Virtual push-forwards

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Let $p\colon F \to G$ be a morphism of DM stacks of positive *virtual* relative dimension k and let $\gamma \in A^k(F)$. We give sufficient conditions for $p_*(\gamma \cdot [F]^{\mathrm{virt}})$ to be a multiple of $[G]^{\mathrm{virt}}$. We show an analogue of the conservation of number for virtually smooth families. We show implications to Gromov–Witten invariants and give a new proof of a theorem of Marian, Oprea and Pandharipande [19] which compares the virtual classes of moduli spaces of stable maps and moduli spaces of stable quotients.

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1 Introduction

Virtual fundamental classes have been introduced by Li–Tian [16] and Behrend–Fantechi [2] and in the past fifteen years have become a useful tool when one has to deal with badly behaved (that is, singular, with several components of possibly different dimensions) moduli spaces. One of the main problems when working with virtual fundamental classes is that in certain situations they fail to behave as fundamental classes do. One easy example is the following. Let $p\colon F \to G$ be a finite morphism of DM stacks and suppose that F and G have pure dimension. Let $G_1, \dots, G_s$ denote the irreducible components of G. Then we have that

$$p_*[F] = n_1[G_1] + \cdots + n_s[G_s]$$

for some $n_1, \dots, n_s \in \mathbb{Q}$. On the contrary, given a morphism $p\colon F \to G$ of stacks which possess virtual classes of the same virtual dimension, we have no reasons to believe that the following relation holds

$$p_*[F]^{\mathrm{virt}} = n_1[G_1] + \cdots + n_s[G_s],$$

where $G_1, \dots, G_s$ are cycles on G such that $[G]^{\mathrm{virt}} = [G_1] + \cdots + [G_s]$.

In this paper we find sufficient conditions for the above condition to hold. More generally, let $p\colon F \to G$ be a morphism of DM stacks which possess virtual classes of dimension k_1 respectively k_2 such that $k := k_1 - k_2 \geq 0$. We say that p satisfies the *virtual push-forward property* if for any $\gamma \in A^k(F)$ we have that

$$p_*(\gamma \cdot [F]^{\mathrm{virt}}) = n_1[G_1] + \cdots + n_s[G_s]$$

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for some $[G_1], \dots, [G_s]$ such that $[G]^{\mathrm{virt}} = [G_1] + \cdots + [G_s]$ and some $n_1, \dots, n_s \in \mathbb{Q}$. If moreover, the push-forward of $\gamma \cdot [F]^{\mathrm{virt}}$ along p is equal to a scalar multiple of the virtual class of G we say that p satisfies the *strong virtual push-forward property*. The main result (see Theorem 3.13) of this paper is the following.

Theorem *Let $p\colon F \to G$ be a proper morphism of Deligne–Mumford stacks which possess perfect obstruction theories $E^\bullet_F$ and $E^\bullet_G$. If p has a perfect relative obstruction theory compatible with $E^\bullet_F$ and $E^\bullet_G$ and G is connected, then p satisfies the strong virtual push-forward property in* homology.

In Section 2 we list the main notions and results needed in the rest of the paper. We review obstruction theories, normal cones and virtual pull-backs for Chow groups in Manolache [18]. We generalize the construction of virtual pull-backs to groups of algebraic equivalence classes. This is a key ingredient in the proof of the conservation number principle for virtually smooth morphisms (see Definition 3.4).

In Section 3 we first show the virtual push-forward property (see Lemma 3.6). The proof uses arguments present in Lai [14] and the functoriality property of virtual cycles in Kim–Kresch–Pantev [10]. We prove an analogue of the *conservation of number principle* (see Fulton [7]) for virtually smooth morphisms. This is achieved by passing to groups of algebraic equivalence classes. The strong virtual push-forward property is a consequence of the push forward property, the properties of virtual pull-backs for algebraic equivalence classes and the conservation of number principle. The relative version of the strong virtual push-forward property (Proposition 3.14) is a generalization of Costello's virtual push-forward property in [4].

In Section 4 we prove applications of the main results in Section 3. More precisely, we

- show that the virtual Euler characteristic is locally constant in virtually smooth families; this is a consequence of the conservation of number principle for virtually smooth morphisms;
- study the relation between virtual classes of moduli spaces of stable maps $\overline{p}\colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(Y, p_*\beta)$ where $\overline{p}$ is the morphism induced by a smooth fibration $p\colon X \to Y$;
- give a new proof to a theorem of Marian, Oprea, Pandharipande [19] which compares the virtual classes of moduli spaces of stable maps to projective spaces and moduli spaces of stable quotients.

Relation to other work The definition of the (strong) virtual push-forward property was introduced by A Gathmann [8] who studied the relation between the virtual push-forward property for the natural map between the moduli space of stable maps to the projectivization of a split rank two bundle $V \to X$ and the moduli space of stable maps to X. His computation uses localization and it is rather involved. A similar result appears in a work of B Kim [9], who compares a certain intersection product on the moduli space of stable maps to the projectivization of a split bundle over a variety X to the virtual class of the moduli space of stable maps to X. Kim's proof also uses localization. Our virtual push-forward theorem was inspired by these results, but the nature of the proof is completely different. Our proof relies on the functoriality property of virtual cycles of Kim, Kresch and Pantev [10] and the properties of virtual pull-backs (see Manolache [18]). Methods similar to ours appear in [14], where H-H Lai analyzes the map between the moduli space of stable maps to a certain blow-up and the moduli space of stable maps to the base variety. Our Lemma 3.6 is a slight generalization of Lai's results. The relative version of the (strong) virtual push-forward theorem is a generalization of Costello's push-forward formula for virtual cycles [4]. The applications of these results generalize several previous results as follows. The conservation of virtual characteristics for virtually smooth morphisms generalizes results of Fantechi and Göttsche in [6]. The relation between moduli spaces of stable maps to projective bundles on some variety X and moduli spaces of stable maps to X have been studied by Gathmann under very restrictive hypothesis. The relation between moduli spaces of stable quotients and moduli spaces of stable maps has been studied by Popa [21], Marian, Oprea and Pandharipande [19] and Toda [22]. Relations between virtual classes have been proved in [19] and [22] by localization techniques. We give a unified approach to these results.

Notation and conventions We take the ground field to be $\mathbb{C}$. An Artin stack is an algebraic stack in the sense of Laumon and Moret-Bailly [15] of locally finite type over the ground field. Deligne–Mumford stacks will be called in short DM-stacks. Unless otherwise specified we will try to respect the following convention: we will usually denote schemes by X, Y, Z, etc, Deligne–Mumford stacks by F, G, H, etc. and Artin stacks for which we know that they are not Deligne–Mumford stacks (such as the moduli space of genus–g prestable curves or vector bundle stacks) by gothic letters $\mathfrak{M}_g$, $\mathfrak{E}$, $\mathfrak{F}$, etc. By a commutative diagram of stacks we mean a 2–commutative diagram of stacks and by a Cartesian diagram of stacks we mean a 2–Cartesian diagram of stacks. Chow groups for schemes are defined in the sense of Fulton [7]; this definition has been extended to DM stacks (with $\mathbb{Q}$–coefficients) by Vistoli [23] and to algebraic stacks (with $\mathbb{Z}$–coefficients) by Kresch [12]. We will consider Chow groups (of schemes/stacks) with $\mathbb{Q}$–coefficients. By homology we mean Borel–Moore homology (see the Appendix in Coates [3] for a definition for DM stacks). As in most applications we will consider compact stacks this will coincide with singular homology. For a fixed stack F we denote by $\mathcal{D}_F$ the derived category of coherent $\mathcal{O}_F$ modules. For a fixed stack F we denote by L_F its cotangent complex defined in Olsson [20].

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2 Virtual pull-backs

2.1 Preliminaries

We shortly review obstruction theories, normal cones to DM-type morphisms and virtual pull-backs. We refer to Behrend–Fantechi [2] for a more complete treatment of cone stacks and obstruction theories and to Manolache [18] for normal cones and virtual pull-backs.

2.1.1 Obstruction theories

Definition 2.1 Let $E^\bullet \in \mathcal{D}_F^{\leq 0}$. $E^\bullet$ is said to be of perfect amplitude if there exists $n \geq 0$ such that $E^\bullet$ is locally isomorphic to $[E^{-n} \to \cdots \to E^0]$, where for all $i \in \{-n, \dots, 0\}$, E^i is a locally free sheaf.

Definition 2.2 Let $E^\bullet \in \mathcal{D}_F^{\leq 0}$. Then a homomorphism $\Phi\colon E^\bullet \to L_F^\bullet$ in $\mathcal{D}_F^{\leq 0}$ is called an obstruction theory if $h^0(\Phi)$ is an isomorphism and $h^{-1}(\Phi)$ is surjective. If moreover, $E^\bullet$ is of perfect amplitude, then $E^\bullet$ is called a perfect obstruction theory.

Convention 2.3 Unless otherwise stated by a perfect obstruction theory we will always mean of perfect amplitude contained in $[-1, 0]$.

2.1.2 Normal cones to DM-type morphisms

Definition 2.4 A morphism $p\colon F \to G$ of Artin stacks is called of Deligne–Mumford type (or shortly of DM-type) if for any morphism $V \to G$, with V a scheme, $F \times_G V$ is a Deligne–Mumford stack.

Definition 2.5 Let X be a scheme and $\mathcal{F}$ be a coherent sheaf on X. We call $C(\mathcal{F}) :=$ SpecSym$(\mathcal{F})$ an abelian cone over X.

As described in Behrend–Fantechi [2, Section 1], every abelian cone $C(\mathcal{F})$ has a section 0: $X \to C(\mathcal{F})$ and an $\mathbb{A}^1$–action.

Definition 2.6 An $\mathbb{A}^1$–invariant closed subscheme of $C(\mathcal{F})$ that contains the zero section is called a cone over X.

Similarly, in [2, Section 1] Behrend and Fantechi define abelian cone stacks and cone stacks. The following definition of cone-stacks is slightly different from the one in [2].

Definition 2.7 Let F be a stack and let $E^\bullet$ be an element in $\mathcal{D}_F^{\leq 0}$. We call the stack quotient $h^1/h^0(E^{\bullet\vee})$ (in the sense of [2, Section 2]) an abelian cone stack over stack F. If $E^\bullet$ is of perfect amplitude contained in [-1,0], then we call the stack quotient $h^1/h^0(E^{\bullet\vee})$ a vector bundle stack.

A cone stack is a closed substack of an abelian cone stack invariant under the action of $\mathbb{A}^1$ and containing the zero section.

Convention 2.8 From now on, unless otherwise stated, by cones we will mean cone-stacks.

Example 2.9 (i) Let $i\colon X \to Y$ be a closed embedding of schemes. If $\mathcal{I}$ denotes the ideal sheaf of X in Y, then $N_{X/Y} = \operatorname{SpecSym} \mathcal{I}/\mathcal{I}^2$ is called the normal sheaf of X in Y and $C_{X/Y} := \operatorname{Spec} \oplus_{k\geq 0} \mathcal{I}^k/\mathcal{I}^{k+1} \hookrightarrow N_{X/Y}$ is called the normal cone of X in Y.

(ii) If $f\colon F \to G$ is a local immersion of DM-stacks, then Vistoli defines (see Vistoli [23, Definition 1.20]) the normal cone to f as described below. Let us consider a commutative diagram

(1)
$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & V \\ \downarrow & & \downarrow \\ F & \xrightarrow[f]{} & G \end{array}$$

with U and V schemes, the upper horizontal arrow a closed immersion and the vertical arrows étale. Then $C_{F/G}$ is the cone obtained by descent from $C_{U/V}$.

Note that $C_{F/G} \hookrightarrow N_{F/G} = \operatorname{SpecSym} h^{-1}(L_{F/G})$.

Definition 2.10 Let $p\colon F \to G$ be a DM-type morphism and let $L_{F/G} \in \mathcal{D}^{\leq 0}(\mathcal{O}_F)$ be the cotangent complex of p. Then we denote the stack

$$h^1/h^0(L^\vee_{F/G}) := h^1/h^0(\tau_{[0,1]}(L^\vee_{F/G}))$$

(see Behrend–Fantechi [2]) by $\mathfrak{N}_{F/G}$ and we call it the normal sheaf of p.

Proposition 2.11 (Behrend–Fantechi [2]) *Let us consider diagram* (1) *with F a Deligne–Mumford stack, the upper horizontal arrow a closed immersion, $U \to F$ an étale morphism and $V \to G$ a smooth morphism. Then for any U and V as above, there exists a unique cone-stack $\mathfrak{C}_{F/G} \subset \mathfrak{N}_{F/G}$ such that $\mathfrak{C}_{F/G} \times_F U = [C_{U/V}/\widetilde{f}^* T_{V/G}]$.*

Definition 2.12 We call $\mathfrak{C}_{F/G}$ the normal cone to p.

2.1.3 Virtual pull-backs in Chow groups In the following we recall the main results in Manolache [18].

Condition 2.13 We say that a morphism $p\colon F \to G$ of Artin stacks and a vector bundle stack $\mathfrak{E} \to F$ satisfy condition $(\star)$ if we have fixed a closed embedding $\mathfrak{C}_{F/G} \hookrightarrow \mathfrak{E}$.

Convention 2.14 We say in short that the pair $(p, \mathfrak{E})$ satisfies condition $(\star)$.

Remark 2.15 Let us consider a Cartesian diagram

$$\begin{array}{ccc} F' & \longrightarrow & G' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xrightarrow[p]{} & G \end{array}$$

If $\mathfrak{E}$ is a vector bundle on F such that $\mathfrak{C}_{F/G} \hookrightarrow \mathfrak{E}$ is a closed embedding, then $\mathfrak{C}_{F'/G'} \hookrightarrow f^*\mathfrak{E}$ is a closed embedding.

Construction 2.16 Let F be a DM stack and $\mathfrak{E}$ a vector bundle stack of (virtual) rank n on F such that $(p, \mathfrak{E})$ satisfies condition $(\star)$. We construct a pull-back map $p^!_{\mathfrak{E}}\colon A_*(G) \to A_{*-n}(F)$ as the composition

$$A_*(G) \xrightarrow{\sigma} A_*(\mathfrak{C}_{F/G}) \xrightarrow{i_*} A_*(\mathfrak{E}) \xrightarrow{s^*} A_{*-n}(F),$$

where

(1) σ is defined at the level of cycles by $\sigma\big(\sum n_i[V_i]\big) = \sum n_i\big[\mathfrak{C}_{V_i \times_G F/V_i}\big]$,

(2) i_* is the push-forward via the closed immersion i,

(3) s^* is the morphism of Kresch [13, Proposition 5.3.2].

The fact that σ is well defined has been checked in [18].

Definition 2.17 In the notation above, we call $p^!_{\mathfrak{E}}\colon A_*(G) \to A_*(F)$ a *virtual pull-back*. When there is no risk of confusion we will omit the index.

Remark 2.18 Let us consider a Cartesian diagram

$$\begin{array}{ccc} F' & \xrightarrow{q} & G' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xrightarrow{p} & G \end{array}$$

and let $\mathfrak{E}$ be a vector bundle stack of (virtual) rank n on F such that $(p, \mathfrak{E})$ satisfies condition $(\star)$. Then, by Remark 2.15 $q, f^*\mathfrak{E}$ satisfies condition $(\star)$. By Construction 2.16 we obtain morphism $p^!_{\mathfrak{E}}\colon A_*(G') \to A_{*-n}(F')$.

Remark 2.19 Let $p\colon F \to G$ be a morphism of stacks and let $\mathfrak{E}$ be a vector bundle stack such that $(p, \mathfrak{E})$ that satisfies condition $(\star)$ for p. Suppose that $\mathfrak{E} = [E^1/E^0]$, where E^i are vector bundles on F. Let $0\colon F \to \mathfrak{E}$ and $0\colon F \to E^1$ be the zero section embeddings. Let $\mathfrak{C}_{F/G}$ be the normal cone of p and let $C = \mathfrak{C}_{F/G} \times_{\mathfrak{E}} E^1$. Then the closed embedding $\mathfrak{C}_{F/G} \to \mathfrak{E}$ induces a closed embedding $C \to E^1$. The commutativity of pull-backs applied to the commutative diagram

(2)
$$\begin{array}{ccc} F & \xrightarrow{0} & E^1 \\ \downarrow & & \downarrow \\ F & \xrightarrow{0} & \mathfrak{E} \end{array}$$

implies that

$$0^!_{\mathfrak{E}}[\mathfrak{C}_{F/G}] = 0^!_{E^1}[C].$$

Proposition 2.20 (Manolache [18, Proposition 3.11]) *If $F \xrightarrow{p} G$ is a DM-type morphism and there exists a perfect relative obstruction theory $E^\bullet_{F/G}$, then condition $(\star)$ is fulfilled.*

Conversely, if $F \xrightarrow{p} G$ is a morphism that satisfies condition $(\star)$, then there exists a perfect obstruction theory $E^\bullet_{F/G} \to L_{F/G}$ such that $\mathfrak{E} = h^1/h^0(E^{\bullet\vee}_{F/G})$ which is unique up to quasi-isomorphism.

Remark 2.21 If $F \xrightarrow{p} G$ is a DM-type morphism such that there exists a perfect relative obstruction theory $E^\bullet_{F/G}$ and G is a stack of pure dimension, then $p^!_{E^\bullet_{F/G}}([G])$ is a virtual class of F in the sense of Behrend–Fantechi [2].

Remark 2.22 Let us consider a Cartesian diagram of stacks

$$\begin{CD} F' @>>> F \\ @VqVV @VVpV \\ G' @>>i> G. \end{CD}$$

If F, G, G' possess virtual classes and the relative obstruction theory $E^\bullet_{F/G}$ is perfect, then we have an induced virtual class on F', namely

$$[F']^{\text{virt}} := p^![G']^{\text{virt}}.$$

We list the basic properties of virtual pull-backs in [18]. The following theorems are Theorems 4.1, 4.3, 4.4 and 4.8 respectively in [18].

Theorem 2.23 *Consider a fibre diagram of Artin stacks*

$$\begin{CD} F' @>p'>> G' \\ @VgVV @VVfV \\ F @>>p> G \end{CD}$$

and let us assume that $\mathfrak{E}$ is a vector bundle stack of rank d such that $(p, \mathfrak{E})$ satisfies condition $(\star)$ for p.

(i) (Push-forward) *If f is a proper morphism of DM-stacks, $\mathfrak{E} = [E^1/E^0]$, with E^i vector bundles on F and $\alpha \in A_k(G')$, then $p^!_{\mathfrak{E}} f_*(\alpha) = g_* p^!_{\mathfrak{E}}\alpha$ in $A_{k-d}(F)$.*

(ii) (Pull-back) *If f is flat of relative dimension n and $\alpha \in A_k(G)$, then $p^!_{\mathfrak{E}} f^*(\alpha)$ $= g^* p^!_{\mathfrak{E}}\alpha$ in $A_{k+n-d}(F')$*

(iii) (Compatibility) *If $\alpha \in A_k(G')$, then $p^!_{\mathfrak{E}}\alpha = p'^!_{g^*\mathfrak{E}}\alpha$ in $A_{k-d}(F')$.*

Theorem 2.24 (Commutativity) *Consider a fiber diagram of Artin stacks*

$$\begin{CD} F' @>>> G' \\ @VVV @VVqV \\ F @>>p> G \end{CD}$$

such that F' and G' admit stratifications by global quotients. Let us assume p and q are morphisms of DM-type and let $\mathfrak{E}$ and $\mathfrak{F}$ be vector bundle stacks of rank d (respectively e) such that $(p, \mathfrak{E})$ and $(q, \mathfrak{F})$ satisfy condition $(\star)$. Then for all $\alpha \in A_k(G)$,

$$q^!_{\mathfrak{F}} p^!_{\mathfrak{E}}(\alpha) = p^!_{\mathfrak{E}} q^!_{\mathfrak{F}}(\alpha)$$

in $A_{k-d-e}(F')$.

Theorem 2.25 *Let F admit a stratification by global quotients, let $p\colon F \to G$ be a morphism and $\mathfrak{E} \to F$ be a rank–n vector bundle stack on F such that $(p, \mathfrak{E})$ satisfies Condition $(\star)$. Then $p^!_{\mathfrak{E}}$ defines a bivariant class in $A^n(F \to G)$ in the sense of Fulton [7, Definition 17.1].*

Definition 2.26 Let $F \xrightarrow{p} G \xrightarrow{q} \mathfrak{M}$ be DM-type morphisms of Artin stacks. If we are given a distinguished triangle of relative obstruction theories which are perfect in $[-1, 0]$

$$p^* E^{\bullet}_{G/\mathfrak{M}} \xrightarrow{\varphi} E^{\bullet}_{F/\mathfrak{M}} \longrightarrow E^{\bullet}_{F/G} \longrightarrow p^* E^{\bullet}_{G/\mathfrak{M}}[1]$$

with a morphism to the distinguished triangle

$$p^* L_{G/\mathfrak{M}} \longrightarrow L_{F/\mathfrak{M}} \longrightarrow L_{F/G} \longrightarrow p^* L_{G/\mathfrak{M}}[1],$$

then we call $(E^{\bullet}_{F/G}, E^{\bullet}_{G/\mathfrak{M}}, E^{\bullet}_{F/\mathfrak{M}})$ a compatible triple.

Theorem 2.27 (Functoriality) *Consider DM-type morphisms of Artin stacks*

$$F \xrightarrow{p} G \xrightarrow{q} \mathfrak{M}.$$

Let us assume p, q and $q \circ p$ have perfect relative obstruction theories $E^{\bullet}_{F/G}$, $E^{\bullet}_{G/\mathfrak{M}}$ and $E^{\bullet}_{F/\mathfrak{M}}$ respectively and let us denote the associated vector bundle stacks by $\mathfrak{E}_{F/G}$, $\mathfrak{E}_{G/\mathfrak{M}}$ and $\mathfrak{E}_{F/\mathfrak{M}}$ respectively. If $\big(E^{\bullet}_{F/G}, E^{\bullet}_{G/\mathfrak{M}}, E^{\bullet}_{F/\mathfrak{M}}\big)$ is a compatible triple, then for any $\alpha \in A_k(\mathfrak{M})$

$$(q \circ p)^!_{\mathfrak{E}_{F/\mathfrak{M}}}(\alpha) = p^!_{\mathfrak{E}_{F/G}}\big(q^!_{\mathfrak{E}_{G/\mathfrak{M}}}(\alpha)\big).$$

2.2 Virtual pull-backs and algebraic equivalences

In the following we extend the definition of virtual pull-backs to groups of algebraic equivalence classes. Groups of algebraic equivalence classes for schemes and basic constructions such that push-forward and pull-back between groups of algebraic equivalence classes are treated in [7, Chapter 10]. We will follow the ideas and notation in [7].

Let T be an irreducible smooth variety of dimension m. The notation $t\colon \{t\} \to T$ will be used to denote the inclusion of a closed point t in T. If $p\colon \mathcal{F} \to T$ is given, then we denote by $\mathcal{F}_t$ the stack $p^{-1}(t)$. Any $(k+m)$–cycle on $\mathcal{F}$ determines a family of k–cycle classes $\alpha_t \in A_k(\mathcal{F}_t)$ defined by the formula

$$\alpha_t := t^!\alpha.$$

If $p\colon \mathcal{F} \to \mathcal{G}$ is a morphism of stacks over T, we denote by

$$p_t\colon \mathcal{F}_t \to \mathcal{G}_t$$

the induced morphism on the fibers over $t \in T$.

Definition 2.28 Let F be a DM stack. A k cycle a is algebraically equivalent to zero if there is a nonsingular variety T and a cycle $\alpha \in A_{k+m}(F \times T)$, $m = \dim T$, and points $t_1,\ t_2 \in T$ such that

$$a = \alpha_{t_1} - \alpha_{t_2}$$

in $A_k(F)$. Two k cycles are algebraically equivalent if their difference is algebraically equivalent to zero. The group of algebraic equivalence classes will be denoted by $B_*(F)$.

Remark 2.29 We have a natural map $Z_k(F) \to B_k(F)$ which associates to a cycle Z its class $[Z] \in B_k(F)$. Definitions of rational and algebraic equivalence imply that the above map induces a map

$$cl_F\colon A_k(F) \to B_k(F).$$

Proposition 2.30 (i) *Let $p\colon \mathcal{F} \to \mathcal{G}$ be a proper morphism of stacks over T. Then*

$$p_{t*}\alpha_t = (p_*\alpha)_t$$

(ii) *Let $p\colon \mathcal{F} \to \mathcal{G}$ be a flat morphism of DM stacks over T. Then, we have that*

$$p_t^*(\alpha_t) = (p^*(\alpha))_t$$

in $A_*(p^{-1}(\alpha))$.

Proof This is the analogue of [7, Proposition 10.1] for DM stacks. □

Corollary 2.31 (i) *Let $p\colon F \to G$ be a proper morphism of DM stacks. Then the following diagram commutes*

$$\begin{array}{ccc} A_*(F) & \xrightarrow{p_*} & A_*(G) \\ \downarrow{\scriptstyle cl_F} & & \downarrow{\scriptstyle cl_G} \\ B_*(F) & \xrightarrow[p_*]{} & B_*(G). \end{array}$$

(ii) *Let $p\colon F \to G$ be a flat morphism of stacks of relative dimension r. Then the following diagram commutes*

$$\begin{array}{ccc} A_*(G) & \xrightarrow{p_*} & A_{*-r}(F) \\ {\scriptstyle cl_G}\downarrow & & \downarrow{\scriptstyle cl_F} \\ B_*(G) & \xrightarrow[p_*]{} & B_{*-r}(F). \end{array}$$

Proof It follows from the above Proposition. □

Lemma 2.32 *Let $p\colon \mathcal{F} \to \mathcal{G}$ be a morphism of DM stacks over T. If $\mathfrak{E} \to \mathcal{F}$ is a vector bundle-stack which satisfies condition $(\star)$ for p, then $\mathfrak{E}_t$ satisfies condition $(\star)$ for p_t and we have that*

$$p_t{}^!_{\mathfrak{E}_t}(\alpha_t) = \big(p^!_{\mathfrak{E}}(\alpha)\big)_t$$

in $A_*(p^{-1}(\alpha))$.

Proposition 2.33 *Let $p\colon F \to G$ be a morphism of DM stacks and let $\mathfrak{E} \to F$ is a vector bundle-stack which satisfies condition $(\star)$ for p. Then we have a morphism $p^!_{\mathfrak{E}}\colon B_*(G) \to B_{*-r}(F)$ which makes the diagram*

$$\begin{array}{ccc} A_*(G) & \xrightarrow{p^!_{\mathfrak{E}}} & A_{*-r}(F) \\ {\scriptstyle cl_G}\downarrow & & \downarrow{\scriptstyle cl_F} \\ B_*(G) & \xrightarrow[p^!_{\mathfrak{E}}]{} & B_{*-r}(F) \end{array}$$

commute.

Proof This follows from the previous proposition. Let us sketch the proof. We have to show that for any cycle $\alpha \in A_*(G)$ such that $cl_G\alpha = 0$ we have that $cl_F\, p^!\alpha = 0$. Let $\alpha \in A_*(G)$ as above. By the definition of algebraic equivalence, there exists a non-singular variety T of dimension m and $(k+m)$–dimensional subvarieties $\mathcal{V}_i$ of $T \times G$, flat over T and points $t_1,\ t_2 \in T$ such that

$$\alpha = \sum_{i=1}^{r} \big[(\mathcal{V}_i)_{t_1}\big] - \big[(\mathcal{V}_i)_{t_2}\big].$$

By Lemma 2.32 we have that

$$p^!_{t_j}\left(\sum_{i=1}^{r}[\mathcal{V}_i]_{t_j}\right) = \left(p^!\sum_{i=1}^{r}[\mathcal{V}_i]\right)_{t_j}$$

for $j = 1, 2$. This shows that

$$p^!\alpha = \Big(p^! \sum_{i=1}^{r} [\mathcal{V}_i]\Big)_{t_1} - \Big(f^! \sum_{i=1}^{r} [\mathcal{V}_i]\Big)_{t_2}.$$

Let us now analyze the right-hand side. Let $\mathcal{W}_i'$ be cycles representing $p^![\mathcal{V}_i]$. As the pull back via t_j is not influenced by components of $\mathcal{W}_i'$ which do not map dominantly to T we may discard them. Let us call the resulting stacks by $\mathcal{W}_i$. This shows that

$$p^!\alpha = \Big(\sum_{i=1}^{r} [\mathcal{W}_i]\Big)_{t_1} - \Big(\sum_{i=1}^{r} [\mathcal{W}_i]\Big)_{t_2}$$

and therefore $p^!\alpha$ is algebraically equivalent to zero. □

Remark 2.34 As $H_0(G) = B_0(G)$ the above morphism

$$p^!_{\mathfrak{E}}\colon B_*(G) \to B_{*-r}(F)$$

induces a morphism which makes the diagram

$$\begin{array}{ccc} A_0(G) & \xrightarrow{p^!_{\mathfrak{E}}} & A_{0-r}(F) \\ \downarrow & & \downarrow \\ H_0(G) & \xrightarrow[p^!_E]{} & H_{0-r}(F) \end{array}$$

commute.

Remark 2.35 The definition of virtual pull-backs $i\colon X \to Y$ related to the Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle q}\downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

with $X' \to Y'$ a regular embedding and the obstruction bundle $E_{X/Y} := q^* N_{X'/Y'}$ gives rise to a pull-back in homology $i^!_{E_{X/Y}}\colon H_*(Y) \to H_*(X)$ (see Fulton [7, Chapter 19]). We could not construct a similar morphism

$$i^!_{E_{X/Y}}\colon H_*(Y) \to H_*(X)$$

in general.

3 Virtual push-forwards

In this section we consider a proper morphism $p\colon F \to G$ of DM stacks which possess perfect obstruction theories and we analyze the push-forward of the virtual class of F along p. The main result of this section is a generalization of the straightforward fact that given a morphism of schemes $p\colon F \to G$, with F of dimension k_1 and G irreducible of dimension k_2, we have that $p_*(\gamma \cdot [F])$ is a scalar multiple of the fundamental class of G for any $\gamma \in A^{k_1-k_2}(F)$. The main technical tool is an analogue of the conservation of number principle in [7] in the context of virtually smooth morphisms (see Definition 3.4 below).

Let us first formalize this idea.

Definition 3.1 Let $p\colon F \to G$ be a proper morphism of stacks possessing virtual classes $[F]^{\mathrm{virt}} \in A_{k_1}(F)$ and $[G]^{\mathrm{virt}} \in A_{k_2}(G)$ with $k_1 \geq k_2$ and let $[G_1], \dots, [G_s] \in A_{k_2}(G)$ be irreducible cycles such that $[G]^{\mathrm{virt}} = [G_1] + \cdots + [G_s]$. Let $\gamma \in A^{k_3}(F)$, with $k_3 \leq k_1 - k_2$ be a cohomology class. We say that p satisfies the virtual push-forward property for $[F]^{\mathrm{virt}}$ and $[G]^{\mathrm{virt}}$ if the following two conditions hold:

(i) If the dimension of the cycle $\gamma \cdot [F]^{\mathrm{virt}}$ is bigger than the virtual dimension of G then $p_*(\gamma \cdot [F]^{\mathrm{virt}}) = 0$.

(ii) If the dimension of the cycle $\gamma \cdot [F]^{\mathrm{virt}}$ is equal to the virtual dimension of G then $p_*(\gamma \cdot [F]^{\mathrm{virt}}) = n_1[G_1] + \cdots + n_s[G_s]$ for some $n_1, \dots, n_s \in \mathbb{Q}$.

We say the p satisfies the *strong* virtual push-forward property if moreover, the following condition holds

(ii') If the dimension of the cycle $\gamma \cdot [F]^{\mathrm{virt}}$ is equal to the virtual dimension of G then $p_*(\gamma \cdot [F]^{\mathrm{virt}})$ is a scalar multiple of $[G]^{\mathrm{virt}}$.

We say the p satisfies the strong virtual push-forward property in *homology* if for any vector bundle E on F and $\gamma = c_{k_3}(E) \in H^{2k_3}$ there exists $N \in \mathbb{Q}$ such that $p_*(\gamma \cdot [F]^{\mathrm{virt}}) = N[G]^{\mathrm{virt}} \in H_{2k_2}(G)$.

Remark 3.2 The definition of "push-forward property" appears in the work of Gathmann [8] with a minor difference. Gathmann says that a morphism satisfies the push-forward property if it satisfies in our language the strong virtual push-forward property. We prefer this terminology mainly because we would like to say that smooth morphisms satisfy the virtual push-forward property.

Remark 3.3 Let $p\colon F \to G$ be a morphism as above. If G is smooth of the expected dimension, then p satisfies the virtual pushforward property. If G is also irreducible, then p satisfies the strong virtual push-forward property.

Definition 3.4 Let

$$\begin{array}{ccc} F & \xrightarrow{p} & G \\ & \underset{\epsilon_1}{\searrow} & \downarrow{\scriptstyle \epsilon_2} \\ & & \mathfrak{M} \end{array}$$

be a commutative diagram of DM-type morphisms of algebraic stacks, with $\mathfrak{M}$ a stack of pure dimension and let $E^{\bullet}_{F/\mathfrak{M}}$, $E^{\bullet}_{G/\mathfrak{M}}$ be perfect obstruction theories to ϵ_1, ϵ_2 inducing virtual classes of dimensions k_1 respectively k_2 with $k_1 \geq k_2$. If we have a compatible triple $\left(E^{\bullet}_{F/G}, E^{\bullet}_{G/\mathfrak{M}}, E^{\bullet}_{F/\mathfrak{M}}\right)$, such that the relative obstruction theory $E^{\bullet}_{F/G}$ is perfect and p is proper, then we call p a virtually smooth morphism.

Remark 3.5 This definition is very similar to Fantechi–Göttsche [6, Definition 3.14] of *a family of proper virtually smooth schemes*. The main difference is that we do not ask the base G to be smooth.

3.1 Virtual push-forward property

Here we find sufficient conditions for the virtual push-forward property to hold.

Lemma 3.6 *Let $p\colon F \to G$ be a virtually smooth morphism of DM stacks and let $\mathfrak{E}_{F/G}$ be the vector bundle stack associated to the obstruction theory $E^{\bullet}_{F/G}$. If $\mathfrak{E}_G = [E^1/E^0]$ with E^i vector bundles, then p satisfies the virtual push-forward property.*

Proof The proof is a reformulation of Lai's arguments in [14, pages 9–11]. Let

$$\mathfrak{E}_F := h^1/h^0\big({E^{\bullet}}^{\vee}_{F}\big), \quad \mathfrak{E}_G := h^1/h^0\big({E^{\bullet}}^{\vee}_{G}\big) \quad \text{and} \quad \mathfrak{E}_{F/G} := h^1/h^0\big({E^{\bullet}}^{\vee}_{F/G}\big),$$

and let

$$\begin{aligned} 0_F\colon F &\to \mathfrak{E}_F, & 0_G\colon F &\to p^*\mathfrak{E}_G, \\ 0_{F/G}\colon F &\to \mathfrak{E}_{F/G} \qquad \text{and} & (0_{F/G}, 0_G)\colon F &\to \mathfrak{E}_{F/G} \oplus p^*\mathfrak{E}_G \end{aligned}$$

be the zero-section embeddings. Then by the definition of the virtual class we have that $[G]^{\mathrm{virt}} = 0_G^![\mathfrak{C}_G]$ and $[F]^{\mathrm{virt}} = 0_F^![\mathfrak{C}_F]$. Let $\mathfrak{C}' = \mathfrak{C}_{F/\mathfrak{C}_G}$. By the proof of Theorem 2.27 we have a closed embedding $\mathfrak{C}' \hookrightarrow \mathfrak{C}_{F/G} \oplus p^*\mathfrak{C}_G$

$$\begin{aligned}[F]^{\mathrm{virt}} &= 0_{F/G}^![\mathfrak{C}_{F/G}] \\ &= (0_{F/G}, 0_G)^![\mathfrak{C}'] \\ &= 0_G^! j_{\mathfrak{C}_{F/G}}^![\mathfrak{C}']\end{aligned}$$

where j is the embedding $j\colon p^*\mathfrak{C}_G \to \mathfrak{C}_{F/G} \oplus p^*\mathfrak{C}_G$ and $j_{F/G}^!$ is the (virtual) pull-back induced by the vector bundle stack $\mathfrak{C}_{F/G}$. Let us consider

$$[\mathfrak{C}''] := j_{\mathfrak{C}_{F/G}}^![\mathfrak{C}'] \in A_*(p^*\mathfrak{C}_G).$$

Let $\gamma \in A^*(F)$ be an arbitrary cohomology cycle. Then the above computation shows that

$$\gamma \cdot [F^{\mathrm{virt}}] = 0_G^! \pi^* \gamma \cdot [\mathfrak{C}''] \tag{3}$$

where $\pi\colon p^*\mathfrak{C}_G \to F$ denotes the canonical projection. By the definition of $\mathfrak{C}'$ we have a natural morphism $\mathfrak{C}' \to \mathfrak{C}_G$ and by the definition of $\mathfrak{C}''$ we have a natural morphisms $\mathfrak{C}'' \to \mathfrak{C}' \times_{\mathfrak{C}_{F/G} \oplus p^*\mathfrak{C}_G} p^*\mathfrak{C}_G \to \mathfrak{C}'$. Composing the two morphisms we obtain a morphism of stacks

$$\mathfrak{C}'' \to \mathfrak{C}_G. \tag{4}$$

Let $C_G^1 = \mathfrak{C}_G \times_{\mathfrak{C}_G} E^1$ and $C^1 = \mathfrak{C}'' \times_{\mathfrak{C}_G} C_G^1$. This implies that

$$C^1 \simeq \mathfrak{C}'' \times_{\mathfrak{C}_G} E^1.$$

Let $0_G^1\colon G \to E^1$ be the zero section embedding. By Remark 2.19 we have that

$$0_G^![\mathfrak{C}''] = 0_G^{1\,!}[C^1]. \tag{5}$$

The morphism (4) induces a morphism $C^1 \to C_G^1$ such that the diagram

$$\begin{array}{ccc} C^1 & \longrightarrow & p^*E^1 \\ \downarrow & & \downarrow r \\ C_G^1 & \longrightarrow & E^1 \end{array} \tag{6}$$

commutes.

By the commutativity of the pull-back with proper push-forward in the Cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & p^*E^1 \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ G & \xrightarrow[0^1_G]{} & E^1 \end{array}$$

and (3) we obtain that

$$p_*(\gamma\cdot[F]^{\mathrm{virt}}) = 0_G^{1\,!} r_*\pi^*\gamma\cdot[C^1]. \tag{7}$$

The commutativity of diagram (6) implies

$$r_*\pi^*\gamma\cdot[C^1] = \sum n_i[C^1_G]_i, \tag{8}$$

where the sum is taken over all the irreducible components of C^1_G. By Remark 2.19 we have that $[G]^{\mathrm{virt}} = \left(0^1_G\right)^! \sum[C^1_G]_i$. This together with (7), (8) implies

$$p_*(\gamma\cdot[F]^{\mathrm{virt}}) = \sum n_i[G]_i.$$

If $k_3 < k_1 - k_2$, then for dimensional reasons $r_*\pi^*\gamma\cdot[C'] = 0$, and hence $p_*(\gamma\cdot[F]^{\mathrm{virt}}) = 0$. □

Remark 3.7 The reason why we impose $\mathfrak{E}_G$ to be a global quotient is that push-forwards for non-projective morphisms of Artin stacks do not exist. If p is a projective morphism then, this condition is not necessary.

3.2 Conservation of number for virtually smooth morphisms

Let us recall Fulton's principle of conservation of number [7, Proposition 10.2].

Proposition 3.8 *Let $p\colon X \to Y$ be a proper morphism of schemes, and let Y be an m–dimensional irreducible scheme. Let $i\colon P \to Y$ be a regular point in Y, $X_P = p^{-1}P$ and γ be an m–dimensional cycle on X. Then the cycle classes $\gamma_P := i^!\gamma \in A_0(X_P)$ have the same degree.*

In this section we will give a version of this principle in the situation when $p\colon F \to G$ is a virtually smooth morphism.

Let us now state the conservation of number principle for virtually smooth morphisms of DM stacks.

Proposition 3.9 *Let G be a complete connected scheme and let $p\colon F \to G$ be a proper virtually smooth morphism of DM stacks (see Definition 3.4) of virtual relative dimension d. Let $i\colon P \to G$ be a regular point in G and let us consider the Cartesian diagram*

$$\begin{array}{ccc} F_P & \xrightarrow{\;j\;} & F \\ {\scriptstyle p_P}\downarrow & & \downarrow{\scriptstyle p} \\ P & \xrightarrow[\;i\;]{} & G \end{array}$$

where F_P is the fiber of F over P and $p_P\colon F_P \to P$ is the map induced by p. Let E be a vector bundle on F and $\gamma = c_d(E) \in H^{2d}(F)$. Let $[F_P]^{\mathrm{virt}} \in H_{2d}(F_P)$ be the class defined in Remark 2.22. Then, the number

$$\deg(j^*\gamma \cdot [F_P]^{\mathrm{virt}})$$

is constant.

Proof As $A_0(P) \simeq \mathbb{Q}$ we have that

$$p_{P*}(j^*\gamma \cdot [F_P]^{\mathrm{virt}}) = n[P] \tag{9}$$

for some $n \in \mathbb{Q}$. It is enough to show that $i_* p_{P*} j^*\gamma \cdot [F_P]^{\mathrm{virt}}$ does not depend on P. For this, we see that

$$i_* p_{P*}(j^*\gamma \cdot [F_P]^{\mathrm{virt}}) = p_* j_*(j^*\gamma \cdot [F_P]^{\mathrm{virt}}) \tag{10}$$

$$= p_*(\gamma \cdot j_*[F_P]^{\mathrm{virt}}) \tag{11}$$

in $A_*(G)$. By commutativity of pull-backs with projective push-forwards we have that

$$j_* p_P^![P] = p^! i_*[P]$$

in $A_*(F)$. By Corollary 2.31 we obtain the equation above for algebraic equivalence groups. As G is a connected scheme we have that $i_*[P] \in H_0(G)$ does not depend on the smooth point P. This shows that $j_* p_P^![P] \in H_*(F)$ is independent of P and since taking homology classes commutes with push-forwards and intersection with Chern classes (see Fulton [7, Proposition 19.1.2]) we obtain that $p_*(\gamma \cdot j_*[F_P]^{\mathrm{virt}}) \in H_0(G)$ is independent of P. We have thus obtained that the degree of the intersection product $j^*\gamma \cdot [F_P]^{\mathrm{virt}}$ is equal to n for any regular P. □

Remark 3.10 Taking G to be smooth we obtain the conservation of number principle in families of virtually smooth schemes (see Fantechi–Göttsche [6, Definition 3.14]) which is [6, Corollary 3.16].

Remark 3.11 The only point where we need to work with algebraic equivalence classes is the last part of the proof of the above theorem. For any connected G we have that $B_0(G) = \mathbb{Q}$, but this is usually no longer true for the corresponding Chow group.

The proof of Proposition 3.9 implies the following statement.

Lemma 3.12 *Let $p\colon F \to G$ be a proper virtually smooth morphism of DM stacks with G a compact stack. Let E be a vector bundle on F and $\gamma = c_d(E) \in H^{2d}(F)$ and let $cl_G\colon A_0(G) \to H_0(G)$ denote the natural cycle map. Then*

$$cl_G(p_*(\gamma \cdot (p^! i_*[P]))) \in H_0(G)$$

does not depend on P.

Proof As in the proof of Proposition 3.9, we have that

$$cl_G(p_*(\gamma \cdot (p^! i_*[P]))) = n cl_G(i_*[P])$$

in $H_0(G)$ for some $n \in \mathbb{Q}$. Since taking homology classes commutes with push-forwards and intersection with Chern classes we obtain that

$$p_*(\gamma \cdot cl_G(p^! i_*[P])) = n cl_G(i_*[P])$$

in $H_0(G)$. By Remark 2.34 this implies that

$$p_*(\gamma \cdot (p^! cl_G(i_*[P])) = n cl_G(i_*[P]). \tag{12}$$

in $H_0(G)$. Let us show that the left hand side of equation (12) does not depend on the point P. Let Q be any other point of G. As G is connected we have that $[Q] = r[P] \in H_0(G)$ for some $r \in Q^*$. By linearity of pull-backs and push-forwards we obtain that

$$\begin{aligned} p_*(\gamma \cdot (p^! i_*[Q])) &= r p_*(\gamma \cdot (p^! i_*[P])) \\ &= r n i_*[P] \\ &= n i_*[Q]. \end{aligned}$$

This completes the proof. □

3.3 Strong virtual push-forward property

Here we find sufficient conditions for the strong virtual push-forward property to hold.

Theorem 3.13 *Let $p\colon F \to G$ be a proper virtually smooth morphism of DM stacks and let $\mathfrak{E}_{F/G}$ be the vector bundle stack associated to the obstruction theory $E^\bullet_{F/G}$. If $\mathfrak{E}_{F/G}$ is isomorphic to a global quotient of vector bundles $[E^1/E^0]$ and G is connected, then p satisfies the strong virtual push-forward property in homology.*

Proof By Lemma 3.6 we have that

$$p_*(\gamma\cdot[F]^{\mathrm{virt}}) = n_1[G_1] + \cdots + n_s[G_s] \tag{13}$$

for some $n_1,\dots,n_s \in \mathbb{Q}$ and $[G_1],\dots,[G_s] \in A_{k_2}(G)$ such that $G_1,\dots,G_s$ are irreducible and such that

$$[G]^{\mathrm{virt}} = [G_1] + \cdots + [G_s].$$

We are left to show that all the n_i's are equal. Let $m_1,\dots,m_s$ be the geometric multiplicity of $G_1,\dots,G_s$. Then $[G]^{\mathrm{virt}} = m_1[G_1^{\mathrm{r}}] + \cdots + m_s[G_s^{\mathrm{r}}]$, where G_i^{r} is the reduced stack associated to G_i and therefore $[C'] = \sum_{i=1}^s m_i[C_i']$, where $C_i' := C_{F/C_{G_i}}$. By equation (7) we have that $p_*(\gamma\cdot[F]^{\mathrm{virt}}) = \big(0_G^1\big)^! r_*\pi^*\gamma\cdot\big(\sum_{i=1}^s m_i[C_i^1]\big)$. With this we have shown that it is enough to show the statement for G reduced.

Let us consider the Cartesian diagram

$$\begin{array}{ccc} F_P & \xrightarrow{\ j\ } & F \\ {\scriptstyle p_P}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ P & \xrightarrow[\ i\ ]{} & G \end{array} \tag{14}$$

where P is a general point in G and F_P is the fiber of p over P. As G is reduced we may assume that P is a smooth point and therefore i is a regular embedding. By the commutativity of pull-backs with proper push-forwards we have that

$$p_{P*}(i^!(\gamma\cdot[F]^{\mathrm{virt}})) = i^!(p_*(\gamma\cdot[F]^{\mathrm{virt}})).$$

Equation (13) implies $p_{P*}(i^!(\gamma\cdot[F]^{\mathrm{virt}})) = i^!\sum_i n_i[G_i]$. Without loss of generality we may assume that P is a point of G_1. With this we obtain that

$$p_{P*}(j^*\gamma\cdot i^![F]^{\mathrm{virt}}) = n_1[P]. \tag{15}$$

On the other hand by the commutativity of pull-backs we have that

$$i^!p^![G]^{\mathrm{virt}} = p_P^! i^![G]^{\mathrm{virt}} = p_P^![P]. \tag{16}$$

By the functoriality property of pull-backs we have that

$$i^!p^![G]^{\mathrm{virt}} = i^![F]^{\mathrm{virt}}. \tag{17}$$

Equations (15), (16) and (17) imply that

$$p_{P*}(j^*\gamma \cdot (p^!_P[P])) = n_1[P]. \tag{18}$$

Pushing forward (18) via i and using that push-forwards commute with pull-backs we obtain

$$p_*(\gamma \cdot (p^! i_*[P])) = n_1 i_*[P]$$

in $A_0(G)$. Passing to homology groups and using Lemma 3.12 we obtain that n_1 does not depend on P. This completes the proof. □

Proposition 3.14 *Let us consider a commutative diagram*

$$\begin{array}{ccc} F & \xrightarrow{p} & G \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \nu} \\ \mathfrak{M}_1 & \xrightarrow[\mu]{} & \mathfrak{M}_2 \end{array}$$

where

(1) μ *is a morphism of smooth stacks*

(2) *the vertical arrows have (relative) perfect obstruction theories* E_ϵ, E_ν

(3) *we have a morphism* φ: $p^*E_\nu \to E_\epsilon$ *such that the diagram*

$$\begin{array}{ccc} p^*E_\nu & \longrightarrow & E_\epsilon \\ \downarrow & & \downarrow \\ p^*L_\nu & \longrightarrow & L_\epsilon \end{array}$$

is commutative and the cone of φ *is a perfect complex.*

Then, p *satisfies the virtual push-forward property. If moreover* G *is connected, the* p *satisfies the strong virtual push-forward property in homology.*

Proof The morphism obtained from the following composition

$$E_\epsilon[-1] \to L_\epsilon[-1] \to \epsilon^* L_{\mathfrak{M}_1}$$

can be completed to a triangle

$$E_\epsilon[-1] \to \epsilon^* L_{\mathfrak{M}_1} \to E_F \to E_\epsilon,$$

and by [18] E_F is an obstruction theory for F. Similarly we have a triangle

$$E_\nu[-1] \to \nu^* L_{\mathfrak{M}_2} \to E_G \to E_\nu.$$

By the fact that we have a natural morphism $\mu^* L_{\mathfrak{M}_2} \to L_{\mathfrak{M}_1}$ and the axioms of triangulated categories we obtain a morphism $p^* E_G \to E_F$. We denote the cone of this morphism by E_p and by [18] we see that E_p is an obstruction theory for p. By the octahedron axiom we obtain that the triangle

$$\epsilon^* L_\mu \to E_p \to F \to \epsilon^* L_\mu[1] \tag{19}$$

is distinguished (see the Stacks Project [5, Proposition 4.21] for a complete proof). From the long exact sequence in cohomology and the fact that $h^{-2}(F) = 0$ we obtain that E_p is a perfect obstruction theory for the morphism p. This shows that we can apply Theorem 3.13 to the composition of morphisms

$$F \xrightarrow{p} G \xrightarrow{\nu} \mathfrak{M}_2. \qquad \square$$

Corollary 3.15 *Let us consider a Cartesian diagram of stacks*

$$\begin{array}{ccc} F' & \xrightarrow{j} & F \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ G' & \xrightarrow[i]{} & G \end{array}$$

such that p is a virtually smooth morphism of DM stacks, G is connected G' admits a virtual class $[G']^{\mathrm{virt}}$, then q satisfies the strong virtual push-forward property in homology for $[F']^{\mathrm{virt}} := p^![G']^{\mathrm{virt}}$ (see Remark 2.22) and $[G']^{\mathrm{virt}}$.

Proof Let E be a vector bundle on F and $\gamma = c_i(E)$ with i as in Theorem 3.13. By Lemma 3.6 we have that

$$q_*(j^*\gamma \cdot [F']^{\mathrm{virt}}) = n_1[G']_1^{\mathrm{virt}} + \cdots + n_s[G']_s^{\mathrm{virt}}$$

for some $n_1, \ldots, n_s \in \mathbb{Q}$. We have to show that all n_i's are equal.

As in the proof of the theorem we may assume that G and G' are reduced. Let us consider the Cartesian diagram

$$\begin{array}{ccccc} F_P & \xrightarrow{j_P} & F' & \xrightarrow{j} & F \\ {\scriptstyle q_P}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ P & \longrightarrow & G' & \xrightarrow[i]{} & G \end{array}$$

where P is any closed point. Then, by Theorem 3.13, we have that

$$p_*(\gamma \cdot [F]^{\mathrm{virt}}) = n[G]^{\mathrm{virt}}$$

for some $n \in \mathbb{Q}$. Also, by the proof of 3.13, we have that

$$q_{P*}(j^* j_P^* \gamma \cdot [F_P]^{\mathrm{virt}}) = n[P].$$

Looking now at the diagram on the left, and assuming that P is a smooth point of G' we obtain the following by Theorem 2.23

$$q_*(j^* \gamma \cdot [F']^{\mathrm{virt}}) = q_{P*}(\gamma \cdot [F_P]^{\mathrm{virt}}).$$

As G' is reduced, we have that the generic point is smooth and hence the above equation holds for a dense open subset of G'. Combining this equation with the previous, we obtain that $q_*(j^* \gamma \cdot [F']^{\mathrm{virt}}) = n[G']^{\mathrm{virt}}$. □

4 Applications

4.1 Virtual Euler characteristics in virtually smooth families

As a consequence of the conservation of number principle we give a proof of the fact that the virtual Euler characteristic is constant in virtually smooth families (see Definition 3.4). This statement is a generalization of [6, Proposition 4.14] of Fantechi and Göttsche.

Definition 4.1 Let $p\colon F \to G$ be a morphism of proper stacks with a perfect obstruction theory $E_{F/G}$ which admits a global resolution of $E_{F/G}$ as a complex of vector bundles $[E^1 \to E^0]$ (for example, if F can be embedded as closed substack in a separated stack which is smooth over G.) We denote by $[E_0 \to E_1]$ the dual complex and by d the expected dimension $d := \mathrm{rk}\, E_{F/G} = \mathrm{rk}\, E^0 - \mathrm{rk}\, E^1$. We denote the class $[E_0] - [E_1] \in K^0(F)$ by $T_{F/G}^{\mathrm{virt}}$ and we call it the virtual relative tangent of f.

Definition 4.2 Let F be a proper DM stack with a perfect obstruction theory. We define the relative virtual Euler characteristic of F to be the top virtual Chern number $e^{\mathrm{virt}}(F) := \deg(c_d(T_F^{\mathrm{virt}}) \cdot [F]^{\mathrm{virt}})$.

Remark 4.3 The definition is consistent with [6, Definition 4.2] by the Hopf Index Theorem [6, Corollary 4.8].

Proposition 4.4 *Let G be a connected stack of pure dimension and let $p\colon F \to G$ be a morphism of stacks with $E_{F/G}$ a perfect obstruction theory for p. Then, all the fibers of p have the same virtual Euler characteristic.*

Proof We use Proposition 3.9 with $\alpha := c_d\big(T_{F/G}^{\mathrm{virt}}\big)$. □

Remark 4.5 Taking G to be smooth we obtain that the virtual Euler characteristic is constant in a family of virtually smooth schemes. This is a different proof of [6, Proposition 4.14].

4.2 Virtual push-forward and Gromov–Witten invariants

We briefly recall moduli spaces of stable maps and the construction of their virtual classes. We prove relations between virtual classes of moduli spaces of maps to X and Y, where $X \to Y$ is a smooth fibration.

The standard obstruction theory for the moduli space of stable maps Let us fix notations. Let X be a smooth projective variety and $\beta \in A_1(X)$ a homology class of a curve in X. We denote by $\overline{M}_{g,n}(X,\beta)$ the moduli space of stable genus–g, n–pointed maps to X of homology class β and by $\mathfrak{M}_{g,n}$ the Artin stack of prestable curves. Let

$$\epsilon_X\colon \overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$$

be the morphism that forgets the map (and does not stabilize the pointed curve) and

$$\pi_X\colon \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$$

the morphism that forgets the last marked point and stabilizes the result. Then it is a well-known fact that

$$E^\bullet_{\overline{M}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}} := (R^\bullet \pi_{X*} \operatorname{ev}_X^* T_X)^\vee$$

defines an obstruction theory for the morphism ϵ_X, where ev_X indicates the evaluation map at the last marked point $\operatorname{ev}_X\colon \overline{M}_{g,n+1}(X,\beta) \to X$ (see Behrend [1]). We call

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}} := (\epsilon_X)^!_{\mathfrak{E}_{\overline{M}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}}}[\mathfrak{M}_{g,n}]$$

the virtual class of $\overline{M}_{g,n}(X,\beta)$.

Remark 4.6 Let $p\colon X \to Y$ be a morphism of smooth algebraic varieties. Let $\beta \in H_2(X)$ and $g,\ n$ be any natural numbers such that

- either $g \geq 2$
- or $g < 2$ and $p_*\beta \neq 0$
- or $g = 1,\ p_*\beta = 0$ and $n \geq 1$, either $g = 0,\ p_*\beta = 0$ and $n \geq 3$.

Then p induces a morphism of stacks

$$\overline{p}\colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(Y, p_*\beta)$$
$$\big(\widetilde{C}, x_1, \dots, x_n, \widetilde{f}\big) \mapsto \big(C, x_1, \dots, x_n, p \circ \widetilde{f}\big)$$

where C is obtain by $\widetilde{C}$ by contracting the unstable components of $f := p \circ \widetilde{f}$.

Convention 4.7 For simplicity, we will denote obstruction theories of a morphism $p\colon F \to G$ by E_p. For example, we will write E_{ϵ_X} instead of $E^{\bullet}_{\overline{M}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}}$.

Convention 4.8 In the following, every time we write

$$\overline{p}\colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(Y, p_*\beta)$$

we will assume that $\overline{M}_{g,n}(Y, p_*\beta)$ is non-empty.

Costello's construction In the following we use a construction of Costello [4, Section 2]. Let us shortly present how his construction applies to our case. In [4, Section 2], Costello introduces an Artin stack $\mathfrak{M}_{g,n,\beta}$, where β is an additional labeling of each irreducible components of a marked curve of genus g by the elements of a semigroup. We will take this semigroup to be $H_2(X)$, for some smooth variety X. In [4] it is shown that the forgetful map $\mathfrak{M}_{g,n,\beta} \to \mathfrak{M}_{g,n}$ is étale and that the natural forgetful map

$$\epsilon_X\colon \overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$$

factors through $\epsilon_{X,\beta}\colon \overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n,\beta}$. Therefore, the perfect relative obstruction theory of $\overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$

$$(\mathcal{R}^{\bullet}\pi_* \operatorname{ev}_X^* T_X)^{\vee}$$

is also a relative obstruction theory of the morphism $\epsilon_{X,\beta}$. We thus obtain a virtual class $([\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}})' := \epsilon^!_{X,\beta}[\mathfrak{M}_{g,n,\beta}]$. It can be easily seen that

$$([\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}})' = [\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}}.$$

This construction has the advantage that for a given map $p\colon X \to Y$ we have a commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}(X,\beta) & \xrightarrow{\epsilon_{X,\beta}} & \mathfrak{M}_{g,n,\beta} \\ \Big\downarrow{\scriptstyle \overline{p}} & & \Big\downarrow{\scriptstyle \psi} \\ \overline{M}_{g,n}(Y, p_*\beta) & \xrightarrow[\epsilon_{Y,p_*\beta}]{} & \mathfrak{M}_{g,n,p_*\beta} \end{array} \tag{20}$$

where $\psi(C)$ contracts the unstable components C_i such that the label β_i satisfies $p_*\beta_i = 0$ and changes the label on each irreducible component by $p_*\beta_i$.

Proposition 4.9 *If $p\colon X \to Y$ is a smooth morphism, then p induces a morphism*

$$\overline{p}^* E^\bullet_{\overline{M}_{g,n}(Y,p_*\beta)/\mathfrak{M}_{g,n}} \longrightarrow E^\bullet_{\overline{M}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}}$$

whose cone is a perfect complex concentrated in $[-1,0]$.

Proof By the discussion in the above paragraph we have that

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}} = \epsilon^!_{X,\beta}[\mathfrak{M}_{g,n,\beta}]$$

and similarly

$$[\overline{M}_{g,n}(Y,p_*\beta)]^{\mathrm{virt}} = \epsilon^!_{Y,p_*\beta}[\mathfrak{M}_{g,n,p_*\beta}].$$

Let us consider the following exact sequence

$$T_{X/Y} \to T_X \to p^*T_Y$$

and let us look at the induced distinguished triangle

$$\mathcal{R}^\bullet\pi_*\,\mathrm{ev}_X^*\,T_{X/Y} \to \mathcal{R}^\bullet\pi_*\,\mathrm{ev}_X^*\,T_X \to \mathcal{R}^\bullet\pi_*\,\mathrm{ev}_X^*\,p^*T_Y \to \mathcal{R}^\bullet\pi_*\,\mathrm{ev}_X^*\,T_{X/Y}[1]. \tag{21}$$

By cohomology and base change we have that $\mathcal{R}^\bullet\pi_*\mathrm{ev}_X^*p^*T_Y \simeq p^*\mathcal{R}^\bullet\pi_*\mathrm{ev}_Y^*T_Y$. In the notation of the beginning of the section we can rewrite triangle (21) as

$$p^*E_{\epsilon_{Y,p_*\beta}} \to E_{\epsilon_{X,\beta}} \to (\mathcal{R}^\bullet\pi_*\,\mathrm{ev}_X^*\,T_{X/Y})^\vee \to p^*E_{\epsilon_{Y,p_*\beta}}[1]. \tag{22}$$

Let us note that all complexes are perfect. This shows the claim. □

Proposition 4.10 *Let $p\colon X \to \mathbb{P}^r$ be a smooth morphism. If $\overline{p}$ has positive virtual relative dimension, then $\overline{p}\colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(\mathbb{P}^r, p_*\beta)$ satisfies the strong push-forward property in homology.*

Proof By Kim–Pandharipande [11] $\overline{M}_{g,n}(\mathbb{P}^r, p_*\beta)$ is connected, and $\overline{p}$ satisfies the strong virtual push-forward property by Proposition 3.14. □

Proposition 4.11 *Let $L_1,\dots,L_s$ be very ample line bundles on a smooth projective variety X and let us consider a projective bundle $p\colon \mathbb{P}_X(\oplus L_i) \to X$. Then the induced morphism $\overline{p}\colon \overline{M}_{g,n}(\mathbb{P}_X(\oplus L_i),\beta) \to \overline{M}_{g,n}(X, p_*\beta)$ satisfies the strong push-forward property in homology.*

Proof Let us consider $j_i\colon X \to \mathbb{P}^{r_i}$ to be the embedding of X into a projective space induced by the line bundle L_i. Then we have a Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}_X(\oplus L_i) & \longrightarrow & \mathbb{P}_{\mathbb{P}^{r_1}\times\cdots\times\mathbb{P}^{r_s}}(\oplus\mathcal{O}(1)) \\ \downarrow & & \downarrow \\ X & \xrightarrow[j_1\times\cdots\times j_s]{} & \mathbb{P}^{r_1}\times\cdots\times\mathbb{P}^{r_s} \end{array}$$

The conclusion follows by the above proposition and Corollary 3.15. □

4.3 Stable maps and stable quotients

In this section we want to analyze the push forward of the virtual class of the moduli space of stable maps $\overline{M}_{g,n}(\mathbb{G}(1,r),d)$ along the morphism

$$c\colon \overline{M}_{g,n}(\mathbb{G}(1,r),d) \to \overline{Q}_{g,n}(\mathbb{G}(1,r),d)$$

which was introduced by Marian, Oprea and Pandharipande [19]. Let us briefly recall the basic definitions.

Stable quotients Let $(\widehat{C}, p_1, \dots, p_n)$ be a nodal curve of genus g with n distinct markings which are different from the nodes. A quotient on $\widehat{C}$

$$0 \longrightarrow \widehat{S} \longrightarrow \mathcal{O}_{\widehat{C}}^{\oplus r} \xrightarrow{q} \widehat{Q} \longrightarrow 0$$

is called *quasi-stable* if $\widehat{Q}$ is locally free at nodes or markings. Let k be the rank of $\widehat{S}$. A quotient $(\widehat{C}, p_1, \dots, p_n, q)$ is called stable if

$$\omega_{\widehat{C}}(p_1 + \cdots + p_n) \otimes (\wedge^k \widehat{S}^\vee)^\epsilon$$

is ample on $\widehat{C}$ for every strictly positive $\epsilon \in \mathbb{Q}$.

Remark 4.12 The space of stable quotients $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ is another compactification of the space of genus g curves (with r markings) in the Grassmannian $\mathbb{G}(k,r)$. This can be easily seen from the universal property of the tautological sequence on the Grassmannian: to give a curve $C \overset{i}{\hookrightarrow} \mathbb{G}(k,r)$ is equivalent to giving a quotient

$$0 \to i^*S \to \mathcal{O}_C^{\oplus r} \to i^*Q \to 0,$$

where

$$0 \to S \to \mathcal{O}^{\oplus r} \to Q \to 0$$

is the tautological sequence on the Grassmannian.

4.3.1 Obstruction theory As the moduli space of stable maps, the moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ has a morphism $\epsilon_Q\colon \overline{Q}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{M}_{g,n}$ to the Artin stack of nodal curves. Let $\pi_Q\colon \widehat{\mathcal{C}}_Q \to \overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ be the universal curve over $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ and let

$$0 \to \widehat{\mathcal{S}}_Q \to \mathcal{O}_{\widehat{\mathcal{C}}_Q}^{\oplus r} \to \widehat{\mathcal{Q}}_Q \to 0$$

be the universal sequence on $\widehat{\mathcal{C}}_Q$. Then the complex

$$E^\bullet_{\overline{Q}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{M}} = R\pi_{Q*} R\operatorname{Hom}(\widehat{\mathcal{S}}_Q, \widehat{\mathcal{Q}}_Q)$$

is a dual obstruction theory relative to ϵ_Q. We call

$$[\overline{Q}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} := (\epsilon_Q)^!_{\mathfrak{E}_{\overline{Q}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{M}}}[\mathfrak{M}_{g,n}]$$

the virtual class of $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$.

Moduli of bundles over prestable curves Let $\mathfrak{Bun}_{g,n}(k,d)(B)$ be the category whose objects are pairs $(\mathcal{C},\mathcal{S})$, where

- $\mathcal{C} \to B$ is a family of prestable curves of genus g with n sections
- $\mathcal{S}$ is a vector bundle of rank k and degree $-d$.

Isomorphisms: An isomorphism

$$\phi\colon (\mathcal{C},\mathcal{S}) \to (\mathcal{C}',\mathcal{S}')$$

is an automorphism of curves

$$\phi\colon \mathcal{C} \to \mathcal{C}'$$

such that

(1) $\phi(p_i) = p'_i$ for all i, and

(2) $\phi^*\mathcal{S}' = \mathcal{S}$.

By Lieblich [17] we have that coherent sheaves on $\mathfrak{M}_{g,n+1}$ over $\mathfrak{M}_{g,n}$ form an Artin stack $Coh_{\mathfrak{M}_{g,n+1}/\mathfrak{M}_{g,n}}$. It can be easily seen that $\mathfrak{Bun}_{g,n}(k,d)$ is a substack of $Coh_{\mathfrak{M}_{g,n+1}/\mathfrak{M}_{g,n}}$. Let $\mathcal{S}$ denote the universal bundle on the universal curve on $\mathfrak{Bun}_{g,n}(k,d)$. We will also consider moduli spaces of vector bundles on curves with stability conditions as follows.

Construction 4.13 Let $\epsilon > 0$ be fixed real number. Let $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ be the substack of $\mathfrak{Bun}_{g,n}(k,d)$ such that the line bundle

$$(\wedge^k \mathcal{S}^\vee)^{\otimes\epsilon} \otimes \omega\Big(\sum p_i\Big) \tag{23}$$

is ample. As ampleness is an open condition, $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ is an open substack of $\mathfrak{Bun}_{g,n}(k,d)$.

Remark 4.14 Let

$$\phi\colon \mathfrak{Bun}_{g,n}(k,d) \to \mathfrak{M}_{g,n}$$

be the morphism which forgets the bundle. The morphism ϕ is smooth as the relative obstruction in a point (C,S) is

$$\mathrm{Ext}^2_C(S,S) = 0.$$

This shows that $\mathfrak{Bun}_{g,n}(k,d)$ is smooth of pure dimension

$$\begin{aligned}\mathrm{Ext}^1(S,S) - \mathrm{Ext}^0(S,S) + 3g - 3 + n &= k(g-1) - \deg(S \otimes S^\vee) + 3g - 3 + n \\ &= k^2(g-1) + 3g - 3 + n.\end{aligned}$$

Lemma 4.15 *Let $\mathcal{C}$ be the universal curve over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. Then there exists a rational tail free curve $\widehat{\mathcal{C}}$ and a projection $p\colon \mathcal{C} \to \widehat{\mathcal{C}}$ over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$.*

Proof Let $\pi\colon \mathcal{C} \to B, \mathcal{S}$ be the tautological bundle on the tautological curve of $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. On each irreducible component of the locus consisting of rational tails the restriction of $\wedge^k \mathcal{S}^\vee$ has positive degree a. Set

$$\mathcal{L} = (\wedge^k \mathcal{S}^\vee)^{\otimes\epsilon} \otimes_a \omega\Big(\sum p_i\Big)^{\epsilon a}$$

where the tensor product runs over all the values of a. As $(\wedge^k \mathcal{S}^\vee)^{\otimes\epsilon} \otimes \omega\big(\sum p_i\big)$ is ample and there are no markings on the rational tails we have that $\mathcal{L}$ is trivial on the locus consisting of rational tails and π relatively ample on the complement of this locus. This shows that $\mathcal{L}^m$ is base point free for a sufficiently large m. Let

$$\widehat{\mathcal{C}} = \mathrm{Proj} \oplus_l \mathcal{L}^{ml}.$$

As $\mathcal{L}^m$ is π–relatively base point free it determines a morphism $p\colon \mathcal{C} \to \widehat{\mathcal{C}}$. We have that $\widehat{\mathcal{C}} \to B$ is a family of genus g curves and as $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ is reduced, we obtain that $\widehat{\mathcal{C}}$ is flat over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. □

Obstruction theories relative to moduli spaces of bundles In the following we define obstruction theories relative to $\mathfrak{Bun}_{g,n}(k,d)$. Let $\pi_M\colon \mathcal{C}_M \to \overline{M}_{g,n}(\mathbb{G}(k,r),d)$ be the universal curve over $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$. Let

$$0 \to \mathcal{S}_M \to \mathcal{O}^{\oplus r} \to \mathcal{Q}_M \to 0 \tag{24}$$

on $\mathcal{C}_M$ be the pull back of the tautological sequence on $\mathbb{G}(k,r)$ under the evaluation morphism $\mathrm{ev}_{n+1}\colon \mathcal{C}_M \to \mathbb{G}(k,r)$. The map

$$\nu_M\colon \overline{M}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{Bun}_{g,n}(k,d)$$

induces a morphism between cotangent complexes and thus we obtain a distinguished triangle

$$\nu_M^* L_{\mathfrak{Bun}_{g,n}(k,d)} \to L_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)} \to L_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{Bun}_{g,n}(k,d)}.$$

Tensoring (24) with $\mathcal{S}_M^\vee$ we obtain an exact sequence

$$0 \to \mathcal{S}_M \otimes \mathcal{S}_M^\vee \to (\mathcal{S}_M^\vee)^{\oplus r} \to \mathcal{Q}_M \otimes \mathcal{S}_M^\vee \to 0$$

which induces a distinguished triangle

$$R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} \to R^\bullet \pi_{M*}\mathcal{Q}_M \otimes \mathcal{S}_M^\vee \to R^\bullet \pi_{M*}\mathcal{S}_M \otimes \mathcal{S}_M^\vee[1].$$

By the cohomology and base change theorem we obtain that

$$\nu_M^* T_{\mathfrak{Bun}_{g,n}(k,d)} = R^\bullet \pi_{M*}\mathcal{S}_M \otimes \mathcal{S}_M^\vee[1].$$

This shows that we have the commutative diagram

$$\begin{array}{ccccc}
T_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{Bun}_{g,n}(k,d)} & \longrightarrow & T_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)} & \longrightarrow & \nu_M^* T_{\mathfrak{Bun}_{g,n}(k,d)} \\
\downarrow & & \downarrow & & \| \\
R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} & \longrightarrow & R^\bullet \pi_{M*}\mathcal{Q}_M \otimes \mathcal{S}_M^\vee & \longrightarrow & R^\bullet \pi_{M*}\mathcal{S}_M \otimes \mathcal{S}_M^\vee[1]
\end{array}$$

and therefore $R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r}$ is a dual relative obstruction theory for ν_M.

In a completely analogous manner we obtain that $R^\bullet \pi_{M*}(\widehat{\mathcal{S}}_Q^\vee)^{\oplus r}$ is a dual relative obstruction theory for ν_Q.

4.3.2 Comparison between virtual fundamental classes

Proposition 4.16 *When $k=1$ there exists a map*

$$c\colon \overline{M}_{g,n}(\mathbb{G}(1,r),d) \to \overline{Q}_{g,n}(\mathbb{G}(1,r),d)$$

extending the isomorphism on smooth curves.

Proof This has been proved by Marian, Oprea and Pandharipande [19] and a similar situation appears in Popa–Roth [21]. Let us shortly sketch the proof. Let $(\pi_{\mathcal{C}}, f)\colon \mathcal{C} \to B \times \mathbb{G}(1,r)$ be a family of stable maps to $\mathbb{G}(1,r)$. By Remark 4.12, this comes with an exact sequence

$$0 \to f^*\mathcal{S} \to \mathcal{O}_{\mathcal{C}}^{\oplus r} \to f^*\mathcal{Q} \to 0.$$

Let $\pi_{\widehat{\mathcal{C}}}\colon \widehat{\mathcal{C}} \to B$, be the family of curves obtained by contracting all rational trees with no marked points and let $p\colon \mathcal{C} \to \widehat{\mathcal{C}}$ be the contracting morphism. In the following, we will give a canonical way to associate a quasi-stable quotient to the family $\pi_{\widehat{\mathcal{C}}}$. Let $D \subset \mathfrak{M}_{g,n+1}$ be the divisor whose general point is a nodal curve with one irreducible rational curve attached in one point to a genus g curve. Let $\mathcal{O}_{\overline{M}_{g,n+1}(\mathbb{G}(1,r),d)}(E) = \epsilon_M^*\mathcal{O}_{\mathfrak{M}_{g,n+1}}(D)$. The line bundle $\mathcal{O}_{\overline{M}_{g,n+1}(\mathbb{G}(1,r),d)}(E)$ has degree -1 when restricted to the general fiber of the induced map from E to $\overline{M}_{g,n}(\mathbb{G}(1,r),d)$ (see Popa–Roth [21]). We attach the weight δ to such an E if the degree of $\mathcal{S}$ restricted to the general fiber in D is $-\delta$. We consider the bundle

$$\mathcal{S}' := \mathcal{S} \otimes_{\delta} \mathcal{O}(-\delta E_{\delta})$$

where the tensor product is taken over all possible δ and all E_{δ} which satisfy the condition above. Then $\mathcal{S}'$ is trivial along the rational tails and it can be showed that $\widehat{\mathcal{S}} = p_*\mathcal{S}'$ is a stable quotient. □

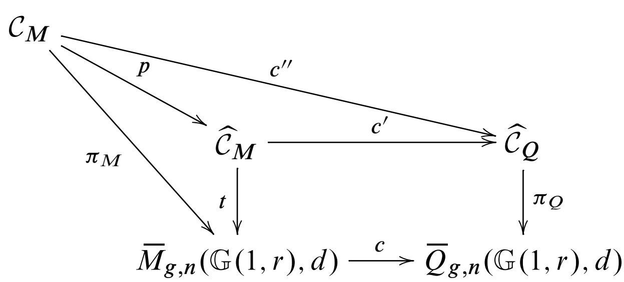
Remark 4.17 Let $(C, x_1, \dots, x_n, f)$ be a stable map to $\mathbb{G}(1,n)$, $\widehat{C}$ be the curve obtained by contracting the rational tails of C and x_i the points on C where the rational trees glue to $\widehat{C}$. Let S^1 be the restriction of S to $\widehat{C}$. The map c associates to a map $f\colon C \to \mathbb{G}(1,n)$, the curve $\widehat{C}$ and the exact sequence

$$0 \to S^1\Big(-\sum d_i x_i\Big) \to \mathcal{O}_{\widehat{C}}^n \to \widehat{Q} \to 0,$$

where d_i is the degree of f on the tree C_i.

Remark 4.18 It can be easily seen that we have a commutative diagram with the right-down square Cartesian

(25)



with $p\colon \mathcal{C}_M \to \widehat{\mathcal{C}}_M$ be the morphism which contracts rational tails. The proof of Proposition 4.16 shows that $c'^*\widehat{\mathcal{S}}_Q = p_*\mathcal{S}_M(-\delta E)$ which is equivalent to $c'^*\widehat{\mathcal{S}}_Q^\vee = p_*\mathcal{S}_M^\vee(\delta E)$. This shows that we have a natural morphism

$$\rho\colon\ p_*(\mathcal{S}_M^\vee) \to c'^*\widehat{\mathcal{S}}_Q^\vee.$$

Lemma 4.19 *The morphism*

$$\rho\colon\ p_*(\mathcal{S}_M^\vee) \to c'^*\widehat{\mathcal{S}}_Q^\vee$$

on $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$ *induces a morphism*

$$R^\bullet\pi_{M*}\big(\mathcal{S}_M^\vee\big)^{\oplus r} \to c^*R^\bullet\pi_{Q*}\big(\widehat{\mathcal{S}}_Q^\vee\big)^{\oplus r}.$$

Proof From the commutativity of diagram (25) we have that

$$R^\bullet\pi_{M*}\mathcal{S}_M^\vee \simeq R^\bullet(t\circ p)_*\mathcal{S}_M^\vee. \tag{26}$$

Using now cohomology and base change in diagram (25) we obtain that

$$c^*R^\bullet\pi_{Q*}\widehat{\mathcal{S}}_Q^\vee \simeq R^\bullet t_*c'^*\widehat{\mathcal{S}}_Q^\vee. \tag{27}$$

By (26) and (27) we see that $\rho\colon\ p_*\big(\mathcal{S}_M^\vee\big) \to c'^*\widehat{\mathcal{S}}_Q^\vee$ induces a morphism

$$R^\bullet\pi_{M*}\big(\mathcal{S}_M^\vee\big)^{\oplus r} \to c^*R^\bullet\pi_{Q*}\big(\widehat{\mathcal{S}}_Q^\vee\big)^{\oplus r}.$$ □

Lemma 4.20 *Let F be the cone of the morphism*

$$R^\bullet\pi_{M*}\big(\mathcal{S}_M^\vee\big)^{\oplus r} \to c^*R^\bullet\pi_{Q*}\big(\widehat{\mathcal{S}}_Q^\vee\big)^{\oplus r}.$$

Then, F is a perfect complex.

Proof Let us consider

$$f\colon (C,x_1,\dots,x_n) \to \mathbb{G}$$

a stable map, $p\colon C \to \widehat{C}$ the morphism contracting the rational tails and let $p_1,\dots,p_s$ be the gluing points of the rational tails with the rest of the curve. We need to show that the morphism

$$H^1(C,f^*S^\vee) \to H^1(\widehat{C},\widehat{S}^\vee)$$

is surjective. By the definition of $\widehat{S}$ we have that

$$H^1(\widehat{C},\widehat{S}^\vee) \simeq H^1\Big(\widehat{C},S^\vee|_{\widehat{C}}\Big(\sum d_ip_i\Big)\Big).$$

Since

$$H^1(C,f^*S^\vee) \simeq H^1\big(\widehat{C},f^*S^\vee|_{\widehat{C}}\big)$$

we need to show that

$$H^1\big(\widehat{C}, f^*S^\vee|_{\widehat{C}}\big) \to H^1\Big(\widehat{C}, f^*S^\vee|_{\widehat{C}}\Big(\sum d_i p_i\Big)\Big)$$

is surjective. As the quotient of the morphism $f^*S^\vee|_{\widehat{C}} \to f^*S^\vee|_{\widehat{C}}\big(\sum d_i p_i\big)$ is supported on the points p_i, it has no higher cohomology. This shows that the above morphism is surjective. □

Proposition 4.21 *We have that*

$$c_*\big[\overline{M}_{g,n}(\mathbb{G}(1,n),d)\big]^{\mathrm{virt}} = \big[\overline{Q}_{g,n}(\mathbb{G}(1,n),d)\big]^{\mathrm{virt}}.$$

Proof We fix $\epsilon > 2$. Let us consider the following commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}(\mathbb{G}(1,n),d) & \xrightarrow{\ c\ } & \overline{Q}_{g,n}(\mathbb{G}(1,n),d) \\ {\scriptstyle \nu_M}\big\downarrow & & \big\downarrow{\scriptstyle \nu_Q} \\ \mathfrak{Bun}^{\epsilon}_{g,n}(1,d) & \xrightarrow[\ p\ ]{} & \mathfrak{Bun}^{\epsilon}_{g,n}(1,d) \end{array} \tag{28}$$

where p is the map contracting the rational tails.

As c is surjective and $\overline{M}_{g,n}(\mathbb{G}(1,r),d)$ is connected, we get that $\overline{Q}_{g,n}(\mathbb{G}(1,r),d)$ is connected. By Proposition 3.14, Lemma 4.19 and Lemma 4.20 we obtain that there exists $N \in Q$ such that

$$c_*\big[\overline{M}_{g,n}(\mathbb{G}(1,r),d)\big]^{\mathrm{virt}} = N\big[\overline{Q}_{g,n}(\mathbb{G}(1,r),d)\big]^{\mathrm{virt}}.$$

As the $\overline{M}_{g,n}(\mathbb{G}(1,r),d)$ and $\overline{M}_{g,n}(\mathbb{G}(1,r),d)$ are isomorphic on an open set and have compatible obstruction theories we have that $N = 1$. □

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