Lagrangian topology and enumerative geometry

PAUL BIRAN
OCTAV CORNEA

We analyze the properties of Lagrangian quantum homology (in the form constructed in our previous work, based on the pearl complex) to associate certain enumerative invariants to monotone Lagrangian submanifolds. The most interesting such invariant is given as the discriminant of a certain quadratic form. For 2–dimensional Lagrangians it corresponds geometrically to counting certain types of configurations involving pseudoholomorphic disks that are associated to triangles on the respective surface. We analyze various properties of these invariants and compute them and the related structures for a wide class of toric fibers. An appendix contains an explicit description of the orientation conventions and verifications required to establish quantum homology and the related structures over the integers.

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1 Introduction

The main motivation for this paper is the search for enumerative invariants for Lagrangian submanifolds. One of the simplest fundamental questions in this topic can be formulated as follows. Fix a closed, connected Lagrangian submanifold L^n inside some symplectic manifold (M^{2n}, ω). Fix an almost complex structure J on M that is compatible with ω and let $P, Q, R \in L$ be three distinct points.

Problem Estimate the number $n_{PQR}(L, J)$ of disks $u\colon (D^2, \partial D^2) \to (M, L)$ that are J–holomorphic (in the sense that $\overline{\partial}_J(u) = 0$; see McDuff and Salamon [39]) and that go through P, Q, R in this order.

It is easily seen that for this question to make sense one should restrict to generic almost complex structures J and, to ensure that the number in question is finite, we have to consider only those disks u belonging to homotopy classes $\lambda \in \pi_2(M, L)$ so that the Maslov index $\mu(\lambda)$ of λ equals $2n$. The count providing the number $n_{PQR}(L, J) \in \mathbb{Z}$ takes into account appropriate orientations. Ideally, one would like to obtain more refined estimates by evaluating the numbers $n_{PQR}(L, J; \lambda)$ of disks u as above that belong to each specific class λ.

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1.1 An enumerative invariant

We work in this paper under the restriction that $L \subset M$ is a monotone Lagrangian, oriented and endowed with a fixed spin structure.

It is easy to see that the numbers $n_{PQR}(L, J)$ above are in general not invariant, as they depend on the choice of the points P, Q, R as well as on J. Thus, it is natural to investigate whether this lack of invariance can possibly be compensated by some more complicated enumerative "counts".

The origin of the present paper lies precisely in such a formula (closely related to expressions first detected in our paper [12]).

Assume that L is the 2–torus $\mathbb{T}^2$. Fix a triangle PQR on the torus. By this we mean three distinct points $P, Q, R \in L$ together with a smooth oriented path $\overrightarrow{PQ}$ starting from P and ending at Q as well as similar paths connecting Q to R and R to P. Fix also a generic almost complex structure J. Let n_P be the number of J–holomorphic disks of Maslov index 2 that go through P and cross transversely the edge $\overrightarrow{QR}$ (this number takes into account orientations – it is defined with more precision in Section 6.2). Define similarly the numbers n_Q and n_R.

We will see that if the Floer homology $HF(L, L) \neq 0$, then the expression

$$\Delta = 4n_{PQR} + n_P^2 + n_Q^2 + n_R^2 - 2n_P n_Q - 2n_Q n_R - 2n_R n_P \tag{1}$$

is independent of the triangle P, Q, R as well as of J.

1.2 Formula (1): its meaning and generalizations

In the paper we investigate the invariant Δ and the meaning of formula (1), besides, of course, proving this formula. We also provide a more general and conceptual perspective on other enumerative expressions in arbitrary dimensions. To summarize, we will see that:

- Δ coincides with the discriminant of a certain quadratic form that can be read off from the quantum homology product of L.
- Δ is actually the unique (symmetric) polynomial, enumerative invariant that can be extracted from the quantum product. Interestingly, this uniqueness is a consequence of the classification of polynomial invariants associated to quadratic forms as in Hilbert [34].
- Δ and/or other invariants like it, as well as formulae like (1), exist for more general Lagrangians and in arbitrarily high dimensions.

- There are refinements of these formulae that take into account the specific homotopy classes $\lambda \in \pi_2(M, L)$. They allow for these invariants to be written as expressions with coefficients in the ring of regular functions $\mathcal{R}$ of certain algebraic subvarieties of the variety of representations $\pi_2(M, L) \to \mathbb{C}^*$.
- In the case of toric fibres the ring of representation point of view is particularly useful as it relates Δ to the quantum Euler class of the ambient manifold.

In a number of examples, we also compute the relevant invariants explicitly over $\mathcal{R}$. Some remarkable numerical identities follow.

1.3 Structure of the paper

We now describe more thoroughly our approach and the structure of the paper.

Section 2 summarizes the main properties of Lagrangian quantum homology $QH(L)$ of L as described in our paper [14] together with a number of its algebraic properties. In particular, we recall that $QH(L)$ is a ring – we will denote the respective multiplication by $*$. We also fix a few basic orientation conventions. To avoid disrupting the natural flow of the paper, a complete and more technical discussion of orientations is postponed to Appendix A.

In Section 3 we consider the representation variety

$$\mathcal{R}ep(L) = \{\rho \colon \pi_2(M, L) \to \mathbb{C}^*\}.$$

We show that for a certain algebraic subset $\mathcal{W} \subset \mathcal{R}ep(L)$ the regular functions on $\mathcal{W}$, $\mathcal{O}(\mathcal{W})$, can be used as coefficient ring for quantum homology with the effect that the resulting object $Q^+H(L; \mathcal{W})$ is isomorphic as a vector space to the singular homology of L taken with the appropriate coefficients. The $+$ in Q^+H reflects the fact that quantum homology as constructed in [14] has some strong positivity features due to the fact that the various quantum structures are defined by using unperturbed J–holomorphic objects. A key consequence of positivity is that the algebra $Q^+H(L; \mathcal{W})$ is a deformation of singular homology – viewed as algebra with the intersection product.

In Section 4 we make use of this setting to define a quadratic form associated to the quantum product with coefficients in $\mathcal{O}(\mathcal{W})$ and its associated discriminant Δ. In the case of the 2–torus this will later be seen to be precisely the term on left hand side of (1).

Section 5 is based on the remark that the isomorphism between $Q^+H(L; \mathcal{W})$ and singular homology that was mentioned above is *not* canonical. In particular, if a specific isomorphism between quantum homology and singular homology is used to expand the quantum product with respect to a singular basis, $\mathbf{a} = (a_1, \dots, a_i, \dots)$, as

$$a_i * a_j = \sum_s k_s^{i,j} a_s t^{\epsilon'(i,j,s)}, \tag{2}$$

then the resulting structural constants $k_s^{i,j}$ are not invariants – they depend on J as well as on the other data used to define the various structures involved (here $\epsilon'(i,j,s)$ are appropriate integers – see Section 5.2; t is a formal deformation variable used in the definition of the quantum homology $Q^+H(L)$). Notice that this lack of invariance of the $k_s^{i,j}$'s is in marked contrast with the closed case where the same type of expansion of the classical quantum product produces structural constants that are identified with triple Gromov–Witten invariants.

On the other hand, the deformation equivalence class of $Q^+H(L; \mathcal{W}_2)$ (as deformation of the singular intersection algebra) is *invariant*. Thus, in searching for invariant enumerative expressions, it is natural to look for polynomial invariants in the $k_s^{i,j}$'s that only depend on this equivalence class. This type of invariants is introduced in Section 5 and most of the section is spent discussing them from a variety of points of view. It is also noticed that Δ as defined in Section 4, is a particular such invariant. Conceptually, one way to view this is by the prism of Hochschild cohomology. Indeed, this cohomology classifies algebra deformations and we notice that there is a natural map that associates to specific Hochschild cohomology classes (of the correct degrees) equivalence classes of quadratic forms. As we will see, Δ is simply the associated discriminant for these forms.

In Section 6 we start by revisiting formula (1) from a related but slightly different perspective. It turns out that Δ, as defined in Section 4 only depends on counts of J–holomorphic disks of Maslov class 2. Thus formula (1) can be viewed as a *splitting formula* expressing counts of Maslov 4 disks in terms of counts of configurations involving only Maslov 2 disks. The first part of Section 6 contains a general definition of such splitting formulae and a proof that they exist for monotone Lagrangians of arbitrary dimensions (under very mild assumptions). We also notice that, as illustrated by formula (1), there is a close relationship between the invariant polynomials described in Section 5 and these splitting formulae. The second part of Section 6 contains the proof of (a more general version) of (1).

As mentioned before, the role of the discriminant Δ is central in our study especially for Lagrangian tori. In view of this, in Section 7 we focus on a variety of further properties for Lagrangian tori that appear as fibres of the moment map in toric manifolds. An extensive study of Floer theory of such tori has been carried out by several authors, eg Cho [15; 16], Cho and Oh [18], Fukaya, Oh, Ohta and Ono [27; 25] and Auroux [6; 7; 8].

We build on these works and exemplify our theory on the case of toric fibres. In particular, we describe a relation between our machinery and the Frobenius structure on the quantum homology of the ambient toric manifold and see that the quantum Euler class viewed in the appropriate context can be identified with our discriminant Δ. Finally, Section 8 contains a series of explicit computations mostly for toric fibers.

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2 Setting

All our symplectic manifolds will be implicitly assumed to be connected and tame (see Audin, Lalonde and Polterovich [5]). The main examples of such manifolds are closed symplectic manifolds, manifolds which are symplectically convex at infinity as well as products of such. We denote by $\mathcal{J}$ the space of ω–compatible almost complex structures on M for which $(M, g_{\omega,J})$ is geometrically bounded, where $g_{\omega,J}$ is the associated Riemannian metric.

Lagrangian submanifolds $L \subset (M, \omega)$ will be assumed to be connected and closed. Write $H_2^D = H_2^D(M, L) = \text{image}\,(\pi_2(M, L) \longrightarrow H_2(M, L))$ for the image of the Hurewicz homomorphism. Denote by $\mu\colon H_2^D \longrightarrow \mathbb{Z}$ the Maslov index and by $N_L = \min\{\mu(A) \mid \mu(A) > 0\}$ the minimal Maslov number, so that $\mu(H_2^D) = N_L\mathbb{Z}$. Since Maslov numbers come in multiples of N_L we put $\bar{\mu} := (1/N_L)\mu$.

Denote by $\omega\colon H_2^D \longrightarrow \mathbb{R}$ the homomorphism induced by integration of ω. We will mostly assume that our Lagrangians are *monotone*, that is there exists a constant $\tau > 0$ such that

$$\omega(A) = \tau\mu(A) \quad \forall\, A \in H_2^D(M, L), \tag{3}$$

and moreover that $N_L \geq 2$.

2.1 Coefficient rings

Our ground ring will be denoted by K. We will mostly take $K = \mathbb{C}$, $\mathbb{Q}$, or $\mathbb{Z}$ and sometimes $\mathbb{Z}_2$. In case $K \neq \mathbb{Z}_2$ we implicitly assume that our Lagrangian L is orientable and spin and fix an orientation and a spin structure on L.

The following rings will be used frequently in the sequel: $\Lambda = K[t^{-1}, t]$, $\Lambda^+ = K[t]$. We grade these rings by setting $|t| = -N_L$. Next consider the group ring $K[H_2^D]$ whose elements we write as "polynomials" in the variable T, ie $P(T) = \sum_{A \in H_2^D} a_A T^A$, with $a_A \in K$. We grade this ring by setting $|T^A| = -\mu(A)$.

The most important ring for our considerations will be $\widetilde{\Lambda}^+$ which is defined as

$$\widetilde{\Lambda}^+ = \left\{ P(T) \in K[H_2^D] \;\middle|\; P(T) = a_0 + \sum_{A, \mu(A) > 0} a_A T^A \right\}.$$

Note that the degree 0 component of $\widetilde{\Lambda}^+$ is just K (ie constants) while that of $K[H_2^D]$ is the whole of $K[\ker \mu]$. We denote by $\widetilde{\Lambda}^{>0}$ the elements $P(T) \in \widetilde{\Lambda}^+$ with $\mu(A) > 0$ for every A (ie linear combinations of elements whose degree is negative).

In what follows we will work with $\widetilde{\Lambda}^+$–algebras. By this we mean commutative, graded rings $\mathcal{R}$ which are also graded algebras over $\widetilde{\Lambda}^+$. This structure is typically specified by a graded morphism of rings $\widetilde{\Lambda}^+ \longrightarrow \mathcal{R}$.

2.2 Lagrangian quantum homology and quantum structures

The pearl complex, Lagrangian quantum homology and its associated quantum structures have been described in detail in our works [13; 14; 12]. We refer the reader to these papers for the detailed constructions. Here we just set up the notation and recall the main properties of this homology. In addition, we explain how to carry out the construction over general ground rings K, other that $\mathbb{Z}_2$. This requires orienting the moduli spaces of pearly trajectories and is explained in detail in Appendix A.

Let $\mathcal{R}$ be an $\widetilde{\Lambda}^+$–algebra. Fix a triple $\mathscr{D} = (f, (\cdot,\cdot), J)$ where $f\colon L \longrightarrow \mathbb{R}$ is a Morse function $(\cdot,\cdot)$ is a Riemannian metric on L and J an ω–compatible almost complex structure on M. Denote by

$$\mathcal{C}(\mathscr{D}) = K\langle \mathrm{Crit} f \rangle \otimes \mathcal{R}, \quad d\colon \mathcal{C}_*(\mathscr{D}) \longrightarrow \mathcal{C}_{*-1}(\mathscr{D}),$$

the pearl complex with coefficients in $\mathcal{R}$. This complex is defined for generic $\mathscr{D}$, its homology does not depend on $\mathscr{D}$ and is denoted by $QH(L; \mathcal{R})$.

2.2.1 Product Recall that $QH(L;\mathcal{R})$ has the structure of an associative (but not necessarily commutative) ring with unity:

$$QH_i(L;\mathcal{R}) \otimes_{\mathcal{R}} QH_j(L;\mathcal{R}) \longrightarrow QH_{i+j-n}(L;\mathcal{R}), \quad \alpha \otimes \beta \longmapsto \alpha * \beta, \tag{4}$$

where $n = \dim L$. The unity lies in $QH_n(L;\mathcal{R})$ and is denoted by $[L]$ (in analogy to the fundamental class in singular homology).

2.2.2 Module structure Denote by $QH(M;\mathcal{R})$ the quantum homology of the (ambient) symplectic manifold (M,ω) endowed with the quantum product $*$. The extension of the coefficients to $\mathcal{R}$ is induced by the composition of the natural maps $\pi_2(M) \longrightarrow \pi_2(M,L) \longrightarrow H_2^D(M,L)$; see [14] for details. Then $QH(L;\mathcal{R})$ becomes an algebra over $QH(M;\mathcal{R})$ in the sense that there exists a canonical map

$$QH_i(M;\mathcal{R}) \otimes_{\mathcal{R}} QH_j(L;\mathcal{R}) \longrightarrow QH_{i+j-2n}(L;\mathcal{R}), \quad a \otimes \alpha \longmapsto a * \alpha, \tag{5}$$

which turns $QH(L;\mathcal{R})$ into an algebra over the ring $QH(M;\mathcal{R})$.

2.2.3 Inclusion We also have a quantum version of the map induced in homology by the inclusion $L \longrightarrow M$. This is a map

$$i_L\colon QH_*(L;\mathcal{R}) \longrightarrow QH_*(M;\mathcal{R}) \tag{6}$$

which extends the classical inclusion on the chain level. The map i_L is a $QH(M;\mathcal{R})$–module morphism.

2.2.4 Minimal models It is important throughout the paper that all the structures above are defined over $\widetilde{\Lambda}^+$ and that, at the chain level, they are deformations of the respective Morse-theoretic structures. The Morse theoretic structures (on the chain level) are obtained from the ones defined above by specializing to $T = 0$. For an algebraic structure defined over $V \otimes \Lambda^+$ where V is some K–vector space we will refer to the algebraic object obtained by specializing to $T = 0$ as the "Morse level" or "classical" associated structure.

A very useful consequence of positivity is the existence of minimal models whose definition and properties we now recall.

If f is a perfect Morse function, in the sense that the differential of its Morse complex is trivial, then the pearl complex is quite efficient for computations. However, not all manifolds admit perfect Morse functions. The existence of the minimal models allows to reduce algebraically the pearl complex to such a minimal form whenever the base ring K is a field. We recall the relevant result from [14].

Proposition 2.2.1 [14] *Let K be a field. For any monotone Lagrangian L there exists a complex $\mathcal{C}_{\min}(L) = (H_*(L;K) \otimes \widetilde{\Lambda}^+, \delta)$, with*

$$\delta\colon H_*(L;K) \otimes \widetilde{\Lambda}^+ \to H_*(L;K) \otimes \widetilde{\Lambda}^{>0}$$

so that, for any triple $\mathscr{D} = (f, (\cdot,\cdot), J)$ such that $\mathcal{C}(\mathscr{D})$ is defined, there are chain morphisms $\phi\colon \mathcal{C}(\mathscr{D}) \to \mathcal{C}_{\min}(L)$ and $\psi\colon \mathcal{C}_{\min}(L) \to \mathcal{C}(\mathscr{D})$ that both induce isomorphisms in quantum homology as well as in Morse homology and verify $\phi \circ \psi = id$. The complex $\mathcal{C}_{\min}(L)$ with these properties is unique up to (a generally noncanonical) isomorphism and is called the minimal pearl complex of L. The maps ψ, ϕ are called structural maps associated to $\mathscr{D}$.

All the algebraic structures described before (product, module structure etc.) can be transported and computed on the minimal complex. For instance, the product is the composition

$$\mathcal{C}_{\min}(L) \otimes \mathcal{C}_{\min}(L) \xrightarrow{\psi_1 \otimes \psi_2} \mathcal{C}(\mathscr{D}_1) \otimes \mathcal{C}(\mathscr{D}_2) \xrightarrow{*} \mathcal{C}(\mathscr{D}_3) \xrightarrow{\phi_3} \mathcal{C}_{\min}(L), \tag{7}$$

where ϕ_i, ψ_i are structural maps associated to the data set $\mathscr{D}_i$.

Remark 2.2.2 If the Lagrangian L admits perfect Morse functions, then any pearl complex associated to such a function is a minimal pearl complex over any ring K (not only when K is a field). Moreover, any two such minimal models are related by canonical comparison maps. This means for instance that for tori we may choose to work over $\mathbb{Z}$.

2.3 Additional conventions

2.3.1 Orientations of the pearly moduli spaces In order to define the pearl complex over a general ground ring we need to orient the moduli space of pearl trajectories. These are a combination of moduli spaces of gradient trajectories arising from Morse theory together with moduli spaces of J–holomorphic disks. The precise orientation conventions are described in detail in Appendix A, we only mention here some of the very basic choices used later in the paper.

Throughout the paper, by a Lagrangian $L \subset (M, \omega)$ we mean an oriented Lagrangian submanifold together with a fixed spin structure.

Denote by $D \subset \mathbb{C}$ the closed unit disk. We orient its boundary ∂D by the counterclockwise orientation. Denote by $G = \mathrm{Aut}(D)$ the group of biholomorphisms of the disk, and by $H \subset G$ the subgroup of elements that preserve the two points $-1, +1 \in \partial D$. We orient both G and H as described in Section A.1.10.

Fix a generic almost complex structure $J \in \mathcal{J}$. Let $B \in H_2^D$. Denote by $\widetilde{\mathcal{M}}(B, J)$ the space of (parametrized) J–holomorphic disks $u\colon (D, \partial D) \longrightarrow (M, L)$ with $u_*([D]) = B$. It is well known by the work of Fukaya, Oh, Ohta and Ono [26] that a spin structure on L induces orientations on the moduli spaces $\widetilde{\mathcal{M}}(B, J)$. The groups G and H act on $\widetilde{\mathcal{M}}(B, J)$ by $\sigma \cdot u = u \circ \sigma^{-1}$, and similarly on $\widetilde{\mathcal{M}}(B, J) \times \partial D$ by $\sigma \circ (u, z) = (u \circ \sigma^{-1}, \sigma(z))$. The following spaces will play an important role in the sequel:

$$\mathcal{M}_2(B, J) = \widetilde{\mathcal{M}}(B, J)/H, \quad (\widetilde{\mathcal{M}}(B, J) \times \partial D)/G.$$

Both spaces come with an orientation induced from those of $\widetilde{\mathcal{M}}(B, J)$ and of G and H as described in Section A.1.3 and Section A.1.10. There are natural evaluation maps that will be used frequently in the sequel:

$$\begin{aligned} &e_{-1}\colon \mathcal{M}_2(B, J) \longrightarrow L, && [u] \longmapsto u(-1), \\ &e_{+1}\colon \mathcal{M}_2(B, J) \longrightarrow L, && [u] \longmapsto u(+1), \\ &\mathrm{ev}\colon (\widetilde{\mathcal{M}}(B, J) \times \partial D)/G \longrightarrow L, && (u, z) \longmapsto u(z). \end{aligned} \tag{8}$$

See Section A.1.11 for more details concerning the orientations of these spaces.

2.3.2 The intersection product In the sequel we will use a version of the classical intersection product on singular homology which we denote by

$$H_i(L) \otimes H_j(L) \longrightarrow H_{i+j-n}(L), \quad a \otimes b \longmapsto a \cdot b.$$

We remark here that our convention for this operation is somewhat nonstandard concerning signs and orientations. Our intersection product is characterized by the following property: if $a = [A]$, $b = [B]$, where $A, B \subset L$ are two transverse oriented submanifolds, then $a \cdot b = [B \cap A]$, (*not* $A \cap B$!), where $\cap$ stands for oriented intersection (see Section A.1.7). When a and b have complementary dimensions we will also use their *intersection number* which we denote $\#(a \cap b) = \#(A \cap B)$. (Thus in this case $a \cdot b = \#(B \cap A)[\mathrm{pt}] = (-1)^{(n-i)(n-j)} \#(A \cap B)[\mathrm{pt}]$, where $n = \dim L$, $i = \dim A$, $j = \dim B$.) Also by abuse of notation, when $i + j = n$ we will sometimes view $(- \cdot -)$ as a $\mathbb{Z}$–valued pairing and write $a \cdot b \in \mathbb{Z}$, instead of $a \cdot b \in H_0(L) = \mathbb{Z}[\mathrm{pt}]$.

In favorable situations the product mentioned in Section 2.2.1 can be considered as a deformation of the above version of the classical intersection product on the singular homology. (See Section A.2.2.) The signs defining this product were so chosen in order to make duality more natural (see Section A.2.6).

Analogous remarks apply also to the module structure from Sections 2.2.2 and A.2.3 (both the classical and the quantum operations).

2.4 Twisted coefficients

The most relevant ground ring here will be $K=\mathbb{C}$, though one could work with $K=\mathbb{Q}$ or $K=\mathbb{Z}$ too.

Let $\rho\colon H_2^D \longrightarrow \mathbb{C}^*$ be a homomorphism. This induces a structure of a $\widetilde{\Lambda}^+$–algebra on the ring $\Lambda^+ = K[t]$ induced by the morphism $\widetilde{\Lambda}^+ \longrightarrow \Lambda^+$ defined by

$$\widetilde{\Lambda}^+ \ni T^A \longmapsto \rho(A)t^{\bar{\mu}(A)} \in \Lambda \quad \forall A \in H_2^D. \tag{9}$$

In order to emphasize the dependence on ρ in this algebra structure we will sometimes write $(\Lambda^\rho)^+$ rather than Λ^+. Similarly, we denote Λ^ρ the $\widetilde{\Lambda}^+$–algebra structure induced on $K[t^{-1},t]$ by (9). With these conventions we now have the Lagrangian quantum homology $QH(L;(\Lambda^\rho)^+)$ together with the quantum operations as described in the previous section as well as the corresponding structures over Λ^ρ. The differential of the pearl complex with coefficients in $(\Lambda^\rho)^+$ is denoted by d^ρ and we adopt a similar notation for all further structures over these twisted coefficients.

A particular important case comes from representations of $H_1(L;\mathbb{Z})$. More specifically, let $\rho'\colon H_1(L;\mathbb{Z}) \longrightarrow \mathbb{C}^*$ be a homomorphism. Then we can take in the preceding construction $\rho = \rho' \circ \partial\colon H_2^D \longrightarrow \mathbb{C}^*$, where $\partial\colon H_2^D \longrightarrow H_1(L;\mathbb{Z})$ is the connectant map.

From now on we will use the following notation. We abbreviate $H_1 = H_1(L;\mathbb{Z})$. For an abelian group H we write $\mathrm{Hom}_0(H,\mathbb{C}^*)$ for the group of homomorphisms $\rho\colon H \longrightarrow \mathbb{C}^*$ that are trivial (ie equal 1) on all torsion elements in H. We denote by $H_{\mathrm{free}} = H/\mathrm{Torsion}(H)$ the free part of H. Clearly there exists a (noncanonical) isomorphism $\mathrm{Hom}_0(H,\mathbb{C}^*) \cong (\mathbb{C}^*)^{\times r}$, where $r = \mathrm{rank}(H_{\mathrm{free}})$.

Remark 2.4.1 While we will not use noncommutative representations in this paper, much of the discussion described below can be adapted to coefficients twisted by representations of $\pi_2(M,L)$ with values in some not necessarily commutative Lie group.

2.4.1 Relation to Floer homology Twisting the coefficients using representations ρ has a counterpart in Floer homology. Recall from [14] that for the ring $\mathcal{R}=\Lambda$, or more generally for $\mathcal{R}$'s that are $K[H_2^D]$–algebras, there is a canonical isomorphism $QH(L;\mathcal{R}) \cong HF(L,L;\mathcal{R})$. Note that if we take here $\mathcal{R}=\Lambda^\rho$ then $HF(L,L;\Lambda^\rho)$ can be naturally identified with $HF((L,E_\rho),(L,E_\rho))$ which is a version of Floer homology in which the coefficients are twisted in a flat complex line bundle $E_\rho \longrightarrow L$. The relation of E_ρ to ρ is that $\rho\colon H_1 \longrightarrow \mathbb{C}^*$ determines the holonomies of the corresponding flat connection along loops in L. Incorporating flat bundles into Floer homology was first introduced by Kontsevich [36] (see also Fukaya [23]) in the context of homological mirror symmetry. Due to considerations coming from physics only unitary bundles were considered (ie $\rho\colon H_1 \longrightarrow S^1$). More recently it was discovered by Cho [17] that working with nonunitary bundles makes sense and is actually very useful. This point of view was further developed and generalized by Fukaya, Oh, Ohta and Ono [27].

2.5 Elementary enumerative invariants

As discussed in Section 1 the purpose of the paper is to discuss enumerative invariants that can be extracted from the algebraic structures before. While later in the paper we will mainly concentrate on the quantum product we now make explicit some simpler such invariants.

2.5.1 Disks of Maslov 2 Consider the number of disks of Maslov class 2, through a generic point of L and in a given homotopy class $B \in \pi_2(M, L)$. Thus, with the notation of Section 2.3, we are talking about the degree of the evaluation map $\mathrm{ev}\colon (\widetilde{\mathcal{M}}(B, J) \times \partial D)/G \to L$, $(u, z) \to u(z)$ under the assumption $\mu(B) = 2$. This degree is well-defined and independent of J precisely because we work under the assumption $N_L \geq 2$ which avoids any bubbling for disks of Maslov index 2. This point of view is formalized in the "superpotentials" of Section 3.3.

2.5.2 Invariants related to the quantum inclusion Assume L satisfies $QH(L) \neq 0$ and that $2n = 4$. Fix two points $P \in M \setminus L$ and $Q \in L$. The number of J–holomorphic disks $u\colon (D, \partial D) \to (M, L)$ with $\mu(u) = 4$ and $u(-1) = Q$, $u(0) = P$ (for a generic J) is independent of J and P and Q. By contrast with the remark in Section 2.5.1 an argument is necessary here. One way to see this is by using the quantum inclusion of Section 2.2.3. We will assume for this argument the definition of the quantum inclusion in terms of the pearl complex (the relevant moduli spaces are recalled in Section A.2.3). We first remark that for $\dim(L) = 2$ we have that $QH(L) \neq 0$ implies that $QH(L) \cong H(L) \otimes \Lambda$. We now consider a Morse function $f\colon L \to \mathbb{R}$ with a single minimum $x_0 = Q$ together with a Riemannian metric $(\cdot\,,\cdot)$ and a generic almost complex structure J. To define the quantum inclusion we also need a metric on M and a Morse function $h\colon M \to \mathbb{R}$. We assume that h has a single maximum $y_4 = P$. With these assumptions, the quantum homology class $[x_0]$ is defined. However, in general, this class is not independent of the choice of $\mathscr{D} = (f, (\cdot\,,\cdot), J)$. For a second choice of data $\mathscr{D}' = (f', (\cdot\,,\cdot), J')$ where f' is a Morse function with a single minimum x'_0, the relation between the two classes is $[x_0] = [x'_0] + q[L]t$ for some $q \in K$. Here $[L]$ represents the fundamental class of L (this class is well defined and independent of the choices of data $\mathscr{D}$). It is easy to see that $i_L([L])$ coincides with the classical singular inclusion $i_*([L]) \in H_2(M;K)$. This implies that $i_L([x_0]) = i_L([x_0']) + q i_*([L])t$. Now write

$$i_L([x_0]) = [\mathrm{pt}] + k_1 a t + k_2[M]t^2, \quad i_L([x_0']) = [\mathrm{pt}] + k_1' a' t + k_2'[M]t^2,$$

where $k_i, k_i' \in K$, $a, a' \in H(M)$ and $[M]$ represents the fundamental class of M. It easily follows that $k_2 = k_2'$, thus k_2 is an invariant of L (ie independent of $\mathscr{D}$ and of h). On the other hand, using the chain level definition of i_L for the data $\mathscr{D}$, we see that k_2 equals the number of J–holomorphic disks with $\mu = 4$ through x_0 and y_4.

The argument above does not work anymore in higher dimensions: both the equality $k_2 = k_2'$ and the interpretation of k_2 are not necessarily valid anymore. However, in Proposition 7.3.3 we will see that – by a different and much more subtle argument – the quantum inclusion of the point $[x_0]$ is independent of $\mathscr{D}$ for monotone toric fibers. Moreover, it follows from Section 7.3 that for monotone toric fibres in dimension $2n = 4$, the invariant k_2 can be computed via the Batyrev–Givental isomorphism (see formula (66) after Proposition 7.3.3).

3 Wide varieties

Let $L \subset M$ be a Lagrangian submanifold and $\mathcal{R}$ a $\widetilde{\Lambda}^+$–algebra. Following [14; 13] we say that L is *$\mathcal{R}$–wide* if there exists an isomorphism $QH(L;\mathcal{R}) \cong H(L;\mathcal{R})$ between the quantum homology and the singular homology of L, taken with coefficients in $\mathcal{R}$. (Note that we do not require existence of a canonical isomorphism here.) At the other extreme we have *$\mathcal{R}$–narrow* Lagrangians. By this we mean Lagrangians L with $QH(L;\mathcal{R}) = 0$.

We now consider the moduli of all representations ρ which make a given Lagrangian wide. More precisely define

(10) $$\mathcal{W}_2 = \{\rho \in \mathrm{Hom}_0(H_2^D, \mathbb{C}^*) \mid L \text{ is } \Lambda^\rho\text{–wide}\}.$$

Similarly, put

(11) $$\mathcal{W}_1 = \{\rho' \in \mathrm{Hom}_0(H_1, \mathbb{C}^*) \mid \rho' \circ \partial \in \mathcal{W}_2\}.$$

We call $\mathcal{W}_2$ and $\mathcal{W}_1$ the *wide varieties* associated to L. The connectant $\partial\colon H_2^D \longrightarrow H_1$ induces a map $\partial_{\mathcal{W}}\colon \mathcal{W}_1 \longrightarrow \mathcal{W}_2$.

Note that both $\mathrm{Hom}_0(H_2^D, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\times r}$ and $\mathrm{Hom}_0(H_1, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\times l}$ are complex algebraic varieties (in fact algebraic groups isomorphic to complex tori), where $r = \mathrm{rank}(H_2^D)_{\mathrm{free}}$, $l = \mathrm{rank}(H_1)_{\mathrm{free}}$.

3.1 The wide varieties are algebraic

Proposition 3.1.1 *For any monotone Lagrangian and with the notation above the sets $\mathcal{W}_2$ and $\mathcal{W}_1$ are algebraic subvarieties of $\mathrm{Hom}_0(H_2^D, \mathbb{C}^*)$ and $\mathrm{Hom}_0(H_1, \mathbb{C}^*)$ respectively. Moreover, the map $\partial_{\mathcal{W}}\colon \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ is a morphism of algebraic varieties.*

Proof We first treat the case when L admits a perfect Morse function. Let $f\colon L \longrightarrow \mathbb{R}$ be a perfect Morse function. Add to it a Riemannian metric $(\cdot,\cdot)$ and an almost complex structure $J \in \mathcal{J}$ so that the triple $\mathscr{D} = (f, (\cdot,\cdot), J)$ is regular. Since f is perfect we have an isomorphism of graded vector spaces $\mathcal{C}(\mathscr{D}; \widetilde{\Lambda}^+) \cong (H(L;\mathbb{C}) \otimes \widetilde{\Lambda}^+)_*$ and $\mathcal{C}_*(\mathscr{D}; \Lambda) \cong (H(L;\mathbb{C}) \otimes \Lambda)_*$. For dimension reasons it follows that L is Λ^ρ–wide if and only if the twisted pearly differential vanishes: $d^\rho(x) = 0$ for every $x \in \mathrm{Crit}(f)$. Notice that the differential d of the complex $\mathcal{C}(\mathscr{D}; \widetilde{\Lambda}^+)$ applied on $x \in \mathrm{Crit}(f)$ is of the form

$$dx = \sum_{A,y} m_{xy}(A) y T^A, \quad m_{xy}(A) \in \mathbb{Z},$$

so that $m_{xy}(A)$ vanishes whenever $|x| - |y| + \mu(A) \neq 1$. The twisted differential d^ρ is then written

$$d^\rho x = \sum_{A,y} m_{xy}(A)\rho(A) y t^{\bar{\mu}(A)},$$

in other words $d^\rho = d \otimes_\rho K[t]$.

Pick a basis $A_1, \cdots, A_r$ for $(H_2^D)_{\mathrm{free}}$. This yields an identification $\mathrm{Hom}_0(H_2^D, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\times r}$. Use this identification to write ρ as a tuple $(z_1, \dots, z_r) \in (\mathbb{C}^*)^{\times r}$ so that if $A \in H_2^D$ is given by $A = \sum a_i A_i$, then $\rho(A) = \prod z_i^{a_i}$. Thus the condition $d^\rho(x) = 0$ translates into a polynomial equation in $z_1, \dots, z_r$. As there are finitely many critical points x we get a system with finite number of algebraic equations for $\mathcal{W}_2$. The proof for $\mathcal{W}_1$ is similar.

We now turn to the general case – when perfect Morse functions might not exist. We will make use of Proposition 2.2.1 by replacing in the argument above the complex $\mathcal{C}(\mathscr{D}; \widetilde{\Lambda}^+)$ with a minimal pearl complex $\mathcal{C}_{\mathrm{min}}(L) = (H(L;K) \otimes \widetilde{\Lambda}^+, \delta)$. Similarly, we replace the twisted pearl complex associated to ρ and $\mathscr{D}$ with the complex $\mathcal{C}^\rho_{\mathrm{min}}(L) = \mathcal{C}_{\mathrm{min}}(L) \otimes \Lambda^\rho$. The differential of this complex, d^ρ_{min}, verifies $d^\rho_{\mathrm{min}} = \delta \otimes \Lambda^\rho$. Again for degree reasons, L is Λ^ρ wide if and only if the differential d^ρ_{min} in the complex $\mathcal{C}^\rho_{\mathrm{min}}(L)$ vanishes. We can then apply the argument above by using any fixed basis of $H_*(L;\mathbb{C})$ in the place of the set of critical points of f. □

Versions of the moduli spaces $\mathcal{W}_2$ have already been considered by Cho [17]. An analogue of $\mathcal{W}_1$ (but with Novikov ring valued representations) has played a central role in the work of Fukaya, Oh, Ohta and Ono [27]. Our approach below is somewhat different than these works. We will not use the varieties $\mathcal{W}$ in order to study Lagrangian intersections, but rather in order to construct new invariants of Lagrangians.

3.2 Regular functions and wide rings of coefficients

From now on we will implicitly assume that $\mathcal{W}_i$ (for either $i = 1$ or 2) is not an empty set.

Denote by $\mathcal{O}(\mathcal{W}_1)$ and $\mathcal{O}(\mathcal{W}_2)$ the rings of global algebraic functions on $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively. We do not grade these rings. Given $A \in H_2^D$, denote by $f_A \in \mathcal{O}(\mathcal{W}_2)$ the function defined by $f_A(\rho) := \rho(A)$. Consider now the map

$$q\colon \widetilde{\Lambda}^+ \longrightarrow \mathcal{O}(\mathcal{W}_2) \otimes \Lambda, \quad q(T^A) := f_A t^{\bar{\mu}(A)}. \tag{12}$$

It is easy to check that q is graded homomorphism of rings hence $\mathcal{O}(\mathcal{W}_2) \otimes \Lambda$ becomes a $\widetilde{\Lambda}^+$–algebra. In a similar way we can define such a structure on $\mathcal{O}(\mathcal{W}_1) \otimes \Lambda$.

With this setup we can define $QH(L; \mathcal{O}(\mathcal{W}_i) \otimes \Lambda^+)$, $i = 1, 2$ (and similarly for Λ). It easily follows from the definitions that

$$QH(L; \mathcal{O}(\mathcal{W}_i) \otimes \Lambda^+) \cong H(L; \mathcal{O}(\mathcal{W}_i) \otimes \Lambda^+) \tag{13}$$

and similarly for Λ. Note that these isomorphisms are not canonical.

Next, we have all the quantum operations with coefficients in $\mathcal{R}_i^+ = \mathcal{O}(\mathcal{W}_i) \otimes \Lambda^+$ as described in (4), (5) and (6) and similarly for $\mathcal{R}_i = \mathcal{O}(\mathcal{W}_i) \otimes \Lambda$.

To shorten notation we will write from now on

$$Q^+H(L; \mathcal{W}_i) := QH(L; \mathcal{O}(\mathcal{W}_i) \otimes \Lambda^+), \quad QH(L; \mathcal{W}_i) := QH(L; \mathcal{O}(\mathcal{W}_i) \otimes \Lambda). \tag{14}$$

3.3 The superpotential

Here we assume that $L^n \subset M^{2n}$ is a monotone Lagrangian with $N_L = 2$.

Pick a generic almost complex structure $J \in \mathcal{J}$. Using the same notation as in Section 2.3 (see also Section A.1) let $B \in H_2^D$ with $\mu(B) = 2$ and denote by $\widetilde{\mathcal{M}}(B, J)$ the space of J–holomorphic disks $u\colon (D, \partial D) \longrightarrow (M, L)$ with $u_*([D]) = B$, and by $G = \mathrm{Aut}(D) \cong PSL(2, \mathbb{R})$ the group of biholomorphisms of the disk. Consider now the space of disks with one marked point on the boundary, ie $(\widetilde{\mathcal{M}}(B, J) \times \partial D)/G$, where G acts as follows $\sigma \cdot (u, z) = (u \circ \sigma^{-1}, \sigma(z))$, for $\sigma \in G$. By standard arguments (see eg [14]) it follows that $(\widetilde{\mathcal{M}}(B, J) \times \partial D)/G$ is a smooth compact manifold without boundary and of (real) dimension n. Moreover by our assumptions on L (ie L is oriented, spin and with a prescribed choice of spin structure) the latter moduli space is also oriented. Consider the evaluation map

$$\mathrm{ev}\colon (\widetilde{\mathcal{M}}(B,J)\times \partial D)/G \longrightarrow L, \quad \mathrm{ev}(u,z)=u(z).$$

We denote by $\nu(B)\in\mathbb{Z}$ the degree of this map. Standard arguments then show that $\nu(B)$ does not depend on J but only on B. Moreover, there can be at most a finite number of classes $B\in H_2^D$ with $\nu(B)\neq 0$. Put

$$\mathcal{E}_2=\{B\in H_2^D \mid \nu(B)\neq 0\}.$$

Define now the function

$$\mathscr{P}\colon \mathrm{Hom}_0(H_1,\mathbb{C}^*)\longrightarrow \mathbb{C}, \quad \mathscr{P}(\rho)=\sum_{B\in\mathcal{E}_2}\nu(B)\rho(\partial B). \tag{15}$$

This function (and other analogous versions of it) is called the *Landau–Ginzburg superpotential*. It plays an important role in the theory of mirror symmetry for toric varieties. Its relation to Lagrangian Floer theory was first noticed by Hori and Vafa [35; 44] and further explored by Cho and Oh [18] and by Fukaya, Oh, Ohta and Ono [26; 27].

3.3.1 Explicit formulae for $\mathcal{W}_1$ We will now write $\mathscr{P}$ in coordinates. Fix a basis

$$\mathbf{e}=\{e_1,\dots,e_l\}$$

for $H_1(L;\mathbb{Z})_{\mathrm{free}}$. For $a\in H_1(L;\mathbb{Z})_{\mathrm{free}}$, denote by $(a)=((a)_1,\dots,(a)_l)\in\mathbb{Z}^{\times l}$ the vector of coordinates of a with respect to the basis $\mathbf{e}$ so that $a=(a)_1e_1+\dots+(a)_le_l$. Using the basis $\mathbf{e}$ we can identify $\mathrm{Hom}_0(H_1,\mathbb{C}^*)\cong(\mathbb{C}^*)^{\times l}$. With these choices fixed, we write an element $\rho\in\mathrm{Hom}_0(H_1,\mathbb{C}^*)$ as $(z_1,\dots,z_l)\in(\mathbb{C}^*)^{\times l}$, where $z_j=\rho(e_j)$. In these coordinates (15) becomes

$$\mathscr{P}(z_1,\dots,z_l)=\sum_{B\in\mathcal{E}_2}\nu(B)z_1^{(\partial B)_1}\cdots z_l^{(\partial B)_l}. \tag{16}$$

The relevance of $\mathscr{P}$ in our context is that we can describe the wide variety $\mathcal{W}_1$ by means of the derivatives of $\mathscr{P}$. To see this fix a basis $\mathbf{C}=\{C_1,\dots,C_l\}$ for $H_{n-1}(L;\mathbb{Z})$ which is dual to $\mathbf{e}$ in the sense that $C_i\cdot e_j=\delta_{i,j}$, where $\cdot$ is the intersection pairing (see Section 2.3.2). Now let $\mathcal{C}_{\min}(L)=H_*(L;\mathbb{C})\otimes\Lambda^+$ be a minimal pearl complex as provided by Proposition 2.2.1. Let $\mathcal{C}^{\rho}_{\min}(L)=\mathcal{C}_{\min}(L)\otimes\Lambda^{\rho}$ and denote by $d^{\rho}_{\min}$ the differential of this last complex. Of course, in case L admits a perfect Morse f function we can simply take instead of $\mathcal{C}_{\min}(L)$ the pearl complex of f and $d^{\rho}_{\min}$ coincides in this case with the differential d^{ρ} of the pearl complex of f twisted by the representation ρ. We can write the (twisted) pearl differential

$$d^{\rho}_{\min}\colon (H(L;\mathbb{C})\otimes\Lambda)_*\longrightarrow (H(L;\mathbb{C})\otimes\Lambda)_{*-1}.$$

Proposition 3.3.1 *For $\rho = (z_1, \dots, z_l)$ we have:*

(1) $d^{\rho}_{\min}(C_j) = z_j \dfrac{\partial \mathcal{P}}{\partial z_j}[L]t$.

(2) *If $QH(L; \Lambda^{\rho}) \neq 0$ then ρ is a critical point of $\mathcal{P}$.*

In particular, $\mathcal{W}_1 \subset \mathrm{Crit}(\mathcal{P})$. Moreover, if the cohomology ring $H^*(L;\mathbb{R})$ (with the classical cup product) is generated by $H^1(L;\mathbb{R})$ then $\mathcal{W}_1 = \mathrm{Crit}(\mathcal{P})$.

Proof In case L admits a perfect Morse function, the proof of (1) follows immediately from the definition of the pearl complex together with our orientation conventions. Concerning the orientations the main point to verify here is the following. Given $B \in H_2^D$, denote

$$Q = \mathcal{M}_2(B, J)\ {}_{e_{+1}}\times_i \{m\}.$$

Here we use the fiber product and its orientation as defined in Section A.1.8, and $i\colon \{m\} \to L$ stands for the inclusion of a point. Consider now the evaluation map $e_{-1}\colon Q \longrightarrow L$. Then we have in homology

$$(e_{-1})_*([\overline{Q}]) = (-1)^n \nu(B)\partial B, \tag{17}$$

where $\partial\colon H_2(M, L; \mathbb{Z}) \to H_1(L; \mathbb{Z})$ is the connectant. This can be checked by a straightforward computation based on the conventions described in Appendix A.

If L does not admit a perfect Morse function we use a minimal pearl complex together with its structural maps ϕ and ψ as in Proposition 2.2.1:

$$\mathcal{C}(\mathcal{D}) \xrightarrow{\phi} \mathcal{C}_{\min}(L) \xrightarrow{\psi} \mathcal{C}(\mathcal{D}),$$

where $\mathcal{D}$ is a generic set of data required to define the pearl complex. By using the fact that both these maps induce an isomorphism in Morse homology the result is again immediate.

To prove (2), recall that $[L] \in \mathcal{C}_{\min}(L)$ is a cycle whose homology class is the unity of the ring $QH(L; \Lambda^{\rho})$. Thus $QH(L; \Lambda^{\rho}) \neq 0$ if and only if $[L]$ is not a boundary. In view of (1), if $QH(L; \Lambda^{\rho}) \neq 0$ we must have

$$\frac{\partial \mathcal{P}}{\partial z_j}(\rho) = 0$$

for every j.

The last statement follows immediately from the following fact: *If $H^*(L)$ is generated by $H^1(L)$ then L is either Λ^{ρ}–narrow or Λ^{ρ}–wide. Moreover, the second case occurs if and only if $d^{\rho} = 0$ on $H_{n-1}(L)$.* The proof of this can be essentially found in [14] where it is proved for the ground ring $K = \mathbb{Z}_2$ and without any representations ρ, but the same proof with obvious changes extends to our setting. □

Remark 3.3.2 Both varieties $\mathcal{W}_1$ and $\mathcal{W}_2$ are arithmetic in the sense that in some coordinate system they are cut by a system of equations with integral coefficients.

Remark 3.3.3 Given that $QH(L;\mathcal{W}_i)$ is isomorphic to Floer homology with coefficients in $\mathcal{O}(\mathcal{W}_i)\otimes\mathbb{C}[t,t^{-1}]$ (as discussed in Section 2.4.1) and in view of equation (13) it follows that for a Lagrangian with $\mathcal{W}\neq\varnothing$ any L' transverse to L and Hamiltonian isotopic to it, intersects L' in at least $\sum_i \dim(H_i(L;\mathbb{C}))$ intersection points. Note that when L is a torus, checking that $\mathcal{W}\neq\varnothing$ can be done by verifying that $\mathrm{Crit}(\mathscr{P})\neq\varnothing$, according to Proposition 3.3.1.

We now turn to the relation between the quantum product and the superpotential. Recall that when L is $\mathcal{R}$–wide, $QH(L;\mathcal{R})$ is not in general *canonically* isomorphic to $H(L;\mathbb{C})\otimes\mathcal{R}$. However, there exist canonical embeddings $H_i(L;K)\hookrightarrow QH_i(L;\mathcal{R})$ for every $n-N_L+1\leq i\leq n$. (See [14, Section 4.5 and Proposition 4.5.1].) As $N_L\geq 2$, we view $H_{n-1}(L;K)$ as a subspace of $QH_{n-1}(L;\mathcal{R})$ and $H_n(L;K)$ as a subspace of $QH_n(L;\mathcal{R})$. The following proposition gives information on the quantum product of elements in this special subspace in terms of the superpotential.

Proposition 3.3.4 *Consider $H_{n-1}(L;\mathbb{C})$ as a subset of $QH_{n-1}(L;\mathcal{W}_1)$. Then*

$$C_i * C_j + C_j * C_i = (-1)^n z_i z_j \frac{\partial^2 \mathscr{P}}{\partial z_i \partial z_j}[L]t, \tag{18}$$

where $[L]\in H_n(L;\mathbb{C})\subset QH_n(L;\mathcal{W}_1)$ is the unity. In other words, for every $\rho=(z_1,\dots,z_l)\in\mathcal{W}_1$ we have the identity (18), where $C_i, C_j, [L]$ are all viewed as elements of $QH(L;\Lambda^\rho)$.

Proof In case L admits perfect Morse functions this follows from the definition of the quantum product together with our orientation conventions. Indeed, in this case, assume that f, f', f'' are three perfect Morse functions and that a, b are critical points of f and f' of index $n-1$, and let w be the maximum of f''. The critical points a, b are canonically identified with singular homology classes in $H_{n-1}(L;\mathbb{C})$ and obviously w is canonically identified with the fundamental class $[L]$. The product in question (defined over $\widetilde{\Lambda}^+$) is given by

$$a*b = a\cdot b + \sum_{\lambda\in\mathcal{E}_2} k_{ab}(\lambda)[L]T^\lambda,$$

where $a\cdot b$ is the singular intersection product and $k_{ab}(\lambda)\in\mathbb{Z}$ is the number of J–holomorphic disks u in the class λ that pass through w and intersect the unstable manifolds of a and of b is such a way that along the boundary of the disk the order of the intersection points is w, $W^u_{f'}(b)\cap u(\partial D)$, $W^u_f(a)\cap u(\partial D)$. Obviously, the order requirement shows that this intersection condition is not purely homological: a different choice of functions f and f' might change the coefficient $k_{ab}(\lambda)$ here. However, the sum $a*b+b*a$ is invariant as the order is now irrelevant. From this description the formula claimed is obvious for $i\neq j$ – it simply claims that if for $\lambda\in H_2^D$ with $\mu(\lambda)=2$ we have algebraically $\nu(\lambda)$ disks in the class λ passing through w then the contribution of these disks to $C_i*C_j+C_j*C_i$ is $\tau\nu(\lambda)(\partial\lambda)_i(\partial\lambda)_j$, where $(\partial\lambda)_i$ are the coefficients of $\partial\lambda$ in the basis $\mathbf{e}$, ie $\partial\lambda=\sum_i(\partial\lambda)_i e_i$, and $\tau=\pm 1$ is a sign that depends on the orientations conventions. In other words,

$$C_i*C_j+C_j*C_i=\tau\sum_{\substack{\lambda\in H_2^D\\ \mu(\lambda)=2}}\nu(\lambda)(\partial\lambda)_i(\partial\lambda)_j z_1^{(\partial\lambda)_1}\cdots z_l^{(\partial\lambda)_j}[L]t.$$

When $i=j$, we note that contribution of the disks in the class λ to $2C_i*C_i$ is $\tau\nu(\lambda)(\partial\lambda)_i^2$. We now use that fact that $\rho\in\mathcal{W}_1$ hence $\partial\mathcal{P}/\partial z_k=0$ for every k, and so $\sum_k\nu(\lambda)(\partial\lambda)_k=0$ by the point (1) in Proposition 3.3.1.

This implies the claimed formula up to showing that $\tau=(-1)^n$. In turn, this is a simple consequence of the orientation conventions for the quantum product (see Section A.2.2) and Equation (17) (with λ instead of B).

In case L does not admit perfect Morse functions the proof uses minimal pearl complexes in a rather straightforward way. □

Remark 3.3.5 It might seem slightly surprising that the coefficient $k_{ab}(\lambda)$ above is not necessarily invariant but still the quantum product

$$QH(L;\widetilde{\Lambda}^+)\otimes QH(L;\widetilde{\Lambda}^+)\longrightarrow QH(L;\widetilde{\Lambda}^+)$$

is well defined. The explanation is that while $H_{n-1}(L;K)$ is canonically embedded in $QH(L;\widetilde{\Lambda}^+)$ this is no longer true for $H_{n-2}(L;K)$. Clearly $a\cdot b$ belongs precisely to $H_{n-2}(L;K)$ and so, even if $a*b$ is well defined and independent of choices, the class $a\cdot b$ is not canonically identified with a quantum class. This is why the coefficient $k_{ab}(\lambda)$ is also, in general, not independent of the choices of f, f', f''. On the other hand, the independence of $k_{ab}(\lambda)+k_{ba}(\lambda)$ of all choices can also be seen as an immediate consequence of the fact that $a\cdot b+b\cdot a=0$.

3.3.2 Relation to previous works

The relation of the superpotential to the nonvanishing of Floer homology was first pointed out in the physics literature in Hori and Vafa [44]. Versions of Propositions 3.3.1 and 3.3.4 were later proved by Cho [16] and Cho and Oh [18] for the case of Lagrangian torus fibres in toric manifolds and in the setting of Floer homology. The toric case has been further studied by Fukaya, Oh, Ohta and Ono [27; 25].

3.3.3 Different versions of the superpotential Different authors use different versions of the superpotential functions, as well as different coordinate systems on $\mathrm{Hom}_0(H_2^D, \mathbb{C}^*)$. For example, Fukaya, Oh, Ohta and Ono use in [27] coordinates $x_1, \dots, x_l$ whose relation to ours is that $x_i = \log z_i$ (so that the x coordinates are defined only modulo some periods). The superpotential is then written as

$$\mathcal{P}'(x_1, \dots, x_l) = \sum_{\lambda, \mu(\lambda)=2} \nu(\lambda) e^{x_1(\partial\lambda)_1 + \cdots + x_l(\partial\lambda)_l}.$$

The formula at point (1) of Proposition 3.3.1 then becomes

$$d^{\rho}_{\min}(C_j) = \frac{\partial \mathcal{P}'}{\partial x_j}[L]t.$$

Similarly, formula (18) becomes now

$$C_i * C_j + C_j * C_i = (-1)^n \frac{\partial^2 \mathcal{P}'}{\partial x_i \partial x_j}[L]t.$$

Other authors, such as Auroux [6; 7; 8], work only with unitary representations, ie $\mathrm{Hom}_0(H_1, S^1)$ but allow the Lagrangian L to move in a family of special Lagrangian submanifolds. The superpotential is in this case a function of two sets of real variables: the representation and the parameter of the Lagrangian. However, these two sets of variables can be put together to form a complex system of coordinates in which the superpotential becomes holomorphic. The relation between this superpotential and ours is rather straightforward.

There is a more general but also less transparent definition of a superpotential that also expresses $\mathcal{W}_2$ in a way similar to the one described above. Moreover, this description also works for $N_L > 2$. We indicate it here.

Let $C_1, \dots, C_k$ be a basis of $H_{n-N_L+1}(L; K)$, and $f_1, \dots, f_s$ a basis for $(H_2^D)_{\mathrm{free}}$. Fix a point P in L. Define a function $\mathcal{P}\colon (\mathbb{C}^*)^{\times k} \times (\mathbb{C}^*)^{\times s}$ by

$$\mathcal{P}(z_1, \dots, z_k, w_1, \dots, w_s) = \sum_{\substack{\alpha \in (H_2^D)_{\mathrm{free}};\\ \mu(\alpha) = N_L}} z_1^{r_1(\alpha)} \cdots z_k^{r_k(\alpha)} w_1^{(\alpha)_1} \cdots w_s^{(\alpha)_s}. \tag{19}$$

The exponents $r_i, (\alpha)_j \in \mathbb{Z}$ are related to α as follows: $\alpha = \sum_j (\alpha)_j f_j$ and $r_i(\alpha)$ is the intersection number of the homology class C_i with the class $D_\alpha \in H_{N_L-1}(L; K)$ which is defined as follows. Put $Q = \mathcal{M}_2(\alpha, J)_{e_{+1}} \times_i \{P\}$, where $i\colon \{P\} \to L$ is the inclusion (see Section A.1.8 for the definitions of the orientation on the fiber product).

The closure of Q is an oriented compact manifold $\bar{Q}$ without boundary. Moreover, the second evaluation map e_{-1} extends to $\bar{Q}$. We define $D_\alpha = (e_{-1})_*[\bar{Q}]$.

This potential is independent of P as well as of (the generic choice of) J. For convenience we put $\mathbf{z} = (z_1, \dots, z_k)$.

Similarly to the case $N_L = 2$ previously discussed we have that if the real cohomology of L is generated as an algebra by $H^{<N_L}(L; \mathbb{R})$, then

$$\mathcal{W}_2(L) = \{\rho \in \mathrm{Hom}_0(H_2^D, \mathbb{C}^*) \mid d_{\mathbf{z}}\mathcal{P}(1, \dots, 1; \rho(f_1), \dots, \rho(f_s)) = 0\}.$$

4 Quadratic forms

Let $L^n \subset M^{2n}$ be a monotone Lagrangian. We continue to assume that L is oriented and spin and fix once and for all a spin structure. We will introduce now a quadratic form associated to L from which we can derive new invariants. The construction works best when $N_L = 2$, so we first describe it in this case and then do the general case.

4.1 The case $N_L = 2$

Let $\mathcal{R}$ be a $\widetilde{\Lambda}^+$–algebra for which L is $\mathcal{R}$–wide. Assume that $\mathcal{R}_k = 0$ for every $k > 0$ and for simplicity assume also that $\mathcal{R}_{-1} = 0$ (the assumption on $\mathcal{R}_{-1}$ is not really necessary; see Remark 4.2.1). Assume in addition that $\mathcal{R}_0$ is a free K–module. (eg $\mathcal{R} = R \otimes_K \Lambda^+$, where R is an ungraded free K–module and also a K–algebra.)

We have a canonical isomorphism

$$QH_{n-1}(L; \mathcal{R}) \cong H_{n-1}(L; K) \otimes_K \mathcal{R}_0, \tag{20}$$

as well as a canonical exact sequence

$$0 \longrightarrow [L]\mathcal{R}_{-2} \xrightarrow{i} QH_{n-2}(L; \mathcal{R}) \xrightarrow{\pi} H_{n-2}(L; K) \otimes_K \mathcal{R}_0 \longrightarrow 0. \tag{21}$$

See [14, Section 4.5] for the details. From now on we will make the identification (20) and also view $[L]\mathcal{R}_{-2}$ as a subspace of $QH_{n-2}(L; \mathcal{R})$ via i. A simple computation shows that

$$\pi(a * b) = a \cdot b \quad \forall\, a, b \in H_{n-1}(L; K),$$

where $\cdot$ is the classical intersection product (see Section 2.3.2). In particular we have $\pi(a * a) = 0$ for every $a \in H_{n-1}(L; K)$, and so we can define a map

$$\widetilde{\varphi}\colon H_{n-1}(L; K) \longrightarrow \mathcal{R}_{-2} \quad \text{by} \quad a * a = \widetilde{\varphi}(a)[L]. \tag{22}$$

Obviously $\widetilde{\varphi}$ is a quadratic form, ie it is homogeneous of degree 2 over K.

We now restrict to the case $\mathcal{R} = R \otimes \Lambda^+$ with R some K–algebra which as module over K is free. In this case $\mathcal{R}_{-2} = tR$ and $\widetilde{\varphi}$ induces an R–valued quadratic form φ by putting $\varphi = t^{-1}\widetilde{\varphi}$.

A particular case of interest will be $K = \mathbb{C}$, $R = \mathcal{O}(\mathcal{W})$ where from now on we denote by $\mathcal{W}$ any of the wide varieties $\mathcal{W}_1$ or $\mathcal{W}_2$. In this case we denote the resulting quadratic form by

$$\varphi_{\mathcal{W}}\colon H_{n-1}(L;\mathbb{C}) \longrightarrow \mathcal{O}(\mathcal{W}).$$

We can also specialize to a particular $\rho \in \mathcal{W}$, ie compose with the evaluation morphism $e_\rho\colon \mathcal{O}(\mathcal{W}) \longrightarrow \mathbb{C}$, $e_\rho(f) = f(\rho)$. We write $\varphi_\rho = e_\rho \circ \varphi_{\mathcal{W}}$.

There is an important integral structure in this picture. Consider the inclusion of $H_{n-1}(L;\mathbb{Z})$ in $H_{n-1}(L;\mathbb{C})$. The restriction of the quadratic form $\varphi_{\mathcal{W}}$ to $H_{n-1}(L;\mathbb{Z})$, which will still be denoted by $\varphi_{\mathcal{W}}$ will play an important role in the sequel.

Remark 4.1.1 Whenever the trivial representation $\rho_0 \equiv 1$ is in $\mathcal{W}$ the quadratic form φ_{ρ_0} is integral. By this we mean that its restriction to $H_{n-1}(L;\mathbb{Z})$ gives values in $\mathbb{Z}$.

It often happens that the variety $\mathcal{W}_1$ is 0–dimensional (eg when the superpotential $\mathscr{P}$ has isolated critical points. See Proposition 3.3.1.) It follows from Remark 3.3.2 that in such cases for every $\rho \in \mathcal{W}_1$ the image $\rho(H_1(L;\mathbb{Z}))$ lies inside a number field $F \subset \mathbb{C}$. It easily follows that for every $\rho \in \mathcal{W}_1$ the restriction of the quadratic form φ_ρ to $H_{n-1}(L;\mathbb{Z})$ gives values in the same field F.

4.2 The case of $N_L > 2$

The definition in this case is based on viewing $\widetilde{\varphi}$ as a secondary operation in the sense that it is defined precisely when the square of the intersection product vanishes. Now assume that $N_L > 2$. We continue to assume that L is $\mathcal{R}$–wide, that $\mathcal{R}_k = 0$ for all $k > 0$ and that $\mathcal{R}_0$ is a free K–module. In addition assume that $\mathcal{R}_{-l} = 0$ for every $1 \leq l \leq 2s-1$ (see Remark 4.2.1).

Recall that N_L is even because L is orientable and write $N_L = 2s$. Notice that we still have a canonical isomorphism $H_{n-s}(L;K) \otimes_K \mathcal{R}_0 \cong QH_{n-s}(L;\mathcal{R})$. Denote by $H^{\sqrt{0}}_{n-s}(L;K)$ the cone consisting of those elements $x \in H_{n-s}(L;K)$ with $x \cdot x = 0$ where $\cdot$ is the intersection product. We now define

$$\widetilde{\varphi}^s\colon H^{\sqrt{0}}_{n-s}(L;K) \to \mathcal{R}_{-2s}$$

by the relation

$$x * x = \widetilde{\varphi}^s(x)[L] \quad \forall\, x \in H^{\sqrt{0}}_{n-s}(L;K).$$

Note that $H^{\sqrt{0}}_{n-s}(L;K)$ is in general only a cone (over K) and might fail to be a K–module. Still, in some cases (eg when $s=$ odd), $H^{\sqrt{0}}_{n-s}(L;K)$ is a K–module. In the general case $\widetilde{\varphi}^s$ restricts to a quadratic form on any subset of $H^{\sqrt{0}}_{n-s}(L;K)$ which is a K–submodule.

As in the case $N_L = 2$ for $K = \mathbb{C}$, $\mathcal{R} = \mathcal{O}(\mathcal{W}) \otimes \Lambda^+$ we obtain a quadratic form $\varphi_{\mathcal{W}} = t^{-1}\widetilde{\varphi}^s$ with values in $\mathcal{O}(\mathcal{W})$. We can also restrict $\varphi_{\mathcal{W}}$ to $H^{\sqrt{0}}_{n-s}(L;\mathbb{Z})$.

Remark 4.2.1 (1) The operation defined above seems to be the first step in a sequence of higher order operations, each defined whenever the previous ones vanish. While these higher order operations are of interest we will not further discuss them here.

(2) The quadratic forms discussed here first appeared in Cho [16] for $N_L = 2$, L a toric fibre and the trivial representation.

(3) The assumptions that $\mathcal{R}_{-1} = 0$ when $N_L = 2$, or more generally that $\mathcal{R}_{-l} = 0$ for $1 \leq l \leq 2s-1$ when $N_L = 2s$, are not really necessary in order to define the quadratic form φ. (And similarly for the assumption that $\mathcal{R}_0$ is a free K–module.) The point is that we still have an inclusion $H_{n-s}(L;K) \otimes_K \mathcal{R}_0 \longrightarrow (H(L;\mathcal{R}))_{n-s} \cong QH_{n-s}(L;\mathcal{R})$ (where the inclusion and the last isomorphism are canonical). Moreover, by inspecting the definition of the quantum product on the chain level it follows that for $x \in H^{\sqrt{0}}_{n-s}(L;K)$ we still have that $x * x$ is a multiple of $[L]$, hence we can write $x * x = \widetilde{\varphi}^s(x)[L]$, with $\widetilde{\varphi}^s(x) \in \mathcal{R}$. For degree reasons it now follows that $\widetilde{\varphi}^s(x) \in \mathcal{R}_{-2s}$.

Nevertheless, we keep these assumptions for the simplicity of the exposition and also because the rings we are interested in satisfy these assumptions anyway.

4.3 The discriminant

Let F be a finitely generated free abelian group and $\mathcal{A}$ a commutative ring. Let $\varphi\colon F \longrightarrow \mathcal{A}$ be a quadratic form. Recall that φ has a well defined invariant $\Delta \in \mathcal{A}$ called the discriminant which is defined as follows. Pick a basis for F and represent φ by a symmetric matrix A in that basis. Then the discriminant of φ,

$$\Delta_\varphi = -\det(A),$$

does not depend on the choice of the basis because any automorphism of F has $\det = \pm 1$.

Now let L be a Lagrangian with $N_L = 2$ as in Section 4.1. (A similar computation is possible for $N_L > 2$). The discriminant Δ of the quadratic form $\varphi_{\mathcal{W}}$ is an element of $\mathcal{O}(\mathcal{W})$. We denote its value at ρ by $\Delta(\rho) \in \mathbb{C}$.

To compute Δ explicitly fix a basis $\mathbf{C} = \{C_1, \dots, C_l\}$ for $H_{n-1}(L;\mathbb{Z})$. Define functions $a_{ij} \in \mathcal{O}(\mathcal{W})$ by the relations

$$C_i * C_j + C_j * C_i = a_{ij}[L]t.$$

Then we clearly have

$$\varphi_{\mathcal{W}}(X_1C_1 + \dots + X_lC_l) = \frac{1}{2}\sum_{i,j} a_{ij}X_iX_j, \quad \Delta = -\det(a_{ij}). \tag{23}$$

The minus sign in front of the determinant appears here in order to make our discriminant compatible with conventions common in number theory. In the same spirit, we take in the determinant the constants a_{ij} (instead of $\frac{1}{2}a_{ij}$) so that whenever the trivial representation $\rho_0 \equiv 1$ is in $\mathcal{W}_1$ the discriminant $\Delta(\rho_0)$ will be an integer.

When $\mathcal{W} = \mathcal{W}_1$ we can express Δ in terms of the super potential as follows. We now use the notation from Section 3.3.1. Fix a basis $\mathbf{e} = \{e_1, \dots, e_l\}$ for $H_1(L;\mathbb{Z})_{\text{free}}$ and a basis $\mathbf{C} = \{C_1, \dots, C_l\}$ for $H_{n-1}(L;\mathbb{Z})$ which is dual to $\mathbf{e}$ as in Section 3.3.1. Write $\rho = (z_1, \dots, z_l)$ with respect to $\mathbf{e}$. Then in view of formulas (18) and (23) we have

$$\Delta(z_1, \dots, z_l) = (-1)^{ln+1} z_1^2 \cdots z_l^2 \det\left(\frac{\partial^2 \mathcal{P}}{\partial z_i \partial z_j}\right). \tag{24}$$

5 The deformation viewpoint

Let L be a monotone Lagrangian which is $\mathcal{R}$–wide, where $\mathcal{R}$ is of the following kind: $\mathcal{R} = R \otimes_K \Lambda^+ = R[t]$ for some K–algebra R. For simplicity we assume that R is free as a K–module. We grade t as usual, $|t| = -N_L$, but do not grade R. Of course, $\mathcal{R}$ is also assumed to be endowed with a $\widetilde{\Lambda}^+$–algebra structure, but we do not make any special assumptions on it. For example we can take $\mathcal{R} = K[t]$ with the $\widetilde{\Lambda}^+$–algebra structure given by (9) for some $\rho \in \mathcal{W}_2$ (we often denote this ring also by $(\Lambda^\rho)^+$ to emphasize its $\widetilde{\Lambda}^+$–algebra structure coming from ρ). Another example is $\mathcal{R} = \mathcal{O}(\mathcal{W})[t]$ with the $\widetilde{\Lambda}^+$–algebra structure given by (12).

As L is wide there exists an isomorphism $QH(L;\mathcal{R}) \cong H(L;K) \otimes \mathcal{R}$ and, as mentioned before, usually there is no canonical one. On the other hand, there is a distinguished class of isomorphisms $QH(L;\mathcal{R}) \longrightarrow H(L;K) \otimes \mathcal{R}$ which we now describe.

For simplicity we will assume from now on that L admits a perfect Morse function. If this is not the case, the use of minimal models allows essentially the same results to be formulated in full generality (we remark however that the actual construction of the maps ψ and ϕ from Proposition 2.2.1 is required, this construction appears in [14, pages 2929–2933]).

5.1 The quantum product as deformation of the intersection product

Let $\mathscr{D} = (f, (\cdot,\cdot), J)$ be a regular triple consisting of a perfect Morse function $f\colon L \longrightarrow \mathbb{R}$, a Riemannian metric $(\cdot,\cdot)$ on L and an almost complex structure $J \in \mathcal{J}$. Denote by $\mathrm{CM}(\mathcal{F})$ the Morse complex (with coefficients in K) associated to the pair $\mathcal{F} = (f, (\cdot,\cdot))$. Denote by $\mathcal{C}(\mathscr{D};\mathcal{R})$ the pearl complex. Note that the Morse differential on $\mathrm{CM}(\mathcal{F})$ vanishes (since f is perfect). The differential of the pearl complex vanishes too because L is wide. It follows that the obvious map

$$\widetilde{h}_{\mathscr{D}}\colon \mathcal{C}(\mathscr{D};\mathcal{R}) \longrightarrow \mathrm{CM}(\mathcal{F}) \otimes_K \mathcal{R} \quad \text{induced by} \quad \widetilde{h}_{\mathscr{D}}(x) = x \quad \forall\, x \in \mathrm{Crit}(f)$$

is a chain map (in fact a chain isomorphism). Let $h_{\mathscr{D}}\colon QH(L;\mathcal{R}) \longrightarrow H(L;K) \otimes \mathcal{R}$ denote the induced map in homology. The isomorphism $h_{\mathscr{D}}$ is of course not canonical; it depends on $\mathscr{D}$. Denote by $\mathcal{K}$ the set of all isomorphisms $QH(L;\mathcal{R}) \longrightarrow H(L;K) \otimes \mathcal{R}$ obtained in this way from all possible triples $\mathscr{D}$.

Proposition 5.1.1 *Elements of $\mathcal{K}$ have the following properties:*

(1) *Every $h_{\mathscr{D}} \in \mathcal{K}$ sends the unity of $QH(L;\mathcal{R})$ to the unity $[L]$ of $H(L;K)$.*

(2) *For every two elements $h_{\mathscr{D}}, h_{\mathscr{D}'} \in \mathcal{K}$ we have*

$$h_{\mathscr{D}'} \circ {h_{\mathscr{D}}}^{-1} = \mathrm{id} + \phi_1 t + \phi_2 t^2 + \cdots,$$

where $\phi_k\colon H_*(L;K) \otimes R \longrightarrow H_{*+kN_L}(L;K) \otimes R$, $k \geq 1$. In other words, $h_{\mathscr{D}'} \circ h_{\mathscr{D}}^{-1}$ is a deformation of the identity.

Proof Let $\mathscr{D}' = (f', (\cdot,\cdot)', J')$ be another triple with f' a perfect Morse function and put $\mathcal{F}' = (f', (\cdot,\cdot)')$. Denote by $F_0\colon \mathrm{CM}(\mathcal{F}) \longrightarrow \mathrm{CM}(\mathcal{F}')$ the comparison map between the Morse complexes and by $F\colon \mathcal{C}(\mathscr{D}) \longrightarrow \mathcal{C}(\mathscr{D}')$ the comparison between the pearl complexes. We have

$$F(x) = F_0(x) + F_1(x)t + F_2(x)t^2 + \cdots \quad \forall\, x \in \mathrm{Crit}(f),$$

for some maps $F_k\colon CM_*(\mathcal{F}) \otimes R \longrightarrow CM_{*+kN_L}(\mathcal{F}') \otimes R$. See [13; 14] for more details. Notice that the comparison chain morphism F is defined by using appropriate homotopies relating the data $\mathscr{D}$ and $\mathscr{D}'$ and is unique, in general, only up to chain homotopy. In this case however, the differentials of the two involved complexes vanish so that F itself is canonical. □

For further use denote

$$\mathcal{G}_L = \{h_{\mathcal{D}'} \circ h_{\mathcal{D}}^{-1} \mid \mathcal{D}, \mathcal{D}' \text{ generic triples}\} \subset \mathrm{Aut}(H(L;K) \otimes \mathcal{R}).$$

This is a subgroup of the group of automorphisms of the $\mathcal{R}$–module $H(L;K)\otimes\mathcal{R}$. It corresponds to the subgroup generated by all morphisms associated to changes in choices of data $\mathcal{D}$.

5.1.1 General deformation theory The previous considerations fit into the general framework of classical deformation theory of algebras (see for example Gerstenhaber [29]). Algebras in this section are assumed to be associative, unital, but not necessarily commutative.

Let $(A,\cdot)$ be an algebra over the commutative ring R (which is also a K–algebra). We denote by $-\cdot-$ the product of A. A deformation of A is a structure of an algebra over $R[t]$ on the module $A\otimes_R R[t]$

$$(A\otimes_R R[t]) \otimes_{R[t]} (A\otimes_R R[t]) \longrightarrow (A\otimes_R R[t]), \quad x\otimes y \longmapsto x * y,$$

which satisfies the following conditions:

1. $A\otimes_R R[t]$ endowed with $*$ is an (associative unital) algebra over $R[t]$.
2. $1\in A$ continues to be the unit for $*$.
3. $*$ reduces to product $\cdot$ for $t=0$.

Sometimes instead of denoting the product on A by $x\cdot y$ and a deformation of it by $x*y$ we will write $m_0(x,y)$ and $m(x,y)$ respectively.

We will also need a graded version of the story. Our algebra $A=\bigoplus_{k\geq 0} A^k$ will be cohomologically graded and the ring R should be regarded as having degree 0 with respect to A, ie R is mapped by a morphism of rings to the center of A in degree 0, $R\longrightarrow Z(A^0)\subset A^0$. Let $d\in\mathbb{Z}$. We will consider deformations $*$ of A where the formal parameter t has degree $|t|=d$. We denote the set of such deformations by $\widetilde{\mathrm{Def}}_d(A)$. Denote by $\mathrm{Iso}_d(A)$ the group consisting of all $R[t]$–linear, degree preserving, module isomorphisms $\phi\colon A\otimes_R R[t]\longrightarrow A\otimes_R R[t]$ that have the form

$$\phi(x) = x + \phi_1(x)t + \phi_2(x)t^2 + \cdots \quad \forall\, x\in A, \quad \text{where } \phi_k\colon A^* \longrightarrow A^{*-dk}.$$

Two deformations $m',m''\in\widetilde{\mathrm{Def}}_d(A)$ are said to be equivalent if they are related by an element of $\mathrm{Iso}_d(A)$, ie there exists $\phi\in\mathrm{Iso}_d(A)$ such that $\phi(m''(x,y)) = m'(\phi(x),\phi(y))$ for every $x,y\in A\otimes_R R[t]$. Denote by $\mathrm{Def}_d(A)=\widetilde{\mathrm{Def}}_d(A)/\mathrm{Iso}_d(A)$ the set of equivalence classes of deformations of A. Similarly, when grading is not relevant we have $\widetilde{\mathrm{Def}}(A)$, $\mathrm{Iso}(A)$ and $\mathrm{Def}(A)=\widetilde{\mathrm{Def}}(A)/\mathrm{Iso}(A)$.

We will also use a slight modification of this construction. Assume $G \subset \mathrm{Iso}_d(A)$ is a subgroup. We then denote by

$$\mathrm{Def}_d^G(A) = \widetilde{\mathrm{Def}}_d(A)/G$$

the equivalence classes of deformations of A with respect to conjugation by elements of G.

5.1.2 The main example Let L be a monotone Lagrangian and $\mathcal{R} = R[t]$ as explained at the beginning of Section 5 so that L is $\mathcal{R}$–wide. Let A be the singular homology algebra of L (tensored with R), $A = H(L; K) \otimes_K R$, endowed with the intersection product $\cdot$. We grade A cohomologically, ie we put $A^i = H_{n-i}(L; K) \otimes_K R$ and here the degree of t is N_L (note that the unity $1 \in A^0$ corresponds in the homological notation to $[L]$).

Next consider the quantum homology $QH(L; \mathcal{R})$. For convenience, we grade it here cohomologically too, namely $QH^i(L; \mathcal{R}) := QH_{n-i}(L; \mathcal{R})$ and whenever working with QH^* we change the degree of t to be N_L rather then $-N_L$.

Recall the set of isomorphisms $\mathcal{K}$ introduced at the beginning of Section 5. Pick $h \in \mathcal{K}$. By transferring the quantum product $*$, via h, from $QH(L; \mathcal{R})$ to $A \otimes_R R[t] = H(L; K) \otimes_K R[t]$ we obtain a deformation $*_h \in \widetilde{\mathrm{Def}}_{N_L}(A)$ of the intersection product $-\cdot-$. This is so because of point (1) of Proposition 5.1.1 and because the quantum product $*$ operation is obviously a deformation of the intersection product $-\cdot-$ operation on the chain level.

It follows from point (2) of Proposition 5.1.1 that $\mathcal{G}_L \subset \mathrm{Iso}_{N_L}(A)$ and so we have quotient maps

$$\widetilde{\mathrm{Def}}_{N_L}(A) \xrightarrow{\Psi_1} \mathrm{Def}_{N_L}^{\mathcal{G}_L}(A) \xrightarrow{\Psi_2} \mathrm{Def}_{N_L}(A).$$

We denote

$$*_L^{\mathcal{G}} = \Psi_1(*_h) \quad \text{and} \quad *_L = \Psi_2(*_L^{\mathcal{G}}).$$

By the preceding discussion neither $*_L^{\mathcal{G}}$ nor $*_L$ depend on the choice of $h \in \mathcal{K}$. In other words, $(QH(L; \mathcal{R}), *)$ provides us with a well defined *class* of deformations of the classical ring $(H(L) \otimes R, \cdot)$.

Notice that $*_L$ belongs to a purely algebraic object: indeed $\mathrm{Def}_{N_L}(A)$ only depends on the algebra structure of $A = H(L; R)$ and not on any properties of the specific Lagrangian embedding $L \subset M$. By contrast, $\mathrm{Def}_{N_L}^{\mathcal{G}_L}(A)$ depends on this embedding because $\mathcal{G}_L$ is strongly depended on it – for instance, if L is exact, then $\mathcal{G}_L$ reduces to the identity element.

5.2 Invariant polynomials in the structural constants of the quantum product

We pursue the discussion in Section 5.1.2. In particular, we continue to write the various structures with cohomological grading. We use the same assumptions on K, R and $R[t]$ as at the beginning of Section 5. The main examples we have in mind are when K is a field, or when $K=\mathbb{Z}$. Moreover, for simplicity we will also assume that $H(L;K)$ is a free K–module. (If this is not the case we can always replace $H(L;K)$ by its free part over K, $H(L;K)_{\mathrm{free}}$, and the discussion below continues to hold with minor modifications.) As for the K–algebra R we will assume for simplicity that K is *embedded* in R.

To shorten notation we set $\widetilde{A}=A\otimes_R R[t]=H(L;K)\otimes_K R[t]$. Note that $\widetilde{A}=H(L;K)\oplus(H(L;K)\otimes_K tR[t])$. Denote by $\mathrm{pr}_q\colon \widetilde{A}\longrightarrow H(L;K)\otimes_K tR[t]$ the projection on the second factor. Put

$$\mathcal{H}=\hom^0_K(H(L;K)\otimes H(L;K),\widetilde{A}),$$
$$\mathcal{H}_q=\hom^0_K(H(L;K)\otimes H(L;K),H(L;K)\otimes_K tR[t]),$$

where $\hom^0_K$ stands for degree preserving K–linear homomorphisms. For degree reasons both $\mathcal{H}$ and $\mathcal{H}_q$ are free R–modules of finite rank. The projection pr_q induces a map $q\colon\mathcal{H}\longrightarrow\mathcal{H}_q$. As explained above an element $h\in\mathcal{K}$ induces an associative product $*_h\colon\widetilde{A}\otimes\widetilde{A}\to\widetilde{A}$. In particular we also get an element which we still denote $*_h\in\mathcal{H}$. We denote its image in $\mathcal{H}_q$ by $q(*_h)$.

Let U be a finite rank free K–module and $V=U\otimes_K R$. By a polynomial on V with coefficients in K we mean a function $P\colon V\longrightarrow R$ for which there is a basis of U, $u_1,\dots,u_l$ such that P can be written as a polynomial with coefficients in K in the R–basis $u_1\otimes 1,\dots,u_l\otimes 1$ of V. Clearly this notion does not depend on the choice of the basis for U. We denote these polynomials by $K[V]$.

Consider now polynomials $P\in K[\mathcal{H}]$ (where $\mathcal{H}$ is written as $U\otimes_K R$ in an obvious way). The purpose of this section is to discuss polynomials P which have the property that

$$P(*_h)=P(*_{h'})\quad\text{for every } h,h'\in\mathcal{K}.$$

Such polynomials will be called invariant polynomials. Next let $\sigma\in\mathrm{Aut}^0_K H(L;K)$ be a degree preserving automorphism. Clearly each such automorphism σ induces an automorphism $\sigma_{\mathcal{H}}\in\mathrm{Aut}_K(\mathcal{H})$. We say that a polynomial P is a symmetric polynomial invariant if P is an invariant polynomial and moreover P remains invariant under composition with $\sigma_{\mathcal{H}}$ for every $\sigma\in\mathrm{Aut}^0_K H(L;K)$. We will be particularly interested in invariant polynomials (symmetric or not) that capture information on the quantum part of the product, namely polynomials P that factor through $q\colon \mathcal{H} \to \mathcal{H}_q$, ie there exists a polynomial $Q \in K[\mathcal{H}_q]$ such that $P(*_h) = Q(q(*_h))$. We will call them Lagrangian quantum polynomials. Finally, we will be interested also in universal invariant polynomials for L, namely those that do not depend on the particular Lagrangian embedding of L.

We will now describe these notions in detail by using coordinates. While the notation in coordinates might appear heavy, it is more useful for applications and computations.

Fix a basis $\mathbf{a} = (a_i)_{i \in I}$ for $H^*(L; K)$ and put $\epsilon(i, j, s) = |a_i| + |a_j| - |a_s|$. We will assume further that the basis $\mathbf{a}$ is ordered in such a way that the first element is $a_0 = 1 \in H^0(L; K)$, the next elements form an ordered basis of $H^1(L; K)$ the ones after that form a basis for $H^2(L; K)$ etc. Obviously, any graded change of basis leaves the $\epsilon(i, j, s)$ invariant.

Any associative product $* \in \widetilde{\mathrm{Def}}_{N_L}(A)$ is characterized by constants $k_s^{i,j} \in R$ given by

$$a_i * a_j = \sum_{\{ s \,|\, N_L \text{ divides } \epsilon(i,j,s),\, \epsilon(i,j,s) \geq 0\}} k_s^{i,j} a_s t^{\epsilon(i,j,s)/N_L}. \tag{25}$$

The fact that the group $\mathcal{G}_L$ is in general nontrivial implies that for a product $*_h$ associated to an element $h \in \mathcal{K}$, the constants $k_s^{i,j}$ depend on h (and thus on $\mathcal{D} = (f, (\cdot,\cdot), J)$) and not only on $*_L^{\mathcal{G}}$. At the same time in the case of quantum homology of the ambient manifold M the structural constants of the quantum product are in fact triple Gromov–Witten invariants (see eg McDuff and Salamon [39]). This suggests that even if these structural constants are not themselves invariant in our Lagrangian setting, it might very well happen that – as a "next best case" – there exist invariants that are polynomial expressions in these constants.

Define

$$\mathcal{I}_L = \{(i, j, s) \in I \times I \times I \mid \epsilon(i, j, s) \geq 0\,,\ N_L \text{ divides } \epsilon(i, j, s)\}. \tag{26}$$

Notice that the number of elements of $\mathcal{I}_L$ only depends on $H(L; K)$ and N_L (and not on the actual basis $\mathbf{a}$). We let $K[z_r; r \in \mathcal{I}_L]$ be the polynomial ring with coefficients in K and variables z_r, $r = (i, j, s) \in \mathcal{I}_L$. Given any polynomial $P \in K[z_r; r \in \mathcal{I}_L]$ and any product $* \in \widetilde{\mathrm{Def}}_{N_L}(A)$ we can evaluate P on the structural constants associated to this product in the basis $\mathbf{a}$: we assign to $z_{(i,j,s)}$ the value $k_s^{i,j} \in R$. We denote the value of P computed in this way by $P(*; \mathbf{a}) \in R$ and we call it the value of P on the product $*$ in the basis $\mathbf{a}$.

Definition 5.2.1 Fix a smooth closed and oriented manifold L_0 endowed with a spin structure. Let $N \geq 2$ be an integer. Let $i\colon L_0 \hookrightarrow M$ be an R–wide monotone Lagrangian embedding with minimal Maslov number N. Put $L = i(L_0)$.

(i) A *Lagrangian polynomial invariant* for L is a polynomial $P \in K[z_r; r \in \mathcal{I}_L]$ so that for every $h \in \mathcal{K}$, the value $P(*_h; \mathbf{a})$ is independent of h for any basis $\mathbf{a}$ (in other words $P(*_h; \mathbf{a})$ only depends on P, $*_L^{\mathcal{G}}$ and $\mathbf{a}$).

(ii) A universal Lagrangian polynomial invariant of L_0 is a polynomial P as in point i which has the property that it is a polynomial invariant for every wide Lagrangian embedding $i\colon L_0 \hookrightarrow M$ (in any M) as above.

Polynomials as above are called *symmetric* if the value $P(*_h; \mathbf{a})$ is independent of the basis $\mathbf{a}$. They are called *quantum* if they depend only on $z_{(i,j,s)}$ with $\epsilon(i,j,s) > 0$.

Example 5.2.2 We start with the trivial example. Notice that the structural constants $k_s^{i,j}$ for $\epsilon(i,j,s) = 0$ are simply the structural constants of the algebra A and thus do not depend on $\mathcal{D}$. Thus, any polynomial $P \in K[z_r : r \in \mathcal{I}_L,\ \epsilon(r) = 0]$ is invariant (and even universal). From now on we will refer to this example as being *trivial* and we will eliminate it from any further discussion by focusing on quantum polynomial invariants.

Example 5.2.3 For this example it is relevant to work with $K = \mathbb{Z}$. Furthermore, we assume $N_L = 2$ and put $l = \operatorname{rank} H^1(L;\mathbb{Z})$. This means that for a basis $\mathbf{a}$ as before, the first element is $a_0 = 1$ and the next elements, $a_1, \ldots, a_l$, form a basis of $H^1(L;\mathbb{Z})$. We consider the elements of $(i,j,0) \in \mathcal{I}_L$ with $1 \le i,j \le l$ (hence $\epsilon(i,j,0) = 2$) and for each such element we define polynomials

$$P_{ij} = z_{(i,j,0)}, \quad \overline{P}_{ij} = P_{ij} + P_{ji}, \quad P_\Delta = -\det(\overline{P}_{ij}).$$

The point of this example is to discuss the invariance of these polynomials.

Let $h \in \mathcal{K}$ with associated product $*_h$. Then we have:

(i) $P_{ij}(*_h; \mathbf{a}) = c_{ij} \in R$ where $a_i *_h a_j = a_i \cdot a_j + c_{ij} t$ (recall that we are using cohomological notation).

(ii) $\overline{P}_{ij}(*_h; \mathbf{a}) = a_{ij} \in R$ where $a_i *_h a_j + a_j *_h a_i = a_{ij} t$ (compare with (23) from Section 4).

(iii) $P_\Delta(*_h; \mathbf{a}) = \Delta$ with Δ the discriminant from Section 4.

This shows that the polynomials $\overline{P}_{ij}$ are universal quantum invariants because by evaluation they provide the coefficients of the quadratic form discussed in Section 4, and this quadratic form is invariant with respect to $\mathcal{D}$. However, the $\overline{P}_{ij}$ are not symmetric polynomials since the coefficients of the quadratic form depend on the basis in which it is written. On the other hand, for obvious reasons, P_Δ is a universal, symmetric, Lagrangian quantum invariant.

Note that in contrast to $\overline{P}_{ij}$, the polynomials P_{ij} are not quantum invariants in general, as the example of the 2–dimensional Clifford torus in $\mathbb{C}P^2$ shows.

Here are the details. We can choose in this case a basis $a_1, a_2 \in H^1(L;\mathbb{Z})$ with $a_1 \cdot a_2 = \mathrm{PD}[\mathrm{pt}]$. For a specific choice of auxiliary data we then have $a_1 *_h a_2 = \mathrm{PD}[\mathrm{pt}] + t$, $a_2 *_h a_1 = -\mathrm{PD}[\mathrm{pt}]$, $a_1 *_h a_1 = a_2 *_h a_2 = t$; see in [14, Theorem 2.3.2]. From this it immediately follows that $P_{1,1}$ and $P_{2,2}$ are quantum (nonsymmetric) invariants. On the other hand $P_{1,2}$ and $P_{2,1}$ are not invariants since there exists $h' \in \mathcal{K}$ such that $h' \circ h^{-1}(\mathrm{PD}[\mathrm{pt}]) = \mathrm{PD}([\mathrm{pt}]) + \kappa t$ with $\kappa \neq 0$. A fast way to see that is the following. We can present the Clifford torus as a quotient of a rectangle $([0,1]\times[0,1])/\!\sim$ with pairs of parallel sides identified and in such a way that for the standard complex structure J_0 of $\mathbb{C}P^2$ the boundaries of the three J_0–holomorphic disks through the point $(0,0)$ are given (up to orientation) by the two edges $[0,1]\times 0$, $0\times[0,1]$ and the diagonal of the rectangle. Now choose two perfect Morse functions $f, g\colon L \to \mathbb{R}$ with critical points $\mathrm{Crit}(f) = \{x_0, x_1', x_1'', x_2\}$, $\mathrm{Crit}(g) = \{y_0, y_1', y_1'', y_2\}$ (where the subscripts stand for the Morse indices). Choose x_2, y_2 to be close enough one to the other and similarly for the pairs x_1', y_1' and x_1'', y_1'', but choose x_0 and y_0 to lie in different connected components of $\mathrm{Int}([0,1]\times[0,1]) \setminus \text{diagonal}$. Put $\mathscr{D} = (f, (\cdot,\cdot), J_0)$ and $\mathscr{D} = (f', (\cdot,\cdot), J_0)$. It is easy to see that the comparison map $\Psi_{\mathscr{D}',\mathscr{D}}\colon \mathcal{C}(\mathscr{D}) \to \mathcal{C}(\mathscr{D}')$ satisfies $\Psi_{\mathscr{D}',\mathscr{D}}(x_2) = y_2$, $\Psi_{\mathscr{D}',\mathscr{D}}(x_1') = y_1'$, $\Psi_{\mathscr{D}',\mathscr{D}}(x_1'') = y_1''$. On the other hand from the formulae in Section 6.2.3 we get that $\Psi_{\mathscr{D}',\mathscr{D}}(x_0) = y_0 \pm y_2 t$ (see formula (53) and Figure 4). It follows that $h_{\mathscr{D}'} \circ h_{\mathscr{D}}^{-1}(\mathrm{PD}[\mathrm{pt}]) = \mathrm{PD}([\mathrm{pt}]) \pm t$. In particular $P_{1,2}(*_{h_{\mathscr{D}'}}) \neq P_{1,2}(*_{h_{\mathscr{D}}})$ and similarly for $P_{2,1}$.

Remark 5.2.4 The argument above also shows that for the Clifford torus $L \subset \mathbb{C}P^2$ we have $\mathscr{G}_L = \mathrm{Iso}_2(A)$, where $A = H(L;K)\otimes_K R$. Indeed, for degree reasons

$$\mathrm{Iso}_2(A) = \big\{\phi \mid \phi([L]) = [L],\ \phi(a) = a,\ \forall\, a \in H_1(L;K),\ \phi([\mathrm{pt}]) = [\mathrm{pt}] + r[L]t, r \in \mathbb{Z}\big\}.$$

This group is generated by the automorphism ϕ' corresponding to $r = 1$, namely $\phi'([\mathrm{pt}]) = [\mathrm{pt}] + [L]t$. At the same time we have seen above that for the Clifford torus $\phi' \in \mathscr{G}_L$, which implies that $\mathrm{Iso}_2(A) \subset \mathscr{G}_L$.

Remark 5.2.5 For $K = \mathbb{Z}$, the polynomial P_Δ is (up to composition with a polynomial of one variable) the *only* universal symmetric quantum invariant that depends only on the $z_{(i,j,0)}$'s with $\epsilon(i,j,0) = 2$. Indeed, any polynomial quantum invariant depending on the variables $z_{(i,j,0)}$'s with $\epsilon(i,j,0) = 2$ is a polynomial in the $\overline{P}_{ij}$'s. In other words, it is a polynomial in the coefficients of the quadratic form φ defined in (22) of Section 4. By definition, the values of this polynomial in the coefficients of φ should be independent of the basis in which φ is expressed. On the other hand it is known since the work of Hilbert [34] that the ring of polynomial invariants of a quadratic form is generated by a single element which can be taken to be the discriminant.

5.2.1 The variety of algebras We describe here a more conceptual point of view on the invariant polynomials introduced in the previous section. We continue to use the notation from Section 5.1.2 and Section 5.2 and, in particular, continue to use cohomological notation. A survey of deformation theory from this perspective can be found in Makhlouf [37] for instance.

We begin by noticing that the set $\widetilde{\mathrm{Def}}_{N_L}(A)$ of deformations of the intersection product on $A = H(L; R)$ has the structure of an algebraic set. Indeed, fix a basis $\mathbf{a} = (a_i)_{i \in I}$ for $H(L; K)$. The structural constants $k_s^{i,j} \in R$ associated to any element $\nu \in \widetilde{\mathrm{Def}}_{N_L}(A)$ by writing the product structure in the basis $\mathbf{a}$ as in (25) verify a series of algebraic equations. First, we have linear equations reflecting the fact that the product is graded:

$$k_s^{i,j} = 0 \quad \text{if } \epsilon(i,j,s) \le 0 \text{ or } N_L \text{ does not divide } \epsilon(i,j,s). \tag{27}$$

Next, the existence of a unit translates to

$$k_j^{0,i} = k_j^{i,0} = \delta_{i,j} \quad \forall\, i,j \in I. \tag{28}$$

The fact that the operation is a deformation of the intersection product in A gives

$$k_s^{i,j} = v_s^{i,j} \quad \text{if } \epsilon(i,j,s) = 0, \tag{29}$$

where $v_s^{i,j}$ are the structural constants of the intersection product in A. Finally we have some quadratic equations that reflect the associativity of the product:

$$\sum_s k_s^{i,j} k_m^{s,l} = \sum_r k_r^{j,l} k_m^{i,r} \quad \forall\, i,j,l,m \in I. \tag{30}$$

Consider variables $z_s^{i,j} \in R$ with $i,j,s \in I$ and define the algebraic set $\mathcal{V}_{N_L}(A)$ by demanding that the $z_s^{i,j}$'s verify (27), (28) and (30). Clearly this set is independent of the basis $\mathbf{a}$. Denote by $\mathcal{V}_{N_L}(A; \mathbf{a})$ the algebraic set obtained by demanding that the $z_s^{i,j}$'s verify additionally (29). We have an identification

$$\Psi_{\mathbf{a}} \colon \widetilde{\mathrm{Def}}_{N_L}(A) \to \mathcal{V}_{N_L}(A; \mathbf{a}) \subset \mathcal{V}_{N_L}(A).$$

The group $\mathcal{G}_L$ acts on $\mathcal{V}_{N_L}(A)$ and this action restricts to an action on $\mathcal{V}_{N_L}(A; \mathbf{a})$ for each basis $\mathbf{a}$.

Given that R is a K–algebra, there is a canonical embedding $K[z_r; r \in \mathcal{I}_L] \to R[z_r; r \in \mathcal{I}_L]$ so that to any polynomial in $K[z_r; r \in \mathcal{I}_L]$ we can associate one in $R[z_r; r \in \mathcal{I}_L]$. In this language, a Lagrangian polynomial invariant is a polynomial in $K[z_r; r \in \mathcal{I}_L]$ whose associated regular function on $\mathcal{V}_{N_L}(A)$ is constant on the

$\mathcal{G}_L$–orbit of $*_h \in \mathcal{V}_{N_L}(A;\mathbf{a})$ for all $h \in \mathcal{K}$ and such that this holds for each basis $\mathbf{a}$. It is symmetric if the value of the respective constant is independent of the basis $\mathbf{a}$.

Remark 5.2.6 An important point which is an immediate consequence of the discussion in this section is that two Lagrangian invariant polynomials $P_1, P_2 \in K[z_r : r \in \mathcal{I}_L]$ as defined in Definition 5.2.1 can be equal, $P_1 = P_2$, as regular functions on $\mathcal{V}_{N_L}(A)$ without being the same polynomials: the polynomial expressions of P_1 and P_2 can be different but, due to the relations among the variables $k_s^{i,j}$, the respective regular functions may agree. Notice also that if we have an equality $P_1 = P_2$ over $\mathcal{V}_{N_L}(A)$ for two polynomials in $K[z_r; r]$ and we know that just one of the polynomials is invariant, then the second one is necessarily also invariant.

5.3 Hochschild cohomology

The classical algebraic approach to deformation theory is via Hochschild cohomology. We recall it here. We use the standard Hochschild cohomology theory for associative algebras [29]. We start with a brief description of this classical construction.

Let A be a graded algebra over a commutative ring R. As before we view R as having degree 0 with respect to A, ie R is mapped by a morphism of rings to the center of A in degree 0, $R \longrightarrow Z(A^0) \subset A^0$.

The Hochschild complex of A (with coefficients in A) is defined by

$$C^k(A;A) = \mathrm{Hom}_R(A^{\otimes k}, A)$$

endowed with the differential $d\colon C^k(A;A) \longrightarrow C^{k+1}(A;A)$:

$$df(a_1 \otimes \cdots \otimes a_{k+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{k+1}) + \sum_{i=1}^{k} (-1)^i f(a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{k+1}) + (-1)^{k+1} f(a_1 \otimes \cdots \otimes a_k) a_{k+1}. \tag{31}$$

The homology of this cochain complex is called the Hochschild cohomology of A (with coefficients in A) and is denoted by $HH^*(A;A)$. The second A here should be regarded as the "coefficients module". It can be replaced by any A–module M yielding $HH^*(A;M)$, but we will not need this in the sequel.

We incorporate the grading into this construction (without modifying the formula for the differential). Simply consider for every $k, l \in \mathbb{Z}$ the submodule

$$C^{k,l}(A;A) = \mathrm{Hom}^l_R(A^{\otimes k}, A) \subset C^k(A;A),$$

where Hom^l_R stands for R–linear homomorphisms that shift degree by l. Here, this means that $f \in C^{k,l}(A;A)$ if f is R–linear and for every k homogeneous elements $a_1,\dots,a_k \in A$ we have

$$|f(a_1\otimes\cdots\otimes a_k)| = |a_1\otimes\cdots\otimes a_k| + l = \sum_{i=1}^{k}|a_i| + l.$$

Clearly $d(C^{k,l}) \subset C^{k+1,l}$. Put

$$HH^{k,l}(A;A) = \frac{\ker(d|_{C^{k,l}})}{d(C^{k-1,l})}.$$

Classical deformation theory provides a map

$$T^1\colon \mathrm{Def}_d(A) \longrightarrow HH^{2,-d}(A;A). \tag{32}$$

The definition of T^1 is straightforward. Given a deformation $* \in \widetilde{\mathrm{Def}}_d(A)$, we can write

$$x * y = x\cdot y + m_1(x,y)t + \cdots \quad \forall\, x,y \in A.$$

A simple computation shows that $m_1 \in \mathrm{Hom}^{-d}_R(A\otimes A, A)$ is a Hochschild cycle (this is due to the associativity of $*$), hence we have an element $[m_1] \in HH^{2,-d}(A;A)$. Moreover, equivalent deformation $*' \sim *$ give rise to cohomologous cycles: $[m_1'] = [m_1] \in HH^{2,-d}(A;A)$. Thus setting $T^1([*]) = [m_1]$ provides a well defined map.

5.3.1 Quadratic forms and Hochschild cohomology Let A be an R–algebra and $S \subset A$ an R–submodule. Denote by $Q^2(S,R)$ the space of R–valued quadratic forms $\varphi\colon S \longrightarrow R$. Put $S_{\sqrt{0}} = \{s \in S \mid s\cdot s = 0\} \subset S$, and consider the restriction map rest: $Q^2(S,R) \longrightarrow \mathrm{Func}(S_{\sqrt{0}}, R)$ to the space of R–valued functions on the set $S_{\sqrt{0}}$. Denote by $Q^2_0(S,R)$ the image of this map.

Assume from now on that our graded R–algebra A is nontrivial only in degrees between 0 and n. Moreover assume that $A^0 = R$. Then we have a map

$$\Theta\colon HH^{2,-d}(A;A) \longrightarrow Q^2_0(A^{d/2}, R), \tag{33}$$

defined as follows. Let $\alpha \in HH^{2,-d}(A;A)$. Choose a cocycle $f_\alpha \in C^{2,-d}(A;A)$ so that $[f_\alpha] = \alpha$ and view f_α as a map $f_\alpha\colon A\otimes A \longrightarrow A$ of degree $-d$. Consider the quadratic form $\widehat{f_\alpha}\colon A^{d/2} \longrightarrow R$, defined by $\widehat{f_\alpha}(a) := f(a\otimes a) \in A^0 = R$. Finally, define $\Theta(\alpha) = \mathrm{rest}(\widehat{f_\alpha})$, where rest is the restriction map $Q^2(A^{d/2},R) \longrightarrow Q^2_0(A^{d/2},R)$.

We claim that the map Θ is well defined. To see this it is enough to show that if $f = dg$, where $g \in C^{1,-d}(A;A)$ then $f(a\otimes a) = 0$ for every $a \in A^{d/2}$ with $a\cdot a = 0$.

Indeed, let $a \in (A^{d/2})_{\sqrt{0}}$. By the definition of the Hochschild differential we have

$$f(a \otimes a) = dg(a \otimes a) = a \cdot g(a) - g(a \cdot a) + g(a) \cdot a.$$

But $g\colon A \longrightarrow A$ has degree $--d$ hence $g(a) \in A^{-d/2} = 0$ and by assumption we also have $a \cdot a = 0$. It follows that $f(a \otimes a) = 0$. This proves that Θ is well defined.

Consider now the composition

(34) $$\Gamma\colon \mathrm{Def}_d(A) \longrightarrow Q_0^2(A^{d/2}, R), \quad \Gamma = \Theta \circ T^1.$$

In case $(A^{d/2})_{\sqrt{0}} = A^{d/2}$, Γ assigns to every graded deformation equivalence class of A a quadratic form on $A^{d/2}$.

Example 5.3.1 Let $L^n \subset M^{2n}$ be a monotone Lagrangian with $N_L = 2s$ and a nonempty wide variety $\mathcal{W}$. Take

$$R = \mathcal{O}(\mathcal{W}) \quad \text{and} \quad A^* = H_{n-*}(L; \mathbb{C}) \otimes \mathcal{O}(\mathcal{W}).$$

As explained in Section 5.1.2, $Q^+H(L; \mathcal{W}) = QH(L; R[t])$ gives rise to a class of deformations $*_L \in \mathrm{Def}_d(A)$. Applying the map T^1 we obtain an invariant $T^1(*_L) \in HH^{2,-d}(A; A)$.

Next, applying the map Γ to $*_L$ we obtain a quadratic form $\Gamma(*_L) \in Q_0^2(A^s, R)$ on $A^s_{\sqrt{0}} \cong (H_{n-s}(L; \mathbb{C}) \otimes \mathcal{O}(\mathcal{W}))^{\sqrt{0}}$ with values in $\mathcal{O}(\mathcal{W})$. A straightforward computation shows that this $\Gamma(*_L)$, when restricted to $H^{\sqrt{0}}_{n-s}(L; \mathbb{C})$, coincides with the quadratic form $\varphi^s_{\mathcal{W}}\colon H^{\sqrt{0}}_{n-s}(L; \mathbb{C}) \longrightarrow \mathcal{O}(\mathcal{W})$ constructed in Section 4.2.

Example 5.3.2 We have a particular interest in the free graded exterior algebra $\Lambda_n(R)$ which is generated as algebra by n generators $a_1, \ldots, a_n \in \Lambda^1_n(R)$ which we will think of as the singular cohomology of the n–torus (with coefficients in R). We put $A = \Lambda_n(R)$, $d = 2$ and consider the resulting map

$$\Gamma\colon \mathrm{Def}_2(A) \to Q^2(A^1, R).$$

Lemma 5.3.3 *Let K be a field of characteristic 0 or $\mathbb{Z}$. For $A = \Lambda_n(R)$, $n \geq 2$, the map Γ is an isomorphism.*

Proof Let $* \in \widetilde{\mathrm{Def}}_2(A)$. The quadratic form $\Gamma(*)$ can be described as follows. Pick a basis $a_1, \ldots a_n \in A^1$ and notice that as $\Lambda_n(R)$ is an exterior algebra we have

(35) $$a_i * a_j + a_j * a_i = \alpha_{ij} t \quad \text{for some } \alpha_{ij} \in R \quad \forall\, 1 \leq i, j \leq n.$$

The quadratic form in question is

$$\Gamma(*)(X_1a_1 + \dots X_na_n) = \frac{1}{2}\sum_{i,j}\alpha_{ij}X_iX_j.$$

The argument is based on a simple remark concerning Clifford algebras. First, recall the definition of the Clifford algebra. Let $Q = (q_{ij})$ be a symmetric $n \times n$ matrix with coefficients in R. The Clifford algebra associated to Q is by definition

$$\mathrm{Cliff}(Q) = \big(F_n(R) \otimes_R R[t]\big)/I,$$

where $|t| = 2$, $F_n(R)$ is the free, noncommutative algebra generated by $a_1, a_2, \dots, a_n$ over R (all of degree 1) and I is the ideal generated by the relations

$$a_ia_j + a_ja_i = 2q_{ij}t.$$

For degree reasons this algebra is a deformation of $\Lambda_n(R)$ endowed with the standard exterior product structure.

Coming back to our situation we see that $(A[t], *)$ can be described as the Clifford algebra associated to the quadratic form $\Gamma(*)$. (This has been previously remarked by Cho in [16].) More precisely, if we take Q to be the matrix corresponding to $\Gamma(*)$ in the basis $a_1, \dots, a_n$ (ie take $q_{ij} = \frac{1}{2}\alpha_{ij}$; if $K = \mathbb{Z}$ we use here the "twos out" convention for integral quadratic forms so that Q can contain half-integers) then the morphism of algebras

$$c_{Q,*}\colon \mathrm{Cliff}(Q) \longrightarrow (A[t], *) \quad \text{given by setting} \quad c_{Q,*}(a_i) = a_i.$$

is in fact an isomorphism, as can be verified by dimension counting, degree by degree.

We now use this remark to show both the surjectivity and injectivity of the map Γ. The surjectivity is clear by the construction of $\mathrm{Cliff}(Q)$: it is a deformation of the exterior algebra and it is sent by Γ to the quadratic form associated to Q.

To show that Γ is also injective assume that $\Gamma(*) = \Gamma(*')$ for some $*, *' \in \mathrm{Def}_2(A)$. Let Q be the $n \times n$ matrix corresponding to $\Gamma(*) = \Gamma(*') \in Q^2(A^1, R)$ in the basis $a_1, \dots, a_n$. We have an isomorphism of algebras

$$\xi = c_{Q,*'} \circ c_{Q,*}^{-1}\colon (A[t], *) \longrightarrow (A[t], *').$$

This composition is the identity on $A^{\leq 1}$. Together with the fact that ξ is an isomorphism of algebras this implies immediately that ξ is an equivalence of deformations and finishes the proof. □

An interesting consequence of this Lemma is obtained if we assume that for some wide n–dimensional Lagrangian torus $L \subset M$ the group of geometric equivalences $\mathcal{G}_L$ coincides with the group of algebraic equivalences $\mathrm{Iso}_2(A)$, for $A = H(T^n; R) = \Lambda_n(R)$. In this case we have two identifications

$$\mathrm{Def}_{N_L}^{\mathcal{G}_L}(A) \to \mathrm{Def}_{N_L}(A) \to Q^2(A^1, R).$$

This implies that any symmetric quantum polynomial invariant of L has to agree as regular function over $\mathcal{V}_2(A)$ with an expression that can be read off from the coefficients of the quadratic form $\Gamma(*_L)$. Moreover, by Remark 5.2.5 it verifies $P = \mathcal{F}(P_\Delta)$ for some polynomial of a single variable $\mathcal{F}$, where the equality here is over $\mathcal{V}_2(A)$. Now such Lagrangian tori do exist. For example, by Remark 5.2.4 for the 2–dimensional Clifford torus $L \subset \mathbb{C}P^2$ we do indeed have $\mathcal{G}_L = \mathrm{Iso}_2(A)$. Another way to formalize this is the following.

Corollary 5.3.4 *For the torus $\mathbb{T}^2$ any universal Lagrangian symmetric quantum polynomial invariant P agrees (as a regular function, in the sense of Remark 5.2.6) with a polynomial belonging to the subring generated by the discriminant.*

We expect this corollary to remain true also for higher dimensional tori.

Remark 5.3.5 The information contained in the superpotential from Section 3.3 can be encoded in a representation of the moduli spaces $\widetilde{\mathcal{M}}(\lambda, J)$ with values in the free loop space $\Lambda(L) = L^{S^1}$. By taking the sum of the cycles represented by all these moduli spaces one gets a homology class $\alpha \in H_*(\Lambda(L); K)$. There exists a well known isomorphism ϕ constructed by Jones (see eg Cohen and Jones [19]) between $H_*(\Lambda(L), K)$ and the Hochschild cohomology $HH^*(C^\cdot(L); C^\cdot(L))$ where $C^\cdot(-)$ is the singular cochain complex. (Note however that one has to adjust the grading and the sign conventions here; for example see Felix, Thomas and Vigué-Poirrier [22].) In favorable cases we also have an isomorphism $q\colon HH^*(C^\cdot(L); C^\cdot(L)) \approx HH^*(H^\cdot(L; K), H^\cdot(L; K))$. We point out here that, for instance, for Lagrangian tori if we project the class $q \circ \phi(\alpha)$ onto $HH^{2,-2}$ we obtain precisely the Hochschild cohomology class $T^1(*_L)$ that is associated to the quantum product when viewed as deformation of the intersection product.

6 The discriminant and enumerative geometry

The purpose of this section is to use the machinery introduced before to address the problem described at the beginning of Section 1. Thus we consider one of the simplest, nontrivial, enumerative problem in Lagrangian topology: counting J–holomorphic disks $u\colon (D,\partial D)\to (M,L)$ passing through 3 distinct points in L. As before, we will assume the closed Lagrangian $L^n\subset (M^{2n},\omega)$ to be monotone with $N_L\geq 2$.

Ideally, one would like to be able to estimate the number of disks in question by separating them according to their homotopy class – this is where the wide varieties will be of help.

6.1 Holomorphic disks through three points

As in the introduction, let $P,Q,R\in L$ be three distinct points. We are interested in the number of disks u of Maslov index $2n$ passing, in order, through P, Q and R. We will count these disks with coefficients in $\mathcal{O}(\mathcal{W})$ where $\mathcal{W}$ is one of our two wide varieties for L. This will lead to more refined formulae than working only over Λ^+.

To be more precise, given a class $\lambda\in H_2^D$ with $\mu(\lambda)=2n$ consider the map

$$\mathbf{e}\colon \widetilde{\mathcal{M}}(\lambda,J)\longrightarrow L\times L\times L,\quad \mathbf{e}(u)=\big(u(e^{-4\pi i/3}),u(e^{-2\pi i/3}),u(1)\big), \tag{36}$$

where $\widetilde{\mathcal{M}}(\lambda,J)$ is the moduli space of parametrized J–disks in the homotopy class λ. Note that both the source and target of this map are $3n$ dimensional. Standard arguments show that once we fix the points P,Q,R then for generic J the tuple (P,Q,R) is a regular value of this map and moreover the set $\mathbf{e}^{-1}(P,Q,R)$ is finite (although the space $\widetilde{\mathcal{M}}(\lambda,J)$ is not compact). We associate to each $u\in\mathbf{e}^{-1}(P,Q,R)$ a sign $\varepsilon(u;P,Q,R)=\pm 1$ by comparing orientations via $\mathbf{e}$. For $\rho\in\mathrm{Hom}_0(H_2^D,\mathbb{C}^*)$ define now

$$n_{PQR}(\rho)=\sum_{\{\lambda\,|\,\mu(\lambda)=2n\}}\ \sum_{u\in\mathbf{e}^{-1}(P,Q,R)}\varepsilon(u;P,Q,R)\rho(\lambda). \tag{37}$$

The numbers $n_{PQR}(\rho)$ are neither invariant with respect to P,Q,R nor to J.

6.1.1 Splitting polynomials The approach to estimating n_{PQR} that we will discuss here is based on the following simple idea: instead of showing that $n_{-,-,-}$ is a numerical invariant (which it is not) show that there exists a polynomial $S\in K[\xi_1,\dots,\xi_q]$ and a subvariety $\mathcal{W}\subset\mathrm{Hom}_0(H_2^D,\mathbb{C}^*)$ both independent of J,P,Q,R so that

$$n_{PQR}(\rho)=S(\xi_1,\dots\xi_q)\ ,\quad \xi_j=\#_\rho(\mathcal{M}_j)\quad \forall\rho\in\mathcal{W}. \tag{38}$$

Here $\mathcal{M}_j$ is a 0–dimensional moduli space of pearl-like trajectories involving only disks of Maslov index *at most* $2n-2$. Of course, the number $\#_\rho(\mathcal{M}_j)$ depends on the various data involved (eg Morse functions, metric and almost complex structure), however the equations defining $\mathcal{M}_j$ are fixed. By definition, the counting giving the ξ_j is given by

$$\#_\rho(\mathcal{M}_j) = \sum_\lambda \#(\mathcal{M}_j(\lambda))\rho(\lambda),$$

where $\mathcal{M}_j(\lambda)$ are the configurations in $\mathcal{M}_j$ whose total homology class is λ.

A polynomial S as above is called a *splitting polynomial over $\mathcal{W}$*. Equation (38) can be interpreted as an equality in $\mathcal{O}(\mathcal{W})$.

As we will see next such splitting polynomials often exist. As in Section 5 we will assume here that L admits a perfect Morse functions but if this is not the case the minimal model technique from Section 2.2.4 may be used instead with minor modifications.

Theorem 6.1.1 *Monotone Lagrangians L with $N_L \geq 2$ that are not rational homology spheres admit splitting polynomials S over their wide varieties $\mathcal{W}_i(L)$, $i = 1, 2$. Moreover, there are such splitting polynomials that are universal in the sense that they are independent of the particular Lagrangian embedding of L.*

As we will see in the proof, this is a rather immediate reflection of three facts: Poincaré duality in singular homology, the fact that $Q^+H(L;\mathcal{W}_i)$ – as defined in Section 3.2 – is a deformation of the singular homology algebra as discussed in Section 5, and finally the fact that the quantum product is an associative operation. Splitting polynomials are closely related to the invariant polynomials in Section 5.2. We prefer to avoid making explicit use of invariant polynomials in the proof of the theorem but we refer to Remark 6.1.3(i). for further discussion of this relationship.

6.1.2 Proof of Theorem 6.1.1 For simplicity we assume that $N_L = 2$ (the arguments for $N_L > 2$ are similar). We will use in this proof homological notation.

Recall that we have assumed that L admits a perfect Morse function, hence $H_*(L;\mathbb{Z})$ is free. Fix a basis $\mathbf{a} = (a_0, a_1, \ldots, a_m)$ for $H_*(L;\mathbb{Z})$, consisting of elements of pure degree and so that $a_0 = [\mathrm{pt}]$, $|a_i| \leq |a_j|$ for every $i < j$ and $a_m = [L]$.

Pick two generic perfect Morse functions f, g on L a Riemannian metric $(\cdot,\cdot)$ and an almost complex structure J on M so that the pearl complexes associated to $\mathcal{D}_f = (f, (\cdot,\cdot), J)$ and to $\mathcal{D}_g = (g, (\cdot,\cdot), J)$ are well defined as well as the chain level quantum product. We also require that the minimum of f is $x_0 = Q$, the maximum of f is $x_m = R$ and the minimum of g is $y_0 = P$.

Denote by $\mathcal{W}$ be the wide variety of L (either $\mathcal{W}_1$ or $\mathcal{W}_2$). The data $\mathcal{D}_f$ and $\mathcal{D}_g$ give us two identifications

$$h_f, h_g\colon Q^+H(L;\mathcal{W}) \longrightarrow H(L;\mathbb{C}) \otimes \mathcal{O}(\mathcal{W}) \otimes \mathbb{C}[t].$$

For $c \in H(L;\mathbb{C})$ we write $c^f = h_f^{-1}(c)$, $c^g = h_g^{-1}(c) \in Q^+ H(L;\mathcal{W})$. The relation between a_i^f and a_i^g is given by

$$a_i^f = a_i^g + \sum_{j>i} \sigma_j^i a_j^g t^{r_j^i}, \quad \text{with } \sigma_j^i \in \mathcal{O}(\mathcal{W}),\ r_j^i \geq 1. \tag{39}$$

Moreover, $r_j^i \leq n/2$ for every i, j and the coefficients σ_j^i are all determined by counting pearly moduli spaces involving only configurations of disks with total Maslov index $\leq n$. This follows from the comparison maps described in Section A.2.5 below. Recall also that $a_m = [L]$ is transformed canonically to the unit of $Q^+ H(L;\mathcal{W})$ and we have $a_m^f = a_m^g$. We therefore denote the latter by a_m too.

Next, given $\alpha, \beta \in H(L;\mathbb{Z})$, denote by $x_\alpha \in \mathbb{Z}\langle \mathrm{Crit}(f)\rangle$ the linear combination of critical points representing in Morse homology the class α. Similarly, denote by $y_\beta \in \mathbb{Z}\langle \mathrm{Crit}(g)\rangle$ the Morse cycle representing β.

Recall the chain level product $\mathcal{C}(\mathscr{D}_f;\mathcal{O}(\mathcal{W})[t]) \otimes \mathcal{C}(\mathscr{D}_g;\mathcal{O}(\mathcal{W})[t]) \longrightarrow \mathcal{C}(\mathscr{D}_f;\mathcal{O}(\mathcal{W})[t])$. We will denote it here by $x \otimes y \longmapsto x \mathbin{\widetilde{*}} y$, for $x \in \mathrm{Crit}(f)$, $y \in \mathrm{Crit}(g)$, in order to distinguish it from the induced product on homology which is denoted by $*$. The relation between $*$ and $\widetilde{*}$ is given by $\alpha^f * \beta^g = [x_\alpha \mathbin{\widetilde{*}} y_\beta]$. Of course, in order to calculate $\alpha^f * \beta^f$ (rather than $\alpha^f * \beta^g$) one needs now to appeal to formula (39).

The following lemma follows immediately from the discussion above.

Lemma 6.1.2 *Let $\alpha, \beta \in \mathbf{a}$. Write*

$$\alpha^f * \beta^f = \sum_{i=0}^{m} s_i a_i^f t^{\nu_i}, \quad s_i \in \mathcal{O}(\mathcal{W}),\ s_i \neq 0. \tag{40}$$

Then the following holds:

1. *$\nu_i \leq n$ for every i. Moreover, if $\nu_i = n$ for some i, then $i = m$ and $\alpha = \beta = a_0$.*
2. *The coefficients s_i for $i < m$ are all determined by counting pearly moduli spaces that involve configurations of disks with total Maslov index strictly smaller than $2n$. This continues to hold also for s_m if $\alpha \neq a_0$ or $\beta \neq a_0$.*

For a class $c \in Q^+ H(L;\mathcal{W})$ denote by $\langle c, a_m t^k\rangle_f$ the coefficient of $a_m t^k$ when writing c in the basis $a_0^f, \ldots, a_m^f$.

Consider now the product $a_0^f * a_0^g$. By the definition of the product we have

$$\langle a_0^f * a_0^g, a_m t^n\rangle_f = \epsilon' n_{PQR} + \epsilon'' \theta_{PQR}$$

for some $\epsilon', \epsilon'' \in \{-1, 1\}$ and where θ_{PQR} counts pearly configurations as in Section A.2.2 in which more than a single J–holomorphic disk is present (by contrast, n_{PQR}

counts the configurations given by a single disk through the three points). The total Maslov number of the configurations counted by θ is $2n$ and as there are more than two disks present, each such disk is of Maslov at most $2n-2$. Thus in order to prove our theorem, it is enough to show that $\langle a_0^f * a_0^g, a_m t^n\rangle_f$ can be determined as a polynomial expression in variables that count pearly configurations with total Maslov index $< 2n$.

For this purpose write $a_0^f = a_0^g + \sum_{j\geq 1} \sigma_j^0 a_j^g t^{r_j^0}$ with $1 \leq r_j^0 \leq n/2$, as in (39). We have

$$\langle a_0^f * a_0^f, a_m t^n\rangle_f = \langle a_0^f * a_0^g, a_m t^n\rangle_f + \sum_{j\geq 1} \sigma_j^0 \langle a_0^f * a_j^g, a_m t^{n-r_j^0}\rangle_f.$$

All the elements in the second summand are determined by configurations with total Maslov $\leq 2n-2$ (note that $r_j^0 \geq 1$). Thus it is enough to prove the same assertion for the left hand side $\langle a_0^f * a_0^f, a_m t^n\rangle_f$.

We now use the assumption that L is not a rational homology sphere. Under this assumption it is possible to choose the basis $\mathbf{a}$ so that there exist $a, b \in \mathbf{a}$ with $0 < |a|, |b| < m$ and $a \cdot b = a_0$. We now have

$$a^f * b^f = a_0^f + E(t)t,$$

where $E(t)$ is a polynomial in t with coefficients are linear combinations of $a_1^f, \ldots, a_m^f$, but $E(t)$ has no term containing a_0^f. Note also that the second summand here is $E(t)t$, hence it has no term of degree 0 with respect to t. It follows now that

$$\begin{aligned}\langle (a^f * b^f) * (a^f * b^f), a_m t^n\rangle_f \\ = \langle a_0^f * a_0^f, a_m t^n\rangle_f + \langle a_0^f * E_1(t) + E_1(t) * a_0^f, a_m t^{n-1}\rangle_f \\ + \langle E(t) * E(t), a_m t^{n-2}\rangle_f.\end{aligned}$$

The last two summands on the right-hand side are clearly determined by configurations with total Maslov $\leq 2n-2$ hence we are reduced to showing that the same holds for the left-hand side.

We now use the fact that the quantum product is associative. This implies that

$$(a^f * b^f) * (a^f * b^f) = ((a^f * b^f) * a^f) * b^f = (a_0^f * a^f + E(t)t * a^f) * b^f. \tag{41}$$

By Lemma 6.1.2 $a_0^f * a^f$ has no term with t^n and the same holds also for $E(t) * a^f$. Moreover when writing $a_0^f * a^f$ and $E(t) * a^f$ in the basis $\mathbf{a}^f$ all the coefficients are determined by configurations with Maslov $\leq 2n-2$. By Lemma 6.1.2 again, the same holds also for $(a_0^f * a^f + E(t)t * a^f) * b^f$. It follows that all the coefficients (and in particular that of $a_m t^n$ in $(a^f * b^f) * (a^f * b^f)$ depend on configurations of Maslov $\leq 2n-2$. This concludes the proof. □

Remark 6.1.3 (i) Splitting polynomials of the type constructed above have a close relationship with the invariant polynomials discussed in Section 5.2. Indeed, in the language of that section, suppose that the homology basis (a_i) is so that $a_0 = [pt]$, $a_s = [L]$. Any invariant polynomial of the form $F = k_s^{0,0} + F'$ with F' depending on variables different from $k_s^{0,0}$ produces a splitting polynomial S. To see this we first express the coefficients $k_l^{i,j}$ in terms of the coefficients $w_l^{i,j}$ of the "geometric" product $\widetilde{*}$. We then express F' in the $w_l^{i,j}$'s (there are also other variables appearing here as in (39)) thus obtaining a new polynomial S'. We then define $S = -S' - \epsilon\theta + F(*)$ for a suitable $\epsilon = \pm 1$ and with $\theta = \theta_{PQR} + \cdots$. Here $\cdots$ stands for other terms resulting from the expression of $k_s^{0,0}$ in terms of the w's and $F(*)$ is the value of the invariant polynomial on the product $*$. The construction in the proof of the theorem is precisely of this type with F a particular polynomial deduced from the associativity relation as it appears in (41).

(ii) It would be interesting to know what is the "simplest" (in some sense yet to be defined) splitting polynomial S that one can produce by these methods.

6.2 Lagrangian 2–tori

In case of the 2–torus all the discussion above becomes much simpler and more elegant. Moreover, we will deduce a splitting formula in terms of some configurations that have some nice geometric meaning.

Consider three distinct points $P, Q, R \in L$. Choose a smooth oriented path $\overrightarrow{PQ}$ starting from P and ending at Q. Similarly connect Q to R and R to P by such paths, denoted $\overrightarrow{QR}$ and $\overrightarrow{RP}$ respectively. We will refer to this triple of points connected by these curves as a "triangle" on the tours.

We will use now the notation from Section 3.3, in particular the set of classes $\mathcal{E}_2$ and the evaluation map $\mathrm{ev} : (\widetilde{\mathcal{M}}(B, J) \times \partial D)/G \longrightarrow L$. By taking J generic we may assume that all three points P, Q, R are regular values of ev and moreover that ev is transverse to the curve $\overrightarrow{QR}$. Given $(u, z) \in \mathrm{ev}^{-1}(P)$ set $\varepsilon(u, z; P) = \pm 1$ according to whether ev preserves or reverses orientations at (u, z). Let $\rho \in \mathcal{W}$. Define now the following (complex) number

$$n_P(\rho) = \sum_{B \in \mathcal{E}_2} \sum_{(u,z) \in \mathrm{ev}^{-1}(P)} \rho(B)\varepsilon(u, z; P)\#\big(u(\partial D) \cap \overrightarrow{QR}\big), \tag{42}$$

where $\#\big(u(\partial D) \cap \overrightarrow{QR}\big)$ stands for the intersection number between the oriented curves $u(\partial D)$ and $\overrightarrow{QR}$. The number $n_P(\rho)$ can be thought of as the number of J–holomorphic disks of Maslov index 2 whose boundaries pass through P and the "edge" QR of the triangle PQR, only that the count of the disks is weighted by the representation ρ. We also have the numbers $n_Q(\rho)$ and $n_R(\rho)$ analogously defined.

Remark 6.2.1 The number $n_P(\rho)$ does not depend on the choice of the path $\overrightarrow{QR}$ connecting Q to R, but only on the points Q and R. The reason for this is that the 1–dimensional cycle

$$\sum_{B\in\mathcal{E}_2}\ \sum_{(u,z)\in\mathrm{ev}^{-1}(P)} \rho(B)\varepsilon(u,z;P)u(\partial D)$$

is null homologous in $H_1(L;\mathbb{C})$. Indeed, it has been shown in [14, Section 4.2] that if this cycle is not null-homologous, then the associated quantum homology vanishes (the proof was in fact only done for ρ the identity representation but it is immediate to see that the argument also applies to any other representation). In our case we are only considering this cycle for $\rho\in\mathcal{W}$ so that this forces the respective 1–cycle to vanish in homology.

Thus the intersection number of this cycle with the path $\overrightarrow{QR}$ depends only on its end points Q and R. Nevertheless, $n_P(\rho)$ is far from being an invariant in any sense since it depends on the choice of the almost complex structure J as well as on the points P, Q, R.

As in the previous section we are interested to evaluate the number n_{PQR} of Maslov $2n=4$ disks through P, Q, R. Similarly to n_P, the number n_{PQR} is not an invariant either.

6.2.1 Triangles on the torus and the discriminant To simplify notation we omit the ρ's from the notation, ie abbreviate $n_{PQR}=n_{PQR}(\rho)$, $n_P=n_P(\rho)$, $n_Q=n_Q(\rho)$, $n_R=n_R(\rho)$.

Theorem 6.2.2 *Let PQR be a triangle on L. Then for every $\rho\in\mathcal{W}$ we have*

$$\Delta(\rho)=4n_{PQR}+n_P^2+n_Q^2+n_R^2-2n_Pn_Q-2n_Qn_R-2n_Rn_P. \qquad (43)$$

The proof of this result is contained in the next couple of sections. The first expresses the discriminant as a polynomial in certain coefficients appearing in the expansion of the Lagrangian quantum product. The second section continues with the combinatorial work needed to relate these coefficients to the enumerative expressions (43).

6.2.2 The discriminant and higher quantum products We continue here with the assumption that $L \subset M$ is a 2–dimensional Lagrangian torus with $N_L = 2$. We also assume that the wide variety $\mathcal{W}_2$ is not empty.

Let $\mathcal{W}$ be any of the wide varieties, $\mathcal{W}_1$ or $\mathcal{W}_2$. Working with the ring $\mathcal{R} = \mathcal{O}(\mathcal{W}) \otimes \Lambda^+$ we obtain from (21)

$$0 \longrightarrow \mathcal{O}(\mathcal{W})[L]t \xrightarrow{i} Q^+H_0(L;\mathcal{W}) \xrightarrow{\pi} H_0(L;\mathbb{C}) \otimes \mathcal{O}(\mathcal{W}) \longrightarrow 0. \tag{44}$$

Choose $\widetilde{p} \in Q^+H_0(L;\mathcal{W})$ with $\pi(\widetilde{p}) = [\mathrm{pt}] \in H_0(L;\mathbb{C})$. Then $\{\widetilde{p}, [L]t\}$ forms a basis for $Q^+H_0(L;\mathcal{W})$, so we can write

$$\widetilde{p} * \widetilde{p} = \sigma \widetilde{p}t + \tau[L]t^2, \tag{45}$$

where $\sigma, \tau \in \mathcal{O}(\mathcal{W})$. The coefficients σ, τ depend on $\widetilde{p}$ as follows. If we replace $\widetilde{p}$ by $\widetilde{p}\,' = \widetilde{p} + r[L]t$, for some $r \in \mathcal{O}(\mathcal{W})$ then the corresponding coefficients σ' and τ' change as follows:

$$\sigma' = \sigma + 2r, \quad \tau' = \tau - \sigma r - r^2. \tag{46}$$

This can be verified by a direct computation from (45). Thus neither σ nor τ are invariants. However it is easy to see that

$$\sigma^2 + 4\tau$$

is invariant in the sense that it does not depend on $\widetilde{p}$ – this is precisely an example of a universal, symmetric polynomial Lagrangian invariant. In view of Corollary 5.3.4 we expect it to be related to the discriminant. Indeed:

Proposition 6.2.3 *We have the following identity in $\mathcal{O}(\mathcal{W})$: $\Delta = \sigma^2 + 4\tau$.*

Proof Choose a basis $\{C_1, C_2\}$ for $H_1(L;\mathbb{Z})$ such that $C_1 \cdot C_2 = [\mathrm{pt}]$. Write

$$\begin{aligned} &C_1 * C_1 = \tfrac{1}{2}a_{11}[L]t, \qquad C_2 * C_2 = \tfrac{1}{2}a_{22}[L]t, \\ &C_1 * C_2 = \widetilde{p} + a'[L]t, \qquad C_2 * C_1 = -\widetilde{p} + a''[L]t, \quad a_{12} = a' + a'', \end{aligned} \tag{47}$$

with $a_{11}, a_{22}, a', a'' \in \mathcal{O}(\mathcal{W})$. Then $\widetilde{p} = C_1 * C_2 - a'[L]t = -C_2 * C_1 + a''[L]t$, hence

$$\begin{aligned} \widetilde{p} * \widetilde{p} &= -C_1 * C_2 * C_2 * C_1 + a'' C_1 * C_2 t + a' C_2 * C_1 t - a'a''[L]t^2 \\ &= (a'' - a')\widetilde{p}t + \big(-\tfrac{1}{4}a_{11}a_{22} + a'a''\big)[L]t^2. \end{aligned}$$

Thus $\sigma = a'' - a'$ and $\tau = a'a'' - \frac{1}{4}a_{11}a_{22}$. It immediately follows that

$$\sigma^2 + 4\tau = a_{12}^2 - a_{11}a_{22} = -\det(a_{ij}) = \Delta.$$

□

6.2.3 Enumerative expressions for σ and τ and proof of Theorem 6.2.2 We will relate the two coefficients σ and τ above to the enumerative expressions $n_{\boldsymbol{P}}, n_{\boldsymbol{Q}}, n_{\boldsymbol{R}}, n_{\boldsymbol{PQR}}$. Theorem 6.2.2 will then follow immediately from Proposition 6.2.3.

We will use here a method described in [12; 13]. This consists in picking two perfect Morse function $f, g\colon L \to \mathbb{R}$ with pairwise distinct critical points, a Riemannian metric $(\cdot,\cdot)$ on L as well as an almost complex structure J which is sufficiently generic so that all pearl complexes, products etc are defined. These functions are required to satisfy a number of additional properties as described below.

Let x_0 be the minimum of f, let x_2 be the maximum of f, let y_0 be the minimum of g and similarly let y_2 be the maximum of g. We may assume that y_2 is as close as we want to x_2 in L. We also assume that the choices of f, g as well as that of the Riemannian metric $(\cdot,\cdot)$ are such that $y_0 = P$, $x_0 = Q$, $x_2 = R$ and the edge $\overrightarrow{RP}$ is the unique flow line of $-\nabla f$ going from x_2 to y_0, and (after slightly rounding the corner at P) the edge $\overrightarrow{PQ}$ is the unique flow line of $-\nabla f$ going from y_0 to x_0. Moreover, the edge $\overrightarrow{QR}$ contains the point y_2 and it consists of two pieces: one is the unique flow line of $-\nabla g$ going from y_2 to x_0 – the orientation of this flow line is opposite that of $\overrightarrow{QR}$; the second consists of a very short flow line, γ, of $-\nabla g$ joining y_2 to x_2 – the orientation of this flow line coincides with that of $\overrightarrow{QR}$. The points y_2 and x_2 are taken close enough so that no J–holomorphic disk of Maslov index 2 passing through y_0 intersects γ (as the number of these disks is finite this is not restrictive). Put $\mathscr{D} = (f, (\cdot,\cdot), J)$ and $\mathscr{D}' = (g, (\cdot,\cdot), J)$ and consider the chain

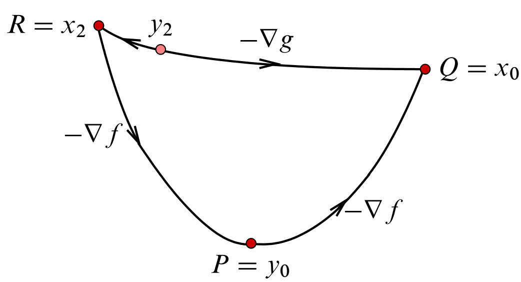

Figure 1. The triangle PQR as drawn by negative flow lines of f and g

level quantum product

$$\mathcal{C}(\mathscr{D}) \otimes \mathcal{C}(\mathscr{D}') \longrightarrow \mathcal{C}(\mathscr{D}), \quad x \otimes y \longmapsto x * y.$$

By abuse of notation we denote by the same $*$ also the induced product in homology. We work here with coefficients in $\mathcal{O}(\mathcal{W}) \otimes \mathbb{C}[t]$, where $\mathcal{W}$ is one of the wide varieties $\mathcal{W}_1$

or $\mathcal{W}_2$. Recall that we also have the comparison map $\Psi_{\mathscr{D}',\mathscr{D}}\colon \mathcal{C}(\mathscr{D}) \longrightarrow \mathcal{C}(\mathscr{D}')$ whose definition is described in Section A.2.5.

Put $\widetilde{p} = [x_0] \in Q^+ H_0(L;\mathcal{W})$, and write $\widetilde{p} * \widetilde{p} = \sigma \widetilde{p} t + \tau [L] t^2$ as in (45). Consider now the chain level product $x_0 * y_0$, and write

$$x_0 * y_0 = \alpha x_0 t + \beta x_2 t^2 \quad \text{for some } \alpha, \beta \in \mathcal{O}(\mathcal{W})$$

The relation between x_0 and y_0 is given by $\Psi_{\mathscr{D}',\mathscr{D}}$, namely

$$\Psi_{\mathscr{D}',\mathscr{D}}(x_0) = y_0 + \kappa y_2 t \quad \text{for some } \kappa \in \mathcal{O}(\mathcal{W}).$$

It follows that

$$\widetilde{p} * \widetilde{p} = [x_0] * [x_0] = [x_0] * ([y_0] + \kappa [L] t) = (\alpha + \kappa) \widetilde{p} t + \beta [L] t^2,$$

hence we have

$$\sigma = \alpha + \kappa, \qquad \tau = \beta. \tag{48}$$

We will now compute α, κ and β explicitly. We begin with β. By the definition of the chain level product (see Section A.2.2) we have $\beta = \beta_I + \beta_{II}$, where β_I counts configurations as in the left part of Figure 2 with $\mu(\lambda) = 4$ and β_{II} counts the configurations drawn in the right-hand side of that figure with $\mu(\lambda_1) = \mu(\lambda_2) = 2$.

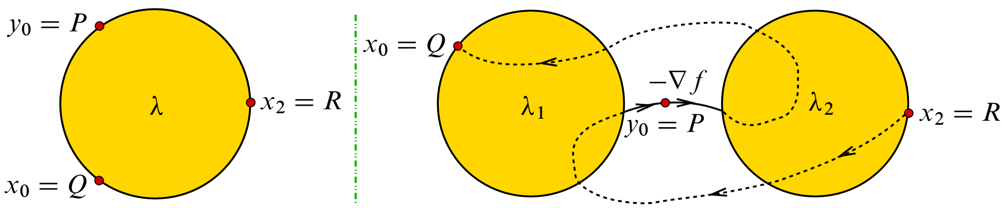

Figure 2. Configurations contributing to β_I and β_{II}

To compute the precise values of β_I and β_{II} we use the definition of the quantum product from Section A.2.2. We have

$$\begin{aligned} \beta_I &= \sum_{\lambda, \mu(\lambda)=4} \# \Big(\{x_0\} \times_L \big(\{y_0\} \times_L \widetilde{\mathcal{M}}(\lambda) \times_L \{x_2\} \big) \Big) \rho(\lambda) \\ &= \sum_{\lambda, \mu(\lambda)=4} \# \Big(\widetilde{\mathcal{M}}(\lambda) \times_{L \times L \times L} \{(P, Q, R)\} \Big) \rho(\lambda) = \# \mathbf{e}^{-1}(P, Q, R) = n_{PQR}. \end{aligned} \tag{49}$$

(Recall that $\mathbf{e}$ and n_{PQR} were defined in (36) and (37).) In the middle two equalities we have used identities (83), (81), (82) from Section A.1.8. The fact that no new signs appear follows from these identities and the fact that all spaces involved in the above fiber products are even dimensional.

We now compute β_{II}. For λ_1, λ_2 with $\mu(\lambda_1) = \mu(\lambda_2) = 2$ put

$$\beta'_{II,\lambda_1} = \#\Big(\{x_0\} \times_L \big(\mathcal{M}_2(\lambda_1) \times \mathbb{R}_+\big) \times_L \{y_0\}\Big),$$
$$\beta''_{II,\lambda_2} = \#\Big((\{y_0\} \times \mathbb{R}_+) \times_L \mathcal{M}_2(\lambda_2) \times_L \{x_2\}\Big).$$

It follows easily from the definition of the quantum product that

$$(50)\quad \beta_{II} = \sum_{\lambda_1,\lambda_2} \beta'_{II,\lambda_1}\beta''_{II,\lambda_2}\rho(\lambda_1)\rho(\lambda_2) = \Big(\sum_{\lambda_1} \beta'_{II,\lambda_1}\rho(\lambda_1)\Big)\Big(\sum_{\lambda_2} \beta''_{II,\lambda_2}\rho(\lambda_2)\Big),$$

where the sums are over all λ_1, λ_2 with $\mu(\lambda_1) = \mu(\lambda_2) = 2$. A straightforward computation shows that

$$\beta'_{II,\lambda_1} = -\sum_{(u,z)\in \mathrm{ev}^{-1}(Q)} \epsilon(u,z;Q)\#(u(\partial D) \cap \overrightarrow{RP}),$$

where we use here the notation from the beginning of Section 6.2. It follows from (42) that $\sum_{\lambda_1} \beta'_{II,\lambda_1}\rho(\lambda_1) = -n_Q$. A similar computation gives $\sum_{\lambda_2} \beta''_{II,\lambda_2}\rho(\lambda_2) = n_R$. Substituting all this into (50) gives $\beta_{II} = -n_Q n_R$, hence

$$(51)\qquad \beta(\rho) = n_{PQR} - n_Q n_R.$$

We now turn to computing α. For a class A with $\mu(A) = 2$ put

$$\alpha_{I,A} = \{x_0\} \times_L \big(\mathcal{M}_2(A) \times \mathbb{R}_+\big) \times_L \big(\{y_0\} \times_L L \times_L W^s_f(x_0)\big),$$
$$\alpha_{II,A} = \{x_0\} \times_L L \times_L \big(\{y_0\} \times_L (\mathcal{M}_2(A) \times \mathbb{R}_+) \times_L W^s_f(x_0)\big).$$

Then by the definition of the quantum product we have (see Figure 3)

$$\alpha = \sum_A \alpha_{I,A}\rho(A) + \sum_A \alpha_{II,A}\rho(A).$$

A straightforward computation shows that

$$\alpha_{I,A} = \{x_0\} \times_L \big(\mathcal{M}_2(A) \times \mathbb{R}_+\big) \times_L \{y_0\} = -\sum_{(u,z)\in \mathrm{ev}^{-1}(Q)} \epsilon(u,z;Q)\#(u(\partial D) \cap \overrightarrow{RP}).$$

Summing over the A's and weighting by ρ we obtain that $\sum_A \alpha_{I,A}\rho(A) = -n_Q$. A similar computation gives $\sum_A \alpha_{II,A}\rho(A) = n_P$. It follows that

$$(52)\qquad \alpha = n_P - n_Q.$$

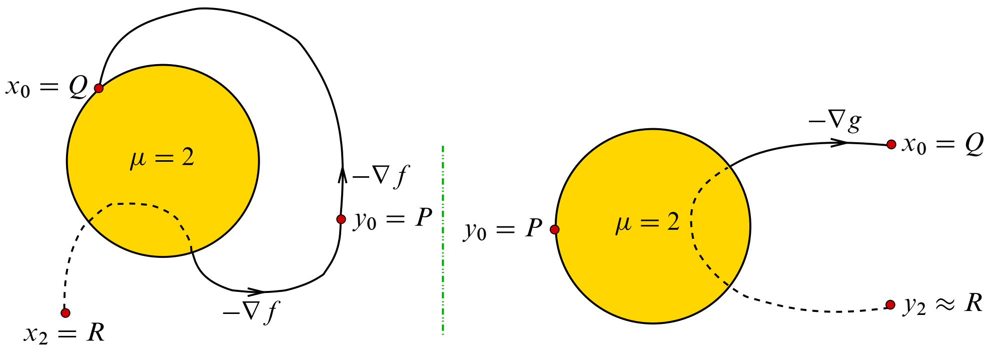

Figure 3. Configurations contributing to α_I and α_{II}

It remains to compute κ. A priori there are two types of configurations that might contribute to κ, both depicted in Figure 4. However, due to our choices of f, g, the configuration on the right part of Figure 4 cannot exist. To see this, first note that since g is generic we may assume that y_2 lies on the unstable submanifold of x_2 (with respect to $-\nabla f$). Next, since y_2 was chosen to be very close to x_2 we conclude that x_2 must lie somewhere on the part of the $(-\nabla f)$ trajectory that connects the holomorphic disk through y_0 and y_2. But this is impossible since x_2 is a maximum hence there are no $(-\nabla f)$ trajectories entering x_2.

We are thus left only with the configuration on the left part of Figure 4. According to Section A.2.5 these are computed by

$$\kappa = \sum_A \#\big(\{x_0\} \times_L (L \times \mathbb{R}_+) \times_L \mathcal{M}_2(A) \times_L \{y_2\}\big)\rho(A). \tag{53}$$

As y_2 was chosen close enough to R, a straightforward computation gives $\kappa = -n_R$.

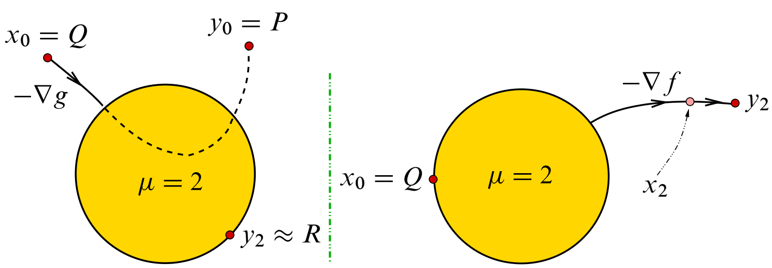

Figure 4. Configurations a priori contributing to κ. The one on the right is impossible.

Substituting this together with (52) and (51) into (48) we get

$$\sigma = n_P - n_Q - n_R, \quad \tau = n_{PQR} - n_Q n_R. \tag{54}$$

By Proposition 6.2.3 we obtain

$$\Delta = \sigma^2 + 4\tau = 4n_{PQR} + n_P^2 + n_Q^2 + n_R^2 - 2n_P n_Q - 2n_Q n_R - 2n_R n_P.$$

The proof of Theorem 6.2.2 is complete. □

Remark 6.2.4 Knowing the precise signs ($\pm$) appearing in the expressions for σ and τ is not really necessary in order to prove Theorem 6.2.2. Here is the shortcut. It is enough to prove that there exist $\epsilon_i \in \{-1, 1\}$, $i \in \{0, 1, 2, 3\}$ so that

$$\tau = n_{PQR} + \epsilon_0 n_Q n_R, \quad \sigma = \epsilon_1 n_P + \epsilon_2 n_Q + \epsilon_3 n_R.$$

Then by Proposition 6.2.3 we get

$$\Delta = 4n_{PQR} + 4\epsilon_0 n_Q n_R + (\epsilon_1 n_P + \epsilon_2 n_Q + \epsilon_3 n_R)^2.$$

We already know that Δ is an invariant and, in particular, it is left invariant by circular permutations of P, Q, R. This immediately implies that ϵ_1, ϵ_2, ϵ_3 can not all have the same sign and so we may assume that just one of them is negative and the other two positive. If either one of ϵ_2, ϵ_3 is negative this circular symmetry can not be satisfied. So $\epsilon_1 = -1$. Again for symmetry reasons this implies $\epsilon_0 = -1$ and proves the claim.

6.3 Modulo–2 invariants

More can be said about the discriminant as well as the enumerative counts introduced in Section 6.2 after reduction modulo 2 (and modulo 4). In the following theorem we focus for simplicity on the trivial representation. We denote by $\Delta = \Delta(1) \in \mathbb{Z}$ the discriminant computed at the trivial representation $\rho \equiv 1$. Similarly, we denote by $n_P, n_Q, n_R, n_{PQR} \in \mathbb{Z}$ the numbers defined in Section 6.2, and by $\sigma, \tau \in \mathbb{Z}$ be the structural constants defined at (45), all computed at $\rho \equiv 1$.

Theorem 6.3.1 *Let $L^2 \subset M^4$ be a wide Lagrangian torus with $N_L = 2$, where by "wide" we mean here that the trivial representation $\rho \equiv 1$ belong to $\mathcal{W}_2$. Then:*

(1) $\Delta \equiv \sigma \equiv n_P + n_Q + n_R \pmod 2$.

(2) $\Delta \pmod 4$ *admits only the values* 0 *or* 1.

Moreover, if $\Delta \equiv 1 \pmod 2$ then:

(i) *The value of $\tau \pmod 2$ is invariant in the sense that it does not depend on the choice of the element $\widetilde{p}$ in (45).*

(ii) $n_P n_Q \equiv n_Q n_R \equiv n_R n_P \pmod 2$.

(iii) *The value of $n_{PQR} + n_P n_Q \pmod 2$ is invariant, ie does not depend neither on P, Q, R nor on the almost complex structure. This number is congruent to $\tau \pmod 2$.*

Proof Recall from Proposition 6.2.3 that $\Delta = \sigma^2 + 4\tau$. Hence $\Delta \equiv \sigma \pmod 2$. The fact that $\sigma \equiv n_P + n_Q + n_R \pmod 2$ follows from (54). Next note that $\Delta \equiv \sigma^2 \pmod 4$, hence the latter can obtain only the values 0 and $1 \pmod 4$. This proves the first two statements in the theorem.

To prove the other statements, assume now that $\Delta \equiv 1 \pmod 2$, or equivalently that $\sigma \equiv 1 \pmod 2$. The fact that $\tau \pmod 2$ is invariant follows immediately from formulae (46). This proves (i).

To prove the identity (ii) note that if $\sigma \equiv 1 \pmod 2$ then either the three numbers $n_P, n_Q, n_R \pmod 2$ are all 1, or exactly two of them are 0 and one of them is 1. In both cases the identity in (ii) holds.

Finally, point (iii) follows from the arguments of Section 6.2.3. See (54) as well as [13, Theorem 7.2.2]. □

Remarks 6.3.2 (1) Some of the statements in Theorem 6.3.1 (eg point (iii)) do not seem to follow by just reducing $\pmod 2$ the identity (43), but rather reveal more geometric information on the structure of the "constants" n_P, n_Q, n_R and n_{PQR}.

(2) It seems that one could get more general congruences by allowing every representation $\rho \in \mathcal{W}_1$ (not just the trivial one). The point is that all the calculations involving an element $\rho \in \mathcal{W}_1$ can be done in a number field (ie a finite extension of $\mathbb{Q}$) and the values of $\Delta(\rho)$ and the constants $n_P(\rho), n_Q(\rho), n_R(\rho), n_{PQR}(\rho)$ belong to the ring of integers of this field. One expects some congruence relations (with respect to some ideal in this ring) to hold between these numbers.

7 Toric fibers

Here we work out in detail the theory discussed in the previous sections for the special case of Lagrangian tori that arise as fibres of the moment map in a toric manifold. Below we will use in an essential way previous results of Cho [15], Cho and Oh [18] and Fukaya, Oh, Ohta and Ono [16; 27; 25] on Floer theory of torus fibres in toric manifolds. The reader is referred to these papers for more details. For the foundations of symplectic toric manifolds see Audin [3], and for an algebro-geometric account see Fulton [28].

7.1 Setting

Let (M^{2n}, ω) be a closed monotone toric manifold. Denote by

$$\mathfrak{m}\colon M \longrightarrow \mathrm{Lie}(\mathbb{T}^n)^* = \mathbb{R}^n$$

the moment map and by $P = \text{image}(\mathfrak{m})$ the moment polytope. The symplectic manifold (M, ω) admits a canonical ω–compatible (integrable) complex structure J_0 which turns (M, J_0) into a complex algebraic manifold. We will refer to J_0 as the *standard complex structure*.

Denote by $F_1, \dots, F_r$ the codimension–1 facets of P and by $\overrightarrow{v_1}, \dots, \overrightarrow{v_r} \in \mathbb{Z}^n$ the normal integral primitive vectors to the facets $F_1, \dots, F_r$ respectively, pointing inwards P. Note that the number of codimension–1 facets is $r = n + b_2(M)$. The fibres $\mathfrak{m}^{-1}(p)$, $p \in P$, are Lagrangian tori. There is a (unique) special point $p_* \in P$ for which the Lagrangian torus $L = \mathfrak{m}^{-1}(p_*)$ is monotone (see eg [18; 17; 27]). Furthermore, we have $N_L = 2$. Note also that $H_2^D \cong \pi_2(M, L)$, and that $r = \text{rank}\, H_2^D$. We denote by $\Sigma_i = \mathfrak{m}^{-1}(F_i) \subset M$. These turn out to be smooth J_0–complex hypersurfaces in M and their sum $[\Sigma_1] + \cdots + [\Sigma_r]$ represents the Poincaré dual of the first Chern class c_1 of M (which is by assumption a positive multiple of $[\omega]$).

Since L is an orbit of the $\mathbb{T}^n$–action we have a canonical identification $H_1(L; \mathbb{Z}) = H_1(\mathbb{T}^n; \mathbb{Z})$ and we denote by $\mathbf{e} = \{e_1, \dots, e_n\}$ the standard basis corresponding to this identification.

7.1.1 Holomorphic disks Due to the $\mathbb{T}^n$–action the Lagrangian torus L comes with a preferred orientation as well as a spin structure. Fixing these two, one can endow the space of holomorphic disks with boundary on L with a canonical orientation (see [26; 16] for more details).

We start with a description, due to Cho and Oh [18], of the subset $\mathcal{E}_2 \subset H_2^D$ of classes that can be represented by J–holomorphic disks with Maslov index 2 for generic J (as well as for J_0). We use here the notation from Section 3.3.

Proposition 7.1.1 (Cho–Oh [18]) *The set $\mathcal{E}_2$ consists of exactly $r = \text{rank}\, H_2^D$ classes $\mathcal{E}_2 = \{B_1, \dots, B_r\}$ with the following properties:*

(1) $\#(B_i \cdot \Sigma_j) = \delta_{i,j}$ *for every i, j.*
(2) *The set $\mathcal{E}_2$ is a $\mathbb{Z}$–basis for H_2^D.*
(3) *Denote by $\partial\colon H_2^D \longrightarrow H_1(L; \mathbb{Z})$ the boundary operator. Then writing ∂B_i in the basis $\mathbf{e}$ we have $(\partial B_i) = \overrightarrow{v_i}$ for every $1 \leq i \leq r$.*
(4) *For every i, $\nu(B_i) = 1$.*

Furthermore, the standard complex structure J_0 is regular for all classes $B \in H_2^D$ with $\mu(B) = 2$. Moreover given a generic point $x \in L$ there exist precisely r J_0–holomorphic disks $u_i\colon (D, \partial D) \longrightarrow (M, L)$, $i = 1, \dots, r$, up to parametrization, with $\mu([u_i]) = 2$ and $u(1) = x$. These disks satisfy $u_i([D]) = B_i$, $i = 1, \dots, r$. The image of $\mathfrak{m} \circ u_i$ is a straight segment going from p_* to a point on the facet F_i.

7.1.2 The superpotential, the wide variety and the discriminant The following is an immediate corollary of Proposition 7.1.1:

Corollary 7.1.2 (Cho–Oh [18]) *The superpotential $\mathscr{P}$ has the following form in the coordinates induced by the basis* **e** *(see Section 3.3.1):*

$$\mathscr{P}(z_1, \dots, z_n) = \sum_{i=1}^{r} z^{\overrightarrow{v_i}}, \tag{55}$$

where for a vector $\overrightarrow{v} = (v^1, \dots, v^n) \in \mathbb{Z}^n$, $z^{\overrightarrow{v}}$ stands for the monomial $z^{\overrightarrow{v}} = z_1^{v^1} \cdots z_n^{v^n}$.

Since $H^1(L;\mathbb{R})$ generates $H^*(L;\mathbb{R})$ (with respect to the cup product) we obtain from Proposition 3.3.1:

Corollary 7.1.3 (Cho–Oh [18]; see also Fukaya–Oh–Ohta–Ono [27]) *We have*

$$\mathcal{W}_1 = \operatorname{Crit}(\mathscr{P}),$$

where $\mathcal{W}_1 \subset \operatorname{Hom}(H_1, \mathbb{C}^*)$ is the wide variety as defined in Section 3.

Choose a basis $\mathbf{C} = \{C_1, \dots, C_n\}$ for $H_{n-1}(L;\mathbb{Z})$ which is dual to **e** as in Section 3.3.1. We view $H_{n-1}(L;\mathbb{C})$ as a subset of $QH_{n-1}(L;\mathcal{W}_1)$ as explained in Section 3.3.1 just before the statement of Proposition 3.3.4. The following corollary immediately follows from Proposition 3.3.4.

Corollary 7.1.4 (Cho [16])

$$C_i * C_j + C_j * C_i = (-1)^n \Big(\sum_{k=1}^{r} v_k^i v_k^j \, z^{\overrightarrow{v_k}} \Big) [L] t \quad \forall z \in \mathcal{W}_1.$$

We now turn to the quadratic form defined in Section 4 and its discriminant. Substituting (55) in (24) we get

$$\Delta(z_1, \dots, z_n) = (-1)^{n+1} \det \Big(\sum_{k=1}^{r} v_k^i v_k^j \, z^{\overrightarrow{v_k}} \Big)_{i,j} \quad \forall z \in \mathcal{W}_1. \tag{56}$$

However, there is a nicer formula for the discriminant which we now present. For a subset of indices $I \subset \{1, \dots, r\}$ with $\#I = n$, say $I = \{i_1, \dots, i_n\}$, define an $(n \times n)$–matrix A_I whose rows consists of the vectors $\overrightarrow{v_{i_1}}, \dots, \overrightarrow{v_{i_n}}$, and a vector $\overrightarrow{v_I}$ which is the sum of the $\overrightarrow{v_{i_k}}$'s, ie

$$(57)\qquad A_I = \begin{pmatrix} \text{———} & \overrightarrow{v_{i_1}} & \text{———} \\ & \vdots & \\ \text{———} & \overrightarrow{v_{i_k}} & \text{———} \\ & \vdots & \\ \text{———} & \overrightarrow{v_{i_n}} & \text{———} \end{pmatrix}, \qquad \overrightarrow{v_I} = \sum_{i \in I} \overrightarrow{v_i}.$$

Note that $\det(A_I)^2$ does not depend on the ordering of the indices i_j in the set I.

Proposition 7.1.5 *The discriminant verifies the following formula:*

$$(58)\qquad \Delta(z_1, \dots, z_n) = (-1)^{n+1} \sum_{\substack{I \subset \{1,\dots,r\} \\ \#I = n}} z^{\overrightarrow{v_I}} \det(A_I)^2 \quad \forall z \in \mathcal{W}_1.$$

The proof follows by direct computation by expanding the determinant in (56).

7.2 Formulae for $\mathcal{W}_2$

Recall from Proposition 7.1.1 that set $\mathcal{E}_2 = \{B_1, \dots, B_r\}$ forms a $\mathbb{Z}$–basis for H_2^D. Using this basis we can identify $\mathrm{Hom}(H_2^D, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\times r}$. An element of the latter space $\xi = (\xi_1, \dots, \xi_r)$ will be identified with the representation ρ that satisfies $\rho(B_k) = \xi_k$, $k = 1, \dots, r$.

We continue to work with the basis $\mathbf{e} = \{e_1, \dots, e_n\}$ for H_1 introduced in Section 7.1 and the dual basis $\mathbf{C} = \{C_1, \dots, C_{n-1}\}$ for $H_{n-1}(L; \mathbb{Z})$.

With this notation the following is a straightforward calculation which results from Proposition 7.1.1.

Proposition 7.2.1 (1) *The wide variety $\mathcal{W}_2$ is cut by the following system of n linear equation (with r unknowns):*

$$\mathcal{W}_2 = \left\{ \sum_{k=1}^{r} v_k^j \xi_k = 0 \;\middle|\; j = 1, \dots, n \right\}.$$

Here, v_k^j is the j–th component of the vector $\overrightarrow{v_k}$, ie $\overrightarrow{v_k} = (v_k^1, \dots, v_k^n)$.

(2) *The natural map $\partial_{\mathcal{W}} \colon \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ induced by the boundary map $\partial \colon H_2^D \longrightarrow H_1$ is given by*

$$\partial_{\mathcal{W}}(z_1, \dots, z_n) = (z^{\overrightarrow{v_1}}, \dots, z^{\overrightarrow{v_r}}).$$

(3) *The product of elements of* $\mathbf{C}$ *satisfies*

$$C_i * C_j + C_j * C_i = (-1)^n \Big(\sum_k^r v_k^i v_k^j \xi_k \Big) [L]\, t.$$

(4) *The discriminant is given by*

$$\Delta(\xi_1, \dots, \xi_k) = (-1)^{n+1} \det \Big(\sum_k^r v_k^i v_k^j \xi_k \Big)_{i,j}.$$

7.3 Wide varieties and quantum homology of the ambient manifold

Here we further study the other quantum structures, such as the quantum algebra and quantum inclusion, and their relations to the wide varieties on toric manifolds.

Let $L = \mathfrak{m}^{-1}(p_*) \subset M$ be the monotone torus fibre in a monotone toric manifold. Assume that the wide variety $\mathcal{W}_1$ is not empty. By Corollary 7.1.3 the wide variety $\mathcal{W}_1$ coincides with the variety of critical points of the superpotential function $\mathcal{P}$, $\mathcal{W}_1 = \mathrm{Crit}(\mathcal{P})$, hence the ring or global algebraic functions $\mathcal{O}(\mathcal{W}_1)$ can be written as

$$\mathcal{O}(\mathcal{W}_1) = \frac{\mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]}{\langle \partial_{z_1} \mathcal{P}, \dots, \partial_{z_n} \mathcal{P} \rangle}, \tag{59}$$

where the denominator stands for the ideal generated by the partial derivatives of $\mathcal{P}$. This ring, or rather localizations of it, plays an important role in singularity theory and is sometime called the Jacobian ring of $\mathcal{P}$.

Interestingly, this ring appears in the symplectic picture also from a different angle. Denote by $QH(M;\Lambda)$ the quantum homology of the *ambient* manifold with coefficients in $\Lambda = \mathbb{C}[t, t^{-1}]$, where for compatibility with the Lagrangian picture we put $|t| = -N_L = -2$. It is well known that the classes $[\Sigma_i] = \mathfrak{m}^{-1}(F_i) \in QH_{2n-2}(M;\Lambda)$, $i = 1, \dots, r$, generate $QH(M;\Lambda)$ with respect to the quantum product $*$; see Batyrev [10] and McDuff and Salamon [39]. It turns out that $QH(M;\Lambda)$ is isomorphic as a ring to $\mathcal{O}(\mathcal{W}_1) \otimes \Lambda$. More precisely:

Theorem 7.3.1 (Batyrev; Givental; Fukaya–Oh–Ohta–Ono) *There exists an isomorphism of rings*

$$I\colon QH(M;\Lambda) \longrightarrow \mathcal{O}(\mathcal{W}_1) \otimes \Lambda \tag{60}$$

which satisfies $I([\Sigma_i]) = z^{\vec{v_i}}\, t$. *This isomorphism shifts degrees by* $-2n$. *Here, the grading on the right-hand side comes from the* Λ*–factor only (ie* $\mathcal{O}(\mathcal{W}_1)$ *is not graded).*

This theorem was first suggested by Givental [30] and by Batyrev [10] and has been verified since then at different levels of rigor. A rather rigorous and conceptual proof has been recently carried out by Fukaya, Oh, Ohta and Ono [27]. See also Ostrover and Tyomkin [40] for a more algebraically oriented proof. It is important to note that the isomorphism (60) does not send $QH(M;\Lambda^+)$ onto $\mathcal{O}(\mathcal{W}_1)\otimes\Lambda^+$ but rather into a subring of the latter.

We now consider the quantum module structure on $QH(L)$. Recall from Section 2.2 that for a $\widetilde{\Lambda}^+$–algebra $\mathcal{R}$, $QH(L;\mathcal{R})$ is a module (in fact an algebra) over $QH(M;\mathcal{R})$. We will use here $\mathcal{R}=\mathcal{O}(\mathcal{W}_1)\otimes\Lambda$, but a similar discussion holds for $\mathcal{O}(\mathcal{W}_1)\otimes\Lambda^+$ too. For $a\in QH_j(M;\mathcal{O}(\mathcal{W}_1)\otimes\Lambda)$ and $\alpha\in QH_k(L;\mathcal{W}_1)$ (see the notation in (14)) we denote by $a*\alpha\in QH_{j+k-2n}(L;\mathcal{W}_1)$ the quantum module action. Using the embedding $\Lambda=1\otimes\Lambda\subset\mathcal{O}(\mathcal{W}_1)\otimes\Lambda$, we have a natural inclusion $QH(M;\Lambda)\subset QH(M;\mathcal{O}(\mathcal{W}_1)\otimes\Lambda)$. We will now take a closer look at the induced module operation

$$QH(M;\Lambda)\otimes_\Lambda QH(L;\mathcal{W}_1)\longrightarrow QH(L;\mathcal{W}_1),\quad a\otimes\alpha\longmapsto a*\alpha. \tag{61}$$

Note that $QH(L;\mathcal{W}_1)$ is also a module over $\mathcal{O}(\mathcal{W}_1)\otimes\Lambda$ in an obvious way. For $c\in\mathcal{O}(\mathcal{W}_1)\otimes\Lambda$, $\alpha\in QH(L;\mathcal{W}_1)$ we denote this module operation as $c\alpha$. It turns out that this module structure and the preceding ones are in fact compatible:

Proposition 7.3.2 *For every $a\in QH(M;\Lambda)$ and $\alpha\in QH(L;\mathcal{W}_1)$ we have*

$$a*\alpha=I(a)\alpha. \tag{62}$$

The same continues to hold if we replace Λ by Λ^+ and $QH(L;\mathcal{W}_1)$ by $Q^+H(L;\mathcal{W}_1)$.

Proof Since $QH(M;\Lambda)$ is generated by the classes $[\Sigma_i]=\mathfrak{m}^{-1}(F_i)$, $i=1,\dots,r$, it is enough to check (62) for $a=[\Sigma_i]$. Since $*$ is an algebra action, it is also enough to restrict to the case $\alpha=[L]$ which is the unity of $QH(L;\mathcal{W}_1)$.

Next note that $[\Sigma_i]*[L]$ lies in the image of the natural map $Q^+H(L;\mathcal{W}_1)\longrightarrow QH(L;\mathcal{W}_1)$ hence it is enough to show that (62) holds in $Q^+H(L;\mathcal{W}_1)$.

Recall from (21) that we have the following exact sequence:

$$0\longrightarrow\mathcal{O}(\mathcal{W}_1)[L]t\xrightarrow{i}Q^+H_{n-2}(L;\mathcal{W}_1)\xrightarrow{\pi}H_{n-2}(L;\mathbb{C})\otimes\mathcal{O}(\mathcal{W}_1)\longrightarrow 0. \tag{63}$$

Moreover, by the definition of the quantum module action, $\pi([\Sigma_i]*[L])=[\Sigma_i]\cdot[L]$, where $[\Sigma_i]\cdot[L]$ stand for the classical intersection product in singular homology. But L is disjoint from Σ_i, hence $\pi([\Sigma_i]*[L])=[\Sigma_i]\cdot[L]=0$. It follows from (63) that $[\Sigma_i]*[L]=c[L]t$ for some function $c\in\mathcal{O}(\mathcal{W}_1)$.

To determine c note that if we work with coefficients in $\widetilde{\Lambda}^+$ we have

(64) $$[\Sigma_i] * [L] = \sum_{B \in \mathcal{E}_2} \#(B \cdot \Sigma_i)\nu(B)T^B[L].$$

Substituting the information from Proposition 7.1.1 into (64) we immediately obtain

$$[\Sigma_i] * [L] = z^{\overrightarrow{v_i}}[L]t.$$

Since $z^{\overrightarrow{v_i}}t = I([\Sigma_i])$ this concludes the proof. □

Next we consider the quantum inclusion map. Let $f\colon L \longrightarrow \mathbb{R}$ be a perfect Morse function having exactly one minimum, $x_0 \in L$. Let $(\cdot,\cdot)$ be a Riemannian metric on L and J an ω–compatible almost complex structure on M. Put $\mathscr{D} = (f,(\cdot,\cdot),J)$ and assume the elements of this triple have been chosen to be generic so that the pearl complex $\mathcal{C}(\mathscr{D};\mathcal{O}(\mathcal{W}_1)\otimes\Lambda)$ is well defined. Under these assumptions $x_0 \in \mathcal{C}(\mathscr{D};\mathcal{O}(\mathcal{W}_1)\otimes\Lambda)$ is a cycle and we denote by $[x_0] \in QH(L;\mathcal{W}_1)$ its homology class. Note that in general $[x_0]$ strongly depends on the choice of $\mathscr{D}$. (See [14, Section 4.5].) Nevertheless, it turns out that its image under the quantum inclusion (6) is well defined.

Proposition 7.3.3 *Let $a_1,\dots,a_m \in H_*(M;\mathbb{C})$ be elements of pure degree which consist of a basis for the total homology $H_*(M;\mathbb{C})$. Denote by $a_1^\#,\dots,a_m^\#$ the dual basis with respect to intersection product. Then*

(65) $$i_L([x_0]) = \sum_{i=1}^{m} I(a_i^\#)a_i \in QH(M;\mathcal{O}(\mathcal{W}_1)\otimes\Lambda).$$

Note that we can always take $a_1 = [\mathrm{pt}] \in H_0(M;\mathbb{C})$ to be the class of a point and $a_m = [M] \in H_{2n}(M;\mathbb{C})$ to be the fundamental class. We will then have $a_1^\# = [M]$ and $a_m^\# = [\mathrm{pt}]$ and formula (65) becomes

(66) $$i_L([x_0]) = [\mathrm{pt}] + \sum_{i=2}^{m-1} I(a_i^\#)a_i + I([\mathrm{pt}])[M].$$

To prove Proposition 7.3.3 we will use the augmentation map $\epsilon_L\colon QH(L;\mathcal{R}) \longrightarrow \mathcal{R}$, defined for every $\widetilde{\Lambda}^+$–algebra $\mathcal{R}$. The precise definition and properties of this map can be found in [14] (see eg Theorem A in that paper). The augmentation map ϵ_L is induced by a map $\widetilde{\epsilon_L}\colon \mathcal{C}(f,\rho,J;\mathcal{R}) \longrightarrow \mathcal{R}$ which is defined as follows. Assume that f has a unique minimum x_0, then $\widetilde{\epsilon_L}(x_0) = 1$ and for every $x \in \mathrm{Crit}(f)$, $x \neq x_0$, $\widetilde{\epsilon_L}(x) = 0$. It satisfies the identity

(67) $$\langle \mathrm{PD}(b), i_L(\beta)\rangle = \epsilon_L(b * \beta) \quad \forall\, b \in H_*(M;\mathbb{C}) \subset QH(M;\mathcal{R}),\ \beta \in QH(L;\mathcal{R}),$$

where PD stand for Poincaré duality and $\langle\cdot,\cdot\rangle$ for the obvious $\mathcal{R}$–linear extension of the Kronecker pairing.

We are now ready to prove Proposition 7.3.3.

Proof of Proposition 7.3.3 Write $i_L([x_0]) = \sum_{i=1}^m \varphi_i a_i$, with $\varphi_i \in \mathcal{O}(\mathcal{W}_1) \otimes \Lambda$. Apply now formula (67) with $b = a_j^{\#}$, $\beta = [x_0]$. We obtain

$$\varphi_j = \langle \mathrm{PD}(a_j^{\#}), i_L([x_0]) \rangle = \epsilon_L(a_j^{\#} * [x_0]) = \epsilon_L(I(a_j^{\#})[x_0]) = I(a_j^{\#}),$$

where the second to last equality follows from Proposition 7.3.2. □

Remark 7.3.4 In a similar way one can prove that

$$I(a) = \langle \mathrm{PD}(a), i_L([x_0]) \rangle \quad \forall\, a \in H_*(M; \mathbb{C}). \tag{68}$$

7.4 The Frobenius structure and the quantum Euler class

The quantum homology $QH(M;\Lambda)$ has the structure of a Frobenius algebra. In this section we explain how to translate this structure via the isomorphism I to the Jacobian ring $\mathcal{O}(\mathcal{W}_1) \otimes \Lambda$. We remark that this translation has been recently established by Fukaya, Oh, Ohta and Ono [24]. Below we explain our point of view on the subject and how it is related to our theory.

7.4.1 Generalities on Frobenius algebras We first recall some basic notions about Frobenius algebras. The reader is referred to Abrams [2] and the references therein for the general theory of Frobenius algebras.

Let $\mathcal{A}$ be an algebra over a ring $\mathcal{R}$ and assume that $\mathcal{A}$ is a free finite-rank module over $\mathcal{R}$. A Frobenius structure on $\mathcal{A}$ is an $\mathcal{R}$–linear map $F\colon \mathcal{A} \longrightarrow \mathcal{R}$ such that the associated bilinear pairing

$$\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \longrightarrow \mathcal{R}, \quad a \otimes b \longmapsto F(ab),$$

is nonsingular in the sense that the induced map $\mathcal{A} \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{R})$, $a \longmapsto F(a \cdot -)$, is invertible (or put in different terms, the associated matrix of the pairing is invertible in some basis of $\mathcal{A}$ over $\mathcal{R}$). Of course, the associated bilinear pairing of a Frobenius structures can be viewed as a generalization of the notion of Poincaré duality. Note that when the ring $\mathcal{R}$ is not a field some authors (eg Abrams [2]) use the notion of Frobenius extension rather than Frobenius structure.

To a Frobenius structure one can associate an invariant called the Euler class, introduced by Abrams [2]. This is defined as follows. Pick a basis $a_1, \dots, a_m$ of $\mathcal{A}$ over $\mathcal{R}$.

Let $a_1^\vee, \dots, a_m^\vee$ be the dual basis with respect to the Frobenius pairing. The Euler class $\mathscr{E}(\mathcal{A}, F)$ is defined as

$$\mathscr{E}(\mathcal{A}, F) = \sum_{i=1}^{m} a_i a_i^\vee. \tag{69}$$

It is straightforward to check that $\mathscr{E}(\mathcal{A}, F)$ does not depend on the choice of the basis. The importance of the Euler class comes form the following theorem.

Theorem 7.4.1 (Abrams [2; 1]) *Let $\mathcal{A}$ be a finite dimensional algebra over a field $\mathcal{R}$ of characteristic 0. Then:*

(1) *For every two Frobenius structures F' and F'' on $\mathcal{A}$ there exists an invertible element $u \in \mathcal{A}$ such that $F'' = uF'$. Moreover we have $\mathscr{E}(\mathcal{A}, F'') = u^{-1}\mathscr{E}(\mathcal{A}, F'')$. Thus the Euler class does not depend on the Frobenius structure up to multiplication by an invertible element. In particular, whether or not the Euler class is a zero divisor, or whether or not it is invertible, does not depend on the particular choice of the Frobenius structure.*

(2) *Suppose that the Euler class of some (hence for every) Frobenius structure on $\mathcal{A}$ is not a zero divisor. Then the Euler classes determine the Frobenius structures on $\mathcal{A}$ in the sense that there exists a unique Frobenius structure F on $\mathcal{A}$ with a given Euler class.*

(3) *The algebra $\mathcal{A}$ is semisimple if and only if the Euler class $\mathscr{E}(\mathcal{A}, F)$ is invertible for some Frobenius structures F on $\mathcal{A}$.*

We also have the following result that will be relevant for our purposes.

Theorem 7.4.2 (Scheja–Storch [42]) *Let $(\mathcal{A}, F)$ be a Frobenius algebra over a field $\mathcal{R}$ of characteristic 0. Suppose that $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{R}[x_1, \dots, x_r]/\mathcal{I}$ for some ideal $\mathcal{I}$ which is generated by r elements $f_1, \dots, f_r \in \mathcal{R}[x_1, \dots, x_r]$. Put*

$$J = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \in \mathcal{A}.$$

If $J \neq 0$ then $J = u\mathscr{E}(\mathcal{A}, F)$ for some invertible element $u \in \mathcal{A}$.

7.4.2 The main examples Here are two examples that are relevant in our context. The first one is classical. Let M be a closed manifold and $\mathcal{R}$ any ring. Assume for simplicity that $H_i(M;\mathcal{R}) = 0$ for every odd i. Let $\mathcal{A} = H_*(M;\mathcal{R})$ endowed with the intersection product $\cdot$. Write

$$H_*(M;\mathcal{R}) = \mathcal{R}[\mathrm{pt}] \bigoplus \bigoplus_{j=1}^{\dim M/2} H_{2j}(M;\mathcal{R}),$$

where [pt] is the class of a point. The Frobenius structure F is defined by the projection onto the $\mathcal{R}[\mathrm{pt}]$ factor. In other words, $F(a)$ is defined to be the coefficient of a at [pt]. The associated bilinear pairing is precisely the intersection pairing. A simple computation shows that the Euler class $\mathscr{E} = \mathscr{E}(\mathcal{A}, F)$ in this case is exactly $\chi(M)[\mathrm{pt}]$.

The second example, which is the one we will focus on, is the quantum cohomology of a symplectic manifold M. We assume that (M,ω) is a closed monotone symplectic manifold and that $H_i(M;\mathbb{C}) = 0$ for every odd i. Put $\mathcal{A} = QH(M;\Lambda)$ endowed with the quantum product $*$ and let $\mathcal{R} = \Lambda$. The Frobenius structure is taken as in the preceding example, ie for $a \in QH(M;\Lambda)$ we set $F(a) \in \Lambda$ to be the coefficient of a at [pt]. We denote it from now on by F_Q to emphasize the relation to quantum homology. The fact that this is indeed a Frobenius structure is not immediate. It is proved eg in [2].

We now turn to the Euler class of the Frobenius structure F_Q on the quantum homology. We denote it for simplicity by $\mathscr{E}_Q$ and call it the *quantum Euler class*. Under the assumptions that (M,ω) is monotone and $H_{\mathrm{odd}}(M;\mathbb{C}) = 0$ we have the following:

Lemma 7.4.3 *Let* $\mathbf{a} = \{a_1, \ldots, a_m\}$ *be a basis for* $H_*(M;\mathbb{C})$ *consisting of elements of pure degree and* $\mathbf{a}^\# = \{a_1^\#, \ldots, a_m^\#\} \in H_*(M;\mathbb{C})$ *be the dual basis with respect to the* classical *intersection product. Then* $\mathbf{a}^\#$ *is also a dual basis with respect to the quantum product* $*$. *In particular we have*

$$\mathscr{E}_Q = \sum_{i=1}^{m} a_i * a_i^\#. \tag{70}$$

This class belongs to $QH_0(M;\Lambda)$ *and is a deformation of the classical Euler class, ie* $\mathscr{E}_Q = \chi(M)[\mathrm{pt}] + \mathrm{ho}(t)$, *where* $\mathrm{ho}(t)$ *stands for higher order terms in* t.

The proof can be found in [12, Proposition 6.5.7 and the proof of Proposition 6.5.8].

Note that Λ is not a field hence Theorems 7.4.1 and 7.4.2 do not apply for $QH(M;\Lambda)$. To go around this difficulty we can work with the completion $\widehat{\Lambda} = \mathbb{C}[t^{-1}, t]]$ consisting of formal Laurent series in t with finitely many terms having negative powers of t. Note that $\widehat{\Lambda}$ is a field. We can define in a straightforward way $QH(M;\widehat{\Lambda})$, endowed with the quantum product and we have an inclusion of rings $QH(M;\Lambda) \subset QH(M;\widehat{\Lambda})$. Obviously the preceding Frobenius structure F_Q extends to $QH(M;\widehat{\Lambda})$ and the quantum Euler class remains exactly the same.

7.4.3 Back to toric manifolds We now return to the case of toric manifolds.

Suppose that the superpotential $\mathscr{P}\colon (\mathbb{C}^*)^{\times n} \longrightarrow \mathbb{C}$ is a Morse function (in the holomorphic sense), ie it has only isolated critical points and at each such point the holomorphic Hessian is nondegenerate. In this case $\mathcal{W}_1$ is a scheme consisting of a finite number of points each coming with multiplicity 1. Therefore $\mathcal{O}(\mathcal{W}_1) = \bigoplus_{z \in \mathcal{W}_1} \mathbb{C}$, hence by the isomorphism (60) from Theorem 7.3.1 the quantum cohomology $QH(M;\Lambda)$ splits as

$$QH(M;\Lambda) \cong \bigoplus_{z \in \mathcal{W}_1} \Lambda,$$

and similarly for $QH(M;\widehat{\Lambda})$. It follows that $QH(M;\widehat{\Lambda})$ is semisimple. It turns out that the converse direction is also true, hence $QH(M;\widehat{\Lambda})$ is semisimple if and only if $\mathscr{P}$ is Morse (see [40] for the proof and for more on semisimplicity of QH for toric manifolds).

We now address the question of how does the isomorphism I from Theorem 7.3.1 translate the quantum Frobenius structure from $QH(M;\widehat{\Lambda})$ to $\mathcal{O}(\mathcal{W}_1) \otimes \widehat{\Lambda}$. Theorem 7.4.2 provides a partial answer. Write

$$\mathcal{O}(\mathcal{W}_1) \otimes \widehat{\Lambda} = \widehat{\Lambda}[z_1, u_1, \dots, z_n, u_n]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by

$$\partial_{z_1}\mathscr{P}, \dots, \partial_{z_n}\mathscr{P}, z_1u_1 - 1, \dots, z_nu_n - 1.$$

Applying Theorem 7.4.2 we obtain that there exists an invertible element $u \in QH(M;\widehat{\Lambda})$ such that

$$I(\mathscr{E}_Q) = uz_1 \cdots z_n \det\left(\frac{\partial^2 \mathscr{P}}{\partial z_i \partial z_j}\right)_{i,j}.$$

Since $z_1 \cdots z_n$ is invertible we obtain from (24) that there exists an invertible element $v \in QH(M;\widehat{\Lambda})$ such that

$$I(\mathscr{E}_Q) = v\Delta,$$

where Δ is the discriminant introduced in Section 4.3. Since $I(\mathscr{E}_Q)$ has degree $-2n$ so must have v. Since v has pure degree it follows that both v as well as its inverse v^{-1} in fact lie in $QH(M;\Lambda)$ (ie we do not need the larger field of coefficients $\widehat{\Lambda}$). These considerations are still far from determining the precise value of v. The following theorem provides this additional information.

Theorem 7.4.4 *Suppose that $\mathcal{P}$ is Morse. Then:*

(1) *$I(\mathcal{E}_Q) = (-1)^{n+1}\Delta t^n$, where $\Delta \in \mathcal{O}(\mathcal{W}_1)$ is the discriminant introduced in (24) of Section 4.3.*

(2) *Via the isomorphism I, the quantum Frobenius structure on $\mathcal{O}(\mathcal{W}_1) \otimes \Lambda$ has the form*

$$F_Q(I^{-1}(\sigma)) = \frac{(-1)^{n+1}}{t^n} \sum_{z \in \mathcal{W}_1} \frac{\sigma(z)}{\Delta(z)} \quad \forall\, \sigma \in \mathcal{O}(\mathcal{W}_1) \otimes \Lambda. \tag{71}$$

Theorem 7.4.4 (stated in a slightly different form) has been recently proved by Fukaya, Oh, Ohta and Ono [24] by methods of Floer theory. It seems to be known for a long time to specialists in quantum homology theory. In fact, Givental has pointed out to us [32] that this theorem follows from his work [31, Proposition 1.1]. It is not difficult to verify Theorem 7.4.4 by direct computation on all toric monotone 4–manifolds (see [11] for all these calculations; a few examples are in Section 8). We sketch in Section 7.4.4 a more conceptual proof of this Theorem.

Denote by $F_1, \dots, F_r$ the codimension–1 facets of the moment polytope $P = \mathrm{image}(\mathfrak{m})$ and by $\vec{v}_1, \dots, \vec{v}_r$ the inwards pointing normal integral primitive vectors to these facets as in Section 7.1. For a subset of indices $I \subset \{1, \dots, r\}$ write $\vec{v_I} = \sum_{i \in I} \vec{v_i}$. The following identities, which seem to bear some arithmetic nature, follow immediately from Theorem 7.4.4.

Corollary 7.4.5 *Assume that $\mathcal{P}$ is Morse. Let $I \subset \{1, \dots, r\}$ be a subset of indices. If $\#I < n$ then*

$$\sum_{z \in \mathcal{W}_1} \frac{z^{\vec{v_I}}}{\Delta(z)} = 0. \tag{72}$$

If $\#I = n$ then

$$\sum_{z \in \mathcal{W}_1} \frac{z^{\vec{v_I}}}{\Delta(z)} = \begin{cases} 0 & \text{if } \bigcap_{i \in I} F_i = \varnothing, \\ (-1)^{n+1} & \text{if } \bigcap_{i \in I} F_i \neq \varnothing. \end{cases} \tag{73}$$

Proof Write $\Sigma_i = \mathfrak{m}^{-1}(F_i)$. The Σ_i's are codimension–2 symplectic submanifolds of M. Recall that by the isomorphism I of Theorem 7.3.1 we have $I([\Sigma_i]) = z^{\vec{v_i}} t$, hence

$$I(*_{i \in I}[\Sigma_i]) = z^{\vec{v_I}} t^{\#I}.$$

By formula (71), the value of the sum

$$\sum_{z \in \mathcal{W}_1} \frac{z^{\vec{v_I}}}{\Delta(z)} \tag{74}$$

is determined by the value of $F_Q(*_{i\in I}[\Sigma_i])$, ie by whether or not $*_{i\in I}[\Sigma_i]$ contains [pt]. But by degree reasons the coefficient of [pt] in $*_{i\in I}[\Sigma_i]$ is the same as the coefficient of [pt] in $\cdot_{i\in I}[\Sigma_i]$ where $\cdot$ is the classical intersection product. The rest of the proof now follows from basic intersection properties of the Σ_i's. □

Finally, putting together Theorem 7.4.4 with formulae (56), (58) we obtain the following:

Corollary 7.4.6 *Suppose that $\mathcal{P}$ is Morse. Then the quantum Euler class admits the following expressions:*

$$\mathcal{E}_Q = \sum_{\substack{I\subset\{1,\dots,r\}\\ \#I=n}} (*_{i\in I}[\Sigma_i])\det(A_I)^2, \tag{75}$$

where A_I is defined in (57) and $*$ stands for the quantum product.

$$\mathcal{E}_Q = \det\Big(\sum_{k=1}^{r} v_k^i v_k^j[\Sigma_k]\Big)_{i,j}, \tag{76}$$

where the determinant here should be evaluated in the quantum homology ring.

7.4.4 Further remarks on Theorem 7.4.4 and its proof Note that by Theorem 7.4.1 the Frobenius structure F_Q is determined by its associated Euler class $\mathcal{E}_Q$. Therefore point (2) of Theorem 7.4.4 follows immediately from point (1). The next Proposition shows that $\mathcal{E}_Q$ is indeed very much related to the "Lagrangian picture".

Consider the morphism

$$j_L\colon QH_*(M;\mathcal{O}(\mathcal{W}_1)\otimes\Lambda)\longrightarrow QH_{*-n}(L;\mathcal{W}_1),\quad a\longmapsto a*[L].$$

Consider $[x_0]\in QH_*(L;\mathcal{W}_1)$ as in the discussion before Proposition 7.3.3. We have:

Proposition 7.4.7 $j_L\circ i_L([x_0]) = I(\mathcal{E}_Q)[L]$.

Proof This follows at once from Propositions 7.3.3 and 7.3.2. □

Thus, the proof of Theorem 7.4.4 reduces to showing that

$$j_L\circ i_L([x_0]) = (-1)^{n+1}\Delta t^n[L]. \tag{77}$$

We sketch here our argument for this identity in dimension $2n=4$. Recall from [13, Section 8.3] that given two Lagrangians L and L' there is a particular formula allowing to express $j_{L'}\circ i_L$. In our case, we ultimately want to study $L=L'$ so it is sufficient to assume that L is Hamiltonian isotopic to L' (and L is transverse to L') so that the formula has the form

$$j_{L'} \circ i_L - \chi_{L,L'} = \Phi_{L,L'} \circ d + d' \circ \Phi_{L,L'}. \tag{78}$$

We now explain the formula (78). We will then notice that from this formula we can easily deduce a closely related one that directly computes $j_L \circ i_L$ in terms of some pearly like configurations. Identity (77) follows from further identities involving these configurations.

The notation in (78) is as follows: $(\mathcal{C}(L; f), d)$ and $(\mathcal{C}(L'; f'), d')$ are pearl complexes for L and L' (we assume appropriate Riemannian metrics fixed on L and L'), $\Phi_{L,L'}$ is a certain chain homotopy and $\chi_{L,L'}$ is a chain map that we will describe in more detail below. In our case we may assume that f and f' are perfect Morse functions so that $d = 0 = d'$ because L and L' are wide tori. Thus we deduce $j_{L'} \circ i_L([x_0]) = \chi_{L,L'}([x_0])$. The map $\chi_{L,L'}$ is described in [13, Section 8.3]. Explicitly, it is defined as follows. For $x \in \operatorname{Crit} f$,

$$\chi_{L,L'}(x) = \sum_{p,y} \#(\mathcal{N}(p, p; x, y)) y t^{k_y},$$

where $y \in \operatorname{Crit}(f')$, $p \in L \cap L'$, $|y| - 2k_y = |x| - 4$ and the moduli spaces $\mathcal{N}(p, p; x, y)$ are formed by configurations (u, v, v') where: u is a Floer strip joining the intersection point p to itself and with $u(\mathbb{R} \times \{0\}) \subset L$, $u(\mathbb{R} \times \{1\}) \subset L'$; v is a chain of pearls on L joining x to the point $u(0, 0)$; v' is a chain of pearls on L' joining x to the point $u(0, 1)$. Because $\chi_{L,L'}$ is a chain map it is easily seen that we may apply the PSS construction to return from L' back to L. This gives rise to another map $\overline{\chi}_{L,L}$ with two properties:

- It verifies a formula similar to (78) except that only involving L:

$$j_L \circ i_L = \overline{\chi}_{L,L}. \tag{79}$$

- The definition of $\overline{\chi}_{L,L}$ is similar to that of $\chi_{L,L'}$ with the following modifications: v is a string of pearls associated to the function $f \colon L \to \mathbb{R}$, v' is a string of pearls associated to the function $f' \colon L \to \mathbb{R}$, u is now also a string of pearls associated to a third function f'' and joining a critical point $p \in \operatorname{Crit}(f'')$ *to the same* p. The incidence conditions among these three strings of pearls are that there is a disk in u (possibly trivial) so that v ends at $u(-i)$ and v' ends at $u(i)$.

In our case, we are interested in the case when $x = x_0 = \min(f)$. For degree reasons we see that the only term that matters corresponds to $y = z_2 = \max(f')$. Moreover, there is a single disk involved which is of Maslov class 4. In short, $j_L \circ i_L([x_0])$ is estimated by the number of elements in the moduli space $W(f'', x_0, z_2, J)$ of configurations formed by a single J–holomorphic disk u of Maslov class 4 and so that $u(-i) = x_0$, $u(+i) = z_2$ and there is a critical point $p \in \mathrm{Crit}(f'')$ with the property that a negative gradient trajectory of f'' exiting p reaches $u(-1)$ and there is a negative gradient trajectory of f'' that carries $u(+1)$ to p again (any one of these trajectories can also be degenerate). The next step is to include $W(f'', x_0, z_2, J)$ as boundary in a 1–dimensional moduli space whose other end had $-\Delta$ elements. The first step is rather easy – the 1–dimensional moduli space in question, $W'(f'', x_0, z_2, J)$, corresponds to gluing at the point p – so that the configurations contained in this moduli space are like the ones in $W(f'', x_0, z_2, J)$ except that the two flow lines there are replaced by a single one that joins $u(+1)$ to $u(-1)$ without breaking at p. Finally, it is essentially a delicate combinatorial verification – that we will not include here – to see that the number of the other boundary components of $W'(f'', x_0, z_2, J)$ gives precisely $-\Delta$.

8 Examples

Here we work out examples of the various objects and invariants constructed in the previous sections, mainly in the case of toric manifolds. We use here the notation introduced in Section 7 and in particular for $\mathcal{W}_1$ we use the coordinates $(z_1, \dots, z_r)$ introduced in Section 3.3.1 and for $\mathcal{W}_2$ we use the coordinates $(\xi_1, \dots, \xi_r)$ introduced in Section 7.2.

8.1 The complex projective space

Consider $\mathbb{C}P^n$ endowed with its standard Fubini–Study Kähler structure ω_{FS} normalized so that $\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = 1$. Consider the Hamiltonian torus action $(\theta_1, \dots, \theta_n) \cdot [z_0 : \cdots : z_n] = [z_0 : e^{-2\pi i \theta_1} : \cdots : e^{-2\pi i \theta_n} z_n]$. The moment polytope is the standard simplex

$$P = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; 0 \leq x_k \ \forall k, \sum_{i=1}^{n} x_i \leq 1 \right\}.$$

It has $n+1$ codimension–1 facets with normal vectors $\vec{v_i} = (0, \dots, 1, \dots, 0)$ (where the 1 is in the i–th coordinate), $i = 1, \dots, n$ and $\overrightarrow{v_{n+1}} = (-1, \dots, -1)$. (See eg [3; 38]). The monotone torus

$$L = \mathfrak{m}^{-1}\left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right) = \{[z_0 : \ldots : z_n] \mid |z_0| = \ldots = |z_n|\}$$

is the Clifford torus. The wide variety $\mathcal{W}_2$ is given in this case by

$$\mathcal{W}_2 = \{(\xi, \dots, \xi) \mid \xi \in \mathbb{C}^*\} \cong \mathbb{C}^*.$$

The superpotential is

$$\mathcal{P}(z_1, \dots, z_n) = \sum_{i=1}^{n} z_i + \frac{1}{z_1 \cdots z_n}.$$

A simple computation shows that $\mathcal{P}$ is Morse. The wide variety $\mathcal{W}_1$ consists of the following $n+1$ points:

$$\mathcal{W}_1 = \{(z, \dots, z) \mid z^{n+1} = 1\},$$

and each of them comes with multiplicity 1. The quadratic form (see (23)) $\varphi_{\mathcal{W}}$ is given in the basis $\{C_1, \dots, C_n\}$ by

$$\varphi_{\mathcal{W}}(X_1, \dots, X_n) = \xi\Big(\sum_{i=1}^{n} X_i^2 + \sum_{i<j} X_i X_j\Big), \ \forall \xi \in \mathcal{W}_2.$$

A simple computation shows that the discriminant of the quadratic form (on $\mathcal{W}_2$ and $\mathcal{W}_1$ respectively) is

$$\Delta(\xi) = (-1)^{n+1}(n+1)\xi^n \quad \forall \xi \in \mathcal{W}_2, \quad \Delta(z) = (-1)^{n+1}(n+1)z^n \quad \forall z \in \mathcal{W}_1.$$

Denote by $H = [\mathbb{C}P^{n-1}] \in QH_{2n-2}(\mathbb{C}P^n; \Lambda)$ the class of a linear hyperplane and by $[\mathbb{C}P^l] \in QH_{2l}(M; \Lambda)$ the class of a linear projective l–dimensional plane. The quantum homology of $\mathbb{C}P^n$ is given by

$$H^{*k} = \begin{cases} [\mathbb{C}P^{n-k}] & \text{if } 0 \le k \le n, \\ [\mathbb{C}P^n]\, t^{n+1} & \text{if } k = n+1. \end{cases}$$

A simple computation shows that the quantum Euler class equals the topological one:

$$\mathcal{E}_Q = \mathcal{E}_{\text{top}} = (n+1)[\text{pt}].$$

The ring $\mathcal{O}(\mathcal{W}_1)$ is

$$\mathcal{O}(\mathcal{W}_1) \cong \mathbb{C}[z^{\pm 1}]/\langle z^{n+1} = 1\rangle,$$

and the isomorphism I from (60) satisfies $I([\mathbb{C}P^l]) = z^{n-l} t^{n-l}$. One can easily verify that $I(\mathcal{E}_Q) = -t^n \Delta(z)$.

The identities of Corollary 7.4.5 now read

$$\frac{1}{n+1} \sum_{\{z \mid z^{n+1}=1\}} z^k = \begin{cases} 0 & \text{if } 1 \le k \le n, \\ 1 & \text{if } k = n+1. \end{cases}$$

Finally, the quantum inclusion of $[x_0]$ is given by

$$i_L([x_0]) = [\mathrm{pt}] + \sum_{k=1}^{n} z^k [\mathbb{C}P^k] t^k \quad \forall z \in \mathcal{W}_1.$$

Next we will exemplify our theory on $S^2 \times S^2$ and on the blow up of $\mathbb{C}P^2$ at one point. The other (nonlinear) toric surfaces (ie the blow up of $\mathbb{C}P^2$ at two and at three points) are treated in detail in the expanded version of this paper [11].

8.2 $S^2 \times S^2$

Consider $M = S^2 \times S^2$ with the balanced symplectic form $\omega = \omega_{S^2} \oplus \omega_{S^2}$ and with the obvious Hamiltonian torus action coming from circle actions on both factors. The moment polytope is

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, \ \ 0 \leq x_2 \leq 1\}.$$

The monotone torus is $L = \mathfrak{m}^{-1}(\frac{1}{2}, \frac{1}{2})$ which is the product of two equators coming from each S^2–factor. The integral normal vectors to the four facets are $\overrightarrow{v_1} = (1,0)$, $\overrightarrow{v_2} = (0,1)$, $\overrightarrow{v_3} = (-1,0)$, $\overrightarrow{v_4} = (0,-1)$. The wide variety $\mathcal{W}_2$ is given by

$$\mathcal{W}_2 = \{(\xi_1, \xi_2, \xi_1, \xi_2) \mid \xi_1, \xi_2 \in \mathbb{C}^*\} \cong \mathbb{C}^* \times \mathbb{C}^*.$$

The superpotential is

$$\mathcal{P}(z_1, z_2) = z_1 + z_2 + \frac{1}{z_1} + \frac{1}{z_2}.$$

This function is Morse and its critical points are

$$\mathcal{W}_1 = \{(1,1), (1,-1), (-1,1), (-1,-1)\}.$$

The quadratic form is

$$\varphi_{\mathcal{W}}(X_1, X_2) = \xi_1 X_1^2 + \xi_2 X_2^2, \ \forall (\xi_1, \xi_2) \in \mathbb{C}^* \times \mathbb{C}^*.$$

The discriminant on $\mathcal{W}_2$ and $\mathcal{W}_1$, respectively, is

$$\Delta(\xi_1, \xi_2) = -4\xi_1\xi_2, \quad \Delta(z_1, z_2) = -4z_1z_2.$$

To describe the quantum homology $QH(M;\Lambda)$ of M, put $A = [S^2 \times \mathrm{pt}]$, $B = [\mathrm{pt} \times S^2] \in QH_2(M;\Lambda)$. Then we have

$$A * B = \mathrm{pt}, \quad A * A = B * B = [M] t^2.$$

The isomorphism I satisfies

$$I(A) = z_2 t, \quad I(B) = z_1 t, \quad I([\mathrm{pt}]) = z_1 z_2 t^2.$$

The quantum Euler class equals in this case to the topological one: $\mathcal{E}_Q = 4[\mathrm{pt}]$. The quantum inclusion satisfies

$$i_L([x_0]) = [\mathrm{pt}] + z_1 At + z_2 Bt + z_1 z_2 [M] t^2.$$

The arithmetic identities of Corollary 7.4.5 can be verified by a straightforward direct substitution.

8.3 Blow ups of $\mathbb{C}P^2$

Consider the standard Hamiltonian torus action on $\mathbb{C}P^2$ and let p be a fixed point of the action. This action has exactly three fixed points p_1, p_2, p_3. By blowing up $p_1, \ldots, p_k$, $1 \leq k \leq 3$, we obtain a manifold M_k which can be endowed with a monotone symplectic form ω in such a way that the torus action on $\mathbb{C}P^2$ lifts to a Hamiltonian torus action on M_k (see [3; 38] for details). Denote by $E_i \in H_2(M_k;\mathbb{Z})$ the exceptional divisor over p_i and by $L \in H_2(M_k;\mathbb{Z})$ the homology class of a projective line not passing through the exceptional divisors. We denote by $[M_k] \in H_4(M_k;\mathbb{Z})$ the fundamental class. The Poincaré dual of the cohomology class of ω satisfies $\mathrm{PD}[\omega] = L - \frac{1}{3}\sum_{i=1}^{k} E_i$. We will now work out in detail the case $k=1$. The cases $k = 2, 3$ are treated in detail in the expanded version of this paper [11].

8.4 The blow-up of $\mathbb{C}P^2$ at one point

Denote by $M_1 = \mathrm{Bl}_{p_1}(\mathbb{C}P^2)$ the blow-up of $\mathbb{C}P^2$ at p_1. The moment polytope and the normal vectors to the facets are depicted in Figure 5. Note that

$$[\mathfrak{m}^{-1}(F_1)] = E, \quad [\mathfrak{m}^{-1}(F_2)] = L - E, \quad [\mathfrak{m}^{-1}(F_3)] = L, \quad [\mathfrak{m}^{-1}(F_4)] = L - E.$$

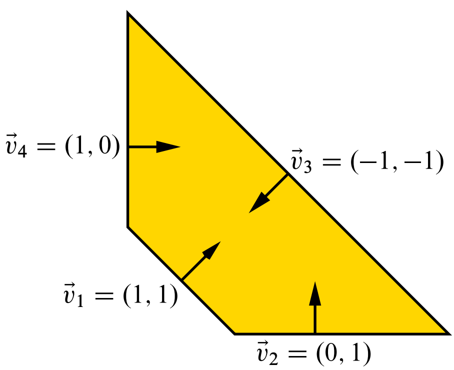

Figure 5. The moment polytope of the blow-up of $\mathbb{C}P^2$ at one point

The wide variety $\mathcal{W}_2$ is

$$\mathcal{W}_2 = \{(\xi_1, \xi_2, \xi_1 + \xi_2, \xi_2) \mid \xi_1, \xi_2 \in \mathbb{C}^*, \xi_1 \neq -\xi_2\}.$$

Note that the trivial representation $(1,1,1,1)$ does not belong to $\mathcal{W}_2$, so L is narrow with respect to this representation. The superpotential is

$$\mathcal{P}(z_1, z_2) = z_1 + z_2 + z_1 z_2 + \frac{1}{z_1 z_2}.$$

The wide variety $\mathcal{W}_1$ consists of 4 points, all with multiplicity 1, and is given by

$$\mathcal{W}_1 = \{(z, z) \mid z^4 + z^3 - 1 = 0\}.$$

The ring of functions over $\mathcal{W}_1$ is therefore

$$\mathcal{O}(\mathcal{W}_1) \cong \mathbb{C}[z, z^{-1}]/\langle z^4 + z^3 - 1 = 0\rangle.$$

The quadratic form is

$$\varphi_{\mathcal{W}}(X_1, X_2) = (\xi_1 + \xi_2) X_1^2 + (2\xi_1 + \xi_2) X_1 X_2 + (\xi_1 + \xi_2) X_2^2, \ \forall (\xi_1, \xi_2) \in \mathbb{C}^* \times \mathbb{C}^*.$$

The discriminant on $\mathcal{W}_2$ and $\mathcal{W}_1$ respectively is

$$\Delta(\xi_1, \xi_2) = -(4\xi_1\xi_2 + 3\xi_2^2), \quad \Delta(z) = -z^2(4z + 3).$$

The quantum product is given by (see Crauder and Miranda [20])

$$E * E = -[\mathrm{pt}] + Et + [M_1] t^2, \quad E * L = [M_1] t^2, \quad L * L = [\mathrm{pt}] + [M_1] t^2.$$

The quantum Euler class is

$$\mathcal{E}_Q = 4[\mathrm{pt}] - Et.$$

The isomorphism I is given by

$$I(L) = \frac{1}{z^2} t, \quad I(E) = z^2 t, \quad I([\mathrm{pt}]) = \left(\frac{1}{z^4} - 1\right) t^2.$$

The fact that $I(\mathcal{E}_Q) = -\Delta t^2$ on $\mathcal{W}_1$ can be verified here by a direct (though long) computation.

The quantum inclusion satisfies

$$i_L([x_0]) = [\mathrm{pt}] + \frac{1}{z^2} Lt - z^2 Et + \left(\frac{1}{z^4} - 1\right)[M_1] t^2.$$

We now turn to the arithmetic identities of Corollary 7.4.5. In the following identity $a(z)$ stands for the function z^k, where $-2 \leq k \leq 3$. We have

$$\sum_{\{z : z^4+z^3-1=0\}} \frac{a(z)}{4z^3+3z^2} = \begin{cases} 0 & \text{if } a(z) \text{ is one of } 1, z, z^2, 1/z^2, \\ 1 & \text{if } a(z) \text{ is one of } z^3, 1/z. \end{cases} \tag{80}$$

These identities seem nontrivial to obtain by a direct computation, though they can be verified using a numerical mathematical program such as Matlab, Mathematica or Octave. An alternative elementary (albeit nondirect) verification of these identities via computations of residues of rational functions, has been recently pointed out to us by Granville [33].

Appendix A Orientations

A.1 Orientations – general conventions

In order to define the pearl complex over a general ground ring we now describe how to orient the moduli space of pearl trajectories.

Remark A.1.1 A couple of statements in our earlier paper [14], Corollary 7.02 and Remark 6.3.3, were stated and proved there over $\mathbb{Z}$ under the explicit assumption – not verified in [14] – that the pearl machinery is compatible with orientations. This is precisely the compatibility verified in this Appendix.

Below we denote orientations on vector spaces or manifolds V by o_V. We often denote dimensions of manifolds V by $|V|$.

A.1.1 Exact sequences Let

$$0 \longrightarrow F \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$$

be a short exact sequence of finite dimensional vector spaces. Orientations on any two of these spaces induces an orientation on the third as follows. Pick a right inverse $s \colon B \longrightarrow E$ of p, so that $E = s(B) + i(F)$. We require that $o_E = s(o_B) + i(o_F)$. Clearly the definition is independent of the choice of s. Thus we orient exact sequence by reading them from "right to left" rather than vice-versa. We remark that this is consistent with the standard orientation on products, ie $o_{(B\times F)} = o_B + o_F$.

A.1.2 Fibrations Orienting exact sequence implies a convention for the orientation of fibrations. Namely, let $\pi\colon E \longrightarrow B$ be a (locally trivial smooth) fibration with fiber F. Given orientations on two of F, E, B we orient the third according to the exact sequence

$$0 \longrightarrow TF \xrightarrow{Di} TE \xrightarrow{Dp} TB \longrightarrow 0,$$

where i is the inclusion of the fiber in E.

A.1.3 Group actions and quotients A special important case of orientations on fibrations is the following. Let X be an oriented manifold and K an oriented Lie group acting freely on X. We orient the quotient space X/K by viewing $X \longrightarrow X/K$ as a fibration. Equivalently, we use the exact sequence $0 \longrightarrow T_x(K\cdot x) \longrightarrow T_x X \longrightarrow T_{[x]}(X/K) \longrightarrow 0$.

A.1.4 Orienting boundaries of manifolds Let W be an oriented manifold with boundary, then the orientation of ∂W is such that $\vec{n} + o_{\partial W} = o_W$, where $\vec{n}$ is an exterior pointing vector to ∂W.

A.1.5 Normal bundles Let W be an oriented manifold and $V \subset W$ an oriented submanifold. We orient the normal bundle ${}^{\nu}V = TW/TV$ of V by the exact sequence $0 \longrightarrow TV \longrightarrow TW \longrightarrow {}^{\nu}V \longrightarrow 0$, or by abuse of notation $o_{({}^{\nu}V)} + o_V = o_W$.

A.1.6 Preimages Let U and W be oriented manifolds and $V \subset W$ an oriented submanifold. Let $f\colon U \longrightarrow W$ be a map transverse to V. We orient $f^{-1}(V)$ as follows. We first orient the normal bundle of $f^{-1}(V)$ in U, by pulling back the orientation of ${}^{\nu}V$ via the isomorphism $Df\colon {}^{\nu}f^{-1}(V) \longrightarrow {}^{\nu}V$. The orientation on ${}^{\nu}f^{-1}(V)$ induces an orientation on $f^{-1}(V)$.

A.1.7 Intersections If U, V are two transverse oriented submanifolds of an oriented manifold W. We orient $U \cap V$ via the exact sequence $0 \longrightarrow T(U\cap V) \longrightarrow TW \longrightarrow {}^{\nu}U \oplus {}^{\nu}V \longrightarrow 0$. In other words we have ${}^{\nu}U \oplus {}^{\nu}V \oplus T(U \cap V) = TW$ as oriented vector spaces.

A.1.8 Fiber products Here we use a convention taken from [26], though our presentation is somewhat different. Let $e_i\colon V_i \to X$, $i = 1, 2$, be two transverse smooth maps, where V_1, V_2, X are oriented manifolds. Denote by $\Delta \subset X \times X$ the diagonal. We denote by $V_1 \times_X V_2$ the submanifold $(e_1, e_2)^{-1}(\Delta) \subset V_1 \times V_2$ endowed with the following orientation – which is, in general, *different* from the standard preimage orientation. At the level of tangent spaces there exists an exact sequence

$$0 \longrightarrow K \longrightarrow TV_1 \oplus TX \oplus TV_2 \xrightarrow{h} TX \oplus TX \longrightarrow 0,$$

where $h(v_1, x, v_2) = (De_1(v_1) - x, x - De_2(v_2))$, and K is the kernel of h. Note that K is canonically identified with the tangent space of $(e_1, e_2)^{-1}(\Delta)$ under the map $(v_1, x, v_2) \to (v_1, v_2)$. Following our conventions above, the kernel K above inherits an orientation from those of V_1, V_2, X. The *fiber product* orientation of $V_1 \times_X V_2$ is induced by that of K. We will sometimes denote this fiber product also by $V_1 \,{}_{e_1}\!\times_{e_2} V_2$ in case we need to make explicit the maps e_1, e_2.

It is easy to see that our fiber product convention coincides with that in [26]. In case V_1 and V_2 are oriented submanifolds of X and the two evaluations are just the respective inclusions one can check that, as oriented submanifolds, $V_1 \times_X V_2 = V_2 \cap V_1$.

The motivation for introducing the fiber product orientation is that it verifies an important associativity property. If $e_1\colon U \to X$, $e_2\colon V \to X$, $f_1\colon V \to Y$, $f_2\colon W \to Y$ are smooth maps with the appropriate transversality conditions, then we have an oriented equality

$$(U \times_X V) \times_Y W = U \times_X (V \times_Y W).$$

This is easily seen by noticing that both orientations can be viewed as induced by the kernel orientation in the short exact sequence

$$0 \longrightarrow K \longrightarrow TU \oplus TX \oplus TV \oplus TY \oplus TW \xrightarrow{h'} TX \oplus TX \oplus TY \oplus TY \longrightarrow 0$$

with $h'(u, x, v, y, w) = (De_1(u) - x, x - De_2(v), Df_1(v) - y, y - Df_2(w))$. Obviously, a similar formula remains valid for longer iterated fiber products.

Let us mention a few other useful identities, which can be derived by straightforward computations. The first one deals with switching the factors in the fiber product:

$$U \times_X V = (-1)^{(|U|-|X|)(|V|-|X|)} V \times_X U. \tag{81}$$

Let $f\colon U \to X$ be a smooth map which is transverse to a submanifold $V \subset X$. Then

$$f^{-1}(V) = (-1)^{(|U|-|X|)(|V|-|X|)} U \times_X V = V \times_X U, \tag{82}$$

where $f^{-1}(V)$ is oriented as in Section A.1.6.

Next, let $e_U\colon U \to X_1 \times X_2$, $e_V\colon V \to X_1$, $e_W\colon W \to X_2$ be smooth maps (satisfying appropriate transversality conditions). Then

$$U \times_{X_1 \times X_2} (V \times W) = (-1)^{|X_2|(|X_1|+|V|)} (U \times_{X_1} V) \times_{X_2} W. \tag{83}$$

See [26, Chapter 8] for a proof. Here the first fiber product on the right-hand side involves the map $\mathrm{pr}_{X_1} \circ e_U\colon U \to X_1$ and the second fiber product on the right uses the map induced from $\mathrm{pr}_{X_2} \circ e_U\colon U \to X_2$.

Another important feature of the fiber product is its behavior with respect to taking boundaries (see [26]). Let U, V be oriented manifolds possibly with boundary and X an oriented manifold without boundary. Let $e\colon U \longrightarrow X$, $f\colon V \longrightarrow X$ be two transverse maps. Then we have the following "Leibniz" formula for fiber products:

$$\partial(U \times_X V) = (\partial U) \times_X V \coprod (-1)^{|X|-|U|} U \times_X \partial V. \tag{84}$$

A.1.9 Lagrangian submanifolds Throughout the paper, by a Lagrangian $L \subset (M,\omega)$ we mean an oriented Lagrangian submanifold together with a fixed spin structure.

A.1.10 The group of biholomorphisms of the disk $\mathrm{Aut}(D)$ Denote by $D \subset \mathbb{C}$ the closed unit disk. We orient its boundary ∂D by the counterclockwise orientation.

Denote by $G = \mathrm{Aut}(D)$ the group of biholomorphisms of the disk. We orient G as follows. Every element in G can be written uniquely as

$$\sigma_{\theta,\alpha}(z) = e^{i\theta}\frac{z+\alpha}{1+\bar{\alpha}z}, \quad \text{with } \theta \in [0,2\pi),\ \alpha \in \mathrm{Int}\, D.$$

This gives an identification between G and $[0,2\pi) \times \mathrm{Int}\, D$ by which we orient G.

Denote by $H \subset G$ the subgroup of elements that preserve the two points $-1, +1 \in \partial D$. This 1–dimensional subgroup consists of the elements $\sigma_{0,\alpha}$ with $\alpha \in (-1,1)$. We orient H by the orientation of the interval $(-1,1)$. With this choice we have $\sigma_{0,\alpha}(0) \longrightarrow +1$ (respectively -1) when $\alpha \longrightarrow +1$ (respectively -1). Note that our conventions here are somewhat different from those of [26]. Namely, our orientations of G and H agree with those of [26], but in [26] these groups act on D from the right (by $z \cdot g := g^{-1}(z)$), whereas we use the obvious action from the left.

A.1.11 Moduli spaces of holomorphic disks Fix a generic almost complex structure $J \in \mathcal{J}$. Let $B \in H_2^D$. Denote by $\widetilde{\mathcal{M}}(B,J)$ the space of (parametrized) J–holomorphic disks $u\colon (D,\partial D) \longrightarrow (M,L)$ with $u_*([D]) = B$. It is well-known by the work [26] that a spin structure on L induces orientations on the moduli spaces $\widetilde{\mathcal{M}}(B,J)$. Given $\zeta \in D$ (resp. ∂D) we denote by $e_\zeta\colon \widetilde{\mathcal{M}}(B,J) \longrightarrow M$ (resp. L) the evaluation map given by $e_\zeta(u) = u(\zeta)$.

Let $p,q \geq 0$ and consider the space of (parametrized) J–holomorphic disks with p–marked points on the boundary and q marked points in the interior: $\widetilde{\mathcal{M}}_{p,q}(B,J) = \widetilde{\mathcal{M}}_{p,q}(B,J) \times T_{p,q}$, where $T_{p,q} \subset (\partial D)^{\times p} \times (\mathrm{Int}\,(D))^{\times q}$ is the open set consisting of all tuples of points $(\underline{z},\underline{\xi}) = (z_1,\ldots,z_p,\xi_1,\ldots,\xi_q)$ with the properties that the z_i's are all distinct, the ξ_j are all distinct and in addition if $p \geq 3$ the points $z_1,\ldots,z_q$ are required to be in cyclic order along ∂D with respect to the standard (counterclockwise) orientation. As $T_{p,q}$ is an open subset of $(\partial D)^{\times p} \times (\mathrm{Int}\,(D))^{\times q}$ it inherits an orientation from the latter. Apart from that we will require that $B \neq 0$ when $p \leq 2$ and $q = 0$ or when $p = 0$ and $q = 1$.

We let $G = \mathrm{Aut}(D)$ (as well as subgroups of it) act on $\widetilde{\mathcal{M}}_{p,q}(B,J)$ as follows. If $\sigma \in G$ and $(u, z_1, \dots, z_p, \xi_1, \dots, \xi_q) \in \widetilde{\mathcal{M}}_{p,q}(B,J)$ define

$$\sigma \cdot (u, z_1, \dots, z_p, \xi_1, \dots, \xi_q) = (u \circ \sigma^{-1}, \sigma(z_1), \dots, \sigma(z_p), \sigma(\xi_1), \dots, \sigma(\xi_q)).$$

We denote the space of disks with marked points by $\mathcal{M}_{p,q}(B,J) = \widetilde{\mathcal{M}}_{p,q}(B,J)/G$, with the orientation induced from the preceding conventions. This space comes with evaluation maps $E_{i,-}\colon \mathcal{M}_{p,q}(B,J) \longrightarrow L$ and $E_{-,j}\colon \mathcal{M}_{p,q}(B,J) \longrightarrow M$ defined by $E_{i,-}[u, \underline{z}, \underline{\xi}] = u(z_i)$ and $E_{-,j}[u, \underline{z}, \underline{\xi}] = u(\xi_j)$.

In what follows it will be often useful to deal with quotients by the group $H \subset G$ of those elements that fix the points $-1, 1 \in D$, namely with $\widetilde{\mathcal{M}}(B,J)/H$. Recall that we have oriented H in Section A.1.10 above. The space $\widetilde{\mathcal{M}}(B,J)/H$ comes with two evaluation maps $e_{-1}, e_{+1}\colon \widetilde{\mathcal{M}}(B,J)/H \longrightarrow L$, defined by $e_{-1}[u] = u(-1)$ and $e_{+1}[u] = u(+1)$.

With these conventions it is not hard to verify that the following maps are orientation preserving diffeomorphisms:

$$\begin{aligned} &\widetilde{\mathcal{M}}(B,J)/H \longrightarrow \mathcal{M}_{2,0}(B,J), && [u] \longmapsto [u, 1, -1], \\ &\widetilde{\mathcal{M}}(B,J) \longrightarrow \mathcal{M}_{1,1}(B,J), && u \longmapsto [u, 1, 0], \\ &\widetilde{\mathcal{M}}(B,J) \longrightarrow \mathcal{M}_{3,0}(B,J), && u \longmapsto [u, 1, e^{2\pi i/3}, e^{4\pi i/3}]. \end{aligned} \tag{85}$$

In view of the first map above we will identify $\mathcal{M}_{2,0}(B,J)$ with $\widetilde{\mathcal{M}}(B,J)/H$ and view e_{-1}, e_{+1} as maps defined on $\mathcal{M}_{2,0}(B,J)$.

To simplify the notation, when $q = 0$, we will sometimes write $\mathcal{M}_p(B,J)$ instead of $\mathcal{M}_{p,0}(B,J)$. We will especially use $\mathcal{M}_2(B,J)$.

A.1.12 Bubbling and gluing Let $B, B', B'' \in H_2^D$ with $B = B' + B''$. Consider the fiber product

$$\mathcal{M}_2(B',J)_{e_{+1}} \times_{e_{-1}} \mathcal{M}_2(B'',J),$$

where $e_{\pm 1}$ are the evaluation maps at $\pm 1 \in \partial D$. By compactness, gluing, as well as further regularity assumptions, this spaces can be embedded into the main stratum of the boundary of the compactification of the space $\mathcal{M}_2(B,J)$:

$$\mathcal{M}_2(B',J)_{e_{+1}} \times_{e_{-1}} \mathcal{M}_2(B'',J) \hookrightarrow \partial \overline{\mathcal{M}_2(B,J)}. \tag{86}$$

This embedding is so that the pair of marked points $-1 \in \mathrm{dom}(u')$ and $+1 \in \mathrm{dom}(u'')$ with $(u', u'') \in \mathcal{M}_2(B', J) \times \mathcal{M}_2(B'', J)$ corresponds after gluing to the pair of marked points $-1, +1 \in \partial D$ in the domain of the glued disk $u' \#_\tau u'' \in \mathcal{M}_2(B, J)$ for all gluing parameters τ.

The embedding (86) is in general not orientation preserving. In fact the orientations on the left and right hand sides differ by $(-1)^{n-1}$. This can be proved by a direct computation based on [26]. We write this fact as

$$\partial_{\text{bubble}} \overline{\mathcal{M}_2(B, J)} = \coprod_{B' + B'' = B} (-1)^{n-1} \mathcal{M}_2(B', J)_{e_{+1}} \times_{e_{-1}} \mathcal{M}_2(B'', J). \tag{87}$$

There is a slight abuse of notation here, since the right hand side is just part of the boundary of $\mathcal{M}_2(B, J)$. However for the purpose of the pearl complex the other boundary components are not relevant. We will also write $\partial^{(B', B'')}_{\text{bubble}} \overline{\mathcal{M}_2(B, J)}$ for the boundary component in (87) that corresponds to bubbling of the type (B', B'').

Remark A.1.2 There is a subtle difference between our conventions for gluing and those in [26]. In our case for the first moduli space in the fiber product we evaluate at the point $+1$ and for the second at the point -1 while [26] use the opposite convention. Furthermore, our conventions for the orientation on H are opposite to theirs. These different sign conventions turn out to cancel each other in this case, hence our sign $(-1)^{n-1}$ coincides with the one that appears in [26].

A.1.13 Orientations in Morse theory There are several different orientation conventions regarding Morse theory (see eg Audin and Damian [4], Banyaga and Hurtubise [9], Salamon [41] and Schwarz [43]). Since none of the conventions we could find in the literature is completely compatible with ours we will now describe our approach in some detail.

Let V be an oriented manifold, $f\colon V \longrightarrow \mathbb{R}$ a Morse function and $(\cdot,\cdot)$ a Riemannian metric. Stable and unstable submanifolds are always taken with respect to the *negative* gradient flow of f which we denote by $\Phi_t\colon V \longrightarrow V$. For every $x \in \mathrm{Crit}(f)$ fix an orientation on the unstable submanifold $W^u(x)$. This induces an orientation on the stable submanifolds $W^s(x)$ by requiring that $o_{W^s(x)} + o_{W^u(x)} = o_V$.

Assume now that the pair $(f, (\cdot,\cdot))$ is Morse–Smale. Given $x, y \in \mathrm{Crit}(f)$ we have the following spaces of gradient trajectories connecting x to y:

$$\widetilde{m}(x, y) = W^s(y) \cap W^u(x), \quad m(x, y) = \widetilde{m}(x, y)/\mathbb{R},$$

where $\mathbb{R}$ acts on $\widetilde{m}(x, y)$ by $t \cdot p = \Phi_t(p)$. All spaces here are oriented by the conventions we have described so far. The Morse complex (with coefficients in $\mathbb{Z}$) is now defined by $CM = \mathbb{Z}\langle \mathrm{Crit}(f)\rangle$, $\partial\colon CM_* \longrightarrow CM_{*-1}$, where

$$\partial(x) = \sum_{|y|=|x|-1} \# m(x,y) y \quad \forall\, x \in \mathrm{Crit}(f).$$

A.1.14 Some useful identities for boundaries We start with two useful formulae for the boundary of the stable and unstable submanifolds of critical points. Recall that these manifold admit a natural compactification in terms of stable and unstable submanifolds of lower indices. Here are the signs that appear in these boundaries. Let $(f,(\cdot,\cdot))$ be a Morse–Smale pair as in Section A.1.13. Let $x \in \mathrm{Crit}(f)$ and $x' \in \mathrm{Crit}(f)$. Then the part of the boundary of $\overline{W^u(x)}$ that involves the critical point x' satisfies

$$\partial \overline{W^u(x)} = (-1)^{|x|-|x'|-1} m(x,x') \times W^u(x'). \tag{88}$$

In particular, when $|x'| = |x|-1$ we have

$$\partial \overline{W^u(x)} = m(x,x') \times W^u(x'). \tag{89}$$

Similarly, if $y, y' \in \mathrm{Crit}(f)$ with $|y'| = |y|+1$ then the part of the boundary of $\overline{W^s(y)}$ that involves y' satisfies

$$\partial \overline{W^s(y)} = (-1)^{|V|-|y|} m(y',y) \times W^s(y'). \tag{90}$$

Of course, in the identities (88), (89), (90) the boundaries $\partial\overline{W^u(x)}$, $\partial\overline{W^s(y)}$ should not be regarded as subsets of V but rather as boundaries of cells that come with appropriate attaching maps to a cell decomposition of V.

For completeness, here is a proof of these identities. To derive identity (88) write $\partial\overline{W^u(x)} = \varepsilon m(x,x') \times W^u(x')$ for some $\varepsilon \in \{-1,1\}$. Denote by $T_f \in T_{x'}V$ the direction in which a $-\nabla f$ trajectory $\gamma \in m(x,x')$ arrives from x to x'. We view T_f as a vector pointing to the exterior of $\partial W^u(x)$.

By definition we have $T_f + \varepsilon o_{m(x,x')} + o_{W^u(x')} = o_{W^u(x)}$, hence

$$o_{({}^\nu W^u(x))} + T_f + \varepsilon o_{m(x,x')} + o_{W^u(x')} = o_V. \tag{91}$$

From the definition of $m(x,x')$ we have

$$o_{({}^\nu W^s(x'))} + o_{({}^\nu W^u(x))} + o_{m(x,x')} + T_f = o_V. \tag{92}$$

Since ${}^\nu W^s(x') = (-1)^{|x'|(|V|-|x'|)} W^u(x')$ we obtain from (92) that

$$(-1)^{|x'|(|V|-|x'|)} o_{W^u(x')} + o_{({}^\nu W^u(x))} + o_{m(x,x')} + T_f = o_V. \tag{93}$$

Comparing (91) with (93) we arrive to $\varepsilon = (-1)^{|x|-|x'|-1}$. Identity (90) can be derived in a similar way. □

Next we derive some general formulas for boundaries of moduli spaces of gradient trajectories "connecting" two manifolds. Consider two oriented manifolds X and Y with maps $e_X\colon X \longrightarrow L$ and $e_Y\colon Y \longrightarrow L$. Let Φ_t be the negative gradient flow of f and consider the map $e'_X\colon X \times \mathbb{R}_+ \longrightarrow L$, given by $(x,t) \longmapsto \Phi_t \circ e_X(x)$. Finally, consider the fiber product $Z = (X \times \mathbb{R}_+) \times_L Y$, where the first factor is mapped to L by e'_X and the second one by e_Y. One might think of Z as the space of gradient trajectories connecting X to Y. Ignoring orientations for a moment, we note that part of the boundary of Z is formed by broken trajectories, ie by elements of the space $(X \times_L W^s(z)) \times (W^u(z) \times_L Y)$, where $z \in \mathrm{Crit}(f)$. Here the (un)stable submanifolds are mapped to L by inclusion and X, Y, by the maps e_X, e_Y respectively. We denote this component of the boundary by $\partial_z((X \times \mathbb{R}_+) \times_L Y)$. Taking now orientations into account we have the following identity:

$$\partial_z\big((X \times \mathbb{R}_+) \times_L Y\big) = (-1)^{|X|}\big(X \times_L W^s(z)\big) \times \big(W^u(z) \times_L Y\big). \tag{94}$$

The proof can be done by a straightforward computation analogous to the one used to prove identity (88).

Another boundary component of $(X \times \mathbb{R}_+) \times_L Y$ arises when the gradient trajectory between X and Y shrinks to zero length. Ignoring orientations, the corresponding part of the boundary can be written as $X \times_L Y$, where X, Y are mapped to L by e_X, e_Y respectively. We denote it by $\partial_{\text{shrink}}\big((X \times \mathbb{R}_+) \times_L Y\big)$. Taking orientations into account, one obtains the following identity:

$$\partial_{\text{shrink}}\big((X \times \mathbb{R}_+) \times_L Y\big) = (-1)^{|X|+1}(X \times_L Y). \tag{95}$$

A.2 Orientation conventions for the pearl complex

Our purpose now is to describe the orientation conventions for the various pearly moduli spaces needed. With the conventions that we will describe, the various algebraic structures described in Section 2.2 verify the usual identities in noncommutative differential graded homological algebra. We will only justify here some of these facts, they are relatively straightforward but tedious exercises. We remark that for the constructions below to work with our orientation conventions it is important that the algebraic structures discussed here are only defined by counting elements of 0–dimensional moduli spaces. In our case, the main equations of interest concern the product from (4) that verifies *at the chain level* the equation

$$d(x * y) = d(x) * y + (-1)^{n-|x|} x * d(y) \tag{96}$$

and the module action from Section 2.2.2 that verifies a similar identity. Besides this we claim that the other identities: $d^2 = 0$, associativity of the product etc are all verified with signs as well.

A.2.1 Orienting the space of pearly trajectories We first recall that a string of pearls associated to the data $\mathcal{D} = (f, (\cdot,\cdot), J)$ and joining two points $x, y \in \mathrm{Crit}(f)$ can be viewed as a sequence $(a, u_1, t_1, u_2, t_2, \ldots, u_k, b)$ where $a \in W^u(x)$, $b \in W^s(y)$, $u_i \in \mathcal{M}_2(B_i, J)$, $B_i \neq 0$, $t_i \in \mathbb{R}_+$, subject to the following incidence conditions $\Phi_{t_i}(u_i(+1)) = u_{i+1}(-1)$ for $1 \leq i < k$, $u_1(-1) = a$, $u_k(+1) = b$. Here Φ_t is the negative gradient flow of f. Appropriate genericity conditions are required to insure the transversality of the relevant evaluation maps. The resulting pearl moduli space is denoted $\mathcal{P}(x, y; \mathcal{D}; (B_1, \ldots, B_k))$. When $k = 1$ we also allow $B_1 = 0$ and put $\mathcal{P}(x, y; \mathcal{D}, 0) = m(x, y)$ ie the space of gradient trajectories going from x to y as in Section A.1.13 above.

All orientation conventions described below are established by assuming that we restrict attention only to the moduli spaces involving absolutely distinct sequences of simple disks in the sense of [14; 12].

The moduli space $\mathcal{P}(x, y; \mathcal{D}; (B_1, \ldots, B_k))$ is thus a subset of

$$W^u(x) \times (\mathcal{M}_2(B_1, J) \times \mathbb{R}_+) \times \cdots \times \mathcal{M}_2(B_k, J) \times W^s(y)$$

obtained from a multidiagonal in $L \times L^{\times 2k} \times L$ by taking the preimage by a suitable evaluation map. However, this procedure will not be used in order to orient these spaces. For the purpose of orientations we describe $\mathcal{P}$ as an iterated fiber product.

Let $B_1, \ldots, B_k$, $k \geq 1$, be a sequence of classes in H_2^D with $B_j \neq 0$ for all j. Consider the fiber product

$$\begin{aligned} \mathcal{P}(x, y; \mathcal{D}; (B_1, \ldots, B_k)) = W^u(x) &\times_L (\mathcal{M}_2(B_1, J) \times \mathbb{R}_+) \times_L \\ &\cdots \times_L (\mathcal{M}_2(B_i, J) \times \mathbb{R}_+) \times_L \\ &\cdots \times_L \mathcal{M}_2(B_k, J) \times_L W^s(y), \end{aligned} \tag{97}$$

where the first and last factor here are mapped into L by inclusion. The i–th moduli space ($i < k$) is mapped to the term L on its left by $(u_i, t) \mapsto e_{-1}(u_i) = u_i(-1)$, and to the term L on its right by $(u_i, t) \mapsto \Phi_t \circ e_{+1}(u_i) = \Phi_t(u_i(+1))$. The second to last factor $\mathcal{M}_2(B_k, J)$ is mapped to the L on its left by e_{-1} and to the L on its right by e_{+1}. When $B = 0$ we simply put $\mathcal{P}(x, y; \mathcal{D}; 0) = m(x, y)$ without any orientation adjustment. Next, for a fixed $0 \neq B \in H_2^D$, the disjoint union of all the moduli spaces $\mathcal{P}(x, y; \mathcal{D}; (B_1, \ldots, B_k))$ such that $B = \sum B_i$ is denoted by $\mathcal{P}(x, y; \mathcal{D}, B)$. Sometimes we will omit $\mathcal{D}$ from the notation. We also put $\delta(x, y; B) = |x| - |y| - 1 + \mu(B)$ which is the virtual dimension of $\mathcal{P}(x, y; \mathcal{D}; B)$.

Fix a $\widetilde{\Lambda}^+$–algebra $\mathcal{R}$ with its structural morphism $q\colon \widetilde{\Lambda}^+ \to \mathcal{R}$. The differential on the pearl complex $\mathcal{C}(\mathscr{D})$ (mentioned at the beginning of Section 2.2) is defined as follows. For $x \in \mathrm{Crit}(f)$,

$$dx = \sum_{y;\,|y|=|x|-1} \#\mathcal{P}(x,y;\mathscr{D};0)\,y \;+ \sum_{\substack{y,B\neq 0;\\ \delta(x,y;B)=0}} (-1)^{|y|}\,\#\mathcal{P}(x,y;\mathscr{D};B)\,y\,q(T^B). \tag{98}$$

Notice that the first summand coincides with the Morse differential. Note also the $(-1)^{|y|}$ sign standing in front of the elements in the second summand. This sign is needed in order to make d be a differential (ie $d^2=0$) and is implied by our sign conventions for the moduli spaces. See Remark A.2.1 for more on that.

Showing that $d^2=0$ reduces to the verifications in the $\mathbb{Z}_2$ case as described in [14] together with two points having to do with the orientation conventions. The first concerns the coherence of the orientation conventions with respect to bubbling and, respectively, with respect to the contraction of a flow line joining two consecutive disks. The claim in this case is that a configuration that appears with a certain sign by bubbling, also appears by the contraction of a flow line but with a reversed sign. The second has to do with the signs that appear at the breaking of a 1–dimensional pearl moduli space at a critical point of f: we need to make sure that these signs are the correct ones so that $d^2=0$. We now intend to explain why our conventions take care of these two points.

For the first point, let us analyze the boundary points of a 1–dimensional moduli space of pearly trajectories $\mathcal{P}(x,y;\mathscr{D};(B_1,\dots,B_k))$ that appear when a gradient trajectory between the i–th disk and the $(i+1)$–st disk $(1\le i\le k-1)$ shrinks to zero length. The relevant part of the fiber product in (97) is the space

$$\mathcal{P}_i = (\mathcal{M}_2(B_i,J)\times\mathbb{R}_+)\times_L \mathcal{M}_2(B_{i+1},J).$$

Applying formula (95) we get

$$\partial_{shrink}\,\overline{\mathcal{P}_i} = (-1)^n\,\mathcal{M}_2(B_i,J)\times_L\mathcal{M}_2(B_{i+1},J).$$

Note that $\dim\mathcal{M}_2(B_i,J)+1 = n+\mu(B_i)\equiv n \pmod 2$, since $\mu(B_i)$ is even because L is orientable. Next, by formula (87) we have that the component of the boundary of $\overline{\mathcal{M}_2(B_i+B_{i+1},J)}$ that corresponds to bubbling into two disks of classes B_i, B_{i+1} is

$$\partial^{(B_i,B_{i+1})}_{\text{bubble}}\,\overline{\mathcal{M}_2(B_i+B_{i+1},J)} = (-1)^{n-1}\mathcal{M}_2(B_i,J)\times_L\mathcal{M}_2(B_{i+1},J).$$

Applying the Leibniz formula for fibre products (84) it follows that bubbling and shrinking of a gradient trajectory between two disks come with opposite signs in boundaries of 1–dimensional spaces of pearly trajectories. Now fix $B \neq 0$. Summing this up over all $k \geq 1$ and $(B_1, \dots, B_k)$ with $\sum B_i = B$ we obtain that

$$\#\partial_{\text{bubble}}\mathcal{P}(x, y; \mathscr{D}; B) + \#\partial_{\text{shrink}}\mathcal{P}(x, y; \mathscr{D}; B) = 0. \tag{99}$$

Of course other bubbles might a priori occur (such as side bubbling, or sphere bubbles) but they actually do not appear when L is monotone (see [14; 13]). This concludes the first point in the proof that $d^2 = 0$.

We now come to the second point in the proof. By the results of [14; 12] when the virtual dimension is $\delta(x, y; B) = 1$, the spaces $\mathcal{P}(x, y; \mathscr{D}; B)$ admit a compactification into a 1–dimensional manifold with boundary. Moreover, the boundary of this compactification consists of precisely the following three types of spaces:

$$\begin{aligned}\partial\overline{\mathcal{P}(x, y; \mathscr{D}; B)} \\ = \partial_{\text{bubble}}\mathcal{P}(x, y; \mathscr{D}; B) \coprod \partial_{\text{shrink}}\mathcal{P}(x, y; \mathscr{D}; B) \coprod \partial_{\text{break}}\mathcal{P}(x, y; \mathscr{D}; B),\end{aligned} \tag{100}$$

where ∂_{break} stands for breaking of a pearly trajectory at a critical point which we now elaborate more about. Let $\mathbf{B} = (B_1, \dots, B_k)$ be such that $\sum B_j = B$, and consider the space $\mathcal{P} = \mathcal{P}(x, y; \mathscr{D}; \mathbf{B})$. We assume that its dimension is 1, namely $\delta(x, y; B) = 1$. There are three types of places where the gradient trajectory might break at. The first is at a critical point x' between x and the first disks B_1. The second possibility is at a critical point z between two consecutive disks B_i and B_{i+1}. The last possibility is that this occurs at a critical point y' between the last disk B_k and the point y. Applying the Leibniz formula (84) together with formulae (89), (94), (90) we obtain

$$\begin{aligned}\partial_{x'}\mathcal{P} &= m(x, x') \times \mathcal{P}(x', y; \mathscr{D}; \mathbf{B}),\\ \partial_z\mathcal{P} &= (-1)^{|x|+1}\mathcal{P}(x, z; \mathscr{D}; (B_1, \dots, B_i)) \times \mathcal{P}(z, y; \mathscr{D}; (B_{i+1}, \dots, B_k)),\\ \partial_{y'}\mathcal{P} &= -\mathcal{P}(x, y'; \mathscr{D}; \mathbf{B}) \times m(y', y).\end{aligned} \tag{101}$$

Recall also by our conventions $m(x, x') = \mathcal{P}(x, x'; \mathscr{D}; 0)$ and similarly for $m(y', y)$. The union of the spaces in (101) over all relevant x', z, y', i, k and $(B_1, \dots, B_k)$ with $\sum B_j = B$ form the space $\partial_{\text{break}}\mathcal{P}(x, y; \mathscr{D}; B)$.

We are now ready to show that $d^2(x) = 0$ for every $x \in \text{Crit}(f)$. We will work here with the ring $\widetilde{\Lambda}^+$, which implies that the same statement holds for every $\widetilde{\Lambda}^+$–algebra. Fix $y \in \text{Crit}(f)$ and $B \in H_2^D$ so that $\delta(x, y; B) = 1$. We have to show that the coefficient of yT^B in $d \circ d(x)$, which we denote by $\langle d^2(x), yT^B\rangle$ is 0. Clearly, if $B = 0$ this amounts to showing that the Morse differential squares to 0 which is well known, thus we assume that $B \neq 0$. A simple computation now shows that

$$\begin{aligned}(102)\quad \langle d^2(x), yT^B\rangle = \sum_{|x'|=|x|-1} & (-1)^{|y|}\#\mathcal{P}(x,x';0)\#\mathcal{P}(x',y;B) \\ &+ \sum_{\substack{z,A;\\ \delta(x,z;A)=0\\ A\neq B}} (-1)^{|z|+|y|}\#\mathcal{P}(x,z;A)\#\mathcal{P}(z,y;B-A) \\ &+ \sum_{y';\delta(x,y';B)=0} (-1)^{|y'|}\#\mathcal{P}(x,y';B)\#\mathcal{P}(y',y;0).\end{aligned}$$

Applying (101) we now arrive to

$$\begin{aligned}(103)\quad \langle d^2(x), yT^B\rangle = \sum_{|x'|=|x|-1} & (-1)^{|y|}\#\partial_{x'}\mathcal{P}(x,y;B) \\ &+ \sum_{\substack{z,A;\\ \delta(x,z;A)=0\\ A\neq B}} (-1)^{|z|+|y|+|x|+1}\#\partial_z\mathcal{P}(x,y;B) \\ &+ \sum_{y';\delta(x,y';B)=0} (-1)^{|y'|+1}\#\partial_{y'}\mathcal{P}(x,y;B).\end{aligned}$$

Note that for the z's that appear in the second summand we have $|z|+|x|+1 \equiv 0 \pmod 2$, hence $(-1)^{|z|+|y|+|x|+1} = (-1)^{|y|}$. Similarly, for the third summand we have $(-1)^{|y'|+1} = (-1)^{|y|}$. Thus we obtain

$$\langle d^2(x), yT^B\rangle = (-1)^{|y|}\,\#\partial_{\mathrm{break}}\mathcal{P}(x,y;B) = (-1)^{|y|}\,\#\partial\overline{\mathcal{P}(x,y;B)} = 0,$$

where the second to last equality follows from (99) and (100). This concludes the verification that $d^2 = 0$.

Remark A.2.1 Here we explain in a more conceptual way the role of the sign $(-1)^{|y|}$ in (98). This sign naturally appears from slightly different moduli spaces than $\mathcal{P}(x,y;\mathscr{D};B)$. For every $x \in \mathrm{Crit}(f)$ denote by $S^u(x)$ the unstable sphere corresponding to x. This can be thought of as small radius (or infinitesimal) sphere inside $W^u(x)$ oriented as the boundary of small disk around the critical point which lies inside $W^u(x)$ (recall that $W^u(x)$ is oriented). Similarly we have the stable sphere $S^s(y)$ for every $y \in \mathrm{Crit}(f)$. Consider now the moduli space

$$\begin{aligned}(104)\quad & \mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};(B_1,\dots,B_k)) \\ &= (S^u(x)\times\mathbb{R}_+)\times_L(\mathcal{M}_2(B_1,J)\times\mathbb{R}_+)\times_L\cdots\times_L(\mathcal{M}_2(B_i,J)\times\mathbb{R}_+)\times_L \\ &\qquad \cdots\times_L(\mathcal{M}_2(B_k,J)\times\mathbb{R}_+)\times_L S^s(y)\,,\end{aligned}$$

where the first factor is mapped to L by $(p,t)\mapsto \Phi_t(p)$, and the last one by inclusion. The only difference between $\mathcal{P}$ and $\mathcal{P}_{\mathrm{sph}}$ is the first factor in the fiber product as well as the last two ones. For $B\neq 0$, we define $\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};B)$ to be the union of all $\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};(B_1,\dots,B_k))$ over all $(B_1,\dots,B_k)$ with $\sum B_j=B$. When $B=0$ we put $\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D},0)=(S^u(x)\times\mathbb{R}_+)\times_L S^s(y)$. The relation between these spaces and the one we have used so far is given by

$$\begin{aligned}\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};0)&=(-1)^{n+|x|-|y|-1}\,\mathcal{P}(x,y;\mathscr{D};0)=(-1)^{n+|x|-|y|-1}\,m(x,y),\\ \mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};B)&=(-1)^{n+1+|x|}\,\mathcal{P}(x,y;\mathscr{D};B)\quad\text{when } B\neq 0.\end{aligned}\tag{105}$$

In particular, when $\delta(x,y;B)=0$ we have

$$\begin{aligned}\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};0)&=(-1)^n m(x,y),\\ \mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};B)&=(-1)^{n+|y|}\,\mathcal{P}(x,y;\mathscr{D};B)\quad\text{when } B\neq 0.\end{aligned}$$

Thus our differential (98) can be written also as

$$d(x)=(-1)^n\sum_{\substack{y,B;\\ \delta(x,y;B)=0}}\#\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};B)\,y\,T^B.$$

Moreover, the spaces $\mathcal{P}_{\mathrm{sph}}$ behave better with respect to breaking at critical points, at least as far as orientations go. In fact, if $\delta(x,y;B)=1$ we have

$$\partial_{\mathrm{break}}\big(\mathcal{P}_{\mathrm{sph}}(x,y;\mathscr{D};B)\big)\coprod_{\substack{v\in\mathrm{Crit}(f),\,B'+B''=B;\\ \delta(x,v;B')=0}}(-1)^{n+1}\,\mathcal{P}_{\mathrm{sph}}(x,v;\mathscr{D};B')\times\mathcal{P}_{\mathrm{sph}}(v,y;\mathscr{D};B'').$$

This together with (99) immediately implies that $d^2=0$.

Although the spaces $\mathcal{P}_{\mathrm{sph}}$ seem more natural from the point of view of orientations we have chosen not to explicitly work with them. One reason is that they seem less convenient for the purpose of the other quantum operations (eg the quantum product). Another drawback is that one has to redefine these spaces in some situations, eg when x is a minimum the unstable sphere $S^u(x)$ is, naively speaking, void. Another case is when the holomorphic disks in $\mathcal{M}_2(B_1)$ come closer to the point x than $S^u(x)$ (or even touch that point).

A.2.2 Orientations for the quantum product The various operations described earlier in this section are modeled on trees with nodes of valence at most four. In other, words they correspond to strings of pearls that possibly meet a disk with at most three entries and one exit.

As an example we now focus on the quantum product (see [14; 12] for a complete definition of the product). Fix three Morse functions f, f', f'' and the pearl data

$\mathcal{D} = (f, (\cdot,\cdot), J)$, $\mathcal{D}' = (f', (\cdot,\cdot)', J)$, $\mathcal{D}'' = (f'', (\cdot,\cdot)'', J)$. Let $v \in \mathrm{Crit}(f)$, $w \in \mathrm{Crit}(f')$, $y \in \mathrm{Crit}(f'')$. The coefficient of y in the product $v * w$ is the sum over all classes $B, B', B'', \lambda \in H_2^D$ of the number of configurations in the moduli space $\mathcal{P}(v, w, y; B, B', B'', \lambda)$ given as an iterated fiber product that we now make explicit.

Given data $\mathcal{D} = (f, (\cdot,\cdot), J)$, $x \in \mathrm{Crit}(f)$ and $(B_1, \dots, B_k)$ with $B_i \neq 0$ we first define the *unstable* pearl moduli space $\mathcal{P}^u(x; \mathcal{D}; (B_1, \dots, B_k))$ to be the following iterated fiber product (together with its orientation):

$$W^u(x) \times_L (\mathcal{M}_2(B_1, J) \times \mathbb{R}_+) \times_L \cdots \times_L (\mathcal{M}_2(B_i, J) \times \mathbb{R}_+) \times_L \cdots \times_L (\mathcal{M}_2(B_k, J) \times \mathbb{R}_+).$$

Given $B \neq 0$ we denote by $\mathcal{P}^u(x; \mathcal{D}; B)$ the union of all $\mathcal{P}^u(x; \mathcal{D}; (B_1, B_2, \dots, B_k))$ with $\sum B_i = B$. In case $B = 0$ we just put $\mathcal{P}^u(x; \mathcal{D}; 0) = W^u(x)$ (again, as oriented manifolds). This is similar to (97) with the exception that the last fiber product is missing here and is replaced by the term $\mathbb{R}_+$. The space $\mathcal{P}^u(x; \mathcal{D}; B)$ comes with an evaluation map

$$e^u_B \colon \mathcal{P}^u(x; \mathcal{D}; B) \to L$$

whose restriction to $\mathcal{P}^u(x; \mathcal{D}; (B_1, B_2, \dots, B_k))$ is induced from the evaluation on the last factor

$$\mathcal{M}_2(B_k, J) \times \mathbb{R}_+ \longrightarrow L, \quad (u, t) \longmapsto \Phi_t(u(+1)).$$

For $B = 0$ we take this evaluation map to be the inclusion $W^u(x) \longrightarrow L$.

Similarly, we define the moduli space $\mathcal{P}^s(x; \mathcal{D}; B)$ whose components are defined when $B \neq 0$ by the fiber product

$$(L \times \mathbb{R}_+) \times_L (\mathcal{M}_2(B_1, J) \times \mathbb{R}_+) \times_L \cdots \times_L (\mathcal{M}_2(B_i, J) \times \mathbb{R}_+) \times_L \cdots \times_L (\mathcal{M}_2(B_k, J) \times_L W^s(x)),$$

and $\mathcal{P}^s(x; \mathcal{D}, 0) = W^s(x)$ when $B = 0$. In addition, there is an evaluation map $e^s_B \colon \mathcal{P}^s(x; \mathcal{D}; B) \to L$ whose restriction to the component written above is induced from the identity $L \to L$ defined on the leftmost term in the product.

Next, consider the parametrized moduli space $\widetilde{\mathcal{M}}(\lambda, J)$ together with the following three evaluation maps $e_{\zeta_j} \colon \widetilde{\mathcal{M}}(\lambda, J) \longrightarrow L$ where $\zeta_j = e^{-2j\pi i/3}$, $j = 1, 2, 3$. Finally, we define the space $\mathcal{P}(v, w, y; B, B', B'', \lambda)$ by the fiber product

$$\mathcal{P}^u(v; \mathcal{D}; B)_{e^u_B} \times_{e'_{\zeta_1}} \big(\mathcal{P}^u(w; \mathcal{D}'; B')_{e^u_{B'}} \times_{e_{\zeta_2}} \widetilde{\mathcal{M}}(\lambda, J)_{e_{\zeta_3}} \times_{e^s_{B''}} \mathcal{P}^s(y; \mathcal{D}''; B'')\big), \tag{106}$$

where in this formula the map e'_{ζ_1} is induced on the fiber product in the brackets by the evaluation e_{ζ_1} originally defined on $\widetilde{\mathcal{M}}(\lambda, J)$. Note that the dimension of $\mathcal{P}(v, w, y; B, B', B'', \lambda)$ is $|v| + |w| - |y| - n + \mu(B) + \mu(B') + \mu(B'') + \mu(\lambda)$.

For $y \in \mathrm{Crit}(f)$, $C \in H_2^D$ such that $|y| - \mu(C) = |v| + |w| - n$. The coefficient of yT^C in the product $v * w$ is given by

$$\sum_{B+B'+B''+\lambda=C} \#\mathcal{P}(v, w, y; B, B', B'', \lambda).$$

By using similar arguments as those used above in the verification showing $d^2 = 0$ (see Section A.2.1), it is easy to see that the product defined by these moduli spaces verifies the Leibniz formula (96) and, moreover, that the classical term in this definition coincides with the Morse intersection product (on the chain level). Furthermore similar arguments show that the induced product on homology makes $QH(L;\mathcal{R})$ a unital associative ring.

A.2.3 Orientations for the quantum module structure Similar conventions are used to define the orientations required for the module structure from Section 2.2.2. Explicitly, let $h\colon M \to \mathbb{R}$ be a Morse function and fix a metric $(\cdot,\cdot)_M$ on M so that the pair $(h, (\cdot,\cdot)_M)$ is Morse–Smale. Fix a pearl data on L, $\mathscr{D} = (f, (\cdot,\cdot), J)$. Let $a \in \mathrm{Crit}(h)$ and $x \in \mathrm{Crit}(f)$. Let $y \in \mathrm{Crit}(f)$ and $C \in H_2^D$ with $|y| - \mu(C) = |a| + |x| - 2n$. The coefficient of yT^C in the product $a * x$ is given by counting elements in moduli spaces of the form

$$W^u(a)_i \times_{e_0'} \big(\mathcal{P}^u(x;\mathscr{D}';B')_{e^u_{B'}} \times_{e_{-1}} \widetilde{\mathcal{M}}(\lambda, J)_{e_{+1}} \times_{e^s_{B''}} \mathcal{P}^s(y;\mathscr{D}'';B'')\big), \tag{107}$$

for all B', B'', λ with $B' + B'' + \lambda = C$. Here $i : W^u(a) \longrightarrow M$ is the inclusion, e_0' is the map induced from $e_0\colon \widetilde{\mathcal{M}}(\lambda, J) \longrightarrow M$, $e_0(u) = u(0)$, and $e_{\pm 1}\colon \widetilde{\mathcal{M}}(\lambda, J) \longrightarrow L$ are the evaluation maps $e_{\pm 1}(u) = u(\pm 1)$.

Proving this operation induces on $QH(L;\mathcal{R})$ a structure of a module over $QH(M;\mathcal{R})$ is based on arguments similar to the ones in Section A.2.1.

A.2.4 Orientations for the quantum inclusion Here we fix our conventions for the quantum inclusion $i_L\colon QH(L;\mathcal{R}) \to QH(M;\mathcal{R})$ recalled in Section 2.2.3. The basic data is similar here as in the case of the module multiplication: besides $\mathscr{D}$ we also fix the Morse–Smale pair $(h, (\cdot,\cdot)_M)$ on M. We fix $x \in \mathrm{Crit}(f)$. Let $a \in \mathrm{Crit}(h)$ and $B \in H_2^D$ with $|a| - \mu(B) = |x|$. The coefficient of aT^B in the expression of $i_L(x)$ is given by counting elements in moduli spaces of the form

$$\mathcal{P}^u(x;\mathscr{D};B')_{e^u_{B'}} \times_{e_{-1}} \widetilde{\mathcal{M}}(\lambda, J)_{e_0} \times_i W^s(a), \tag{108}$$

where $i\colon W^s(a) \to M$ is the inclusion. It is not difficult to see that with our conventions this defines a chain morphism whose classical part coincides with the usual Morse (or singular homology) inclusion.

A.2.5 Invariance of the structures We now shortly discus the proof of the invariance of all these structures with respect to changes in the data $\mathscr{D}$. This is based on constructing comparison chain maps associated to any two pairs of data. In turn, to construct such comparison maps there are two distinct methods each perfectly similar to those described over $\mathbb{Z}_2$ as in [14, Section 3.2-e] and, respectively, in the proof of Proposition 4.4.1 in the same paper. The first method is based on a cone construction naturally appearing in a pearly version of Morse cobordisms. It provides a quasi-isomorphism (canonical up to chain homotopy) $\Psi_{\mathscr{D}',\mathscr{D}}\colon \mathcal{C}(L;\mathscr{D};\mathcal{R}) \longrightarrow \mathcal{C}(L;\mathscr{D}';\mathcal{R})$ for *any two* tuples of data $\mathscr{D}$, $\mathscr{D}'$. In view of the previous subsections, the right convention to orient the relevant moduli spaces in this case is rather straightforward and we omit the details.

The second method is less general in the sense that it allows to compare the pearl complexes associated only to two tuples $\mathscr{D}$, $\mathscr{D}'$ having the same almost complex structure J and moreover the two Morse functions should be mutually in general position. The resulting chain map $\phi_{\mathscr{D}',\mathscr{D}}$ coincides in homology with the one provided by the general method $\Psi_{\mathscr{D}',\mathscr{D}}$. As we use explicitly in the paper only the second construction we indicate briefly the orientation conventions in that case. Let $\mathscr{D} = (f, (\cdot,\cdot), J)$ and $\mathscr{D}' = (f', (\cdot,\cdot), J)$ with f and f' in general position. The map $\phi_{\mathscr{D}',\mathscr{D}}\colon \mathcal{C}(L;\mathscr{D}) \to \mathcal{C}(L;\mathscr{D}')$ is defined by counting elements in the moduli spaces of the form

$$\Phi(x, y;\mathscr{D}, B;\mathscr{D}', B') = \mathcal{P}^u(x;\mathscr{D}; B) \times_L \mathcal{P}^s(y;\mathscr{D}'; B').$$

The evaluation maps here are the obvious ones. The chain map $\phi_{\mathscr{D}',\mathscr{D}}$ is now defined by

$$\phi_{\mathscr{D}',\mathscr{D}}(x) = \sum_{\substack{y,B,B';\\ |y|=|x|+\mu(B+B')}} \#\Phi(x, y;\mathscr{D}, B;\mathscr{D}', B')\, yT^{B+B'}.$$

By the same type of arguments as above, it is easy to see that this definition provides a chain map that induces an isomorphism in homology and that this definition provides the usual Morse comparison map in the classical case.

A.2.6 Orientation conventions for duality This is a topic that has been discussed in [14, Section 4.4] but only over $\mathbb{Z}_2$ hence in the absence of orientations. We fix a ground ring K (it will be here a field or $\mathbb{Z}$). We now recall some notation from [14] and adapt it to the present setting.

Assume that $\mathcal{R}$ is a commutative $\widetilde{\Lambda}^+$–algebra and suppose that $(\mathcal{C}, \partial)$ is a free $\mathcal{R}$–chain complex (see [14, Section 2.2.1] for the precise definition). Thus $\mathcal{C} = \mathcal{R} \otimes G$ for some graded free K–module G. To the chain complex $(\mathcal{C}, \partial)$ we associate the following two closely related complexes:

(a) $(\mathcal{C}^{\odot}, \partial^*)$; $\mathcal{C}^{\odot} = \hom_{\mathcal{R}}(\mathcal{C}, \mathcal{R})$ with the following grading. For $g \in \mathcal{C}^{\odot}$ define $|g| = k$ if $g(\mathcal{C}_i) \subset \mathcal{R}_{i+k}$. The differential ∂^* is given by

$$\langle \partial^* g, x\rangle = -(-1)^{|g|}\langle g, \partial x\rangle. \tag{109}$$

Clearly, $\mathcal{C}^{\odot}$ is a chain complex and we have an isomorphism of graded modules $\mathcal{C}^{\odot} \cong \mathcal{R} \otimes \hom_K(G, K)$.

(b) $(\mathcal{C}^*, \partial^*)$; $\mathcal{C}^q = \mathcal{C}^{\odot}_{-q}$ and the differential ∂^* coincides with the differential of $\mathcal{C}^{\odot}$ but it has now degree $+1$ so that $\mathcal{C}^*$ is a cochain complex. The cohomology of $\mathcal{C}$ is by definition $H^k(\mathcal{C}) = H^k(\mathcal{C}^*)$. Notice that $\mathcal{C}^* = \mathcal{R}^{\mathrm{inv}} \otimes \hom_K(G, K)^{\mathrm{inv}}$ where for a graded vector space A, A^{inv} is the graded vector space so that for $a \in A^{\mathrm{inv}}$, $|a| = -\deg_A(a)$.

Remark A.2.2 (a) The identification between chain complexes (C_k, d_k) and cochain complexes (C^k, d^k), $C^k = C_{-k}$, $d^k = d_{-k}$ that appears at point b. is standard in homological algebra but we have preferred to make it explicit here by means of the functor $(-)^{\mathrm{inv}}$.

(b) The sign that appears in the definition of ∂^* in formula (109) is the only addition to the notation in [14, Section 4.4] (where we worked over $\mathbb{Z}_2$). This sign appears in other situations in algebraic topology as well. For instance, let $(S_\bullet X, \delta)$ be the standard singular chain complex of a space X. In the definition of the singular cohomology of X the literature contains essentially two variants for the differential: one is the adjoint of δ, without the signs in (109), and the other is given by formula (109). Many authors use the signed formula at least as soon as they deal with products and duality (see for instance Dold [21]). The advantage of this formula is that the Kronecker pairing $S^\bullet(N) \otimes S_\bullet(N) \to \mathbb{Z}$ is a chain map. If X is an oriented manifold and once Poincaré duality is defined by $(-) \cap [X]$, the intersection product is the dual of the cup-product, and both the intersection product and the cup-product verify the respective Leibniz formulas with the usual signs. These are the conventions concerning classical algebraic topology that we also use in this paper.

(c) Clearly, in our situation equation (109) insures that the pairing $\mathcal{C}^{\odot} \otimes_{\mathcal{R}} \mathcal{C} \to \mathcal{R}$ is a chain map (where the differential on $\mathcal{R}$ is trivial).

For a complex $\mathcal{C}$ we denote by $s^n\mathcal{C}$ its n–fold suspension: this coincides with $\mathcal{C}$ but is graded so that the degree of x in $s^n\mathcal{C}$ is $n + deg_{\mathcal{C}}(x)$, in other words, $(s^n\mathcal{C})_k = \mathcal{C}_{k-n}$. The differential on this complex remains the same. In particular, $H_k(s^n\mathcal{C}^{\odot}) \cong H_{k-n}(\mathcal{C}^{\odot}) = H^{n-k}(\mathcal{C}^*)$.

With the conventions above the results stated in [14, Proposition 4.4.1] remain true. Namely, there exists a degree preserving morphism of chain complexes

$$\eta\colon \mathcal{C}(L;\mathscr{D}) \longrightarrow s^n(\mathcal{C}(L;\mathscr{D})^{\odot}) \tag{110}$$

that induces an isomorphism in homology. Thus there is an induced isomorphism

$$QH_k(L;\mathcal{R}) \cong QH^{n-k}(L;\mathcal{R}).$$

The proof of this fact is basically the same as that of [14, Proposition 4.4.1]: η is written as the composition of two morphisms, each of them inducing an isomorphism in homology. The first is the comparison chain morphism between $\mathcal{C}(L; f, (\cdot,\cdot), J)$ and $\mathcal{C}(L; -f, (\cdot,\cdot), J)$. The second is the identification

$$\Theta\colon \mathcal{C}(L; -f, (\cdot,\cdot), J) \cong s^n(\mathcal{C}(L; f, (\cdot,\cdot), J)^{\odot}),$$

where Θ is induced by

$$\mathrm{Crit}(-f) \ni x \longmapsto (x')^* \in \hom_K(K\langle \mathrm{Crit}(f)\rangle, K).$$

Here we have denoted by x' the point $x \in \mathrm{Crit}(-f)$ viewed as critical point of f and by $(x')^*$ the dual of x' with respect to the basis $\{y'\}_{y' \in \mathrm{Crit}(f)}$ of $K\langle \mathrm{Crit}(f)\rangle$. The orientations of the stable and unstable manifolds of $-f$ are related to those for f as follows. First we orient the stable submanifolds of $-f$ by requiring that $W^s_{-f}(x) = W^u_f(x)$. Next, in order to orient the unstable submanifolds of $-f$ we apply to $-f$ the standard orientation conventions. Namely we require that $T_x W^s_{-f}(x) \oplus T_x W^u_{-f}(x) = T_x(L)$ at each $x \in \mathrm{Crit}(-f)$.

With these conventions one obtains the following identities. Let $x, y \in \mathrm{Crit}(-f)$, $B \in H_2^D$ with $\delta(x, y; B) = 0$. Put $\mathscr{D} = (-f, (\cdot,\cdot), J)$ and $\mathscr{D}' = (f, (\cdot,\cdot), J)$ and denote the differentials of the complexes $\mathcal{C}(\mathscr{D})$, $\mathcal{C}(\mathscr{D}')$ by d and d' respectively. Then

$$\begin{aligned} \mathcal{P}(x, y; \mathscr{D}; 0) &= -(-1)^{|x'|}\mathcal{P}(y', x'; \mathscr{D}'; 0), \\ \mathcal{P}(x, y; \mathscr{D}; B) &= (-1)^{|y|+1}\mathcal{P}(y', x'; \mathscr{D}'; B). \end{aligned} \tag{111}$$

This implies the coefficient of y in $d(x)$ satisfies $d(x)|_y = -(-1)^{|x'|}d(y')|_{x'}$. This immediately implies that Θ is a chain map.

Proof of identities (111) The first identity follows easily from the fact that

$$W^u_f(x') = W^s_{-f}(x), \quad W^s_f(x') = (-1)^{|x|(n-|x|)} W^u_{-f}(x). \tag{112}$$

We now turn to the proof of the second identity in (111). We start with a useful identity. Let X, Y be oriented manifolds and $e_X\colon X \to L$, $e_Y\colon Y \to L$ be two smooth maps.

Let $\Psi\colon L\times\mathbb{R}\to\mathbb{R}$ the flow of a (time independent) vector field. Define

$$\Psi_X\colon X\times\mathbb{R}\to L,\quad \Psi_X(x,t)=\Psi(e_X(x),t),$$
$$\overline{\Psi}_Y\colon Y\times\mathbb{R}\to L,\quad \overline{\Psi}_Y(y,t)=\Psi(e_Y(y),-t).$$

Define $\widetilde{\sigma}\colon (X\times\mathbb{R}_+)\times L\times Y\longrightarrow X\times L\times(Y\times\mathbb{R}_+)$ by $\widetilde{\sigma}(x,t,l,y)=(x,\Psi(l,-t),y,t)$. A simple computation shows that $\widetilde{\sigma}$ induces an orientation preserving diffeomorphism

$$\sigma\colon (X\times\mathbb{R}_+)\ {}_{\Psi_X}\!\times_{e_Y} Y\longrightarrow (-1)^{n+|Y|}X\ {}_{e_X}\!\times_{\overline{\Psi}_Y}(Y\times\mathbb{R}_+). \tag{113}$$

We will now apply identity (113) to our purposes. We will take Ψ to be the flow of the vector field $-\nabla(-f)=\nabla f$. Consider the fiber product in (97) which defines $\mathcal{P}(x,y;\mathcal{D};(B_1,\dots,B_k))$. (Note however that $\mathcal{D}$ now involves the function $-f$ rather than f.) For brevity we write $\overrightarrow{B}=(B_1,\dots,B_k)$. Applying identity (113) repeatedly on each of the middle $(k-1)$ factors, starting with $(\mathcal{M}_2(B_{k-1})\times\mathbb{R}_+)\times_L\mathcal{M}_2(B_k,J)$ and moving on to the right we get

$$\begin{aligned}\mathcal{P}(x,y;\mathcal{D};\overrightarrow{B})=(-1)^{k-1}\,W^u(x)\times_L\mathcal{M}_2(B_1,J)\times_L(\mathcal{M}_2(B_2,J)\times\mathbb{R}_+)\times_L\\ \cdots\times_L(\mathcal{M}_2(B_i,J)\times\mathbb{R}_+)\times_L\\ \cdots\times_L(\mathcal{M}_2(B_k,J)\times\mathbb{R}_+)\times_L W^s(y).\end{aligned} \tag{114}$$

The next manipulation is to reverse the order in which we take the fiber products of the $\mathcal{M}_2(B_i)$'s to

$$(\mathcal{M}_2(B_k,J)\times\mathbb{R}_+)\times_L\cdots\times_L(\mathcal{M}_2(B_i,J)\times\mathbb{R}_+)\times_L\cdots\times_L\mathcal{M}_2(B_1,J).$$

A simple computation shows that this has no effect on orientations. Next we also switch the $W^u(x)$ and $W^s(y)$ factors which gives us an additional $(-1)^{|y|n+n+1}$ sign, hence

$$\begin{aligned}\mathcal{P}(x,y;\mathcal{D};\overrightarrow{B})=(-1)^{k+n+|y|n}\,W^s(y)\ {}_i\!\times_{e_{+1}}(\mathcal{M}_2(B_k,J)\times\mathbb{R}_+)\ {}_{\overline{e}'_{-1}}\!\times_{e_{+1}}\\ \cdots{}_{\overline{e}'_{-1}}\!\times_{e_{+1}}(\mathcal{M}_2(B_i,J)\times\mathbb{R}_+)\ {}_{\overline{e}'_{-1}}\!\times_{e_{+1}}\\ \cdots\ {}_{\overline{e}'_{-1}}\!\times_{e_{+1}}\mathcal{M}_2(B_1,J)\ {}_{e_{-1}}\!\times_i W^u(x).\end{aligned} \tag{115}$$

Here, i stands for the inclusions, $\overline{e}'_{-1}(u,t)=\Psi(e_{-1}(u),-t)$ and we have denoted (by abuse of notation) $e_{+1}(u,t)=e_{+1}(u)=u(+1)$.

The marked points for the maps involved in the product (115) are not in the "standard" order. Namely, in order to arrive to $\mathcal{P}(y',x';\mathcal{D};(B_k,\dots,B_1))$ we need to have ${}_i\!\times_{e_{-1}}$ and ${}_{e_{+1}}\!\times_i$ for the first and last products and ${}_{\overline{e}'_{+1}}\!\times_{e_{-1}}$ for the products in the middle, where $\overline{e}'_{+1}(u,t)=\Psi(e_{+1}(u),-t)$. To rectify this, note that the diffeomorphism $\tau\colon\mathcal{M}_2(A)\to\mathcal{M}_2(A)$, defined by $\tau([u,z_1,z_2])=[u,z_2,z_1]$ satisfies $\tau\circ e_{\pm1}=e_{\mp1}$ and that τ is orientation reversing. (Here we are using the first identification in (85) by which e_{+1} is identified with $E_{1,-}$ and e_{-1} with $E_{2,-}$.) Applying the diffeomorphism τ on each of the k middle factors transforms the maps and marked points to the standard order, but adds a $(-1)^k$ sign to the product.

Finally, we replace the first and last factors in the product by $W^u(y')$ and $W^s(x')$ respectively, which by (112) adds another $(-1)^{|x|(n-|x|)}$ to the total sign. Putting everything together we obtain

$$\mathcal{P}(x, y; \mathscr{D}; (B_1, \ldots, B_k)) = (-1)^{\nu} \mathcal{P}(y', x'; \mathscr{D}'; (B_k, \ldots, B_1)),$$

where $\nu = k + n + |y|n + k + |x|(n - |x|) \equiv |x| \bmod 2 \equiv (|y| + 1) \bmod 2$. □

Finally, with the above conventions it is not difficult to verify in the present context the following formula:

$$\langle \mathrm{PD}(h), x \rangle = \epsilon(h * x), \ \forall h \in H_*(M, K), \ x \in QH(L). \tag{116}$$

This formula appeared in [14], as "formula (6)" in the point iii of Theorem A in that paper (which was proved there only over $\mathbb{Z}_2$). More specifically: Here $\langle \, , \rangle$ is the Kronecker product, $- * -$ is the module operation discussed in Section 2.2.2 and ϵ is the augmentation defined in [14]. Recall that for a pearl complex $\mathcal{C}(L; f, (\cdot, \cdot), J)$ where f has a unique minimum, the augmentation ϵ is induced by the map that sends the minimum to $1 \in \mathcal{R}$ (and sends all the other critical points to 0).

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Department of Mathematics, ETH-Zürich
Rämistrasse 101, CH-8092 Zürich, Switzerland

Department of Mathematics and Statistics, University of Montreal
CP 6128, succ. Centre-ville Montréal QC H3C 3J7, Canada

biran@math.ethz.ch, cornea@dms.umontreal.ca

http://www.dms.umontreal.ca/~cornea/

Proposed: Leonid Polterovich
Seconded: Yasha Eliashberg, Simon Donaldson

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