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Geodesic flow for CAT(0)–groups

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We associate to a CAT(0)–space a flow space that can be used as the replacement for the geodesic flow on the sphere tangent bundle of a Riemannian manifold. We use this flow space to prove that CAT(0)–group are transfer reducible over the family of virtually cyclic groups. This result is an important ingredient in our proof of the Farrell–Jones Conjecture for these groups.

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Introduction

In Bartels–Lück [1] we introduced the concept of *transfer reducible groups* with respect to a family of subgroups. This definition is somewhat technical and recalled as Definition 0.4 below. We showed that groups that are transfer reducible over the family of virtually cyclic subgroups satisfy the Farrell–Jones Conjecture with coefficients in an additive category. For further explanations about the Farrell–Jones Conjecture we refer for instance to [1], Bartels–Lück–Reich [3] and Lück–Reich [8], where more information about the applications, history, literature and status is given.

By a CAT(0)–group we mean a group G that admits a cocompact proper action by isometries on a finite dimensional CAT(0)–space. The following is our main result in this paper and has already been used in [1].

Main Theorem *Every* CAT(0)*–group is transfer reducible over the family of virtually cyclic subgroups.*

A similar result for hyperbolic groups has been proven in [1, Proposition 2.1] using the technical paper Bartels–Lück–Reich [2], where an important input is the flow space for hyperbolic groups due to Mineyev [9]. The methods for hyperbolic groups cannot be transferred directly to CAT(0)–groups, but the general program is the same and carried out in this paper.

An important step in the proof of the theorem above is the construction of a flow space $FS(X)$ associated to CAT(0)–spaces X, which is a replacement for the geodesic flow

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on the sphere tangent bundle of a Riemannian manifold of non-positive curvature. In particular, the dynamic of the flow on $FS(X)$ is similar to the geodesic flow. As in the hyperbolic case, the flow space $FS(X)$ is not a bundle over X.

In Section 1 we assign to any metric space X its flow space $FS(X)$ (see Definition 1.2). Elements in $FS(X)$ are generalized geodesics, that is, continuous maps $c\colon \mathbb{R} \to X$ such that either c is constant or there exists $c_-, c_+ \in \overline{\mathbb{R}} = \mathbb{R} \amalg \{-\infty, \infty\}$ with $c_- < c_+$ such that c is locally constant outside the interval (c_-, c_+) and its restriction to (c_-, c_+) is an isometric embedding. The flow on $FS(X)$ is given by $\Phi_\tau(c)(t) := c(t+\tau)$. The topology on $FS(X)$ is the topology of uniform convergence on compact subsets; this is also the topology associated to a natural metric on $FS(X)$. Many properties of X can be transported to $FS(X)$. For example, if a group G acts by isometries on X, then there is an induced isometric action on $FS(X)$. If the action on X is cocompact, then the induced action is also cocompact.

In Sections 2, 4 and 3 we study properties of $FS(X)$ under the assumption that X is a CAT(0)–space. The main observation in Section 2 is that the endpoint evaluation maps $c \mapsto c(\pm\infty)$ from $FS(X)$ to the bordification $\overline{X}$ of X are continuous on the complement of the subspace $FS(X)^{\mathbb{R}}$ of constant generalized geodesics. These are used in Proposition 2.6 to give coordinates on $FS(X) - FS(X)^{\mathbb{R}}$ and allow a detailed study of the topology of $FS(X) - FS(X)^{\mathbb{R}}$. We also discuss in Section 2.5 the case where X is a non-positively curved manifold. In Section 3 we prove our main flow estimates for $FS(X)$. These are crucial for our main result and differ from the corresponding estimates in the hyperbolic case. In the hyperbolic case the flow acts contracting on geodesics that determine the same point at infinity. This is not true in the CAT(0)–situation, for instance on flats the flow acts as an isometry. This problem is overcome by using a variant of the focal transfer (formulated as a homotopy action) from Farrell–Jones [7]. In Section 4 we assume that G acts properly and by isometries on X and study the periodic orbits of $FS(X)$ with respect to the induced action. In Theorem 4.2 we construct certain open covers for the subspace $FS(X)_{\leq\gamma}$ of $FS(X)$ consisting of all geodesics of period $\leq \gamma$. The dimension of this cover is uniformly bounded and the cover is long in the sense that for every $c \in FS(X)_{\leq\gamma}$ there is a member U of this cover that contains $\Phi_{[-\gamma,\gamma]}(c)$. (In fact, U will contain even $\Phi_{\mathbb{R}}(c)$.) This result is much harder than the corresponding result in the hyperbolic case, because in the CAT(0)–case the periodic orbits are no longer discrete, but appear in continuous families.

Section 5 contains the final preparation for the proof of our main theorem. We show in Proposition 5.11 that the existence of a suitable flow space for a group G implies that G is transfer reducible over a given family. This result depends very much on the long thin covers for flow spaces from Bartels–Lück–Reich [2].

In Section 6 we put our previous results together and prove our main theorem. It is only here that we assume that the action of G on the CAT(0)–space X is cocompact. All previous results are formulated without this assumption. This forces for example the appearance of a compact set K in Theorem 4.2 and in Section 4.2. There are of course prominent groups, for instance $SL_n(\mathbb{Z})$, that are naturally equipped with an isometric proper action on a CAT(0)–space, where the action is not cocompact. We hope that the level of generality in Sections 1 to 5 will be useful to prove the Farrell–Jones Conjecture for some of these groups.

Conventions

Let H be a (discrete) group that acts on a space Z. We will say that the action is *proper*, if for any $x \in X$ there is an open neighborhood U such that $\{h \in H \mid h \cdot U \cap U \neq \varnothing\}$ is finite. If Z is locally compact, this is equivalent to the condition that for any compact subset $K \subset Z$ the set $\{h \in H \mid h \cdot K \cap K \neq \varnothing\}$ is finite.

We will say that the action is *cocompact* if the quotient space $H \backslash Z$ is compact. If Z is locally compact, this is equivalent to the existence of a compact subset $L \subseteq Z$ such that $G \cdot L = Z$.

Let X be a topological space. Let $\mathcal{U}$ be an open covering. Its *dimension* $\dim(\mathcal{U}) \in \{0, 1, 2, \dots\} \amalg \{\infty\}$ is the infimum over all integers $d \geq 0$ such that for any collection U_0, U_1, ..., U_{d+1} of pairwise distinct elements in $\mathcal{U}$ the intersection $\bigcap_{i=0}^{d+1} U_i$ is empty. An open covering $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$. The *(topological) dimension* (sometimes also called *covering dimension*) of a topological space X

$$\dim(X) \in \{0, 1, 2, \dots\} \amalg \{\infty\}$$

is the infimum over all integers $d \geq 0$ such that any open covering $\mathcal{U}$ possesses a refinement $\mathcal{V}$ with $\dim(\mathcal{V}) \leq d$.

For a metric space Z, a subset $A \subseteq Z$ and $\epsilon > 0$, we set

$$B_\epsilon(A) := \{z \in Z \mid \exists z' \in A \text{ with } d_Z(z, z') < \epsilon\}.$$
$$\overline{B}_\epsilon(A) := \{z \in Z \mid \exists z' \in A \text{ with } d_Z(z, z') \leq \epsilon\}.$$

We abbreviate $B_\varepsilon(z) = B_\varepsilon(\{z\})$ and $\overline{B}_\varepsilon(z) = \overline{B}_\varepsilon(\{z\})$. A metric space Z is called *proper* if for every $R > 0$ and every point $z \in Z$ the closed ball $\overline{B}_R(z)$ of radius R around z is compact. A map is called *proper* if the preimage of any compact subset is again compact.

A *family of subgroups of a group* G is a set of subgroups closed under conjugation and taking subgroups. Denote by $\mathcal{VC}$yc the family of virtually cyclic subgroups.

Homotopy actions and S–long covers

Next we explain the notion of transfer reducible. In Bartels–Lück [1] we introduced the following definitions in order to formulate conditions on groups that imply that a group satisfies the Farrell–Jones conjecture in K– and L–theory.

Definition 0.1 (Homotopy S–action) Let S be a finite subset of a group G. Assume that S contains the trivial element $e \in G$. Let X be a space.

(i) A *homotopy S–action (φ, H) on X* consists of continuous maps $\varphi_g\colon X \to X$ for $g \in S$ and homotopies $H_{g,h}\colon X \times [0,1] \to X$ for $g, h \in S$ with $gh \in S$ such that $H_{g,h}(-,0) = \varphi_g \circ \varphi_h$ and $H_{g,h}(-,1) = \varphi_{gh}$ holds for $g, h \in S$ with $gh \in S$. Moreover, we require that $H_{e,e}(-,t) = \varphi_e = \mathrm{id}_X$ for all $t \in [0,1]$.

(ii) Let (φ, H) be a homotopy S–action on X. For $g \in S$ let $F_g(\varphi, H)$ be the set of all maps $X \to X$ of the form $x \mapsto H_{r,s}(x,t)$ where $t \in [0,1]$ and $r, s \in S$ with $rs = g$.

(iii) Let (φ, H) be a homotopy S–action on X. For $(g,x) \in G \times X$ and $n \in \mathbb{N}$, let $S^n_{\varphi,H}(g,x)$ be the subset of $G \times X$ consisting of all (h,y) with the following property: There are $x_0, \dots, x_n \in X$, $a_1, b_1, \dots, a_n, b_n \in S$ and $f_1, \widetilde{f}_1, \dots, f_n, \widetilde{f}_n\colon X \to X$, such that $x_0 = x$, $x_n = y$, $f_i \in F_{a_i}(\varphi, H)$, $\widetilde{f}_i \in F_{b_i}(\varphi, H)$, $f_i(x_{i-1}) = \widetilde{f}_i(x_i)$ and $h = g a_1^{-1} b_1 \dots a_n^{-1} b_n$.

(iv) Let (φ, H) be a homotopy S–action on X and $\mathcal{U}$ be an open cover of $G \times X$. We say that $\mathcal{U}$ is *S–long with respect to* (φ, H) if for every $(g,x) \in G \times X$ there is $U \in \mathcal{U}$ containing $S^{|S|}_{\varphi,H}(g,x)$ where $|S|$ is the cardinality of S.

Definition 0.2 (N–dominated space) Let X be a metric space and $N \in \mathbb{N}$. We say that X is *controlled N–dominated* if for every $\varepsilon > 0$ there is a finite CW–complex K of dimension at most N, maps $i\colon X \to K$, $p\colon K \to X$ and a homotopy $H\colon X \times [0,1] \to X$ between $p \circ i$ and id_X such that for every $x \in X$ the diameter of $\{H(x,t) \mid t \in [0,1]\}$ is at most ε.

Definition 0.3 (Open $\mathcal{F}$–cover) Let Y be a G–space. Let $\mathcal{F}$ be a family of subgroups of G. A subset $U \subseteq Y$ is called an *$\mathcal{F}$–subset* if

(i) for $g \in G$ we have $g(U) = U$ or $U \cap g(U) = \emptyset$, where $g(U) := \{gx \mid x \in U\}$;

(ii) the subgroup $G_U := \{g \in G \mid g(U) = U\}$ lies in $\mathcal{F}$.

An *open $\mathcal{F}$–cover* of Y is a collection $\mathcal{U}$ of open $\mathcal{F}$–subsets of Y such that the following conditions are satisfied:

(i) $Y = \bigcup_{U \in \mathcal{U}} U$;

(ii) for $g \in G$, $U \in \mathcal{U}$ the set $g(U)$ belongs to $\mathcal{U}$.

Definition 0.4 (Transfer reducible) Let G be a group and $\mathcal{F}$ be a family of subgroups. We will say that G is *transfer reducible* over $\mathcal{F}$ if there is a number N with the following property:

For every finite subset S of G there are

- a contractible compact controlled N–dominated metric space X;
- a homotopy S–action (φ, H) on X;
- a cover $\mathcal{U}$ of $G \times X$ by open sets,

such that the following holds for the G–action on $G \times X$ given by $g \cdot (h, x) = (gh, x)$:

(i) $\dim \mathcal{U} \le N$;

(ii) $\mathcal{U}$ is S–long with respect to (φ, H);

(iii) $\mathcal{U}$ is an open $\mathcal{F}$–covering.

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1 A flow space associated to a metric space

Summary In this section we introduce the flow space $FS(X)$ for arbitrary metric spaces. We show that $FS(X)$ is a proper metric space if X is a proper metric space (see Proposition 1.9). If X comes with a proper cocompact isometric G–action, then $FS(X)$ inherits a proper cocompact isometric G–action (see Proposition 1.11).

Definition 1.1 Let X be a metric space. A continuous map $c\colon \mathbb{R} \to X$ is called a *generalized geodesic* if there are $c_-, c_+ \in \overline{\mathbb{R}} := \mathbb{R} \coprod \{-\infty, \infty\}$ satisfying

$$c_- \le c_+, \qquad c_- \ne \infty \qquad \text{and} \qquad c_+ \ne -\infty,$$

such that c is locally constant on the complement of the interval $I_c := (c_-, c_+)$ and restricts to an isometry on I_c.

The numbers c_- and c_+ are uniquely determined by c, provided that c is not constant.

Definition 1.2 Let (X, d_X) be a metric space. Let $FS = FS(X)$ be the set of all generalized geodesics in X. We define a metric on $FS(X)$ by

$$d_{FS(X)}(c,d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}}\, dt.$$

Define a flow

$$\Phi\colon FS(X) \times \mathbb{R} \to FS(X)$$

by $\Phi_\tau(c)(t) = c(t+\tau)$ for $\tau \in \mathbb{R}$, $c \in FS(X)$ and $t \in \mathbb{R}$.

The integral $\int_{-\infty}^{+\infty} \frac{d_X(c(t),d(t))}{2e^{|t|}}\, dt$ exists since $d_X(c(t), d(t)) \le 2|t| + d_X(c(0), d(0))$ by the triangle inequality. Obviously $\Phi_\tau(c)$ is a generalized geodesic with

$$\Phi_\tau(c)_- = c_- - \tau \quad \text{and} \quad \Phi_\tau(c)_+ = c_+ - \tau,$$

where $-\infty - \tau := -\infty$ and $\infty - \tau := \infty$.

We note that any isometry $(X, d_X) \to (Y, d_Y)$ induces an isometry $FS(X) \to FS(Y)$ by composition. In particular, the isometry group of (X, d_X) acts canonically on $FS(X)$. Moreover, this action commutes with the flow.

For a general metric space X all generalized geodesics may be constant. Later we will consider the case where X is a CAT(0)–space, but in the remainder of this section we will consider properties of $FS(X)$ that do not depend on the CAT(0)–condition.

Lemma 1.3 *Let (X, d_X) be a metric space. The map Φ is a continuous flow and we have for $c, d \in FS(X)$ and $\tau, \sigma \in \mathbb{R}$*

$$d_{FS(X)}\big(\Phi_\tau(c), \Phi_\sigma(d)\big) \le e^{|\tau|} \cdot d_{FS(X)}(c,d) + |\sigma - \tau|.$$

Proof Obviously $\Phi_\tau \circ \Phi_\sigma = \Phi_{\tau+\sigma}$ for $\tau, \sigma \in \mathbb{R}$ and $\Phi_0 = \mathrm{id}_{FS(X)}$. The main task is to show that $\Phi\colon FS(X) \times \mathbb{R} \to FS(X)$ is continuous.

We estimate for $c \in FS(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned} d_{FS(X)}\big(c, \Phi_\tau(c)\big) &= \int_{\mathbb{R}} \frac{d_X\big(c(t), c(t+\tau)\big)}{2e^{|t|}}\, dt \\ &\le \int_{\mathbb{R}} \frac{|\tau|}{2e^{|t|}}\, dt \\ &= |\tau| \cdot \int_{\mathbb{R}} \frac{1}{2e^{|t|}}\, dt \\ &= |\tau|. \end{aligned}$$

We estimate for $c, d \in FS(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned} d_{FS(X)}\big(\Phi_\tau(c), \Phi_\tau(d)\big) &= \int_{\mathbb{R}} \frac{d_X\big(c(t+\tau), d(t+\tau)\big)}{2e^{|t|}}\, dt \\ &= \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t-\tau|}}\, dt \\ &\le \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t|-|\tau|}}\, dt \\ &= e^{|\tau|} \cdot \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t|}}\, dt \\ &= e^{|\tau|} \cdot d_{FS(X)}(c, d). \end{aligned}$$

The two inequalities above together with the triangle inequality imply for $c, d \in FS(X)$ and $\tau, \sigma \in \mathbb{R}$

$$\begin{aligned} d_{FS(X)}\big(\Phi_\tau(c), \Phi_\sigma(d)\big) &= d_{FS(X)}\big(\Phi_\tau(c), \Phi_{\sigma-\tau} \circ \Phi_\tau(d)\big) \\ &\le d_{FS(X)}\big(\Phi_\tau(c), \Phi_\tau(d)\big) + d_{FS(X)}\big(\Phi_\tau(d), \Phi_{\sigma-\tau} \circ \Phi_\tau(d)\big) \\ &\le e^{|\tau|} \cdot d_{FS(X)}(c, d) + |\sigma - \tau|. \end{aligned}$$

This implies that Φ is continuous at (c, τ). □

The following lemma relates distance in X to distance in $FS(X)$.

Lemma 1.4 *Let $c, d\colon \mathbb{R} \to X$ be generalized geodesics. Consider $t_0 \in \mathbb{R}$.*

(i) $d_X\big(c(t_0), d(t_0)\big) \le e^{|t_0|} \cdot d_{FS}(c, d) + 2$;

(ii) *If $d_{FS}(c, d) \le 2e^{-|t_0|-1}$, then*

$$d_X\big(c(t_0), d(t_0)\big) \le \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{FS}(c, d)}.$$

In particular, $c \mapsto c(t_0)$ defines a uniform continuous map $FS(X) \to X$.

Proof We abbreviate $D := d_X(c(t_0), d(t_0))$. Since c and d are generalized geodesics, we conclude using the triangle inequality for $t \in \mathbb{R}$

$$d_X(c(t), d(t)) \geq D - d_X\big(c(t_0), c(t)\big) - d_X\big(d(t_0), d(t)\big) \geq D - 2 \cdot |t - t_0|.$$

This implies

$$\begin{aligned} d_{FS(X)}(c,d) &= \int_{-\infty}^{+\infty} \frac{d_X(c(t), d(t))}{2e^{|t|}}\, dt \\ &\geq \int_{-D/2+t_0}^{D/2+t_0} \frac{D - 2\cdot |t - t_0|}{2e^{|t|}}\, dt \\ &= \int_{-D/2}^{D/2} \frac{D - 2\cdot |t|}{2e^{|t+t_0|}}\, dt \\ &\geq \int_{-D/2}^{D/2} \frac{D - 2\cdot |t|}{2e^{|t|+|t_0|}}\, dt \\ &= e^{-|t_0|} \cdot \int_{-D/2}^{D/2} \frac{D - 2\cdot |t|}{2e^{|t|}}\, dt \\ &= e^{-|t_0|} \cdot \int_{0}^{D/2} \big(D - 2t\big) \cdot e^{-t}\, dt \\ &= e^{-|t_0|} \cdot \big[(-D + 2 + 2t) \cdot e^{-t}\big]_0^{D/2} \\ &= e^{-|t_0|} \cdot \big(2 \cdot e^{-D/2} + D - 2\big). \end{aligned} \tag{1.5}$$

Since $e^{-|t_0|} \cdot \big(2 \cdot e^{-D/2} + D - 2\big) \geq e^{-|t_0|} \cdot (D - 2)$, assertion (i) follows. It remains to prove assertion (ii).

Consider the function $f(x) = e^{-x} + x - 1 - \frac{x^2}{2e}$. We have $f'(x) = -e^{-x} + 1 - \frac{x}{e}$ and $f''(x) = e^{-x} - e^{-1}$. Hence $f''(x) \geq 0$ for $x \in [0,1]$. Since $f'(0) = 0$, this implies $f'(x) \geq 0$ for $x \in [0,1]$. Since $f(0) = 0$, this implies $f(x) \geq 0$ for $x \in [0,1]$. Setting $x = D/2$ we obtain

$$\frac{(D/2)^2}{2e} \leq e^{-D/2} + D/2 - 1$$

for $D \in [0,2]$. From (1.5) we get

$$d_{FS}(c,d) \geq 2e^{-|t_0|} \cdot \big(e^{-D/2} + D/2 - 1\big).$$

Therefore

$$\frac{D^2}{4e^{|t_0|+1}} \leq d_{FS(X)}(c,d) \quad \text{if } D \leq 2.$$

Consider the function $g(x) = e^{-x} + x - 1$. Since $g'(x) = -e^{-x} + 1 > 0$ for $x \geq 1$, we conclude $g(x) > g(1) = e^{-1}$ for $x > 1$. Hence (1.5) implies

$$d_{FS(X)}(c,d) > 2e^{-|t_0|-1} \quad \text{if } D > 2.$$

Hence we have

$$d_X\big(c(t_0), d(t_0)\big) = D \;\leq\; \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{FS}(c,d)}.$$

if $d_{FS}(c,d) \leq 2 \cdot e^{-|t_0|-1}$. □

Lemma 1.6 *Let (X, d_X) be a metric space. The maps*

$$FS(X) - FS(X)^{\mathbb{R}} \to \overline{\mathbb{R}}, \quad c \mapsto c_-;$$
$$FS(X) - FS(X)^{\mathbb{R}} \to \overline{\mathbb{R}}, \quad c \mapsto c_+,$$

are continuous.

Proof By an obvious symmetry it suffices to consider the second map. Let $c \in FS(X) - FS(X)^{\mathbb{R}}$. Let $\alpha_0 \in \mathbb{R}$ with $\alpha_0 < c_+$. We will first show that there is ε_0 such that $d_+ > \alpha_0$ if $d_{FS}(c,d) < \varepsilon_0$. Pick $s_0, t_0 \in \mathbb{R}$ such that $\alpha_0 < s_0 < t_0 < c_+$ and $c(s_0) \neq c(t_0)$. By Lemma 1.4 (ii) there is $\varepsilon_0 > 0$ such that $d_{FS}(c,d) < \varepsilon_0$ implies

$$\max\big\{d_X\big(c(s_0), d(s_0)\big), d_X\big(c(t_0), d(t_0)\big)\big\} < \frac{d_X\big(c(s_0), c(t_0)\big)}{3}.$$

For d with $d_{FS}(c,d) < \varepsilon_0$ we have $d_X\big(d(s_0), d(t_0)\big) > d_X\big(c(s_0), c(t_0)\big)/3$ by the triangular inequality and in particular, $d(s_0) \neq d(t_0)$. This implies $d_+ > s_0$ and therefore $d_+ > \alpha_0$. If $c_+ = +\infty$, then this shows that the second map is continuous at c.

If $c_+ < \infty$, then we need to show in addition that for a given $\alpha_1 > c_+$, there is $\varepsilon_1 > 0$ such that $d_+ < \alpha_1$ for all d with $d_{FS}(c,d) < \varepsilon_1$. Note that the previous argument also implied that $d_- < t_0 < c_+$ if $d_{FS}(c,d) < \varepsilon_0$ (because then $d(s_0) \neq d(t_0)$). Pick now $s_1, t_1 \in \mathbb{R}$ with $c_+ < s_1 < t_1 < \alpha_1$. By Lemma 1.4 (ii) there is $\varepsilon_1 \in \mathbb{R}$ satisfying $0 < \varepsilon_1 < \varepsilon_0$ such that $d_{FS}(c,d) < \varepsilon_1$ implies

$$\max\big\{d_X\big(c(s_1), d(s_1)\big), d_X\big(c(t_1), d(t_1)\big)\big\} < \frac{t_1 - s_1}{2}.$$

Because $c(s_1) = c(t_1)$ we get $d_X\big(d(s_1), d(t_1)\big) < t_1 - s_1$ for d with $d_{FS}(c,d) < \varepsilon_1$. This implies $d_+ < t_1$ or $d_- > s_1$, because otherwise $d_X\big(d(s_1), d(t_1)\big) = t_1 - s_1$. However, $d_- < c_+ < s_1$ because $\varepsilon_1 < \varepsilon_0$. Thus $d_+ < t_1 < \alpha_1$. □

Proposition 1.7 *Let (X, d_X) be a metric space. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $FS(X)$. Then it converges uniformly on compact subsets to $c \in FS(X)$ if and only if it converges to c with respect to $d_{FS(X)}$.*

Proof From Lemma 1.4 (ii) we conclude that convergence with respect to $d_{FS(X)}$ implies uniform convergence on compact subsets.

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $FS(X)$ that converges uniformly on compact subsets to $c \in FS(X)$. Let $\varepsilon > 0$. Pick $\alpha \geq 1$ such that $\int_\alpha^\infty \frac{2t+\varepsilon}{e^t} dt < \varepsilon$. Because of uniform convergence on $[-\alpha, \alpha]$, there is n_0 such that $d_X(c_n(t), c(t)) \leq \varepsilon/\alpha$ for all $n \geq n_0$, $t \in [-\alpha, \alpha]$. In particular, $d_X(c_n(t), c(t)) \leq \varepsilon + 2|t|$ for all t, provided $n \geq n_0$. Thus for $n \geq n_0$

$$\begin{aligned} d_{FS}(c_n, c) &= \int_{-\infty}^{\infty} \frac{d_X\big(c_n(t), c(t)\big)}{2e^{|t|}} dt \\ &\leq \int_{-\alpha}^{\alpha} \frac{\varepsilon/\alpha}{2e^{|t|}} dt + 2\int_\alpha^\infty \frac{\varepsilon + 2t}{2e^{|t|}} dt \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

This shows $c_n \to c$ with respect to d_{FS}, because ε was arbitrary. □

Lemma 1.8 *Let (X, d_X) be a metric space. The flow space $FS(X)$ is sequentially closed in the space of all maps $\mathbb{R} \to X$ with respect to the topology of uniform convergence on compact subsets.*

Proof Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of generalized geodesics that converges uniformly on compact subsets to $f\colon \mathbb{R} \to X$. We have to show that f is a generalized geodesic. By passing to a subsequence we can assume that either $c_n \in FS^{\mathbb{R}}$ for all n, or $c_n \notin FS^{\mathbb{R}}$ for all n. In the first case $f \in FS^{\mathbb{R}}$. Thus it remains to treat the second case. In this case we have well-defined sequences $(c_n)_-$ and $(c_n)_+$. After passing to a further subsequence we can assume that these sequences converge in $[-\infty, \infty]$. Thus there are $\alpha_-, \alpha_+ \in [-\infty, \infty]$ such that $(c_n)_\pm \to \alpha_\pm$ as $n \to \infty$. We will show that f is a generalized geodesic with $f_\pm = \alpha_\pm$ or $f \in FS^{\mathbb{R}}$. Clearly, $d_X(f(s), f(t)) \leq |t - s|$ for all s, t.

If $\alpha_- > -\infty$, then we have to show that $f(s) = f(t)$ for all $s < t \leq \alpha_-$. Pick $\varepsilon > 0$. There is n_0 such that $|f(\tau) - c_n(\tau)| < \varepsilon$ for all $\tau \in [s - \varepsilon, t]$, $n \geq n_0$. Since $(c_n)_- \to \alpha_-$, there is $k \geq n_0$ such that $s - \varepsilon, t - \varepsilon \leq (c_k)_-$. Thus

$$\begin{aligned} d_X(f(s), f(t)) &\leq d_X(f(s-\varepsilon), f(t-\varepsilon)) + 2\varepsilon \\ &\leq d_X(c_k(s-\varepsilon), c_k(t-\varepsilon)) + 4\varepsilon \\ &= 4\varepsilon. \end{aligned}$$

Because ε is arbitrary, we conclude $f(s) = f(t)$. If $\alpha_+ < \infty$, then a similar argument shows that $f(s) = f(t)$ for all $s, t \geq \alpha_+$. If $\alpha_- = \alpha_+$, then $f \in FS(X)^{\mathbb{R}}$ and we are done. It remains to treat the case $\alpha_- < \alpha_+$. We have to show that $d_X(f(s), f(t)) = t - s$ for all $s, t \in \mathbb{R}$ with $\alpha_- \leq s < t \leq \alpha_+$. Pick $\varepsilon > 0$, such that $2\epsilon < t - s$. There is n_0 such that $|f(\tau) - s_n(\tau)| < \varepsilon$ for all $\tau \in [s,t]$, $n \geq n_0$. Since $(c_n)_\pm \to \alpha_\pm$, there is $k \geq n_0$ such that $(c_k)_- \leq s + \varepsilon < t - \varepsilon \leq (c_k)_+$. Thus

$$\begin{aligned} d_X(f(s), f(t)) &\geq d_X(f(s+\varepsilon), f(t-\varepsilon)) - 2\varepsilon \\ &\geq d_X(c_k(s+\varepsilon), c_k(t-\varepsilon)) - 4\varepsilon \\ &= ((t-\varepsilon) - (s+\varepsilon)) - 4\varepsilon \\ &= t - s - 6\varepsilon. \end{aligned}$$

This implies $d_X(f(s), f(t)) = t - s$, because ε was arbitrarily small. □

Proposition 1.9 *If (X, d_X) is a proper metric space, then $(FS(X), d_{FS(X)})$ is a proper metric space.*

Proof Let $R > 0$ and $c \in FS(X)$. It suffices to show that the closed ball $\overline{B}_R(c)$ in $FS(X)$ is sequentially compact since any metric space satisfies the second countability axiom. Let $(c_n)_n \in \mathbb{N}$ be a sequence in $\overline{B}_R(c)$. By Lemma 1.4 (i) there is $R' > 0$ such that $c_n(0) \in \overline{B}_{R'}(c(0))$. By assumption $\overline{B}_{R'}(c(0))$ is compact. Now we can apply the Arzelà–Ascoli theorem (see for example Bridson–Haefliger [4, I.3.10, page 36]). Thus after passing to a subsequence there is $d\colon \mathbb{R} \to X$ such that $c_n \to d$ uniformly on compact subsets. Lemma 1.8 implies $d \in FS(X)$. □

Lemma 1.10 *Let (X, d_X) be a proper metric space and $t_0 \in \mathbb{R}$. Then the evaluation map $FS(X) \to X$ defined by $c \mapsto c(t_0)$ is uniformly continuous and proper.*

Proof The map is uniformly continuous by Lemma 1.4 (ii). To show that is also proper, it suffices by Proposition 1.9 to show that preimages of closed balls have finite diameter. If $d_X(c(t_0), d(t_0)) \leq r$, then $d_X(c(t), d(t)) \leq r + 2|t - t_0|$. Thus

$$d_{FS}(c,d) \leq \int_{\mathbb{R}} \frac{r + 2|t-t_0|}{2e^{|t|}} dt \quad \text{provided} \quad d_X\big(c(t_0), d(t_0)\big) \leq r. \qquad \square$$

Proposition 1.11 *Let G act isometrically and proper on the proper metric space (X, d_X). Then the action of G on $(FS(X), d_{FS})$ is also isometric and proper. If the action of G on X is in addition cocompact, then this is also true for the action on $FS(X)$.*

Proof The action of G on $FS(X)$ is isometric. The map $FS(X) \to X$ defined by $c \mapsto c(0)$ is G–equivariant, continuous and proper by Lemma 1.10. The existence of such a map implies that the G–action on $FS(X)$ is also proper. This also implies that the action of G on $FS(X)$ is cocompact, provided that the action on X is itself cocompact. □

Lemma 1.12 *Let (X, d_X) be a metric space. Then $FS(X)^{\mathbb{R}}$ is closed in $FS(X)$.*

Proof Note that $FS(X)^{\mathbb{R}}$ is the space of constant generalized geodesics. Let $c \in FS(X) - FS(X)^{\mathbb{R}}$. Pick $t_0, t_1 \in \mathbb{R}$ such that $c(t_0) \neq c(t_1)$. Set $\delta := d_X(c(t_0), c(t_1))/2$. For $x \in X$ then $d_X(x, c(t_0)) \geq \delta$ or $d_X(x, c(t_1)) \geq \delta$. Denote by c_x the constant generalized geodesic at x. If $d_X(x, c(t_0)) \geq \delta$, then $d_X(x, c(t)) \geq \delta/2$ if $t \in [t_0 - \delta/2, t_0 + \delta/2]$. Thus in this case

$$d_{FS}(c_x, c) \geq \int_{t_0-\delta/2}^{t_0+\delta/2} \frac{\delta/2}{2e^{|t|}} dt =: \varepsilon_0.$$

Similarly,

$$d_{FS}(c_x, c) \geq \int_{t_1-\delta/2}^{t_1+\delta/2} \frac{\delta/2}{2e^{|t|}} dt =: \varepsilon_1,$$

if $d_X(x, c(t_1)) \geq \delta/2$. Hence $B_\varepsilon(c) \cap FS(X)^{\mathbb{R}} = \varnothing$ if we set $\varepsilon := \min\{\varepsilon_0/2, \varepsilon_1/2\}$. □

Notation 1.13 Let X be a metric space. For $c \in FS(X)$ and $T \in [0, \infty]$, define $c|_{[-T,T]} \in FS(X)$ by

$$c|_{[-T,T]}(t) := \begin{cases} c(-T) & \text{if } t \leq -T; \\ c(t) & \text{if } -T \leq t \leq T; \\ c(T) & \text{if } t \geq T. \end{cases}$$

Obviously $c|_{[-\infty,\infty]} = c$ and if $c \notin FS(X)^{\mathbb{R}}$ and $(-T, T) \cap (c_-, c_+) \neq \varnothing$ then $\big(c|_{[-T,T]}\big)_- = \max\{c_-, -T\}$ and $\big(c|_{[-T,T]}\big)_+ = \min\{c_+, T\}$.

We denote by

$$FS(X)_f := \big\{c \in FS(X) - FS(X)^{\mathbb{R}} \bigm| c_- > -\infty, c_+ < \infty\big\} \cup FS(X)^{\mathbb{R}}$$

the subspace of finite geodesics.

Lemma 1.14 *Let (X, d_X) be a metric space. The map $H\colon FS(X) \times [0, 1] \to FS(X)$ defined by $H_\tau(c) := c|_{[\ln(\tau), -\ln(\tau)]}$ is continuous and satisfies $H_0 = \mathrm{id}_{FS(X)}$ and $H_\tau(c) \in FS(X)_f$ for $\tau > 0$.*

Proof Observe that for $c \in FS(X)$ and $T, T' \geq 0$

$$d_X\big(c|_{[-T,T]}(t), c|_{[-T',T']}(t)\big) \leq |T - T'| \quad \text{for all } t \in \mathbb{R}.$$

Recall from Proposition 1.7 that the topology on $FS(X)$ is the topology of uniform convergence on compact subsets. Let $c_n \to c$ uniformly on compact subsets, and $\tau_n \to \tau$. Let $\alpha > 0$. We need to show that $c_n|_{[\ln(\tau_n), -\ln(\tau)]} \to c|_{[\ln(\tau), -\ln(\tau)]}$ uniformly on $[-\alpha, \alpha]$.

Consider first the case $\tau = 0$. Then $c = c|_{[\ln(\tau), -\ln(\tau)]}$. Moreover, $-\ln(\tau_n) > \alpha$ for sufficient large n. Thus $c_n(t) = c_n|_{[\ln(\tau_n), -\ln(\tau)]}(t)$ for such n and $t \in [-\alpha, \alpha]$. This implies the claim for $\tau = 0$.

Next consider the case $\tau \in (0, 1]$. Let $\varepsilon > 0$. There is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} d_X(c_n(t), c(t)) &\leq \varepsilon \text{ if } n \geq n_0, t \in [-\alpha, \alpha]; \\ |\ln(\tau) - \ln(\tau_n)| &\leq \varepsilon \text{ if } n \geq n_0. \end{aligned}$$

Then for $t \in [-\alpha, \alpha]$, $n \geq n_0$,

$$\begin{aligned} d_X\big(c_n|_{[\ln(\tau_n), -\ln(\tau_n)]}(t), c|_{[\ln(\tau), -\ln(\tau)]}(t)\big) &\leq d_X\big(c_n|_{[\ln(\tau_n), -\ln(\tau_n)]}(t), c_n|_{[\ln(\tau), -\ln(\tau)]}(t)\big) \\ &\quad + d_X\big(c_n|_{[\ln(\tau), -\ln(\tau)]}(t), c|_{[\ln(\tau), -\ln(\tau)]}(t)\big) \\ &\leq 2\varepsilon. \end{aligned}$$

This implies the claim in the second case because ε was arbitrary. □

2 The flow space associated to a CAT(0)- space

Summary In this section we study $FS(X)$ further in the case where X is a CAT(0)–space. Let $\overline{X}$ be the bordification of X. We construct an injective continuous map from $FS(X) - FS(X)^{\mathbb{R}}$ to $\overline{\mathbb{R}} \times \overline{X} \times X \times \overline{X} \times \overline{\mathbb{R}}$ which is a homeomorphism onto its image (see Proposition 2.6). It sends a generalized geodesic c to $\big(c_-, c(-\infty), c(0), c(\infty), c_+\big)$, where $c(-\infty)$ and $c(\infty)$ are the two endpoints of c. This is used to show that $FS(X) - FS(X)^{\mathbb{R}}$ has finite dimension if X has (see Proposition 2.9), and that $FS(X) - FS(X)^{\mathbb{R}}$ is locally connected (see Proposition 2.10). We will relate our construction to the geodesic flow on the sphere tangent bundle of a simply connected Riemannian manifold with non-positive sectional curvature in Section 2.5.

For the definition of a CAT(0)–space we refer to Bridson–Haefliger [4, II.1.1, page 158], namely to be a geodesic space all of whose geodesic triangles satisfy the CAT(0)–inequality. We will follow the notation and the description of the *bordification* $\overline{X} =$

$X \cup \partial X$ of a CAT(0)–space X given in [4, Chapter II.8]. The definition of the topology of this bordification is briefly reviewed in Remark 2.3. In this section we will use the following convention.

Convention 2.1 *Let*

- X *be a complete* CAT(0)*–space;*
- $\overline{X} := X \cup \partial X$ *be the bordification of* X, *see [4, Chapter II.8].*

2.1 Evaluation of generalized geodesics at infinity

Definition 2.2 For $c \in FS(X)$ we set $c(\pm\infty) := \lim_{t\to\pm\infty} c(t)$, where the limit is taken in $\overline{X}$.

Since X is by assumption a CAT(0)–space, we can find for $x_- \in X$ and $x_+ \in \overline{X}$ a generalized geodesic $c\colon \mathbb{R} \to X$ with $c(\pm\infty) = x_\pm$ (see [4, II.8.2, page 261]). It is not true in general that for two different points x_- and x_+ in ∂X there is a geodesic c with $c(-\infty) = x_-$ and $c(\infty) = x_+$.

Remark 2.3 (Cone topology on $\overline{X}$) A *generalized geodesic ray* is a generalized geodesic c that is either a constant generalized geodesic or a non-constant generalized geodesic with $c_- = 0$. Fix a base point $x_0 \in X$. For every $x \in \overline{X}$, there is a unique generalized geodesic ray c_x such that $c(0) = x_0$ and $c(\infty) = x$, see [4, II.8.2, page 261]. Define for $r > 0$

$$\rho_r = \rho_{r,x_0}\colon \overline{X} \to \overline{B}_r(x_0)$$

by $\rho_r(x) := c_x(r)$. The sets $(\rho_r)^{-1}(V)$ with $r > 0$, V an open subset of $\overline{B}_r(x_0)$ are a basis for the cone topology on $\overline{X}$, see [4, II.8.6, page 263]. A map f whose target is $\overline{X}$ is continuous if and only if $\rho_r \circ f$ is continuous for all r. The cone topology is independent of the choice of base point, see [4, II.8.8, page 264].

Lemma 2.4 *The maps*

$$FS(X) - FS(X)^{\mathbb{R}} \to \overline{X}, \quad c \mapsto c(-\infty);$$
$$FS(X) - FS(X)^{\mathbb{R}} \to \overline{X}, \quad c \mapsto c(\infty),$$

are continuous.

The proof of this Lemma depends on the following result.

Lemma 2.5 *Given $\varepsilon > 0$, $a > 0$ and $s > 0$, there exists a constant $T = T(\varepsilon, a, s) > 0$ such that the following in true: if x, $x' \in X$ with $d_X(x, x') \leq a$, if $c\colon \mathbb{R} \to X$ is a generalized geodesic ray with $c(0) = x$, and if $\sigma_t\colon [0, d(x', c(t)] \to X$ is the geodesic from x' to $c(t)$, then $d_X(\sigma_t(s), \sigma_{t+t'}(s)) < \varepsilon$ for all $t \geq T$ and all $t' \geq 0$.*

Proof In [4, II.8.3, page 261] this is proven under the additional assumptions that c is a geodesic ray and that $d_X(x, x') = a$. But the proof given in [4] can be adapted as follows to give our more general result.

The argument given in [4] can be applied without change to show that there is T such that $d_X(\sigma_t(s), \sigma_{t+t'}(s)) < \varepsilon$ for all $t \geq T$, $t' \geq 0$ provided that $t + t' \leq c_+$. (This is needed to deduce that $|t - a| \leq d_X(x', c(t))$ and that $|t + t' - a| \leq d_X(x', c(t + t'))$.)

It remains to treat the case where $t \geq T$, $t' \geq 0$ and $t + t' \geq c_+$. If $t \geq c_+$, then $\sigma_t = \sigma_{t+t'}$ (because $c(t + t') = c(t) = c(c_+)$) and there is nothing to show. Thus we can assume $t \leq c_+$. Set $t'' := c_+ - t$. Then $t + t'' \leq c_+$, $t'' \geq 0$ and $\sigma_{t+t'} = \sigma_{t+t''}$ (because $c(t + t') = c(t + t'') = c(c_+)$). Thus

$$d_X(\sigma_t(s), \sigma_{t+t'}(s)) = d_X(\sigma_t(s), \sigma_{t+t''}(s)) < \varepsilon. \qquad \square$$

Proof of Lemma 2.4 By an obvious symmetry it suffices to consider the second map. Let $c \in FS(X) - FS(X)^{\mathbb{R}}$. Set $s_0 := \max\{0, c_-\} + 1$ and $x_0 := c(s_0)$. Then

$$\widetilde{c}\colon \mathbb{R} \to X, \quad t \mapsto \begin{cases} c(t + s_0) & t \geq 0; \\ c(s_0) & t \leq 0, \end{cases}$$

is a generalized geodesic ray starting at x_0. We need to show that the map $f_r\colon FS(X) \to \overline{B}_r(x_0)$ defined by $f_r(d) := \rho_r(d(\infty))$ is for all $r > 0$ continuous at c, see Remark 2.3. Note that $f_r(c) = \widetilde{c}(r) = c(s_0 + r)$.

Let $\varepsilon > 0$ be given. By Lemma 1.6 there is δ_0 such that $d_- < s_0$ for all d with $d_{FS}(c, d) < \delta_0$. In particular, we obtain for any such generalized geodesic ray d a generalized geodesic ray $\widetilde{d}$ by putting $\widetilde{d}(t) = d(t + s_0)$ for $t \geq 0$ and $\widetilde{d}(t) = d(s_0)$ for $t \leq 0$.

For $t > 0$ and $d \in FS$ with $d_{FS}(c, d) < \delta_0$ denote by $\sigma_t^d\colon [0, d_X(x_0, d(s_0 + t))] \to X$ the geodesic from x_0 to $d(s_0 + t)$. By Lemma 2.5 there is a number T (not depending on d!) such that $d_X(\sigma_t^d(r), \sigma_{t+t'}^d(r)) < \varepsilon$ for all $t' > 0$, $t \geq T$, provided that $d_X(d(s_0), x_0) \leq 1$. We extend σ_t^d to a generalized geodesic ray by setting $\sigma_t^d(s) := d(s_0 + t)$ for $s > d_X(x_0, d(s_0 + t))$ and $\sigma_t^d(s) := x_0$ for $s < 0$. The unique generalized geodesic ray $c_{d(\infty)}$ from x_0 to $d(\infty)$ can be constructed as the limit of the σ_t^d, see [4, Proof of (8.2), page 262]. It follows that $\sigma_t^d(r) \to c_{d(\infty)}(r)$ as $t \to \infty$.

By definition of ρ_r we have $c_{d(\infty)}(r) = f_r(d)$. Therefore $d_X(\sigma_T(r), f_r(d)) \leq \varepsilon$, provided that $d_X(d(s_0), c(s_0)) \leq 1$.

By Lemma 1.4 (ii) there exists $0 < \delta_1 < \delta_0$ such that

$$d_X(c(s_0), d(s_0)) < 1 \quad \text{and} \quad d_X(c(s_0+T), d(s_0+T)) < \varepsilon$$

if $d_{FS}(c,d) < \delta_1$. Consider the triangle whose vertices are $x_0 = c(s_0)$, $c(s_0+T)$ and $d(s_0+T)$. Recall that σ_T^d is side of this triangle that connects x_0 to $d(s_0+T)$. Using the CAT(0)–condition in this triangle it can be deduced that if $d_{FS}(c,d) < \delta_1$ then

$$d_X(c(s_0+r), \sigma_T^d(r)) < 2\varepsilon.$$

Therefore for $d_{FS}(c,d) < \delta_1$ we conclude

$$\begin{aligned} d_X(f_r(c), f_r(d)) &= d_X(c(r+s_0), c_{d(\infty}(r)) \\ &\leq d_X(c(r+s_0), \sigma_T^d(r)) + d_X(\sigma_T^d(r), c_{d(\infty)}(r)) \\ &< 3\varepsilon. \end{aligned}$$

Because ε was arbitrary this implies that f_r is continuous at c. □

2.2 Embeddings of the flow space

Proposition 2.6 *If X is proper as a metric space, then the map*

$$E\colon FS(X) - FS(X)^{\mathbb{R}} \to \bar{R} \times \bar{X} \times X \times \bar{X} \times \bar{R}$$

defined by $E(c) := (c_-, c(-\infty), c(0), c(\infty), c_+)$ is injective and continuous. It is a homeomorphism onto its image.

Proof Lemmas 1.4 (ii), 1.6 and 2.4 imply that E is continuous.

Next we show that E is injective. Let $c \in FS(X) - FS(X)^{\mathbb{R}}$. If $t_0 \in [c_-, c_+]$, then $t \mapsto c(t+t_0)$, $t \geq 0$ is the unique [4, II.8.2, page 261] (generalized) geodesic ray from $c(t_0)$ to $c(\infty)$, and similarly $t \mapsto c(t_0 - t)$, $t \geq 0$ is the unique (generalized) geodesic ray from $c(t_0)$ to $c(-\infty)$. Let c, $d \in FS(X) - FS(X)^{\mathbb{R}}$, with $c(\pm\infty) = d(\pm\infty)$, $c_\pm = d_\pm$. Then c and d will agree if and only if $c(t) = d(t)$ for some $t \in [c_-, c_+]$, $t \neq \pm\infty$. If $E(c) = E(d)$, then there is such a t: if one of c_- and c_+ is real (not $\pm\infty$), then it can be used, otherwise $t = 0$ works.

It remains to show that the inverse E^{-1} to E defined on the image of E is continuous. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $FS(X) - FS(X)^{\mathbb{R}}$ and $c \in FS(X) - FS(X)^{\mathbb{R}}$ such that $E(c_n) \to E(c)$ as $n \to \infty$. We have to show that $c_n \to c$ as $n \to \infty$. We proceed by contradiction and assume this fails. Then there is a subsequence c_{n_k} and $\rho > 0$

such that $d_{FS}(c, c_{n_k}) > \rho$ for all k. We can pass to this subsequence and assume $d_{FS}(c, c_n) > \rho$ for all n. We have $c_n(0) \to c(0)$ as $n \to \infty$. The evaluation at $t = 0$ is a proper map $FS(X) \to X$ by Lemma 1.10. Thus we can pass to a further subsequence and assume that $c_n \to d$ as $n \to \infty$ with $d \in FS(X)$.

We claim that $d \notin FS(X)^{\mathbb{R}}$. Because $c \notin FS(X)^{\mathbb{R}}$ we have either $c(-\infty) \neq c(0)$ or $c(\infty) \neq c(0)$. By symmetry we may assume $c(-\infty) \neq c(0)$. We consider now two different cases.

Case 1 $(c_- \neq -\infty)$ Then $c_- \in \mathbb{R}$ and we can consider the evaluation at c_-. We have $(c_n)_- \to c_-$, $c_n(-\infty) \to c(-\infty)$ and $c_n(0) \to c(0)$ since $E(c_n) \to E(c)$ as $n \to \infty$. Moreover $c_n(c_-) \to d(c_-)$ and $c_n(0) \to d(0)$ since $c_n \to d$ as $n \to \infty$. Therefore $c(0) = d(0)$ and we get

$$\begin{aligned} d_X\big(d(c_-), c(c_-)\big) &\leq d_X\big(d(c_-), c_n(c_-)\big) + d_X\big(c_n(c_-), c_n((c_n)_-)\big) \\ &\quad + d_X\big(c_n((c_n)_-), c(c_-)\big) \\ &\leq d_X\big(d(c_-), c_n(c_-)\big) + \big|c_- - (c_n)_-\big| + d_X\big(c_n(-\infty), c(-\infty)\big) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $d(c_-) = c(c_-) = c(-\infty) \neq c(0) = d(0)$. Therefore $d \notin FS(X)^{\mathbb{R}}$.

Case 2 $(c_- = -\infty)$ Because $(c_n)_+ \to c_+ \neq -\infty$, there is $K > 0$ such that $-K < (c_n)_+$ for all n. Since $(c_n)_- \to c_- = -\infty$ we have $(c_n)_- < -2K$ for all sufficiently large n. Then

$$\begin{aligned} d_X\big(c_n(-2K), c_n(0)\big) &= d_X\big(c_n(-2K), c_n(-K)\big) + d_X\big(c_n(-K), c_n(0)\big) \\ &\geq d_X\big(c_n(-2K), c_n(-K)\big) \\ &= K \end{aligned}$$

for sufficiently large n. Using Lemma 1.10 we conclude $d_X\big(d(-2K), d(0)\big) \geq K$. Therefore $d \notin FS(X)^{\mathbb{R}}$. This finishes the proof of the claim.

Because $d \in FS(X) - FS(X)^{\mathbb{R}}$ we can apply E to d and deduce $E(d) = E(c)$ from continuity of E. Thus $c = d$ because E is injective. This contradicts $d_{FS}(c, c_n) > \rho$ for all n and finishes the proof. □

Recall that $FS(X)_f$ is the subspace of finite geodesics, see Notation 1.13.

Proposition 2.7 *Assume that X is proper as a metric space. Then the map*

$$E_f\colon FS(X)_f - FS(X)^{\mathbb{R}} \to \mathbb{R} \times X \times X$$

defined by $E_f(c) = \big(c_-, c(-\infty), c(\infty)\big)$ *is a homeomorphism onto its image*

$$\operatorname{im} E_f = \{(r,x,y) \mid x \neq y\}.$$

In particular, $FS(X)_f - FS(X)^{\mathbb{R}}$ *is locally path connected.*

Proof Lemmas 1.6 and 2.4 imply that E_f is continuous. The map E_f is injective with the stated image because of existence and uniqueness of geodesics between points in X, see [4, II.1.4, page 160],

Next we show that the induced map

$$E_f\colon FS(X)_f - FS(X)^{\mathbb{R}} \to \{(r,x,y) \mid x \neq y\}$$

is proper. Let $K \subset \{(r,x,y) \mid x \neq y\}$ be compact. We will show that ${E_f}^{-1}(K)$ is sequentially compact. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in ${E_f}^{-1}(K)$. After passing to a subsequence, we may assume that $E_f(c_n)$ converges in K. Thus $(c_n)_- \to t_0 \in \mathbb{R}$, $c_n(-\infty) \to x_- \in X$, $c_n(\infty) \to x_+ \in X$, and $x_- \neq x_+$. We have

$$\begin{aligned} d_X(c_n(t_0), x_-) &\leq d_X(c_n(t_0), c_n(-\infty)) + d_X(c_n(-\infty), x_-) \\ &\leq |t_0 - (c_n)_-| + d_X(c_n(-\infty), x_-). \end{aligned}$$

Thus $c_n(t_0) \to x_-$ as $n \to \infty$. Using Lemma 1.10 we deduce that $(c_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $FS(X)$, that we will again denote by c_n. So now $c_n \to c$ in $FS(X)$ for some $c \in FS(X)$.

We will show next that $c \notin FS^{\mathbb{R}}$. We have

$$(c_n)_+ - (c_n)_- = d_X(c_n(\infty), c_n(-\infty)) \to d_X(x_-, x_+),$$

as $n \to \infty$. Thus $(c_n)_+ \to t_1 := t_0 + d_X(x_-, x_+)$. We have

$$\begin{aligned} d_X(c_n(t_1), x_+) &\leq d_X(c_n(t_1), c_n(\infty)) + d_X(c_n(\infty), x_+) \\ &\leq |t_1 - (c_n)_+| + d_X(c_n(\infty), x_+). \end{aligned}$$

Thus $c_n(t_1) \to x_+$ as $n \to \infty$. From Lemma 1.10 we conclude

$$c(t_1) = \lim c_n(t_1) = x_+ \neq x_- = c(t_0).$$

Thus $c(t_1) \neq c(t_0)$ and $c \in FS - FS^{\mathbb{R}}$.

Now $E_f(c_n) \to E_f(c)$ as E_f is continuous. Therefore $c_- = t_0$, $c(-\infty) = x_-$ and $c(\infty) = x_+$. Thus $c \in {E_f}^{-1}(K)$. Hence $E_f\colon FS(X)_f - FS(X)^{\mathbb{R}} \to \{(r,x,y) \mid x \neq y\}$ is an injective continuous proper maps of metric spaces. This implies that it is a homeomorphism (see Steenrod [12, 2.2 and 2.7]).

By the existence of geodesics the image of E_f is locally path connected. Hence $FS(X)_f - FS(X)^{\mathbb{R}}$ is locally path connected. □

2.3 Covering dimension of the flow space

We will need the following elementary fact.

Lemma 2.8 *If X is proper as a metric space and its covering dimension* $\dim X$ *is* $\leq N$, *then* $\dim \overline{X} \leq N$.

Proof Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of $\overline{X}$. Recall from Remark 2.3 that a basis for the topology on $\overline{X}$ is given by sets of the form $\rho_r^{-1}(W)$ for $r \geq 0$ and open $W \subseteq \overline{B}_r(x_0)$, where we fix a base point x_0.) Thus for every $x \in \overline{X}$ there are r_x, $W_x \subseteq \overline{B}_{r_x}(x_0)$ and $U_x \in \mathcal{U}$ such that $x \in \rho_{r_x}^{-1}(W_x) \subset U_x$. Because $\overline{X}$ is compact a finite number of the sets $\rho_{r_x}^{-1}(W_x)$ cover $\overline{X}$. Note that $\rho_r = \rho_r|_{\overline{B}_{r'}(x_0)} \circ \rho_{r'}$, if $r' > r$. Therefore we can refine $\mathcal{U}$ to a finite cover $\mathcal{V}$, such that there is r and a finite cover $\mathcal{W}$ of $\overline{B_r(x_0)}$ such that

$$\mathcal{V} = \rho_r^{-1}(\mathcal{W}) := \{\rho_r^{-1}(W) \mid W \in \mathcal{W}\}.$$

The result follows because $\overline{B_r(x_0)}$ is closed in X and thus $\dim \overline{B_r(x_0)} \leq \dim X$. □

Proposition 2.9 *Assume that X is proper and that* $\dim X \leq N$. *Then*

$$\dim\bigl(FS(X) - FS(X)^{\mathbb{R}}\bigr) \leq 3N + 2.$$

Proof The image of any compact subset under a continuous map is compact and a bijective continuous map with a compact subset as source and Hausdorff space as target is a homeomorphism. Hence every compact subset K of $FS(X) - FS(X)^{\mathbb{R}}$ is homeomorphic to a compact subset of $\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$ by Proposition 2.6 and hence its topological dimension satisfies

$$\dim(K) \leq \dim\bigl(\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}\bigr) = 2\dim(\overline{R}) + 2\dim(\overline{X}) + \dim(X) \leq 3N + 2.$$

because of Lemma 2.8. Since $FS(X)$ is a proper metric space by Proposition 1.9, it is locally compact and can be written as the countable union of compact subspaces and hence contains a countable dense subset. This implies that $FS(X)$ has a countable basis for its topology. Since $FS(X) - FS(X)^{\mathbb{R}}$ is an open subset of $FS(X)$, the topological space $FS(X) - FS(X)^{\mathbb{R}}$ is locally compact and has a countable basis for its topology. Now $\dim\bigl(FS(X) - FS(X)^{\mathbb{R}}\bigr) \leq 3N + 2$ follows from Munkres [10, Chapter 7.9, Exercise 9, page 315]. □

2.4 The flow space is locally connected

A topological space Y is called *semi-locally path-connected* if for any $y \in Y$ and neighborhood V of y there is an open neighborhood U of y such that for every $z \in U$ there is a path w in V from y to z. Recall that Y is called *locally connected* or *locally path-connected* if any neighborhood V of any point $y \in Y$ contains an open neighborhood U of y such that U itself is connected or path-connected respectively. Suppose that Y is semi-locally path-connected. Then any open subset of Y is again semi-locally path-connected and each component of any open subset of Y is an open subset of Y. The latter is equivalent to the condition that Y is locally connected. Hence semi-locally path-connected implies locally connected. The notion of semi-locally path-connected is weaker than the notion of locally path-connected.

Proposition 2.10 *Assume that X is proper as a metric space. Then $FS(X)-FS(X)^{\mathbb{R}}$ is semi-locally path-connected. In particular, $FS(X)-FS(X)^{\mathbb{R}}$ is locally connected.*

Proof Consider $c \in FS(X)-FS(X)^{\mathbb{R}}$ and a neighborhood $V \subseteq FS(X)-FS(X)^{\mathbb{R}}$ of c. By Lemma 1.14 there is a homotopy $H_t\colon FS(X) \to FS(X)$ such that $H_0 = \mathrm{id}$ and $H_t(FS(X)) \subseteq FS(X)_f$ for all $t > 0$. Lemma 1.12 implies that V is also open as a subset of $FS(X)$. Since H is continuous, there is $\delta > 0$ and an open neighborhood $U_1 \subseteq V$ of c such that $H_t(U_1) \subseteq V$ for all $t \in [0,\delta]$. For any $d \in U_1$, $\omega_d(t) := H_{t\delta}(d)$ defines a path in V from d to $H_\delta(d)$. We have $H_\delta(c) \in FS(X)_f$. From Proposition 2.6 we conclude that $FS(X)_f - FS(X)^{\mathbb{R}}$ is open in $FS(X)-FS(X)^{\mathbb{R}}$. By Proposition 2.7 we can find a path-connected neighborhood $W \subseteq V \cap \big(FS(X)_f - FS(X)^{\mathbb{R}}\big)$ of $H_\delta(c)$. Set now $U := U_1 \cap (H_\delta)^{-1}(W)$.

Consider $d \in U$. Then $H_\delta(c)$ and $H_\delta(d)$ both lie in W. Thus there is a path in W from $H_\delta(c)$ to $H_\delta(d)$. This is in particular a path in V, since $W \subseteq V$. Then $\omega := \omega_c * v * \omega_d$ is a path in V from c to d. Hence $FS(X)-FS(X)^{\mathbb{R}}$ is semi-locally path connected. □

2.5 The example of a complete Riemannian manifold with non-positive sectional curvature

Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature. It is a CAT(0)–space with respect to the metric coming from the Riemannian metric (see Bridson–Haefliger [4, I.A.6, page 173]). Let STM be its sphere tangent bundle. For every $x \in M$ and $v \in ST_xM$ there is precisely one geodesic $c_v\colon \mathbb{R} \to M$ for which $c_v(0) = x$ and $c_v'(0) = v$ holds. Given a geodesic $c\colon \mathbb{R} \to M$ in M and $a_-, a_+ \in \overline{\mathbb{R}}$ with $a_- \le a_+$, define the generalized geodesic

$c_{[a_-,a_+]}\colon \mathbb{R} \to M$ by sending t to $c(a_-)$ if $t \le a_-$, to $c(t)$ if $a_- \le t \le a_+$, and to $c(a_+)$ if $t \ge a_+$. Obviously $c_{[-\infty,\infty]} = c$. Let $d\colon \mathbb{R} \to M$ be a generalized geodesic with $d_- < d_+$. Then there is precisely one geodesic $\widehat{d}\colon \mathbb{R} \to M$ with $\widehat{d}_{[d_-,d_+]} = d$.

Define maps

$$\alpha\colon STM \times \big\{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\big\} \to FS(M), \quad (v, a_i, a_+) \mapsto c_v|_{[a_-,a_+]};$$

$$\beta\colon FS(M) \to STM \times \big\{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\big\}, \qquad c \mapsto (\widehat{c}\,'(0), c_-, c_+).$$

Then α and β are to another inverse homeomorphisms. They are compatible with the flow on $FS(M)$ of Definition 1.2, if one uses on $STM \times \big\{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\big\}$ the product flow given by the geodesic flow on STM and the flow on $\overline{R}$ which is at time t given by the homeomorphism $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ sending $s \in \mathbb{R}$ to $s - t$, $-\infty$ to $-\infty$, and ∞ to ∞.

3 Dynamic properties of the flow space

Summary In Definition 3.2 we introduce the homotopy action that we will use to show that CAT(0)–groups are transfer reducible over $\mathcal{VC}$yc. It will act on a large ball in X. (The action of G on the bordification $\overline{X}$ is not suitable, because it has to large isotropy groups.) In Propositions 3.5 and 3.8 we study the dynamics of the flow with respect to the homotopy action. In the language of Section 5 this shows that $FS(X)$ admits contracting transfers.

Throughout this section we fix the following convention.

Convention 3.1 *Let*

- (X, d_X) *be a* CAT(0)*–space which is proper as a metric space;*
- $x_0 \in X$ *be a fixed base point;*
- G *be a group with a proper isometric action on* (X, d_X).

For $x, y \in X$ and $t \in [0,1]$ we will denote by $t \cdot x + (1-t) \cdot y$ the unique point z on the geodesic from x to y such that $d_X(x,z) = t d_X(x,y)$ and $d_X(z,y) = (1-t) d_X(x,y)$. For $x, y \in X$ we will denote by $c_{x,y}$ the generalized geodesic determined by $(c_{x,y})_- = 0$, $c(-\infty) = x$ and $c(\infty) = y$. (By [4, II.1.4(1), page 160] and Proposition 1.7, $(x,y) \mapsto c_{x,y}$ defines a continuous map $X \times X \to FS(X)$. Note that $g \cdot c_{x,y} = c_{gx,gy}$.)

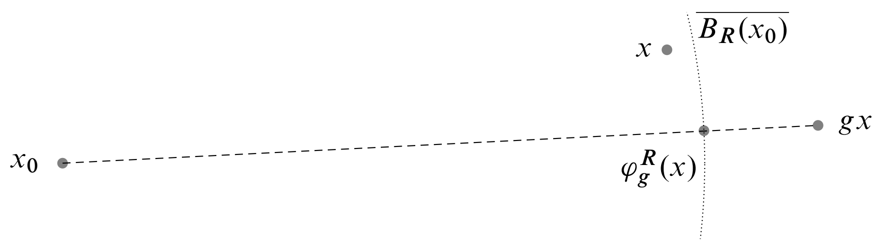
3.1 The homotopy action on $\overline{B}_R(x)$

Recall that for $r > 0$ and $z \in X$ we denote by $\rho_{r,z}\colon X \to \overline{B}_r(z)$ the canonical projection along geodesics, that is, $\rho_{r,z}(x) = c_{z,x}(r)$, see also Remark 2.3. Note that $g \cdot \rho_{r,z}(x) = \rho_{r,gz}(gx)$ for $x, z \in X$ and $g \in G$.

Definition 3.2 (The homotopy S–action on $\overline{B}_R(x_0)$) Let $S \subseteq G$ be a finite subset of G with $e \in G$ and $R > 0$. Define a homotopy S–action (φ^R, H^R) on $\overline{B}_R(x)$ in the sense of Definition 0.1 (i) as follows. For $g \in S$, we define the map

$$\varphi_g^R\colon \overline{B}_R(x_0) \to \overline{B}_R(x_0)$$

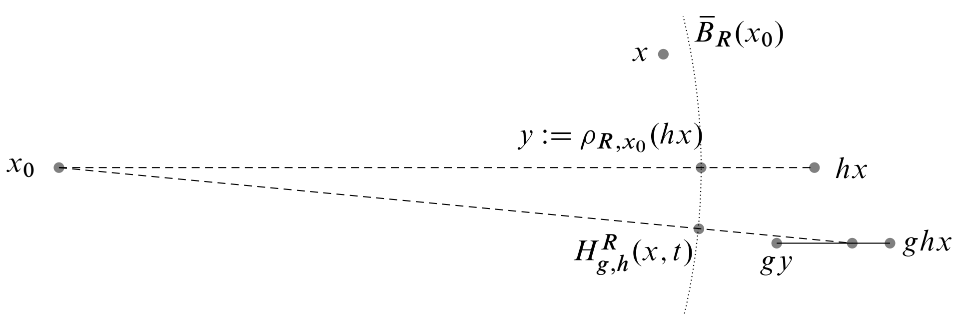
by $\varphi_g^R(x) := \rho_{R,x_0}(gx)$.



For $g, h \in S$ with $gh \in S$ we define the homotopy

$$H_{g,h}^R\colon \varphi_g^R \circ \varphi_h^R \simeq \varphi_{gh}^R$$

by $H_{g,h}^R(x,t) := \rho_{R,x_0}\big(t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R,x_0}(hx))\big)$.



Remark 3.3 Notice that $H_{g,h}^R$ is indeed a homotopy from $\varphi_g^R \circ \varphi_h^R$ to φ_{gh} since

$$\begin{aligned} H_{g,h}^R(x,0) &= \rho_{R,x_0}\big(0 \cdot (ghx) + 1 \cdot (g \cdot \rho_{R,x_0}(hx))\big) \\ &= \rho_{R,x_0}\big(g \cdot \rho_{R,x_0}(hx)\big) \\ &= \varphi_g^R \circ \varphi_h^R(x), \end{aligned}$$

and

$$\begin{aligned} H^R_{g,h}(x,1) &= \rho_{R,x_0}\big(1\cdot(ghx) + 0\cdot(g\cdot\rho_{R,x_0}(hx)\big) \\ &= \rho_{R,x_0}(ghx) \\ &= \varphi^R_{gh}(x). \end{aligned}$$

It turns out that the more obvious homotopy given by convex combination $(x,t)\mapsto t\cdot\varphi^R_{gh}(x)+(1-t)\cdot\varphi^R_g\circ\varphi^R_h(x)$ is not appropriate for our purposes.

Definition 3.4 (The map ι) Define the map

$$\iota\colon G\times X\to FS(X)$$

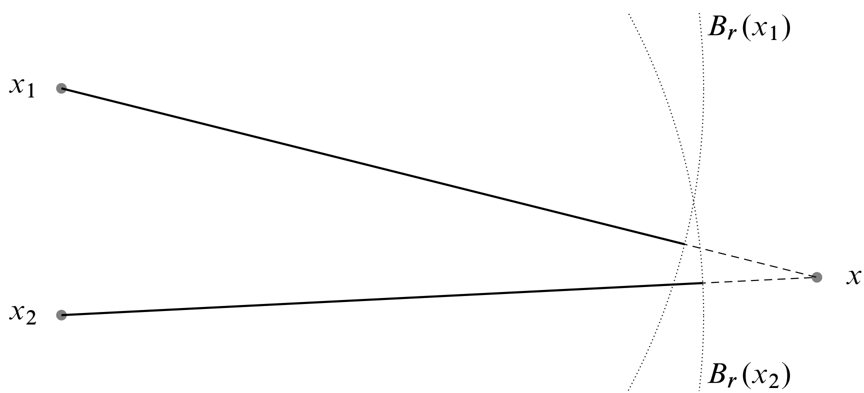
as follows. For $(g,x)\in G\times X$ let $\iota(g,x):=c_{gx_0,gx}$.

The map ι is G–equivariant for the action an $G\times X$ defined by $g\cdot(h,x)=(gh,x)$.

3.2 The flow estimate

Proposition 3.5 *Let $\beta, L>0$. For all $\delta>0$ there are $T,r>0$ such that for $x_1,x_2\in X$ with $d_X(x_1,x_2)\le\beta$, $x\in\overline{B}_{r+L}(x_1)$ there is $\tau\in[-\beta,\beta]$ such that*

$$d_{FS}\big(\Phi_T(c_{x_1,\rho_{r,x_1}(x)}),\Phi_{T+\tau}(c_{x_2,\rho_{r,x_2}(x)})\big)\le\delta.$$



The proof depends on the next lemma.

Lemma 3.6 *Let $r',L,\beta>0$, $r''>\beta$. Set $T:=r''+r'$ and $r:=r''+2r'+\beta$. Let $x_1,x_2\in X$ such that $d_X(x_1,x_2)\le\beta$. Let $x\in\overline{B}_{r+L}(x_1)$. Set $\tau:=d_X(x_2,x)-d_X(x_1,x)$. Then for all $t\in[T-r',T+r']$*

(i) $d_X(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) \leq \frac{2\cdot\beta\cdot(L+2r'+\beta)}{r''}$;

(ii) $c_{x_1,\rho_{r,x_1}(x)}(t) = c_{x_1,x}(t)$ *and* $c_{x_2,\rho_{r,x_2}(x)}(t+\tau) = c_{x_2,x}(t+\tau)$.

Proof (i) Let $t \in [T-r', T+r']$. Note that $|\tau| \leq \beta$. From $T - r' = r'' > \beta$ we conclude $t, t+\tau > 0$. If $t \geq d_X(x, x_1)$, then $c_{x_1,x}(t) = x = c_{x_2,x}(t+\tau)$ and the assertion follows in this case. Hence we can assume $0 < t < d_X(x, x_1)$. A straight forward computation shows that $0 < t+\tau < d_X(x, x_2)$ and $d_X(c_{x_1,x}(t), x) = d_X(c_{x_2,x}(t+\tau), x)$. We get $r'' = T - r' \leq t < d_X(x, x_1)$. Applying the CAT(0)–condition to the triangle Δ_{x,x_1,x_2} we deduce that

$$d_X(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) \leq \frac{2\cdot d_X(x_1,x_2)\cdot(d_X(x,x_1)-t)}{d_X(x,x_1)} \leq \frac{2\cdot\beta\cdot(d_X(x,x_1)-t)}{d_X(x,x_1)}.$$

Combining this with $d_X(x, x_1) > r''$ and $d_X(x,x_1) - t \leq (r+L) - (T-r') = r'' + 2r' + \beta + L - r'' - r' + r' = 2r' + \beta + L$ we obtain the asserted inequality.

(ii) We have $t \leq T + r' = 2r' + r'' = r - \beta$ and $t \geq T - r' = r'' > \beta$. Thus $t, t+\tau \in [0, r]$. Hence $c_{x_1,\rho_{r,x_1}(x)}(t) = c_{x_1,x}(t)$ and $c_{x_2,\rho_{r,x_2}(x)}(t+\tau) = c_{x_2,x}(t+\tau)$. □

Proof of Proposition 3.5 Let $\delta > 0$ be given. Pick $r' > 0$, $r'' > \beta$, $1 > \delta' > 0$ such that

$$\int_{-\infty}^{-r'} \frac{1+|t|}{e^{|t|}} dt \leq \frac{\delta}{3}, \qquad \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt \leq \frac{\delta}{3} \qquad \text{and} \qquad \frac{2\cdot\beta(L+2r'+\beta)}{r''} \leq \delta'.$$

Set $r := 2r' + r'' + \beta$ and $T := r' + r''$. Let $x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \beta$. Let $x \in \overline{B}_r(x_1)$ be given. Set $\tau := d_X(x_2, x) - d_X(x_1, x)$. Then $|\tau| \leq d_X(x_2, x_1) \leq \beta$. Using Lemma 3.6 we conclude that for all $t \in [-r', r']$

$$\begin{aligned} d_X\big(c_{x_1,\rho_{r,x_1}(x)}(T+t), c_{x_2,\rho_{r,x_2}(x)}(T+t+\tau)\big) &= d_X\big(c_{x_1,x}(T+t), c_{x_2,x}(T+t+\tau)\big) \\ &\leq \frac{2\cdot\beta(L+2r'+\beta)}{r''} \leq \delta'. \end{aligned}$$

Thus

$$
\begin{aligned}
&d_{FS}\big(\Phi_T(c_{x_1,\rho_{r,x_1}(x)}), \Phi_{T+\tau}(c_{x_2,\rho_{r,x_2}(x)})\big) \\
&\quad = \int_{-\infty}^{\infty} \frac{d_X\big(c_{x_1,\rho_{r,x_1}(x)}(T+t), c_{x_2,\rho_{r,x_2}(x)}(T+t+\tau)\big)}{2e^{|t|}} dt \\
&\quad \le \int_{-\infty}^{-r'} \frac{2|t|+\delta'}{2e^{|t|}} dt + \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt + \int_{r'}^{\infty} \frac{\delta'+2|t|}{2e^{|t|}} dt \\
&\quad \le \int_{-\infty}^{-r'} \frac{|t|+1}{e^{|t|}} dt + \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt + \int_{r'}^{\infty} \frac{1+|t|}{e^{|t|}} dt \\
&\quad \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.
\end{aligned}
$$

This completes the proof of Proposition 3.5. □

Lemma 3.7 *Let $\varepsilon > 0$, $\beta > 0$. Then there is $\delta > 0$ such that for all $|\tau| \le \beta$*

$$d_{FS}(c_0, c_1) \le \delta \implies d_{FS}\big(\Phi_\tau(c_0), \Phi_\tau(c_1)\big) \le \varepsilon$$

for c_0, $c_1 \in FS(X)$.

Proof This follows from Lemma 1.3. □

Proposition 3.8 *Let S be a finite subset of G (containing e). Then there is $\beta > 0$ such that the following holds:*

For all $\delta > 0$ there are $T, R > 0$ such that for every $(a,x) \in G \times \overline{B}_R(X)$, $s \in S$, $f \in F_s(\varphi^R, H^R)$ there is $\tau \in [-\beta, \beta]$ such that

$$d_{FS}\big(\Phi_T(\iota(a,x)), \Phi_{T+\tau}(\iota(as^{-1}, f(x)))\big) \le \delta.$$

Proof Pick β such that $\frac{\beta}{2} \ge d_X(sx_0, x_0)$ for all $s \in S$. Let $L := \beta$. Let $\delta > 0$ be given. By Lemma 3.7 there is $\frac{\delta}{2} > \delta_0 > 0$ such that for $|\tau'| \le \beta$

$$d_{FS}(c_0, c_1) \le \delta_0 \implies d_{FS}\big(\Phi_{\tau'}(c_0), \Phi_{\tau'}(c_1)\big) \le \frac{\delta}{2} \tag{3.9}$$

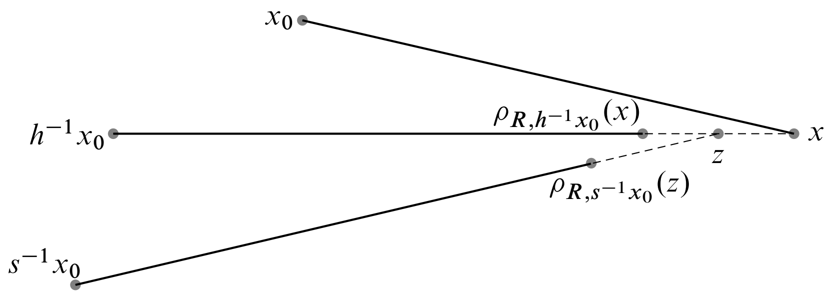
for c_0, $c_1 \in FS(X)$. By Proposition 3.5 there are T, $R > 0$ such that for $x, x_1, x_2 \in X$ with $d_X(x_1, x_2) \le \beta$ and $d_X(x, x_1) \le R + L$ there is $\tau = \tau(x, x_1, x_2) \in [-\frac{\beta}{2}, \frac{\beta}{2}]$ such that

$$d_{FS}\big(\Phi_T(c_{x_1,\rho_{R,x_1}(x)}), \Phi_{T+\tau}(c_{x_2,\rho_{R,x_2}(x)})\big) \le \delta_0.$$

Let $(a,x) \in G \in \overline{B}_R(X)$, $s \in S$, $f \in F_s(\varphi^R, H^R)$. Note that

$$d_{FS}\big(\Phi_T(\iota(a,x)), \Phi_{T+\tau}(\iota(as^{-1}, f(x)))\big) = d_{FS}\big(\Phi_T(\iota(e,x)), \Phi_{T+\tau}(\iota(s^{-1}, f(x)))\big)$$

because ι and Φ are G–equivariant and d_{FS} is G–invariant. Thus it suffices to consider the case $a = e$. Then there are $t \in [0,1]$ and $g, h \in S$ such that $s = gh$ and $f(x) = H^R_{g,h}(x,t) = \rho_{R,x_0}\big(t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R,x_0}(hx))\big)$. Therefore $s^{-1} f(x) = \rho_{R,s^{-1}x_0}\big(t \cdot x + (1-t) \cdot \rho_{R,h^{-1}x_0}(x)\big)$. Set $z := t \cdot x + (1-t) \cdot \rho_{R,h^{-1}x_0}(x)$. Then $\iota(s^{-1}, f(x)) = c_{s^{-1}x_0, \rho_{R,s^{-1}x_0}(z)}$.



We have $d_X(x, x_0) \le R$. Moreover, $d_X(z, h^{-1}x_0) \le d_X(x, h^{-1}x_0) \le d_X(x, x_0) + d_X(x_0, h^{-1}x_0) \le R + L$. Therefore, we can set $\tau_1 := \tau(x, x_0, h^{-1}x_0)$, $\tau_2 := \tau(z, h^{-1}x_0, s^{-1}x_0)$ and $\tau := \tau_1 + \tau_2$. Note that $|\tau| \le \beta$, since $|\tau_i| \le \frac{\beta}{2}$. We have

$$d_{FS}\big(\Phi_T(c_{h^{-1}x_0, \rho_{R,h^{-1}x_0}(z)}), \Phi_{T+\tau_2}(c_{s^{-1}x_0, \rho_{R,s^{-1}x_0}(z)})\big) \le \delta_0$$

and therefore by (3.9)

$$d_{FS}\big(\Phi_{T+\tau_1}(c_{h^{-1}x_0, \rho_{R,h^{-1}x_0}(z)}), \Phi_{T+\tau_1+\tau_2}(c_{s^{-1}x_0, \rho_{R,s^{-1}x_0}(z)})\big) \le \frac{\delta}{2}.$$

Thus

$$\begin{aligned} & d_{FS}\big(\Phi_T(\iota(e,x)), \Phi_{T+\tau}(\iota(s^{-1}, f(x)))\big) \\ &= d_{FS}\big(\Phi_T(c_{x_0,x}), \Phi_{T+\tau}(c_{s^{-1}x_0, \rho_{R,s^{-1}x_0}(z)})\big) \\ &\le d_{FS}\big(\Phi_T(c_{x_0,x}), \Phi_{T+\tau_1}(c_{h^{-1}x_0, \rho_{R,h^{-1}x_0}(x)})\big) \\ &\quad + d_{FS}\big(\Phi_{T+\tau_1}(c_{h^{-1}x_0, \rho_{R,h^{-1}x_0}(z)}), \Phi_{T+\tau_1+\tau_2}(c_{s^{-1}x_0, \rho_{R,s^{-1}x_0}(z)})\big) \\ &\le \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

This completes the proof. □

4 Orbits with bounded G–period

Summary Let $FS(X)_{\leq\gamma}$ be the part of $FS(X)$ that consists of in some sense periodic orbits, namely, those generalized geodesics for which there exists for every $\epsilon > 0$ an element $\tau \in (0, \gamma + \epsilon]$ and $g \in G$ such that $g \cdot c = \Phi_\tau(c)$ holds (see (5.4)). Our main result here is Theorem 4.2 that asserts that there is a cover of uniformly bounded dimension for $FS(X)_{\leq\gamma}$ that is long in the direction of the flow. To this end we study hyperbolic elements in G and their axis. These come in parallel families (called FS_a below) that project to convex subspaces of X. We construct the desired cover first for the FS_a by considering the quotient Y_a of FS_a by the flow. One difficulty here is that the group G_a that naturally acts on Y_a, does so with infinite isotropy. The isotropy groups here are virtually cyclic and this forces the appearance of the family $\mathcal{VC}$yc in Theorem 4.2 and in our main result.

Throughout this section we fix the following convention.

Convention 4.1 *Let*

- (X, d_X) *be a* CAT(0)*–space which is proper as a metric space and has finite covering dimension;*
- G *be a group with a proper isometric action on* (X, d_X).

The following is the main result of this section.

Theorem 4.2 *There is a natural number* M *such that for every* $\gamma > 0$ *and every compact subset* K *of* X *there exists a collection* $\mathcal{V}$ *of subsets of* $FS(X)$ *satisfying:*

(i) *Each element* $V \in \mathcal{V}$ *is an open* $\mathcal{VC}yc$*–subsets of the* G*–space* $FS(X)$ *(see Definition 0.3);*

(ii) $\mathcal{V}$ *is* G*–invariant; that is, for* $g \in G$ *and* $V \in \mathcal{V}$ *we have* $g \cdot V \in \mathcal{V}$*;*

(iii) $G \backslash \mathcal{V}$ *is finite;*

(iv) $\dim \mathcal{V} \leq M$*;*

(v) *there is* $\varepsilon > 0$ *with the following property: for* $c \in FS_{\leq\gamma}$ *such that* $c(t) \in G \cdot K$ *for some* $t \in \mathbb{R}$ *there is* $V \in \mathcal{V}$ *such that* $B_\varepsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq V$.

4.1 Hyperbolic isometries of spaces

We recall some basic facts from Bridson–Haefliger [4, Chapter II.6] about isometries of a CAT(0)–space. Let $\gamma\colon X \to X$ be an isometry. The *displacement function* of γ is defined by

$$d_\gamma\colon X \to [0,\infty), \quad x \mapsto d_X(\gamma x, x).$$

The *translation length* of γ is defined by

$$l(\gamma) := \inf\{d_\gamma(x) \mid x \in X\}.$$

Define a subspace of X by

$$\operatorname{Min}(\gamma) := \{x \in X \mid d_\gamma(x) = l(\gamma)\}.$$

We call γ *elliptic* if γ has a fixed point and *hyperbolic* if the displacement function d_γ attains a strictly positive minimum, or, equivalently, $l(\gamma) > 0$ and $\operatorname{Min}(\gamma) \neq \varnothing$.

Lemma 4.3 *Let $\gamma\colon X \to X$ be an isometry.*

(i) *If $\alpha\colon X \to X$ is an isometry, then $l(\gamma) = l(\alpha\gamma\alpha^{-1})$ and $\operatorname{Min}(\alpha\gamma\alpha^{-1}) = \alpha(\operatorname{Min}(\gamma))$.*

(ii) *$\operatorname{Min}(\gamma)$ is a closed convex set.*

(iii) *The isometry γ is hyperbolic if and only if it possesses an* axis, *that is, there is a geodesic $c\colon \mathbb{R} \to X$ and $\tau > 0$ such that $\gamma \circ c(t) = c(t+\tau)$ holds for $t \in \mathbb{R}$. In this case $\tau = l(\gamma)$.*

(iv) *Two axes c, d for the hyperbolic isometry γ are* parallel, *that is, $d_X(c(t), d(t))$ is constant in $t \in \mathbb{R}$. The union of the images $c(\mathbb{R})$ of all axes c for γ is $\operatorname{Min}(\gamma)$.*

Proof See [4, II.6.2(2), page 229] for (i). See [4, II.6.2(3), page 229] for (ii). See [4, II.6.8(1), page 231] for (iii). See [4, II.6.8(3), page 231] for (iv). □

We emphasize that an axis for a hyperbolic element γ is a (parametrized) geodesic $c\colon \mathbb{R} \to X$ and not only $c(\mathbb{R})$. So two hyperbolic elements γ_1 and γ_2 have a common axis if there exists a geodesic $c\colon \mathbb{R} \to X$ such that $\gamma_1 \cdot c(t) = c(t + l(\gamma_1))$ and $\gamma_2 \cdot c(t) = c(t + l(\gamma_2))$ holds for all $t \in \mathbb{R}$. We denote by $l(g)$ the translation length of the isometry $X \to X$ given by multiplication with $g \in G$. We will say that g is hyperbolic if this isometry is hyperbolic.

Lemma 4.4 *Let $c\colon \mathbb{R} \to X$ be a geodesic. Put*

$$\begin{aligned} G_{\Phi_{\mathbb{R}}(c)} &= \{g \in G \mid g(\Phi_{\mathbb{R}}(c)) \\ &= \Phi_{\mathbb{R}}(c)\} \\ &= \big\{g \in G \mid \exists \tau \in \mathbb{R} \text{ with } gc(t) = c(t+\tau) \text{ for all } t \in \mathbb{R}\big\} \end{aligned}$$

and

$$G_{c(\mathbb{R})} := \big\{g \in G \mid g \cdot c(\mathbb{R}) = c(\mathbb{R})\big\}.$$

Then $G_{\Phi_{\mathbb{R}}(c)}$ is virtually cyclic of type I and $G_{c(\mathbb{R})}$ is virtually cyclic.

Proof The group $G_{c(\mathbb{R})}$ acts properly and isometrically on $\mathbb{R}$ since $c(\mathbb{R})$ is isometric to $\mathbb{R}$. The isometry group of $\mathbb{R}$ fits into the exact sequence

$$1 \longrightarrow \mathbb{R} \xrightarrow{i} \mathrm{Isom}(\mathbb{R}) \xrightarrow{p} \{\pm 1\} \longrightarrow 1,$$

where i sends a real number r to the isometry $t \mapsto t + r$ and p sends an isometry to 1 if it is strictly monotone increasing and to -1 otherwise. Since $G_{c(\mathbb{R})}$ acts properly on $\mathbb{R}$, the obvious homomorphism $G \to \mathrm{Isom}(\mathbb{R})$ has a finite kernel and its image is a discrete subgroup of $\mathrm{Isom}(\mathbb{R})$. This implies that $G_{c(\mathbb{R})}$ is virtually cyclic.

Since the action of $G_{\Phi_{\mathbb{R}}(c)}$ on c is by translations, the same argument shows that $G_{\Phi_{\mathbb{R}}(c)}$ is virtually cyclic of type I. □

4.2 Axes in the flow space

Throughout this subsection we fix a compact subset K of X.

Notation 4.5 Let $\gamma > 0$.

(i) Let

$$G^{hyp}_{\leq\gamma} \subseteq G$$

be the set of all hyperbolic $g \in G$ of translation length $l(g) \leq \gamma$ such that some axis c for g intersects $G \cdot K$.

Consider the equivalence relation $\sim$ on $G^{hyp}_{\leq\gamma}$ for which $g \sim g'$ if and only if there exists parallel axes c_g and $c_{g'}$ for g and g'. (This relation is transitive by Lemma 4.3 (iv) and because parallelism is an equivalence relation for geodesics in CAT(0)–spaces by the Flat Strip Theorem [4, II.2.13, page 182].) Put

$$A_{\leq\gamma} := G^{hyp}_{\leq\gamma}/\sim.$$

The conjugation action of G on G restricts to an action on $G^{hyp}_{\leq\gamma}$ and descends to an action of G on $A_{\leq\gamma}$, see Lemma 4.3 (i). For $a \in A_{\leq\gamma}$ we set

$$G_a := \{g \in G \mid g \cdot a = a\}.$$

(ii) For $a \in A_{\leq\gamma}$ let

$$FS_a \subseteq FS(X)$$

denote the subspace of $FS(X)$ that consists of all geodesics $c\colon \mathbb{R} \to X$, that are an axis for some $g \in a$ and intersect $G \cdot K$. We remark that $c \in FS(X)$ is an axis for g if and only if $\Phi_\tau(c) = gc$ for some $\tau > 0$ and in this case $\tau = l(g)$, see Lemma 4.3 (iii). Define

$$p_a\colon FS_a \to X, \quad c \mapsto c(0).$$

We denote by

$$Y_a := FS_a/\Phi$$

the quotient of FS_a by the action of the flow Φ. Let

$$q_a\colon FS_a \to Y_a$$

be the canonical projection. The action of G on $FS(X)$ restricts to an action of G_a on FS_a. Because p_a is G_a–equivariant and because the flow Φ commutes with the G–action on $FS(X)$ we obtain an action of G_a on Y_a and q_a is G_a–equivariant for this action.

Lemma 4.6 *Let (Z, d_Z) be a proper metric space with a proper isometric action of a group H. If $(z_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are sequences in Z and H such that $z_n \to z$ and $h_n z_n \to z'$ converge in Z, then $\{h_n \mid n \in \mathbb{N}\}$ is finite and for every $h \in H$ such that $h_n = h$ for infinitely many n we have $hz = z'$.*

Proof Let $n_0 > 0$ such that $d_Z(z, z_n) < 1$ and $d_Z(z', h_n z_n) < 1$ for all $n \geq n_0$. Thus $d_Z(h_n z, z') < 2$ for all $n \geq n_0$. Thus $d_Z((h_{n_0})^{-1} h_n z, z) < 4$ for all $n \geq n_0$. Thus $\{(h_{n_0})^{-1} h_n \mid n \geq n_0\}$ is finite, because the action is proper. Therefore $\{h_n \mid n \in \mathbb{N}\}$ is finite. If $h_n = h$ for infinitely many $n \in \mathbb{N}$, then $hz = \lim_{n\to\infty} hz_n = \lim_{n\to\infty} h_n z_n = z'$. □

Corollary 4.7 *Let (Z, d_Z) be a proper metric space with a proper isometric action of a group H. If $L \subset Z$ is compact, then $H \cdot L \subset Z$ is closed.*

Proof Let $h_n z_n \to z$ with $h_n \in H$ and $z_n \in L$. After passing to a subsequence we have $z_n \to z'$. Lemma 4.6 implies that we can pass to a further subsequence for which $h_n = h$ is constant. Thus $z \in h \cdot L \subset H \cdot L$. □

Lemma 4.8 *There is a compact subset* $K_\gamma \subseteq X$ *such that* $c(0) \in G \cdot K_\gamma$ *for all* $c \in FS_a$, $a \in A_{\leq\gamma}$.

Proof Set $K_\gamma := \bar{B}_\gamma(K)$. This is compact because K is compact and X is proper as a metric space. For $a \in A_{\leq\gamma}$, $c \in FS_a$ there are $g \in a$, $t \in [0,\gamma]$ such that $c(s+t) = gc(s)$ for all $s \in \mathbb{R}$. Because c intersects $G \cdot K$, this implies that there is $s_0 \in [0,\gamma]$ such that $c(s_0) \in G \cdot K$. Thus $c(0) \in G \cdot \bar{B}_\gamma(K) = G \cdot K_\gamma$. □

Lemma 4.9 *Let* $\gamma > 0$. *Let* $(c_n)_{n\in\mathbb{N}}$ *be a sequence in* $\bigcup_{a\in A_{\leq\gamma}} FS_a$ *that converges to* c *in* $FS(X)$. *Then there are* $g \in G^{hyp}_{\leq\gamma}$ *and an infinite subset* $I \subseteq \mathbb{N}$ *such that* c *and all* c_i, $i \in I$ *are axes for* g *and intersect* $G \cdot K$, *where* K *is the fixed compact subset of* X. *In particular,* $c \in FS_a$ *and* $c_i \in FS_a$ *for all* $i \in I$, *where* $a \in A_{\leq\gamma}$ *is the class of* g.

Proof There are $g_n \in G$, $t_n \in [0,\gamma]$ such that $g_n c_n = \Phi_{t_n}(c_n)$. We can pass to a subsequence and assume that $t_n \to t_0$. Then $g_n c_n = \Phi_{t_n}(c_n) \to \Phi_{t_0}(c)$ (using Lemma 1.3). Since $FS(X)$ is proper and G acts properly on $FS(X)$ (see Propositions 1.9 and 1.11) we can apply Lemma 4.6 and assume after passing to a further subsequence that $g_n = g$ is constant. Then $gc = \lim g_n c_n = \lim \Phi_{t_n} c_n = \Phi_{t_0}(c)$. It remains to show that c intersects $G \cdot K$.

For each n there is s_n in $\mathbb{R}$ such that $c_n(s_n) \in G \cdot K$. Since $g_n c_n(s) = c_n(s+t_n)$ and $t_n \in [0,\gamma]$, we can arrange $s_n \in [0,\gamma]$ for all $n \geq 0$. By passing to a subsequence we can arrange that $s_n \to s_0$ for $n \to \infty$ for some $s_0 \in [0,\gamma]$. Then $c_n(s_n) \to c(s_0)$ (using Proposition 1.7). By Corollary 4.7 $G \cdot K$ is closed. Thus $c(s) \in G \cdot K$. □

Lemma 4.10 *Let* $\gamma > 0$. *Then*

(i) $G \backslash A_{\leq\gamma}$ *is finite;*

(ii) $G \cdot FS_a \subseteq FS(X)$ *is closed for all* $a \in A_{\leq\gamma}$; *there is* $K_a \subseteq G \cdot FS_a$ *compact such that* $G \cdot K_a = G \cdot FS_a$;

(iii) *there is* $\varepsilon > 0$ *such that* $d_X(FS_a, FS_b) > \varepsilon$ *for all* $a \neq b \in A_{\leq\gamma}$;

(iv) *consider* $c \in FS(X)_{\leq\gamma} - FS(X)^{\mathbb{R}}$ *such that* $c(t) \in K$ *for some* $t \in \mathbb{R}$, *where* K *is the fixed compact subset of* X. *Then there are* $a \in A_{\leq\gamma}$ *and* $y \in Y_a$ *such that* $\Phi_{\mathbb{R}}(c) = q_a^{-1}(y)$.

Proof (i) We proceed by contradiction and assume that there are $a_1, a_2, \ldots$ in $A_{\leq\gamma}$ such that $Ga_i \cap Ga_j = \varnothing$ if $i \neq j$. Then there are $g_i \in a_i$ and $c_i \in FS(X)_{a_i}$ such that c_i is an axis for g_i. After replacing a_i by $h_i a_i$, g_i by $h_i g_i h_i^{-1}$ and c_i by $h_i c_i$ for suitable $h_i \in G$ we can assume that $c_i(0) \in K_\gamma$, where K_γ is the compact subset of X from Lemma 4.8. Lemma 1.10 implies that $\widehat{K}_\gamma := \{c \in FS(X) \mid c(0) \in K_\gamma\}$ is also compact. Thus we can pass to a subsequence and arrange that $c_i \to c \in FS(X)$. Lemma 4.9 yields a contradiction.

(ii) By Lemma 4.9 $G \cdot FS_a \subseteq FS(X)$ is closed. Thus $K_a := \widehat{K}_\gamma \cap G \cdot FS_a$ is compact. Now we get $G \cdot FS_a = G \cdot \widehat{K}_\gamma \cap G \cdot FS_a = G \cdot (\widehat{K}_\gamma \cap G \cdot FS_a) = G \cdot K_a$.

(iii) We proceed by contradiction and assume that for every n, there are $a_n \neq b_n \in A_{\leq\gamma}$ and $c_n \in FS_{a_n}$, $d_n \in FS_{b_n}$ such that $d_{FS}(c_n, d_n) < 1/n$. Because of (i) and (ii) there are $h_n \in G$ such that a subsequence of $h_n c_n$ converges to some $c \in FS(X)$. After replacing a_n by $h_n a_n$, b_n by $h_n b_n$, c_n by $h_n c_n$, d_n by $h_n d_n$ and after passing to a suitable subsequence we can thus assume that $c_n \to c \in FS(X)$. Then also $d_n \to c$. Using Lemma 4.9 for the sequence $(c_n)_{n\in\mathbb{N}}$ we can after passing to a subsequence assume that $a_n = a$ is constant and that $c \in FS_a$. Using Lemma 4.9 for the sequence $(d_n)_{n\in\mathbb{N}}$ we can after passing to a further subsequence assume that $b_n = b$ is constant and $c \in FS_b$. Thus $FS_b \cap FS_a \neq \varnothing$. From the definition of $A_{\leq\gamma}$ it is immediate that this implies $a = b$, contradicting $a_n \neq b_n$.

(iv) Let $c \in FS(X)_{\leq\gamma} - FS(X)^{\mathbb{R}}$. Then we can find for each natural number n a real number τ_n and $g_n \in G$ satisfying $\gamma + \frac{1}{n} > \tau_n > 0$ and $g_n c(t) = c(t + \tau_n)$. In particular, c is a geodesic that is an axis for each g_n and $l(g_n) = \tau_n$. After passing to a subsequence we can assume that $\tau_n \to \tau_0$. Thus, $d_X(g_n c(0), c(\tau_0)) \to 0$. Because of Lemma 4.6 $l(g_n) = \tau_n = \tau_0$ for infinitely many n. For such an n we have $g_n \in G^{hyp}_{\leq\gamma}$. Now assume additionally that $c(t) \in K$ for some t. If a is the equivalence class of such a g_n, then $c \in FS_a$ and $\Phi_{\mathbb{R}}(c) = q_a^{-1}(q_a(c))$. □

Proposition 4.11 *Let $\gamma > 0$ and $a \in A_{\leq\gamma}$. Then*

- (i) *$p_a\colon FS_a \to X$ is an isometric embedding with closed image;*
- (ii) *there is a G_a–invariant metric d_a on Y_a that generates the topology; with this metric Y_a is a proper metric space;*
- (iii) *there is $\tau_a\colon FS_a \to \mathbb{R}$, such that $c \mapsto (q_a(c), \tau_a(c))$ defines an isometry $FS_a \to Y_a \times \mathbb{R}$ which is compatible with the flow, that is, $\tau_a(\Phi_t(c)) = \tau_a(c) + t$ for $t \in \mathbb{R}$, $c \in FS_a$;*
- (iv) *for $y \in Y_a$, $G_y := \{g \in G_a \mid gy = y\}$ is virtually cyclic of type I and $G_a y \subseteq Y_a$ is discrete;*

Proof (i) If c and d are parallel, then $d_X(c(t), d(t))$ is constant by definition. An easy computation shows $d_X(c(0), d(0)) = d_{FS}(c, d)$. Thus p_a is an isometry. It remains to show that $p_a(FS_a)$ is closed. Let $c_n \in FS_a$ such that $c_n(0) \to x \in X$.

Because $c \mapsto c(0)$ is proper (Lemma 1.10) we can pass to a subsequence and assume that $c_n \to c$ in $FS(X)$. Then $c \in FS_a$ by Lemma 4.9 and $c(0) = x$.

(ii) and (iii) Let FS_a^+ be the subset of all d that are parallel to some (and therefore all) $c \in FS_a$. Define

$$p_a^+\colon FS_a^+ \to X, \quad c \mapsto c(0).$$

By the argument from the proof of assertion (i) $p_a^+\colon FS_a^+ \to X$ is an isometric embedding. It follows from [4, II.2.14, page 183] that there is convex subspace Y_a^+ of X (which is therefore a CAT(0)–space) and an isometry $\psi\colon Y_a^+ \times \mathbb{R} \to FS_a^+$ such that $\Phi_t(\psi(y,s)) = \psi(y, s+t)$ for $s, t \in \mathbb{R}$ and $y \in Y_a^+$. This identifies Y_a with a subspace of Y_a^+ and provides the metric on Y_a. If $g \in G_a$, then the action of g on X permutes the images of geodesics $c \in FS_a^+$. It follows from [4, I.5.3(4), page 56] that the induced action of g on Y_a^+ (and therefore also the induced action on Y_a) is isometric. Since X is proper by assumption, FS_a is proper by assertion (i). Since FS_a is isometric to $Y_a \times \mathbb{R}$, the metric space Y_a is proper.

(iv) By Lemma 4.4 G_y is virtually cyclic of type I. We proceed by contradiction to show that $G_a y$ is discrete. Assume that there are $g_n \in G_a$, $n \in \mathbb{N}$ such that $g_n y \neq g_m y$ if $n \neq m$ and $g_n y \to y_0$. Pick $c \in FS_a$ such that $q_a(c) = y$ and $\tau_a(c) = 0$. Pick $g \in a \subseteq G^{hyp}_{\leq \gamma}$ such that c is an axis for g. Then $gc = \Phi_{l(g)}(c)$, see Lemma 4.3 (iii). Thus $\tau_a(g_n g g_n^{-1}(g_n c)) = l(g) + \tau_a(g_n c)$ and $(g_n g g_n^{-1}) g_n y = g_n y$. By replacing g_n by $(g_n g g_n^{-1})^{l_n} g_n$ for some $l_n \in \mathbb{Z}$ we can arrange that $\tau_a(g_n c) \in [0, l(g)]$. By passing to a subsequence we can arrange that the sequence $\tau_a(g_n c)$ converges. Since $q_a(g_n c) = g_n c$ converges to y_0, we conclude from (iii) that $g_n c \to d$ as $n \to \infty$ for some $d \in FS(X)$. Now Lemma 4.6 implies that $g_n c = g_m c$ for infinitely many n, m. This contradicts $g_n y \neq g_m y$ for $n \neq m$. □

In the proofs of the next two results we will denote by $\pi_a\colon Y_a \to G_a \backslash Y_a$ the quotient map. We point out that π_a is open, since for any open subset $V \subseteq Y_a$ the subset $\pi_a^{-1}(\pi_a(V)) = \bigcup_{g \in G_a} g \cdot V$ is open.

Lemma 4.12 *We have*

$$\dim(G_a \backslash Y_a) \leq \dim(X).$$

Proof Let $K_a \subseteq G \cdot FS_a$ be compact such that $G \cdot K_a = G \cdot FS_a$, see Lemma 4.10 (ii). Using Lemma 4.10 (iii) we conclude that there is a compact subset $K'_a \subseteq FS_a$ such that $FS_a = G_a \cdot K'_a$. Let $K''_a \subseteq Y_a$ be the compact subset $q_a(K'_a)$. Define $U := B_1(K''_a)$. Then U is an open subset of Y_a with $Y_a = G_a \cdot U$. Let $i\colon U \to Y_a$ be the inclusion. Since U is open, i is open. As pointed out above, $\pi_a\colon Y_a \to G_a \backslash Y_a$ is open. Hence the composite $\pi_a \circ i\colon U \to G_a \backslash Y_a$ is open and surjective. Since Y_a is a proper metric space by Proposition 4.11 (ii) and $K''_a \subseteq Y_a$ is compact, the set $\overline{B}_1(K''_a)$ is compact subset of Y_a. Since for every $y \in Y$ the orbit $G_a y \subseteq Y_a$ is discrete by Proposition 4.11 (iv), the intersection $\overline{B}_1(K'_a) \cap G_a y$ and hence also the intersection $U \cap G_a y$ is finite. Hence the composite $\pi_a \circ i\colon U \to G_a \backslash Y_a$ is finite-to-one. Its source is a metric as U is a subspace of the metric space Y_a. Since G_a acts isometrically on Y_a and for every $y \in Y$ the orbit $G_a y \subseteq Y_a$ is discrete by Proposition 4.11 (iv), the quotient $G_a \backslash Y_a$ is also a metric space. Since every metric space is paracompact by Stone's Theorem (see Munkres [10, Chapter VI, Theorem 4.3, page 256], the composite $\pi_a \circ i\colon U \to G_a \backslash Y_a$ is a finite-to-one open surjective map of paracompact spaces. Hence we conclude from Nagami [11, 4.1, page 35]

$$\dim(U) = \dim(G_a \backslash Y_a).$$

Since Y_a is a proper metric space by Proposition 4.11 (ii) it is locally compact and can be written as the countable union of compact subspaces and hence contains a countable dense subset. Hence the open subset U is locally compact and has a countable basis for its topology. Since any compact subset $L \subset U$ is a closed subset of X and satisfies $\dim(L) \leq \dim(X)$, we conclude $\dim(U) \leq \dim(X)$ from Munkres [10, Chapter 7.9, Exercise 9, page 315]. This finishes the proof of Lemma 4.12. □

Proposition 4.13 *Let $\gamma > 0$ and $a \in A_{\leq\gamma}$. There is an open $\mathcal{VC}yc$–cover $\mathcal{V}_a$ of Y_a such that*

(i) $\dim \mathcal{V}_a \leq \dim X$;

(ii) *$\mathcal{V}_a$ is G_a–invariant, that is, $g \cdot V \in \mathcal{V}_a$ if $g \in G_a$ and $V \in \mathcal{V}_a$;*

(iii) *$G_a \backslash \mathcal{V}_a$ is finite.*

Proof Because of Proposition 4.11 (ii) and (iv) for any $y \in Y_a$, the open ball of sufficiently small radius is $\mathcal{VC}$yc–neighborhood for y. Pick for each y such a ball V_y. Because $\pi_a\colon Y_a \to G_a \backslash Y_a$ is open, $\{\pi_a(V_y) \mid y \in Y_a\}$ is an open cover of $G_a \backslash Y_a$. By Lemma 4.12 it has a refinement $\mathcal{W}$ whose dimension is bounded by $\dim X$. The G_a–action on Y_a is cocompact because it is cocompact on FS_a, see Lemma 4.10 (ii). Therefore $G_a \backslash Y_a$ is compact. Thus we may assume that $\mathcal{W}$ is finite. For any $W \in \mathcal{W}$ pick $y_W \in FS_a$ such that $W \subseteq \pi_a(V_{y_W})$. Now define $\mathcal{V}_a := \{\pi_a^{-1}(W) \cap g V_{y_W} \mid W \in \mathcal{W}, g \in G_a\}$. This is an open $\mathcal{VC}$yc–cover because each V_y is an open $\mathcal{VC}$yc–set. Its dimension is bounded by $\dim X$, because the dimension of $\mathcal{W}$ is bounded by $\dim X$ and because for $g \in G_a$ and $y \in Y$ we have either $V_y = g V_y$ or $V_y \cap g V_y = \varnothing$. It is G_a–invariant because each $\pi_a^{-1}(W)$ is G_a–invariant. Finally, $G_a \backslash \mathcal{V}_a$ is finite because $\mathcal{W}$ is finite. □

4.3 The cover $\mathcal{V}$

Lemma 4.14 *Let (Z, d_Z) be a metric space with an action of a group H by isometries. Let A be a H–invariant subspace. For $\varnothing \neq U \subsetneq A$, we define*

$$Z(U) := \{z \in Z \mid d_Z(z, U) < d_Z(z, A - U)\}$$

and set $Z(A) := Z$, $Z(\varnothing) := \varnothing$. Then for $U, V \subseteq A$,

(i) *$Z(U)$ is open in Z;*

(ii) *$Z(U \cap V) = Z(U) \cap Z(V)$;*

(iii) *$Z(U) \cap A = U$ holds if and only if U is open in A;*

(iv) *for all $g \in H$ we have $Z(gU) = gZ(U)$.*

Proof (i) Either $Z(U)$ is $\varnothing$ or Z or can be written as the preimage of $(0, \infty)$ for a continuous function on Z. Hence $Z(U)$ is open for every $U \subseteq Z$.

(ii) This is obviously true if $U = A$, $U = \varnothing$, $V = \varnothing$ or $V = A$ holds, so we can assume without loss of generality $\varnothing \neq U \neq A$ and $\varnothing \neq V \neq A$ in the sequel. Recall that $d_Z(z, U) := \inf\{d_Z(z, u) \mid u \in U\}$ for $z \in Z$ and $\varnothing \neq U \subseteq Z$. One easily checks that $d_Z(z, U), d_Z(z, V) \leq d_Z(z, U \cap V)$ and $d_Z(z, A - (U \cap V)) = \min\{d_Z(z, A - U), d_Z(z, A - V)\}$ hold for $z \in Z$ and open subsets $U, V \subseteq A$. This implies $Z(U \cap V) \subseteq Z(U) \cap Z(V)$.

It remains to show $Z(U) \cap Z(V) \subseteq Z(U \cap V)$. Let $z \in Z(U) \cap Z(V)$. Because of $d_Z(z, U) < d_Z(z, A - U)$ there is $u \in U$ with $d_Z(z, u) < d_Z(z, A - U)$. Because of $d_Z(z, V) < d_Z(z, A - V)$ there is $v \in V$ with $d_Z(z, v) < d_Z(z, A - V)$. If $u \notin V$ then $d_Z(z, v) < d_Z(z, u)$. If $v \notin U$ then $d_Z(z, u) < d_Z(z, v)$. In particular we have $u \in V$ or $v \in U$.

Suppose that $u \notin V$. Then $v \in U$ and $d_Z(z, v) < d_Z(z, u) < d_Z(z, A - U)$. Thus

$$\begin{aligned} d_Z(z, V \cap U) \leq d_Z(z, v) < \min\{&d_Z(z, A - U), d_Z(z, A - V)\} \\ &= d_Z(z, A - (U \cap V)) \end{aligned}$$

and $z \in Z(U \cap V)$. Analogously one shows $z \in Z(U \cap V)$ if $v \notin U$. It remains to treat the case, where $u \in V$ and $v \in V$, or, equivalently, where $u, v \in U \cap V$. We may assume without loss of generality $d_Z(z, v) \leq d_Z(z, u)$. Then $d_Z(z, v) \leq d_Z(z, u) < d_Z(z, A - U)$. Thus

$$\begin{aligned} d_Z(z, V \cap U) \leq d_Z(z, v) < \min\{&d_Z(z, A - U), d_Z(z, A - V)\} \\ &= d_Z(z, A - (U \cap V)) \end{aligned}$$

and $z \in Z(U \cap V)$. This proves $Z(U \cap V) = Z(U) \cap Z(V)$.

(iii) If $U = \varnothing$ or $U = A$, this is obvious so that it suffices to treat the case $\varnothing \neq U \neq A$. If $Z(U) \cap A = U$, then U is open in A, because $Z(U)$ is open in Z by (i).

Assume that U is open in A. Consider $z \in U$. Since $U \subseteq A$ is open, there exists $\epsilon > 0$ such that $\{y \in Z \mid d_Z(z,y) < \epsilon\} \cap A \subseteq U$. Hence $d_Z(x,z) \geq \epsilon$ for $x \in A - U$ and hence $d_Z(z, A-U) \geq \epsilon > 0 = d_Z(z,U)$. Therefore $z \in Z(U) \cap A$. Consider $z \in Z(U) \cap A$. Since $d_Z(z,U) < d_Z(z,A-U)$ implies $d_Z(z,A-U) > 0$ and hence $z \notin A - U$, we conclude $z \in U$. This proves $Z(U) \cap A = U$.

(iv) This is obvious as H acts by isometries. □

Lemma 4.15 *Let K be a compact subset of X. There is $\varepsilon_{\mathbb{R}} > 0$ and a G–invariant cofinite collection $\mathcal{V}_{\mathbb{R}}$ of open $\mathcal{F}in$–subsets of $FS(X)$ such that*

(i) $\dim \mathcal{V}_{\mathbb{R}} \leq \dim X$;

(ii) *if the image of $c \in FS(X)^{\mathbb{R}}$ as a generalized geodesic in X (which is a point) intersects (and is therefore contained in) $G \cdot K$, then there is $U \in \mathcal{V}_{\mathbb{R}}$ such that $B_{\varepsilon_{\mathbb{R}}}(c) = B_{\varepsilon_{\mathbb{R}}}(\Phi_{\mathbb{R}}(c)) \subseteq U$.*

Proof The argument is very similar to the proof of Proposition 4.13.

For every $x \in X$ we can choose a $\mathcal{F}$in–neighborhood V_x. Because the quotient map $\pi\colon X \to G\backslash X$ is open, $\{\pi(V_x) \mid x \in X\}$ is an open cover for $G\backslash X$. Arguing as in Lemma 4.12 we see that $\dim G\backslash X \leq \dim X$. Therefore $\{\pi(V_x) \mid x \in X\}$ can be refined to an cover $\mathcal{W}$ of $G\backslash X$ whose dimension is at most $\dim X$. For every $W \in \mathcal{W}$ pick $x_W \in X$ with $W \subseteq \pi(V_{x_W})$ and set $\mathcal{W}_0 := \{\pi^{-1}(W) \cap gV_{x_W} \mid W \in \mathcal{W}, g \in G\}$. This is an open $\mathcal{F}$in–cover, because each V_x is an open $\mathcal{F}$in–set. Its dimension is bounded by $\dim X$, because the dimension of $\mathcal{W}$ is bounded by $\dim X$ and because for each $g \in G$ and $x \in X$ we have either $V_x = gV_x$ or $V_x \cap gV_x = \varnothing$.

For each $W \in \mathcal{W}_0$ we set $W_{FS} := \{c \in FS(X) \mid c(0) \in W\}$. Because $c \mapsto c(0)$ is proper by Lemma 1.10, $\{c \in FS(X) \mid c(0) \in K\}$ is compact. Thus there is $\varepsilon_{\mathbb{R}} > 0$ and a finite subcollection $\mathcal{W}_1 \subseteq \mathcal{W}_0$ such that for every $c \in FS(X)$ with $c(0) \in K$ we have $B_{\varepsilon_{\mathbb{R}}}(c) \subseteq W_{FS}$ for some $W \in \mathcal{W}_1$. Thus $\mathcal{V}_{\mathbb{R}} := \{gW_{FS} \mid W \in \mathcal{W}_1, g \in G\}$ has the claimed properties. □

Proof of Theorem 4.2 Let $R \subseteq A_{\leq\gamma}$ be a subset that contains exactly one element from each orbit of the G–action. Then R is finite by Lemma 4.10 (i). For each $a \in R$, let $\mathcal{V}_a$ be a covering of Y_a satisfying the assertions from Proposition 4.13. Let $\mathcal{W}_a := (q_a)^{-1}\mathcal{V}_a = \{q_a^{-1}(V) \mid V \in \mathcal{V}_a\}$. For $b \in A_{\leq\gamma}$ pick $a \in R$ and $g \in G$, such that $ga = b$ and set $\mathcal{W}_b := g(\mathcal{W}_a) = \{gW \mid W \in \mathcal{W}_a\}$. (This does not depend on the choice of g, as q_a is G_a–equivariant and $\mathcal{V}_a$ is G_a–invariant.) By Lemma 4.10 (iii) there is $\delta > 0$ such that, if we set $U_a := B_\delta(FS_a)$, then $U_a \cap U_b = \varnothing$ for $a \neq b \in A_{\leq\gamma}$. We now use Lemma 4.14 to extend the $W \in \mathcal{W}_a$ to open subsets of $FS(X)$ and define the collection $\mathcal{U}$ by

$$\mathcal{U} := \bigcup_{a \in A_{\leq\gamma}} \{Z_a(W) \cap U_a \mid W \in \mathcal{W}_a\},$$

where $Z_a(W) := \{c \in FS(X) \mid d_{FS}(c, W) < d_{FS}(c, FS(X) - W)\}$. Define the desired collection of open subsets of $FS(X)$ by

$$\mathcal{V} := \mathcal{U} \cup \mathcal{V}_{\mathbb{R}},$$

where $\mathcal{V}_{\mathbb{R}}$ is from Lemma 4.15. It remains to show, that $\mathcal{V}$ has the desired properties.

(i) The members of each $\mathcal{V}_a$ are open $\mathcal{VC}$yc–sets with respect to the G_a–action by Proposition 4.13. The q_a are continuous and G_a–equivariant by construction. Thus the members of each $\mathcal{W}_a$ are also open $\mathcal{VC}$yc–sets with respect to the G_a–action. For each a, FS_a is a G_a–set. Because the U_a are mutually disjoint, each U_a is an G_a–set as well. By Lemma 4.14 each $Z_a(W) \cap U_a$ with $W \in \mathcal{W}_a$ is an open $\mathcal{VC}$yc–subset of $FS(X)$ with respect to the G–action.

(ii) Each $\mathcal{V}_a$ is G_a–invariant. The union of the $\mathcal{W}_a$ is G–invariant, because the q_a are G_a–equivariant. Thus by Lemma 4.14 the collection of all $Z_a(W)$ is G–invariant. The collection of the U_a is G–invariant. Therefore $\mathcal{U}$ is G–invariant. Since $\mathcal{U}_{\mathbb{R}}$ is G–invariant this implies that $\mathcal{V}$ is G–invariant.

(iii) Each $G_a \backslash \mathcal{V}_a$ is finite by Proposition 4.13 (iii). Therefore $G \backslash \mathcal{U}$ is finite. Since $G \backslash \mathcal{U}_{\mathbb{R}}$ is finite, $G \backslash \mathcal{V}$ is finite.

(iv) All the U_a are mutually disjoint. For each a, $\dim \mathcal{W}_a = \dim \mathcal{V}_a$. Using Proposition 4.13 (i) and Lemma 4.14 we get therefore

$$\begin{aligned} \dim \mathcal{U} &= \max_{a \in A} \dim\{Z_a(W) \cap U_a \mid W \in \mathcal{W}_a\} \\ &= \max_{a \in R} \dim \mathcal{V}_a \leq \dim X. \end{aligned}$$

Put $M := 1 + 2\dim(X)$. This number is independent of γ and

$$\dim(\mathcal{V}) \leq 1 + \dim(\mathcal{V}_{\mathbb{R}}) + \dim(\mathcal{U}) \leq M.$$

(v) Let $c \in FS(X)$ such that c intersects $G \cdot K$. If $c \in FS(X)^{\mathbb{R}}$, then $B_{\varepsilon_{\mathbb{R}}}(\Phi_{\mathbb{R}}(c)) = B_{\varepsilon_{\mathbb{R}}}(c) \subseteq U$ for some $U \in \mathcal{V}_{\mathbb{R}}$, where $\varepsilon_{\mathbb{R}}$ is from Lemma 4.15. Hence it remains to show that there is $\varepsilon_{\mathcal{U}}$ such that for any $c \in FS(X)_{\leq\gamma} - FS(X)^{\mathbb{R}}$ such that c intersects

$G \cdot K$, there is $U \in \mathcal{U}$ satisfying $B_{\varepsilon_{\mathcal{U}}}(\Phi_{[-\gamma,\gamma]}(c)) \subseteq U$ because then we can take $\epsilon = \min\{\epsilon_{\mathbb{R}}, \epsilon_{\mathcal{U}}\}$.

Now suppose the desired $\epsilon_{\mathcal{U}}$ does not exist, that is, for $n \geq 1$ there are $c_n \in FS(X)_{\leq \gamma} - FS(X)^{\mathbb{R}}$, $d_n \in FS(X)$ and $t_n \in \mathbb{R}$ such that c_n intersects $G \cdot K$, $d_{FS}(\Phi_{t_n}(c_n), d_n) < 1/n$ and $d_n \notin U$ for all $U \in \mathcal{U}$ that contain $\Phi_{[-\gamma,\gamma]}(c_n)$. Choose $a_n \in R$ with $c_n \in G \cdot FS_{a_n}$. As R is finite, we arrange by passing to a subsequence that there is $a \in R$ with $a_n = a$ for all n.

Because of Lemma 4.10 (ii) there are $g_n \in G_a$ such that all $g_n c_n$ are contained in a compact set $K_a \subset G \cdot FS_a$. After passing to a subsequence and replacing c_n by $g_n c_n$ and d_n by $g_n d_n$ we can assume that $c_n \to c$ for some $c \in K_a \subset G \cdot FS_a$. Choose $g \in G$ with $g^{-1}c \in FS_a$. Choose $V \in \mathcal{V}_a$ with $q_a(g^{-1}c) \in V$. Then $g^{-1}c \in q_a^{-1}(V)$ and hence $\Phi_{\mathbb{R}}(g^{-1}c) \subset q_a^{-1}(V)$. Thus $\Phi_{\mathbb{R}}(c) \subseteq U$ for some $U \in \mathcal{U}$, namely for $U = g \cdot Z_a(W) \cap U_{ga}$, where we set $W := q_a^{-1}(V)$.

Passing to a further subsequence we can arrange that $t_n \to t_0$ for some $t_0 \in [-\gamma, \gamma]$. Then also $d_n \to \Phi_{t_0}(c)$. Hence there is n_0 such that $d_n \in U$ for $n \geq n_0$, because $d_n \to \Phi_{t_0}(c) \in U$.

The set $\Phi_{[-\gamma,\gamma]}(c)$ is a compact subset of the open set U. Hence we can find $\delta > 0$ such that $B_\delta\big(\Phi_{[-\gamma,\gamma]}(c)\big) \subseteq U$. We have $d_X\big(\Phi_t(c), \Phi_t(c_n)\big) \leq e^\tau \cdot d_{FS(X)}(c_n, c)$ for $n \geq 1$ and $t \in [-\tau, \tau]$ by Lemma 1.3. Since there exists n_1 such that $d_{FS(X)}(c_n, c) < e^{-\tau} \cdot \delta$ for $n \geq n_1$, we get $\Phi_{[-\gamma,\gamma]}(c_n) \subseteq U$ for $n \geq n_1$, a contradiction. This finishes the proof of Theorem 4.2. □

5 Flow spaces and S–long covers

Summary In Definitions 5.5 and 5.9 we formulate two conditions for a flow space FS with an action of a group G. The first condition asks for the existence of long covers of uniformly bounded dimension for a subset of FS that contains the periodic orbits of the flow and is large in the sense that its complement is cocompact for the action of G. (In our application later the action of G on FS will be cocompact and we will get the second part of this condition for free; we expect however that this condition can also be verified in situations where the action is not cocompact.) The second condition concerns the dynamic of the flow with respect to a suitable homotopy action. Our main result is Proposition 5.11 which asserts that G is transfer reducible, provided the two conditions are satisfied.

In this section we fix the following convention, compare Bartels–Lück–Reich [2, Convention 1.3].

Convention 5.1 *Let*

- G *be a group;*
- $\mathcal{F}$ *be a family of subgroups of* G*;*
- (FS, d_{FS}) *be a locally compact metric space with a proper isometric* G*–action;*
- $\Phi\colon FS \times \mathbb{R} \to FS$ *be a flow.*

We assume that the following conditions are satisfied:

- Φ *is* G*–equivariant;*
- $FS - FS^{\mathbb{R}}$ *is locally connected;*
- $k_G := \sup\{|H| \mid H \subseteq G$ *subgroup with finite order* $|H|\} < \infty$*;*
- $\dim(FS - FS^{\mathbb{R}}) < \infty$*;*
- *the flow is uniformly continuous in the following sense: for* $\alpha > 0$ *and* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$d_{FS}(z, z') \leq \delta, \tau \in [-\alpha, \alpha] \Longrightarrow d_{FS}(\Phi_\tau(z), \Phi_\tau(z')) \leq \varepsilon. \tag{5.2}$$

For a subset $I \subseteq \mathbb{R}$ we set $\Phi_I(z) := \{\Phi_t(z) \mid t \in I\}$. For $z \in FS$ we define its *G–period*

$$\mathrm{per}^G_\Phi(z) = \inf\{\tau \mid \tau > 0,\ \exists\, g \in G \text{ with } \Phi_\tau(z) = gz\} \quad \in [0, \infty],$$

where the infimum over the empty set is defined to be ∞. Obviously $\mathrm{per}^G_\Phi(z) = 0$ if and only if $z \in FS^{\mathbb{R}}$. Because the flow Φ commutes with action of G, the function $\mathrm{per}^G_\Phi(z)$ is constant on orbits of the flow. If $L \subseteq FS$ is such an orbit of the flow Φ, define its *G–period* by

$$\mathrm{per}^G_\Phi(L) := \mathrm{per}^G_\Phi(z)$$

for any choice of $z \in FS$ with $L = \Phi_{\mathbb{R}}(z)$. For $\gamma \geq 0$ put

$$FS_{>\gamma} := \{z \in FS \mid \mathrm{per}^G_\Phi(z) > \gamma\}; \tag{5.3}$$

$$FS_{\leq\gamma} := \{z \in FS \mid \mathrm{per}^G_\Phi(z) \leq \gamma\}. \tag{5.4}$$

Definition 5.5 We will say that *FS admits long $\mathcal{F}$–covers at infinity and periodic flow lines* if the following holds:

There is $M > 0$ such that for every $\gamma > 0$ there is a collection $\mathcal{V}$ of open $\mathcal{F}$–subsets of FS and $\varepsilon > 0$ satisfying:

(i) $\mathcal{V}$ is G–invariant: $g \in G$, $V \in \mathcal{V} \Longrightarrow gV \in \mathcal{V}$;

(ii) $\dim \mathcal{V} \leq M$;

(iii) there is a compact subset $K \subseteq FS$ such that
- $FS_{\leq \gamma} \cap G \cdot K = \varnothing$;
- for $z \in FS - G \cdot K$ there is $V \in \mathcal{V}$ such that $B_\varepsilon\big(\Phi_{[-\gamma,\gamma]}(z)\big) \subset V$.

Theorem 5.6 *There is $M \in \mathbb{N}$ such that the following holds:*

For any $\alpha > 0$ there is $\gamma > 0$ such that for any compact subset K of $FS_{>\gamma}$ there is a collection of open $\mathcal{VC}yc$–subsets of FS such that

(i) *$\mathcal{V}$ is G–invariant: $g \in G$, $V \in \mathcal{V} \Longrightarrow gV \in \mathcal{V}$;*

(ii) *$\dim \mathcal{V} \leq M$;*

(iii) *$G \backslash \mathcal{V}$ is finite;*

(iv) *for every $z \in G \cdot K$ there is $V \in \mathcal{V}$ such that $\Phi_{[-\alpha,\alpha]}(z) \subset V$.*

Proof This follows from the techniques used and developed in Bartels–Lück–Reich [2, Sections 2–5], but is unfortunately not stated in precisely this form.

The main input is [2, Proposition 4.1]. In this reference it is assumed that the action of G on FS is cocompact, but this is not used in its proof, mainly because the statement concerns only a cocompact part of the flow space. (In [2] cocompactness of the action is used to conclude that the flow space is locally compact; note that we assumed this in Convention 5.1.) Therefore [2, Proposition 4.1] is valid in the present situation as well. Theorem 5.6 can be deduced from this using the argument in [2, page 1848]. □

Theorem 5.7 *Assume that FS admits long $\mathcal{F}$–covers at infinity and periodic flow lines and that $\mathcal{F}$ contains the family $\mathcal{VC}yc$ of virtually cyclic subgroups.*

Then there is $\widehat{N} \in \mathbb{N}$ such that for every $\alpha > 0$ there exists an open $\mathcal{F}$–cover $\mathcal{U}$ of FS of dimension at most $\widehat{N}$ and $\epsilon > 0$ (depending on α) such that the following holds:

(i) *For every $z \in FS$ there is $U \in \mathcal{U}$ such that $B_\epsilon\big(\Phi_{[-\alpha,\alpha]}(z)\big) \subseteq U$.*

(ii) *$G \backslash \mathcal{U}$ is finite.*

Proof Let M_0 be the number M appearing in Definition 5.5 and M_1 be the number M appearing in Theorem 5.6. We set $\widehat{N} := M_0 + M_1 + 1$. Theorem 5.6 also provides a number γ depending on α. We can assume that $\gamma \geq \alpha$. Since FS admits long $\mathcal{F}$–covers at infinity and periodic flow lines we can find a collection $\mathcal{V}_0$ of open $\mathcal{F}$–subsets of FS and $\varepsilon_0 > 0$ such that (i) to (iii) from Definition 5.5 hold.

Next we apply Theorem 5.6 and obtain a collection $\mathcal{V}_1$ of open $\mathcal{VC}$yc–subsets of FS and a compact subset $K \subseteq FS$ such that (i) to (iv) from Theorem 5.6 hold. A simple compactness argument, provided in Lemma 5.8 below, shows that there is $\varepsilon_1 > 0$ such that for every $z \in G \cdot K$ there is $V \in \mathcal{V}_1$ such that

$$B_{\varepsilon_1}(\Phi_{[-\alpha,\alpha]}(z)) \subset V.$$

Now set $\varepsilon := \min\{\varepsilon_0, \varepsilon_1\}$ and $\mathcal{U} := \mathcal{V}_0 \cup \mathcal{V}_1$. □

Lemma 5.8 *Assertion* (iv) *in Theorem 5.6 can be strengthened to*

(iv′) *There exists $\varepsilon > 0$ such that for any $z \in G \cdot K$ there is $V \in \mathcal{V}$ such that $B_\varepsilon(\Phi_{[-\alpha,\alpha]}(z)) \subset V$.*

Proof Suppose that there is no such ε. Then we can find a sequence $(z_n)_{n\geq 1}$ of points in $G \cdot K$ such that $B_{1/n}\big(\Phi_{[-\alpha,\alpha]}(z_n)\big) \not\subseteq U$ holds for every $n \geq 1$ and every $U \in \mathcal{V}$. Because $\mathcal{V}$ is G–invariant and Φ is G–equivariant, we can assume without loss of generality that $z_n \in K$ for all n. After passing to a subsequence we can assume that $z_n \to z$ as $n \to \infty$. Choose $V \in \mathcal{V}$ with $\Phi_{[-\alpha,\alpha]}(z) \subseteq V$. Since $\Phi_{[-\alpha,\alpha]}(z)$ is compact and V is open, we can find $\mu > 0$ with $B_\mu\big(\Phi_{[-\alpha,\alpha]}(z)\big) \subseteq V$. Because of the uniform continuity of the flow, we can find $\delta > 0$ such that $d_{FS}(\Phi_\tau(z), \Phi_\tau(z')) < \mu/2$ holds for all $\tau \in [-\alpha, \alpha]$ provided that $d_{FS}(z, z') \leq \delta$ is true. Since $\lim_{n\to\infty} z_n = z$, we can find $n \geq 1$ such that $d_{FS}(z, z_n) < \delta$ and $1/n < \mu/2$ hold. This implies $d_{FS}(\Phi_\tau(z), \Phi_\tau(z_n)) < \mu/2$ for all $\tau \in [-\alpha, \alpha]$. Thus

$$B_{1/n}\big(\Phi_{[-\alpha,\alpha]}(z_n)\big) \subseteq B_\mu\big(\Phi_{[-\alpha,\alpha]}(z)\big) \subseteq V,$$

contradicting the assumption. □

Definition 5.9 We will say that *FS admits contracting transfers* if for every finite subset S of G (containing e) there exists $\beta > 0$ and $\widetilde{N} \in \mathbb{N}$ such that the following holds:

For every $\delta > 0$ there is

(i) $T > 0$;

(ii) a contractible compact controlled $\widetilde{N}$–dominated space X;

(iii) a homotopy S–action (φ, H) on X;

(iv) a G–equivariant map $\iota\colon G \times X \to FS$ (where we use the left action $g \cdot (h, x) = (gh, x)$ on $G \times X$),

such that

(5.10) for every $(g,x)\in G\times X$, $s\in S$, $f\in F_s(\varphi,H)$ there is $\tau\in[-\beta,\beta]$ such that

$$d_{FS}\Big(\Phi_T(\iota(g,x)),\Phi_{T+\tau}(\iota(gs^{-1},f(x)))\Big)\leq\delta.$$

Proposition 5.11 *If FS satisfies the assumptions appearing in Convention 5.1, admits long covers at infinity and periodic orbits (see Definition 5.5), and admits contracting transfers (see Definition 5.9), then G is transfer reducible over the family $\mathcal{F}$ in the sense of Definition 0.4.*

The proof of Proposition 5.11 will use the following the lemma.

Lemma 5.12 *Let $\beta>0$, $T>0$ and $\epsilon>0$. Let S be a finite subset of G (containing e). Set $n:=|S|$ and $\alpha:=2n\beta$. Pick δ such that (5.2) holds. Let (φ,H) be a homotopy S–action on a compact metric space X and let $\iota\colon G\times X\to FS$ be a G–equivariant map. Assume 5.10 holds.*

Then

(5.13) *for every $(g,x)\in G\times X$ and $(h,y)\in S^n_{\varphi,H}(g,x)$ (see Definition 0.1) there is $\tau\in[-\alpha,\alpha]$ such that*

$$d_{FS}\left(\Phi_T(\iota(g,x)),\Phi_{T+\tau}(\iota(h,y))\right)\leq 2n\epsilon.$$

Proof If $(h,y)\in S^n_{\varphi,H}(g,x)$, then there are $x_0,\dots,x_n\in X$, $a_1,b_1,\dots,a_n,b_n\in S$, $f_1,\widetilde{f}_1,\dots,f_n,\widetilde{f}_n\colon X\to X$, such that $x_0=x$, $x_n=y$, $f_i\in F_{a_i}(\varphi,H)$, $\widetilde{f}_i\in F_{b_i}(\varphi,H)$, $f_i(x_{i-1})=\widetilde{f}_i(x_i)$ and $h=ga_1^{-1}b_1\dots a_n^{-1}b_n$. Set $g_i:=ga_1^{-1}b_1\dots a_i^{-1}b_i$. By 5.10 there are $\tau_1,\widetilde{\tau}_1,\dots,\tau_n,\widetilde{\tau}_n\in[-\beta,\beta]$ such that

$$d_{FS}\big(\Phi_T(\iota(g_{i-1},x_{i-1})),\Phi_{T+\tau_i}(\iota(g_{i-1}a_i^{-1},f_i(x_{i-1})))\big)\leq\delta;$$
$$d_{FS}\big(\Phi_T(\iota(g_i,x_i)),\Phi_{T+\widetilde{\tau}_i}(\iota(g_ib_i^{-1},\widetilde{f}_i(x_i)))\big)\leq\delta$$

for $i=1,\dots,n$. Put $\sigma_0=0$ and $\sigma_i:=(-\widetilde{\tau}_1+\tau_1)+\dots+(-\widetilde{\tau}_i+\tau_i)$ for $i=1,2,\dots,n$. Since $\sigma_i\in[-\alpha,\alpha]$, we conclude for $i=1,\dots,n$ from (5.2)

$$d_{FS}\big(\Phi_{T+\sigma_{i-1}}(\iota(g_{i-1},x_{i-1})),\Phi_{T+\tau_i+\sigma_{i-1}}(\iota(g_{i-1}a_i^{-1},f_i(x_{i-1})))\big)\leq\epsilon;$$
$$d_{FS}\big(\Phi_{T+\sigma_i}(\iota(g_i,x_i)),\Phi_{T+\widetilde{\tau}_i+\sigma_i}(\iota(g_ib_i^{-1},\widetilde{f}_i(x_i)))\big)\leq\epsilon.$$

Since $g_{i-1}a_i^{-1}=g_ib_i^{-1}$, $f_i(x_{i-1})=\widetilde{f}_i(x_i)$ and $T+\tau_i+\sigma_{i-1}=T+\widetilde{\tau}_i+\sigma_i$ we conclude for $i=1,\dots,n$ from the triangle inequality

$$d_{FS}\big(\Phi_{T+\sigma_{i-1}}(\iota(g_{i-1},x_{i-1})),\Phi_{T+\sigma_i}(\iota(g_i,x_i))\big)\leq 2\epsilon.$$

Using the triangle inequality we obtain

$$\begin{aligned} d_{FS}(\Phi_T(\iota(g,x)),\Phi_{T+\sigma_n}(\iota(h,y))) &= d_{FS}(\Phi_T(\iota(g_0,x_0)),\Phi_{T+\sigma_n}(\iota(g_n,x_n))) \\ &\le \sum_{i=1}^{n} d_{FS}(\Phi_{T+\sigma_{i-1}}(\iota(g_{i-1},x_{i-1})),\Phi_{T+\sigma_i}(\iota(g_i,x_i))) \\ &\le \sum_{i=1}^{n} 2\epsilon \\ &\le 2n\epsilon. \end{aligned}$$

This completes the proof. □

Proof of Proposition 5.11 Let S be a finite subset of G. Let $\widehat{N}$ be the number from Theorem 5.7. Let $\widetilde{N}$ and β be the numbers (depending on S) appearing in Definition 5.9. Put $\alpha := 2\beta|S|$. By Theorem 5.7 there is an open $\mathcal{F}$–cover $\mathcal{U}$ of FS of dimension at most $\widehat{N}$ and $\epsilon_0 > 0$ with the property that for every $z \in FS$ there is $U_z \in \mathcal{U}$ such that

$$B_{\epsilon_0}\big(\Phi_{[-\alpha,\alpha]}(z)\big) \subseteq U_z.$$

Put $\epsilon := \frac{\epsilon_0}{2|S|}$. Pick $\delta > 0$ such that (5.2) holds. By assumption (see Definition 5.9), there are $T > 0$, a contractible compact controlled $\widetilde{N}$–dominated space X, a G–equivariant map $\iota\colon G{\times}X \to FS$ and a homotopy S–action (φ, H) on X such that 5.10 holds. Using Lemma 5.12 we conclude that for every $(g,x) \in G{\times}X$ we have

$$\Phi_T(\iota(h,y)) \in B_{\epsilon_0}\big(\Phi_{[-\alpha,\alpha]}(\Phi_T(\iota(g,x)))\big)$$

for all $(h,y) \in S^n_{\varphi,H}(g,x)$, $n \le |S|$. Hence we get $\Phi_T(\iota(h,y)) \in U_{\Phi_T(\iota(g,x))}$ for all $(h,y) \in S^n_{\varphi,H}(g,x)$, $n \le |S|$. This implies that $\mathcal{V} := \{(\Phi_T \circ \iota)^{-1}(U) \mid U \in \mathcal{U}\}$ is S–long with respect to (φ,H). Finally, $\dim \mathcal{V} \le \dim \mathcal{U} \le \widehat{N}$. Thus G is transfer reducible over $\mathcal{F}$, where we use $N := \max\{\widehat{N}, \widetilde{N}\}$. □

6 Non-positively curved groups are transfer reducible

In this section we prove our Main Theorem as stated in the introduction. Let G be a group with an isometric cocompact proper action on a finite dimensional CAT(0)–space X. We need to show that G is transfer reducible over the family $\mathcal{VC}$yc of virtually cyclic subgroups, see Definition 0.4. To this end we will show that Proposition 5.11 applies, where $\mathcal{F} = \mathcal{VC}$yc.

6.1 $\overline{B}_R(x)$ is finitely dominated

Fix a base point $x_0 \in X$. For $r \geq 0$ let

$$\rho_{r,x_0}\colon \overline{X} \to \overline{B}_r(x_0)$$

be the natural projection introduced in Remark 2.3. The map ρ_{r,x_0} is the identity on $\overline{B}_r(x_0)$. If $x \in \overline{X}$ with $x \notin B_r(x_0)$, then $\rho_{r,x_0}(x) = c_{x_0,x}(r)$, where $c_{x_0,x}\colon \mathbb{R} \to X$ is the generalized geodesic uniquely determined by $c_- = 0$, $c(-\infty) = x_0$ and $c(\infty) = x$ (see Section 3.1).

Lemma 6.1 *The space X is a Euclidean neighborhood retract, that is, there is a natural number N, a closed subset $A \subseteq \mathbb{R}^N$, an open neighborhood U of A in $\mathbb{R}^N$ and a map $r\colon U \to A$ such that $r|_A = \mathrm{id}_A$ and X is homeomorphic to A. The number N can be chosen to be $2 \cdot \dim(X) + 1$.*

Proof Since X is proper as metric space, it is locally compact. Since any two points in X can be joint by a unique geodesic, X is connected and locally contractible. Hence X has a countable basis for its topology (see Munkres [10, Chapter 6.5, Exercise 2, page 261]). Obviously X is Hausdorff. By assumption $\dim(X) < \infty$. We conclude from [10, Chapter 7.9, Exercise 10, page 315] that X is homeomorphic to a closed subset A of $\mathbb{R}^N$ for $N = 2 \cdot \dim(X) + 1$. Now apply Dold [5, Chapter IV.8, Proposition 8.12, page 83]. □

Lemma 6.2 *The space $\overline{B}_R(x_0)$ is a compact contractible metric space which is controlled $(2 \cdot \dim(X) + 1)$–dominated (in the sense of Definition 0.2).*

Proof Since X is proper as metric space by assumption, the closed ball $\overline{B}_R(x_0)$ is compact. The space $\overline{B}_R(x_0)$ inherits from X a metric and is contractible.

Because of Lemma 6.1 we can find an open subset $U \subset \mathbb{R}^{2 \cdot \dim(X)+1}$ and maps $i\colon X \to U$ and $r\colon U \to X$ with $r \circ i = \mathrm{id}_X$. Since U is a smooth manifolds, it can be triangulated and hence is a simplicial complex of dimension $(2 \cdot \dim(X) + 1)$. Since $\overline{B}_R(x_0)$ is compact, $i\big(\overline{B}_R(x_0)\big)$ is compact and hence contained in a finite subcomplex $K \subseteq U$. Since K and hence $r(K)$ are compact, we can find $C > 0$ with $r(K) \subseteq \overline{B}_{R+C}(x_0)$. Let

$$i'\colon \overline{B}_R(x_0) \to K$$

be the map defined by i. Let

$$r'\colon K \to \overline{B}_R(x_0)$$

be the composite

$$K \xrightarrow{r|_K} \overline{B}_{R+C}(x_0) \xrightarrow{\rho_{R,x_0}|_{\overline{B}_{R+C}(x_0)}} \overline{B}_R(x_0).$$

Then $r' \circ i' = \mathrm{id}_{\overline{B}_R(x_0)}$ and K is a finite $(2 \cdot \dim(X) + 1)$–dimensional simplicial complex. This implies that $\overline{B}_R(x_0)$ is controlled $(2 \cdot \dim(X) + 1)$–dominated. □

6.2 Convention 5.1 applies to $FS(X)$

Let $FS(X)$ be the flow space for X from Definition 1.2. We will show that this flow space satisfies the conditions from Convention 5.1. By Proposition 1.11 the action of G on $FS(X)$ is isometric, proper and cocompact. In particular, the flow space $FS(X)$ is locally compact. By Proposition 2.10 the flow space $FS(X)$ is locally connected. By Bridson–Haefliger [4, II.2.8(2), page 179] there is $k_G < \infty$ such that finite subgroups of G have order at most k_G. By Proposition 2.9 $\dim FS(X) - FS(X)^{\mathbb{R}}$ is finite. The uniform continuity of the flow follows from Lemma 1.3.

6.3 Long $\mathcal{VC}$yc–covers for $FS(X)$ at periodic flow lines

We need to show that the flow space $FS(X)$ admits long $\mathcal{VC}$yc–covers at infinity and periodic flow lines in the sense of Definition 5.5. We take for M the number appearing in Theorem 4.2. Let $\gamma > 0$ be given. Because the action of G on $FS(X)$ is cocompact we conclude from Theorem 4.2 that there is $\varepsilon > 0$ and a G–invariant cofinite collection $\mathcal{V}$ of open $\mathcal{VC}$yc–subsets of $FS(X)$ such that

- $\dim \mathcal{V} \leq M$;
- for any $c \in FS(X)_{\leq\gamma}$ there is $V \in \mathcal{V}$ such that $B_{2\varepsilon}(\Phi_{[-\gamma,\gamma]}(c)) \subseteq V$.

Put $S := \{c \in FS \mid \exists U \in \mathcal{V} \text{ such that } \overline{B}_\varepsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq U\}$. Note that S is G–invariant, because $\mathcal{V}$ is. Moreover, $FS(X)_{\leq\gamma} \subseteq S$. It remains to show, that there is a compact subset $K \subseteq FS(X)$ such that $FS(X) - S = G \cdot K$. Because the action of G on $FS(X)$ is cocompact (Proposition 1.11) and $FS(X)$ is locally compact (Proposition 1.9) it suffices to show that S is open.

Choose $U \in \mathcal{V}$ with $\overline{B}_\varepsilon(\Phi_{[-\gamma,\gamma]}(c_0)) \subseteq U$. It suffices to show that there is $\delta > 0$ such that $\overline{B}_\varepsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq U$ for all $c \in FS(X)$ with $d_{FS}(c, c_0) < \delta$. We proceed by contradiction and assume that there is no such δ. Then there are c_n, $d_n \in FS(X)$, $t_n \in [-\gamma,\gamma]$ such that $d_{FS}(c_n, c_0) < 1/n$, $d_{FS}(d_n, \Phi_{t_n}(c_n)) \leq \varepsilon$ but $d_n \notin U$. In particular $c_n \to c_0$. By passing to a subsequence we can assume that $t_n \to t \in [-\gamma,\gamma]$. Thus $\Phi_{t_n}(c_n) \to \Phi_t(c_0)$. It follows that $d_{FS}(d_n, \Phi_t(c_0))$ is bounded independent of n.

Because $FS(X)$ is a proper metric space (Proposition 1.9) we can pass to a further subsequence and assume that $d_n \to d$. Then $d_{FS}(d, \Phi_t(c_0)) \leq \varepsilon$. Thus $d \in U$. But this implies $d_n \in U$ for sufficiently large n, a contradiction.

This shows that the flow space $FS(X)$ admits long $\mathcal{VC}$yc–covers at infinity and periodic flow lines.

6.4 Contracting transfers for $FS(X)$

Finally we need to show that $FS(X)$ admits contracting transfers in the sense of Definition 5.9. Let S be a finite subset of G (containing e). Let β be the number appearing in Proposition 3.8. Set $\widetilde{N} := 2\dim X + 1$. Let $\delta > 0$ be given. Let $T, R > 0$ be the numbers coming from Proposition 3.8. Then $\overline{B}_R(x_0)$ is compact, contractible and controlled $\widetilde{N}$–dominated by Lemma 6.2. We use the homotopy S action (φ, H) on $B_R(x_0)$ from Definition 3.2 and the map $\iota\colon G \times B_r(x_0) \to FS(X)$ obtained by restriction from the map from Definition 3.4. It follows from Proposition 3.8 that 5.10 holds.

Thus Proposition 5.11 applies and we conclude that G is transfer reducible over $\mathcal{VC}$yc as claimed in our main Main Theorem in the introduction.

Remark 6.3 Our argument proves a little more than is stated in our main Theorem. Namely, the cover we construct is in addition cofinite for the action of G, that is, $G \backslash \mathcal{U}$ is finite. This follows from Theorem 4.2 (iii) and Theorem 5.6 (iii).

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