On the Taylor tower of relative K–theory

AYELET LINDENSTRAUSS
RANDY MCCARTHY

For R a discrete ring, M a simplicial R–bimodule, and X a simplicial set, we construct the Goodwillie Taylor tower of the reduced K–theory of parametrized endomorphisms $\widetilde{K}(R; \widetilde{M}[X])$ as a functor of X. Resolving general R–bimodules by bimodules of the form $\widetilde{M}[X]$, this also determines the Goodwillie Taylor tower of $\widetilde{K}(R; M)$ as a functor of M. The towers converge when X or M is connected. This also gives the Goodwillie Taylor tower of $\widetilde{K}(R \ltimes M) \simeq \widetilde{K}(R; B.M)$ as a functor of M.

For a functor with smash product F and an F–bimodule P, we construct an invariant $W(F; P)$ which is an analog of $\mathrm{TR}(F)$ with coefficients. We study the structure of this invariant and its finite-stage approximations $W_n(F; P)$ and conclude that the functor sending $X \mapsto W_n(R; \widetilde{M}[X])$ is the n–th stage of the Goodwillie calculus Taylor tower of the functor which sends $X \mapsto \widetilde{K}(R; \widetilde{M}[X])$. Thus the functor $X \mapsto W(R; \widetilde{M}[X])$ is the full Taylor tower, which converges to $\widetilde{K}(R; \widetilde{M}[X])$ for connected X.

Introduction

In this paper, we calculate the Goodwillie Taylor tower (evaluated at $*$) of the parametrized K–theory of endomorphisms $\widetilde{K}(R; \widetilde{M}[X])$ as a functor of X. Here R is a discrete ring, M a simplicial R–bimodule and X a simplicial set. The R–bimodule $\widetilde{M}[X]$ is the diagonal of the bisimplicial R–bimodule obtained by taking a reduced $(M \cdot * \sim 0)$ free M–module on X in each simplicial degree. If N is a discrete R–bimodule, the parametrized (over N) K–theory of endomorphisms $K(R; N)$ is the K–theory of the category whose objects are pairs (P, f) with P a finitely generated projective right R–module and $f\colon P \to P \otimes_R N$ a map of right R modules, and whose maps are maps of the modules P which induce commutative diagrams; for a simplicial N, $K(R; N)$ is calculated degreewise. The reduced version is then

$$\widetilde{K}(R; N) = \mathrm{hofib}(K(R; N) \to K(R; 0)) = \mathrm{hofib}(K(R; N) \to K(R)).$$

The Goodwillie Taylor tower turns out to be given by $W(R; \widetilde{M}[X])$, where for an R–bimodule N, $W(R; N)$ is a generalization involving coefficients in a bimodule other than R of $\mathrm{TR}(R) = \mathrm{holim}_{m \in \mathbb{N}^\times} \mathrm{THH}(R)^{C_m}$ (the limit is taken over restriction maps $\mathrm{THH}(R)^{C_n} \to \mathrm{THH}(R)^{C_m}$ defined whenever m divides n). The invariant TR was used as an intermediate stage in a topological cyclic homology calculation by Bökstedt, Hsiang and Madsen [3], and given the name TR by Hesselholt and Madsen [13]. In the $N = R$ case, to get a simplicial C_m action on $\mathrm{THH}(R)$ one needs to take its m–th edgewise subdivision. For general N, we take $U^m(R; N)$ to be a cyclic derived tensor product over R of m copies of N; this can be realized by (compatibly) replacing m evenly spaced copies of R in each simplicial degree of $\mathrm{sd}^m \mathrm{THH}(R)$ with copies of N. And then we let

$$W(R; N) = \mathrm{holim}_{m \in \mathbb{N}^\times} U^m(R; N)^{C_m},$$

with the limits taken over restriction maps, as before. The n–th stage of the Goodwillie Taylor tower of $X \mapsto \widetilde{K}(R; \widetilde{M}[X])$ at $*$ is exactly

$$W_n(R; \widetilde{M}[X]) = \mathrm{holim}_{m \in \{1, 2, \ldots, n\}} U^m(R; \widetilde{M}[X])^{C_m},$$

so our result extends that of Dundas and McCarthy [5], identifying $\mathrm{THH}(R; \widetilde{M}[X]) = U^1(R; \widetilde{M}[X])$ with the first stage of the Goodwillie Taylor tower of $X \mapsto \widetilde{K}(R; \widetilde{M}[X])$ and confirming Tom Goodwillie's conjecture that topological Hochschild homology $\mathrm{THH}(R)$ is the same as the stable K–theory $K^s(R) = \mathrm{hocolim}_n \Omega^{n+1} \widetilde{K}(R \ltimes \widetilde{R}[S^n])$.

By resolving any connected simplicial bimodule N in terms of bimodules of the form $\widetilde{M}[X]$ with X connected, it can be deduced (see the authors' paper [16, Section 3] for details) that for any connected simplicial R–bimodule N,

$$\widetilde{K}(R; N) \simeq W(R; N).$$

Since the Goodwillie Taylor tower at $*$ of a homotopy functor is determined by its values on connected objects, and since Corollary 5.9 shows that

(0-1) $$U^n(R; N)_{hC_n} \simeq \mathrm{hofib}[W_n(R; N) \to W_{n-1}(R; N)]$$

is a homogenous degree n functor, we get that $W_n(R; -)$ is the n–th stage of the Goodwillie Taylor tower of the functor $N \mapsto \widetilde{K}(R; N)$ from simplicial R–bimodules to spectra. Using the equivalence $K(R; B.N) \xrightarrow{\simeq} K(R \ltimes N)$ from [5, Theorem 4.1], then, we know that the n–th stage of the Goodwillie Taylor tower at 0 of the functor $N \mapsto K(R \ltimes N)$ is $W_n(R; B.N)$.

A more fundamental question would be to understand the Goodwillie Taylor tower of the functor $A \mapsto \widetilde{K}(A) = \mathrm{hofib}(K(A) \to K(R))$ from simplicial augmented R–algebras to spectra. One would expect it to be more approachable on free augmented algebras,

that is, on tensor algebras $A = T_R M$ of an R–bimodule M over R (assuming M is flat over R—otherwise, we should look at derived tensor algebras; see [16]). Since $\widetilde{K}(A) = \widetilde{K}(\mathrm{id}(A))$ one could hope to further simplify the problem by replacing the identity map on simplicial augmented R–algebras by its first stage Goodwillie Taylor approximation $P_1(\mathrm{id})$. Note that $P_1(\mathrm{id})(A) = A/I^2$, if I is the augmentation ideal of A: this follows by the universal property discussed after the proof of Proposition 1.6 in Goodwillie [10], since if the augmentation $A \to R$ is k–connected, that is, I is k–connected, then the map $A \to A/I^2$ will be at least $2k$–connected. But in the free case, if $A = T_R M$ then $A/I^2 = R \ltimes M$, and so one would want to study the Goodwillie Taylor tower of $\widetilde{K}(R \ltimes M)$, which is what we do here.

The intermediate goal of studying the values that the Goodwillie Taylor tower at R of $A \mapsto \widetilde{K}(A)$ takes on augmented R–algebras of the form $A = T_R M$ is accomplished, using the results of this paper, in [16]—in terms of the analytic parallel, knowing the coefficients in the Taylor series of a composition $f(a_1 z)$ of an analytic function f (whose Taylor series is not known ahead of time) with the linear part $a_1 z$ of an analytic function $g(z) = \sum_{i=1}^{\infty} a_1 z^i$ (whose Taylor series is known) determines the coefficients in the Taylor series of $f \circ g$. We show there that the Goodwillie Taylor tower $\Sigma W(R; -)$ converges for connected M, so when M is connected we get an equivalence

$$(0\text{-}2) \qquad\qquad \Sigma W(R; M) \simeq \widetilde{K}(T_R M).$$

The question of determining the Goodwillie Taylor tower of $A \mapsto \widetilde{K}(A)$ for general augmented R–algebras remains.

From the calculational point of view, the fact that for any simplicial R–bimodule M,

$$(0\text{-}3) \qquad\qquad \widetilde{K}(R \ltimes M) \simeq \widetilde{K}(R; B.M) \simeq W(R; B.M)$$

(note that $B.M$ *is* connected) is useful: for example, in [17] we use this and Lars Hesselholt and Ib Madsen's calculation of $W(\mathbb{F}_p; \mathbb{F}_p) = \mathrm{TR}(\mathbb{F}_p)$ to completely calculate $\widetilde{K}(\mathbb{F}_p \ltimes (\bigoplus_{i=1}^{n} \mathbb{F}_p))_p^{\wedge}$. Equation (0-3) is an absolute calculation of relative algebraic K–theory in terms of invariants related to Hochschild homology; so is the related equation (0-2). They are not calculations via the cyclotomic trace of [3], which lands in TC, but see the introduction of [16] for a comparison of the two methods for understanding $\widetilde{K}(T_R M)$ for M connected.

In terms of understanding the K–theory of endomorphisms, our work follows the program begun by Gert Almkvist in 1974 by relating the K–theory of endomorphisms of R with the big Witt vectors over R [1]. For R commutative, he defines an injection from $\pi_0(\widetilde{K}(R; R))$ onto a dense subring of the ring of big Witt vectors over R.

Hesselholt shows [12] that for an associative ring R, $\pi_0(\mathrm{TR}(R; p)) = \pi_0(W^{(p)}(R; R))$ (the 0–th homotopy group of versions of both invariants calculated by taking limits only over powers of p) agrees with the ring of p–typical Witt vectors of R. For p–adic rings R, Hesselholt and Madsen [14] relate the higher homotopy groups of $W^{(p)}(R; R)$ to higher cohomology of a de Rham–Witt complex of R. In [11], Hesselholt shows that for commutative rings R, there exists an initial big Witt complex over R which, when evaluated at the set consisting of a positive integer r and its divisors is closely related to $\mathrm{THH}(R)^{C_r} = U^r(R; R)^{C_r}$. Our map $\beta\colon \widetilde{K}(R; M) \to W(R; M)$ of Main Theorem 9.2 has a similar flavor to Almkvist's, sending a homomorphism to all its powers, but using Waldhausen's S–construction to make it into a spectrum level map. The map β cannot be an equivalence in general—Almkvist's result precludes it already for $R = M$—but we show that it is an equivalence for connected R–bimodules M.

In Section 1 of the paper, we define functors with smash product over a category and their bimodules. In Section 2 we define $U^n(F; P)$ to be an analog of $\mathrm{sd}^n \mathrm{THH}(F)$ with n bimodule coefficient coordinates. Note that the construction we generalize is not Marcel Bökstedt's original construction in [2] of the topological Hochschild homology of an FSP but rather the construction by Dundas and McCarthy in [5; 6] of the topological Hochschild homology of an FSP over a category. This will be important in defining the map from K–theory in Section 9. In [6], it is shown that the two variants agree for THH; Proposition 6.13 below shows the analogous equivalence for our generalization U. As in [3], when m divides n we have restriction maps $\mathrm{Res}^{n/m}\colon U^n(F; P)^{C_n} \to U^m(F; P)^{C_m}$, which we use in Section 4 to define $W_n(F; P)$ and $W(F; P)$.

Our main theorem, Main Theorem 9.2, states that a natural transformation

$$\beta\colon \widetilde{K}(R; \widetilde{M}[-]) \to W(R; \widetilde{M}[-])$$

which we construct induces an equivalence between the two functors on connected X. Moreover, by Corollary 9.3 $W_n(R; \widetilde{M}[-])$ is the n–th stage of the Goodwillie calculus Taylor tower of the functor $\widetilde{K}(R; \widetilde{M}[-])$, with the tower structure maps the same as those induced on the homotopy inverse limits by the restriction of categories from $\{1, 2, \ldots, n-1, n\}$ to $\{1, 2, \ldots, n-1\}$.

To obtain this result, we start in Section 5 to use analysis like that done by Goodwillie [8] for THH and TR to calculate $\mathrm{hofib}[W_n(F; P) \to W_{n-1}(F; P)]$ for a general FSP F and an F–bimodule P. Corollary 5.9 gives equation (0-1).

In Section 8, this analysis is used to show that each layer $\mathrm{hofib}[W_n(R; \widetilde{M}[-]) \to W_{n-1}(R; \widetilde{M}[-])]$ is a homogenous degree n functor (this is basically Corollary 8.2):

we get that

$$U^n(R; \widetilde{M}[X]) \simeq U^n(R; M) \wedge \left(\overset{n}{\bigwedge} X \right).$$

This implies already that the $W_n(R; \widetilde{M}[-])$ are n–excisive functors which are the n–th stage of the Taylor tower of their homotopy inverse limit $W(R; \widetilde{M}[-])$.

Section 6 generalizes results from [6], allowing us to interchange U and W of a ring, which can be viewed as an FSP defined over a point (the approach which is easier to calculate with), and of that ring's corresponding FSP defined over the category $\mathcal{P}_R$ (the approach we to define the map β of Main Theorem 9.2). In the latter case, it also allows us to use a subset of the maps one would normally use that still stabilize by Waldhausen's S–construction to give all of $U^n(\underline{\mathcal{P}_R}; \underline{M})$, which is useful in Section 10 and Section 11.

Section 7 discussed the Goodwillie calculus properties of our functors, most importantly: showing that $W(R; \widetilde{M}[-])$ is 0–analytic (Proposition 7.14). It is already known by [19, Proposition 3.2] that the functor $\widetilde{K}(R; \widetilde{M}[-])$ is 0–analytic as well.

Thus in Section 9, after constructing the natural transformation β, we can use a variant of Goodwillie's Theorem 5.3 from [9]: it states that if there is a natural transformation between two ρ–analytic functors F and G which induces an equivalence of the differentials at every space X, then for $(\rho+1)$–connected maps $X \to Y$, there is a Cartesian square

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & G(Y). \end{array}$$

The variant is simply the observation that the proof in [9] requires an equivalence of the differentials only on ρ–connected X. We will want to apply it for $\rho = 0$, $Y = *$, and our two functors above to get that for 0–connected X (that is, X for which $X \to *$ is 1–connected),

$$\beta \colon \widetilde{K}(R; \widetilde{M}[X]) \xrightarrow{\simeq} W(R; \widetilde{M}[X]).$$

It follows from [5] that β induces an equivalence of the differentials at $*$, and the remainder of the paper uses that result to show an equivalence of the differentials at arbitrary 0–connected spaces X. Section 9 concludes by reducing our Main Theorem 9.2 to Technical Lemma 9.4, which states that for a ring R and simplicial R–bimodules M and N with N k–connected, β induces a $2k$–connected map

$$\mathrm{hofib}\big(\widetilde{K}(R; B_{\cdot}M \oplus B_{\cdot}N) \to \widetilde{K}(R; B_{\cdot}M) \big)$$

$$\to \mathrm{hofib}\big(W(R; B_{\cdot}M \oplus B_{\cdot}N) \to W(R; B_{\cdot}M) \big).$$

(The lemma is actually stated in terms of $W(\underline{\mathcal{P}_R}; -)$ rather than $W(R; -)$, but the two agree by Proposition 6.13.)

The final two sections of the paper prove Technical Lemma 9.4. The basic strategy is to write

$$K(R; B.M \oplus B.N) \simeq K(R \ltimes (M \oplus N)) = K((R \ltimes M) \ltimes N) \simeq K(R \ltimes M; B.N)$$

and observe that by [5], the homotopy fiber of the map from this to $K(R \ltimes M) \simeq K(R; B.M)$ admits a $2k$–connected map to $\mathrm{THH}(R \ltimes M; B.N)$. Using the multi-linearity of each $\mathrm{THH}_r(R \ltimes M; B.N)$ in the r FSP coordinates (here r indicates simplicial dimension in the simplicial spectrum THH), and careful analysis of dimensions, in Corollary 10.5 we obtain a decomposition

$$\mathrm{THH}(R \ltimes M; B.N) \simeq \prod_{a=0}^{\infty} U^a(R; B.M, \ldots, B.M, B.N).$$

It is much easier to see that up to order $2k + 1$, $W(R; B.M \oplus B.N)$ also decomposes into the same product; this is Lemma 10.3.

Thus Section 10 establishes the plausibility of Technical Lemma 9.4, by demonstrating that the domain and range there are indeed equivalent up to dimension $2k$. Section 11 then traces the actual map β through, to show that *it* induces a $2k$–equivalence, as desired.

1 Functors with smash products

We want to generalize $\mathrm{THH}(R; M)$ to functors $U^n(R; M)$ which are n copies of M, tensored (in a derived sense, over R) cyclically
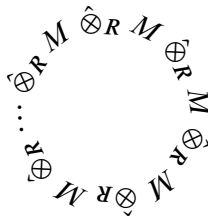
where the cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$ acts by rotation. $\mathrm{THH}(R; M)$ can be thought of as the $n = 1$ case of this cyclic tensor. We use Bökstedt's notion of an FSP from [2], but generalize it to work over categories as was done in [6].

Let $\mathcal{S}_*$ denote the category of pointed simplicial sets.

Definition 1.1 A *functor with stabilization* is a functor F from $\mathcal{S}_*$ to $\mathcal{S}_*$ together with a natural transformation

$$\lambda_{X,Y} \colon X \wedge F(Y) \to F(X \wedge Y)$$

such that

(i) $\lambda_{S^0,X} \colon S^0 \wedge F(X) \to F(S^0 \wedge X)$ is the obvious isomorphism for all $X \in \mathcal{S}_*$.

(ii) $\lambda_{X,Y \wedge Z} \circ (\mathrm{id}_X \wedge \lambda_{Y,Z}) = \lambda_{X \wedge Y, Z}$ for all $X, Y, Z \in \mathcal{S}_*$.

(iii) If X is n–connected, then $F(X)$ is n–connected.

(iv) Let $\sigma_X \colon F(X) \to \Omega F(\Sigma X)$ be the adjoint to $\lambda_{S^1,X}$. Then the following limit system stabilizes for each n:

$$\pi_n|F(X)| \overset{\sigma_X}{\to} \pi_n \Omega|F(\Sigma X)| \overset{\sigma_{\Sigma X}}{\to} \pi_n \Omega^2 |F(\Sigma^2 X)| \to \cdots$$

Definition 1.2 Let $\mathcal{O}$ be a set. A *functor with stabilization over $\mathcal{O}$* is a functor F from $\mathcal{S}_* \times \mathcal{O} \times \mathcal{O}$ to $\mathcal{S}_*$, such that for all $A, B \in \mathrm{obj}(\mathcal{O})$, $F_{A,B}( ) = F(-, A, B)$ is a functor with stabilization.

For F a functor with stabilization, we let $\mathbf{F}$ be the spectrum with $\mathbf{F}(m) = F(S^m)$ and structure maps given by σ_{S^m} of condition (iii) above. We call $\mathbf{F}$ the *spectrum associated to F*. We let $\pi_i(F) = \pi_i \mathbf{F} = \lim_{n \to \infty} \pi_i \Omega^n F(S^n)$. We say that F is n–*connected* if $\pi_i(F) = 0$ for $i \leq n$. Thus by condition (iii), every functor with stabilization is -1–connected and hence bounded below. A functor with stabilization over $\mathcal{O}$ is n–connected if for every $A, B \in \mathcal{O}$, $F_{A,B}$ is n–connected.

For F a functor with stabilization over $\mathcal{O}$ and F' a functor with stabilization over $\mathcal{O}'$, a morphism $\eta \colon F \to F'$ is a set map $\mathcal{O} \to \mathcal{O}'$ and natural transformations of functors with stabilization $\eta_{A,B} \colon F_{A,B} \to F'_{\eta(A),\eta(B)}$ for all $A, B \in \mathcal{O}$.

Definition 1.3 A *functor with smash product over $\mathcal{O}$* (or just FSP) is a functor F with stabilization over $\mathcal{O}$ together with natural transformations for all $A, B, C \in \mathcal{O}$:

$$1_{A;X} \colon X \to F_{A,A}(X),$$

$$\mu_{A,B,C;X,Y} \colon F_{B,C}(X) \wedge F_{A,B}(Y) \to F_{A,C}(X \wedge Y)$$

such that

$$\mu(\mu \wedge \mathrm{id}) = \mu(\mathrm{id} \wedge \mu),$$

$$\mu(1_{A;X} \wedge 1_{A;Y}) = 1_{A;X \wedge Y},$$

$$\lambda_{A,B;X,Y} = \mu_{A,A,B;X,Y}(1_{A;X} \wedge \mathrm{id}_{F_{A,B}(Y)}).$$

For F and F' FSP's over $\mathcal{O}$ and $\mathcal{O}'$ respectively, a morphism η from F to F' as FSP's is a set map $\tilde{\eta}: \mathcal{O} \to \mathcal{O}'$ and morphisms of functors with stabilizations $\eta_{A,B}: F_{A,B} \to F'_{\tilde{\eta}(A),\tilde{\eta}(B)}$ which strictly commute with the natural transformations μ and μ'. We have *not* assumed a morphism of FSP's preserves the unit. We will say a morphism is *unital* if it does.

Examples 1.4 (i) For the applications in these notes, we will primarily be interested in the following type of FSP: Let $\mathcal{A}$ be a *linear* category (its Hom sets are abelian groups and composition is bilinear). For any two objects $A, B \in \operatorname{obj}(\mathcal{A})$, we define the FSP $\underline{\mathcal{A}}$ by

$$(X, A, B) \mapsto \operatorname{Hom}_{\mathcal{A}}(A, B) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}[X],$$

where $\tilde{\mathbb{Z}}[X] = \mathbb{Z}[X]/\mathbb{Z}[*]$. The multiplication is given by sending smash to tensor followed by composition:

$$\underline{\mathcal{A}}_{B,C}(X) \wedge \underline{\mathcal{A}}_{A,B}(Y) \to \operatorname{Hom}_{\mathcal{A}}(B, C) \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(A, B) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}[X \wedge Y]$$
$$\to \underline{\mathcal{A}}(A, C)(X \wedge Y)$$

and the unit at any $A \in \mathcal{A}$ is given by the inclusion

$$X \to \underline{\mathcal{A}}_{A,A}(X)$$
$$x \mapsto \operatorname{id}_A \otimes 1 \cdot x.$$

(ii) For F an FSP over $\mathcal{O}$, we can form a linear category with objects the set $\mathcal{O}$ and whose Hom sets are $\pi_0 F_{A,B}$. In this way every small linear category can be thought of as a special case of an FSP. In particular, for a discrete ring R, we can regard it as the linear category with a single object $*$ and $\operatorname{Hom}(*, *) = R$, and thus as an FSP as in (i) above, which we will denote R like the original ring.

(iii) For $\mathcal{B}$ any category, we can form an FSP over $\operatorname{obj}(\mathcal{B})$ by sending

$$(X, A, B) \mapsto \operatorname{Hom}_{\mathcal{B}}(A, B)_+ \wedge X$$

(where $_+$ denotes a disjoint basepoint) and using composition of morphisms to define μ. Conversely, given F an FSP over $\mathcal{O}$ we can form a category with objects the set $\mathcal{O}$ with $\operatorname{Hom}(A, B)$ equal to the set (obtained by forgetting the topology) $F_{A,B}(S^0)$.

Definition 1.5 Let F be an FSP over $\mathcal{O}$ and T a functor with stabilization over $\mathcal{O}$. A *left F–module structure* on T is a natural transformation

$$l_{A,B,C;X,Y}: F_{B,C}(X) \wedge T_{A,B}(Y) \to T_{A,C}(X \wedge Y)$$

such that

$$l(\mu \wedge \mathrm{id}) = l(\mathrm{id} \wedge l),$$

$$\lambda_{A,B;X,Y} = l_{A,B,B;X,Y}(1_{B;X} \wedge \mathrm{id}_{T_{A,B}(Y)}).$$

The notion of right F–module $r_{A,B,C;X,Y}$ is defined similarly, with the analog of the second property defined using the order switching map $T_2\colon Y \wedge T_{A,B}(X) \to T_{A,B}(X) \wedge Y$ and the map $T_1\colon T_{A,B}(X \wedge Y) \to T_{A,B}(Y \wedge X)$ induced by switching the coordinates, $\lambda_{A,B;Y,X} = T_1 \circ r_{A,A,B;X,Y}(\mathrm{id}_{T_{A,B}(X)} \wedge 1_{A;Y}) \circ T_2$.

Definition 1.6 A *bimodule* over F is a functor T with stabilization over $\mathcal{O}$ together with a structure of left and right module over F such that

$$l_{A,C,D;X,Y \wedge Z}(\mathrm{id}_{F_{C,D}(X)} \wedge r_{A,B,C;Y,Z}) = r_{A,B,D;X \wedge Y,Z}(l_{B,C,D;X,Y} \wedge \mathrm{id}_{F_{A,B}(Z)}),$$

where r is the structure of right module over T.

For F and F' FSP's over $\mathcal{O}$ and $\mathcal{O}'$, respectively, and P and P' bimodules of F and F', a *map* $(f; g)$ from $(F; P)$ to $(F'; P')$ is a pair of morphisms such that

(i) $f\colon F \to F'$ is a unital map of FSP's.

(ii) $g\colon P \to P'$ is a map of functors with stabilization over the same set map $\mathcal{O} \to \mathcal{O}'$ as f, which is a map of F–bimodules if we use f to make P' into an F–bimodule.

Examples 1.7 (i)   Let $\mathcal{A}$ be a linear category and T any bilinear functor from $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ to abelian groups. We can form the bimodule $\underline{T}$ of $\underline{\mathcal{A}}$ by

$$(A, B, X) \mapsto T(A, B) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X].$$

(ii)   If R is a ring and M an R–bimodule, in Examples 1.4 above we discussed how we can regard R as an FSP over a category with a single element $*$ with morphism set R. Then M is a functor from the product of this category with its opposite to abelian groups, and so gives rise as in example (i) to a bimodule over the FSP $\underline{R}$, which we will denote $\underline{M}$.

(iii)   As a special case of (i) above, if G_1 and G_2 are two functors of linear categories $\mathcal{A} \to \mathcal{B}$, we can form the $\underline{\mathcal{A}}$–bimodule

$$(A, B, X) \mapsto \mathcal{B}(G_1(A), G_2(B)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X].$$

(iv)   If F is an FSP over $\mathcal{O}$, P an F–bimodule, and Y a finite pointed simplicial set, we can construct another F–bimodule $P \otimes Y$ by letting

$$(P \otimes Y)_{A,B}(X) = |P_{A,B}(X) \wedge Y.|$$

with F acting through its action on P; all we are doing here is taking a smash product with the realization of $Y.$.

(v) In the same setting as (iv), if we know that for every $A, B \in \mathcal{O}$ and $X \in \mathcal{S}_*$, $F_{A,B}(X)$ and $P_{A,B}(X)$ have abelian group structures compatible with all the FSP and bimodule structure maps, then we can construct yet another F–bimodule $\widetilde{P}[Y]$ by letting

$$\widetilde{P}[Y]_{A,B}(X) = P_{A,B}(X) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[Y].$$

The F–bimodule structure, as before, involves only P. In degree n, $(P \otimes Y)_{A,B}(X)$ has $P_{A,B}(X) \wedge Y_n$, and $\widetilde{P}[Y]_{A,B}(X)$ has $P_{A,B}(X)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[Y_n]$; thus there is an inclusion map $P \otimes Y \hookrightarrow \widetilde{P}[Y]$. Note that it induces a stable equivalence on the associated spectra (stabilizing, of course, in the X coordinate. The Y coordinate is part of the definition of the bimodule).

2   The construction of U^n

Let I be the category whose objects are the natural numbers considered as ordered sets ($n = \{1 < 2 < \cdots < n\}$) and whose morphisms are all injective maps. For any $X \in I$ we denote by $|X|$ the cardinality of X and for any $\underline{X} = (X_0, \ldots, X_j) \in I^{j+1}$ we let $\sqcup \underline{X}$ denote $X_0 \sqcup X_1 \sqcup \ldots \sqcup X_j$, where $\sqcup$ means *concatenation*.

For E a functor from the small category $\mathcal{C}$ to pointed spaces, we let $\mathrm{hocolim}_{C \in \mathcal{C}} E(C)$ be a functorial choice for constructing the homotopy colimit of the functor E. We recall the following lemma of Bökstedt [2] (see also Madsen [18]).

Lemma   *Let* $\mathbf{N}$ *be any subcategory of* I *with the same set of objects but with exactly one morphism between any pair of objects* n *and* m *where* $n \leq m$. *Let* G *be a functor from* I^{j+1} *to spaces. If the connectivity of the maps* $G(n_0, \ldots, n_j) \to G(m_0, \ldots, m_j)$ *for maps in* I^{j+1} *tends to infinity uniformly with* Σn_i, *then the inclusion* $\mathrm{hocolim}_{\mathbf{N}^{j+1}} G \to \mathrm{hocolim}_{I^{j+1}} G$ *is a (weak) homotopy equivalence.*

Example 2.1   We note that there is a functor from I to pointed spaces given by sending the ordered finite set X to S^X–the $|X|$–sphere obtained by smashing together copies of S^1 indexed by the elements of X. Given any functor with stabilization F, we can define a functor from I to pointed spaces by sending the ordered set X to $\mathrm{map}(S^X, F(S^X))$. Given an injective map $\alpha\colon X \to Y$, let β be some isomorphism of Y such that $\alpha = \beta \circ \mathrm{inc}$ where inc is the ordered inclusion of X into the first $|X|$ terms of Y. The map α_* from $\mathrm{map}(S^X, F(S^X))$ to $\mathrm{map}(S^Y, F(S^Y))$ is given by

taking $f \in \mathrm{map}(S^X, F(S^X))$ to the composite

$$
\begin{array}{ccc}
S^Y & \xrightarrow{\;\;\alpha_*(f)\;\;} & F(S^Y) \\
\downarrow{\scriptstyle\beta^{-1}} & & \uparrow{\scriptstyle F(\beta)} \\
S^X \wedge S^{Y\backslash X} \xrightarrow{f\wedge\mathrm{id}} F(S^X) \wedge S^{Y\backslash X}\lambda & \longrightarrow & F(S^X \wedge S^{Y\backslash X}).
\end{array}
$$

Let $\vec{k} = (k_1, \ldots, k_n)$ where the k_i are nonnegative integers. We set

$$I^{\vec{k}+n} = I^{k_1+1} \times \cdots \times I^{k_n+1}$$

and for $\underline{X} \in I^{\vec{k}+n}$ and (i,j) integers such that $0 \le j \le k_i$ we let $X_{i,j} \in I$ be in the $(j+1)$–st position of the I^{k_i} component of $\underline{X}$. We let

$$\mathcal{A}^{\vec{k}+n} = \mathcal{A}^{k_1+1} \times \cdots \times \mathcal{A}^{k_n+1}$$

and for $\underline{A} \in \mathcal{A}^{\vec{k}+n}$ and (i,j) integers such that $0 \le j \le k_i$ we let $A_{i,j}$ be in the $(j+1)$–st position of the $\mathcal{A}^{k_i+1}$ component of $\underline{A}$.

Definition 2.2 Let F be an FSP over a skeletally small category $\underline{\mathcal{A}}$ and $\underline{P}(1), \ldots, P(n)$ be a sequence of F–bimodules. We define the functor V from $I^{\vec{k}+n} \times \mathcal{A}^{\vec{k}+n}$ to spaces by setting $V(\underline{X}; \underline{A})$ to be

$$
\begin{array}{l}
P(1)_{A_{1,1},A_{1,0}}(S^{X_{1,0}}) \wedge F_{A_{1,2},A_{1,1}}(S^{X_{1,1}}) \wedge \cdots \wedge F_{A_{2,0},A_{1,k_1}})(S^{X_{1,k_1}} \wedge \\[4pt]
P(2)_{A_{2,1},A_{2,0}}(S^{X_{2,0}}) \wedge F_{A_{2,2},A_{2,1}}(S^{X_{2,1}}) \wedge \cdots \wedge F_{A_{3,0},A_{2,k_2}}(S^{X_{2,k_2}}) \wedge \\[4pt]
\qquad\qquad\qquad \vdots \quad \vdots \\[4pt]
P(n)_{A_{n,1},A_{n,0}}(S^{X_{n,0}}) \wedge F_{A_{n,2},A_{n,1}}(S^{X_{n,1}}) \wedge \cdots \wedge F_{A_{1,0},A_{n,k_n}}(S^{X_{n,k_n}}),
\end{array}
$$

and let

$$V(\underline{X}, \mathcal{A}) = \bigvee_{\underline{A} \in \mathcal{A}^{\vec{k}+n}} V(\underline{X}; \underline{A}).$$

If $\mathcal{A}$ is not itself small, we use $\mathcal{A}$'s skeleton rather than $\mathcal{A}$ itself here. In light of Example 6.3 below, which says that for a small category $\mathcal{A}$, U^n taken over a subcategory which is equivalent to all of $\mathcal{A}$ is C_n–equivariantly homotopy equivalent to U^n taken over $\mathcal{A}$, this should not lead to problems when we map between small and large categories.

Given any two pointed spaces S and T, we write $\underline{\mathrm{Map}}(S,T)$ for the functor with stabilization $X \mapsto \mathrm{map}(S, X \wedge T)$.

Definition 2.3 Let F be an FSP over a category $\mathcal{A}$ and let $P(1), \ldots, P(n)$ be a sequence of F–bimodules. We use the above construction of $\underline{\mathrm{Map}}$ to define an n–simplicial functor with stabilization $U^n(F; P(1), \ldots, P(n))$ which is given in simplicial dimension $\vec{k} = (k_1, \ldots, k_n)$ by

$$\mathrm{hocolim}_{\underline{X} \in I^{\vec{k}+n}} \underline{\mathrm{Map}}\big((S^{\sqcup \underline{X}}), V(\underline{X}, \mathcal{A})\big).$$

The stabilization maps for the homotopy colimit are given like those in Example 2.1. The boundary maps and degeneracy maps in each simplicial dimension are the Hochschild-type maps, with concatenations of the X's and the FSP multiplication and left and right actions for the boundary maps, and the FSP unit map for the degeneracies.

Definition 2.4 Given an FSP F over a category $\mathcal{A}$, and a left F–module P and a right F–module Q, we define a simplicial functor with stabilization over $\mathcal{A}$, $P \widehat{\otimes}_F Q$, as follows: Given $B, C \in \mathcal{A}$, we define for each $k \in I$ a functor $W_{B,C}$ from $I^{k+2} \times \mathcal{A}^{k+1}$ to spaces by

$$W_{B,C}(\underline{X}; \underline{A}) = P_{A_1, B}(S^{X_0}) \wedge F_{A_2, A_1}(S^{X_1}) \wedge \cdots \wedge F_{A_{k+1}, A_k}(S^{X_k}) \wedge Q_{C, A_{k+1}}(S^{X_{k+1}}),$$

and let $P \widehat{\otimes}_F Q$ to be the simplicial functor with stabilization over $\mathcal{A}$ defined on $B, C \in \mathcal{A}$ in simplicial dimension $[k]$ by

$$\mathrm{hocolim}_{\underline{X} \in I^{k+2}} \underline{\mathrm{Map}}\bigg((S^{\sqcup \underline{X}}), \bigvee_{\underline{A} \in \mathcal{A}^{k+1}} W_{B,C}(\underline{X}; \underline{A})\bigg).$$

The stabilization maps for the homotopy colimit are given like those in Example 2.1. The face operators are induced by bar-construction type natural transformations constructed like those for $U^1(F; P)$.

Note that if P and Q are F–bimodules, then $P \widehat{\otimes}_F Q$ is naturally a simplicial F–bimodule. Since the construction $\widehat{\otimes}_F$ is natural, we can iterate it to form $P \widehat{\otimes}_F Q \widehat{\otimes}_F R$, a bisimplicial F–bimodule. If $f \colon P \to P'$ is a map of right F–bimodules which is an equivalence, then $f \widehat{\otimes}_F \mathrm{id}_Q$ is an equivalence also (by the realization lemma). Right multiplication induces a map of simplicial right F–modules $P \widehat{\otimes}_F F \to P$ (where P is the trivial multisimplicial object with structure maps all equal to the identity) which is an equivalence (a simplicial contraction is given by the unused degeneracy operator in $U^1(F; P)$).

Definition 2.5 We define $\widehat{\otimes}_F^n$ to be the functor from F–bimodules to n–fold simplicial F–bimodules given by

$$\overbrace{P \widehat{\otimes}_F \cdots \widehat{\otimes}_F P}^{n \text{ times}}.$$

Lemma 2.6 *There is a homotopy equivalence of n–fold simplicial functors with stabilization*

$$U^n(F; P(1), \ldots, P(n)) \xrightarrow{\simeq} U^{n-1}(F; P(1) \hat{\otimes}_F P(2), P(3), \ldots, P(n)).$$

By iteration of this equivalence, there is for each $m|n$ an equivalence of n–fold simplicial functors with stabilization

$$U^n(F; P(1), \ldots, P(n)) \xrightarrow{\alpha_m} U^{n/m}(F; Q(1), \ldots, Q(n/m)),$$

where $Q(i) = P(m(i-1)+1) \hat{\otimes}_F \cdots \hat{\otimes}_F P(m(i-1)+m)$.

Proof This is a formal consequence of the definitions, using the fact that homotopy colimits commute with one another and the fact that suspension

$$\text{hocolim}_{\underline{X} \in I^{\vec{k}+n}} \underline{\text{Map}} \left(S^{\sqcup \underline{X}}, \bigvee_{\underline{A} \in \mathcal{A}^{\vec{k}+n}} V(\underline{X}; \underline{A}) \right)$$

$$\rightarrow \text{hocolim}_{\underline{X} \in I^{\vec{k}+n}} \underline{\text{Map}} \left(S^{\sqcup \underline{X}} \wedge S^Y, \bigvee_{\underline{A} \in \mathcal{A}^{\vec{k}+n}} V(\underline{X}; \underline{A}) \wedge S^Y \right)$$

induces a homotopy equivalence when evaluated at any space. □

3 First properties of U^n

We now establish some elementary first properties of U^n. Since for a fixed FSP F over a category $\mathcal{O}$, a morphism of F–bimodules is just a map of functors with stabilization at each $(A, B) \in \mathcal{O} \times \mathcal{O}$ which strictly commutes with the left and right actions of F, it follows immediately from the definitions that $U^n(F; \,)$ is a functor from the n–fold product category of F–bimodules to spectra.

Lemma 3.1 *If $f(i): P(i) \rightarrow P'(i)$ is an m–connected map of bimodules, then the induced map of functors with stabilization*

$$U^n(f): U^n(F; \ldots, P(i), \ldots) \rightarrow U^n(F; \ldots, P'(i), \ldots)$$

is m–connected.

Proof Since the associated spectrum of U^n is an Ω–spectrum, it suffices to show $U^n(f)(S^0)$ is m–connected. A map of simplicial spaces which is m–connected in each simplicial dimension is m–connected upon realization (essentially because homotopy colimits preserve connectivity) and hence it suffices to show that $U^n(f)(S^0)_{[k]}$ is

m–connected for all $k \geq 0$. If $f\colon X \to X'$ is a q–connected map of (pointed) spaces, then $\mathrm{id} \wedge f\colon Y \wedge X \to Y \wedge X'$ is $(p+q+1)$–connected for any p–connected space Y. This, the fact that our functors with stabilization preserve connectivity, and the fact that homotopy colimits preserve connectivity complete the argument. $\qquad\square$

Corollary 3.2 *The functor $U^n(F; P(1), \ldots, P(n))$ is a reduced homotopy functor in each variable $P(i)$.*

Remark 3.3 It will often be useful to replace a given functor with stabilization F by an equivalent one (one whose associated spectrum is stably equivalent to that of the original F) whose associated spectrum is an Ω–spectrum. Define a new functor with stabilization

$$\Omega^\infty F = \mathrm{hocolim}_{X \in I} \underline{\mathrm{Map}}(S^X, F(S^X))$$

with $\lambda_{Z,Y}$ defined by the natural composite

$$
\begin{aligned}
Z \wedge \Omega^\infty F(Y) &= Z \wedge \mathrm{hocolim}_{X \in I} \underline{\mathrm{Map}}(S^X, F(Y \wedge S^X)) \\
&\cong \mathrm{hocolim}_{X \in I} Z \wedge \underline{\mathrm{Map}}(S^X, F(Y \wedge S^X)) \\
&\to \mathrm{hocolim}_{X \in I} \mathrm{map}(S^X, Z \wedge F(Y \wedge S^X)) \\
&\xrightarrow{\lambda} \mathrm{hocolim}_{X \in I} \underline{\mathrm{Map}}(S^X, F(Z \wedge Y \wedge S^X)) \\
&= \Omega^\infty F(Z \wedge Y).
\end{aligned}
$$

The natural map $F \to \Omega^\infty F$ of functors with stabilization gives a stable equivalence of the associated spectra by the condition of Definition 1.1 (iii). If F was an FSP, then we can make $\Omega^\infty F$ an FSP by defining $\mu_{\Omega^\infty F}$ to be the composite

$$
\begin{aligned}
\Omega^\infty &F(Z) \wedge \Omega^\infty F(Y) \\
&= \mathrm{hocolim}_{X \in I} \underline{\mathrm{Map}}(S^X, F(Z \wedge S^X)) \wedge \mathrm{hocolim}_{X' \in I} \underline{\mathrm{Map}}(S^{X'}, F(Y \wedge S^{X'})) \\
&\xrightarrow{\alpha} \mathrm{hocolim}_{(X,X') \in I \times I} \underline{\mathrm{Map}}(S^{X \sqcup X'}, F(Z \wedge S^X) \wedge F(Y \wedge S^{X'})) \\
&\xrightarrow{\mu_F} \mathrm{hocolim}_{(X,X') \in I \times I} \underline{\mathrm{Map}}(S^{X \sqcup X'}, F(Z \wedge S^X \wedge Y \wedge S^{X'})) \\
&\xrightarrow{\beta} \mathrm{hocolim}_{(X,X') \in I \times I} \underline{\mathrm{Map}}(S^{X \sqcup X'}, F(Z \wedge Y \wedge S^{X \sqcup X'})) \\
&\xrightarrow{\gamma} \mathrm{hocolim}_{\widetilde{X} \in I} \underline{\mathrm{Map}}(S^{\widetilde{X}}, F(Z \wedge Y \wedge S^{\widetilde{X}})) \\
&= \Omega^\infty F(Z \wedge Y),
\end{aligned}
$$

where α is obtained by smashing maps, β by switching factors and γ is induced by the concatenation functor $\sqcup\colon I \times I \to I$. The multiplication map $\mu_{\Omega^\infty F}$ is strictly

associative; the natural map $F \to \Omega^\infty F$ is an equivalence of FSP's, which are both unital if we define the units in $\Omega^\infty F$ to be the images of those of F.

If P is a right/left/bimodule of F, then $\Omega^\infty P$ is again a right/left/bimodule of $\Omega^\infty F$ (defined as we did for FSP's above) and the natural map $(F, P) \to (\Omega^\infty F, \Omega^\infty P)$ preserves the right/left/bimodule structure.

Corollary 3.4 *The natural map of functors with stabilization*

$$U^n(F; P(1), \dots, P(n)) \to U^n(\Omega^\infty F; \Omega^\infty P(1), \dots, \Omega^\infty P(n))$$

is an equivalence. Thus, we can always replace the FSP and its associated bimodules with equivalent ones whose associated spectra are Ω–spectra.

Lemma 3.5 *If each bimodule $P(i)$ is m_i–connected, then $U^n(F; P(1), \dots, P(n))$ is $(\sum_{i=1}^n m_i + (n-1))$–connected.*

Proof It suffices to prove the result after taking Ω^∞ of F and the $P(i)$'s. Thus, we may assume that $P(i)(S^X)$ is $(m_i + |X|)$–connected for all i and $X \in I$. Recall that if T_i are x_i–connected for $1 \le i \le n$, then $T_1 \wedge \cdots \wedge T_n$ is $(\sum_{i=1}^n x_i + (n-1))$–connected. Thus, $V(\underline{X}, \underline{A})$ will be $(|\underline{X}| + \sum_{i=1}^n m_i + (n-1))$–connected for all $\underline{X}$. Since homotopy colimits preserve connectivity, we see that $U^n(F; P(1), \dots, P(n))$ is (at least) $(\sum_{i=1}^n m_i + (n-1))$–connected. $\square$

4 The construction of W_n and W

In this section, we will only be considering $U^n(F; P(1), \dots, P(n))$ as a simplicial functor with stabilization by taking the diagonal of the n–dimensional multisimplicial functor with stabilization defined in Definition 2.3. We first note that there is an isomorphism

$$t: U^n(F; P(1), \dots, P(n)) \cong U^n(F; P(n), P(1), \dots, P(n-1))$$

induced in each simplicial dimension k by precomposing each map with the map induced on the smash product of spheres by τ^{-1}, where $\tau: I^{(k+1,\dots,k+1)} \to I^{(k+1,\dots,k+1)}$ is the functor

$$\tau(\underline{X}) = \begin{pmatrix} X_{n,0}, \dots, X_{n,k} \\ X_{1,0}, \dots, X_{1,k} \\ \vdots \quad \vdots \\ X_{n-1,0}, \dots, X_{n-1,k} \end{pmatrix},$$

and postcomposing with the analogous permutations

$$V(\underline{X};\underline{A}) \to \begin{pmatrix} P(n)_{A_{n,1},A_{n,0}}(S^{X_{n,0}}) \wedge \cdots \wedge F_{A_{1,0},A_{n,k}}(S^{X_{n,k}}) \\ P(1)_{A_{1,1},A_{1,0}}(S^{X_{1,0}}) \wedge \cdots \wedge F_{A_{2,0},A_{1,k}}(S^{X_{1,k}}) \\ \vdots \quad \vdots \\ P(n-1)_{A_{n-1,1},A_{n-1,0}}(S^{X_{n-1,0}}) \wedge \cdots \wedge F_{A_{n,0},A_{n-1,k_n}}(S^{X_{n-1,k}}) \end{pmatrix}.$$

It is sometimes convenient to work with a slightly modified version of U^n we which we will temporarily write as $\tilde{U}^n$. The only difference between $\tilde{U}^n$ and U^n is that for $\tilde{U}^n$, one takes the homotopy colimit over the diagonal of $(I^{k+1})^{\times n}$ in simplicial dimension k, rather than all of $(I^{k+1})^{\times n}$ as we do for U^n. The face and degeneracy maps restrict to this sublimit system, and so does the simplicial isomorphism t. The natural map of simplicial functors with stabilization $\tilde{U}^n \to U^n$ determined by the inclusion of the subcategory is always an equivalence, by finality and the realization lemma.

Definition 4.1 For F an FSP and P a bimodule, we define $U^n(F;P)$ to be the simplicial functor with stabilization with C_n–action (given by t) $\tilde{U}^n(F;P,\ldots,P)$. Thus, U^n is a functor from the category of pairs $(F;P)$ (an FSP F and an F–bimodule P) with morphisms of pairs as in Definition 1.6 to simplicial functors with stabilization with C_n–action.

Remark 4.2 $U^1(F;P)$ is just $\mathrm{THH}(F;P)$: the topological Hochschild homology of F with coefficients in P as defined in [6] which was a straightforward generalization of the definition found in Pirashvili and Waldhausen [20] to FSP's with several objects. The spectrum $\tilde{U}^n(F;F,\ldots,F)$ is isomorphic (as simplicial functors with stabilization with C_n–action) to the n–th edgewise subdivision of $\mathrm{THH}(F;F)$.

Lemma 4.3 *For every $m|n$, the n–fold simplicial isomorphism*

$$\alpha_m\colon U^n(F;P,\ldots,P) \to U^{n/m}(F;P^{\hat{\otimes} m},\ldots,P^{\hat{\otimes} m})$$

of Lemma 2.6 is $C_{n/m}$–equivariant if we take diagonals. The composite of the equivalence $\tilde{U}^n(F;P,\ldots,P) \xrightarrow{\simeq} U^n(F;P)$ with the diagonal of α_m factors to produce a $C_{n/m}$–equivariant map:

$$
\begin{array}{ccc}
\tilde{U}^n(F;P,\ldots,P) & \xrightarrow{\tilde{\alpha}_m} & \tilde{U}^{m/n}(F;P^{\hat{\otimes}_F^m},\ldots,P^{\hat{\otimes}_F^m}) \\
\Big\downarrow {\scriptstyle =} & & \Big\downarrow {\scriptstyle =} \\
U^n(F;P) & \xrightarrow{\alpha_m} & U^{n/m}(F;P^{\hat{\otimes}_F^m})
\end{array}
$$

which is an equivalence and natural with respect to morphism pairs.

Proof This is formally true from the definitions. Given a simplicial C_n–set X_*, the C_n fixed point space of the realization of X_* is homeomorphic to the realization of the simplicial set $X_*^{C_n}$ obtained by taking fixed points degreewise. The same is true for a simplicial C_n–CW–complex (or anything C_n–equivariantly homotopy equivalent to one in each degree) and hence one can compute $U^n(F; P)(X)^{C_m}$ degreewise for every $m|n$. Similarly, since our model for homotopy colimits is given by simplicial spaces, we see that if E is a functor from a small category $\mathcal{C}$ to C_n–equivariant CW–complexes (or spaces C_n–equivariantly equivalent to them) then $(\operatorname{hocolim}_{C \in \mathcal{C}} E(C))^{C_n} \cong \operatorname{hocolim}_{C \in \mathcal{C}} E(C)^{C_n}$. Thus, $U^n(F; P)^{C_m}$ for $m|n$ is naturally homeomorphic to the realization of

$$[k] \mapsto U^n(F; P)_k^{C_m} \cong \operatorname{hocolim}_{\underline{X} \in I^{k+1}} \underline{\operatorname{Map}}\left((S^{\sqcup \underline{X}})^{\wedge n}, \bigvee_{\underline{A} \in (\mathcal{A}^{k+1})^{\times n}} V(\underline{X}^{\times n}; \underline{A})\right)^{C_m}.$$

We note further, that with the specified C_n–action, we have natural $C_n/C_m \cong C_{n/m}$ equivariant homeomorphisms

$$((S^{\sqcup \underline{X}})^{\wedge n})^{C_m} \cong (S^{\sqcup \underline{X}})^{\wedge n/m},$$

$$\text{(4-1)} \qquad \left(\bigvee_{\underline{A} \in (\mathcal{A}^{k+1})^{\times n}} V(\underline{X}^{\times n}; \underline{A})\right)^{C_m} \cong \bigvee_{\underline{A} \in (\mathcal{A}^{k+1})^{\times n/m}} V(\underline{X}^{\times n/m}; \underline{A}).$$

We recall that if G is a group with normal subgroup H, and X and Y are G–spaces, then there is a continuous map from $\operatorname{Hom}(X, Y)^G$ to $\operatorname{Hom}(X^H, Y^H)^{G/H}$ given by the restriction from X to X^H,

$$\operatorname{Hom}(X, Y)^G \overset{\operatorname{res}^H}{\to} \operatorname{Hom}(X^H, Y)^G = \operatorname{Hom}(X^H, Y^H)^{G/H}. \qquad \square$$

Definition 4.4 For r, s and t integers greater than 0, we let

$$\operatorname{Res}^r \colon U^{rst}(F; P)^{C_{rs}} \to U^{st}(F; P)^{C_s}$$

be the map of simplicial functors with stabilization with C_t–action defined degreewise by applying res^{C_r} and making the appropriate identifications by equation (4-1). Thus,

$$\operatorname{Res}^1 = \operatorname{id},$$

$$\operatorname{Res}^r \operatorname{Res}^s = \operatorname{Res}^{r \cdot s} = \operatorname{Res}^s \operatorname{Res}^r.$$

Lemma 4.5 *For r, s and t integers greater than 0 the following diagram commutes:*

$$
\begin{array}{ccc}
U^{rst}(F;P)^{C_{rs}} & \xrightarrow{\mathrm{Res}^r} & U^{st}(F;P)^{C_s} \\
\downarrow{\scriptstyle \alpha_t} & & \downarrow{\scriptstyle \alpha_t} \\
U^{rs}(F;P^{\widehat{\otimes}t})^{C_{rs}} & \xrightarrow{\mathrm{Res}^r} & U^s(F,P^{\widehat{\otimes}t})^{C_s}.
\end{array}
$$

Proof As in the previous discussion, we check the claim simplicially.

$$
U^{rst}(F;P)_k^{C_{rs}} \cong \mathrm{hocolim}_{\underline{X} \in I^{k+1}} \underline{\mathrm{Map}}\left((S^{\sqcup \underline{X}})^{\wedge rst}, \bigvee_{\underline{A} \in (\mathcal{A}^{k+1})^{\times rst}} V(\underline{X}^{\times rst}; \underline{A}) \right)^{C_{rs}}.
$$

We consider the model of $P^{\widehat{\otimes}t}$ arising from the diagonal of the $(t-1)$–simplicial construction $(\cdots((P \widehat{\otimes}_F P) \widehat{\otimes}_F P) \widehat{\otimes}_F \cdots) \widehat{\otimes}_F P$. Also, we consider the simplicial model of $U^{rs}(F;P^{\widehat{\otimes}t})$ which is the diagonal on the simplicial structure of U and all the simplicial structures of the $P^{\widehat{\otimes}t}$ simultaneously. Then α_t is induced by grouping together the first $(t-1)(k+1)+1$ coordinates in each $t(k+1)$–tuple of coordinates, and sending them to the corresponding coordinate in $P^{\widehat{\otimes}t}$.

Therefore, if instead of looking at C_{rs}–equivariant maps on the full $(S^{\sqcup \underline{X}})^{\wedge rst}$ we look at their restrictions to the C_r fixed point in the domain, which must land in that part of the range whose coordinates repeat themselves in r blocks of $ts(k+1)$, the effect of grouping together before or after the r–fold repetition is the same. $\qquad \square$

Let $\mathbb{N}^\times$ be the natural numbers $\{1, 2, \ldots\}$ as a partially ordered set with $n < m \Leftrightarrow m|n$. For F an FSP and P an F bimodule, we have a functor from $\mathbb{N}^\times$ to functors with stabilization sending every natural number n to $U^n(F;P)^{C_n}$ and every morphism $n < m$ to $\mathrm{Res}^{n/m}$.

Definition 4.6 Let F be an FSP and P an F–bimodule. For $\mathcal{M}$ a subcategory of $\mathbb{N}^\times$, we set

$$
W_{\mathcal{M}}(F;P) = \mathrm{holim}_{n \in \mathcal{M}} \ U^n(F;P)^{C_n}.
$$

We will use simplified notation for various distinguished subcategories of $\mathbb{N}^\times$ as follows. First, we set

$$
W(F;P) = W_{\mathbb{N}^\times}(F;P).
$$

We let $\{\leq n\}$ be the full subcategory of $\mathbf{N}^\times$ generated by $\{1, 2, \ldots, n\}$ and write

$$
W_n(F;P) = W_{\{\leq n\}}(F;P).
$$

We let (p) be the full subcategory of $\mathbb{N}^\times$ generated by the powers of p, $(p) = \{1, p, p^2, \ldots\}$ and write

$$
W^{(p)}(F;P) = W_{(p)}(F;P)
$$

(we use this notation so we can have room for a subscript). We also look at the full subcategory of (p) generated by $\{1, p, \ldots, p^n\}$ and write

$$W_n^{(p)}(F; P) = W_{\{1,p,\ldots,p^n\}}(F; P).$$

Definition 4.7 For n a natural number, we let

$$W_n(F; P) \overset{R_n}{\to} W_{n-1}(F; P), \quad W_n^{(p)}(F; P) \overset{R_n^{(p)}}{\to} W_{n-1}^{(p)}(F; P)$$

be the natural maps obtained by restriction to subcategories. Thus,

$$W(F; P) = \mathrm{holim}_{\infty \leftarrow n} W_n(F; P), \quad W^{(p)}(F; P) = \mathrm{holim}_{\infty \leftarrow n} W_n^{(p)}(F; P)$$

with structure maps given by the R_n's and $R_n^{(p)}$'s, respectively.

5 The fiber of R_n and $R_n^{(p)}$

Our goal in this section is to identify the fiber of the maps R_n and $R_n^{(p)}$ up to natural equivalence with $U_{hC_n}^n$ and $U_{hC_{p^n}}^{p^n}$. This was essentially done by Goodwillie in the appendix to his MSRI notes [8]. Since these MSRI notes are not published, in this section we reproduce what is needed from them (Definition 5.4, Proposition 5.5 and Theorem 5.6 below) to establish the result. We have modified some of the constructions found in [8] to make the proofs more transparent.

Let G be a group. Recall that for X a (pointed) space with G–action, the *homotopy orbit* space of X is $X_{hG} = X \wedge_G EG_+ = (X \wedge EG_+)_G$; this is the homotopy colimit of the diagram consisting of all the elements of G acting on X. Therefore if $f : X \to Y$ is an n–connected G–equivariant map then f_{hG} is also n–connected. We note that if F is a functor with stabilization with G–action, then $X \mapsto F(X)_{hG}$ is again naturally a functor with stabilization.

The *homotopy fixed-point* space of X is $X^{hG} = \mathrm{map}_G(EG_+, X) = \mathrm{map}_*(EG_+, X)^G$. If $f : X \to Y$ is a G–equivariant map which is also an equivalence, then f^{hG} is also an equivalence but $(\ )^{hG}$ does *not* preserve connectivity in general. Thus, if F is a functor with stabilization then $X \mapsto F(X)^{hG}$ satisfies Definition 1.1(i) but not necessarily Definition 1.1(ii), (iii) or (iv) and hence is not again a functor with stabilization.

Definition 5.1 A *functor with structure* will be a functor F from $\mathcal{S}_*$ to $\mathcal{S}_*$ together with a natural transformation

$$\lambda_{X,Y} : X \wedge F(Y) \to F(X \wedge Y)$$

which satisfies Definition 1.1(i) but not necessarily Definition 1.1(ii), (iii) or (iv). A *functor with structure over* $\mathcal{O}$ for a set $\mathcal{O}$ is a functor from $\mathcal{S}_* \times \mathcal{O} \times \mathcal{O}$ to $\mathcal{S}_*$ such that for all $A, B \in \mathcal{O}$, $F_{A,B}(\ )$ is a functor with structure.

Definition 5.2 Let G be a group and F a functor with stabilization with G–action. We define the *homotopy orbits* of F to be the functor with stabilization

$$F_{hG}\colon X \mapsto \operatorname{hocolim}_m \Omega^m [F(\Sigma^m X)]_{hG}$$

and the *homotopy fixed-points* of F to be the functor with structure

$$F^{hG}\colon X \mapsto \operatorname{hocolim}_m \Omega^m [\operatorname{hocolim}_\ell \Omega^\ell F(\Sigma^{m+\ell} X)]^{hG}.$$

Remark 5.3 Since homotopy orbits are themselves a homotopy direct limit, the obvious map $|(F_*)_{hG}| \to |(F_*)|_{hG}$ is an equivalence; but the analogous map $|(F_*)^{hG}| \to |F_*|^{hG}$ is generally not an equivalence. That is, homotopy fixed points do not in general commute with realizations.

We will now assume that G is a *finite* group, and define the *Tate map*, a chain of natural maps of functors with structure from F_{hG} to F^{hG}. Before doing that, we establish a sequence of natural equivalences

$$(G_+ \wedge F)_{hG} \simeq \Omega^\infty F \simeq (G_+ \wedge F)^{hG}.$$

For X a G–space, we let γ be the G–equivariant map

$$G_+ \wedge X \cong \bigvee_G X \overset{\operatorname{incl}}{\to} \prod_G X \cong \operatorname{map}(G_+, X),$$

that is,

$$\gamma(g \wedge x)(u) = \begin{cases} x & \text{if } g = u, \\ * & \text{otherwise.} \end{cases}$$

If X is k–connected, then γ is $(2k-1)$–connected by Blakers–Massey and we obtain the diagram

(5-1)

$$
\begin{array}{ccc}
(G_+ \wedge X)_G & \overset{\cong}{\longleftarrow} X \overset{\cong}{\longrightarrow} & \operatorname{map}(G_+, X)^G \\
\simeq \uparrow & & \downarrow \simeq \\
(G_+ \wedge X)_{hG} & & \operatorname{map}(G_+, X)^{hG} \\
\gamma_{hG} \downarrow & & \uparrow \gamma^{hG} \\
\operatorname{map}(G_+, X)_{hG} & & (G_+ \wedge X)^{hG}.
\end{array}
$$

Since γ is $(2k-1)$–connected, so is γ_{hG}; but we do not know anything about the connectivity of γ^{hG}. If, however, F is functor with stabilization with G–action then γ induces an equivalence on the spectra associated to $G_+ \wedge F \to \mathrm{map}(G_+, F)$, and then γ^{hG} *is* a stable equivalence, that is, for all X we get that the composite map

$$\text{(5-2)} \qquad \begin{array}{c} [\mathrm{hocolim}_n \Omega^n (G_+ \wedge F(\Sigma^n X))]^{hG} \\ \Big\downarrow \gamma^{hG} \\ [\mathrm{hocolim}_n \Omega^n \ \mathrm{map}(G_+, F(\Sigma^n X)]^{hG} \\ \Big\downarrow \cong \\ [\mathrm{hocolim}_n \ \mathrm{map}(G_+, \Omega^n F(\Sigma^n X)]^{hG} \end{array}$$

is an equivalence. We also note that if G is a finite group, the natural G–equivariant map

$$\text{(5-3)} \qquad \mathrm{hocolim}_n \ \mathrm{map}(G_+, \Omega^n F(\Sigma^n X)) \xrightarrow{\simeq} \mathrm{map}(G_+, \mathrm{hocolim}_n \ \Omega^n F(\Sigma^n X))$$

is an equivalence. Thus, we can assemble all these remarks to obtain the following sequence of natural equivalences of spaces for G finite:

$$\text{(5-4)} \qquad \begin{array}{c} (G_+ \wedge F)_{hG}(X) = \mathrm{hocolim}_n \Omega^n[(G_+ \wedge F(\Sigma^n X)_{hG}] \\ \text{by (5-1)} \Big\downarrow \simeq \\ \Omega^\infty F(X) = \mathrm{hocolim}_n \Omega^n F(\Sigma^n X)] \\ \text{by (5-1)} \Big\downarrow \simeq \\ [\mathrm{map}(G_+, \mathrm{hocolim}_n \Omega^n F(\Sigma^n X)]^{hG} \\ \text{by (5-3)} \Big\uparrow \simeq \\ [\mathrm{hocolim}_n \Omega^n \ \mathrm{map}(G_+, F(\Sigma^n X))]^{hG} \\ \text{by (5-2)} \Big\uparrow \simeq \\ (G_+ \wedge F)^{hG}(X) = [\mathrm{hocolim}_n \Omega^n (G_+ \wedge F(\Sigma^n X))]^{hG} \end{array}$$

The map on the last line is an equivalence since the previous lines show we already have an omega-spectrum. The maps in (5-4) assemble (as X varies) into a natural sequence of equivalences of functors with stabilization with G–action.

The Tate map uses (5-4) to construct a "map" from homotopy quotients to homotopy fixed-points for a general functor with stabilization F. It uses the fact that for any

G–space A, there is a homotopy equivalence $EG_+ \wedge A \xrightarrow{\simeq} A$ which is G–equivariant to obtain natural equivalences on G–homotopy orbits and homotopy fixed points. Since EG_+ is the realization of a simplicial G–set $[q] \mapsto \bigwedge^{q+1} G_+$ (the simplicial path space of the bar construction for G, with G acting on the 0–th coordinate),

$$EG_+ \wedge F \cong_G \left| [q] \mapsto \overset{q+1}{\bigwedge} G_+ \wedge F \right|.$$

Definition 5.4 (Goodwillie)   The *Tate "map"* is the following natural diagram:

$$
\begin{array}{c}
F_{hG} \\
\Big\uparrow {\scriptstyle \simeq} \\
(EG_+ \wedge F)_{hG} \\
\Big\downarrow {\scriptstyle \cong} \\
\left| [q] \mapsto (\bigwedge^{q+1} G_+ \wedge F) \right|_{hG} \\
\Big\uparrow {\scriptstyle \cong} \\
\left| [q] \mapsto (\bigwedge^{q+1} G_+ \wedge F)_{hG} \right| \\
{\scriptstyle \text{by (5-4)}} \Big\downarrow {\scriptstyle \simeq} \\
\left| [q] \mapsto (\bigwedge^{q+1} G_+ \wedge F)^{hG} \right| \\
\Big\downarrow \\
\left| [q] \mapsto (\bigwedge^{q+1} G_+ \wedge F) \right|^{hG} \\
\Big\downarrow {\scriptstyle \cong} \\
\left| EG_+ \wedge F \right|^{hG} \\
\Big\downarrow {\scriptstyle \simeq} \\
F^{hG}
\end{array}
$$

(In the middle step, $\bigwedge^{q+1} G_+ \wedge F$ should be viewed as $G_+ \wedge \bigwedge^q G_+ \wedge F$.)

There is one case where the Tate map is easily seen to be an equivalence: when E is a functor with stabilization with G–action and $F = G_+ \wedge E$. This follows from the commuting diagram (using the maps induced by the projection map π which forgets

the EG coordinate; they are equivalences since EG is contractible)

(5-5)

$$
\begin{array}{ccc}
|[q] \mapsto [\bigwedge^{q+1} G_+ \wedge (G_+ \wedge E)]_{hG}| & \xrightarrow{\pi_*} & (G_+ \wedge E)_{hG} \\
\text{by (5-4)} \downarrow \simeq & & \downarrow \simeq \text{ by (5-4)} \\
|[q] \mapsto [\bigwedge^{q+1} G_+ \wedge (G_+ \wedge E)]^{hG}| & & \\
\downarrow & & \\
|[q] \mapsto [\bigwedge^{q+1} G_+ \wedge (G_+ \wedge E)]|^{hG} & \xrightarrow{\pi_*} & (G_+ \wedge E)^{hG}.
\end{array}
$$

Another case which follows from this one is that of functors with stabilization of the form $\mathrm{map}(G_+, E)$, where we have the stable equivalences of homotopy orbits γ_{hG} from (5-1) and of homotopy fixed-points γ^{hG} from (5-2).

Proposition 5.5 (Goodwillie) *If F is either $U \wedge E$ or $\mathrm{map}(U, E)$, where E is a functor with stabilization with G–action and U is a pointed finite free G–space (ie, a simplicial G–set with finitely many nondegenerate nonbasepoint simplices permuted freely by G), then the Tate map for F is an equivalence.*

The proof is by induction over skeleta; the cells attached at stage n are dealt with by applying the above discussion to the case $(E \wedge \bigvee^t S^n) \wedge G_+$ and map $(\bigvee^t S^n \wedge G_+, E)$, using the fact that after we apply Ω^∞, cofibrations become fibrations.

Theorem 5.6 (Goodwillie) *Let U be a free finite based G–complex of dimension n and W a $(n-1)$–connected based G–complex. Then the spectrum associated to the functor with stabilization $\underline{\mathrm{Map}}(U, W)^G$ is naturally equivalent to that associated to $\underline{\mathrm{Map}}(U, W)_{hG}$.*

Proof Consider the diagram

$$
\begin{aligned}
\underline{\mathrm{Map}}(U, W)^G &\xrightarrow{\alpha} \underline{\mathrm{Map}}(U, \mathrm{hocolim}_k \, \Omega^k (S^k \wedge W))^G \\
&\xrightarrow{\beta} \underline{\mathrm{Map}}(U, \mathrm{hocolim}_k \, \Omega^k (S^k \wedge W))^{hG} \\
&\xleftarrow{\gamma} (\mathrm{hocolim}_k \, \Omega^k (\underline{\mathrm{Map}}(U, (S^k \wedge W))^{hG}) \\
&\xrightarrow{\delta} \mathrm{hocolim}_k \, \Omega^k (\underline{\mathrm{Map}}(U, (S^k \wedge W))_{hG}).
\end{aligned}
$$

The first map, α, is induced by the inclusion $W \to \mathrm{hocolim}_k \, \Omega^k (S^k \wedge W)$. Since W is $(n-1)$–connected this map is $(2n-1)$–connected. Since U is a free G–space of dimension n, the map α itself is then $(n-1)$–connected. The second map, β, is

the canonical map from fixed points to homotopy fixed points. It is an equivalence because this is always so for function spaces $\mathrm{map}(U, ?)$ where U and ? are G–spaces and U is free. The third map, γ, is an equivalence because U is a finite complex; this has nothing to do with the G–action. The fourth map, δ, is the Tate map, and is an equivalence by Proposition 5.5. $\square$

Definition 5.7 Let $\mathcal{M} \subset \mathbb{N}^{\times}$ be a full subcategory, and let $M \in \mathcal{M}$. Let $p_1, , \ldots, p_t$ be the distinct prime divisors of M. For $U \subseteq \{1, \ldots, t\}$, we let $\langle U \rangle = \prod_{u \in U} p_u$ (with $\langle \varnothing \rangle = 1$). Assume $M/\langle U \rangle \in \mathcal{M}$ for all $U \subseteq \{1, \ldots, t\}$, and let $\vec{M}$ denote the full subcategory of $\mathcal{M}$ with objects $\{M/\langle U \rangle \,|\, U \subseteq \{1, \ldots, t\}\}$. We define $\vec{M} - M$ to be the full subcategory of $\vec{M}$ generated by all the objects *except* M.

For the rest of this section, we will assume that $\mathcal{M}$ is a full subcategory of $\mathbb{N}^{\times}$ and $M \in \mathcal{M}$ is such that $\mathcal{M}$ is covered by the object of, and compositions of the morphisms of, $\vec{M}$ and $\mathcal{M} - M$ (that is, there does not exist an $M' \in \mathcal{M} - M$ such that $M | M'$).

Since $\vec{M} - M$ is the intersection of $\vec{M}$ and $\mathcal{M} - M$, for any functor F from $\mathcal{M}$ to functors with stabilization the following natural diagram is homotopy cartesian:

$$
\begin{array}{ccc}
\mathrm{holim}_{\mathcal{M}} F & \longrightarrow & \mathrm{holim}_{\mathcal{M}-M} F \\
\downarrow & & \downarrow \\
\mathrm{holim}_{\vec{M}} F & \longrightarrow & \mathrm{holim}_{\vec{M}-M} F
\end{array}
$$

Since M is initial in $\vec{M}$, the natural map $F(M) \to \mathrm{holim}_{\vec{M}} F$ is an equivalence and the homotopy fiber of the composite $F(M) \to \mathrm{holim}_{\vec{M}-M} F$ is naturally equivalent to the total fiber of the t–dimensional cube determined by F on $\vec{M}$ (see [9], Definition 1.1(b).

If we consider the functor $n \mapsto U^n(F; P)^{C_n}$ from Definition 4.1, we see that the homotopy fiber of the restriction map from $W_{\mathcal{M}}(F; P)$ to $W_{\mathcal{M}-M}(F; P)$ is naturally equivalent to the total fiber of the t–dimensional cube determined by this functor on $\vec{M}$. and since this functor takes values in simplicial functors with stabilization, this total fiber is naturally equivalent to the realization of the total fibers computed in each simplicial dimension separately.

Proposition 5.8 If $M \in \mathcal{M}$ is such that M does not divide any elements of $\mathcal{M}$ other than itself, then there is a natural (in F and P) chain of equivalences of functors with stabilization

$$
U^M(F; P)_{hC_M} \simeq \mathrm{hofib}[W_{\mathcal{M}}(F; P) \to W_{\mathcal{M}-M}(F; P)].
$$

Proof For $\underline{X} \in I^{\vec{k}+1}$, set

$$Z = (S^{\underline{X}})^{\wedge M},$$
$$Y = V(\underline{X}, \mathcal{O})^{\wedge M},$$

(recall Definition 2.2), and $x = \sum_{i=0}^{k} |X_i|$.

By the above remark,

$$\mathrm{hofib}[W_{\mathcal{M}}(F; P) \to W_{\mathcal{M}-M}(F; P)] \simeq \mathrm{hofib}[W_{\vec{M}}(F; P) \to W_{\vec{M}-M}(F; P)].$$

This is the total fiber of the t–dimensional cube

$$\mathcal{X}(U) = U^{M/\langle U \rangle}(F; P)^{C_{M/\langle U \rangle}} = \underline{\mathrm{Map}}(Z^{C_{\langle U \rangle}}, Y)^{C_M}$$

with maps by restriction to fixed subsets. By applying the functor $\underline{\mathrm{Map}}(\ , Y)^{C_M}$, we see that this is the same as $\underline{\mathrm{Map}}(\mathcal{U}, Y)^{C_M}$ for $\mathcal{U}$ the total cofiber of the t–dimensional cube with $\mathcal{Y}(U) = Z^{C_{\langle U \rangle}}$ and structure maps given by inclusion. Thus, $\mathcal{U} = Z/Z'$ where Z' is the pushout of the diagram

$$\mathcal{Z}(U) = (S^{\underline{X}})^{\wedge(M/\langle U \rangle)}, \quad U \subseteq \{1, \ldots, n\}, \ U \neq \varnothing,$$

with the maps given by inclusions.

Now $\mathcal{U}$ is a finite free based C_M–space of dimension Mx. Since Y is $(Mx-1)$–connected, $\underline{\mathrm{Map}}(\mathcal{U}, Y)^{C_M}$ is naturally equivalent to $\underline{\mathrm{Map}}(\mathcal{U}, Y)_{hC_M}$ by Theorem 5.6. The quotient map $Z \to \mathcal{U}$ produces a natural map

$$\underline{\mathrm{Map}}(\mathcal{U}, Y)_{hC_M} \xrightarrow{\epsilon} \underline{\mathrm{Map}}(Z, Y)_{hC_M}.$$

Since $\dim(Z') = \max\{Mx/p_i | 1 \leq i \leq n\} = Mx/p$ (for p the smallest prime divisor of M) and $S^\ell \wedge Y$ is $(\ell + Mx - 1)$–connected, the map

$$\underline{\mathrm{Map}}(\mathcal{U}, (S^\ell \wedge Y)) \to \underline{\mathrm{Map}}(Z, (S^\ell \wedge Y))$$

is $(\ell + Mx(1-1/p)-1)$–connected, hence ϵ is $(Mx(1-1/p)-1)$–connected. Since the quotient map from Z to $\mathcal{U}$ is functorial in $I^{\vec{k}+1}$, we can take the homotopy colimit with respect to $I^{\vec{k}+1}$ and hence obtain an equivalence

$$\mathrm{hocolim}_{I^{\vec{k}+1}} \underline{\mathrm{Map}}(\mathcal{U}, (S^\ell \wedge Y)) \to \mathrm{hocolim}_{I^{\vec{k}+1}} \underline{\mathrm{Map}}(Z, (S^\ell \wedge Y)).$$

Thus, we have obtained a natural sequence of equivalences

$$(U^M(F; P)_{[k]})_{hM} \simeq \mathrm{hofib}[W_{\mathcal{M}}(F; P)_{[k]} \to W_{\mathcal{M}-M}(F; P)_{[k]}],$$

and these maps respect the simplicial operators. $\square$

Corollary 5.9 *For any FSP F and bimodule P there is a natural chain of equivalences of functors*

$$U^n(F; P)_{hC_n} \simeq \mathrm{hofib}[W_n(F; P) \xrightarrow{R} W_{n-1}(F; P)],$$

$$U^{p^n}(F; P)_{hC_{p^n}} \simeq \mathrm{hofib}[W_n^{(p)}(F; P) \xrightarrow{R} W_{n-1}^{(p)}(F; P)].$$

Corollary 5.10 *Let P be a $(k-1)$–connected F–bimodule. The natural map*

$$W(F; P) \xrightarrow{R} W_n(F; P)$$

is $((n+1)k-2)$–connected and the natural map

$$W^{(p)}(F; P) \xrightarrow{R^{(p)}} W_n^{(p)}(F; P)$$

is $(p^{n+1}k-2)$–connected.

Proof By Lemma 3.5, $U^n(F; P)$ is $kn+(n-1)$–connected for all $n \geq 1$. Since homotopy orbits preserve connectivity the result follows from Corollary 5.9. □

6 Additional properties of U^n

In this section we will develop several useful properties of U^n (and therefore of W) which will be needed to define the map from algebraic K–theory to W, and which can also be useful in calculations. Two properties of particular importance are: Proposition 6.13 allows us to use interchangeably the definition of $U^n(R; M)$ as in Examples 1.4(ii), defined over a category consisting of a single point, which generalizes the usual definition of $\mathrm{THH}(R; M)$ and is much easier to use for explicit calculations, and the version $U^n(\underline{\mathcal{P}_R}, \underline{M})$ defined over the category of finitely generated projective R–modules, which generalizes the version of THH used in [5; 6]. Corollary 6.17 then allows us to build $U^n(\underline{\mathcal{P}_R}, \underline{M})$ by using Waldhausen's S–construction on a smaller model than the usual one—a model that is more similar to that used in defining $K(R; M)$.

These results are all analogous to results from [6], where they were proved for the THH (ie U^n for $n = 1$) case, with or without coefficients in a bimodule. We will restrict our attention to FSP's over small categories (note that in the [6] nomenclature, what we here call an FSP is a unital ring functor; they reserve the name FSP for the case where the category in question consists of a single point, as in Bökstedt's original definition). Later in the section we will restrict ourselves further to small linear categories (where the homomorphism set between any two objects is an abelian group and composition is bilinear).

The proofs from [6] can be adapted as explained below to give homotopy equivalences between $U^n(F; P_1, \ldots, P_n)$ for different FSP's F over categories $\mathcal{C}$ and F–bimodules $P_1, \ldots, P_n$ (the proofs that the maps are indeed homotopy equivalences work when the bimodules are distinct; we mention only the case $P_1 = \cdots = P_n$, which is what we will mostly use, in order to simplify notation). In all the cases, there are maps inducing these homotopy equivalences which are naturally C_n–equivariant when $P_1 = \cdots = P_n$. This turns out to be enough to show that they are C_n–equivalences: Using the fundamental sequence of Proposition 5.8 (for $\mathcal{M}$ equal to all of n's divisors), by induction on n (since a C_n–equivariant map which is a homotopy equivalence induces a homotopy equivalence on the C_n homotopy quotient) we can see that they induce an equivalence on the C_n fixed points of U^n. Using groupings $U^n(F; P) \simeq U^m(F; P^{\hat{\otimes}_F n/m})$ as in Lemma 2.6 we can get an equivalence of the C_m fixed points for any $m|n$ in a similar way. Once we know that the maps in each of these claims are C_n equivalences, it follows (except in Lemma 6.4 where the map is from a direct limits of U^n's) that they induce equivalences on W and all its variants as well.

If F is an FSP over a small category $\mathcal{C}$ and P an F–bimodule, then for any small category $\mathcal{D}$ and functor $\phi\colon \mathcal{D} \to \mathcal{C}$ we can get an FSP $\phi^* F$ over $\mathcal{D}$ by letting $\phi^* F_{d,d'}(X) = F_{\phi(d),\phi(d')}(X)$ for all $d, d' \in \mathcal{D}$, $X \in \mathcal{S}_*$. We can similarly define the $\phi^* F$–bimodule $\phi^* P$.

Lemma 6.1 *Let $\phi_1, \phi_2\colon \mathcal{D} \to \mathcal{C}$ be two naturally isomorphic functors between small categories, and let F be an FSP over $\mathcal{C}$ and P an F–bimodule. Then the natural isomorphism induces a C_n–homeomorphism*

$$U^n(\phi_1^*(F); \phi_1^*(P)) \xrightarrow{\cong} U^n(\phi_2^*(F); \phi_2^*(P))$$

for all n and therefore a homeomorphism on W.

Proof Analogous to that of [6, Lemma 1.6.2]: the natural isomorphism η induces an equivalence

$$F_{\eta(a)^{-1}, \eta(b)}(\mathrm{id}_X)\colon F_{\phi_1(a), \phi_1(b)}(X) \xrightarrow{\cong} F_{\phi_2(a), \phi_2(b)}(X)$$

for all X, and similarly for P. These are compatible with the multiplicative structure. □

Proposition 6.2 *Let $\phi\colon \mathcal{D} \to \mathcal{C}$ be an equivalence of categories, and let F be an FSP over $\mathcal{C}$ and P an F–bimodule. Then ϕ induces a C_n–equivalence*

$$U^n(\phi^*(F); \phi^*(P)) \xrightarrow{\cong} U^n(F; P)$$

and therefore an equivalence on W.

Proof   Use the natural transformation $\psi\colon \mathcal{C}\to\mathcal{D}$ so that both compositions are naturally isomorphic to the identity and Lemma 6.1, as in the proof of [6, Lemma 1.6.6].   □

Example 6.3   Let $\mathcal{C}$ and $\mathcal{D}$ be small linear categories, with an equivalence of categories $\phi\colon \mathcal{D}\to\mathcal{C}$. Then we can look at the FSPs $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$, as defined in Examples 1.4(i). There is a natural isomorphism between the FSPs $\phi^*\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ on $\mathcal{D}$: Whenever the identity is naturally isomorphic to a functor $F\colon \mathcal{E}\to\mathcal{E}$ then $F^*\colon \mathrm{Hom}_{\mathcal{E}}(e_1, e_2)\to \mathrm{Hom}_{\mathcal{E}}(F(e_1), F(e_2))$ is one-to-one and onto for all $e_1, e_2\in\mathcal{E}$ (because if $H\colon \mathrm{id}_{\mathcal{E}}\to F$ is a natural isomorphism, for every $\alpha\colon e_1\to e_2$, $F(\alpha)=H(e_2)\alpha H(e_1)^{-1}$). So we can use this on $\phi^*\colon \mathrm{Hom}_{\mathcal{D}}(d_1, d_2)\to \mathrm{Hom}_{\mathcal{C}}(\phi(d_1), \phi(d_2))$ and $\psi^*\colon \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)\to \mathrm{Hom}_{\mathcal{D}}(\psi(c_1), \psi(c_2))$ first to establish that ϕ^* is one-to-one on morphism sets and ψ^* onto, and then in the opposite order to establish that ψ^* is one-to-one on morphism sets and ϕ^* onto.

Now say we have two functors $F, G\colon \mathcal{C}\to\mathcal{B}$ and a $\underline{\mathcal{C}}$–bimodule of the form $P_{a,b}(X)=\mathcal{B}(F(a), G(b))\otimes_{\mathbb{Z}}\widetilde{\mathbb{Z}}[X]$ (see Examples 1.7(iii) above). Then ϕ^*P is a bimodule on the same form on $\mathcal{D}$, corresponding to the functors $F\circ\phi$ and $G\circ\phi$, and we get a C_n–equivalence

$$U^n(\underline{\mathcal{D}}; \phi^*P) \xrightarrow{\;\cong\;} U^n(\underline{\mathcal{C}}; P).$$

We now want to show that our construction of U^n commutes with direct limits. Call the category we are working on $\mathcal{C}$, and assume that there is a directed set of subcategories $\mathcal{C}_j$ of $\mathcal{C}$, $j\in J$, such that for any object $c\in\mathcal{C}$ there is $j\in J$ with $c\in\mathcal{C}_j$. If J satisfies this condition, we say that it is a *saturated* directed set in $\mathcal{C}$. Note that this condition is really a condition on the underlying sets of the small categories involved.

Lemma 6.4   *If J is a saturated directed set in $\mathcal{C}$, and F is an FSP on $\mathcal{C}$ with P an F–bimodule, then we have a C_n–equivalence*

$$\lim_{j\in J} U^n(F|_{C_j}; P|_{C_j}) \xrightarrow{\;\cong\;} U^n(F; P).$$

Proof   As in the proof of [6, Lemma 1.6.9], this follows from the fact that any map from a sphere $S^{\sqcup\underline{X}}$ (which is compact) to $V(\underline{X}, \mathcal{C})$ (see Definition 2.3 above) has a compact image, and therefore can intersect only finitely many summands which each involve only finitely many elements of $\mathcal{C}$, by commuting colimits and homotopy colimits. The C_n equivalence of U^n is proved as usual, but note that this lemma does not imply a similar result for W because of the problem of commuting direct and inverse limits.   □

Lemma 6.5 *If we have FSPs F and F' over $\mathcal{C}$ and bimodules P and P' over F and F' , respectively, and a map $(f, g)\colon (F; P) \to (F'; P')$ (see Definition 1.6) so that f and g induce a stable equivalence on the associated spectra $\mathbf{F} \overset{\simeq}{\to} \mathbf{F}'$, $\mathbf{P} \overset{\simeq}{\to} \mathbf{P}'$ (see Definition 1.2), then we have a C_n –equivalence*

$$U^n(F; P) \xrightarrow{(f;g)_*} U^n(F'; P')$$

and therefore an equivalence on W .

Proof Since the associated spectra are all equivalent, we get an equivalence of the k –simplices in U^n for all n . □

Given an FSP F_1 over a small category $\mathcal{C}_1$ with an F_1 –bimodule P_1 , and an FSP F_2 over a small category $\mathcal{C}_2$ with an F_2 –bimodule P_2 , one can define functors with stabilization over $\mathcal{C}_1 \times \mathcal{C}_2$

$$(F_1 \times F_2)_{(a_1,a_2),(b_1,b_2)}(X) = (F_1)_{a_1,b_1}(X) \times (F_2)_{a_2,b_2}(X),$$
$$(F_1 \vee F_2)_{(a_1,a_2),(b_1,b_2)}(X) = (F_1)_{a_1,b_1}(X) \vee (F_2)_{a_2,b_2}(X).$$

Similar definitions can be made for bimodules. Note that $F_1 \times F_2$ is an FSP; $F_1 \vee F_2$ has no unit, so is not an FSP. However the inclusion of the latter in the former induces a stable equivalence of the associated spectra, so if one used the definition of U^n on $F_1 \vee F_2$ with coefficients in $P_1 \vee P_2$, as in Lemma 6.5 above one would get the same thing as $U^n(F_1 \times F_2; P_1 \times P_2)$. One can also define an FSP over $\mathcal{C}_1 \amalg \mathcal{C}_2$

$$(F_1 \amalg F_2)_{a,b}(X) = \begin{cases} (F_1)_{a,b}(X) & \text{if } a, b \in \mathcal{C}_1, \\ (F_2)_{a,b}(X) & \text{if } a, b \in \mathcal{C}_2, \\ * & \text{if } a, b \text{ lie in different } \mathcal{C}_i, \end{cases}$$

and similarly define an $F_1 \amalg F_2$ –bimodule $P_1 \amalg P_2$.

Lemma 6.6 *For FSPs F_1 and F_2 over small categories C_1 and C_2 , respectively, with bimodules P_1 and P_2 we have a C_n –equivalence*

$$U^n(F_1 \amalg F_2; P_1 \amalg P_2) \overset{\simeq}{\to} U^n(F_1; P_1) \times U^n(F_2; P_2)$$

inducing an equivalence on W .

Proof Following the proof of [6, Lemma 1.6.13], by picking $\underline{X}$ and looking at $V(\underline{X}, \mathcal{C}_1 \amalg \mathcal{C}_2)$ calculated with respect to the FSP $F_1 \amalg F_2$ and the bimodule $P_1 \amalg P_2$, shows that only summands which correspond to sequences of elements which are all in $\mathcal{C}_1$ or all in $\mathcal{C}_2$ are nontrivial. Thus for every $\underline{X}$ we have

$$V(\underline{X}, \mathcal{C}_1 \amalg \mathcal{C}_2) = V_{(F_1; P_1)}(\underline{X}, \mathcal{C}_1) \vee V_{(F_2; P_2)}(\underline{X}, \mathcal{C}_2),$$

and so the limit of $\underline{\mathrm{Map}}(S^{\sqcup X}, V(\underline{X}, \mathcal{C}_1 \sqcup \mathcal{C}_2))$ will be weakly equivalent to the limit of $\underline{\mathrm{Map}}(S^{\sqcup X}, V_{(F_1; \overline{P_1})}(\underline{X}, \mathcal{C}_1)) \times V_{(F_2; P_2)}(\underline{X}, \mathcal{C}_2))$. $\square$

Lemma 6.7 *For FSPs F_1 and F_2 over small categories $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, with bimodules P_1 and P_2. Then the projections to both coordinates define a C_n–equivalence*

$$U^n(F_1 \times F_2; P_1 \times P_2) \xrightarrow{\simeq} U^n(F_1; P_1) \times U^n(F_2; P_2)$$

inducing an equivalence on W.

Proof   Call the map induced by the projections (which is a C_n–equivariant map) f; we need to show that it is an equivalence. Like in the proof of [6, Lemma 1.6.15], one can construct a commutative diagram

$$
\begin{array}{ccc}
U^n(F_1 \vee F_2; P_1 \vee P_2) & \xrightarrow{\;\;g\;\;} & U^n(F_1 \sqcup F_2; P_1 \sqcup P_2) \\
\Big\downarrow{\scriptstyle \mathrm{incl}_*} & & \Big\downarrow{\scriptstyle \text{Lemma 6.6}} \\
U^n(F_1 \times F_2; P_1 \times P_2) & \xrightarrow{\;\;f\;\;} & U^n(F_1; P_1) \times U^n(F_2; P_2)r
\end{array}
$$

with the vertical maps weak equivalences. The map g is obtained by restricting the maps

$$V_{(F_1 \times F_2; P_1 \times P_2)}(\underline{X}, \mathcal{C}_1 \times \mathcal{C}_2) \to V_{(F_1; P_1)}(\underline{X}, \mathcal{C}_1) \times V_{(F_2; P_2)}(\underline{X}, \mathcal{C}_2)$$

used to define f (induced by the product of the projections) to

$$V_{(F_1 \vee F_2; P_1 \vee P_2)}(\underline{X}, \mathcal{C}_1 \times \mathcal{C}_2),$$

and observing that they give maps

$$
\begin{aligned}
V_{(F_1 \vee F_2; P_1 \vee P_2)}&(\underline{X}, \mathcal{C}_1 \times \mathcal{C}_2) \\
&\to V_{(F_1; P_1)}(\underline{X}, \mathcal{C}_1) \vee V_{(F_2; P_2)}(\underline{X}, \mathcal{C}_2) = V_{(F_1 \sqcup F_2; P_1 \sqcup P_2)}(\underline{X}, \mathcal{C}_1 \sqcup \mathcal{C}_2).
\end{aligned}
$$

In the opposite direction, a map $i \colon U^n(F_1 \sqcup F_2; P_1 \sqcup P_2) \to U^n(F_1 \vee F_2; P_1 \vee P_2)$ can be defined by first mapping $\mathcal{C}_1 \sqcup \mathcal{C}_2 \to \mathcal{C}_1 \times \mathcal{C}_2$ using some fixed $a \in \mathcal{C}_2$ as a "filler" second coordinate for $\mathcal{C}_1$ and some fixed $b \in \mathcal{C}_1$ as a "filler" first coordinate for $\mathcal{C}_2$, and then mapping F_i or P_i into $F_1 \vee F_2$ or $P_1 \vee P_2$, respectively. Now $g \circ i = \mathrm{id}$, and the proof in [6, 1.6.15] works to show that $i \circ g \cong \mathrm{id}$ (if one ignores the cyclic action, it does not matter whether the coordinates are the FSP or the bimodule). Since the vertical maps are known to be equivalences, the fact that g is an equivalence implies that f is one, too. $\square$

Definition 6.8 Given a functor with stabilization A over $\mathcal{C}$, we can define its $\ell \times \ell$ matrices as a functor with stabilization over $\mathcal{C}^\ell$ in two ways:

$$M_\ell(A)_{(c_1,\dots,c_\ell),(c'_1,\dots,c'_\ell)}(X) = \prod_{r=1}^{\ell} \bigvee_{s=1}^{\ell} A_{c_r,c'_s}(X),$$

$$M_\ell(A)^{\vee}_{(c_1,\dots,c_\ell),(c'_1,\dots,c'_\ell)}(X) = \bigvee_{r=1}^{\ell} \bigvee_{s=1}^{\ell} A_{c_r,c'_s}(X).$$

Now if F is an FSP, $M_\ell(F)$ is an FSP too, using matrix multiplication (see [6, 1.2.6]; since the FSP multiplication sends $F_{b,c}(X) \wedge F_{a,b}(Y) \to F_{a,c}(X \wedge Y)$, one should think of $A_{c_r,c'_s}(X)$ as the (s,r)–th entry in the matrix; $M_\ell(F)^{\vee}$ using the same multiplication does not have a unit but of course the associated spectra are stably equivalent. If F is an FSP and P is an F–bimodule, $M_\ell(P)$ is a $M_\ell(F)$–bimodule.

One can also define upper-triangular matrices (see above comment about indexing)

$$T_\ell(A)_{(c_1,\dots,c_\ell),(c'_1,\dots,c'_\ell)}(X) = \prod_{r=1}^{\ell} \bigvee_{s=1}^{r} A_{c_r,c'_s}(X),$$

$$T_\ell(A)^{\vee}_{(c_1,\dots,c_\ell),(c'_1,\dots,c'_\ell)}(X) = \bigvee_{r=1}^{\ell} \bigvee_{s=1}^{r} A_{c_r,c'_s}(X),$$

and again if F is an FSP then $T_\ell(F)$ is one too, and $T_\ell(A)$, $T_\ell(A)^{\vee}$ are stably equivalent for any A.

Proposition 6.9 (Morita equivalence) *Let F be an FSP on $\mathcal{C}$ and let P be an F–bimodule. Then there is a C_n–equivalence*

$$U^n(F; P) \xrightarrow{\simeq} U^n(M_\ell(F); M_\ell(P))$$

inducing an equivalence on W.

Proof We will define the map which induces this equivalence; it will as usual be C_n–equivariant. The proof that it is a homotopy equivalence is completely analogous to the proof [6, Proposition 1.6.18]. The map is defined by picking some element $c_0 \in \mathcal{C}$, and using it to embed $i\colon \mathcal{C} \to \mathcal{C}^\ell$ by $a \mapsto (a, c_0, \dots, c_0)$ on objects and $f \mapsto (f, \mathrm{id}_{c_0}, \dots, \mathrm{id}_{c_0})$ on morphisms. Then on $\mathcal{C}$ we map F to $i^* M_n(F)^{\vee}$ by including, for every $a, b \in \mathcal{C}$ and any finite simplicial X, $F_{a,b}(X)$ as the $(1,1)$–st coordinate in $M_n(F)^{\vee}_{(a,c_0,\dots,c_0),(b,c_0,\dots,c_0)}$, which in turn includes into $M_n(F)$. $\qquad\square$

Proposition 6.10 *Let F be an FSP on $\mathcal{C}$ and let P be an F–bimodule. Then the inclusion of the diagonal matrices in the upper-triangular ones induces a C_n–equivalence*

$$U^n\left(\prod_{i=1}^{\ell} F; \prod_{i=1}^{\ell} P\right) \xrightarrow{\simeq} U^n(T_\ell(F); T_\ell(P))$$

inducing an equivalence on W.

Proof There is an obvious map from the upper-triangular matrices to the diagonal ones—collapsing all the off-diagonal terms—which shows that the above map must be the inclusion of a retract. The proof that it is in fact an equivalence is analogous to that of [6, Proposition 1.6.20], and is done by replacing $U^n(\prod_{i=1}^{\ell} F; \prod_{i=1}^{\ell} P)$ with the equivalent $U^n(\bigvee_{i=1}^{\ell} F; \bigvee_{i=1}^{\ell} P)$ and $U^n(T_\ell(F); T_\ell(P))$ with the equivalent $U^n(T_\ell(F)^\vee; T_\ell(P)^\vee)$. $\qquad\qquad\square$

Definition 6.11 We now restrict ourselves to small linear categories $\mathcal{C}$, to FSPs of the form $\underline{\mathcal{C}}_{a,b} = \mathcal{C}_{a,b} \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$, as described in Examples 1.4, and to bimodules over them of the form

$$P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$$

for some other linear category $\mathcal{B}$ and two functors $G_1, G_2 \colon \mathcal{C} \to \mathcal{B}$ which respect the linear structure of the morphism sets. In this case we can define FSPs

$$m_\ell(\underline{\mathcal{C}})_{(c_1,\dots,c_\ell),(c_1',\dots,c_\ell')}(X) = \left(\bigoplus_{r=1}^{\ell}\bigoplus_{s=1}^{\ell} \mathcal{C}(c_r, c_s')\right) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X],$$

$$M_\ell(\underline{\mathcal{C}})^{\oplus}_{(c_1,\dots,c_\ell),(c_1',\dots,c_\ell')}(X) = \bigoplus_{r=1}^{\ell}\bigoplus_{s=1}^{\ell} \mathcal{C}(c_r, c_s') \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X].$$

Observe that the two are, in fact, homeomorphic on any X. One can do the same construction for matrices over P of the above form. Observe also that the obvious inclusions $M_\ell(\underline{\mathcal{C}}) \to M_\ell(\underline{\mathcal{C}})^{\oplus}$, $M_\ell(P) \to M_\ell(P)^{\oplus}$ are stable equivalences, and therefore by Lemma 6.5 above we have C_n–equivalences

$$U^n(M_\ell(\underline{\mathcal{C}}); M_\ell(P)) \xrightarrow{\simeq} U^n(M_\ell(\underline{\mathcal{C}})^{\oplus}; M_\ell(P)^{\oplus}) = U^n(m_\ell(\underline{\mathcal{C}}); m_\ell(P)).$$

One can similarly construct upper-triangular versions $t_\ell(\underline{\mathcal{C}})$, $t_\ell(P)$, $T_\ell(\underline{\mathcal{C}})^{\oplus}$, $T_\ell(P)^{\oplus}$ and get

$$U^n(T_\ell(\underline{\mathcal{C}}); T_\ell(P)) \xrightarrow{\simeq} U^n(T_\ell(\underline{\mathcal{C}})^{\oplus}; T_\ell(P)^{\oplus}) = U^n(t_\ell(\underline{\mathcal{C}}); t_\ell(P)).$$

Proposition 6.12 (Cofinality) *Let $\mathcal{D}$ be a small additive category (that is, a small linear category with a notion of $\oplus$ on the objects which corresponds to taking the direct sum of abelian groups on the morphisms, and with a zero object). Let $\mathcal{C}$ be a full subcategory of it which is cofinal, that is, for any $d \in \mathcal{D}$ there is $d' \in \mathcal{D}$ such that $d \oplus d' \in \mathcal{C}$. Let P be an FSP of the form $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ on $\mathcal{D}$ for some additive category functors $G_1, G_2 \colon \mathcal{D} \to \mathcal{B}$. Then the inclusion induces a C_n–equivalence*

$$U^n(\underline{\mathcal{C}}; P|_{\mathcal{C}}) \xrightarrow{\simeq} U^n(\underline{\mathcal{D}}; P)$$

and an equivalence on W.

Proof Since by construction P respects the direct sum structure, the proof of [6, Lemma 2.1.1] works if we use it in some of the coordinates. $\square$

The following proposition connects $U^n(\underline{\mathcal{P}_R}; P)$ with the definition of U of the FSP associated to a ring with coefficients in a bimodule which is analogous to Bökstedt's original definition of THH in [2]. The former will be needed to construct the map from K–theory in Section 9 below; the latter is more compact and easier to use for calculations.

Proposition 6.13 (Another Morita equivalence) *Let R be an associative ring with unit. We can view R as the full subcategory on the rank 1 free module inside $\mathcal{P}_R$, the category of finitely generated projective right R–modules. Let $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ for some additive category functors $G_1, G_2 \colon \mathcal{P}_R \to \mathcal{B}$. Then the inclusion $R \hookrightarrow \mathcal{P}_R$ induces a C_n–equivalence*

$$U^n(\underline{R}; P|_R) \xrightarrow{\simeq} U^n(\underline{\mathcal{P}_R}; P)$$

and an equivalence on W.

Proof Following the proof of [6, Proposition 2.1.5], we let $\mathcal{F}_R$ be the category of finitely generated free right R–modules, and $\mathcal{F}_R^k$ its full subcategory on the modules of rank less than or equal to k. Then the inclusion $m_k(R) \hookrightarrow \mathcal{F}_R^k$, where we regard $m_k(R)$ as the full subcategory on a rank k free module, is an equivalence of categories. Note that on the category with one object, the FSP $m_\ell(\underline{R})$ of Definition 6.11 is the same FSP as what we would call $\underline{m_\ell(R)}$, the one associated to the full subcategory described above.

So we get a commutative diagram of C_n–equivariant maps

$$
\begin{array}{ccc}
U^n(\underline{R}; P|_R) & \xrightarrow{\hspace{3cm}} U^n(\underline{\mathcal{P}_R}; P) \xleftarrow{\text{Proposition 6.12}} U^n(\underline{\mathcal{F}_R}; P|_{\mathcal{F}_R}) \\
\Big\downarrow{\text{Proposition 6.9}} & & \Big\uparrow{\text{Lemma 6.4}} \\
\lim_{k\to\infty} U^n(M_k(\underline{R}); M_k(P|_R)) \xrightarrow{\text{Definition 6.11}} \lim_{k\to\infty} U^n(m_k(\underline{R}); m_k(P|_R)) \xrightarrow{\text{Example 6.3}} \lim_{k\to\infty} U^n(\underline{\mathcal{F}_R^k}; P|_{\mathcal{F}_R^k})
\end{array}
$$

where the labels on the arrows indicate from where it follows that those maps are homotopy equivalences. Thus the unlabeled map must be a homotopy equivalence as well. □

In [5] the authors construct a map from $K(R; M)$ to topological Hochschild homology, and then show that it is the map from $K(R; M)$ to the first layer of its Goodwillie Taylor tower. They start with another functor which maps very naturally to topological Hochschild homology and then look at the functor induced on the Waldhausen S–constructions of domain and range. In Section 9 below, we will follow the same method. So we briefly recall the S–construction from [21; 5].

Given an exact category (an additive category with a compatible notion of exact sequences) $\mathcal{C}$, one can define for any $n \geq 0$ another exact category $S_n\mathcal{C}$ whose objects are sequences of admissible monomorphisms $0 = c_0 \hookrightarrow c_1 \hookrightarrow \cdots \hookrightarrow c_n$ with particular identifications of c_j/c_i for all $i \leq j$, and whose morphisms are commuting diagrams. Assembled over all n, with the obvious composition maps for ∂_i, $0 < i < n$, omission of and quotienting by c_1 for ∂_0, and omission of c_n for ∂_n, these form a simplicial exact category.

If the original category $\mathcal{C}$ is split exact, that is, all exact sequences split in it, then $S_\cdot\mathcal{C}$ is a split simplicial exact category. One can take functors from small categories to spaces or spectra and define them on a simplicial exact category levelwise, and then realize. One can also define the iterated S–construction $S_\cdot^{(k)}\mathcal{C}$ to be the simplicial exact category obtained by taking the diagonal of the k–simplicial exact category one would get by iterating the process k times.

Note that if $\mathcal{C}$ is a small exact category, and we have a bimodule over the FSP $\underline{\mathcal{C}}$ of the form $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ for some exact functors G_1, G_2 from $\mathcal{C}$ to an exact category $\mathcal{B}$, the G_i induce simplicial exact functors $S_\cdot G_i \colon S_\cdot\mathcal{C} \to S_\cdot\mathcal{B}$ and so we can define a $\underline{S_\cdot\mathcal{C}}$ bimodule

$$
S_n P_{\bar{a}, \bar{b}}(X) = S_n \mathcal{B}(S_n G_1(\bar{a}), S_n G_2(\bar{b})) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]
$$

for all $\bar{a}, \bar{b} \in S_n\mathcal{C}$ for all $n \geq 0$.

Proposition 6.14 *Let $\mathcal{C}$ be a split small exact category and let P be a bimodule over the FSP $\underline{C}$ of the form $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ for some exact functors $G_1, G_2 \colon \mathcal{C} \to \mathcal{B}$. Then there is a C_n–equivalence*

$$U^n(\underline{C}; P) \stackrel{\simeq}{\to} \Omega|U^n(\underline{S.C}; S.P)|$$

induced by the adjoint to the map $\Sigma U^n(\underline{C}; P) \to |U^n(\underline{S.C}; S.P)|$ coming from the identification of $S_0\mathcal{C}$ with the trivial category and $S_1\mathcal{C}$ with $\mathcal{C}$ via $\{0 \hookrightarrow c\} \leftrightarrow c$. This equivalence yields an equivalence

$$W(\underline{C}; P) \stackrel{\simeq}{\to} \Omega|W(\underline{S.C}; S.P)|.$$

Proof As in the proof of [6, Proposition 2.1.3], the key point is that because $\mathcal{C}$ is split exact, for any $k \geq 0$, if we look at the functor $\mathcal{C}^k \simeq f \to S_k\mathcal{C}$ defined by $(c_1, \ldots, c_k) \mapsto \{0 \hookrightarrow c_1 \hookrightarrow c_1 \oplus c_2 \hookrightarrow \cdots \hookrightarrow c_1 \oplus \cdots \oplus c_k\}$, it is an equivalence of categories. The morphisms of $S_k\mathcal{C}$ pull back exactly to the upper-triangular matrices so $f^*(\underline{S_k\mathcal{C}}) = t_k(\underline{C})$ of Definition 6.11 above. Similarly, since the G_i are exact functors, they send direct sums to direct sums, and so P preserves direct sums and $f^*(S_k P) = t_k(P)$. We also want to look at the functor $S_k\mathcal{C} \simeq g \to \mathcal{C}^k$ sending $\{0 \hookrightarrow c_1 \hookrightarrow c_2 \hookrightarrow \cdots \hookrightarrow c_k) \mapsto (c_1, c_2/c_1, \ldots, c_k/c_{k-1})$. We get a commutative diagram of C_n–equivariant maps

$$
\begin{array}{ccccc}
U^n(\underline{S_k\mathcal{C}}; S_k P) & \xrightarrow{\;\;g_*\;\;} & U^n(\prod_{i=1}^k \underline{C}; \prod_{i=1}^k P) & \xrightarrow{\text{Lemma 6.7}} & U^n(\underline{C}; P)^k \\
{\scriptstyle f_*}\Big\downarrow {\scriptstyle \text{Example 6.3}} & & \Big\uparrow {\scriptstyle \text{Proposition 6.10}} & & \\
U^n(t_k(\underline{C}); t_k(P)) & \xrightarrow{\text{Definition 6.11}} & U^n(T_k(\underline{C}); T_k(P)) & &
\end{array}
$$

where the labels on the arrows indicate from where it follows that those maps are homotopy equivalences. We deduce that $U^n(\underline{S_k\mathcal{C}}; S_k P) \stackrel{\simeq}{\to} U^n(\underline{C}; P)^k$ for every k.

We will show that if we apply the maps g_*, followed by the projections of Lemma 6.7, levelwise, we get an equivalence $|U^n(\underline{S.C}; S.P)| \stackrel{\simeq}{\to} |B.U^n(\underline{C}; P)|$ compatible with the identifications $U^n(\underline{S_1\mathcal{C}}; S_1 P) = U^n(\underline{C}; P) = B_1 U^n(\underline{C}; P)$, which will complete our proof. (The classifying space $B.$ is taken with respect to the operation induced by the abelian group structure on the morphism sets of $\mathcal{C}$.) To show this, we consider that for any simplicial object $X.$ one can look at its simplicial path space $(\mathcal{P}X).$ defined by $(\mathcal{P}X)_k = X_{k+1}$ with the original $\partial_0, \ldots \partial_k$ and $s_0, \ldots, s_k$ as structure maps. The "extra" degeneracy map s_{k+1} from $(\mathcal{P}X)_k$ to $(\mathcal{P}X)_{k+1}$ allows us to embed the cone on $|(\mathcal{P}X).|$ in $|(\mathcal{P}X).|$, showing that $|(\mathcal{P}X).|$ is contractible. The "extra" boundary map $\partial_{k+1} \colon (\mathcal{P}X)_n \to X_n$ is a simplicial map. Then we have a commutative diagram

for each k

$$U^n(\underline{C}; P) \xrightarrow{(s_0)_*^k} U^n(\underline{PS_kC}; PS_kP) \xrightarrow{(\partial_{k+1})_*} U^n(\underline{S_kC}; S_kP)$$

where the vertical arrows are the maps g_* followed by the projections of Lemma 6.7, so we know that they are all equivalences, and the bottom row is the trivial product fibration, inserting $U^n(\underline{C}; P)$ in the last coordinate. The two fibrations are therefore homotopy equivalent. $\qquad\square$

Proposition 6.15 *Let C be a split small exact category and let P be a bimodule over the FSP $\underline{C}$ of the form $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ for some exact functors $G_1, G_2 \colon \mathcal{C} \to \mathcal{B}$. Then there is a C_n–equivalence*

$$\lim_{k\to\infty} \Omega^k U_0^n(\underline{S^{(k)}C}; S^{(k)}P) \xrightarrow{\simeq} \lim_{k\to\infty} \Omega^k U^n(\underline{S^{(k)}C}; S^{(k)}P).$$

Proof This is analogous to the proof of [6, Proposition 2.2.3], but is done simultaneously in all n blocks. As usual, we will prove that for all n the given map, which respects the C_n action, is a homotopy equivalence, and the C_n–equivalence will follow by induction.

As in [6, Sections 2.0.7 and 2.2.1], we can replace $U^n(\mathcal{C}, P)$ by the simplicial abelian group $R_{\cdot}(\mathcal{C})$ with

$$R_p(\mathcal{C}) = \operatorname{hocolim}_{\underline{X} \in I^{n(p+1)}} \Omega^{\sqcup \underline{X}} \bigoplus_{(c_{1,0},\dots,c_{1,p},c_{2,0},\dots,c_{n,p}) \in \mathcal{C}^{n(p+1)}}$$

$$\big(s\mathcal{B}(G_1(a_{1,0}), G_2(a_{1,-1})) \otimes \widetilde{\mathbb{Z}}[s\mathcal{C}(a_{1,1}, a_{1,0})]$$

$$\otimes \cdots \otimes \widetilde{\mathbb{Z}}[s\mathcal{C}(a_{1,p}, a_{1,p-1})] \otimes \widetilde{\mathbb{Z}}[s\mathcal{B}(G_1(a_{2,0}), G_2(a_{2,-1}))]$$

$$\otimes \cdots \otimes \widetilde{\mathbb{Z}}[s\mathcal{C}(a_{2,p}, a_{2,p-1})] \otimes \cdots \otimes \widetilde{\mathbb{Z}}[s\mathcal{C}(a_{n,p}, a_{n,p-1})]\big),$$

where $s\mathcal{B}$, $s\mathcal{C}$ denote the categories of simplicial objects in $\mathcal{B}$, $\mathcal{C}$,

$$a_{1,-1} = c_{n,p} \otimes \widetilde{\mathbb{Z}}[S^{\sqcup \underline{X}}],$$

for $1 < i \le n$,

$$a_{i,-1} = c_{i-1,p} \otimes \widetilde{\mathbb{Z}}[S^{\sqcup_{(k,l)>(i-1,p)} X_{k,l}}],$$

and for $1 \le i \le n$, $0 \le j \le p$

$$a_{i,j} = c_{i,j} \otimes \widetilde{\mathbb{Z}}[S^{\sqcup_{(k,l)>(i,j)} X_{k,l}}].$$

The ordering on pairs of indices is lexicographic.

For every p, then, R_p clearly satisfies the requirements of [6, Lemma 2.2.2], yielding

$$d_0 + d_2 \simeq d_1 \colon \lim_{k \to \infty} \Omega^k R_p(S^{(k)} S_2 \mathcal{C}) \to \lim_{k \to \infty} \Omega^k R_p(S^{(k)} \mathcal{C}).$$

For every p we have maps

$$R_0(\mathcal{C}) \overset{s_0^p}{\to} R_p(\mathcal{C}) \overset{d_0^p}{\to} R_0(\mathcal{C})$$

with $d_0^p \circ s_0^p = \mathrm{id}_{R_0(\mathcal{C})}$. We will show that $s_0^p \circ d_0^p \simeq \mathrm{id}_{R_p(\mathcal{C})}$ as well, so that the $R_p(\mathcal{C})$ are all homotopy equivalent to $R_0(\mathcal{C})$.

Since $d_0^{p-1} \circ d_i = d_0^p$, $0 \le i < p$, these $d_i \colon R_p(\mathcal{C}) \to R_{p-1}(\mathcal{C})$ are homotopy equivalences compatible with the equivalence $R_p(\mathcal{C}) \to R_0(\mathcal{C})$. Since $d_0^{p+1} \circ s_i = d_0^p$ for all $0 \le i \le p$, $s_i \colon R_p(\mathcal{C}) \to R_{p+1}(\mathcal{C})$ are similarly compatible. One could, similarly to the calculation we are about to make, show that $d_1 \circ d_2 \circ \cdots \circ d_p \colon R_p(\mathcal{C}) \to R_0(\mathcal{C})$ is a homotopy inverse of s_0^p, which implies that $d_p \colon R_p(\mathcal{C}) \to R_{p-1}(\mathcal{C})$ is also a homotopy equivalence compatible with the equivalence $R_p(\mathcal{C}) \to R_0(\mathcal{C})$. Thus

$$|\{p \mapsto R_p(\mathcal{C})\}| = \mathrm{hocolim}\, R_0(\mathcal{C}) \simeq R_0(\mathcal{C}),$$

where the homotopy colimit is taken over the cosimplicial category, and the last equivalence is because all the maps involved are homotopic to the identity and the homotopy colimit of a point over the category is contractible.

To define the homotopy $s_0^p \circ d_0^p \simeq \mathrm{id}_{R_p(\mathcal{C})}$ for a fixed p, note that $R_p(\mathcal{C})$ is generated by products of blocks of maps

$$\alpha_{i,0} \colon G_1(a_{i,0}) \to G_2(a_{i-1,p})$$

$$a_{i,0} \overset{\alpha_{i,1}}{\longleftarrow} a_{i,1} \overset{\alpha_{i,2}}{\longleftarrow} a_{i,2} \overset{\alpha_{i,3}}{\longleftarrow} \cdots \overset{\alpha_{i,p}}{\longleftarrow} a_{i,p}$$

for all $1 \le i \le n$ (defining $a_{-1,p} = a_{n,p}$). We can write such a generator as $\underline{\alpha} = (\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,p}, \alpha_{2,0}, \ldots, \alpha_{n,0}, \alpha_{n,1}, \ldots, \alpha_{n,p})$. Define

$$\beta_{i,j} = \alpha_{i,j} \circ \alpha_{i,j+1} \circ \cdots \circ \alpha_{i,p} \colon a_{i,p} \to a_{i,j-1}$$

for $1 \le j \le p$ and

$$\beta_i = \alpha_{i,0} \circ G_1(\beta_{i,1}) \colon G_1(a_{i,p}) \to G_2(a_{i-1,p}).$$

Then

$$s_0^p \circ d_0^p(\underline{\alpha}) = (\beta_1, \mathrm{id}_{a_{1,p}}, \ldots, \mathrm{id}_{a_{1,p}}, \beta_2, \mathrm{id}_{a_{2,p}}, \ldots, \mathrm{id}_{a_{2,p}}, \ldots, \beta_n, \mathrm{id}_{a_{n,p}}, \ldots, \mathrm{id}_{a_{n,p}}).$$

Now define $t^i_{\mathrm{id}}(\underline{\alpha})$ for $1 \le i \le n$ to consist of

$$
\begin{array}{ccc}
G_2(a_{i-1,p}) & \xleftarrow{\qquad 0 \qquad} & G_1(a_{i,p}) \\
\Big\downarrow {\scriptstyle G_2(i_1)} & & \Big\downarrow {\scriptstyle G_1(i_1)} \\
G_2(a_{i-1,p} \oplus a_{i-1,p}) & \xleftarrow{\; G_2(\Delta)\alpha_{i,0}G_1(\pi_2) \;} & G_1(a_{i,p} \oplus a_{i,0}) \\
\Big\downarrow {\scriptstyle G_2(\pi_2)} & & \Big\downarrow {\scriptstyle G_1(\pi_2)} \\
G_2(a_{i-1,p}) & \xleftarrow{\qquad \alpha_{i,0} \qquad} & G_1(a_{i,0})
\end{array}
$$

$$
\begin{array}{ccccccc}
a_{i,p} & =\!=\!= & a_{i,p} & =\!=\!= & \cdots & =\!=\!= & a_{i,p} \\
\Big\downarrow {\scriptstyle i_1} & & \Big\downarrow {\scriptstyle i_1} & & & & \Big\downarrow {\scriptstyle i_1} \\
a_{i,p} \oplus a_{i,0} & \xleftarrow{\;\mathrm{id}\,\oplus\alpha_{i,1}\;} & a_{i,p} \oplus a_{i,1} & \xleftarrow{\;\mathrm{id}\,\oplus\alpha_{i,2}\;} & \cdots \xleftarrow{\;\mathrm{id}\,\oplus\alpha_{i,p}\;} & & a_{i,p} \oplus a_{i,p} \\
\Big\downarrow {\scriptstyle \pi_2} & & \Big\downarrow {\scriptstyle \pi_2} & & & & \Big\downarrow {\scriptstyle \pi_2} \\
a_{i,0} & \xleftarrow{\; \alpha_{i,1} \;} & a_{i,1} & \xleftarrow{\; \alpha_{i,2} \;} & \cdots \xleftarrow{\; \alpha_{i,p} \;} & & a_{i,p},
\end{array}
$$

where i_1 is the inclusion into the first factor in the direct sum and π_2 the projection into the second. Then the map

$$
\underline{\alpha} \mapsto (t^1_{\mathrm{id}}(\underline{\alpha}), t^2_{\mathrm{id}}(\underline{\alpha}), \ldots, t^n_{\mathrm{id}}(\underline{\alpha}))
$$

induces a map

$$
T_{\mathrm{id}} \colon R_p(\mathcal{C}) \to R_p(S_2\mathcal{C}).
$$

Also, define $t^i_{\beta_i}(\underline{\alpha})$ for $1 \le i \le n$ to consist of

$$
\begin{array}{ccc}
G_2(a_{i-1,p}) & \xleftarrow{\qquad \beta_i \qquad} & G_1(a_{i,p}) \\
\Big\downarrow {\scriptstyle G_2(\Delta)} & & \Big\downarrow {\scriptstyle G_1((\mathrm{id}\,\oplus\beta_{i,1})\Delta)} \\
G_2(a_{i-1,p} \oplus a_{i-1,p}). & \xleftarrow{\; G_2(\Delta)\alpha_{i,0}G_1(\pi_2) \;} & G_1(a_{i,p} \oplus a_{i,0}) \\
\Big\downarrow {\scriptstyle G_2(\mathrm{id},-\mathrm{id})} & & \Big\downarrow {\scriptstyle G_1(\beta_{i,1},-\mathrm{id})} \\
G_2(a_{i-1,p}) & \xleftarrow{\qquad 0 \qquad} & G_1(a_{i,0}),
\end{array}
$$

$$
\begin{array}{ccccccc}
a_{i,p} \cdot & \!\!=\!\!=\!\! & a_{i,p} & \!\!=\!\!=\!\! & \cdots & \!\!=\!\!=\!\! & a_{i,p} \\
{\scriptstyle (\mathrm{id}\,\oplus\,\beta_{i,1})\Delta}\downarrow & & {\scriptstyle (\mathrm{id}\,\oplus\,\beta_{i,2})\Delta}\downarrow & & & & \downarrow{\scriptstyle \Delta} \\
a_{i,p}\oplus a_{i,0} & \xleftarrow{\mathrm{id}\,\oplus\,\alpha_{i,1}} & a_{i,p}\oplus a_{i,1} & \xleftarrow{\mathrm{id}\,\oplus\,\alpha_{i,2}} & \cdots \xleftarrow{\mathrm{id}\,\oplus\,\alpha_{i,p}} & & a_{i,p}\oplus a_{i,p}\cdot \\
{\scriptstyle (\beta_{i,1},\,-\,\mathrm{id})}\downarrow & & {\scriptstyle (\beta_{i,2},\,-\,\mathrm{id})}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{id},\,-\,\mathrm{id})} \\
a_{i,0} & \xleftarrow{\ \alpha_{i,1}\ } & a_{i,1} & \xleftarrow{\ \alpha_{i,2}\ } & \cdots \xleftarrow{\ \alpha_{i,p}\ } & & a_{i,p}\cdot
\end{array}
$$

The map

$$
\underline{\alpha} \mapsto (t^1_{\beta_1}(\underline{\alpha}), t^2_{\beta_2}(\underline{\alpha}), \dots, t^n_{\beta_n}(\underline{\alpha}))
$$

induces a map

$$
T_\beta \colon R_p(\mathcal{C}) \to R_p(S_2\mathcal{C}).
$$

Recall that for $c_1 \hookrightarrow c_2$ with an identification of c_2/c_1 in $S_2\mathcal{C}$, d_0 is c_2/c_1, d_1 is c_2, and d_2 is c_1. Thus

$$
\begin{aligned}
d_0 T_{\mathrm{id}} &= \mathrm{id} & d_0 T_\beta &= 0, \\
d_1 T_{\mathrm{id}} &= d_1 T_\beta, & & \\
d_2 T_{\mathrm{id}} &= 0 & d_2 T_\beta &= s_0^p \circ d_0^p.
\end{aligned}
$$

Thus the induced maps

$$
\lim_{k\to\infty} \Omega^k R_p(S^{(k)}\mathcal{C}) \to \lim_{k\to\infty} \Omega^k R_p(S^{(k)} S_2\mathcal{C}) \xrightarrow{d_j} \lim_{k\to\infty} \Omega^k R_p(S^{(k)}\mathcal{C})
$$

satisfy

$$
\mathrm{id} = d_0 T_{\mathrm{id}} \simeq d_1 T_{\mathrm{id}} = d_1 T_\beta \simeq d_2 T_\beta = s_0^p \circ d_0^p,
$$

where the homotopies use the homotopy $d_0 + d_2 \simeq d_1 \colon \lim_{k\to\infty} \Omega^k R_p(S^{(k)} S_2\mathcal{C}) \to \lim_{k\to\infty} \Omega^k R_p(S^{(k)}\mathcal{C})$ mentioned above, along with the vanishing of $d_2 T_{\mathrm{id}}$, $d_0 T_\beta$. $\square$

Proposition 6.16 *Let $\mathcal{C}$ be a split small exact category and let P be a bimodule over the FSP $\underline{\mathcal{C}}$ of the form $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ for some exact functors $G_1, G_2 \colon \mathcal{C} \to \mathcal{B}$. Then there is a C_n–equivalence*

$$
\lim_{k\to\infty} \Omega^k \bigvee_{\underline{c} \in S^{(k)}\mathcal{C}} U_0^n(\underline{S^{(k)}\mathcal{C}}|_{\underline{c}}; S^{(k)} P|_{\underline{c}}) \xrightarrow{\ \simeq\ } \lim_{k\to\infty} \Omega^k U_0^n(\underline{S^{(k)}\mathcal{C}}; S^{(k)} P).
$$

Proof As in the proof of Proposition 6.15 above, we replace U_0^n by the abelian group R_0; the corresponding replacement for $\bigvee_{c \in \mathcal{C}} U_0^n(\mathcal{C}|_c, P|_c)$ is

$$R'(\mathcal{C}) = \bigoplus_{c \in \mathcal{C}} \operatorname{hocolim}_{\underline{X} \in I^{n(p+1)}} \Omega^{\sqcup \underline{X}} \Big(s\mathcal{B}(G_1(a_1'), G_2(a_0'))$$

$$\otimes \tilde{\mathbb{Z}}[s\mathcal{B}(G_1(a_2'), G_2(a_1')] \otimes \cdots \otimes \tilde{\mathbb{Z}}[s\mathcal{B}(G_1(a_n'), G_2(a_{n-1}')] \Big)$$

$$\simeq \operatorname{hocolim}_{\underline{X} \in I^{n(p+1)}} \Omega^{\sqcup \underline{X}} \bigoplus_{c \in \mathcal{C}} \Big(s\mathcal{B}(G_1(a_1'), G_2(a_0'))$$

$$\otimes \tilde{\mathbb{Z}}[s\mathcal{B}(G_1(a_2'), G_2(a_1')] \otimes \cdots \otimes \tilde{\mathbb{Z}}[s\mathcal{B}(G_1(a_n'), G_2(a_{n-1}')] \Big),$$

where $a_0' = c \otimes \tilde{\mathbb{Z}}[S^{\sqcup \underline{X}}], a_1' = c \otimes \tilde{\mathbb{Z}}[S^{X_2 \sqcup \cdots \sqcup X_n}], \ldots, a_{n-1}' = c \otimes \tilde{\mathbb{Z}}[S^{X_n}], a_n' = c$.

We have the obvious inclusion i and map j

$$i \colon R'(\mathcal{C}) \xrightarrow{i} R_0(\mathcal{C}), \quad R_0(\mathcal{C}) \xrightarrow{j} R'(\mathcal{C})$$

sending each summand corresponding to $(c_1, \ldots, c_n) \in \mathcal{C}^n$ in $R_0(\mathcal{C})$ to the summand corresponding to $c = c_1 \oplus \cdots \oplus c_n$ in $R'(\mathcal{C})$ via the map induced by the obvious inclusions and projections

$$c_j \xrightarrow{i_j} c_1 \oplus \cdots \oplus c_n \xrightarrow{\pi_j} c_j,$$

$$s\mathcal{B}(G_1(\pi_1), G_2(i_n)) \otimes \tilde{\mathbb{Z}}[s\mathcal{B}(G_1(\pi_2), G_2(i_1)] \otimes \cdots \otimes \tilde{\mathbb{Z}}[s\mathcal{B}(G_1(\pi_n), G_2(i_{n-1})].$$

Since R_0 satisfies the requirements of [6, Lemma 2.2.2],

$$d_0 + d_2 \simeq d_1 \colon \lim_{k \to \infty} \Omega^k R_0(S^{(k)} S_2 \mathcal{C}) \to \lim_{k \to \infty} \Omega^k R_0(S^{(k)} \mathcal{C}).$$

Define a map $T \colon R_0(\mathcal{C}) \to R_0(S_2 \mathcal{C})$ by sending

$$G_2(a_0) \xleftarrow{\alpha_1} G_1(a_1) \qquad G_2(a_1) \xleftarrow{\alpha_2} G_1(a_2) \quad \cdots \quad G_2(a_{n-1}) \xleftarrow{\alpha_n} G_1(a_n)$$

to

$$
\begin{array}{ccccccc}
G_2(a_0) & \xleftarrow{\alpha_1} & G_1(a_1) & \quad & G_2(a_1) & \xleftarrow{\alpha_2} & G_1(a_2) & \cdots & G_2(a_{n-1}) & \xleftarrow{\alpha_n} & G_1(a_n) \\
\downarrow{\scriptstyle G_2(i_n)} & & \downarrow{\scriptstyle G_1(i_1)} & & \downarrow{\scriptstyle G_2(i_1)} & & \downarrow{\scriptstyle G_1(i_2)} & & \downarrow{\scriptstyle G_2(i_{n-1})} & & \downarrow{\scriptstyle G_1(i_n)} \\
G_2(a_0') & \xleftarrow{\gamma_1} & G_1(a_1') & & G_2(a_1') & \xleftarrow{\gamma_2} & G_1(a_2') & & G_2(a_{n-1}') & \xleftarrow{\gamma_n} & G_1(a_n') \\
\downarrow{\scriptstyle G_2(p_n)} & & \downarrow{\scriptstyle G_1(p_1)} & & \downarrow{\scriptstyle G_2(p_1)} & & \downarrow{\scriptstyle G_1(p_2)} & & \downarrow{\scriptstyle G_2(p_{n-1})} & & \downarrow{\scriptstyle G_1(p_n)} \\
G_2(a_0'') & \xleftarrow{0} & G_1(a_1'') & & G_2(a_1'') & \xleftarrow{0} & G_1(a_2'') & & G_2(a_{n-1}'') & \xleftarrow{0} & G_1(a_n'')
\end{array}
$$

where $\gamma_j = G_2(i_{j-1})\alpha_j G_1(\pi_j)$ (with i_0 defined as i_n), and for all $0 \le i \le n$ (with 0 cyclically identified with n as an index for the c_i as usual),

$$a_i = c_i \otimes \widetilde{\mathbb{Z}}[S^{\sqcup_{j>i} X_j}],$$

$$a_i' = c \otimes \widetilde{\mathbb{Z}}[S^{\sqcup_{j>i} X_j}],$$

$$a_i'' = \bigoplus_{k \ne i} c_k \otimes \widetilde{\mathbb{Z}}[S^{\sqcup_{j>i} X_j}].$$

By construction we have short exact sequences

$$0 \to a_j \xrightarrow{i_j} a_j' \xrightarrow{p_j} a_j'' \to 0.$$

We have $d_0 T = 0$, $d_2 T = \mathrm{id}$, and $d_1 T = i \circ j$, so on $\lim_{k \to \infty} \Omega^k R_0(S^{(k)} \mathcal{C})$,

$$\mathrm{id} = d_2 T = d_0 T + d_2 T \simeq d_1 T = i \circ j.$$

Now $j \circ i$ is not equal to the identity, but note that the map T above sends $R'(\mathcal{C})$ to $R'(S_2 \mathcal{C})$ and R' also satisfies the conditions of [6, Lemma 2.2.2], so the formula for T also gives a homotopy $j \circ i \simeq \mathrm{id}_{\lim_{k \to \infty} \Omega^k R'(S^{(k)} \mathcal{C})}$. $\qquad\square$

Corollary 6.17 *Let $\mathcal{C}$ be a split small exact category and let P be a bimodule over the FSP $\underline{\mathcal{C}}$ of the form $P_{a,b}(X) = \mathcal{B}(G_1(a), G_2(b)) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$ for some exact functors $G_1, G_2 \colon \mathcal{C} \to \mathcal{B}$. Then there is a C_n–equivalence*

$$\lim_{k \to \infty} \Omega^k \bigvee_{\underline{c} \in S^{(k)} \mathcal{C}} U_0^n(\underline{S^{(k)}\mathcal{C}}|_{\underline{c}}; S^{(k)} P|_{\underline{c}}) \xrightarrow{\simeq} \lim_{k \to \infty} \Omega^k U^n(\underline{S^{(k)}\mathcal{C}}; S^{(k)} P).$$

7 Cubical diagrams and analyticity

In this section we recall some definitions and terminology from [G2] and show that the functor $W(F; P \otimes -)$ is 0–analytic. Working with a functor from spaces to spectra allows us to apply [9; 10] directly, as opposed to studying the functor $W(F; -)$ from F–bimodules to spectra and having to adapt the results to a Goodwillie calculus of functors in that setting.

For S a finite set, let $\mathcal{P}(S)$ be the poset of all subsets of S. An S–cube (or n–cube where $n = |S|$) in a category $\mathcal{C}$ is a functor $\mathcal{X}$ from $\mathcal{P}(S)$ to $\mathcal{C}$. Thus, a 0–cube is an object of $\mathcal{C}$, a 1–cube is a morphism and a 2–cube is a commuting square in $\mathcal{C}$.

Let $\mathcal{X}$ be an S–cube of spaces or spectra. Since $\varnothing$ is initial in S, $\mathcal{X}(\varnothing) \to \mathrm{holim}_S \mathcal{X}$ is an equivalence. Let $\mathcal{P}_0(S)$ be the poset of *nonempty* subsets of S. The *homotopy fiber* of $\mathcal{X}$ is

$$\mathrm{hofib}(\mathcal{X}) = \mathrm{hofib}(\mathcal{X}(\varnothing) \xrightarrow{\alpha} \mathrm{holim}_{\mathcal{P}_0(S)} \mathcal{X}).$$

The cube $\mathcal{X}$ is called k–*Cartesian* if α is k–connected and *Cartesian* if α is an equivalence.

Since S is final in $\mathcal{P}(S)$, $\operatorname{colim}_{\mathcal{P}(S)} \mathcal{X} \to \mathcal{X}(S)$ is an equivalence. Let $\mathcal{P}_1(S)$ be the poset of $T \subseteq S$ such that $T \neq S$. The *homotopy cofiber* of $\mathcal{X}$ is

$$\operatorname{hocofib}(\mathcal{X}) = \operatorname{hocofib}(\operatorname{hocolim}_{\mathcal{P}_1(S)} \mathcal{X} \xrightarrow{\beta} \mathcal{X}(S)).$$

The cube $\mathcal{X}$ is called k–*co-Cartesian* if β is k–connected and *co-Cartesian* if β is an equivalence.

Let F be a fixed FSP. Then the above definitions for functors from spectra to spectra generalize in the obvious way to the category of F–bimodules: Recall that a morphism of F–bimodules is k–connected if for all $A, B \in \mathcal{O}$, the map of associated spectra $f_{A,B}$ is k–connected; we will call a cubical diagram $\mathcal{X}$ of F–bimodules (or functors with stabilization) k–(co)Cartesian if for all $A, B \in \mathcal{O}$ its associated diagram of spectra $\mathcal{X}_{A,B}$ is k–(co-)Cartesian.

Definition 7.1 An S–cube $\mathcal{X}$ of spaces or spectra is *strongly (co-)Cartesian* if each face of dimension ≥ 2 is (co-)Cartesian.

Definition 7.2 Let F be a homotopy functor from $\mathcal{C}$ to $\mathcal{D}$. Then F is n–*excisive*, or satisfies n–*th order excision*, if for every strongly co-Cartesian $(n+1)$–cubical diagram $\mathcal{X} \colon \mathcal{P}(S) \to \mathcal{C}$ the diagram $F(\mathcal{X})$ is Cartesian.

Example 7.3 By [9, Proposition 3.4], if $M \colon \mathcal{C}^{\times r} \to \mathcal{D}$ is n_i–excisive in the i–th variable for all $1 \leq i \leq r$, then the composition with the diagonal inclusion $\mathcal{C} \to \mathcal{C}^{\times r}$ is n–excisive with $n = n_1 + \cdots + n_r$. Thus, by [9, Lemma 3.3], $U^n(F;\ )$ is an n–excisive functor.

Example 7.4 Spectra have the nice property that for n–cubes of spectra, "k–Cartesian" $=$"$(k+n-1)$–co-Cartesian" [9, 1.19] and hence, by definition, the category of F–bimodules (or functors with stabilization) also shares this property. In particular, every co-Cartesian diagram of F–bimodules is Cartesian and thus for any FSP F, the identity functor from F–bimodules to itself is 1–excisive. More generally, given a unital map $f \colon F \to F'$ of FSP's, the functor f^* from F'–bimodules to F–bimodules is 1–excisive.

Example 7.5 The functor Ω^∞ from F–bimodules to $\Omega^\infty F$–bimodules is 1–excisive (since $\operatorname{id} \xrightarrow{\cong} \to \Omega^\infty$). Since F maps to $\Omega^\infty F$, one can regard the resulting $\Omega^\infty F$–bimodules as F–bimodules again. Thus one can replace every strong cofibration square of F–bimodules by an equivalent strong co-Cartesian square of F–bimodules for which the associated spectra sending n to the value of the functor on S^n are Ω–spectra.

Definition 7.6 Given a functor with stabilization Q, we let Q_* denote the homotopy functor $X \mapsto Q(X)$ which is necessarily reduced ($Q(*) \simeq *$). Then $\Omega^\infty Q_*$ is a 1–excisive functor from spaces to spaces which satisfies the *limit axiom*. We recall from [7, 1.7] that a homotopy functor G satisfies the limit axiom if for all spaces X,

$$\mathrm{hocolim}_{Y \subseteq X} G(Y) \xrightarrow{\simeq} G(X)$$

where the limit system runs over all compact CW-subspaces, with maps being the inclusions.

Definition 7.7 (Goodwillie [9]) Let F be a homotopy functor from $\mathcal{C}$ to $\mathcal{D}$. Then F is *stably n–excisive*, or satisfies *stable n–th order excision*, if the following is true for some numbers c and κ:

$E_n(c, \kappa)$: If $\mathcal{X} \colon \mathcal{P}(S) \to \mathcal{C}$ is any strongly co-Cartesian $(n+1)$–cube such that for all $s \in S$ the map $\mathcal{X}(\varnothing) \to \mathcal{X}(s)$ is k_s–connected and $k_s \geq \kappa$, then the diagram $F(\mathcal{X})$ is $(-c + \Sigma k_s)$–Cartesian.

If $E_n(c, \kappa)$ holds for all κ then we simply say that F satisfies $E_n(c)$.

Example 7.8 Every n–excisive functor satisfies $E_n(c)$ for all c.

Example 7.9 If F_* is a simplicial object of functors from spaces or spectra to spectra satisfying $E_n(c, \kappa)$, then the realized functor $|F_*|$ (to spectra) also satisfies $E_n(c, \kappa)$. This is because the realization functor is equivalent to a homotopy colimit which preserves connectivity and commutes with finite homotopy inverse limits (because we are working with spectra).

Example 7.10 For any functor with stabilization P, the functor from spaces to functors with stabilization sending a space X to the functor with stabilization $P \otimes X \colon Y \mapsto P(Y) \wedge X$ is 1–excisive, and the functor from functors with stabilization to spaces defined by $P \mapsto \mathrm{hocolim}_{X \in I} \mathrm{map}(S^X, P(S^X))$ is 1–excisive.

Definition 7.11 (Goodwillie [9]) The functor F is ρ–*analytic* if there is some number q such that F satisfies $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$.

Definition 7.12 Given an F–bimodule P, write $U^n(F; P \otimes -)$ and $W_{\mathcal{M}}(F; P \otimes -)$ for the homotopy functors from spaces to simplicial functors with stabilization defined by composition with the functor $P \otimes -$ (of Example 7.10) from spaces to F–bimodules. We note that $U^n(F; P \otimes -)$ satisfies the limit axiom but $W_{\mathcal{M}}(F; P \otimes -)$ only does if $\mathcal{M}$ is finite or P is 0–connective.

Example 7.13 For any $m|n$ and F–bimodule P, the functor $U^n(F; P \otimes -)_{hC_m}$ from space to functors with stabilization is n–excisive and satisfies $E_k(0)$ for all $k \geq 1$ and hence is -1–analytic.

Proof (after [9, 4.4]) Since homotopy orbits preserve connectivity and commute with finite homotopy inverse limits, it suffices to show the result for $U^n(F; P \otimes -)$.

By Example 7.5, we may assume P is an Ω F–bimodule. Now $U^n(F; P \otimes -)$ is the diagonal on an n–multiexcisive functor so by Example 7.3 it is n–excisive. By [9, Proposition 3.2], being n–excisive implies being k–excisive for all $k \geq n$, which implies (as observed in Example 7.8 above) that $U^n(F; P \otimes -)$ satisfies $E_k(c)$ for all $k \geq n$ and all c. Below we will show that for all $k \geq 1$, $U^n(F; P \otimes -)$ satisfies $E_k(0)$. Once we know that, we can take $q = -n$ and get that for all $k \geq 1$, $U^n(F; P \otimes -)$ satisfies $E_k(-k + n, 0)$ and so it is -1–analytic.

By Example 7.9, to show that $U^n(F; P \otimes -)$ satisfies $E_k(0)$ it suffices to show that $U^n(F; P \otimes -)_{[m]}$ satisfies $E_k(0)$ for all $k \geq 1$ and $m \geq 0$, so we fix m and consider $U^n(F; P \otimes -)_{[m]}$.

Let $\mathcal{X}$ be a strongly co-Cartesian S–cube of F–bimodules such that $\mathcal{X}(\varnothing) \to \mathcal{X}(s)$ is k_s–connected for $s \in S$. Now for any space Z, the spectra

$$U^n(F; P)_{[m]} \wedge \bigwedge^n Z \simeq U^n(F; P \otimes Z)_{[m]}$$

are equivalent. By [9, Example 4.4], $\wedge^n \mathcal{X}$ is $\Sigma k_s + n$–co-Cartesian (reduce to a CW case and consider cells), thus $U^n(F; P \otimes \mathcal{X})_{[m]}$ is a $\Sigma k_s + n$–co-Cartesian diagram of spectra and so a Σk_s–Cartesian diagram of spectra. $\square$

Proposition 7.14 Let $\mathcal{M} \subseteq \mathbb{N}^\times$ be such that for all $M \in \mathcal{M}$, $\vec{M} \subseteq \mathcal{M}$ (notation as in Proposition 5.8). Then for F–bimodules P, $W_{\mathcal{M}}(F; P \otimes -)$ satisfies $E_n(1)$ for all $n \geq 1$ and hence is 0–analytic.

Proof Note, if $\mathcal{M} \subseteq \mathbb{N}^\times$ is finite, then there exists an $M \in \mathcal{M}$ such that $\mathcal{M}$ is covered by $\vec{M}$ and $\mathcal{M} - M$ (as in Proposition 5.8). Thus, by Proposition 5.8 we have a natural fibration

$$U^M(F; P \otimes -)_{hC_M} \to \text{holim}_{\mathcal{M}} U(F; P \otimes -)|_{\mathcal{M}} \xrightarrow{\text{Res}} \text{holim}_{\mathcal{M}-M} U(F; P \otimes -)|_{\mathcal{M}-M}.$$

By Example 7.13, $U^M_{hC_M}$ satisfies $E_n(0)$. Now $\mathcal{M} - M$ is again such that if $M' \in \mathcal{M} - M$ then $\vec{M}' \subseteq \mathcal{M} - M$ and so by induction on the number of objects of $\mathcal{M}$, $\text{holim}_{\mathcal{M}-M} U(F; P \otimes -)|_{\mathcal{M}-M}$ satisfies $E_n(0)$ and hence $\text{holim}_{\mathcal{M}} U(F; P \otimes -)$ satisfies $E_n(0)$ too.

In general, $W_{\mathcal{M}}(F; P \otimes -)$ can be written as

$$W_{\mathcal{M}}(F; P \otimes -) \simeq \operatorname{holim}_{\infty \leftarrow n} W_{\mathcal{M}_n}(F; P \otimes -)$$

for $\mathcal{M}_n$ finite and as above. Now since homotopy inverse limits commute, the homotopy fiber of $W_{\mathcal{M}}(F; P \otimes -)$ on a strongly co-Cartesian $(n+1)$–cube is the homotopy inverse limit of the homotopy fibers of the $W_{\mathcal{M}_n}(F; P \otimes -)$ on that cube. By [4, XI.7.4], there is a natural short exact sequence

$$0 \to \operatorname{\lim^1}_{n \to \infty} \pi_{i+1}(\text{the homotopy fiber of } W_{\mathcal{M}_n}) \to \pi_i(\text{the homotopy fiber of } W_{\mathcal{M}})$$

$$\to \operatorname{\lim}_{n \to \infty} \pi_i(\text{the homotopy fiber of } W_{\mathcal{M}_n}) \to 0,$$

and since the homotopy fibers of the $W_{\mathcal{M}_n}$ are Σk_s–connected, the homotopy fiber of $W_{\mathcal{M}}$ must be $-1 + \Sigma k_s$–connected. So $W_{\mathcal{M}}(F; P \otimes -)$ always satisfies $E_n(1)$. $\quad\square$

Remark 7.15 Certainly Proposition 7.14 holds for more general categories $\mathcal{M}$ but what is proven suffices for our purposes. For example, using $\vec{M}$ and the equivalence

$$U^M()^{C_M} \xrightarrow{\simeq} \operatorname{holim}_{\vec{M}} U(F; P \otimes -)|\vec{M}$$

(M is initial in $\vec{M}$) we see that $U^M(F; P \otimes -)^{C_M}$ also satisfies $E_n(0)$.

8 Taylor towers and $W_{\mathcal{M}}(F; P \otimes -)$

We now want to identify the Taylor tower for $W(F; P \otimes -)$ and $W^{(p)}(F; P \otimes -)$ when P is an F–bimodule. Technically, what we will write down is what one might call the Maclaurin series since it is the Taylor tower at the basepoint. We will be using universal properties and the results of Section 6.

Recall that for a ρ–analytic functor F there is an n–excisive homotopy functor called the n–th degree Taylor polynomial which we write as $P_n F$. We also define

$$D_n F = \operatorname{hofib}(P_n F \xrightarrow{q_n F} P_{n-1} F).$$

Terminology An *admissible sequence* $\{x_1, \ldots, x_n\}$ is a (possibly infinite) strictly increasing sequence of positive integers such that

  (i) $x_1 = 1$.
  (ii) if $m | x_j$ then $x_j / m \in \{x_1, \ldots, x_{j-1}\}$.

Note that if $\{x_1, \ldots, x_n\}$ is an admissible sequence then so is $\{x_1, \ldots, x_{n-1}\}$. Some examples of admissible sequences are: $\{1, 2, 3, \ldots\}$, $\{1, p, p^2, \ldots\}$, $\{1, p, q, pq\}$ where

p and q are prime. Any multiplicatively closed subset of $N^\times$ which contains all its prime divisors and 1 determines an admissible sequence and every admissible sequence is a subsequence of one obtained in this manner.

Given an admissible sequence $\{x_1, \ldots, x_n \ldots\}$, let $\vec{\mathcal{X}}_j \subseteq \mathbb{N}^\times$ be the full subcategory generated by $\{x_1, \ldots, x_j\}$. Thus, $\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$ and $\mathcal{X}_{j+1}$ is covered (as in Proposition 5.8) by $\mathcal{X}_j$ and $\vec{x}_{j+1}$. Let $\mathcal{X}$ be the full subcategory generated by all x_i's.

Proposition 8.1 *Given an admissible sequence $\{x_1, x_2, \ldots\}$ and an F–bimodule P, then*

$$D_n(W_{\mathcal{X}_j}(F; P \otimes -)) \simeq \begin{cases} U^n(F; P \otimes -)_{hC_n} & \text{if } n \in \{x_1, \ldots, x_j\}, \\ * & \text{otherwise,} \end{cases}$$

$$P_n(W_{\mathcal{X}_j}(F; P \otimes -)) \simeq \begin{cases} W_{\mathcal{X}_k}(F; P \otimes -) & \text{where } x_k \leq n < x_{k+1}, k < j, \\ W_{\mathcal{X}_j}(F; P \otimes -) & \text{where } x_j \leq n, \end{cases}$$

$$D_n(W_{\mathcal{X}}(F; P \otimes -)) \simeq \begin{cases} U^n(F; P \otimes -)_{hC_n} & \text{if } n \in \{x_1, \ldots\}, \\ * & \text{otherwise,} \end{cases}$$

$$P_n(W_{\mathcal{X}}(F; P \otimes -)) \simeq W_{\mathcal{X}_k}(F; P \otimes -) \quad \text{where } x_k \leq n < x_{k+1},$$

with structure maps $q_n\colon P_n \to P_{n-1}$ given by restriction to subcategories.

Proof In order to ease notation, we will drop F and P from our notation so that $W_{\mathcal{X}}$ will represent $W_{\mathcal{X}}(F; P \otimes -)$.

We recall from [10] the following facts about P_n (at the space consisting of a single point) and the natural transformation $p_n\colon \text{id} \to P_n$: the natural transformation $F \mapsto P_n F$ preserves equivalences and fibrations of functors. For F ρ–analytic, the natural transformation $p_n F\colon F \to P_n F$ is universal with respect to mapping F to an n–excisive functor which agrees with it to order n, that is, so that for some constant q, $p_n F$ is at least $nk + q + k$–connected on k–connected spaces for $k \geq \rho$ (see [10], just after the proof of Proposition 1.6).

We will use the fibration

$$U^{x_j}_{hC_{x_j}} \simeq \text{hofib}[W_{\mathcal{X}_j} \overset{\text{res}}{\to} W_{\mathcal{X}_{j-1}}]$$

from Proposition 5.8. By [9, Example 3.5], $U^n_{hC_n}$ is an n–excisive functor. If $F'' \to F \to F'$ is a fibration of homotopy functors and both F'' and F' satisfy $E_n(c)$, then F satisfies $E_n(c)$ also. So by induction and Proposition 7.14, $W_{\mathcal{X}_j}$ is 0–analytic and x_j–excisive.

By the universality of $P_n F$, if F is 0–analytic and m–excisive, the natural transformation $F \to P_n F$ is an equivalence for all $n \geq m$ on all 0–connected spaces and so a stable equivalence for those n. Therefore $P_n(W_{\mathcal{X}_j}) \simeq W_{\mathcal{X}_j}$ for all $n \geq x_j$.

Now the connectivity condition in the universal property of P_n shows that $U^n_{hC_n}$ is in fact a *homogenous n–excisive functor*, ie

$$P_k(U^n_{hC_n}) = \begin{cases} U^n_{hC_n} & \text{if } k \geq n, \\ * & \text{if } k < n. \end{cases}$$

Using the fibration of Proposition 5.8 again, this implies that $P_k(\mathrm{res}) \colon P_k(W_{\mathcal{X}_j}) \to P_k(W_{\mathcal{X}_{j-1}})$ is an equivalence for $k < x_j$, so the first two results follow by induction on j.

As in Corollary 5.10, we see that $\mathrm{res} \colon W_{\mathcal{X}} \to W_{\mathcal{X}_j}$ is $(x_{j+1}(k+1)-2)$–connected for X k–connected. Since $x_{j+1} \geq x_j + 1$, this implies $((x_j+1)(k+1)-2)$–connectedness, that is, $(x_j k + x_j - 1 + k)$–connectedness. Thus, by the universal property of p_{x_j}, $\mathrm{res} \colon P_{x_j} W_{\mathcal{X}} \to P_{x_j} W_{\mathcal{X}_j}$ is an equivalence. The formula for the layers $D_n W_{\mathcal{X}}$ now follows by induction since we know the layers $D_n W_{\mathcal{X}_j}$ for all j and n. $\qquad\square$

Corollary 8.2 *For any FSP F and linear bimodule P, one has*

$$D_n(W(F; P \otimes -)) \simeq U^n(F; P \otimes -)_{hC_n},$$

$$P_n(W(F; P \otimes -)) \simeq W_n(F; P \otimes -),$$

$$D_n(W^{(p)}(F; P \otimes -)) \simeq \begin{cases} U^{p^k}(F; P \otimes -)_{hC_{p^k}} & \text{if } n = p^k, \\ * & \text{otherwise,} \end{cases}$$

$$P_n(W^{(p)}(F; P \otimes -)) \simeq W_k^{(p)}(F; P \otimes -) \quad \text{where } p^k \leq n < p^{k+1},$$

with structure maps $q_n \colon P_n \to P_{n-1}$ given by restriction to subcategories.

9   Relating $K(R \ltimes \widetilde{M}[X])$ to $W(\underline{R}; \underline{M} \otimes \Sigma X)$

When R is an associative ring with unit and M is a simplicial R–bimodule, Dundas and McCarthy defined in [5] the invariant $K(R; M)$. For M discrete, this is the algebraic K–theory of the exact category whose objects are (P, α), where P is a finitely generated projective R–module and $\alpha \colon P \to P \otimes_R M$ is a right R–module map, and whose morphisms from (P, α) to (Q, β) are right R–module homomorphisms $f \colon P \to Q$ such that $\beta \circ f = (f \otimes \mathbf{1}_M) \circ \alpha$. For M a simplicial R–bimodule, this definition is applied degreewise. In Theorem 4.1 there, Dundas and McCarthy show that there is a natural homotopy equivalence

$$K(R \ltimes M) \simeq K(R; B.M)$$

where $B.M \simeq \widetilde{M}[S^1]$ is the bar construction on M. The functoriality of this identification means that the direct summand $K(R) = K(R; 0)$ maps compatibly to both sides, so we have a natural equivalence on the reduced theories $\widetilde{K}(R \ltimes M) \simeq \widetilde{K}(R; B.M)$ as well. The argument in [5] goes on to show that topological Hochschild homology $\text{THH}(R; M)$ is the first derivative at the one-point space $*$ of the functor $X \mapsto \widetilde{K}(R; \widetilde{M}[X])$.

Now [19, Proposition 3.2] says that the functor $X \mapsto \widetilde{K}(R; \widetilde{M}[X])$ is 0–analytic. We want to show that for connected X, this functor is naturally homotopy equivalent to the functor $X \mapsto W(R; \widetilde{M}[X]) \simeq W(R; M \otimes X)$, which is also 0–analytic by Proposition 6.14 above. Here R is the FSP associated (as in Examples 1.4(ii)) to the category having a single object and R as its morphism set, and M is the R–bimodule sending $X \mapsto \widetilde{M}[X]$. The equivalence $W(R; M \otimes X) \simeq W(R; \widetilde{M}[X])$ is by Lemma 6.5 above.

It will turn out to be more convenient to map into $W(\underline{\mathcal{P}_R}; \underline{M})$ instead of $W(R; M)$, where $\mathcal{P}_R$ is the category of finitely generated projective right R–modules, and $\underline{M}$ is the $\underline{\mathcal{P}_R}$–bimodule given by

$$\underline{M}_{A,B}(X) = \text{Hom}_{\mathcal{M}_R}(A; B \otimes_R M) \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[X]$$

for all $A, B \in \mathcal{P}_R$, $X \in \mathcal{S}_*$, where $\mathcal{M}_R$ is the category of all right R–modules. The restriction of this $\underline{M}$ on $\mathcal{P}_R$ to the full subcategory on the rank 1 free module (which is isomorphic to the category R) is the M of the previous paragraph (which should really be denoted $\underline{M}$ as well, but in that case we omit the underline to agree with the usual notation for $K(R; M)$).

Lemma 9.1 *Let R be an associative ring with unit and let M be a discrete R–bimodule. Then we can define maps*

$$\widetilde{K}(R; M) \xrightarrow{\beta_n} U^n(\underline{\mathcal{P}_R}; \underline{M})$$

for all $n \geq 1$ such that $\text{Res}^{n/m} \circ \beta_n = \beta_m$ for all $m|n$ and such that β_1 is the map of [5, Theorem 3.4].

Proof As defined in [5, Section 3],

$$K(R; M) = \Omega| \coprod_{\overline{c} \in S.\mathcal{P}_R} \text{Hom}_{S.\mathcal{M}_R}(\overline{c}, \overline{c} \otimes_R M)|.$$

We will let

$$K(R; M) = \coprod_{c \in \mathcal{P}_R} \text{Hom}_{\mathcal{M}_R}(c, c \otimes_R M).$$

It is easy to map

$$K(R; M) \xrightarrow{b_n} U^n(\underline{\mathcal{P}_R}; \underline{M})$$

for all $n \geq 1$ such that $\mathrm{Res}^{n/m} \circ b_n = b_m$ for all $m|n$ and such that b_1 is the map [5] used at this level: send

$$\mathrm{Hom}_{\mathcal{M}_R}(c, c \otimes_R M) \to \underbrace{\mathrm{Hom}_{\mathcal{M}_R}(c, c \otimes_R M) \wedge \cdots \wedge \mathrm{Hom}_{\mathcal{M}_R}(c, c \otimes_R M)}_{n \text{ times}}$$

$$\alpha \mapsto \alpha^{\wedge n},$$

that is, map $\mathrm{Hom}_{\mathcal{M}_R}(c, c \otimes_R M)$ into the multisimplicial degree $(0, 0, \ldots 0)$ part of $U^n(\underline{\mathcal{P}_R}; \underline{M})$ (where there are no $\mathcal{P}_R$ coordinates, only bimodule coordinates), into the term corresponding to $\underline{X} = (0, 0, \ldots, 0)$ (since $\otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}[S^0] = \otimes_{\mathbb{Z}} \mathbb{Z}$ does not do anything to an abelian group) by sending α to n copies of itself.

For each k, on S_k of Waldhausen's S–construction (as reviewed before Proposition 6.14 above) b_n induce maps

$$\coprod_{\overline{c} \in S_k \mathcal{P}_R} \mathrm{Hom}_{S_k \mathcal{M}_R}(\overline{c}, \overline{c} \otimes_R M) \to U^n(\underline{S_k \mathcal{P}_R}; S_k \underline{M})$$

$$\alpha \mapsto \alpha^{\wedge n},$$

and so we get maps

$$K(R; M) = \Omega \Big| \coprod_{\overline{c} \in S.\mathcal{P}_R} \mathrm{Hom}_{S.\mathcal{M}_R}(\overline{c}, \overline{c} \otimes_R M) \Big| \xrightarrow{\beta_n} \Omega |U^n(\underline{S.\mathcal{P}_R}; S.\underline{M})| \simeq U^n(\underline{\mathcal{P}_R}; \underline{M})$$

by the equivalence of Proposition 6.14. These satisfy $\mathrm{Res}^{n/m} \circ \beta_n = \beta_m$ for all $m|n$ and generalize the construction of [5].

Note that by naturality, these β_n send $K(R) = K(R; 0) \to U^n(\underline{\mathcal{P}_R}; \underline{0}) \simeq *$ and so factor through reduced K–theory maps

$$\widetilde{K}(R; M) \xrightarrow{\beta_n} U^n(\underline{\mathcal{P}_R}; \underline{M}). \qquad \qquad \square$$

Recall that for simplicial R–bimodules N, $K(R; N)$ is defined by geometrically realizing the $K(R; N_n)$'s with respect to the maps induced by N's simplicial structure. Since the functors $U^n(F; -)$ commute with realizations, we could do the same for $U^n(\underline{\mathcal{P}_R}; \underline{N})$. Therefore, Lemma 9.1 allows us to define maps

$$\widetilde{K}(R; N) \xrightarrow{\beta_n} U^n(\underline{\mathcal{P}_R}; \underline{N})$$

for any simplicial R–bimodule N, satisfying $\mathrm{Res}^{n/m} \circ \beta_n = \beta_m$ for all $m|n$ and generalizing the construction of [5]. Since the β_n are compatible with the restriction maps, they define a map

$$\tilde{K}(R; N) \xrightarrow{\beta} W(\underline{\mathcal{P}_R}; \underline{N}).$$

We will want to apply this for $N = \widetilde{M}[X]$ for M a discrete R–bimodule and X a finite pointed simplicial set. Note that the FSP $\widetilde{M}[X]$ associated to the simplicial R–bimodule $\widetilde{M}[X]$ is the same as the FSP $\overline{\widetilde{M}[X]}$ of Examples 1.7(iv), and as such has an associated spectrum stably equivalent to that of $\underline{M} \otimes X$ of Example (iii), which has been studied in the previous section. The following Theorem will be proved in several steps.

Main Theorem 9.2 *Let R be an associative ring with unit and let M be a discrete R–bimodule. Then the natural transformation*

$$\tilde{K}(R; \widetilde{M}[X]) \xrightarrow{\beta} W(\underline{\mathcal{P}_R}; \widetilde{\underline{M}}[X])$$

induces an equivalence when X is connected. Since by Proposition 6.13,

$$W(R; \widetilde{M}[X]) \simeq W(\underline{\mathcal{P}_R}; \widetilde{\underline{M}}[X])$$

is a homotopy equivalence (note that in the proof there are maps given in both directions), this gives a homotopy equivalence

$$\tilde{K}(R; \widetilde{M}[X]) \xrightarrow{\simeq} W(R; \widetilde{M}[X])$$

for connected X.

Corollary 9.3 *The Taylor tower of the functor $X \mapsto \tilde{K}(R; \widetilde{M}[X])$ at $*$ has $W_n(R; \widetilde{M}[X])$ as its n–th stage, with the tower maps given by category restriction.*

Proof of Corollary 9.3 In Corollary 8.2 above, we have seen that the $W_n(R; M \otimes -)$ are the finite stages in the Taylor tower of $W(R; M \otimes -)$ at $*$, related by the category restriction maps. By Lemma 6.5, this means that the $W_n(R; \widetilde{M}[X])$ are the finite stages in the Taylor tower of $W(R; \widetilde{M}[X])$ at $*$, related by the same maps. The coefficients in the Taylor tower can be computed by looking arbitrarily close to the space at which we are working. See [10, Remark 1.1] for a discussion of this. So to find the Taylor tower of $X \mapsto \tilde{K}(R; \widetilde{M}[X])$ it is enough to look at connected X, where Main Theorem 9.2 tells us it agrees with $W(R; \widetilde{M}[X])$. □

Proof of Main Theorem 9.2 We will use a variant of [9, Theorem 5.3]. Theorem 5.3 says that if two ρ–analytic functors F and G have a natural transformation between

them which induces an equivalence of the differentials $D_X F \to D_X G$ at every space X, then for $(\rho+1)$–connected maps $X \to Y$ there is a Cartesian diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & G(Y). \end{array}$$

The same proof shows that if two ρ–analytic functors F and G have a natural transformation between them which induces an equivalence of the differentials $D_X F \to D_X G$ at every ρ–connected space X (ie every X for which the map $X \to *$ is $(\rho+1)$–connected—see the comment just after [9, Definition 1.3]), then for every ρ–connected X there is a Cartesian diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ \downarrow & & \downarrow \\ F(*) & \longrightarrow & G(*). \end{array}$$

So we need to show that the natural transformation β induces an equivalence on the derivatives at all 0–connected spaces. Since $\widetilde{K}(R; \widetilde{M}[*]) \simeq W(\underline{\mathcal{P}_R}; \underline{\widetilde{M}}[*]) \simeq *$, this will imply that for any 0–connected X, β is an equivalence.

So let X be connected; we should consider spaces $Y \to X$ over X; but for simplicity of writing, we would like to eliminate the spaces from our calculation. In the following sections, we will prove:

Technical Lemma 9.4 *Let R be an associative ring with unit, and let M, N be two simplicial R–bimodules. If N is k–connected, β induces a $2k$–connected map*

$$\mathrm{hofib}\big(\widetilde{K}(R; B_{\scriptscriptstyle\bullet} M \oplus B_{\scriptscriptstyle\bullet} N) \to \widetilde{K}(R; B_{\scriptscriptstyle\bullet} M)\big)$$

$$\to \mathrm{hofib}\big(W(\underline{\mathcal{P}_R}; \underline{B_{\scriptscriptstyle\bullet} M} \oplus \underline{B_{\scriptscriptstyle\bullet} N}) \to W(\underline{\mathcal{P}_R}; \underline{B_{\scriptscriptstyle\bullet} M})\big).$$

Having this lemma lets us conclude the proof of Main Theorem 9.2: because of the obvious equivalences

$$\widetilde{M}[X \vee A] \cong \widetilde{M}[X] \oplus \widetilde{M}[A]$$

and, for connected spaces Z, $\widetilde{M}[Z] \simeq B_{\scriptscriptstyle\bullet}(\Omega \widetilde{M}[Z])$ (for Ω a simplicial model of the loop space), Technical Lemma 9.4 would tell us that

$$\mathrm{hofib}\big(\widetilde{K}(R; \widetilde{M}[X \vee S^k]) \to \widetilde{K}(R; \widetilde{M}[X])\big)$$

$$\to \mathrm{hofib}\big(W(\underline{\mathcal{P}_R}; \underline{\widetilde{M}}[X \vee S^k]) \to W(\underline{\mathcal{P}_R}; \underline{\widetilde{M}}[X])\big)$$

is $2(k-1)$–connected for all $k \geq 1$. But by the definition of a derivative at a space X of a functor (at the basepoint of X) from [7] (for functors to spaces) and [9, Section 5], the spectrum $k \mapsto \mathrm{hofib}\big(\widetilde{K}(R; \widetilde{M}[X \vee S^k]) \to \widetilde{K}(R; \widetilde{M}[X])\big)$ is equivalent to the derivative of the functor $\widetilde{K}(R; \widetilde{M}[-])$ at X, and similarly the spectrum $k \mapsto \mathrm{hofib}\big(W(\underline{\mathcal{P}_R}; \widetilde{M}[X \vee S^k]) \to W(\underline{\mathcal{P}_R}; \widetilde{M}[X])\big)$ is equivalent to the derivative of the functor $W(\underline{\mathcal{P}_R}; \widetilde{M}[-])$ at X. The $(2(k-1))$–equivalences of the k–th spaces in these two spectra make the two of them equivalent, and thus the two derivatives agree, as we needed to show. $\qquad\square$

10 Proof of Technical Lemma 9.4, Part I: the homotopy fibers are abstractly $2k$–equivalent for k–connected N

In this section we are going to prove that for k–connected N, the two fibers in Technical Lemma 9.4 in are both $2k$–equivalent to $\mathrm{THH}(R \ltimes M, B.N)$. In Section 11 we will prove that the trace map β induces the abstract equivalence obtained in this section.

Lemma 10.1 *Let R be an associative ring with unit, and let M and N be two simplicial R–bimodules, then*

$$K(R \ltimes M; B.N) \simeq \mathrm{hofib}(\widetilde{K}(R; B.M \oplus B.N) \to \widetilde{K}(R; B.M)).$$

Proof Using the isomorphisms $B.M \oplus B.N \simeq B.(M \oplus N)$ and $R \ltimes (M \oplus N) \cong (R \ltimes M) \ltimes N$ the result follows from the commuting diagram whose vertical maps are equivalences by [5, 4.1]

(10-1)

$$
\begin{array}{ccccc}
\mathrm{hofib} & \longrightarrow & \widetilde{K}(R; B.M \oplus B.N) & \longrightarrow & \widetilde{K}(R; B.M) \\
\Big\downarrow{\simeq} & & \Big\downarrow{\simeq} & & \Big\downarrow{\simeq} \\
\widetilde{K}(R \ltimes M; B.N) & \longrightarrow & \widetilde{K}((R \ltimes M) \ltimes N) & \longrightarrow & \widetilde{K}(R \ltimes M)
\end{array}
$$

as well as the identification of the homotopy fiber on the bottom row. $\qquad\square$

We are interested in $\widetilde{K}(R \ltimes M; B.N)$ in a $2k$–connected range when N is k–connected. By [5, Theorem 3.4], the trace map $\widetilde{K}(R \ltimes M; B.N) \to \mathrm{THH}(\underline{R \ltimes M}; \underline{B.N})$ is $(2(k+1))$–connected if N is k–connected. We let γ be the composite obtained

using the natural splitting of the rows in (10-1):

(10-2)

$$\tilde{K}(R, B.(M \oplus N)) \xrightarrow{\simeq} \tilde{K}(R \ltimes (M \oplus N)) \xrightarrow{\cong} \tilde{K}((R \ltimes M) \ltimes N)$$

$$\downarrow \gamma \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathrm{THH}(R \ltimes M; B.N) \xleftarrow{\quad 2(k+1) \quad} \tilde{K}(R \ltimes M; B.N)$$

Proposition 10.2 *For a ring R and simplicial R–bimodules M and N, using the natural map γ from (10-2), we obtain*

$$\tilde{K}(R; B.M \oplus B.N) \xrightarrow{2(k+1)} \mathrm{THH}(R \ltimes M; B.N) \times \tilde{K}(R; B.M).$$

In order to identify the homotopy fiber of the map

$$W(R; B.M \oplus B.N) \to W(R; B.M)$$

we first observe that by the multilinearity of $U^n(F; -)$, for any FSP F and F–bimodules P_0 and P_1, we have a C_n–equivariant decomposition

$$U^n(F; P_0 \oplus P_1) \xrightarrow{\simeq} \prod_{\alpha \in \mathrm{Hom}_{\mathrm{Sets}}(\{1,2,\ldots,n\},\{0,1\})} U^n(F; P_{\alpha(1)}, \ldots, P_{\alpha(n)}).$$

Thus, if P_1 is k–connected we have a $2k$–connected C_n–equivariant map

(10-3) $U^n(F; P_0 \oplus P_1) \to U^n(F; P_0) \times (C_n)_+ \wedge U^n(R; P_0, \ldots, P_0, P_1).$

Taking C_n fixed points, this gives a map

$$U^n(F; P_0 \oplus P_1)^{C_n} \to U^n(F; P_0)^{C_n} \times U^n(R; P_0, \ldots, P_0, P_1).$$

We obtain maps for all m dividing n,

$$U^n(F; P_0 \oplus P_1)^{C_n} \xrightarrow{\mathrm{res}} U^m(F; P_0 \oplus P_1)^{C_m} \to U^m(R; P_0 \ldots, P_0, P_1)$$

and hence maps ϵ_n

(10-4) $U^n(F; P_0 \oplus P_1)^{C_n} \xrightarrow{\epsilon_n} U^n(F; P_0)^{C_n} \times \prod_{m|n} U^m(F; P_0, \ldots, P_0, P_1)$

which take the restriction maps for the fixed points of the U's applied to $P_0 \oplus P_1$ to the restriction maps of the U's applied to P_0 and the projections on the product.

Lemma 10.3   *The map of homotopy inverse limits produced by the ϵ_n's*

$$W(R; B.M \oplus B.N) \xrightarrow{\epsilon} \mathrm{holim}_{\mathbb{N} \times}\left(U^n(R; B.M)^{C_n} \times \prod_{k|n} U^k(R; B.M, \dots, B.M, B.N)\right)$$

$$\Big\downarrow \cong$$

$$W(R; B.M) \times \prod_{a=0}^{\infty} U^a(R; B.M, \dots, B.M, B.N)$$

is $(2(k+1)-1)$–connected if N is k–connected.

Proof   It is enough to show the map for each W_n to $\mathrm{holim}_{\{\leq n\}}$ is $(2(k+1))$–connected. By Corollary 5.9 it is enough to show that the C_n–homotopy orbits of the maps in (10-3) are $(2(k+1))$–connected, which they are since homotopy orbits preserve connectivity.   □

In order to relate $\prod_{a=0}^{\infty} U^a(R; B.M, \dots, B.M, B.N)$ to $\mathrm{THH}(R \ltimes M, B.N)$, a proposition motivated by the result in [15] will be useful. For $R \to S$ a map of FSP's and M an S–bimodule we can form a bisimplicial spectrum

$$[p] \times [q] \mapsto U_q^{p+1}(R; S, \dots, S, M),$$

where the q–direction has the usual (diagonal) simplicial structure of U and the p direction has the evident simplicial structure defined so that

$$U_0^{*+1}(R; S \dots, S, M) \cong \mathrm{THH}(S, M).$$

Proposition 10.4   *If $R \to S$ is a map of FSP's over the same underlying set and M is an S–bimodule, then the natural map*

$$U^1(S; M) \to U^{*+1}(R; S^*, M)$$

given by the inclusion of the zero simplicial dimension

$$U_0^{*+1}(R; S^*, M) = U_*^1(S; M)$$

is an equivalence.

Proof   Since both theories are linear in the bimodule M variable it is enough to show that the map is an equivalence for $M = S \otimes X \otimes S$ for X a spectrum (and the tensor product taken over the sphere spectrum). More general bimodules M can be resolved by bimodules of this form, using the functor $X \mapsto S \otimes X \otimes S$ from spectra to S–bimodules and its adjoint, the forgetful functor. In the case of $M = S \otimes X \otimes S$, one can "break" the circles to get

$$U_*^{n+1}(R; S^n, M) \simeq X \otimes \overbrace{S \,\widehat{\otimes}_R\, S \widehat{\otimes} \cdots \widehat{\otimes}_R\, S}^{n+2 \text{ times}}.$$

The induced structure maps (from the unused simplicial direction) are simply $X \otimes (\,)$ applied to the standard bimodule resolution of S as an R–algebra (ie multiplication on the "insides", no twists) and hence since this is homotopy equivalent to S again (using the extra degeneracy map), we get $X \otimes S$. And $U_*^1(S; M)$ is homotopy equivalent to $X \otimes S$, by a similar argument; this $X \otimes S$ can be embedded in $U_*^1(S; M)$ within M to map to an $X \otimes S$ within M in $U^{*+1}(R; S^*, M)$ which is homotopy equivalent to the whole $U^{*+1}(R; S^*, M)$. $\qquad\square$

Corollary 10.5 *For a ring R and simplicial R–bimodules M and N,*

$$\mathrm{THH}(R \ltimes M; B.N) \simeq \prod_{a=0}^{\infty} U^a(R; B.M, \ldots, B.M, B.N).$$

Proof By Proposition 10.4, we have

$$\mathrm{THH}(R \ltimes M; B.N) \xrightarrow{\simeq} U^{*+1}(R; (R \ltimes M)^*, B.N).$$

Using the multilinearity of the U's, if we let $S^k = \Delta^k / \partial$, we have by the calculations at the end of the next section an equivalence ρ_A of bisimplicial spectra

$$U^{*+1}(R; (R \ltimes M)^*, B.N) \cong \prod_{a=0}^{\infty} U^{a+1}(R; M, \ldots, M, B.N) \otimes S^a.$$

Using the fact that

$$S^a \simeq \overbrace{S^1 \wedge \cdots \wedge S^1}^{a \text{ times}},$$

the fact that $B.M \simeq M \otimes S^1$, and the multilinearity of the U's, we have that each $U^{a+1}(R; M, \ldots, M, B.N) \otimes S^a$ is equivalent to $U^{a+1}(R; B.M, \ldots, B.M, B.N)$ and hence the result. $\qquad\square$

Corollary 10.6 *For a ring R and simplicial R–bimodules M and N, using the map ϵ and the equivalences of Lemma 10.3 and Corollary 10.5, we have a $(2k+1)$–connected map*

$$W(R; B.M \oplus B.N) \xrightarrow{2k+1} \mathrm{THH}(R \ltimes M; B.N) \times W(R; B.M).$$

11 Proof of Technical Lemma 9.4, Part II: the trace induces the $2k$–equivalence

In Section 10 we showed that the two fibers used in Technical Lemma 9.4 agree in a $2k + 1$ range if the bimodule N being considered was k–connected. In this section

we will show that the trace map β defined in Section 9 induces the equivalence on the fibers in a $2k$ range as suggested by Section 10. We now recall, from [19], primarily page 218, and [5, Section 4], details that allow us to construct an unstable model for the composite γ used in Proposition 10.2.

If $\mathcal{P}$ is a category of diagrams of projective right R–modules and $\mathcal{M}$ is the category of diagrams of the same form of right R–modules, we will for brevity write

$$R(P) = \operatorname{Hom}_{\mathcal{P}}(P, P),$$
$$M_P = \operatorname{Hom}_{\mathcal{M}}(P, P \otimes_R M)$$

for any $P \in \mathcal{P}$ and any R–bimodule M.

We recall, using this notation, that for a discrete ring R,

$$\widetilde{K}(R; M) = \text{stabilization w.r.t. Waldhausen's S-construction of } \mathcal{P} \mapsto \bigvee_{P \in \mathcal{P}} M_P$$

on $\mathcal{P}_R$. As in the proof of [5, Theorem 3.4], the point here is that

$$K(R; M) = \Omega \left| \coprod_{\overline{c} \in S.\mathcal{P}_R} \operatorname{Hom}_{S.\mathcal{M}_R}(\overline{c}, \overline{c} \otimes M) \right|,$$
$$K(R) = K(R; 0) = \Omega |S.\mathcal{P}_R|.$$

By [21], the S–construction stabilizing maps induce equivalences

$$\Omega |S.\mathcal{P}_R| \xrightarrow{\simeq} \Omega^2 |S_.^{(2)}\mathcal{P}_R| \xrightarrow{\simeq} \Omega^3 |S_.^{(3)}\mathcal{P}_R| \xrightarrow{\simeq} \cdots$$

and similarly

$$\Omega \left| \coprod_{\overline{c} \in S.\mathcal{P}_R} \operatorname{Hom}_{S.\mathcal{M}_R}(\overline{c}, \overline{c} \otimes M) \right| \xrightarrow{\simeq} \Omega^2 \left| \coprod_{\overline{c} \in S_.^{(2)}\mathcal{P}_R} \operatorname{Hom}_{S_.^{(2)}\mathcal{M}_R}(\overline{c}, \overline{c} \otimes M) \right| \xrightarrow{\simeq} \cdots.$$

As is shown in the proof of [5, Theorem 3.4], for each p the map

$$\operatorname{hofib} \left(\left| \coprod_{\overline{c} \in S_.^{(p)}\mathcal{P}_R} \operatorname{Hom}_{S^{(p)}\mathcal{M}_R}(\overline{c}, \overline{c} \otimes M) \right| \to |S_.^{(p)}\mathcal{P}_R| \right)$$

$$\to \operatorname{hocofib} \left(|S_.^{(p)}\mathcal{P}_R| \xrightarrow{0} \left| \coprod_{\overline{c} \in S_.^{(p)}\mathcal{P}_R} \operatorname{Hom}_{S_.^{(p)}\mathcal{M}_R}(\overline{c}, \overline{c} \otimes M) \right| \right)$$

$$\simeq \left| \bigvee_{\overline{c} \in S_.^{(p)}\mathcal{P}_R} \operatorname{Hom}_{S^{(p)}\mathcal{M}_R}(\overline{c}, \overline{c} \otimes M) \right|$$

is $(2p-3)$–connected (where 0 is the map which sends every $\bar{c}$ to the zero map $\bar{c} \to \bar{c} \otimes M$), and thus the homotopy cofiber spectrum is the same as the spectrum $\widetilde{K}(R; M)$.

By Proposition 6.14 and Corollary 6.17, we know that

$$U^n(R, M)^{C_n} = \text{stabilization w.r.t. the S–construction of } \mathcal{P} \mapsto \bigvee_{P \in \mathcal{P}} U_0^n(\underline{\mathcal{P}}|_P; \underline{M}|_P)^{C_n}$$

on $\mathcal{P}_R$. We observe that for $P \in \mathcal{P}_R$ and N an $(R \ltimes M)$-module which M acts on by the zero action:

$$\text{Hom}_{\mathcal{P}_{R \ltimes M}}(P \otimes_R (R \ltimes M), P \otimes_R (R \ltimes M) \otimes_{R \ltimes M} N) \cong \text{Hom}_{\mathcal{P}_R}(P, P \otimes_R N).$$

In other words,

$$N_{P \otimes_R R \ltimes M} \cong N_P.$$

Similarly,

$$\text{Hom}_{\mathcal{P}_{R \ltimes M}}(P \otimes_R (R \ltimes M), P \otimes_R (R \ltimes M))$$
$$\cong \text{Hom}_{\mathcal{P}_R}(P, P \otimes_R (R \ltimes M)) \cong \text{Hom}_{\mathcal{P}_R}(P, P) \oplus \text{Hom}_{\mathcal{P}_R}(P, P \otimes_R M),$$

and taking into account the way the composition of the two summands works, we get

$$(R \ltimes M)(P \otimes_R (R \ltimes M)) \cong R(P) \ltimes M_P.$$

Given $m \in M_P$, we let $(1, m)$ be the obvious isomorphism in $R(P) \ltimes M_P$. We note that since the product of any two elements in M is zero in $R \ltimes M$, $(1, m) \circ (1, m') = (1, m + m')$, and because the product of any element of M with any element of N is zero, for $n \in N_P$, the two compositions

$$P \otimes_R (R \ltimes M) \xrightarrow{(1,m)} P \otimes_R (R \ltimes M) \xrightarrow{n} P \otimes_R (R \ltimes M) \otimes_{R \ltimes M} N,$$

$$P \otimes_R (R \ltimes M) \xrightarrow{n} P \otimes_R (R \ltimes M) \otimes_{R \ltimes M} N \xrightarrow{(1,m) \otimes \text{id}_N} P \otimes_R (R \ltimes M) \otimes_{R \ltimes M} N$$

are both equal to the map n. Thus, we have a representation

$$M_P \xrightarrow{1+\star} \text{GL}(R(P) \ltimes M_P)$$

whose image acts as the identity on N_P. Here GL is used to indicate the invertible elements.

We obtain a natural simplicial map

$$B_* M_P \times N_P \xrightarrow{B_*(1+\star)} \text{THH}_*(R(P) \ltimes M_P; N_P)$$

where $R(P) \ltimes M_P$ is viewed as an FSP on the category consisting of one point with a morphism for each element of the ring $R(P) \ltimes M_P$, and N_P as a bimodule over it. Note that

$$R(P) \ltimes M_P \cong \underline{\mathcal{P}_{R \ltimes M}}|_{P \otimes_R (R \ltimes M)},$$
$$N_P \cong \underline{N}|_{P \otimes_R (R \ltimes M)}.$$

Lemma 11.1   *The natural transformation*

$$\bigvee_{P \in \mathcal{P}_R} \mathrm{THH}_*(R(P) \ltimes M_P; N_P) \xrightarrow{\star \otimes_R R \ltimes M} \bigvee_{P \in \mathcal{P}_{R \ltimes M}} \mathrm{THH}_*(R(P) \ltimes M_P; N_P)$$

(obtained by tensoring the indexing category $\star \otimes_R R \ltimes M$) is an equivalence after stabilization.

Proof   Since $R \ltimes M$ is an R–bimodule, $\underline{R \ltimes M}$ is clearly a $\underline{\mathcal{P}_R}$–bimodule. But since $R \ltimes M$ is actually an R–algebra, this $\underline{\mathcal{P}_R}$–bimodule is an FSP on $\mathcal{P}_R$ in its own right, and so we can take U's of it. The domain of the map in the statement of the lemma can be written as

$$\bigvee_{P \in \mathcal{P}_R} U_*^1(R(P) \ltimes M_P; N_P) = \bigvee_{P \in \mathcal{P}_R} U_*^1(\underline{R \ltimes M}|_P; \underline{N}|_P)$$
$$= \bigvee_{P \in \mathcal{P}_R} U_0^{*+1}(\underline{\mathcal{P}_R}|_P; \underline{R \ltimes M}|_P^*, \underline{N}|_P),$$

and so stabilizes to $U^{*+1}(\underline{\mathcal{P}_R}; \underline{R \ltimes M}^*, \underline{N})$ by the nonequivarant analog of Corollary 6.17 which allows different bimodules in the $*+1$ bimodule positions (and is proved in exactly the same way).

The target of the map can be written as

$$\bigvee_{P \in \mathcal{P}_{R \ltimes M}} U_*^1(R(P) \ltimes M_P; N_P) = \bigvee_{P \in \mathcal{P}_{R \ltimes M}} U_*^1(\underline{\mathcal{P}_{R \ltimes M}}|_P, \underline{N}|_P)$$
$$= \bigvee_{P \in \mathcal{P}_{R \ltimes M}} U_0^{*+1}(\underline{\mathcal{P}_{R \ltimes M}}|_P; \underline{R \ltimes M}|_P^*, \underline{N}|_P),$$

and similarly stabilizes to $U^{*+1}(\underline{\mathcal{P}_{R \ltimes M}}; \underline{R \ltimes M}^*, \underline{N})$. On $\mathcal{P}_{R \ltimes M}$, $\underline{R \ltimes M}$ is the same as $\underline{\mathcal{P}_{R \ltimes M}}$, but we write it in this way to keep the parallel clear. If we abbreviate the map $\star \otimes_R (R \ltimes M)$ to $t \colon \mathcal{P}_R \to \mathcal{P}_{R \otimes M}$, the map in the statement of the lemma stabilizes to the obvious map induced by t, noting that $\underline{R \ltimes M}$ on $\mathcal{P}_R$ is the same as t^* of $\underline{R \ltimes M}$ on $\mathcal{P}_{R \ltimes M}$ and that $\underline{N}$ on $\mathcal{P}_R$ is the same as t^* of $\underline{N}$ on $\mathcal{P}_{R \ltimes M}$.

Thus the map we are interested in is the bottom horizontal map in the commutative diagram

$$U^{*+1}(\underline{\mathcal{P}_R}|_R; (t^*\underline{R \ltimes M})|_R^*, (t^*\underline{N})|_R) \xrightarrow{t_*} U^{*+1}(\underline{\mathcal{P}_{R\ltimes M}}|_{R\ltimes M}; \underline{R \ltimes M}|_{R\ltimes M}^*, \underline{N}|_{R\ltimes M})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$U^{*+1}(\underline{\mathcal{P}_R}; (t^*\underline{R \ltimes M})^*, t^*\underline{N}) \xrightarrow{\quad t_* \quad} U^{*+1}(\underline{\mathcal{P}_{R\ltimes M}}; \underline{R \ltimes M}^*, N)$$

where the vertical maps are induced by the inclusion. By the nonequivariant analog of Proposition 6.13 which allows different bimodules in the $* + 1$ bimodule positions (and is proved in exactly the same way), these vertical maps are equivalences, so to show that the bottom horizontal map is an equivalence, it suffices to show that the top horizontal map is. The top horizontal map is a map of U's of FSP's and bimodules over a one point set, and can also be written as the map

$$U^{*+1}(R; (R \ltimes M)^*, N) \xrightarrow{t_*} U^{*+1}(R \ltimes M; (R \ltimes M)^*, N)$$

induced by t on the FSP's. It is an equivalence because of the diagram

$$U_0^{*+1}(R; (R \ltimes M)^*, N) =\!=\!= U_0^{*+1}(R \ltimes M; (R \ltimes M)^*, N)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$U^{*+1}(R; (R \ltimes M)^*, N) \xrightarrow{t_*} U^{*+1}(R \ltimes M; (R \ltimes M)^*, N)$$

where the vertical maps are equivalences by the nonequivariant analog of Proposition 10.4 which allows different bimodules in the $* + 1$ bimodule positions (and is proved in exactly the same way). □

Model 11.2 (A model for γ) *The stabilization of the composite*

$$\bigvee_{P \in \mathcal{P}} B_* M_P \times N_P \xrightarrow{\quad 1 + \star \quad} \bigvee_{P \in \mathcal{P}} \mathrm{THH}_*(R(P) \ltimes M_P; N_P)$$

$$\xrightarrow{\quad \star \otimes_R R \ltimes M \quad} \bigvee_{P \in \mathcal{P}_{R\ltimes M}} \mathrm{THH}_*(R(P) \ltimes M_P; N_P)$$

is a natural transformation from $K(\underline{\mathcal{P}_R}; \underline{B.M \oplus N})$ *to* $K(\underline{\mathcal{P}_{R\ltimes M}}; \underline{N})$. *When* N *is replaced by* $B.N$, *this (by [5; 19]) is a model for the natural transformation* γ *used in Proposition 10.2.*

By the discussion in the beginning of Section 9, for any R–bimodule M, the map of spaces (not spectra) from M_P to $U_0^n(\underline{\mathcal{P}_R}, \underline{M})^{C_n}$

$$m \mapsto m \wedge m \wedge \cdots \wedge m$$
$$\in [\,\underline{M}\,|_P(S^0) \wedge \cdots \wedge \underline{M}\,|_P(S^0)]^{C_n}$$
$$\to [\mathrm{holim}_{I^{k+1}} \Omega^{X_0 \sqcup \ldots \sqcup X_k}(\underline{M}\,|_P(S^{X_0}) \wedge \underline{M}\,|_P(S^{X_1}) \wedge \cdots \wedge \underline{M}\,|_P(S^{X_k})]^{C_n}$$

stabilizes to the map $\widetilde{K}(R; M) \to U^n(\underline{\mathcal{P}_R}; \underline{M})^{C_n}$ used in the construction of the trace map β. Using this and the definition of the maps ϵ_n in (10-4) we obtain the following.

Model 11.3 (A model for the W fiber map) *The composite*

$$K(\underline{\mathcal{P}_R}; \underline{M} \oplus \underline{N}) \xrightarrow{\beta} W(\underline{\mathcal{P}_R}; \underline{M} \oplus \underline{N}) \to \prod_{a=1}^{\infty} U^a(\underline{\mathcal{P}_R}; \underline{M}^{a-1}, \underline{N})$$

is equivalent to the stabilization of the natural transformation determined by the product of the maps β_a: $\bigvee_{P \in \mathcal{P}} (\underline{M} \oplus \underline{N})|_P \to \bigvee_{P \in \mathcal{P}} U_0^a(\underline{\mathcal{P}_R}|_P; \underline{M}\,|_P^{a-1}, \underline{N}\,|_P)$ *given by*

$$\beta_a(m \times n) = \overbrace{m \wedge \cdots \wedge m}^{(a-1) \text{ times}} \wedge n.$$

Definition 11.4 Define bisimplicial spectra $U(a)(R(P); M_P, N_P)$ by

$$U(a)_\star^*(R(P); M_P, N_P) = \prod_{1 \le j_1 < \cdots < j_a \le *} U_\star^{*+1}(R(P); F_1, \ldots, F_*, N_P)$$

with

$$F_t = \begin{cases} M_P & \text{if } t \in \{j_1, \ldots, j_a\}, \\ R(P) & \text{otherwise.} \end{cases}$$

In other words, if we break $R(P) \ltimes M_P$ as a bimodule down into $R(P) \oplus M_P$, then $U(a)(R(P); M_P, N_P)$ is the part of $\mathrm{THH}(R(P) \ltimes M_P, N_P)$ which contains exactly a M_P's. The simplicial coordinate $\star$ comes from U's diagonal simplicial structure. The boundary maps from the simplicial coordinate $*$ involve multiplying adjacent bimodule coordinates in U.

By Models 11.2 and 11.3 and the equivalence in Corollary 10.5 it will suffice to prove the following proposition to finish the proof of Technical Lemma 9.4.

Proposition 11.5 *For R a ring and M and N R–bimodules, there are maps ρ_A which make the following diagram commute:*

$$
(11\text{-}1)\quad
\begin{array}{ccc}
B.M_P \times N_P & \xrightarrow{\ \ 1+\star\ \ } & \mathrm{THH}(R(P) \ltimes M_P; N_P) \\
\Big\downarrow{\scriptstyle \prod_a \beta_a} & & \Big\downarrow{\scriptstyle \simeq} \\
\prod_{a=0}^{\infty} U_0^a(R(P); B.M_P,\ldots,B.M_P, N_P) & \xrightarrow{\prod_a \rho_a} & \prod_{a=0}^{\infty} U(a)_0(R(P); M_P, N_P) \\
\Big\downarrow{\scriptstyle \mathrm{incl}} & & \Big\downarrow{\scriptstyle \simeq} \\
\prod_{a=0}^{\infty} U_\star^a(R(P); B.M_P,\ldots,B.M_P, N_P) & \xrightarrow[\simeq]{\prod_a \rho_a} & \prod_{a=0}^{\infty} U(a)_\star(R(P); M_P, N_P).
\end{array}
$$

We recall from Proposition 10.4 the equivalence

$$
\mathrm{THH}_*(R(P) \ltimes M_P, N_P) \cong U_0^{*+1}(R(P); R(P) \ltimes M_P, N_P)
$$
$$
\xrightarrow{\simeq} U_\star^{*+1}(R(P); R(P) \ltimes M_P, N_P).
$$

By the multilinear property of the U, we see that we have a natural equivalence of bisimplicial spectra

$$
U_\star^{*+1}(R(P); R(P) \ltimes M_P, N_P) \xrightarrow{\simeq} \prod_{a=0}^{\infty} U(a)_\star^*(R(P); M_P, N_P).
$$

Lemma 11.6 *For all a, $U(a)_0^*(R(P); M_P, N_P) \xrightarrow{\simeq} U(a)_\star^*(R(P); M_P, N_P)$.*

Proof We have a commutative diagram

$$
\begin{array}{ccc}
U_0^{*+1}(R(P); R(P) \ltimes M_P, N_P) & \xrightarrow{\ \simeq\ } & \prod_{a=0}^{\infty} U(a)_0^*(R(P); M_P, N_P) \\
\Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle \simeq} \\
U_\star^{*+1}(R(P); R(P) \ltimes M_P, N_P) & \xrightarrow{\ \simeq\ } & \prod_{a=0}^{\infty} U(a)_\star^*(R(P); M_P, N_P)
\end{array}
$$

where the horizontal maps are equivalences by the decomposition into homogeneous pieces as explained above and the left vertical map is an equivalence by Proposition 10.4. Thus, the right vertical map which makes the diagram commute, namely the inclusion, is an equivalence. However, this inclusion is a product of maps and hence each of them is an equivalence as well. $\qquad\square$

Composing $1 + \star$ with the projection equivalence from $\mathrm{THH}(R(P) \ltimes M_P, N_P) \xrightarrow{\simeq} \prod_{a=0}^{\infty} U(a)(R(P); M_P, N_P)$ we obtain simplicial maps

$$
(11\text{-}2)\qquad\qquad \eta^a \colon B_* M_P \times N_P \to U(a)_0^*(R(P); M_P, N_P)
$$

In order to better express the maps η^a we make the following observations. Let $\mathrm{Surj}_\Delta([a],[k])$ be the set of surjective monotone maps from $[a]$ to $[k]$. For $\sigma \in \mathrm{Surj}_\Delta([a],[k])$ and $1 \le j \le k$, we will write $\mu_j(\sigma) = \min\{\sigma^{-1}(j)\}$. We have an isomorphism of sets from $\mathrm{Surj}_\Delta([a],[k])$ to $\{1 \le j_1 < \cdots < j_k \le a\}$ given by sending $\sigma \in \mathrm{Surj}_\Delta([a],[k])$ to $\{\mu_1(\sigma), \dots, \mu_k(\sigma)\}$. With these conventions, the maps η^a are the composites

$$(m_1 \times \cdots \times m_k, n)$$

$$\mapsto \prod_{\sigma \in \mathrm{Surj}_\Delta([a],[k])} \sigma^*(m_{\mu_1(\sigma)} \wedge \cdots \wedge m_{\mu_k(\sigma)} \wedge n)$$

$$\in \Omega^{0 \sqcup \dots \sqcup 0} \prod_{1 \le \mu_1(\sigma) < \cdots < \mu_k(\sigma) \le a} (F_1[S^0] \wedge \cdots \wedge F_k[S^0] \wedge N_P[S^0])$$

$$\to \mathrm{holim}_{I^{k+1}} \Omega^{x_0 \sqcup \dots \sqcup x_k} \prod_{1 \le \mu_1(\sigma) < \cdots < \mu_k(\sigma) \le a} (F_1[S^{x_0}] \wedge \cdots \wedge F_k[S^{x_{k-1}}] \wedge N_P[S^{x_k}])$$

$$= U(a)_0^k(R(P); M_P, N_P),$$

where F_t are as, as before, M_P if $t = \mu_j(\sigma)$ for some j and $R(P)$ otherwise, and where $\sigma^*(m_{\mu_1(\sigma)} \wedge \cdots \wedge m_{\mu_k(\sigma)} \wedge n)$ has n in the last coordinate, $m_{\mu_j(\sigma)}$ in the $(\mu_j(\sigma)-1)$–st coordinate, $1 \le j \le a$, and 1_R in the others.

We now define, for all a, bisimplicial maps (which are equivalences)

$$U_\star^a(R(P); (B_* M_P)^a, N_P) \to U(a)_\star^*(R(P); M_P, N_P).$$

In order to do this, we first make a few general remarks about simplicial constructions and then apply them to this specific application.

Let $S^k = \Delta^k/\partial$ as a simplicial set. That is, $\Delta_n^k = \mathrm{Hom}_\Delta([n],[k])$ and ∂ is the usual subsimplicial set determined by the $k-1$ subskeleton. We also have the simplicial set

$$\bigwedge^k S^1 := \overbrace{S^1 \wedge \cdots \wedge S^1}^{k \text{ times}}$$

by which we mean the diagonal of the obvious k–fold multisimplicial set. We know that after realization, this is homeomorphic to S^k but there is not a simplicial map that realizes to a homeomorphism. There are exactly $k!$ simplicial maps from $\bigwedge^k S^1$ to S^k which realize to homotopy equivalences, one for each of the nondegenerate cells of $\bigwedge^k S^1$ in dimension k. We will be wanting to use one of these simplicial maps.

We can first look at S^k. It has one point in every simplicial dimension less than k. In dimension k it has one nondegenerate element which we can identify with the identity

on $[k]$. In dimension n larger than k it has an element for every ordered surjection from $[n]$ to $[k]$ and one other point, $*$. Thus, we can write

$$S_n^k = \mathrm{Surj}_\Delta([n], [k])_+$$

Now we want to write down simplicial maps from $\bigwedge^k S^1$ to S^k which are homotopy equivalences. The critical dimension is k. Given a sequence $(\tau_1, \dots, \tau_k)$ of surjections $[n] \to [1]$, we can associate a sequence of nonzero positive integers by looking at the cardinality of the inverse image of 0, ie $\{|\tau_1^{-1}(0)|, |\tau_2^{-1}(0)|, \dots, |\tau_k^{-1}(0)|\}$. We can also express this as $\{\mu_1(\tau_1), \mu_1(\tau_2), \dots, \mu_1(\tau_k)\}$ since $\mu_1(\tau) = \min(\tau^{-1}(1)) = |\tau^{-1}(0)|$. A sequence

$$(\tau_1, \dots, \tau_k) \in (\bigwedge^k S^1)_k$$

is nondegenerate if and only if the sequence $\{\mu_1(\tau_1), \mu_1(\tau_2), \dots, \mu_1(\tau_k)\}$ has no repeated terms.

We define α to be the following simplicial map from $\bigwedge^k S^1$ to S^k: On the $k-1$ skeleton it must be trivial, in simplicial dimension k it takes the nondegenerate simplex whose sequence is $(1, 2, \dots, k)$ to the identity in $(S^k)_k$ and all others to the basepoint. This determines a map of the k–skeletons and hence by extension of degeneracies a map of the simplicial sets (as both have only degenerate simplices in dimensions greater than k).

We actually need to understand α explicitly in all simplicial dimensions. Following the degeneracies in both settings, we see that α sends a simplex $(\tau_1, \dots, \tau_k) \in (\bigwedge^k S^1)_n$ to the basepoint if the sequence $\{\mu_1(\tau_1), \mu_1(\tau_2), \dots, \mu_1(\tau_k)\}$ associated to it in $\{1, 2, \dots, n\}^k$ is not monotone increasing. If the sequence is monotone increasing, then $(\tau_1, \dots, \tau_k)$ is sent to the monotone surjection $\alpha(\tau_1, \dots, \tau_n) \in \mathrm{Surj}_\Delta([n], [k])$ given by

$$(11\text{-}3) \qquad\qquad \mu_j(\alpha(\tau_1, \dots, \tau_n)) = \mu_1(\tau_j).$$

We now return to constructing bisimplicial equivalences

$$\rho_a \colon U_\star^a(R(P); B_* M_P, \dots, B_* M_P, N_P) \;\to\; U(a)_\star^*(R(P); M_P, N_P).$$

We observe that because U^n commutes with realizations in each of its bimodule variables, we have a simplicial map which is an equivalence in each simplicial dimension:

$$(11\text{-}4) \quad U_\star^a(R(P); B_* M_P, \dots, B_* M_P, N_P)$$

$$\xrightarrow{\;\simeq\;} U_\star^a(R(P); M_P, \dots, M_P, N_P) \otimes \bigwedge^a S^1.$$

Applying the simplicial map α we obtain an equivalence

$$U_\star^a(R(P); M_P, \ldots, M_P, N_P) \otimes \bigwedge\nolimits^a S^1 \xrightarrow{\alpha} U_\star^a(R(P); M_P, \ldots, M_P, N_P) \otimes S^a.$$

We have a bisimplicial map which is an equivalence

$$U_\star^a(R(P); M_P, \ldots, M_P, N_P) \otimes S_\star^a \to U(a)_\star^*(R(P); M_P, N_P),$$

given in each dimension k by

$$
\begin{aligned}
U_k^a(R(P); M_P, &\ldots, M_P, N_P) \otimes S_k^a \\
&= \bigvee_{\sigma \in \mathrm{Surj}_\triangle([k],[a])} U_k^a(R(P); M_P, \ldots, M_P, N_P) \\
&\cong \prod_{\sigma \in \mathrm{Surj}_\triangle([k],[a])} U_k^a(R(P); M_P, \ldots, M_P, N_P) \\
&\to \prod_{\sigma \in \mathrm{Surj}_\triangle([k],[a])} \sigma^*(U_k^a(R(P); M_P, \ldots, M_P, N_P)) \\
&\to U(a)_k^*(R(P); M_P, N_P).
\end{aligned}
$$

(11-5)

The map (11-5) includes into the product of all maps in

$$U_k^{*+1}(R(P); F_1, F_2, \ldots, F_*, N_P),$$

where a of the F_i are M_P and the other $* - a$ are $R(P)$, and the maps are anything within the combined M_P–coordinates and just a smash of the unit map $1_{R(P)}$ in the $R(P)$ coordinates. It is not an equivalence in each separate simplicial degree k, but taken over all k combined, it does induce an equivalence since for every $*$, the inclusion

$$U_\star^{a+1}(R(P); M_P^a, N_P) \xrightarrow{\simeq} U_\star^{*+1}(R(P); F_1, F_2, \ldots, F_*, N_P)$$

is an equivalence, because taking $\otimes_{R(P)}^\wedge R(P)$ has no effect on $R(P)$–bimodules.

We define ρ_a to be the composite of these equivalences.

Proof of Proposition 11.5 We have just defined equivalences ρ_A, so we only need to show that the diagram (11-1) commutes, that is, that $\eta^a = \rho_a \circ \beta_a$ for all a, where η^a is the piece of $1 + \star$ as defined in (11-2) and β_a was defined in the Model 11.3. Let $(m_1, \ldots, m_k, n) \in B_k M_P \times N_P$. Then $\beta_a(m_1, \ldots, m_k, n)$ is the image in the homotopy colimit of the k–simplex (which is the diagonal k in the product of a k–simplices) $(m_1 \times \cdots \times m_k)^{\wedge a} \wedge n$. Via the equivalence in (11-4), this element is mapped to the

product in the homotopy colimit represented by

$$\prod_{\gamma_1,\ldots,\gamma_a\in\mathrm{Surj}_\triangle([k],[1])} (m_{\mu_1(\gamma_1)}\wedge m_{\mu_1(\gamma_2)}\wedge\cdots\wedge m_{\mu_1(\gamma_a)}\wedge n)$$

The simplicial map α will send the nonmonotone factors to a point; in the remaining factors, α composed with the map in (11-5) to the image in the colimit of $\prod_{\sigma\in\mathrm{Surj}_\triangle([k],[a])}\sigma^*(m_{\mu_1(\sigma)}\wedge\cdots\wedge m_{\mu_k(\sigma)}\wedge n)$ which is the image of η^a (the confusing point here is that the indexing set of the k–simplices is written as a product rather than a sum). $\square$

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Department of Mathematics, Indiana University
831 E Third St, Bloomington IN 47405, USA

Department of Mathematics, University of Illinois, Urbana
1409 W Green Street, Urbana IL 61801, USA

alindens@indiana.edu, rmccrthy@illinois.edu