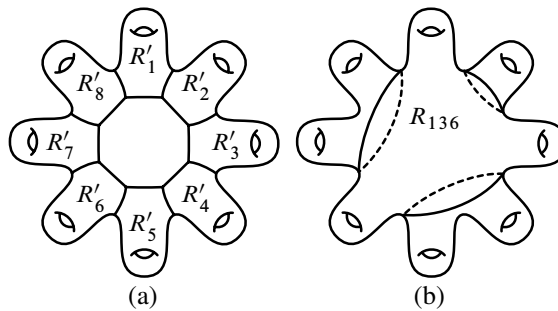


Figure 1: (a) The subsurfaces  $R'_i \cong \Sigma_{1,1}$ . To avoid cluttering the picture, the portion of the boundaries of the  $R'_i$  which lie on the back side the figure are not drawn. (b) A subsurface isotopic to  $R_{136}$

The generating set we construct was conjectured to generate  $\mathcal{I}_{g,n}$  by Brendle and Farb [2]. To describe it, we must introduce some notation. As in Figure 1(a), let  $R'_1, \dots, R'_g$  be  $g$  subsurfaces of  $\Sigma_g$  each homeomorphic to  $\Sigma_{1,1}$  such that the following hold. Interpret all indices modulo  $g$ .

- If  $1 \leq i < j \leq g$  satisfy  $i \notin \{j - 1, j + 1\}$ , then  $R'_i \cap R'_j = \emptyset$ .
- For all  $1 \leq i \leq g$ , the intersection  $R'_i \cap R'_{i+1}$  is homeomorphic to an interval.

For  $1 \leq i < j < k \leq g$ , define a subsurface  $R_{ijk}$  of  $\Sigma_g$  by  $R_{ijk} = \overline{\Sigma_g \setminus \bigcup_{l \neq i,j,k} R'_l}$ . Thus  $R_{ijk}$  is a genus 3 surface with at most 3 boundary components such that  $R'_i, R'_j, R'_k \subset R_{i,j,k}$  (see Figure 1(b)).

If  $S$  is a subsurface of  $\Sigma_g$ , define  $\text{Mod}(\Sigma_g, S)$  to be the subgroup of  $\text{Mod}_g$  consisting of mapping classes that can be realized by homeomorphisms supported on  $S$  and  $\mathcal{I}(\Sigma_g, S)$  to equal  $\mathcal{I}_g \cap \text{Mod}(\Sigma_g, S)$ . The key result for the proof of Theorem A is the following theorem.

**Theorem B** For  $g \geq 3$ , the group  $\mathcal{I}_g$  is generated by the set

$$\bigcup_{1 \leq i < j < k \leq g} \mathcal{I}(\Sigma_g, R_{ijk}).$$

Using Johnson’s work, it is easy to see that  $\mathcal{I}(\Sigma_g, R_{ijk})$  is finitely generated by a generating set with at most 57 generators (see Lemma 2.2). Also, standard techniques (see Lemma 2.1) show that if  $\mathcal{I}_g$  has a generating set with  $k$  elements, then  $\mathcal{I}_{g,1}$  has a generating set with  $k + 2g + 1$  elements. Since there are  $\binom{g}{3}$  subsurfaces  $R_{ijk}$ , Theorem A follows from Theorem B.

**Remark** To illustrate the relative sizes of our generating sets, Johnson’s generating set for  $\mathcal{I}_{20}$  contains more than one trillion elements while our generating set for  $\mathcal{I}_{20}$  has 64980 elements.

**New proof of Johnson’s theorem** Our deduction of Theorem A from Theorem B depends on Johnson’s theorem that  $\mathcal{I}_3$  is finitely generated. However, Hain [6] has recently announced a direct conceptual proof that  $\mathcal{I}_3$  is finitely generated. Hain’s proof uses special properties of the moduli space of genus 3 Riemann surfaces and cannot be easily generalized to  $g > 3$ . Combining this with our paper, we obtain a new proof that  $\mathcal{I}_{g,n}$  is finitely generated for  $g \geq 3$  and  $n \leq 1$ .

Our new proof is more conceptual than Johnson’s original one. To illustrate this, we will sketch Johnson’s proof. He starts by writing down an enormous finite subset  $S \subset \mathcal{I}_{g,n}$  which is known (from work of Powell [15]) to normally generate  $\mathcal{I}_{g,n}$  as a subgroup of  $\text{Mod}_{g,n}$ . Letting  $T$  be a standard generating set for  $\text{Mod}_{g,n}$ , Johnson then proves via a laborious computation that for  $t \in T$  and  $s \in S$ , the element  $ts t^{-1} \in \mathcal{I}_{g,n}$  can be written as a word in  $S$ . This implies that the subgroup  $\Gamma$  of  $\mathcal{I}_{g,n}$  generated by  $S$  is a normal subgroup of  $\text{Mod}_{g,n}$ , and thus that  $\Gamma = \mathcal{I}_{g,n}$ .

**Remark** Our proof of Theorem B appeals to a theorem of [17] whose proof depends on Johnson’s theorem. However, Hatcher and Margalit [12] have recently given a new proof of this result that is independent of Johnson’s work.

**Nature of generators** Some basic elements of  $\mathcal{I}_{g,n}$  are as follows (see, eg [16]). If  $x$  is a simple closed curve on  $\Sigma_{g,n}$ , then denote by  $T_x \in \text{Mod}_{g,n}$  the Dehn twist about  $x$ . If  $x$  is a separating simple closed curve, then  $T_x \in \mathcal{I}_{g,n}$ ; these are called *separating twists*. If  $x$  and  $y$  are disjoint homologous nonseparating simple closed curves, then  $T_x T_y^{-1} \in \mathcal{I}_{g,n}$ ; these are called *bounding pair maps*. Following work

of Birman [1], Powell [15] proved that  $\mathcal{I}_{g,n}$  is generated by bounding pair maps and separating twists for  $g \geq 1$  and  $n \leq 1$  (see [16] and [12] for alternate proofs). Johnson's finite generating set for  $\mathcal{I}_{g,n}$  for  $g \geq 3$  and  $n \leq 1$  consists entirely of bounding pair maps. It follows easily from our proofs of Lemma 2.1 and 2.2 that our generating set consists of bounding pair maps and separating twists; see the remark after Lemma 2.2.

**The handle graph** Our proof of Theorem B is topological. To prove that a group  $G$  is finitely generated, it is enough to find a connected simplicial complex upon which  $G$  acts cocompactly with finitely generated stabilizers. We use a variant on the curve complex. If  $\gamma$  is an oriented simple closed curve on  $\Sigma_g$ , then denote by  $[\gamma] \in H_1(\Sigma_g; \mathbb{Z})$  its homology class. Also, if  $\gamma_1$  and  $\gamma_2$  are isotopy classes of simple closed curves on  $\Sigma_g$ , then denote by  $i_g(\gamma_1, \gamma_2)$  their *geometric intersection number*, ie the minimal possible number of intersections between two curves in the isotopy classes of  $\gamma_1$  and  $\gamma_2$ . Finally, denote by  $i_a(\cdot, \cdot)$  the algebraic intersection pairing on  $H_1(\Sigma_g; \mathbb{Z})$ .

**Definition** Let  $a, b \in H_1(\Sigma_g; \mathbb{Z})$  satisfy  $i_a(a, b) = 1$ . The *handle graph* associated to  $a$  and  $b$ , denoted  $\mathcal{H}_{a,b}$ , is the graph whose vertices are isotopy classes of oriented simple closed curves on  $\Sigma_g$  that are homologous to either  $a$  or  $b$  and where two vertices  $\gamma_1$  and  $\gamma_2$  are joined by an edge exactly when  $i_g(\gamma_1, \gamma_2) = 1$ .

We will show that  $\mathcal{H}_{a,b}/\mathcal{I}_g$  consists of a single edge (see Lemma 5.2) and that  $\mathcal{H}_{a,b}$  is connected for  $g \geq 3$  (see Lemma 3.1).

**A complication** It would appear that we have all the ingredients in place to use the space  $\mathcal{H}_{a,b}$  to prove that  $\mathcal{I}_g$  is finitely generated. However, there is one remaining complication. Namely, we do not know the answer to the following question.

**Question 1.1** For some  $g \geq 4$ , let  $\gamma$  be the isotopy class of a nonseparating simple closed curve on  $\Sigma_g$ . Is the stabilizer subgroup  $(\mathcal{I}_g)_\gamma$  of  $\gamma$  finitely generated?

In other words, we do not know if the vertex stabilizer subgroups of the action of  $\mathcal{I}_g$  on  $\mathcal{H}_{a,b}$  are finitely generated. Nonetheless, in Section 4 we will prove a weaker statement that suffices to prove Theorem B. The proof of Theorem B is in Section 5.

**Smaller generating sets** A positive answer to Question 1.1 would likely lead to a smaller generating set for  $\mathcal{I}_g$ , though of course this depends on the nature of the finite generating sets for the stabilizer subgroups. Let us describe one way this could work. For  $g \geq 3$ , let  $\sigma_g$  be the smallest cardinality of a generating set for  $\mathcal{I}_g$ . Consider

$g \geq 4$ , and fix an edge  $\{\alpha, \beta\}$  of  $\mathcal{H}_{a,b}$ . The proof of Theorem B shows that  $\mathcal{I}_g$  is generated by  $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$ . Let  $S$  be a subsurface of  $\Sigma_g$  such that  $S \cong \Sigma_{g-1,1}$  and  $\alpha \cup \beta \subset \Sigma_g \setminus S$ . We have  $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$  (see Section 2) and  $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\alpha$  and  $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\beta$ . Assume that there exists a finite set  $V_\alpha$  (resp.  $V_\beta$ ) such that  $(\mathcal{I}_g)_\alpha$  (resp.  $(\mathcal{I}_g)_\beta$ ) is generated by  $\mathcal{I}(\Sigma_g, S) \cup V_\alpha$  (resp.  $\mathcal{I}(\Sigma_g, S) \cup V_\beta$ ). The group  $\mathcal{I}_g$  is then generated by  $\mathcal{I}(\Sigma_g, S) \cup V_\alpha \cup V_\beta$ . Lemma 2.1 says that  $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$  can be generated by  $\sigma_{g-1} + 2g + 1$  elements. Moreover, it seems likely that there exists some relatively small  $K$  such that  $|V_\alpha|, |V_\beta| \leq Kg^2$ . This would imply that

$$\sigma_g \leq \sigma_{g-1} + 2g + 1 + 2Kg^2.$$

Iterating this, we would get that

$$\sigma_g \leq \sigma_3 + \sum_{i=4}^g (2i + 1 + 2Ki^2)$$

for  $g \geq 4$ . This bound is cubic in  $g$  (as it needs to be), but as long as  $K$  is not too large it is much smaller than  $57\binom{g}{3}$ .

**Finite presentability** Perhaps the most important open question about the combinatorial group theory of  $\mathcal{I}_g$  is whether or not it is finitely presentable for  $g \geq 3$ . One way of proving that a group  $G$  is finitely presentable is to construct a simply-connected simplicial complex  $X$  upon which  $G$  acts cocompactly with finitely presentable stabilizer subgroups (see, eg [3]). For example, Hatcher and Thurston use this technique in [7] to prove that the mapping class group is finitely presentable.

The handle graph  $\mathcal{H}_{a,b}$  appears to be the first example of a useful space upon which  $\mathcal{I}_g$  acts cocompactly (of course, there are trivial non-useful examples of such spaces; for example, the Cayley graph of  $\mathcal{I}_g$  or a 1-point space). Unfortunately, while  $\mathcal{H}_{a,b}$  is connected for  $g \geq 3$ , it is not simply connected. Indeed, it does not even have any 2-cells (and is not a tree). However, one could probably attach 2-cells to  $\mathcal{H}_{a,b}$  to obtain a simply connected complex upon which  $\mathcal{I}_g$  acts cocompactly. This would not be enough, however – one would also have to prove that the simplex stabilizer subgroups were finitely presentable. In other words, this complex would provide the inductive step in a proof that  $\mathcal{I}_g$  was finitely presentable, but one would still need a base case.

**A complex that does not work** We close this introduction by discussing an approach to Theorem B that does not work. One might think of trying to prove Theorem B using the following complex. Let  $a \in H_1(\Sigma_g; \mathbb{Z})$  be a primitive vector. Define  $\mathcal{C}_a$  to be the graph whose vertices are isotopy classes of oriented simple closed curves  $\gamma$  on  $\Sigma_g$  such that  $[\gamma] = a$  and where two vertices  $\gamma$  and  $\gamma'$  are joined by an edge if

$i_g(\gamma, \gamma') = 0$ . It is known ([17, Theorem 1.9]; see [12] for an alternate proof) that  $\mathcal{C}_a$  is connected for  $g \geq 3$ . Moreover,  $\mathcal{I}_g$  acts transitively on the vertices of  $\mathcal{C}_a$ . However, it does *not* act cocompactly; indeed, there are infinitely many edge orbits. To see this, consider edges  $e_1 = \{\gamma_1, \gamma'_1\}$  and  $e_2 = \{\gamma_2, \gamma'_2\}$  of  $\mathcal{C}_a$ . Assume that there exists some  $f \in \mathcal{I}_g$  such that  $f(e_1) = e_2$ . Since  $\gamma_1$  is homologous to  $\gamma'_1$ , the multicurve  $\gamma_1 \cup \gamma'_1$  divides  $\Sigma_g$  into two subsurfaces  $S_1$  and  $S'_1$ . Similarly,  $\gamma_2 \cup \gamma'_2$  divides  $\Sigma_g$  into two subsurfaces  $S_2$  and  $S'_2$ . Relabeling if necessary, we have  $f(S_1)$  isotopic to  $S_2$  and  $f(S'_1)$  isotopic to  $S'_2$ . Since  $f \in \mathcal{I}_g$ , the images of  $H_1(S_1; \mathbb{Z})$  and  $H_1(S_2; \mathbb{Z})$  in  $H_1(\Sigma_g; \mathbb{Z})$  must be the same, and similarly for  $H_1(S'_1; \mathbb{Z})$  and  $H_1(S'_2; \mathbb{Z})$ . It is easy to see that infinitely many such images occur for different edges of  $\mathcal{C}_a$ , so there must be infinitely many edges orbits. We remark that Johnson proved in [8, Corollary to Lemma 9 on page 250] that the images of  $H_1(S_1; \mathbb{Z})$  and  $H_1(S'_1; \mathbb{Z})$  in  $H_1(\Sigma_g; \mathbb{Z})$  are a complete invariant for the edge orbits.

**Acknowledgments** I wish to thank Tara Brendle, Benson Farb, and Dan Margalit for their help. I also wish to thank an anonymous referee for a very helpful referee report. The author is supported in part by NSF grant DMS-1005318.

## 2 The Torelli group on subsurfaces

We will need to understand how the Torelli group restricts to subsurfaces. For a general discussion of this, see [16]. In this section, we will extract from [16] results on two kinds of subsurfaces. In Section 2.1, we will show how to analyze subsurfaces like the subsurfaces  $R_{ijk}$  from Section 1. In Section 2.2, we will show how to analyze stabilizers of nonseparating simple closed curves (which are supported on the subsurface obtained by taking the complement of a regular neighborhood of the curve).

### 2.1 Analyzing the subsurfaces $R_{ijk}$

We begin by defining groups  $\mathcal{I}_{g,n}$  for  $n \geq 2$ . There is a map  $\text{Mod}_{g,n} \rightarrow \text{Mod}_g$  induced by gluing discs to the boundary components of  $\Sigma_{g,n}$  and extending homeomorphisms by the identity. Define  $\mathcal{I}_{g,n}$  to be the kernel of the resulting action of  $\text{Mod}_{g,n}$  on  $H_1(\Sigma_g; \mathbb{Z})$ . For the case  $n = 1$ , the map  $H_1(\Sigma_{g,1}; \mathbb{Z}) \rightarrow H_1(\Sigma_g; \mathbb{Z})$  is an isomorphism, so this agrees with our previous definition of  $\mathcal{I}_{g,1}$ .

**Remark** In [16], the different definitions of the Torelli group on a surface with boundary are parametrized by partitions of the boundary components. The above definition of  $\mathcal{I}_{g,n}$  corresponds to the discrete partition  $\{\{\beta_1\}, \dots, \{\beta_n\}\}$  of the set  $\{\beta_1, \dots, \beta_n\}$  of boundary components of  $\Sigma_{g,n}$ .

In [16, Theorem 1.2], a version of the Birman exact sequence is proven for the Torelli group. For  $\mathcal{I}_{g,n}$  with  $g \geq 2$ , it takes the form

$$(1) \quad 1 \longrightarrow \pi_1(U\Sigma_{g,n}) \longrightarrow \mathcal{I}_{g,n+1} \longrightarrow \mathcal{I}_{g,n} \longrightarrow 1.$$

Here  $U\Sigma_{g,n}$  is the unit tangent bundle of  $\Sigma_{g,n}$ . The subgroup  $\pi_1(U\Sigma_{g,n})$  of  $\mathcal{I}_{g,n+1}$  is often called the “disc-pushing subgroup” – the mapping class associated to  $\gamma \in \pi_1(U\Sigma_{g,n})$  “pushes” a fixed boundary component around  $\gamma$  while allowing it to rotate. The following is an immediate consequence of (1) and the fact that  $\pi_1(U\Sigma_g)$  can be generated by  $2g + 1$  elements.

**Lemma 2.1**  $\mathcal{I}_{g,1}$  can be generated by  $k + 2g + 1$  elements if  $\mathcal{I}_g$  can be generated by  $k$  elements.

Now assume that  $S \cong \Sigma_{h,n}$  is an embedded subsurface of  $\Sigma_g$  and that all the boundary components of  $S$  are non-nullhomotopic separating curves in  $\Sigma_g$ . For example,  $S$  could be one of the surfaces  $R_{ijk}$  from Section 1. Letting  $\text{Mod}(S)$  be the mapping class group of  $S$ , the induced map  $\text{Mod}(S) \rightarrow \text{Mod}_g$  is an injection. This gives a natural identification of  $\text{Mod}(S)$  with  $\text{Mod}(\Sigma_g, S)$ . The group  $\mathcal{I}(\Sigma_g, S)$  is thus naturally a subgroup of  $\text{Mod}(S) \cong \text{Mod}_{h,n}$ , and in [16, Theorem 1.1] it is proven that  $\mathcal{I}(\Sigma_g, S) = \mathcal{I}_{h,n}$ . Johnson [9] proved that  $\mathcal{I}_3$  can be generated by 35 elements. Applying (1) repeatedly, we see that  $\mathcal{I}_{3,1}$  can be generated by 42 elements,  $\mathcal{I}_{3,2}$  by 49 elements, and  $\mathcal{I}_{3,3}$  by 57 elements. Since  $R_{ijk} \cong \Sigma_{3,k}$  with  $k \leq 3$ , we obtain the following.

**Lemma 2.2** For all  $1 \leq i < j < k \leq g$ , the group  $\mathcal{I}(\Sigma_g, R_{ijk})$  can be generated by 57 elements.

**Remark** It is well-known (see, eg [16, Section 2.1]) that the mapping classes corresponding to the generators of  $\pi_1(U\Sigma_{g,n})$  used to prove Lemmas 2.1 and 2.2 can be chosen to be bounding pair maps and separating twists. Additionally, Johnson’s minimal-size generating set for  $\mathcal{I}_3$  consists entirely of bounding pair maps, so the generating set for  $\mathcal{I}(\Sigma_g, R_{ijk})$  in Lemma 2.2 can be taken to consist of bounding pair maps and separating twists.

## 2.2 Stabilizers of nonseparating simple closed curves

Let  $\gamma$  be a nonseparating simple closed curve on  $\Sigma_g$ . Define  $\Sigma_{g,\gamma}$  to be the result of cutting  $\Sigma_g$  along  $\gamma$ , so  $\Sigma_{g,\gamma} \cong \Sigma_{g-1,2}$ . Letting  $\text{Mod}_{g,\gamma}$  be the mapping class group of  $\Sigma_{g,\gamma}$ , the natural map  $\Sigma_{g,\gamma} \rightarrow \Sigma_g$  induces a map  $i: \text{Mod}_{g,\gamma} \rightarrow \text{Mod}_g$ . Define  $\mathcal{I}_{g,\gamma} = i^{-1}(\mathcal{I}_g)$ . The map  $i$  restricts to a surjection  $\mathcal{I}_{g,\gamma} \rightarrow (\mathcal{I}_g)_\gamma$ , where  $(\mathcal{I}_g)_\gamma$  is the stabilizer subgroup of  $\gamma$ .

**Remark** In the notation of [16], the group  $\mathcal{I}_{g,\gamma}$  corresponds to the Torelli group of  $\Sigma_{g-1,2}$  with respect to the “indiscrete partition”  $\{\{\beta, \beta'\}\}$  of the boundary components  $\beta$  and  $\beta'$  of  $\Sigma_{g,\gamma}$ . Also, the kernel of the map  $\mathcal{I}_{g,\gamma} \rightarrow (\mathcal{I}_g)_\gamma$  is isomorphic to  $\mathbb{Z}$  and is generated by  $T_\beta T_{\beta'}^{-1}$ , where  $T_\beta$  and  $T_{\beta'}$  are the Dehn twists about  $\beta$  and  $\beta'$ , respectively.

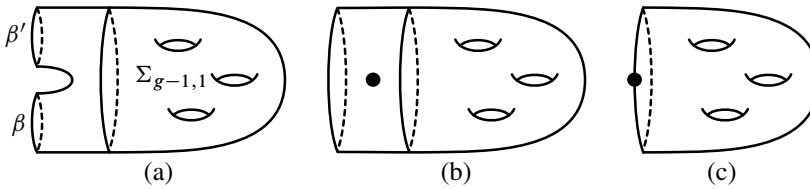


Figure 2: (a) The surface  $\Sigma_{g,\gamma}$  and the subsurface  $\Sigma_{g-1,1}$  of  $\Sigma_{g,\gamma}$  such that the induced map  $\mathcal{I}_{g-1,1} \rightarrow \mathcal{I}_{g,\gamma}$  splits the exact sequence (2) (b) The basepoint for  $\pi_1(\Sigma_{g-1,1})$  is obtained from  $\Sigma_{g,\gamma}$  by collapsing the boundary component  $\beta$  to a point. (c) The surface in b deformation retracts to  $\Sigma_{g-1,1}$  such that the basepoint ends up on the boundary component.

In [16, Theorem 1.2], it is proven that for  $g \geq 2$  there is a short exact sequence

$$(2) \quad 1 \longrightarrow K_{g,\gamma} \longrightarrow \mathcal{I}_{g,\gamma} \longrightarrow \mathcal{I}_{g-1,1} \longrightarrow 1.$$

Here  $K_{g,\gamma} \cong [\pi_1(\Sigma_{g-1,1}), \pi_1(\Sigma_{g-1,1})]$ . This exact sequence splits via the inclusion  $\mathcal{I}_{g-1,1} \hookrightarrow \mathcal{I}_{g,\gamma}$  induced by the inclusion  $\Sigma_{g-1,1} \hookrightarrow \Sigma_{g,\gamma}$  indicated in Figure 2(a). In other words, the following holds.

**Lemma 2.3**  $\mathcal{I}_{g,\gamma} = K_{g,\gamma} \times \mathcal{I}_{g-1,1}$  for  $g \geq 3$  and  $\gamma$  a simple closed nonseparating curve on  $\Sigma_g$ .

The group  $\mathcal{I}_{g-1,1}$  acts on  $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$  as follows. As is clear from [16, Theorem 1.2], the basepoint for  $\pi_1(\Sigma_{g-1,1})$  is as indicated in Figure 2(b). As shown in Figure 2(c), the surface  $\Sigma_{g-1,1}$  deformation retracts onto the surface  $\Sigma_{g-1,1}$  on which  $\mathcal{I}_{g-1,1}$  is supported. After this deformation retract, the basepoint ends up on  $\partial\Sigma_{g-1,1}$ . Summing up,  $\mathcal{I}_{g-1,1}$  acts on  $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$  via the action of  $\text{Mod}_{g-1,1}$  on  $\pi_1(\Sigma_{g-1,1})$ , where the basepoint for  $\pi_1(\Sigma_{g-1,1})$  is on  $\partial\Sigma_{g-1,1}$ .

### 3 The handle graph is connected

In this section, we prove the following.

**Lemma 3.1** Fix  $g \geq 3$ . Let  $a, b \in H_1(\Sigma_g; \mathbb{Z})$  satisfy  $i_a(a, b) = 1$ . Then  $\mathcal{H}_{a,b}$  is connected.



We will need two lemmas. In the first, if  $\epsilon$  is an oriented arc in a surface, then  $\epsilon^{-1}$  denotes the arc obtained by reversing the orientation of  $\epsilon$ .

**Lemma 3.2** *Let the boundary components of  $\Sigma_{g,2}$  be  $\delta_0$  and  $\delta_1$ . Choose points  $v_i \in \delta_i$  for  $i = 0, 1$  and let  $\epsilon$  be an oriented properly embedded arc in  $\Sigma_{g,2}$  whose initial point is  $v_0$  and whose terminal point is  $v_1$ . Then for any  $h \in H_1(\Sigma_{g,2}; \mathbb{Z})$ , there exists an oriented properly embedded arc  $\epsilon'$  in  $\Sigma_{g,2}$  whose initial point is  $v_0$  and whose terminal point is  $v_1$  such that the homology class of the loop  $\epsilon' \cdot \epsilon^{-1}$  is  $h$ .*

**Proof** Gluing  $(\delta_0, v_0)$  to  $(\delta_1, v_1)$ , we obtain a surface  $S \cong \Sigma_{g+1}$ . Let  $\alpha$  and  $*$  be the images of  $\delta_0$  and  $v_0$  in  $S$ , respectively. The image of  $\epsilon$  in  $S$  is an oriented simple closed curve  $\beta$  with  $i_g(\alpha, \beta) = 1$ . There is a natural isomorphism  $H_1(\Sigma_{g,2}; \mathbb{Z}) \cong [\alpha]^\perp$ , where the orthogonal complement is taken with respect to  $i_a(\cdot, \cdot)$ . Under this identification, we can apply [16, Lemma A.3] to find an oriented simple closed curve  $\beta'$  on  $S$  such that  $[\beta'] = [\beta] + h$  and such that  $\alpha \cap \beta' = \{*\}$ . Cutting  $S$  open along  $\alpha$ , the curve  $\beta'$  becomes the desired arc  $\epsilon'$ .  $\square$

**Lemma 3.3** *Let  $a, b \in H_1(\Sigma_g; \mathbb{Z})$  satisfy  $i_a(a, b) = 1$ . Let  $\alpha_1$  and  $\alpha_2$  be disjoint oriented simple closed curves on  $\Sigma_g$  such that  $[\alpha_i] = a$  for  $i = 1, 2$ . There then exists some oriented simple closed curve  $\beta$  on  $\Sigma_g$  such that  $[\beta] = b$  and  $i_g(\alpha_i, \beta) = 1$  for  $i = 1, 2$ .*

**Proof** Let  $\beta'$  be any simple closed curve on  $\Sigma_g$  such that  $i(\alpha_i, \beta') = 1$  for  $i = 1, 2$ . Orient  $\beta'$  so that its intersections with  $\alpha_1$  and  $\alpha_2$  are positive. Let  $X_1$  and  $X_2$  be the two subsurfaces of  $\Sigma_g$  that result from cutting  $\Sigma_g$  along  $\alpha_1 \cup \alpha_2$ . For  $i = 1, 2$ , the surface  $X_i$  has 2 boundary components and the intersection of  $\beta'$  with  $X_i$  is an oriented properly embedded arc  $\epsilon_i$  running between these boundary components. Also, the induced map  $H_1(X_i; \mathbb{Z}) \rightarrow H_1(\Sigma_g; \mathbb{Z})$  is an injection, and we will identify  $H_1(X_i; \mathbb{Z})$  with its image in  $H_1(\Sigma_g; \mathbb{Z})$ . The orthogonal complement to  $a$  with respect to the algebraic intersection pairing is spanned by  $H_1(X_1; \mathbb{Z}) \cup H_1(X_2; \mathbb{Z})$ . Since  $i_a(a, b) = i_a(a, [\beta'])$ , the homology class  $b - [\beta']$  is orthogonal to  $a$ . There thus exist  $h_i \in H_1(X_i; \mathbb{Z})$  for  $i = 1, 2$  such that  $b = [\beta'] + h_1 + h_2$ . Lemma 3.2 says that for  $i = 1, 2$  there exists an oriented properly embedded arc  $\epsilon'_i$  in  $X_i$  with the same endpoints as  $\epsilon_i$  such that the homology class of the loop  $\epsilon'_i \cdot \epsilon_i^{-1}$  equals  $h_i$ . Letting  $\beta$  be the loop  $\epsilon'_1 \cdot \epsilon'_2$ , it follows that  $[\beta] = [\beta'] + h_1 + h_2 = b$ , as desired.  $\square$

**Proof of Lemma 3.1** Let  $\delta$  and  $\delta'$  be vertices of  $\mathcal{H}_{a,b}$ . We will construct a path in  $\mathcal{H}_{a,b}$  from  $\delta$  to  $\delta'$ . Without loss of generality,  $[\delta] = [\delta'] = a$ . By [17, Theorem 1.9] (see [12] for an alternate proof), we can find a sequence

$$\delta = \alpha_1, \alpha_2, \dots, \alpha_n = \delta'$$

of isotopy classes of oriented simple closed curves on  $\Sigma_g$  such that  $[\alpha_i] = a$  for  $1 \leq i \leq n$  and  $i_g(\alpha_i, \alpha_{i+1}) = 0$  for  $1 \leq i < n$  (this is where we use the condition  $g \geq 3$ ). Lemma 3.3 implies that there exist isotopy classes  $\beta_1, \dots, \beta_{n-1}$  of oriented simple closed curves on  $\Sigma_g$  such that  $[\beta_i] = b$  and  $i_g(\alpha_i, \beta_i) = i_g(\alpha_{i+1}, \beta_i) = 1$  for  $1 \leq i < n$ . Since  $\beta_i$  is adjacent to both  $\alpha_i$  and  $\alpha_{i+1}$  in  $\mathcal{H}_{a,b}$ , the desired path from  $\delta$  to  $\delta'$  is thus

$$\delta = \alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \beta_{n-1}, \alpha_n = \delta'. \quad \square$$

### 4 Generating the stabilizer of a nonseparating simple closed curve

Let the subsurfaces  $R'_i$  of  $\Sigma_g$  be as in the introduction. Define  $S_i = \overline{\Sigma_g \setminus R'_i}$ . The goal of this section is to prove the following lemma.

**Lemma 4.1** *Assume that  $g \geq 4$ . Let  $\gamma$  be the isotopy class of a simple closed nonseparating curve on  $\Sigma_g$  that is contained in  $R'_1$ . Then the subgroup  $(\mathcal{I}_g)_\gamma$  of  $\mathcal{I}_g$  stabilizing  $\gamma$  is contained in the subgroup of  $\mathcal{I}_g$  generated by  $\cup_{i=1}^g \mathcal{I}(\Sigma_g, S_i)$ .*

Before proving this, we need a technical lemma. Set  $\pi = \pi_1(\Sigma_{g,1}, *)$ , where  $* \in \partial \Sigma_{g,1}$ . Let  $T'_1, \dots, T'_g$  be disjoint subsurfaces of  $\Sigma_{g,1}$  such that  $T'_i \cong \Sigma_{1,1}$  and  $T'_i \cap \partial \Sigma_{g,1} = \emptyset$  for  $1 \leq i \leq g$  (see Figure 3(a)). Define  $T_i = \overline{\Sigma_{g,1} \setminus T'_i}$ . We have  $T_i \cong \Sigma_{g-1,2}$  and  $* \in T_i$  for  $1 \leq i \leq g$ . The maps  $\pi_1(T_i, *) \rightarrow \pi_1(\Sigma_{g,1}, *)$  and  $H_1(T'_i; \mathbb{Z}) \rightarrow H_1(\Sigma_{g,1}; \mathbb{Z})$  are injective; we will identify  $\pi_1(T_i, *)$  and  $H_1(T'_i; \mathbb{Z})$  with their images in  $\pi_1(\Sigma_{g,1}, *)$  and  $H_1(\Sigma_g; \mathbb{Z})$ , respectively. Define  $K_i = [\pi, \pi] \cap \pi_1(T_i, *)$ . We then have the following.

**Lemma 4.2** *For  $g \geq 3$ , the group  $[\pi, \pi]$  is generated by the  $\mathcal{I}_{g,1}$ -orbits of the set  $\cup_{i=1}^g K_i$ .*

The proof of this will have two ingredients. The first is the following theorem of Tomaszewski. As notation, if  $G$  is a group and  $a, b \in G$ , then  $[a, b] := a^{-1}b^{-1}ab$  and  $a^b := b^{-1}ab$ .

**Theorem 4.3** (Tomaszewski, [20]) *Let  $F_n$  be the free group on  $\{x_1, \dots, x_n\}$ . Then the set*

$$\{[x_i, x_j]^{x_i^{k_i} x_{i+1}^{k_{i+1}} \dots x_n^{k_n}} \mid 1 \leq i < j \leq n \text{ and } k_m \in \mathbb{Z} \text{ for all } i \leq m \leq n\}$$

*is a free basis for  $[F_n, F_n]$ .*

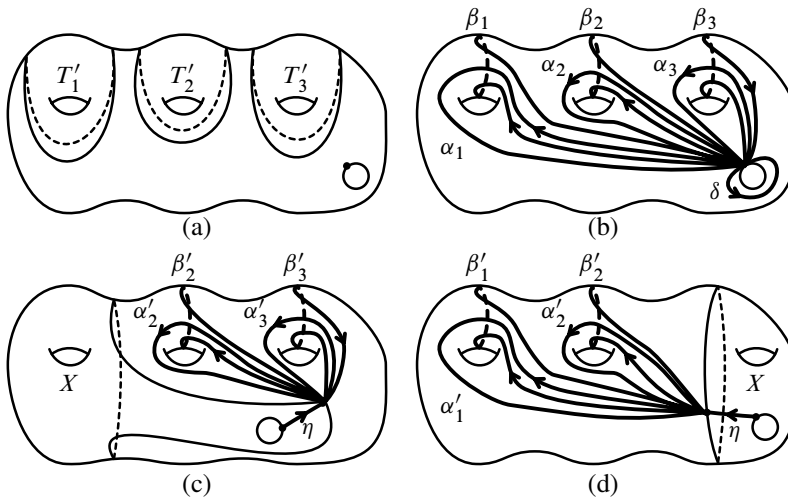


Figure 3: (a) The subsurfaces  $T'_i$  (b) The standard basis for  $\pi$   
 (c) The surface  $X$  when  $i = 1$  (d) The surface  $X$  when  $i = g$

The second is the following lemma about the action of  $\mathcal{I}_{g,1}$  on  $\pi$ . Choose a standard basis  $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$  for  $\pi$  (as in Figure 3(b)) such that  $\alpha_i$  and  $\beta_i$  are freely homotopic into  $T'_i$  for  $1 \leq i \leq g$ . Our proof of Lemma 4.2 would be much simpler if the image of  $\text{Mod}_{g,1}$  in  $\text{Aut}(\pi)$  contained the inner automorphisms – since inner automorphisms act trivially on homology, this would imply that the  $\mathcal{I}_g$ -orbits of  $\{[x, y] \mid x, y \in \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}\}$  generate  $[\pi, \pi]$ . However, the image of  $\text{Mod}_{g,1}$  in  $\text{Aut}(\pi)$  does not contain the inner automorphisms since  $\text{Mod}_{g,1}$  fixes the loop  $\delta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$  depicted in Figure 3(b). The following lemma is a weak replacement for this.

**Lemma 4.4** *Let  $i$  be either 1 or  $g$ . Consider  $h \in H_1(T'_i; \mathbb{Z})$ . There then exists some  $w \in \langle \alpha_i, \beta_i, \delta \rangle$  and  $f \in \mathcal{I}_{g,1}$  such that  $[w] = h$  and such that  $f(a_j) = a_j^w$  and  $f(b_j) = b_j^w$  for  $1 \leq j \leq g$  with  $j \neq i$ .*

**Proof** Let  $X$  be a regular neighborhood of the curves  $\alpha_i \cup \beta_i \cup \partial \Sigma_{g,1}$  depicted in Figure 3(b). Thus  $X \cong \Sigma_{1,2}$ , the surface  $T'_i$  is homotopic into  $X$ , and the image of  $\pi_1(X, *)$  in  $\pi$  is  $\langle \alpha_i, \beta_i, \delta \rangle$ . Let  $Y = \overline{\Sigma_{g,1} \setminus X}$ , so  $Y \cong \Sigma_{g-1,1}$  and  $X \cap Y \cong S^1$ . The key property of  $X$  is as follows (this is where we use the assumption that  $i$  is either 1 or  $g$ ). There exists some  $*' \in X \cap Y$ , a properly embedded arc  $\eta$  in  $X$  from  $*$  to  $*'$ , and elements

$$\{\alpha'_j, \beta'_j \mid 1 \leq j \leq g, j \neq i\} \subset \pi_1(Y, *')$$

such that  $\alpha_j = \eta \cdot \alpha'_j \cdot \eta^{-1}$  and  $\beta_j = \eta \cdot \beta'_j \cdot \eta^{-1}$  for  $1 \leq j \leq g$  with  $j \neq i$ . See Figure 3(c) for the case  $i = 1$  and Figure 3(d) for the case  $i = g$ .

By Lemma 3.2, there exists an oriented properly embedded arc  $\eta'$  in  $X$  whose endpoints are the same as those of  $\eta$  such that the homology class of  $w := \eta \cdot (\eta')^{-1} \in \pi$  in  $H_1(\Sigma_g; \mathbb{Z})$  is  $h$ . Observe that  $w \in \langle \alpha_i, \beta_i, \delta \rangle$ . Also,

$$\eta' \cdot \alpha'_j \cdot (\eta')^{-1} = w^{-1} \cdot \eta \cdot \alpha'_j \cdot \eta^{-1} \cdot w = \alpha_j^w$$

for  $j \neq i$ , and similarly for  $\beta_j$ . It is thus enough find some  $f \in \mathcal{I}(\Sigma_g, X)$  such that  $f(\eta) = \eta'$ .

The “change of coordinates principle” from [5, Section 1.3] implies that there exists some  $f' \in \text{Mod}(\Sigma_g, X)$  such that  $f'(\eta) = \eta'$ . Briefly, an Euler characteristic calculation shows that cutting  $X$  open along either  $\eta$  or  $\eta'$  results in a surface homeomorphic to  $\Sigma_{1,1}$ . Choosing an orientation-preserving homeomorphism between these two cut-open surfaces and gluing the boundary components back together in an appropriate way, we obtain some  $f' \in \text{Mod}(\Sigma_g, X)$  such that  $f'(\eta) = \eta'$ . See [5, Section 1.3] for more details and many other examples of arguments of this form.

The mapping class  $f'$  need not lie in Torelli; however, it satisfies  $f'([\alpha_j]) = [\alpha_j]$  and  $f'([\beta_j]) = [\beta_j]$  for  $j \neq i$  and  $f'(H_1(T'_i; \mathbb{Z})) = H_1(T'_i; \mathbb{Z})$ . Since the image of  $\text{Mod}(T'_i)$  in  $\text{Aut}(H_1(T'_i; \mathbb{Z})) = \text{Aut}(\mathbb{Z}^2)$  is  $\text{SL}_2(\mathbb{Z})$ , we can choose some  $f'' \in \text{Mod}(\Sigma_g, T'_i)$  such that  $f'([\alpha_i]) = f''([\alpha_i])$  and  $f'([\beta_i]) = f''([\beta_i])$ . It follows that  $f := f' \cdot (f'')^{-1}$  lies in  $\mathcal{I}(\Sigma_g, X)$  and satisfies  $f(\eta) = \eta'$ , as desired.  $\square$

**Proof of Lemma 4.2** The generating set for  $[F_n, F_n]$  in Theorem 4.3 depends on an ordering of the generators for  $F_n$ . It seems hard to prove the lemma using the generating set corresponding to the standard ordering

$$(x_1, x_2, \dots, x_{2g}) = (\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$$

of the generators for  $\pi \cong F_{2g}$ . However, consider the following nonstandard ordering on the generators for  $\pi$ :

$$(x_1, x_2, \dots, x_{2g}) = (\alpha_2, \beta_2, \alpha_1, \beta_1, \alpha_3, \beta_3, \alpha_4, \beta_4, \dots, \alpha_g, \beta_g).$$

Let  $S$  be the generating set for  $[\pi, \pi]$  given by Theorem 4.3 using this ordering of the generators. All the elements of  $S$  lie in  $K_2$  except for

$$(3) \quad [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}} \quad \text{and} \quad [\beta_2, \zeta']^{\beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}};$$

here  $\zeta \in \{\beta_2, \alpha_1, \beta_1, \alpha_3, \dots, \beta_g\}$  and  $\zeta' \in \{\alpha_1, \beta_1, \alpha_3, \dots, \beta_g\}$  and  $n_i, m_i \in \mathbb{Z}$ . Letting  $T \subset S$  be the elements in (3), we must show that every  $t \in T$  can be expressed as a product of elements in the  $\mathcal{I}_{g,1}$ -orbit of the set  $\cup_{i=1}^g K_i$ . Consider  $t \in T$ , so either  $t = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}}$  or  $t = [\beta_2, \zeta']^{\beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}}$ . There are two cases.

**Case 1**  $\zeta \notin \{\alpha_1, \beta_1\}$ .

We will do the case where  $t = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}}$ ; the other case is treated in a similar way. Set  $t' = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_3^{n_3} \dots \beta_g^{m_g}}$ , so  $t' \in K_1$ . By Lemma 4.4, there exists some  $w \in \{\alpha_1, \beta_1, \delta\}$  and  $f \in \mathcal{I}_{g,1}$  such that  $[w] = [\alpha_1^{n_1} \beta_1^{m_1}]$  and such that  $f(a_j) = a_j^w$  and  $f(b_j) = b_j^w$  for  $j > 1$ . This implies that  $f(t') = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_3^{n_3} \dots \beta_g^{m_g}} w$ . Now,  $\alpha_3^{n_3} \dots \beta_g^{m_g} w$  and  $\alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}$  are homologous, so there exists some  $\theta \in [\pi, \pi]$  such that  $\alpha_3^{n_3} \dots \beta_g^{m_g} w \theta = \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}$ . Moreover, since  $w \in \langle a_1, b_1, \delta \rangle$  we have  $\theta \in K_2$ . Observe now that

$$\theta^{-1} \cdot f(t') \cdot \theta = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_3^{n_3} \dots \beta_g^{m_g}} w \theta = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \dots \beta_g^{m_g}} = t.$$

We have thus found the desired expression for  $t$ .

**Case 2**  $\zeta' \in \{\alpha_1, \beta_1\}$ .

This case is similar to Case 1. The only difference is that the  $\alpha_g^{n_g} \beta_g^{m_g}$  term of  $t$  is deleted to form  $t'$  instead of the  $\alpha_1^{n_1} \beta_1^{m_1}$  term.  $\square$

**Proof of Lemma 4.1** Let  $I$  be the subgroup of  $\mathcal{I}_g$  generated by  $\cup_{i=1}^g \mathcal{I}(\Sigma_g, S_i)$ . Using the notation of Section 2, there is a surjection  $\rho: \mathcal{I}_{g,\gamma} \rightarrow (\mathcal{I}_g)_\gamma$  induced by a continuous map  $\phi: \Sigma_{g,\gamma} \rightarrow \Sigma_g$ . Define  $X = \phi^{-1}(S_1)$ , so  $X \cong \Sigma_{g-1,1}$ . Letting  $\mathcal{I}(X)$  be the Torelli group of  $X$ , Lemma 2.3 gives a decomposition  $\mathcal{I}_{g,\gamma} = K_{g,\gamma} \times \mathcal{I}(X)$ . Clearly  $\rho(\mathcal{I}(X)) = \mathcal{I}(\Sigma_g, S_1) \subset I$ . Also, Lemma 4.2 implies that  $K_{g,\gamma}$  is generated by the  $\mathcal{I}(X)$ -conjugates of a set  $S \subset K_{g,\gamma}$  such that  $\rho(S) \subset I$ . We conclude that  $\rho(\mathcal{I}_{g,\gamma}) \subset I$ , as desired.  $\square$

## 5 Proof of main theorem

We finally prove our main theorem. The key is the following standard lemma, whose proof is similar to that given in [19, (1) of Appendix to Section 3] and is thus omitted.

**Lemma 5.1** Consider a group  $G$  acting without inversions on a connected graph  $X$ . Assume that  $X/G$  consists of a single edge  $\bar{e}$ . Let  $e$  be a lift of  $\bar{e}$  to  $X$  and let  $v$  and  $v'$  be the endpoints of  $e$ . Then  $G$  is generated by  $G_v \cup G_{v'}$ .

To apply this, we will need the following lemma.

**Lemma 5.2** Let  $a, b \in H_1(\Sigma_g; \mathbb{Z})$  satisfy  $i_a(a, b) = 1$ . Then  $\mathcal{H}_{a,b}/\mathcal{I}_g$  is isomorphic to a graph with a single edge.

The proof is similar to the proofs of [16, Lemma 6.2] and [18, Lemma 6.9], and is thus omitted.

**Proof of Theorem B** Let  $R'_1, \dots, R'_g$  and  $R_{ijk}$  be the subsurfaces of  $\Sigma_g$  from the introduction. Let  $\Gamma$  be the subgroup of  $\mathcal{I}_g$  generated by  $\bigcup_{1 \leq i < j < k \leq g} \mathcal{I}(\Sigma_g, R_{ijk})$ . Our goal is to prove that  $\Gamma = \mathcal{I}_g$ .

The proof will be by induction on  $g$ . The base case  $g = 3$  is trivial, so assume that  $g \geq 4$  and that the theorem is true for all smaller  $g$  such that  $g \geq 3$ . Choose simple closed curves  $\alpha$  and  $\beta$  in  $R'_1$  such that  $i_g(\alpha, \beta) = 1$ . Observe that  $R'_1$  is a closed regular neighborhood of  $\alpha \cup \beta$ . Set  $a = [\alpha]$  and  $b = [\beta]$ . Clearly  $\mathcal{I}_g$  acts on  $\mathcal{H}_{a,b}$  without inversions. Lemmas 3.1 and 5.2 show that the action of  $\mathcal{I}_g$  on  $\mathcal{H}_{a,b}$  satisfies the other conditions of Lemma 5.1. We deduce that  $\mathcal{I}_g$  is generated by the union  $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$  of the stabilizer subgroups of  $\alpha$  and  $\beta$ .

Recall that  $S_i = \overline{\Sigma_g \setminus R'_i}$  for  $1 \leq i \leq g$ . By Lemma 4.1, both  $(\mathcal{I}_g)_\alpha$  and  $(\mathcal{I}_g)_\beta$  are contained in the subgroup generated by  $\bigcup_{i=1}^g \mathcal{I}(\Sigma_g, S_i)$ . We must prove that  $\mathcal{I}(\Sigma_g, S_i) \subset \Gamma$  for  $1 \leq i \leq g$ . We will do the case  $i = g$ ; the other cases are similar. We have a Birman exact sequence

$$1 \longrightarrow \pi_1(U\Sigma_{g-1}) \longrightarrow \mathcal{I}(\Sigma_g, S_g) \longrightarrow \mathcal{I}_{g-1} \longrightarrow 1.$$

By induction, the subset  $\bigcup_{1 \leq i < j < k \leq g-1} \mathcal{I}(\Sigma_g, R_{ijk})$  of  $\mathcal{I}(\Sigma_g, S_g)$  projects to a generating set for  $\mathcal{I}_{g-1}$ . Also, it is clear that the disc-pushing subgroup  $\pi_1(U\Sigma_{g-1})$  of  $\mathcal{I}(\Sigma_g, S_g)$  is generated by elements that lie in  $\bigcup_{1 \leq i < j < g} \mathcal{I}(\Sigma_g, R_{ijg})$ . We conclude that  $\mathcal{I}(\Sigma_g, S_g) \subset \Gamma$ , as desired.  $\square$

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Received: 24 June 2011

Revised: 11 August 2011

