Geometry & Topology 16 (2012) 1171–1203 **1171**

The Dirichlet Problem for constant mean curvature graphs in $\mathbb{M} \times \mathbb{R}$

ABIGAIL FOLHA
HAROLD ROSENBERG

We study graphs of constant mean curvature $H > 0$ in $\mathbb{M} \times \mathbb{R}$ for $\mathbb{M}$ a Hadamard surface, ie a complete simply connected surface with curvature bounded above by a negative constant $-a$. We find necessary and sufficient conditions for the existence of these graphs over bounded domains in $\mathbb{M}$, having prescribed boundary data, possibly infinite.

53A10; 53C42

1 Introduction

Let $\mathbb{M}$ be an Hadamard surface, we study graphs in $\mathbb{M} \times \mathbb{R}$ having constant mean curvature with possible infinite boundary data. We consider domains Ω with piecewise smooth boundary composed of three families $\{A_k\}$, $\{B_l\}$, $\{C_m\}$. We suppose that the curvature of the arcs of these families satisfy $\kappa(A_k) = 2H$, $\kappa(B_l) = -2H$ and $\kappa(C_m) \geq 2H$. We prescribe boundary values $+\infty$ on each A_k, $-\infty$ on each B_l and a continuous function on $\{C_m\}$. The purpose of this paper is to find a smooth function $u\colon \Omega \to \mathbb{R}$, whose graph has constant mean curvature H and boundary data as above. This will be called the Dirichlet Problem (see Definition 2.5).

In Theorems 2.8 and 2.9 we give necessary and sufficient conditions to solve the Dirichlet Problem. They depend on the geometry of the domain. Roughly, they relate the length of the sides $\{A_k\}$, $\{B_l\}$, the curvature H and the length and area of inscribed polygons (see Definition 2.6).

H Jenkins and J Serrin [7] studied this problem for domains contained in $\mathbb{R}^2$ and the curvature $H = 0$. They gave necessary and sufficient conditions for the existence of a solution on this domain in terms of the length of the boundary arcs of the domain and of inscribed polygons. J Spruck [16] worked in domains in $\mathbb{R}^2$ and mean curvature $H > 0$; an important idea introduced in this work was to reflect curves of the family $\{B_l\}$ in order to get convex curves, with respect to the domain. In this work, we give some conditions which assure the existence of curves B_l^* such that the domain bounded by

Published: 23 June 2012 DOI: 10.2140/gt.2012.16.1171

$B_l^* \cup B_l$ is convex, and this will enable us to consider a convex domain, changing B_l by B_l^*; see Section 6. On the other hand, we construct barriers and subsolutions in order to find a graph having constant mean curvature H and finite continuous boundary values over the domain Ω; see Section 4.

Many authors have studied this problem. Consider $H = 0$. H Rosenberg considered the case when $\mathbb{M}$ is the sphere [14]. B Nelli and H Rosenberg [11] worked in the case where $\mathbb{M}$ is hyperbolic space. A Pinheiro [12] obtained a similar result for geodesically convex domains. For hyperbolic space, P Collin and H Rosenberg [1], L Mazet, M Rodríguez and H Rosenberg [10] studied this question for ideal domains. Finally, J Gálvez and H Rosenberg [3] considered $\mathbb{M}$ a surface with negative sectional curvature and solved this problem for unbounded domains. When $H \neq 0$, L Hauswirth, H Rosenberg and J Spruck studied the cases when $\mathbb{M}$ is hyperbolic space and the sphere [5]. A Folha and S Melo generalized this for unbounded domains in hyperbolic space [2].

The paper is organized as follows. The main Theorems are stated in Section 2. In Section 3 we give some definitions and preliminary results, we construct the barriers necessary to assure the continuity of a solution at the boundary. In Section 4 we give conditions for existence of a solution with continuous bounded boundary values in a domain $\Omega \subset \mathbb{M}$. We study Flux Formulas in Section 5, which are the necessary conditions for the existence of a solution to the Dirichlet Problem. In Section 6 we discuss the existence of a curve B^* as in the definition of admissible domain; see Definition 2.2. We prove the existence of an embedded arc joining any two points p, q of $\mathbb{M}$, $p \neq q$, of constant curvature κ, $0 < \kappa < \sqrt{a}$. Finally, in Section 7 we prove Theorems 2.8 and 2.9.

We would like to thank the referee for making many useful suggestions.

2 Statements of results

We consider a simply connected domain $\Omega \in \mathbb{M}$, $\mathbb{M}$ is a Hadamard surface. We will give necessary and sufficient conditions for the existence of constant mean curvature graphs in $\Omega \times \mathbb{R}$ with possible infinite boundary values. These conditions depend on the geometry of the domain Ω, roughly, they involve the area of Ω and the length of its boundary. We give some definitions in order to state the theorems.

Given a function $u\colon \Omega \to \mathbb{R}$ the graph of u, $S = \{(p, u(p));\ p \in \mathbb{M}\}$, has constant mean curvature H with respect to the normal pointing up to S if u satisfies the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=2H, \tag{1}$$

where the divergence and gradient are calculated with respect to the metric of $\mathbb{M}$. If u satisfies this equation in a domain Ω, u is called a solution of (1) in Ω.

Definition 2.1 Let Ω be a simply connected bounded domain in $\mathbb{M}$ and $h\colon \Omega\to\mathbb{R}$ a smooth function.

(1) The function h is a subsolution in Ω of (1) if

$$\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)\geq 2H.$$

(2) The function h is a supersolution in Ω of (1) if

$$\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)\leq 2H.$$

Definition 2.2 (Admissible domain) A bounded domain Ω is an admissible domain if it is simply connected and $\partial\Omega$ consists of three sets of $C^{2,\alpha}$ open arcs $\{A_k\}$, $\{B_l\}$, $\{C_m\}$ and its end points, satisfying $\kappa(A_k)=2H$; $\kappa(B_l)=-2H$ and $\kappa(C_m)\geq 2H$, respectively, (with respect to the interior of Ω). We suppose that no two of the arcs A_k and no two of the arcs B_l have a common endpoint. In addition if the family $\{B_l\}$ is nonempty, we assume that there exists a simply connected domain Ω^* whose boundary is formed by replacing each arc B_l by B_l^*, where B_l^* is a C^2 arc joining the end points of B_l having $\kappa(B_l^*)=2H$ with respect to Ω^*. In addition, we suppose that in Ω (or Ω^* if $\{B_l\}\neq\varnothing$) there is a bounded subsolution of (1).

Remark 2.3 Proposition 3.2 gives the existence of a bounded subsolution of (1) for H small enough (H small in terms of the negative upper bound for the curvature of $\mathbb{M}$).

Remark 2.4 In Section 6 we will give conditions which assure the existence of these curves B_l^*.

Definition 2.5 (Dirichlet Problem) Given an admissible domain Ω, the Dirichlet Problem is to find a solution of (1) in Ω which assumes values $+\infty$ on each A_k, $-\infty$ on each B_l and assigned continuous data on each of the arcs C_m.

Definition 2.6 (Admissible polygon) Let Ω be an admissible domain. We say that $\mathcal{P}$ is an admissible polygon if $\mathcal{P}$ is piecewise smooth consisting of arcs of constant curvature $\kappa = \pm 2H$, these arcs are contained in Ω or in the boundary $\partial\Omega$ and its vertices are among the endpoints of the families $\{A_k\}$, $\{B_l\}$ and $\{C_m\}$.

Definition 2.7 Given an admissible polygon, let

$$\alpha(\mathcal{P}) = \sum_{A_k \in \mathcal{P}} |A_k|, \quad \beta(\mathcal{P}) = \sum_{B_l \in \mathcal{P}} |B_l|,$$

and let $l(\mathcal{P})$ be the perimeter of $\mathcal{P}$, where $|L|$ denotes the length of the curve L. We denote $\Omega_{\mathcal{P}}$ the admissible domain bounded by $\mathcal{P}$.

In the same spirit as H Jenkins and J Serrin [7] we obtain the following theorems.

Theorem 2.8 *Let Ω be an admissible domain, with the family $\{C_m\}$ nonempty. There is a solution to the Dirichlet Problem, if and only if*

$$2\alpha(\mathcal{P}) < l(\mathcal{P}) + 2H\mathcal{A}(\Omega_{\mathcal{P}}), \tag{2}$$

$$2\beta(\mathcal{P}) < l(\mathcal{P}) - 2H\mathcal{A}(\Omega_{\mathcal{P}}), \tag{3}$$

for all admissible polygons $\mathcal{P}$.

If the family $\{C_m\}$ is empty we have the following theorem.

Theorem 2.9 *Let Ω be an admissible domain with the family $\{C_m\}$ empty. There is a solution to the Dirichlet problem, if and only if*

$$\alpha(\partial\Omega) = \beta(\partial\Omega) + 2H\mathcal{A}(\Omega), \tag{4}$$

and for every inscribed polygon $\mathcal{P} \neq \partial\Omega$,

$$2\alpha(\mathcal{P}) < l(\mathcal{P}) + 2H\mathcal{A}(\Omega_{\mathcal{P}}), \tag{5}$$

$$2\beta(\mathcal{P}) < l(\mathcal{P}) - 2H\mathcal{A}(\Omega_{\mathcal{P}}). \tag{6}$$

3 Local barriers

Let $\gamma(t)$ be a complete geodesic in $\mathbb{M}$ with $\langle \gamma'(t), \gamma'(t) \rangle = 1$, $t \in \mathbb{R}$. Then

$$\varphi(s,t) = \exp_{\gamma(t)}(sJ(\gamma'(t))), \quad (s,t) \in \mathbb{R}^2,$$

is a parametrization of $\mathbb{M}$. Where J denotes the standard rotation of $\pi/2$, such that $\{v, Jv\}$, $v \in T_p\mathbb{M}$, is a positive base of $T_p\mathbb{M}$. We note that

$$\langle \partial_s, \partial_s \rangle = 1, \quad \langle \partial_s, \partial_t \rangle = 0 \quad \text{and} \quad \langle \partial_t, \partial_t \rangle = G(s,t).$$

Moreover, since γ is a geodesic and $|\gamma'(t)| = 1$ for all $t \in \mathbb{R}$, we have

$$G(0,t) = 1 \quad \text{and} \quad G_s(0,t) = 0, \tag{7}$$

where G_s is the derivative of G with respect to s.

In this case the induced metric by φ in $\mathbb{M}$, is

$$ds^2 + G(s,t)\, dt^2.$$

We will deal with graphs over simply connected bounded domains, then in this work, Ω denotes a simply connected bounded domain contained in $\mathbb{M}$.

Remark 3.1 Ω is not assumed to be an admissible domain.

Let Ω be a simply connected bounded domain in $\mathbb{M}$. Since Ω is a bounded domain, we can suppose that $\Omega \subset \{(s,t) \in \mathbb{M}\,; s > 0\}$, where we identify $(s,t) \in \mathbb{R}^2$ with $\varphi(s,t) \in \mathbb{M}$. We will consider functions h: $\Omega \to \mathbb{R}$ which do not depend on the parameter t, that is, $h(s,t) = h(s)$. In this case h is a solution in Ω of (1) if

$$\begin{aligned} 2H = \operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) &= \operatorname{div}\left(\frac{h_s \partial_s}{\sqrt{1+h_s^2}}\right) \\ &= \left(\frac{h_s}{\sqrt{1+h_s^2}}\right)_s + \frac{G_s}{2G}\frac{h_s}{\sqrt{1+h_s^2}} \\ &= \frac{h_{ss}(1+h_s^2) - h_{ss}h_s^2}{(1+h_s^2)^{3/2}} + \frac{G_s}{2G}\frac{h_s(1+h_s^2)}{(1+h_s^2)^{3/2}} \\ &= \frac{G_s h_s(1+h_s^2) + 2G h_{ss}}{2G(1+h_s^2)^{3/2}}, \end{aligned}$$

where h_s denotes the derivative of h with respect to s. In particular, h is a subsolution of (1) in Ω if

$$\frac{G_s h_s(1+h_s^2) + 2G h_{ss}}{2G(1+h_s^2)^{3/2}} \geq 2H. \tag{8}$$

If $h_s > 0$, (8) is equivalent to

$$\frac{G_s}{G} \geq \frac{4H(1+h_s^2)^{3/2} - 2h_{ss}}{h_s(1+h_s^2)}. \tag{9}$$

We recall we are supposing that $\mathbb{M}$ has curvature bounded above by $-a,\ a>0$.

Proposition 3.2 Let $\Omega \subset \{(s,t); s>0\}$ be a simply connected bounded domain. For $H \leq \sqrt{a}/2$, there is a subsolution of (1) in Ω.

Remark 3.3 The existence of bounded subsolutions and supersolutions is necessary to show the existence of solutions in a given domain $\Omega \in \mathbb{M}$. One can prove that for each given domain Ω there is a constant $d>0$, big enough, such that there are no graphs over Ω having constant mean curvature $H \geq d$.

We will prove a lemma and then we prove the proposition above.

For $a>0$, we define $\mathbb{H}(-a)=\{(s,t)\in\mathbb{R}^2\}$ with the metric $ds^2+\cosh^2(\sqrt{a}s)\,dt^2$ the hyperbolic space having curvature $-a$.

A similar lemma was proved in [3] for $H=0$.

Lemma 3.4 *Let $\Omega \subset \{(s,t)\in\mathbb{M}; s>0\}$ be a simply connected bounded domain and let $h(s,t)=h(s)$ be a smooth function defined for $s>0$. Suppose $h_s>0$. If h satisfies*

$$\frac{4H(1+h_s^2)^{3/2}-2h_{ss}}{h_s(1+h_s^2)} \leq 2\sqrt{a}\tanh(\sqrt{a}s), \tag{10}$$

then h is a subsolution in Ω of (1).

Proof The Gaussian curvature of Ω is given by

$$K(s,t) = -\frac{1}{4}\left(\frac{G_s}{G}\right)^2 - \frac{1}{2}\left(\frac{G_s}{G}\right)_s \leq -a, \quad (s,t)\in\Omega, \tag{11}$$

since the Gaussian curvature of Ω is bounded above by $-a$.

Using the fact that, if a real function satisfies the equation

$$\left(\frac{G_s}{G}\right)^2 + 2\left(\frac{G_s}{G}\right)_s \geq \left(\frac{\widetilde{G}_s}{\widetilde{G}}\right)^2 + 2\left(\frac{\widetilde{G}_s}{\widetilde{G}}\right)_s,$$

$$\frac{G_s}{G}(s_0) = \frac{\widetilde{G}_s}{\widetilde{G}}(s_0),$$

then

$$\frac{G_s}{G} \geq \frac{\widetilde{G}_s}{\widetilde{G}} \quad \forall\ s>s_0,$$

we conclude by Equations (11) and (7) taking $\widetilde{G}(s)=\cosh^2(\sqrt{a}s)$ that

$$\frac{G_s}{G} \geq \frac{\widetilde{G}_s}{\widetilde{G}} = 2\sqrt{a}\tanh(\sqrt{a}s) \quad \forall\ s>0.$$

Taking $\widetilde{G}(s)=\cosh^2(\sqrt{a}s)$ means that we compare the Gaussian Curvature of $\mathbb{M}$ with that of $\mathbb{H}(-a)$. So if

$$\frac{4H(1+h_s^2)^{3/2}-2h_{ss}}{h_s(1+h_s^2)}\le 2\sqrt{a}\tanh(\sqrt{a}s),$$

then h is a subsolution in Ω of (1). □

Remark 3.5 We just need suppose that $\Omega\subset\{(s,t);0<s<s_0 \text{ and } t_0<t<t_1\}$, and the Gaussian curvature of $\{(s,t);0<s<s_0\ \ t_0<t<t_1\}$ is bounded above by $-a<0$.

We will prove Proposition 3.2 giving an explicit example; see Spruck [17, Example 1.8].

Proof of Proposition 3.2 We will show that there is a function $h(s,t)=h(s)$ which is a solution of (1) in $\mathbb{H}(-a)$. For $\widetilde{G}=\cosh^2(\sqrt{a}s)$, if

$$\begin{aligned}\frac{\widetilde{G}_s h_s(1+h_s^2)+2\widetilde{G}h_{ss}}{2\widetilde{G}(1+h_s^2)^{3/2}}=2H &\Leftrightarrow \frac{1}{\cosh(\sqrt{a}s)}\left(\frac{h_s\cosh(\sqrt{a}s)}{(1+h_s^2)^{1/2}}\right)_s=2H\\ &\Leftrightarrow \frac{h_s\cosh(\sqrt{a}s)}{(1+h_s^2)^{1/2}}=\frac{2H}{\sqrt{a}}\sinh(\sqrt{a}s)+A,\end{aligned}$$

so $H=\sqrt{a}/2$, we have that $h=(1/\sqrt{a})\cosh(\sqrt{a}s)$, is a solution (for $A=0$) of (1) for these equations. Then by Lemma 3.4, $h=(1/\sqrt{-a})\cosh(\sqrt{-a}s)$ is a subsolution of (1) in Ω. □

3.1 The distance function

Let Ω be a simply connected bounded domain with an oriented boundary. Let d be the distance function (with sign) to $\partial\Omega$. Fix a point p in the interior of an arc $E\subset\partial\Omega$. Let E' a neighborhood of p in E, such that E' is contained in the interior of E. Let Ω_0 be the largest open set of points $q\in\Omega$ which have a unique closest point $s\in E'$. In [9, Theorem 1] Y Li and L Nirenberg proved if E' is $C^{k,\alpha}$ then the distance function is in $C^{k-1,\alpha}(\Omega_0\cup\partial\Omega)$, for $k\ge 2,\ 0<\alpha<1$.

Letting $w=h(d)\colon\Omega_0\to\mathbb{R}$, where $h\colon\mathbb{R}\to\mathbb{R}$ is a smooth function, we have

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla h(d)}{\sqrt{1+|\nabla h(d)|^2}}\right)&=\operatorname{div}\left(\frac{h'}{\sqrt{1+(h')^2}}\nabla d\right)\\ &=\left\langle\nabla\left(\frac{h'}{\sqrt{1+(h')^2}}\right),\nabla d\right\rangle+\frac{h'}{\sqrt{1+(h')^2}}\Delta d\\ &=\left\langle\frac{h''}{(1+(h')^2)^{3/2}}\nabla d,\nabla d\right\rangle+\frac{h'}{\sqrt{1+(h')^2}}\Delta d,\end{aligned}$$

and then

$$\operatorname{div}\left(\frac{\nabla h(d)}{\sqrt{1+|\nabla h(d)|^2}}\right)=\frac{h''}{(1+(h')^2)^{3/2}}-\frac{h'}{\sqrt{1+(h')^2}}\mathcal{H}(x), \tag{12}$$

where $\mathcal{H}(x)$ is the curvature of the level set of d passing through $x\in\Omega_0$.

We will construct some local barriers. The function used to make the first barrier appears in Spruck [17] under other hypothesis and the conclusion of the author is different from our conclusion and application. The second one is in [5], where L Hauswirth, H Rosenberg and J Spruck made this barrier for another space, but the same holds for our case.

Lemma 3.6 *Let Γ be a $C^{2,\alpha}$ arc in $\partial\Omega$, with $\kappa(p)\geq 2H$ for $p\in\Gamma$ and let $q\in\Gamma$. There is a neighborhood $\Delta\subset\Omega$ of q, with $\partial\Delta=\Gamma'\cup\eta$ (where $\Gamma'\subset\Gamma$ and η is an arc contained in Ω), such that there is a subsolution w of (1) in $\Delta\cup\partial\Delta$ with $w(q)=0$, $w(p)<0$ for $p\in(\Delta\cup\partial\Delta)-q$ and $w|_\eta=-M$, for $M>0$ sufficiently large.*

Proof Let γ be a $C^{2,\alpha}$ arc of curvature $\kappa(\gamma)\geq 2H-\epsilon,\epsilon>0$ small, such that γ is tangent to Γ at q, $\gamma\subset(\mathbb{M}-\Omega)$ and the curvature vector of γ at q has the same direction as that of Γ at q; see Figure 1. Let d be distance function to γ,

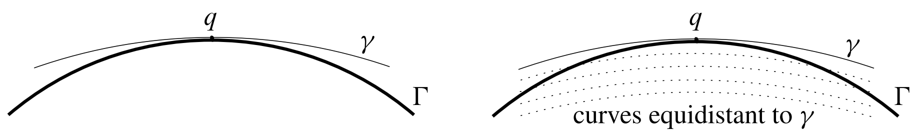

Figure 1

defined on the convex side of γ in a neighborhood of q. We know that d is $C^{1,\alpha}$ in a set $\{p\in\mathbb{M}; 0\leq d(p)\leq\delta_1\}$ for some $\delta_1>0$. Let $0<\delta_2\leq\delta_1$ be sufficiently small so that if $0\leq d(p)<\delta_2$ then $\mathcal{H}(p)\geq 2H-2\epsilon$. We consider the function $h(r)=(e^{AC}/(2C))(e^{-2Cr}-1)$, $r\in[0,\delta]$ and $0<\delta<\delta_2$, where $C>1/A,\ A>\delta/2$ will be chosen later. By Equation (12), for $w=h(d)$ we have

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right)&=\frac{2Ce^{AC-2Cd}}{(1+e^{2(AC-2Cd)})^{3/2}}+\frac{e^{AC-2Cd}}{(1+e^{2(AC-2Cd)})^{1/2}}\mathcal{H}(x)\\&=\frac{e^{AC-2Cd}}{(1+e^{2(AC-2Cd)})^{3/2}}\big(2C+\mathcal{H}(x)(1+e^{2(AC-2Cd)})\big)\\&\geq\frac{e^{AC-2Cd}}{(1+e^{(AC-2Cd)})^{3}}\big(2C+\mathcal{H}(x)(1+e^{2(AC-2Cd)})\big).\end{aligned}$$

Since $e^{(AC-2Cd)} \geq e^{(AC-2C\delta)}$ and $e^{(AC-2Cd)} \leq e^{AC}$, we obtain

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) &\geq \frac{e^{AC-2C\delta}}{(1+e^{AC})^3}\big(2C+\mathcal{H}(x)(1+e^{2(AC-2C\delta)})\big)\\ &\geq \frac{e^{AC-2C\delta}}{8e^{3AC}}\big(2C+\mathcal{H}(x)(1+e^{2(AC-2C\delta)})\big)\\ &\geq \frac{e^{AC-2C\delta}}{8e^{3AC}}\big(2C+(2H-2\epsilon)(1+e^{2(AC-2C\delta)})\big).\end{aligned}$$

We observe that the function

$$F(A,\delta)=\frac{e^{AC-2C\delta}}{8e^{3AC}}\big(2C+(2H-2\epsilon)(1+e^{2(AC-2C\delta)})\big)$$

for $C>6H+6\epsilon$ satisfies $F(0,0)=\frac{1}{8}(2C+(2H-2\epsilon)2)>2H$. Then, for (A,δ) sufficiently small $F(A,\delta)\geq 2H$. So, choosing A and δ small, we have

$$\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right)\geq 2H,$$

that is, w is a subsolution.

Let Δ be the simply connected domain (decreasing δ if necessary) whose boundary is composed of a subarc of Γ which contains q and a subarc of the level curve $\{x\in\Omega; d(x)=\delta\}$; see Figure 2.

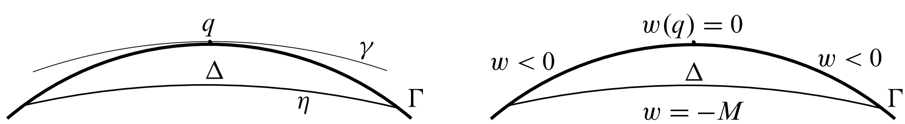

Figure 2

Let $C_0=\max\{1/A,6H+6\epsilon\}$. For

$$M>-\frac{e^{AC_0}}{2C_0}(e^{-2C_0\delta}-1),$$

we can choose $C\geq C_0$ such that

$$h(\delta)=\frac{e^{AC}}{2C}(e^{-2C\delta}-1)=-M. \qquad \square$$

Remark 3.7 Let $\Omega \subset \mathbb{M}$ be a domain and $\Gamma \subset \partial\Omega$ be a $C^{2,\alpha}$ arc with $\kappa(\Gamma) \geq 2H$. Fix a point $q \in \Gamma$, and choose a small compact arc $\Gamma' \subset \Gamma$ containing q and an arc η of a geodesic circle joining the end points of Γ', such that the domain Δ bounded by $\eta \cup \Gamma'$ is contained in Ω. Then, if $f\colon \partial\Delta \to \mathbb{R}$ is a continuous function with $f(q) = 0$ and $f(p) > 0$ for $p \in \partial\Delta - \{q\}$ there is $u \in C^2(\Delta) \cap C^0(\Delta \cup \partial\Delta)$, with

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 < 2H$$

and $u|_{\partial\Delta} = f$. In particular, there is a supersolution w of (1) in Δ with $w(q) = 0$, $w(p) > 0$ for $p \in \Delta \cup \partial\Delta$ and $w|_\eta = M, M > 0$. See Pinheiro [12, Theorem 1.2].

Lemma 3.8 *Let Ω be a domain.*

(i) *If η is a $C^{2,\mu}$ arc of $\partial\Omega$, with $0 \leq \kappa(p) \leq 2\alpha < 2H$, for $p \in \eta$, then for any interior point $p \in \eta$, there is a neighborhood Δ of p in $\Omega \cup \eta$ and a supersolution w^+ of (1) such that*

$$\frac{\partial w^+}{\partial \nu}(q) = +\infty$$

for $q \in \eta \cap \partial\Delta$, where ν is the outer conormal.

(ii) *If η is a $C^{2,\mu}$ arc of $\partial\Omega$, with $\kappa(p) \leq -2\alpha < -2H$ for $p \in \eta$, then there exists a subsolution w^- of (1) with*

$$\frac{\partial w^-}{\partial \nu}(q) = -\infty$$

for $q \in \eta \cap \partial\Delta$, where ν is the outer conormal.

Proof We consider $h(r) = -\sqrt{2r/\epsilon}$. Let d be the distance function to η, since distance function is continuous, we can conclude for $d(x) < \delta,\ \delta > 0$ small that:

(i) For $w^+ = h(d)$, by Equation (12),

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) &= \frac{\epsilon}{(2\epsilon d+1)^{3/2}} + \frac{\mathcal{H}(x)}{(2\epsilon d+1)^{1/2}} \\ &\leq \frac{1}{(2\epsilon d+1)^{1/2}}\left(\frac{\epsilon}{(2\epsilon d+1)} + 2\alpha + \epsilon\right) \\ &\leq \frac{1}{(2\epsilon d+1)^{1/2}}(2\alpha + 2\epsilon) < 2H,\end{aligned}$$

for ϵ, d small enough, which proves (i); see Figure 3.

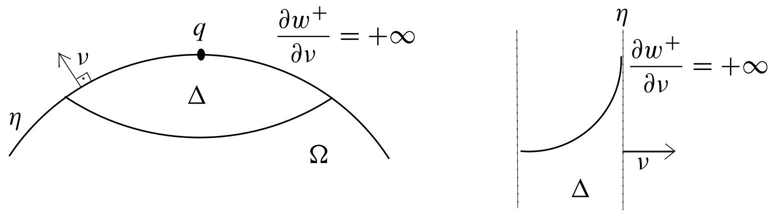

Figure 3

(ii) For $w^- = -h(d)$, by Equation (12),

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) &= -\frac{\epsilon}{(2\epsilon d+1)^{3/2}} - \frac{\mathcal{H}(x)}{(2\epsilon d+1)^{1/2}}\\ &\geq \frac{1}{(2\epsilon d+1)^{1/2}}\left(-\frac{\epsilon}{(2\epsilon d+1)} + 2\alpha - \epsilon\right)\\ &\geq \frac{1}{(2\epsilon d+1)^{1/2}}(2\alpha - 2\epsilon) > 2H,\end{aligned}$$

for $\epsilon > 0, d > 0$ small enough, which proves (ii); see Figure 4. □

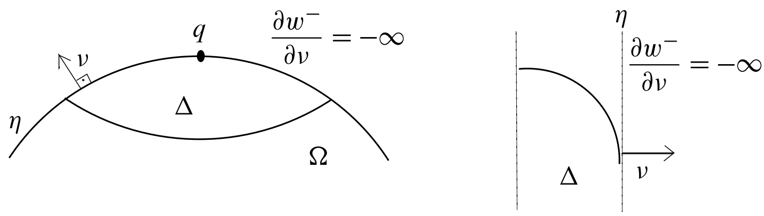

Figure 4

3.2 Maximum principles

We state a maximum principle, which enables us to compare subsolutions, solutions and supersolutions of (1). The proof is based on ideas of L Hauswirth, H Rosenberg and J Spruck [5].

Theorem 3.9 (Maximum Principle) *Let Ω be a simply connected bounded domain in $\mathbb{M}$, with piecewise C^2 boundary. Let u_1 and u_2 be functions defined in Ω which satisfy*

$$\operatorname{div}\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}}\right) \geq 2H \geq \operatorname{div}\left(\frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right)$$

in Ω. Suppose that $\liminf(u_2 - u_1) \geq 0$ for any approach to $\partial\Omega$ with the possible exception of a finite number of points $E = \{P_i\,;\ i = 1, \dots, n\} \subset \partial\Omega$. Then $u_2 \geq u_1$ on $(\Omega \cup \partial\Omega) - E$ with strict inequality unless $u_2 \equiv u_1$.

Proof Let M, ϵ be positive constants, with M large and ϵ small. We define

$$\varphi = \begin{cases} M-\epsilon & \text{if } u_1 - u_2 \geq M, \\ u_1 - u_2 - \epsilon & \text{if } \epsilon \leq u_1 - u_2 \leq M, \\ 0 & \text{if } u_1 - u_2 \leq \epsilon. \end{cases}$$

We have that $0 \leq \varphi \leq M$, φ is Lipschitz, $\nabla\varphi = \nabla u_1 - \nabla u_2$ in the set where $\epsilon < u_1 - u_2 < M$ and $\nabla\varphi = 0$ in the set where $u_1 - u_2 > M$ or $u_1 - u_2 < \epsilon$.

For each i we consider a closed geodesic ball $B_i(\epsilon)$ centered at P_i of radius $\epsilon > 0$ small. For each ϵ, let $\Omega_\epsilon = \Omega - (\bigcup_i B_i(\epsilon))$, $i = 1, \dots, n$, we denote $\partial\Omega_\epsilon = \Gamma_\epsilon \cup \Lambda_\epsilon$, where $\Gamma_\epsilon = \partial\Omega - (\bigcup_i B_i(\epsilon))$ and $\Lambda_\epsilon = \bigcup_i (\partial B_i(\epsilon) \cap \Omega)$.

We have

$$\operatorname{div}\left(\varphi\left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right)\right) = \left\langle \nabla\varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\rangle + \varphi\left(\operatorname{div}\left(\frac{\nabla u_1}{W_1}\right) - \operatorname{div}\left(\frac{\nabla u_2}{W_2}\right)\right),$$

where $W_j = \sqrt{1 + |\nabla u_j|^2}$, $j = 1, 2$. Applying the divergence theorem and the hypothesis $\varphi \equiv 0$ on Γ_ϵ, we have

$$\begin{aligned} &\int_{\Omega_\epsilon} \left\langle \nabla\varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\rangle dV + \int_{\Omega_\epsilon} \varphi\left(\operatorname{div}\left(\frac{\nabla u_1}{W_1}\right) - \operatorname{div}\left(\frac{\nabla u_2}{W_2}\right)\right) dV \\ &= \int_{\Omega_\epsilon} \operatorname{div}\left(\varphi\left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right)\right) dV \\ &= \int_{\partial\Omega_\epsilon} \varphi\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle dV \\ &= \int_{\Lambda_\epsilon} \varphi\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle dV + \int_{\Gamma_\epsilon} \varphi\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle dV \\ &= \int_{\Lambda_\epsilon} \varphi\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle dV, \end{aligned}$$

where ν is the outer conormal to the $\partial\Omega_\epsilon$.

The term in the last equality is bounded above by

$$2M \sum_{i=1}^{n} l(\partial B_i(\epsilon)),$$

where $l(\partial B)$ is the length of B on $\mathbb{M}$. The second term in the first line is nonnegative by hypothesis. The first term in the first line does not vanish except if $\nabla\varphi = \nabla u_1 - \nabla u_2$, and the previous lemma assures that this term is nonnegative. From this we have

$$0 \le \int_{\Omega_\epsilon} \left\langle \nabla\varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\rangle dV \le 2M \sum_{i=1}^{n} l(\partial B_i(\epsilon)).$$

When ϵ goes to zero, Ω_ϵ goes to Ω and $2M\sum_{i=1}^{n} Vol(B_i(\epsilon))$ goes to 0. So the conclusion is that $\nabla u_1 = \nabla u_2$ whenever $u_1 > u_2$. This implies that $u_1 = u_2 + a,\ a > 0$ in each open component of $\{u_1 > u_2\}$. If there is any such nonempty component of this set, then the maximum principle ensures $u_1 = u_2 + a, a > 0$ in Ω, but this contradicts the hypothesis $\liminf(u_2 - u_1) \ge 0$. □

Lemma 3.10 *Let Δ be a domain whose boundary $\partial\Delta = \Gamma' \cup \eta$, Γ', η are closed sets and Γ' is a $C^{2,\alpha}$ arc. Let u and w be functions defined in Δ, $u \in C^2(\Delta) \cap C^1(\Delta \cup \eta)$ and $w \in C^2(\Delta) \cap C^0(\Delta \cup \partial\Delta)$.*

(i) *Suppose*
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \ge \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right)$$
in Δ and
$$\frac{\partial w}{\partial \nu} = +\infty$$
in η, ν the outer conormal to Δ in η. Then, if $\liminf(w-u) \ge 0$ *for any approach to Γ', then $w \ge u$ in Δ.*

(ii) *Suppose*
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \le \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right)$$
in Δ and
$$\frac{\partial w}{\partial \nu} = -\infty$$
in η, ν the outer conormal to Δ in η. Then if $\liminf(w-u) \le 0$ *for any approach to Γ', then $w \le u$ in Δ.*

Proof (i) If $\liminf(w-u) \ge 0$ in η then the conclusion follows from the maximum principle. If this is not the case we consider $v = w + M$, so that $v > u$ in $\Delta \cup \partial\Delta$, now we translate v down until the first contact point, which is a point of η and the conclusion is that $\partial u/\partial\nu \ge +\infty$; see Figure 5(left). This gives us a contradiction.

(ii) Similarly, if $\liminf(w-u) \le 0$ in η then the conclusion follows from the maximum principle. If this is not the case we consider $v = w - M$, so that $v < u$ in $\Delta \cup \partial\Delta$, now we translate v up until the first contact point, which is a point of η and the conclusion is that $\partial u/\partial\nu \le -\infty$; see Figure 5(right). This gives us a contradiction. □

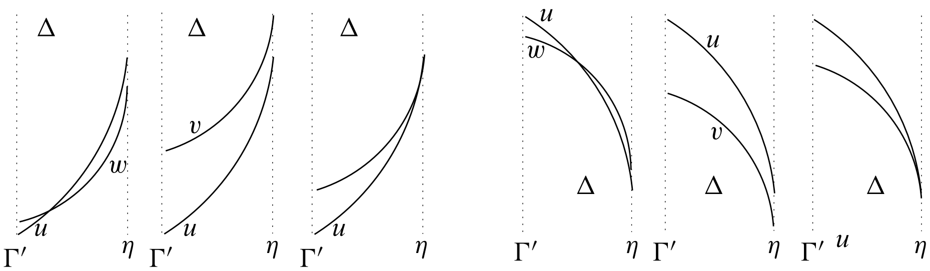

Figure 5

The proof of the following lemma uses Lemma 3.10 and the barriers of Lemma 3.8.

Lemma 3.11 *Let u be a solution of* (1) *in a bounded domain $\Omega \subset \mathbb{M}$ and let $\Gamma \subset \partial\Omega$ be a compact arc. Suppose $m < u < M$ on Γ. Then there is a constant c which depends only on Ω such that for any compact $C^{2,\alpha}$ subarc $\Gamma' \subset \Gamma$,*

(i) *if $\kappa(\Gamma') > -2H$, there is a neighborhood Δ of Γ' in D such that $u \le M + c$ in Δ,*

(ii) *if $\kappa(\Gamma') \ge 2H$ with strict inequality except for isolated points, there is a neighborhood Δ of Γ' in D such that $u \ge m - c$ in Δ.*

Proof (i) Let Γ' be an interior arc of Γ sufficiently small so that the arc η joining its end points satisfies $0 < \kappa(\eta) < 2H$, with respect to the domain Δ bounded by $\Gamma' \cup \eta$. We can assume in Δ there is the supersolution w^+ given by Lemma 3.8 so that $\partial w^+/\partial \nu = +\infty$ on η. Lemma 3.10(i) [for w^+ replacing w] implies that $u \le M - \inf_\Delta\{w^+\}$.

(ii) We take a point p where $\kappa(p) > 2H$, let $\Gamma' \subset \Gamma$ be an arc which contains p. For Γ' small enough there is a curve η joining the end points of Γ' such that $\kappa(\eta) < -2H$ with respect to the domain Δ bounded by Γ' and η. We can suppose Δ sufficiently small such that there is a subsolution w^- given by Lemma 3.8 such that $\partial w^-/\partial\nu = -\infty$ on η. Lemma 3.10(ii) [for w^- replacing w] ensures that $u \ge m - \sup_\Delta w^-$ in Δ. □

4 Existence and uniqueness theorem

We state some results found in Spruck [17] and using the barriers in the previous section and the Perron Method (see Gilbarg and Trudinger [4]) we give the existence of solutions of (1) in simply connected bounded domains with piecewise $C^{2,\alpha}$ boundary.

Theorem 4.1 (Spruck [17]) *Let Ω be a bounded domain with C^2 boundary and $\kappa(\partial\Omega) \ge 2H + \epsilon$, for some $\epsilon > 0$. Then given a continuous function $f\colon \partial\Omega \to \mathbb{R}$, there is a unique solution of* (1) *$u \in (C^2(\Omega) \cap (C^0(\Omega \cup \partial\Omega)))$ in Ω with $u|_{\partial\Omega} = f$.*

Theorem 4.2 (Compactness Theorem) *Let* $\{u_n\}$ *be a uniformly bounded sequence of solutions of* (1) *in a bounded domain* Ω. *Then there is a subsequence which converges uniformly on compact subsets (in the* C^k *topology, for any* k *) to a solution of* (1) *in* Ω.

Theorem 4.2 follows from gradient estimates for a solution of (1) at an interior point $p\in\Omega$; see [17, Theorem 1.1]. The interior gradient estimates enable us to apply the Arzelá–Ascoli Theorem and obtain a convergent subsequence on compact subsets in Ω; Schauder theory guarantees that, in fact, the limit of this subsequence is smooth.

We will show that the condition $\kappa(\partial\Omega)\geq 2H+\epsilon$ can be removed. First, we need to give some definitions.

Let Ω be a domain and $f\colon\partial\Omega\to\mathbb{R}$ be a continuous function, we consider the set $\mathcal{S}_f=\{v\in C^2(\Omega)\cap C^0(\Omega\cup\partial\Omega); v$ is a subsolution of (1) and $v|_{\partial\Omega}\leq f\}$. Let $B\subset\Omega$ be a compact domain having smooth boundary such that $\kappa(\partial B)>2H$ (which is equivalent to $\kappa(\partial B)\geq 2H+\epsilon$, for some $\epsilon>0$, since B has compact boundary). For each $v\in\mathcal{S}_f$ we define the lifting of v in B as

$$V(p)=\begin{cases}\overline{v}(p) & \text{if } p\in B,\\ v(p) & \text{if } p\in\Omega-B.\end{cases}$$

where $\overline{v}$ is the solution of Equation (1) in B given by Theorem 4.1 with boundary values $\overline{v}|_{\partial B}=v|_{\partial B}$.

Now we are able to prove an existence theorem for domains having $C^{2,\alpha}$ boundary and curvature big enough.

Theorem 4.3 *Let* Ω *be a bounded domain with* $C^{2,\alpha}$ *boundary and* $\kappa(\partial\Omega)\geq 2H$. *Suppose that there is a bounded subsolution of* (1) *in* Ω. *Then given a continuous function* $f\colon\partial\Omega\to\mathbb{R}$, *there is a unique solution of* (1) $u\in(C^2(\Omega)\cap(C^0(\Omega\cup\partial\Omega)))$ *in* Ω *with* $u|_{\partial\Omega}=f$.

Proof We observe that there is a minimal surface in Ω which assumes boundary values f; see Pinheiro [12, Theorem 1.2]. Note that this minimal surface is a supersolution of (1) in Ω. Using the Perron Method we will show that there is a solution of (1) in Ω and using the barriers of Lemma 3.6 we will show that this solution has the prescribed boundary values.

We define

$$u=\sup_{\mathcal{S}_f} v.$$

The set $\mathcal{S}_f$ is nonempty and the existence of one supersolution and the maximum principle guarantees that u is well defined. We show that u is in fact a solution in Ω.

Let $p \in \Omega$, let $B = B_p(\epsilon)$ be a geodesic ball centered at p having radius ϵ, we choose ϵ small enough such that $B \cup \partial B$ is contained in Ω and $\kappa(\partial B) > 2H$. Let $\{v_n\}$ be a sequence in $\mathcal{S}_f$ such that $\lim_{n\to\infty} v_n(p) = u(p)$, this sequence exists by definition of u. For each n let V_n be the lifting of v_n in B. We observe that $\{V_n\} \subset \mathcal{S}_f$, $V_n(p) \to u(p)$, moreover, since $\{V_n\} \leq u$ in B, the Compactness Theorem assures that there is a subsequence of $\{V_n\}$, still called $\{V_n\}$ which converges to a solution $\widetilde{u}$ of (1) in B uniformly on compact subsets of B. From the definition of u, and $\{v_n\}$ we have that $\widetilde{u} \leq u$ and $\widetilde{u}(p) = u(p)$. We claim that $u = \widetilde{u}$ in B. If this were not the case, we take a point $q \in B$, such that $u(q) > \widetilde{u}(q)$. This implies that there is a function $\widetilde{v} \in \mathcal{S}_f$, with $\widetilde{u}(q) < \widetilde{v}(q)$. We define another sequence $\{w_n\} \subset \mathcal{S}_f$ as $w_n = \max\{\widetilde{v}, v_n\}$, and we consider W_n its lifting in B. As before we obtain a subsequence of $\{W_n\}$ which converges to a solution w of (1) in B, uniformly over compact subsets of B. By construction, we have $\widetilde{u} \leq w \leq u$ and $\widetilde{u}(p) = w(p) = u(p)$. Since $\widetilde{u}$ and w are solutions of (1) on B, applying the maximum principle, we conclude that $\widetilde{u} = w$ in B. This contradicts the definition of $\widetilde{v}$ and shows that $\widetilde{u} = u$ in B. As p is an arbitrary point we have that u is a solution of (1) in Ω.

Now, we need to show that $u|_{\partial\Omega} = f$. As there is a supersolution of (1) in Ω with boundary values f, we have by the maximum principle, that $u|_{\partial\Omega} \leq f$.

We suppose that f is $C^2(\partial\Omega)$. Observe that the constant M in Lemma 3.6 can be chosen as large as necessary, such that, for a fixed point $p \in \partial\Omega$,

(i) $w(p) + f(p) = f(p)$,

(ii) $w(q) + f(p) = -M + f(p) < u(q)$ for $q \in \Omega \cap \partial\Delta$,

(iii) $w(q) + f(p) < f(q)$ for $q \in (\partial\Omega \cap \partial\Delta - \{p\})$,

with w defined as in Lemma 3.6; the third condition can be obtained choosing M large, since f is C^2 and the function

$$h(r, C) = \frac{e^{AC}}{2C}(e^{-2Cr} - 1)$$

is decreasing in r and C (the function h defines the barrier w; see Lemma 3.6). These barriers enable us to conclude that the solution u has the prescribed boundary values f, if f is $C^2(\partial\Omega)$.

Now, if f is continuous, we consider a sequence of $C^2(\partial\Omega)$ functions $\{f_n\}$ which converges to f and $f_n(p) < f_{n+1}(p)$ for $p \in \partial\Omega$ and $n \in \mathbb{N}$. We proved, that there is, for each $n \in \mathbb{N}$, a solution u_n of (1) in Ω, such that $u_n|_{\partial\Omega} = f_n$. By the maximum principle the sequence $\{u_n\}$ is monotonically increasing. Moreover since there is a minimal surface (supersolution) on Ω having boundary values f, the sequence $\{u_n\}$

converges to a solution u of (1). Furthermore, since $\{f_n\}$ converges monotonically to f, u has boundary values f. □

With this Theorem we can construct an example as in Hauswirth, Rosenberg and Spruck [5] and extend the result above for domains having piecewise $C^{2,\alpha}$ arcs.

Example 4.4 Let $\Gamma \subset \partial\Omega$ be a $C^{2,\alpha}$ arc with $\kappa(\Gamma) = 2H$ and $p \in \Gamma$ an interior point. Let $\Delta \subset B_p(\delta)$ be a convex domain obtained by smoothing the convex domain bounded by Γ and $\partial B_p(\delta)$, where $B_p(\delta)$ is the geodesic ball centered at p having radius $\delta > 0$. We consider smooth boundary data $f \leq 0$ on $\partial\Delta$ with $f \equiv 0$ in a neighborhood of p and $f \equiv -M$ in a neighborhood of $\partial B_p(\delta)$. Then, if there is some subsolution of (1) in Δ, there is a smooth solution w^- of (1) in Δ with boundary values f.

Definition 4.5 For $p \in \partial\Omega$ we define the outer curvature $\widehat{\kappa}(p)$ to be the supremum of all (inward) normal curvatures of C^2 curves passing through p and locally supporting Ω. If no such curve exists we define $\widehat{\kappa}(p)$ to be $-\infty$. Note that $\widehat{\kappa}(p) = \kappa(p)$ at all regular points of $\partial\Omega$.

Using Example 4.4 we obtain the next theorem.

Theorem 4.6 (Existence Theorem) *Let Ω be a domain with piecewise C^2 boundary. Suppose that $\widehat{\kappa}(p) \geq 2H$, $\forall p \in \partial\Omega$, except for a finite set E of exceptional corner points of $\partial\Omega$. We suppose that there is a bounded subsolution of* (1) *in Ω and we prescribe continuous boundary data f. If $E = \varnothing$ there is a unique solution of* (1) *in Ω taking arbitrarily assigned continuous boundary data on $\partial\Omega$. If $E \neq \varnothing$, then there is a unique solution of* (1) *in Ω taking on arbitrarily assigned continuous boundary data on $\partial\Omega - E$.*

Proof We suppose that $E = \varnothing$. We approximate Ω by smooth (convex) domains $\Omega_n \subset \Omega$ satisfying $\kappa(\partial\Omega_n) \geq 2H$ by rounding each corner point of $\partial\Omega$. We extend the boundary data f to a minimal solution in Ω. Let f_n be the restriction of this extension to $\partial\Omega_n$, observe that $\{f_n\}$ converges uniformly to f. Then, Theorem 4.3 gives a unique smooth solution u_n in Ω_n with $u_n = f_n$ in $\partial\Omega_n$ and each u_n is uniformly bounded independent of n (since the minimal solution is a supersolution for (1)). Thus by the Compactness Theorem, a subsequence of u_n converges uniformly on compact subsets to a solution u of (1) in Ω.

It remains to show that $u = f$ in $\partial\Omega$. We fix $p \in \partial\Omega$; $q \in \partial\Omega_n$ with $\mathrm{dist}(p,q) < \delta$. Given ϵ we choose $\delta > 0$ such that $|f_n(x) - f_n(q)| < \epsilon$ and $|f_n(q) - f(p)| < \epsilon$

if $\mathrm{dist}(x;q) < \delta$ and n is large. Now consider an arc of constant curvature $2H$ that supports $\partial\Omega_n$ at q and let $-w(x) = w^+$ and $w(x) = w^-$ be the lower and upper barriers in Δ given by Example 4.4 with $M = 2\sup|u_n|$. Then by the maximum principle,

$$w(x) - 3\epsilon \leq u_n(x) - f(p) \leq -w(x) + 3\epsilon.$$

This enables us to conclude that u is continuous in $\Omega \cup \partial\Omega$ and $u = f$ in $\partial\Omega$.

If $E \neq \varnothing$, the compactness of $\partial\Omega$ and the continuity of f imply that f is bounded. So the slice $\mathbb{M} \times \{d\}$, $d > \sup_{p\in\partial\Omega}|f(p)|$ is a bounded supersolution. By hypothesis there is a bounded subsolution. Then we apply the Perron method and we obtain a solution of (1) in Ω. Moreover, except for the points in E, the solution assumes the boundary data, since the barriers in the previous case hold in $\partial\Omega - E$. □

5 Flux formula

Let $u \in C^2(\Omega) \cap C^1(\Omega \cup \partial\Omega)$ be a solution of (1) in a domain Ω. Integrating (1) over Ω we have

$$2H\mathcal{A}(\Omega) = \int_\Omega \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla|^2}}\right) = \int_{\partial\Omega}\left\langle \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right\rangle,$$

where $\mathcal{A}(\Omega)$ is the area of Ω and ν is the outer conormal. The integral on the right side is called the flux of u across $\partial\Omega$. Let Γ be a subarc of $\partial\Omega$, if u is not differentiable on Γ we can define the flux of u across Γ as follows; see [5].

Definition 5.1 Choose Υ to be an embedded smooth curve in Ω so that $\Gamma \cup \Upsilon$ bounds a simply connected domain Δ_Υ. We then define the flux of u across Γ to be

$$F_u(\Gamma) = 2H\mathcal{A}(\Delta_\Upsilon) - \int_\Upsilon \left\langle \frac{\nabla u}{W}, \nu\right\rangle ds.$$

The last integral is well defined, and $F_u(\Gamma)$ does not depend on the choice of Υ.

The lemmas below can be found in Nelli and Rosenberg [11].

Lemma 5.2 *Let u be a solution of (1) in Ω, a simply connected bounded domain and Γ be a C^1 compact curve in $\Omega \cup \partial\Omega$. Then*

$$F_u(\partial\Omega) = 2H\mathcal{A}(\Omega),$$
$$|F_u(\Gamma)| \leq |\Gamma|.$$

Lemma 5.3 *Let* Ω *be a domain and* $\Gamma\subset\partial\Omega$ *be a compact piecewise* C^2 *arc satisfying* $\kappa(p)\geq 2H$ *for all* $p\in\Gamma$. *Let u be a solution of* (1) *in* Ω *which is continuous on* Γ. *Then*

$$|F_u(\Gamma)|<|\Gamma|.$$

Lemma 5.4 *Let* Ω *be a domain and* $\Gamma\subset\partial\Omega$ *be a compact piecewise* C^2 *arc, let u be a solution of* (1) *in* Ω.

(i) *If* u *tends to* $+\infty$ *on* Γ, *we have* $\kappa(\Gamma)=2H$ *and*

$$F_u(\Gamma)=|\Gamma|.$$

(ii) *If* u *tends to* $-\infty$ *on* Γ, *we have* $\kappa(\Gamma)=-2H$ *and*

$$F_u(\Gamma)=-|\Gamma|.$$

Proof We will prove (i), the other case is similar. We know that stable surfaces having constant mean curvature H have bounded second fundamental form at points a fixed positive distance from their boundary; see Rosenberg, Souam and Toubiana [15]. This implies a stable surface is a graph of bounded geometry over the δ–geodesic disc(in exponential coordinates) centered at the origin of the tangent space at q (the fixed distance from the boundary) and δ does not depend on q. Graphs having mean curvature H are stable, so we can apply the curvature estimates here.

Let $p\in\Gamma$ and let $p_n\to p$ be a sequence of points in Ω. Since u tends to $+\infty$, the curvature estimates guarantees the existence of a $\delta>0$ (independent of n) such that a neighborhood of each $(p_n,u(p_n))$ in the graph of u, is a graph (in geodesic coordinates) over a disk of radius δ centered at the origin of $T_{(p_n,u(p_n))}G(u)$, where $T_{(p_n,u(p_n))}G(u)$ is the tangent plane of $G(u)$ at $(p_n,u(p_n))$ and $G(u)$ is the graph of u. We translate these graphs to the point $(p_n,0)$ and we denote these translated graphs by $G_{p_n}(\delta)$. Let $N((p_n,u(p_n)))$ be the unit normal vector to $G(u)$ at $(p_n,u(p_n))$, after passing to a subsequence, we have $N((p_n,u(p_n)))\to N_\infty$. Let Π be the plane orthogonal to N_∞ whose origin is p. Each $G_{p_n}(\delta)$ is a graph of a function with height and slope uniformly bounded. For n large, $G_{p_n}(\delta)$ is a graph over a disk of radius δ' centered at p (the origin of Π), $0<\delta'\leq\delta$. Since these graphs have height and slope uniformly bounded, they converge to a graph $G_p(\delta')$ defined over a disk of radius δ' centered at the origin of Π.

We want to show that N_∞ is a horizontal vector and that $\kappa(\Gamma)=2H$.

We suppose that N_∞ is not a horizontal vector. This implies that Π is not a vertical plane, so the projection of $G_p(\delta')$ has points inside $\Omega\cup\partial\Omega$ and outside $\Omega\cup\partial\Omega$ this contradicts the fact that $G_p(\delta')$ is a limit of vertical graphs over Ω. This shows that N_∞ is a horizontal vector, so we have the equality

$$\int_\Gamma \left\langle \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu \right\rangle = |\Gamma|.$$

We will show that $\kappa(\Gamma) = 2H$. Let L be a curve tangent to Γ at p with $\kappa(L) = 2H$, with respect to N_∞. Note that the surface $L \times \mathbb{R}$ has curvature H and is tangent at p to the graph $G_p(\delta')$, which also has mean curvature H. Their mean curvature vectors point to the same side, by choices of $N((p_n, u(p_n)))$ and L. We need to show that $G_p(\delta') \subset (L \times \mathbb{R})$. Since $G_p(\delta')$ is tangent to $L \times \mathbb{R}$ at p, if $G_p(\delta')$ is on one side of $L \times \mathbb{R}$, by the maximum principle, we have that $G_p(\delta') \subset (L \times \mathbb{R})$. If this is not the case, $G_p(\delta') \cap (L \times \mathbb{R})$ is composed of k curves passing through p, $k \geq 2$, meeting transversely at p. So in a neighborhood of p these curves separate $G_p(\delta')$ in $2k$ components and the adjacent components lie in alternate sides of $L \times \mathbb{R}$. Moreover the curvature vector alternates from pointing down to pointing up when one goes from the one component to the another. So for n large, this implies that the mean curvature vector to $G_{p_n}(\delta)$ points down and up. Consequently the normal vector to $G_{p_n}(\delta)$ points down and up, this gives us a contradiction since $G_{p_n}(\delta)$ is a graph. Since the sequence $\{p_n\}$ and p are arbitrary we have that $L \subset \Gamma$ and $\kappa(p) = 2H$ for $p \in \Gamma$. □

Lemma 5.5 *Let Ω be a domain and $\Gamma \subset \partial\Omega$ be a compact piecewise C^2 arc, let $\{u_n\}$ be a sequence of solutions of* (1) *in Ω with each u_n continuous on Γ.*

(i) *If the sequence diverges to $+\infty$ uniformly on compact subsets of Γ while remaining uniformly bounded on compact subsets of Ω, we have*

$$\lim_{n\to\infty} F_{u_n}(\Gamma) = |\Gamma|.$$

(ii) *If the sequence diverges to $-\infty$ uniformly on compact subsets of Γ while remaining uniformly bounded on compact subsets of Ω, we have*

$$\lim_{n\to\infty} F_{u_n}(\Gamma) = -|\Gamma|.$$

Proof Let $p \in \Gamma$ and let $\{p_n\}$ be a sequence in Ω, with $p_n \to p$. After passing to a subsequence, we can choose $\delta > 0$ independent of n, such that a neighborhood of $(p_n, u_n(p_n))$ in the graph of u_n is a graph (in geodesic coordinates) over a disk of radius δ centered at the origin of $T_{(p_n,u(p_n))}G(u_n)$, here $T_{(p_n,u_n(p_n))}G(u_n)$ denotes the tangent plane to $G(u_n)$ at $(p_n, u_n(p_n))$and $G(u_n)$ denotes the graph of u_n. As in Lemma 5.4 the conclusion is that (after passing to a subsequence) $N_n(p_n) \to N_\infty$ and N_∞ is a horizontal vector, where $N_n(q)$ is the normal vector to the graph of u_n at the point q. □

Lemma 5.6 *Let Ω be a domain and $\Gamma \subset \partial\Omega$ be a compact piecewise C^2 arc and let $\{u_n\}$ be a sequence of solutions of* (1) *in Ω with each u_n continuous on Γ.*

(i) *If $\kappa(\Gamma) = 2H$ and the sequence diverges to $-\infty$ uniformly on compact subsets of Ω while remaining uniformly bounded on compact subsets of Γ, we have*

$$\lim_{n\to\infty} F_{u_n}(\Gamma) = |\Gamma|.$$

(ii) *If $\kappa(\Gamma) = -2H$ and the sequence diverges to $+\infty$ uniformly on compact subsets of Ω while remaining uniformly bounded on compact subsets of Γ, we have*

$$\lim_{n\to\infty} F_{u_n}(\Gamma) = -|\Gamma|.$$

Proof (i) Let $p \in \Gamma$ and let $p_n \to p$ be a sequence of points in Ω. Suppose $\{u_n\}$ diverges to $-\infty$ in Ω and remains uniformly bounded on Γ. After passing to a subsequence, we can assume that the distance from $(p_n, u_n(p_n))$ to the boundary of $G(u_n)$ is bigger than a fixed constant. Then, curvature estimates guarantees the existence of a $\delta > 0$ (independent of n) such that a neighborhood of each $(p_n, u_n(p_n))$ in the graph of u_n, is a graph (in geodesic coordinates) over a disk of radius δ centered at the origin of $T_{(p_n,u_n(p_n))}G(u_n)$. We translate these graphs to the point $(p_n, 0)$ and we denote these graphs translated by $G_n(\delta)$. Let $N((p_n, u_n(p_n)))$ be the unit normal vector to $G(u_n)$ at $(p_n, u_n(p_n))$, after passing to a subsequence, we have $N((p_n, u_n(p_n))) \to N_\infty$. Let Π be the plane orthogonal to N_∞ whose origin is p. Each $G_n(\delta)$ is a graph of a function with height and slope uniformly bounded. For n large, $G_n(\delta)$ is a graph over a disk of radius δ' centered at p (the origin of Π), $0 < \delta' \leq \delta$. Since these graphs have height and slope uniformly bounded, they converge to a graph $G_p(\delta')$ defined over a disk of radius δ' centered at the origin of Π.

The vector N_∞ is horizontal, if not, Π would not be a vertical plane, so the projection of $G_p(\delta')$ would have points outside Ω which is a contradiction with the fact that $G_p(\delta')$ is the limit of graphs over Ω. Moreover, since $\{u_n\}$ diverges to $-\infty$ in Ω and is bounded in Γ, we have the equality

$$\lim_{n\to\infty} F_{u_n}(\Gamma) = |\Gamma|.$$

Case (ii) is similar. □

Theorem 5.7 (Monotone convergence theorem) *Let $\{u_n\}$ be a monotonically increasing or decreasing sequence of solutions of* (1) *in a bounded domain Ω. If the sequence is bounded at a single point $p \in \Omega$, there exists a neighborhood $U \subset \Omega$ of p, such that $\{u_n\}$ converges to a solution of* (1) *in U. The convergence is uniform on compact subsets of U and the divergence is uniform on compact subsets of $V = \Omega - U$. If V is nonempty, ∂V consists of arcs of curvature $\pm 2H$ and parts of $\partial\Omega$. These arcs are convex to U for increasing sequences and concave to U for decreasing sequences.*

Refer to Jenkins and Serrin [6, Theorems 6.1 and 6.2] and Hauswirth, Rosenberg and Spruck [5, Theorem 6.2] for a proof of Theorem 5.7.

6 The curve B^*

In this section, we will prove that given two points p, q in $\mathbb{M}$ there is a convex domain bounded by two smooth arcs joining p, q having the same constant prescribed curvature for any constant less than $\sqrt{a}$, if the sectional curvature of $\mathbb{M}$ is less than $-a$, $a > 0$.

The manifold $\mathbb{M}$ is oriented by $\{v, J(v)\}$, v a unit vector at p and $J(v)$ rotation of v by $\pi/2$. We say that the curve $C(p, v, \kappa)$ has curvature $\kappa > 0$ at p if the curvature vector of $C(p, v, \kappa)$ has length κ at p and near p, $C(p, v, \kappa)$ is in the sector from v to $J(v)$. When $C(p, v, \kappa)$ is not in this sector, we say the curvature of $C(p, v, \kappa)$ at p is $-\kappa$; see Figure 6.

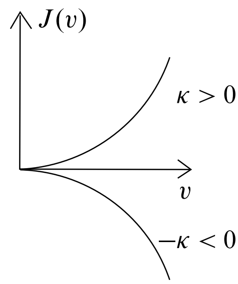

Figure 6

Let $T_p^1\mathbb{M}$ be the set of vectors in the tangent space of $\mathbb{M}$ at p having norm 1. We know there is a unique curve, denoted by $C(p, v, \kappa)$ starting at $p \in \mathbb{M}$, having $v \in T_p^1\mathbb{M}$ as tangent at p, and having constant curvature κ at each point. Denote by $C(p, v, -\kappa)$ the unique curve having curvature $-\kappa$ at each point.

In fact, in the discussion that follows we need not distinguish between $C(p, v, \kappa)$ and $C(p, v, -\kappa)$.

Claim 6.1 The curvature of geodesic circles centered at p is larger than $\sqrt{a}$.

Proof Let $C_r(t)$ be the geodesic circle centered at p having radius $r > 0$. We denote the geodesic curvature of C_r at $C_r(t)$ by $\kappa_g(C_r(t))$. The geodesic curvature of C_r satisfies the equation (see Labourie [8, Propositions 3.1.1 and 3.2.1])

$$\frac{\partial}{\partial r}\kappa_g(C_r(t)) = -K(C_r(t)) - \kappa_g^2(C_r(t)),$$

where $K(C_r(t))$ is the sectional curvature of $\mathbb{M}$ at $C_r(t)$. Since $K<-a$,

$$\frac{\partial}{\partial r}\kappa_g(C_r(t)) > a-\kappa_g^2(C_r(t)),$$
$$\frac{\partial}{\partial r}\kappa_g(C_r(t))+\kappa_g^2(C_r(t)) > a.$$

We observe that

$$\frac{\partial}{\partial r}(\sqrt{a}\coth(\sqrt{a}r)) = a-(\sqrt{a}\coth(\sqrt{a}r))^2,$$

and we conclude

$$\kappa_g(C_r(t)) > \sqrt{a}\coth(\sqrt{a}r) \geq \sqrt{a}.$$ □

Claim 6.2 The curve $C(p,v,\kappa)$, for $0<\kappa<\sqrt{a}$, is embedded.

Proof Suppose $C(p,v,\kappa)$ is not embedded. Let $\eta\subset C(p,v,\kappa)$ be a Jordan curve, smooth except at one point $q\in\eta$, a point of self-intersection of $C(p,v,\kappa)$. Consider $r=\sup\{r_0>0;\eta\cap C_r(q)\neq\varnothing\}$. At a point z of intersection $\eta\cap C_r(q)$, $C_r(q)$ is tangent to η and locally on the concave side of η, where $C_r(q)$ is the geodesic circle centered at q having radius r. This contradicts $\kappa<\sqrt{a}$. □

Claim 6.3 Let C_r be the geodesic circle centered at p having radius $r>0$. If $C_r\cap C(p,v,\kappa)\neq\varnothing$, then $C(p,v,\kappa)$ intersects C_r transversally, when $0<\kappa<\sqrt{a}$.

Proof Let $q\in C_r\cap C(p,v,\kappa)$. Suppose that $C(p,v,\kappa)$ is tangent to C_r at q, we will obtain a contradiction. There are three possibilities, either $C(p,v,\kappa)$ is inside the disc D_r bounded by C_r in a neighborhood of q, or $C(p,v,\kappa)$ is outside D_r in a neighborhood of q or $C(p,v,\kappa)$ has points inside and outside D_r in a neighborhood of q.

If $C(p,v,\kappa)$ is inside D_r in a neighborhood of q and is tangent to C_r at q, then the curvature vector of C_r and $C(p,v,\kappa)$ at q have the same direction and $C(p,v,\kappa)$ is above C_r with respect to the curvature vector; see Figure 7(left). On the other hand, the curvature of C_r is greater than the curvature of $C(p,v,\kappa)$, a contradiction.

If $C(p,v,\kappa)$ is outside D_r in a neighborhood q, we can consider the compact arc η contained in $C(p,v,\kappa)$ joining p to q. As η is compact and is outside D_r in a neighborhood of q, there is a point $q'\in\eta$ such that the distance from p to any point in η is smaller than or equal to the distance from p to q'; see Figure 7(middle). This implies that η(so $C(p,v,\kappa)$) is tangent to $C_{r'}$, $r'>r$ and is inside $D_{r'}$, where $C_{r'}$ is the geodesic circle centered at p passing through q' and $D_{r'}$ the disc bounded by $C_{r'}$. This is impossible.

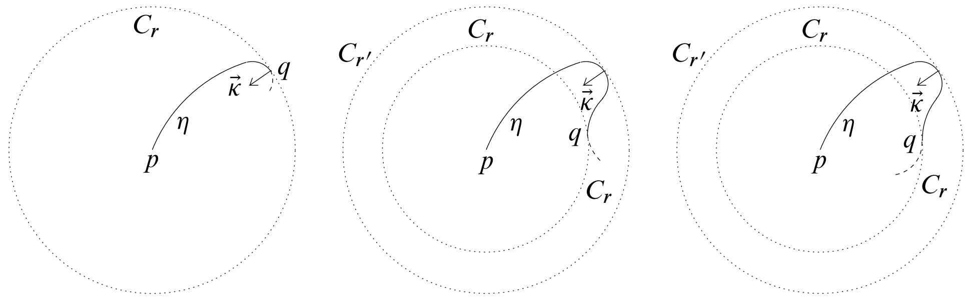

Figure 7

Finally, if $C(p, v, \kappa)$ has points inside and outside D_r in a neighborhood q and is tangent to C_r at q, we consider the compact arc η contained in $C(p, v, \kappa)$ joining p to q; see Figure 7(right). If η is not inside D_r it is possible to find a point q' so that the distance from p to any point in η is smaller than or equal to the distance from p to q', and the contradiction is obtained as in the second case. If η is inside D_r, the curvature vector of η and C_r at q have the same direction, and η is above C_r, this contradicts the comparison principle at the boundary, since the curvature of C_r is larger than the curvature of $C(p, v, \kappa)$. □

Claim 6.4 The intersection $C(p, v_1, \kappa) \cap C(p, v_0, \kappa)$ is the point p, when $v_1 \neq v_0$, $0 < \kappa < \sqrt{a}$.

Proof Suppose that $C(p, v_1, \kappa) \cap C(p, v_0, \kappa) \neq \varnothing$. Let $q \in C(p, v_1, \kappa) \cap C(p, v_0, \kappa)$ such that the compact arcs η_i contained in $C(p, v_i, \kappa)$, $i = 0, 1$, joining p to q satisfy $\eta_1 \cap \eta_0 = \{p, q\}$. Let Δ be the domain bounded by $\eta_1 \cup \eta_0$, suppose that η_1 is convex and η_0 is concave with respect to Δ.

Consider a smooth function $\varphi\colon [0,1] \to T_p\mathbb{M}$, such that, $\varphi(0) = v_0$, $\varphi(1) = v_1$ for $t \in [0,1]$ and $C(p, \varphi(t), \kappa) \cap \Delta \neq \varnothing$; see Figure 8. Since the intersection of $C(p, v_1, \kappa)$ and $C(p, v_0, \kappa)$ is transverse, for each $t \in [0,1]$ there is a point $q(t) \in \eta_1 \cap C(p, \varphi(t), \kappa)$. Observe that $q(0) = q$ and $q(t)$ tends to p, when t tends to 1. Let η_t, the compact arc in $C(p, \varphi(t), \kappa)$ joining p to $q(t)$.

Now we consider a variation of η_1 by equidistant curves. Let $\eta_1(s)$, $s \in [0,1]$ be a parametrization of η_1. The variation of η_1 by equidistant curves is given by

$$(s, \xi) \in [0,1] \times [0,1] \mapsto \exp_{\eta_1(s)}(\xi N(\eta_1(s))),$$

where $N(\eta_1(s))$ is the normal vector to η_1 at $\eta_1(s)$ having the same direction as the curvature vector of η_1 at $\eta_1(s)$. For each $\xi \in [0,1]$, let $\eta^{\xi}(s) := \exp_{\eta_1(s)}(\xi N(\eta_1(s)))$.

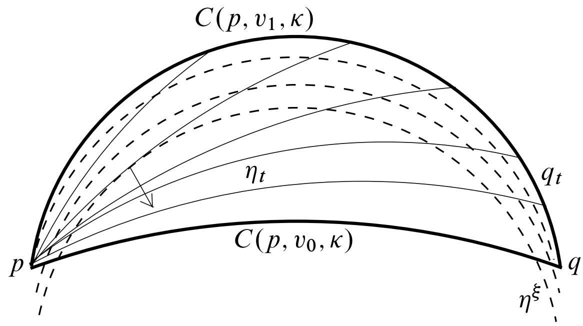

Figure 8

The formula of first variation of curvature (see [8, Propositions 3.1.1 and 3.2.1])

$$\frac{\partial}{\partial\xi}(\kappa_\epsilon(\eta^\xi)(s)) = K(\eta^\xi(s)) + \kappa_g^2(\eta^\xi(s)),$$

where $K(\eta^\xi(s))$ is the sectional curvature of $\mathbb{M}$ at $\eta^\xi(s)$ and $\kappa_g^2(\eta^\xi(s))$ is the geodesic curvature of η^ξ at $\eta^\xi(s)$. Since $K<-a$ and $0<\kappa_g^2(\eta^\xi(s))<a$, we have

$$\frac{\partial}{\partial\xi}(\kappa(\eta^\xi)(s)) < 0,$$

so the geodesic curvature of $\eta^\xi(s)$ is smaller that the geodesic curvature of $\eta^0(s) = \eta_1(s)$. On the other hand η^ξ is tangent to some curve η_t in the convex side of η_t, a contradiction, since $\kappa_g(\eta_t) > \kappa_g(\eta^\xi)$. □

Claim 6.5 The intersection $C(p,v,\kappa)\cap C_r$ is nonempty for every $r>0$ and $0<\kappa<\sqrt{a}$, where C_r is the geodesic circle centered at p having radius r.

Proof Suppose that the set $\Lambda = \{r\in(0,+\infty); C(p,v,\kappa)\cap C_r \neq \varnothing\}$ (for a fixed $v\in T_p^1\mathbb{M}$) is bounded above. Let r_0 be the supremum of Λ. By Claim 6.3, $r_0\notin\Lambda$. Let $n_0\in\mathbb{N}$ be such that $1/n<r_0$. For $n\in\mathbb{N}$, $n>n_0$, there exists a point $q_n\in C(p,v,\kappa)\cap C_r$, for some $r_0-1/n<r<r_0$. The sequence $\{q_n\}$ is in a compact set of $\mathbb{M}$, so after passing to a subsequence, we can assume that $\{q_n\}$ converges to some point $q\in C_{r_0}$. Let η_n be a small connected arc of length 2δ contained in $C(p,v,\kappa)$, such that $q_n\in\eta_n$ and $\eta_n-\{q_n\}$ are two arcs having length δ. Since $\{q_n\}$ converges to q and $C(p,v,\kappa)$ is embedded, the sequence of arcs $\{\eta_n\}$ converges to a arc η having q as an interior point. Moreover, since all arcs η_n are inside D_{r_0} (the disc bounded by C_{r_0}), the arc η is tangent to C_{r_0} at q, this contradicts Claim 6.3. □

We fix $r\in(0,+\infty)$, and we define the function $f_r\colon T_p^1\mathbb{M}\to C_r$ by $f_r(v) = C(p,v,\kappa)\cap C_r$, where C_r is the geodesic circle centered at p having radius r.

Claim 6.6 The function f_r defined above is continuous, if $0 < \kappa < \sqrt{a}$.

Proof We take a sequence of vectors $\{v_n\}$, such that $v_n \to v \in T_p^1\mathbb{M}$. We want to prove that

$$\lim_{n\to+\infty} f_r(v_n) = f(v). \tag{13}$$

Let $q_n = f_r(v_n)$ and θ_n be the angle between C_r and $C(p, v_n, \kappa)$. We will show that $\theta_n \geq \theta > 0$. Let η_n be a compact arc in $C(p, v_n, \kappa)$ of length 2δ containing q_n such that $\eta_n - \{q_n\}$ are two arcs having length δ. If some subsequence (again denoted by $\{\theta_n\}$) of $\{\theta_n\}$ converges to 0, we could find a subsequence of $\{q_n\}$ (denoted by $\{q_n\}$), such that $\{q_n\}$ converges to q and $\{\eta_n\}$ converges to a arc η. Since $\{\theta_n\}$ converges to 0, the limit arc η is tangent to C_r at q. This contradicts Claim 6.3.

This bound on the angles implies a bound for the lengths of $C(p, v_n, \kappa) \cap D_r$, where D_r is the disc bounded by C_r. Then,

$$\lim_{n\to+\infty} f_r(v_n) = \lim_{n\to+\infty} \{C(p, v_n, \kappa) \cap C_r\} = \lim_{n\to+\infty} \{C(p, v_n, \kappa) \cap D_r\} \cap C_r = f(v).$$

□

Definition 6.7 The stability operator for the curves $C(p, v, \kappa)$ is $L = -\Delta - \kappa^2 - K$, where K is the sectional curvature of $\mathbb{M}$. We say that $C(p, v, \kappa)$ is stable (in the strong sense) if for any function u with compact support in $C(p, v, \kappa)$ we have that

$$\int_{C(p,v,\kappa)} -u\Delta u - u^2\kappa^2 - u^2 K \geq 0.$$

Claim 6.8 The curves $C(p, v, \kappa)$ are stable for $0 < \kappa < \sqrt{a}$.

Proof We observe that

$$\int_{C(p,v,\kappa)} -u\Delta u - u^2\kappa^2 - u^2 K \geq \int_{C(p,v,\kappa)} -u\Delta u - u^2(\sqrt{a})^2 + u^2 a = \int_{C(p,v,\kappa)} -u\Delta u \geq 0.$$

So each curve $C(p, v, \kappa)$ is stable. □

Claim 6.9 The image $I = f_r(T_p^1\mathbb{M})$ is an open set on C_r.

Proof Let $q = f_r(v)$ and η be the compact arc in $C(p, v, \kappa)$ joining p to q. The stability of $C(p, v, \kappa)$, Claim 6.8, enables us to apply the Implicit Function Theorem and conclude that there are neighborhoods V of p and U of q in $\mathbb{M}$, such that for each $p' \in V$ and $q' \in U$ there is a curve (varying continuously with p' and q') having curvature κ joining p' to q'. So for every point $q' \in U \cap C_r$, there is a curve having curvature κ joining p to q', which implies $q' = f_r(v')$, for some $v' \in T_p^1\mathbb{M}$. For details, see Rosenberg [13, Theorem 4.2]. □

Proposition 6.10 The map f_r is a homeomorphism.

Proof By Claim 6.6 f_r is continuous, so $f_r(T_p^1\mathbb{M})$ is compact. On the other hand, Claim 6.9 guarantees that $f_r(T_p^1\mathbb{M})$ is open and is not empty by Claim 6.5. The connectedness of C_r implies that $f_r(T_p^1\mathbb{M}) = C_r$. The injectivity of f_r follows from Claim 6.4, so f_r is homeomorphism. □

7 The main theorems

We now prove the main theorems of this work. We have constructed all the necessary tools to prove these theorems. The proofs are similar to the proofs found in Spruck [16].

Proposition 7.1 Let Ω be an admissible domain such that the family $\{B_l\}$ is empty, $\kappa(C_m) > 2H$ and the assigned boundary data f on the arcs $\{C_m\}$ bounded below. Then there is a solution to the Dirichlet Problem if and only if

$$2\alpha(\mathcal{P}) < l(\mathcal{P}) + 2H\mathcal{A}(\Omega_{\mathcal{P}})$$

for all admissible polygons $\mathcal{P}$.

Proof Let $\{u_n\}$ be a sequence of solutions of (1) in Ω defined by

$$u_n = \begin{cases} n & \text{on } \bigcup_k A_k, \\ \min\{n, f\} & \text{on } \bigcup_m C_m. \end{cases}$$

By the maximum principle the sequence $\{u_n\}$ is monotone increasing, we need to show that the divergence set is empty. Observe that by Lemma 3.11 there is a neighborhood of each arc C_m which is contained in the convergence domain. Denoting by $\partial V = \mathcal{P} = \bigcup_k (A_k \cap \mathcal{P}) \cup (\mathcal{P} - \bigcup_k A_k)$ and applying the flux formula on $\mathcal{P}$,

$$\begin{aligned} 2H\mathcal{A}(V) &= \lim_{n\to\infty} F_{u_n}(\mathcal{P}) \\ &= \lim_{n\to\infty} F_{u_n}(\textstyle\bigcup_k (A_k \cap \mathcal{P})) + \lim_{n\to\infty} F_{u_n}(\mathcal{P} - \bigcup_k A_k) \\ &\le \alpha(\mathcal{P}) - l(\mathcal{P} - \textstyle\bigcup_k A_k)) \\ &= 2\alpha(\mathcal{P}) - l(\mathcal{P}). \end{aligned}$$

This contradicts the hypothesis. □

Proposition 7.2 Let Ω be an admissible domain with the family $\{A_k\}$ empty, and $\kappa(C_m) > 2H$ and the assigned boundary data f on the arcs C_m bounded above. Then there is a solution to the Dirichlet Problem if and only if

$$2\beta(\mathcal{P}) < l(\mathcal{P}) - 2H\mathcal{A}(\Omega_{\mathcal{P}})$$

for all admissible polygons $\mathcal{P}$.

Proof Let $\{u_n\}$ be a sequence of solutions of (1) in Ω defined by

$$u_n = \begin{cases} -n & \text{on } \bigcup_l B_l^*, \\ \max\{-n, f\} & \text{on } \bigcup_m C_m \end{cases}$$

By the maximum principle the sequence $\{u_n\}$ is monotone decreasing. We need to show that the divergence set is empty. Observe that by Lemma 3.11 there is a neighborhood of each arc C_m which is contained in the convergence domain, moreover in the domain bounded by $B_l \cup B_l^*$ the sequence is unbounded, and assumes the value $-\infty$ in B_l. As in Proposition 7.1, we denote by $\partial V = \mathcal{P} = \bigcup_l (B_l \cap \mathcal{P}) \cup (\mathcal{P} - \bigcup_l B_l)$. Then applying the flux formula on $\mathcal{P}$,

$$\begin{aligned} 2H\mathcal{A}(V) &= \lim_{n\to\infty} F_{u_n}(\mathcal{P}) \\ &= \lim_{n\to\infty} F_{u_n}(\textstyle\bigcup_l (B_l \cap \mathcal{P})) + \lim_{n\to\infty} F_{u_n}(\mathcal{P} - \textstyle\bigcup_l B_l) \\ &\geq -\beta(\mathcal{P}) + l(\mathcal{P} - \textstyle\bigcup_k B_l)) \\ &= -2\beta(\mathcal{P}) + l(\mathcal{P}), \end{aligned}$$

which contradicts the hypothesis. □

Example 7.3 Let γ be a $C^{2,\alpha}$ arc of curvature $2H$ and p, q two points in γ whose distance is δ with $\delta > 0$ small compared with H. Let A_1 and A_2 be compact $C^{2,\alpha}$ arcs of curvature $2H$ orthogonal to γ at p and q respectively. We assume that the length of A_1 and A_2 is ϵ with ϵ small compared with δ. Let C_1^+ and C_2^+ be two arcs of geodesic circles joining the end points of A_1 and A_2, such that the domain Δ^+ bounded by A_1, A_2, C_1^+, C_2^+ is convex (see Figure 9(left)), for δ small we can suppose these arcs C_1^+ and C_2^+ have curvature greater than $2H$. By the choice of ϵ and δ, if there is some inscribed polygon, the inequality of Proposition 7.1 is satisfied. Then there is a solution in Δ^+ with boundary values $+\infty$ on A_1, A_2 and M on C_1^+, C_2^+, $M \in \mathbb{R}, M > 0$. Similarly, we can consider $C^{2,\alpha}$ arcs B_1, B_2 of curvature $-2H$ orthogonal to γ at p, q respectively, and C_1^-, C_2^- arcs of geodesic circles joining the end points of B_1, B_2 which are convex with respect to the domain Δ^- bounded by B_1, B_2, C_1^-, C_2^- (see Figure 9(right)). If the lengths of B_1, B_2 are small compared with δ and δ is small compared with H, the hypothesis of Proposition 7.2 is satisfied, so there is a solution in Δ^- with boundary values $-\infty$ on B_1, B_2 and $-M$ on C_1^-, C_2^-.

Remark 7.4 With these barriers, Propositions 7.1 and 7.2 are valid under the weaker hypothesis $\kappa(C_m) \geq 2H$.

We now prove the main theorems.

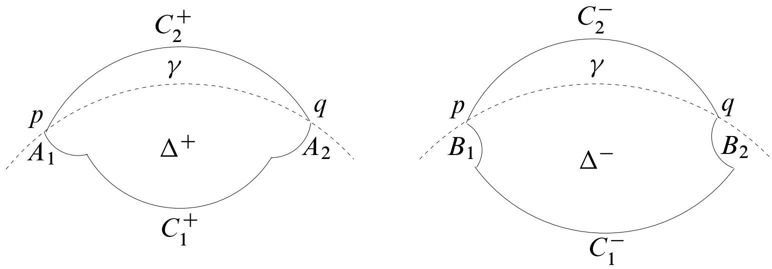

Figure 9

Proof of Theorem 2.8 We consider the sequence $\{u_n\}$ defined by

$$u_n = \begin{cases} n & \text{on } \bigcup_k A_k, \\ -n & \text{on } \bigcup_l B_l^*, \\ f_n & \text{on } \bigcup_m C_m, \end{cases}$$

where f_n is the truncation of f above by n and below by $-n$.

Consider u^+ and u^- the functions defined by

$$u_+ = \begin{cases} +\infty & \text{on } \bigcup_k A_k, \\ 0 & \text{on } \bigcup_l B_l^*, \\ \max\{f,0\} & \text{on } \bigcup_m C_m. \end{cases} \qquad u_- = \begin{cases} 0 & \text{on } \bigcup_k A_k, \\ -\infty & \text{on } \bigcup_l B_l, \\ \min\{f,0\} & \text{on } \bigcup_m C_m, \end{cases}$$

The function u_- exists by Remark 7.4, observe that $\kappa(A_k) = 2H$ and A_k is counting now in the family C_m. In order to show that the function u_+ exists, we have to verify the condition of Proposition 7.1. We denote by $\widetilde{P}$ the polygon obtained from a polygon P by removing the arcs B_l and attaching the arcs B_l^*. If the condition of Proposition 7.1 is not verified for some polygon $\widetilde{P}$ we would have

$$\begin{aligned} 2\alpha(\widetilde{P}) &> l(\widetilde{P}) + 2H\mathcal{A}(\Omega_{\widetilde{P}}) \\ &= l(P) - \beta(P) + \sum_{l;B_l\subset P} |B_l^*| + 2H\mathcal{A}(\Omega_P) + 2H\mathcal{A}(\Omega_{\widetilde{P}} - \Omega_P) \\ &> 2\alpha(P) - \beta(P) + \sum_{l;B_l\subset P} |B_l^*| + 2H\mathcal{A}(\Omega_{\widetilde{P}} - \Omega_P). \end{aligned}$$

Since $2\alpha(\widetilde{P}) = 2\alpha(P)$, we obtain

$$0 > -\beta(P) + \sum_{l;B_l\subset P} |B_l^*| + 2H\mathcal{A}(\Omega_{\widetilde{P}} - \Omega_P). \tag{14}$$

On the other hand, since the domain Δ_l bounded by $B_l \cup B_l^*$ is convex, by Theorem 4.6 there is a solution on Δ_l having continuous boundary values and the flux formulas give us (see Lemma 5.3)

$$2H\mathcal{A}(\Delta_l) > -|B_l^*| - |B_l|.$$

Then

$$2H\mathcal{A}(\Omega_{\widetilde{P}} - \Omega_P) > -\beta(P) - \sum_{l;B_l \subset P} |B_l^*|. \tag{15}$$

Adding (14) and (15) we obtain

$$0 > -2\beta(P).$$

So, in fact, the condition of Proposition 7.1 holds.

By the maximum principle, $u^- < u_n < u^+$ in Ω. Then by the Compactness Theorem there is a subsequence of $\{u_n\}$ which converges to a solution of (1) in Ω. By the definition of each u_n, u has boundary values $+\infty$ on each arc A_k, $-\infty$ on each arc B_l and f on the family $\{C_m\}$.

Conversely, if there is a solution to the Dirichlet problem,

$$\begin{aligned}
2H\mathcal{A}(\Omega_{\mathcal{P}}) &= F_u(\mathcal{P}) \\
&= \sum_{A_k \in \mathcal{P}} F_u(A_k) + \sum_{B_l \in \mathcal{P}} F_u(B_l) + \sum_{C_m \in \mathcal{P}} F_u(C_m) \\
&< \alpha(\mathcal{P}) - \beta(\mathcal{P}) + \sum_{C_m \in \mathcal{P}} |C_m| \\
&= -2\beta(\mathcal{P}) + l(\mathcal{P}),
\end{aligned}$$

(the strict inequality follows from Lemma 5.3), which shows (3). Equation (2) is similar. □

Proof of Theorem 2.9 For each n, let u_n be the solution on Ω^*, given by

$$u_n = \begin{cases} n & \text{on } \bigcup_k A_k, \\ 0 & \text{on } \bigcup_l B_l^*. \end{cases}$$

For each $c \in \mathbb{R}$, $0 < c < n$ fixed, we define the set

$$\begin{aligned}
S_c &= \{p \in \Omega^*; u_n(p) - u_0(p) > c\}, \\
R_c &= \{p \in \Omega^*; u_n(p) - u_0(p) < c\}.
\end{aligned}$$

These sets depend on n, but we will omit this in the notation. Let S_c^i and R_c^i be the components of S_c and R_c which contain A_i and B_i^*, respectively. The maximum principle assures that $S_c=\bigcup_i S_c^i$ and $R_c=\bigcup_i R_c^i$. If c is sufficiently close to n, the sets S_c^i will be distinct and disjoint. We define $\mu(n)$ as the infimum of the constants c such that S_c^i are distinct and disjoint. The sets S_μ^i are distinct and disjoint, and there are indices $i,j,i\neq j$, such that $(S_\mu^i\cup\partial S_\mu^i)\cap(S_\mu^j\cup\partial S_\mu^j)\neq\varnothing$. This implies that given any i, there is a j such that R_μ^i and R_μ^j are disjoint.

We consider the solutions

$$u_+^i=\begin{cases}+\infty & \text{on } A_i,\\ 0 & \text{on } (\bigcup_{k\neq i}A_k)\cup(\bigcup_l B_l^*)\end{cases}\qquad u_-^i=\begin{cases}0 & \text{on } (\bigcup_k A_k)\cup B_i^*,\\ -\infty & \text{on } \bigcup_{l\neq i}B_l.\end{cases}$$

The solution u_+^i exists since using the hypothesis of the theorem we can verify that the conditions on Proposition 7.1 and Remark 7.4 hold (the argument is the same which we use to show that the solution u_- exists in the proof of Theorem 2.8). In order to show that the solution u_+^i exists, we need verify the conditions of Proposition 7.2 and Remark 7.4. We denote by $\widetilde{\Omega}_i$ the domain bounded by $(\bigcup_k A_k)\cup(B_i^*)\cup(\bigcup_{l\neq i}B_l)$. The quantities related to $\widetilde{\Omega}_i$ will be denoted with a tilde. We take some admissible domain $\widetilde{\mathcal{P}}$ on $\widetilde{\Omega}_i$, so if $\widetilde{\mathcal{P}}=\partial\Omega$

$$2\widetilde{\beta}(\widetilde{\mathcal{P}})=2\beta\widetilde{\mathcal{P}}-2|B_i|<l(\widetilde{\mathcal{P}})-2H\mathcal{A}(\Omega)-2|B_i|<l(\widetilde{\mathcal{P}})-2H\mathcal{A}(\Omega).$$

We have to verify the conditions for polygons $\widetilde{\mathcal{P}}$ which contain the domain Δ bounded by $B_i\cup B_i^*$. We consider $\mathcal{P}$ the polygon obtained from $\widetilde{\mathcal{P}}$ by deleting the domain Δ and adding the arc B_i. Then,

$$2\beta(\mathcal{P})=2\widetilde{\beta}(\widetilde{\mathcal{P}})+2|B_i|\leq l(\mathcal{P})-2H\mathcal{A}(\Omega_{\mathcal{P}}),$$

so,

$$\begin{aligned}2\widetilde{\beta}(\widetilde{\mathcal{P}})&\leq\widetilde{l}(\widetilde{\mathcal{P}})+|B_i|-|B_i^*|-2H\mathcal{A}(\Omega_{\widetilde{\mathcal{P}}})+2H\mathcal{A}(\Delta)-2|B_i|\\&<\widetilde{l}(\widetilde{\mathcal{P}})-2H\mathcal{A}(\Omega_{\widetilde{\mathcal{P}}}).\end{aligned}$$

The last inequality follows from the flux formulas applied to Δ (Δ is a convex domain, so there are solutions having continuous boundary values).

We define

$$u_+(p)=\max_i\{u_+^i(p)\}\quad\text{for } p\in\Omega^*,$$
$$u_-(p)=\min_i\{u_-^i(p)\}\quad\text{for } p\in\Omega.$$

We are assuming that there is some subsolution of (1) in Ω^*, then, by the maximum principle, $u_+^i>-N,\ N>0$ for all i. We consider the sequence of solutions $\{v_n\}$, $v_n=u_n-\mu(n)$. We will show that if $M=\sup_{\Omega^*}|u_0|+N$,

$$v_n\leq u_++M\quad\text{in }\Omega,$$
$$v_n\geq u_--M\quad\text{in }\Omega.$$

We suppose that $v_n > u_0$ at $p \in \Omega$. So, $u_n - u_0 > \mu(n)$ at p, then $p \in S_\mu^i$, for some i. By the maximum principle, applied to the domain S_μ^i, we have

$$v_n \leq u_+^i + N + \sup_{S_\mu^i}\{u_0\} \leq u_+ + M \quad \text{at } p.$$

On the other hand, suppose that $v_n < u_0$ at p. Then $u_n - u_0 < \mu(n)$ at p, so $p \in R_\mu^i$ for some i. Let $j = j(i)$ such that $R_\mu^i \cap R_\mu^j = \varnothing$. By the maximum principle applied to the domain R_μ^i, we have

$$v_n \geq u_-^j - \sup_{R_\mu^i}\{u_0\} \geq u_- - M \quad \text{at } p.$$

Then the sequence $\{v_n\}$ is uniformly bounded, so it is convergent. Let $v_n \to u$. We have to show that u has the desired boundary values.

We observe that $\mu(n) \to \infty$, otherwise, we can extract a subsequence of $\{\mu(n)\}$ which converges to some value $\mu < \infty$. By the definition of v_n, the limit u would have boundary values $+\infty$ on the arcs A_i and $-\mu$ on the arcs B_l^*. Applying the flux formulas we obtain that this condition can not occur. So, u assumes the boundary values as prescribed.

Conversely, if such a solution u exists, we have

$$2H\mathcal{A}(\Omega) = F_u(\partial\Omega) = \sum_k F_u(A_k) + \sum_l F_u(B_l) = \alpha(\mathcal{P}) - \beta(\mathcal{P}),$$

which shows Equation (4). The other conditions are similar to the conditions done in Theorem 2.8. □

References

[1] **P Collin**, **H Rosenberg**, *Construction of harmonic diffeomorphisms and minimal graphs*, Ann. of Math. 172 (2010) 1879–1906 MR2726102

[2] **A Folha**, **S Melo**, *The Dirichlet problem for constant mean curvature graphs in $\mathbb{H} \times \mathbb{R}$ over unbounded domains*, Pacific J. Math. 251 (2011) 37–65 MR2794614

[3] **J A Gálvez**, **H Rosenberg**, *Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces*, Amer. J. Math. 132 (2010) 1249–1273 MR2732346

[4] **D Gilbarg**, **N S Trudinger**, *Elliptic partial differential equations of second order*, second edition, Grundl. Math. Wissen. 224, Springer, Berlin (1983) MR737190

[5] **L Hauswirth**, **H Rosenberg**, **J Spruck**, *Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Amer. J. Math. 131 (2009) 195–226 MR2488489

[6] **H Jenkins**, **J Serrin**, *Variational problems of minimal surface type. I*, Arch. Rational mech. Anal. 12 (1963) 185–212 MR0145194

[7] **H Jenkins**, **J Serrin**, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rational Mech. Anal. 21 (1966) 321–342 MR0190811

[8] **F Labourie**, *Un lemme de Morse pour les surfaces convexes*, Invent. Math. 141 (2000) 239–297 MR1775215

[9] **Y Li**, **L Nirenberg**, *Regularity of the distance function to the boundary*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. 29 (2005) 257–264 MR2305073

[10] **L Mazet**, **M M Rodríguez**, **H Rosenberg**, *The Dirichlet problem for the minimal surface equation, with possible infinite boundary data, over domains in a Riemannian surface*, Proc. Lond. Math. Soc. 102 (2011) 985–1023 MR2806098

[11] **B Nelli**, **H Rosenberg**, *Minimal surfaces in* $\mathbb{H}^2\times\mathbb{R}$, Bull. Braz. Math. Soc. 33 (2002) 263–292 MR1940353

[12] **A L Pinheiro**, *A Jenkins–Serrin theorem in* $M^2\times\mathbb{R}$, Bull. Braz. Math. Soc. 40 (2009) 117–148 MR2496117

[13] **H Rosenberg**, *Hypersurfaces of constant curvature in space forms*, Bull. Sci. Math. 117 (1993) 211–239 MR1216008

[14] **H Rosenberg**, *Minimal surfaces in* $\mathbb{M}^2\times\mathbb{R}$, Illinois J. Math. 46 (2002) 1177–1195 MR1988257

[15] **H Rosenberg**, **R Souam**, **E Toubiana**, *General curvature estimates for stable* H*–surfaces in* 3*–manifolds and applications*, J. Differential Geom. 84 (2010) 623–648 MR2669367

[16] **J Spruck**, *Infinite boundary value problems for surfaces of constant mean curvature*, Arch. Rational Mech. Anal. 49 (1972) 1–31 MR0334010

[17] **J Spruck**, *Interior gradient estimates and existence theorems for constant mean curvature graphs in* $M^n\times R$, Pure Appl. Math. Q. 3 (2007) 785–800 MR2351645

Instituto de Matemática – Departamento de Geometria, Universidade Federal Fluminense
R Mário Santos Braga, s/n, Campus do Valonguinho, CEP 24020-140 Niterói, RJ, Brazil

Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina 110, CEP 22460-320 Rio de Janeiro, RJ, Brazil

abigailfolha@vm.uff.br, rosen@impa.br

http://www.math.jussieu.fr/~rosen/

Proposed: Tobias H Colding Received: 21 February 2011
Seconded: John Lott, Yasha Eliashberg Revised: 5 March 2012