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# Entropy zero area preserving diffeomorphisms of $S^2$

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In this paper we formulate and prove a structure theorem for area preserving diffeomorphisms of genus zero surfaces with zero entropy and at least three periodic points. As one application we relate the existence of faithful actions of a finite index subgroup of the mapping class group of a closed surface $\Sigma_g$ on $S^2$ by area preserving diffeomorphisms to the existence of finite index subgroups of bounded mapping class groups $\mathrm{MCG}(S, \partial S)$ with nontrivial first cohomology. In another application we show that the rotation number is defined and continuous at every point of a zero entropy area preserving diffeomorphism of the annulus.

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## 1 Introduction and statement of results

Surface diffeomorphisms with positive entropy have been studied from both the hyperbolic dynamical systems point of view and the Nielsen–Thurston point of view. In this paper we formulate and prove a structure theorem for area preserving diffeomorphisms of genus zero surfaces with zero entropy. The area preserving assumption is a natural one arising in many dynamical systems and it is an essential ingredient for most of the dynamical structure we investigate here. The genus zero assumption is made to simplify the problem. There should be a similar theory for higher genus and much of what we show here may well be true for Hamiltonian diffeomorphisms in higher genus.

If $N$ is a genus zero surface with finitely many boundary components and $F'\colon N \to N$ is a diffeomorphism, then collapsing each component of $\partial N$ to a point produces a homeomorphism $F\colon S^2 \to S^2$ which restricts to a diffeomorphism on the complement of a finite set. For almost all of our analysis we can work directly with $F$ instead of $F'$ and can even forget that $F'$ is smooth but there are two (very important) steps (see Section 4 and Lemma 8.9) when we must remember $F'$ and make use of its smoothness. With this in mind we make the following definitions.

Let $\mu$ be a measure on $S^2$ that is topologically conjugate to the Lebesgue measure. A homeomorphism that preserves $\mu$ is said to *preserve area*. Let $P \subset S^2$ be a (possibly

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empty) finite set and let $N$ be the genus zero surface obtained from $S^2$ by blowing up each element of $P$ to a boundary circle. Inverting this process produces a quotient map $\pi_P\colon N \to S^2$ that restricts to a diffeomorphism from $\mathrm{int}(N)$ to $S^2 \setminus P$ and that maps each component of $\partial N$ to an element of $P$.

Define $\mathrm{Diff}_\mu(S^2, P)$ to be the group of *orientation preserving homeomorphisms of $S^2$ that preserve $\mu$*, that fix each element of $P$ and for which there is a $C^\infty$ diffeomorphism $F'\colon N \to N$ such that $F\pi_P = \pi_P F'$. Note that if $P = \varnothing$ and $\mu$ is a smooth volume form, then $\mathrm{Diff}_\mu(S^2, P)$ is just the group $\mathrm{Diff}_\mu(S^2)$ of $C^\infty$ diffeomorphisms of $S^2$ which preserve $\mu$.

There are certain elements of $\mathrm{Diff}_\mu(S^2, P)$ which are trivial from the point of view of their periodic points. These include $F \in \mathrm{Diff}_\mu(S^2, P)$ of finite order and $F$ for which $\mathrm{Per}(F)$ contains only two points. It is known that an area preserving $F$ must have at least two fixed points (see Simon [26]). In the case that $\mathrm{Per}(F)$ contains exactly two points, those points must be fixed. Blowing up the fixed points as above produces a homeomorphism $F'$ of the closed annulus with every point having the same irrational rotation number (see Theorem 2.3). This is an interesting topic to investigate but is not addressed in this article. For the remainder of this paper we make the following:

**Standing hypothesis** *Assume that $F \in \mathrm{Diff}_\mu(S^2, P)$ has infinite order and entropy zero and that $\mathrm{Per}(F)$ contains at least three points.*

Suppose that $F \in \mathrm{Diff}_\mu(S^2, P)$ has zero topological entropy and that $\mathrm{Fix}(F)$ is the set of fixed points for $F$. To enhance the topology of the ambient surface, we consider $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$ and $f = F|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$.

Disks in this paper are topological objects; they are not assumed to be round. Every $x \in \mathcal{M}$ has a neighborhood $B$ that is a *free disk*, meaning that $B$ is an open disk and that $f(B) \cap B = \varnothing$. A very weak notion of recurrence for a point $x \in \mathcal{M}$ is to require that there be $n \neq 0$ and a free disk $B$ that contains both $x$ and $f^n(x)$. We will call such points *free disk recurrent* and denote the set of these points by $\mathcal{W}_0$. Each periodic point is free disk recurrent; a nonperiodic $x$ is free disk recurrent if and only if there is a free disk $B$ which intersects the orbit of $x$ in at least two points. Clearly, if either the $\alpha$–limit set $\alpha(F, x)$ or the $\omega$–limit set $\omega(F, x)$ contains a point which is not in $\mathrm{Fix}(F)$ then $x \in \mathcal{W}_0$. In particular the set $\mathcal{W}_0$ contains the full measure subset of $\mathcal{M}$ consisting of birecurrent points. The set $\mathcal{W}_0$ is open and dense in $\mathcal{M}$. It is technically useful to work with sets that equal the interior of their closure so we define the larger set $\mathcal{W}$ of *weakly free disk recurrent* points as follows. (We expect that $\mathcal{W}_0 \neq \mathcal{W}$ in general but have not worked out a specific example.)

For sets $A \subset X$ we denote the *interior* of $A$ with respect to $X$ by $\text{int}_X(A)$ and the *closure* of $A$ with respect to $X$ by $\text{cl}_X(A)$. If $X$ is understood then we drop it from the notation and simply write $\text{int}(A)$ and $\text{cl}(A)$.

**Definition 1.1** A point $x \in \mathcal{M}$ is *free disk recurrent* for $f$ provided there exists $n \neq 0$ and a free disk $B$ that contains both $x$ and $f^n(x)$. The set of free disk recurrent points in $\mathcal{M}$ is denoted $\mathcal{W}_0$. If $W_0$ is a component of $\mathcal{W}_0$ and $x \in \mathcal{M}$ is in $\text{int}_{\mathcal{M}}(\text{cl}_{\mathcal{M}}(W_0))$, then we say that $x$ is *weakly free disk recurrent.* The set of weakly free disk recurrent points in $\mathcal{M}$ is denoted $\mathcal{W}$.

The main building block in our structure theorem is a partition of $\mathcal{W}$ into countably many disjoint $f$–invariant annuli.

**Theorem 1.2** *Suppose $F \in \text{Diff}_\mu(S^2, P)$ has entropy zero, infinite order and at least three periodic points. Let $f = F|_{\mathcal{M}}$ where $\mathcal{M} = S^2 \setminus \text{Fix}(F)$. Then there is a countable collection $\mathcal{A}$ of pairwise disjoint open $f$–invariant annuli such that:*

1. *$\mathcal{U} = \bigcup_{U \in \mathcal{A}} U$ is the set $\mathcal{W}$ of weakly free disk recurrent points for $f$.*
2. *$\mathcal{A}$ is the set of maximal $f$–invariant open annuli in $\mathcal{M}$.*
3. *If $z \notin \mathcal{U}$, there are components $F_+(z)$ and $F_-(z)$ of $\text{Fix}(F)$ so that $\omega(F, z) \subset F_+(z)$ and $\alpha(F, z) \subset F_-(z)$.*
4. *For each $U \in \mathcal{A}$ and each component $C_{\mathcal{M}}$ of the frontier of $U$ in $\mathcal{M}$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_{\mathcal{M}}$.*

We emphasize the fact that replacing $F$ by an iterate $F^q$ changes $\mathcal{M}$ and hence changes the annuli of Theorem 1.2.

**Remark 1.3** If $h: S^2 \to S^2$ commutes with $F$ then it preserves $\mathcal{W}$ and hence permutes the open annuli in the family $\mathcal{A}$.

To see how the elements of $\mathcal{A}$ arise, consider the special case that $F$ is the time one map of an area preserving flow $\phi_t$. Given $x \in \mathcal{M}$, choose a free disk neighborhood $B$ of $x$ which is also a flow box for $\phi_t$. It is an easy consequence of the Poincare–Bendixson theorem that if the flow line for $\phi_t$ that contains $x$ returns to $B$ it closes up into a simple closed curve $\rho_x$. In particular, in this case the subsets $\mathcal{W}_0$ and $\mathcal{W}$ are equal and coincide with the union of the periodic orbits of the flow which lie in $\mathcal{M}$. Denote *the isotopy class of* $\rho_x$ in $\mathcal{M}$ by $[\rho_x]$. It is clear that $\rho_x$ depends only on the orbit of $x$ and not $x$ itself and that if $z \in B$ is sufficiently close to $x$ then $\rho_x$ and $\rho_z$ cobound an annulus in $\mathcal{M}$; in particular $[\rho_x] = [\rho_z]$. In this case $U = \{y \in \mathcal{W} : [\rho_y] = [\rho_x]\}$ is the element of $\mathcal{A}$ that contains $x$.

For a second special case suppose that $f$ is isotopic to the identity. Given $x \in \mathcal{W}_0$, choose $B$ and $n$ as in the definition of free disk recurrent. If $f_t\colon \mathcal{M} \to \mathcal{M}$ is an isotopy between $f_0 =$ identity and $f_1 = f$ then the path $\mu_x \subset \mathcal{M}$ defined by $\mu_x(t) = f_t(x)$ connects $x$ to $f(x)$. The path $\mu_x \cdot \mu_{f(x)} \cdots \mu_{f^{n-1}(x)}$ can be closed by adding a path in $B$ connecting $f^n(x)$ to $x$. Up to homotopy in $\mathcal{M}$, this closed path is a multiple of some nonrepeating closed path $\rho_x$. Using the hypothesis that $F$ has entropy zero, one can show (see the authors' paper [11]) that the homotopy class of $\rho_x$ is represented by a simple closed curve (also written $\rho_x$) that is independent of $B, n$ and the choice of isotopy $f_t$. It is easy to see that if $z \in B$ is sufficiently close to $x$ then $[\rho_x] = [\rho_z]$. As in the previous case, $U = \{y \in \mathcal{W} : [\rho_y] = [\rho_x]\}$ is the element of $\mathcal{A}$ that contains $x$.

In the general case, we make use of the fact (see Section 4) that $f$ is isotopic to a composition of Dehn twists along a finite set of simple closed curves $\mathcal{R}$. Cutting along the elements of $\mathcal{R}$ produces a decomposition of $\mathcal{M}$ into subsurfaces $\mathcal{M}_i$ such that $f|_{\mathcal{M}_i}\colon \mathcal{M}_i \to \mathcal{M}$ is isotopic to the inclusion $\mathcal{M}_i \hookrightarrow \mathcal{M}$. The main technical work in this proof is showing that each $\mathcal{M}_i$ is realized, in a suitable sense, by an $f$–invariant subsurface; see Section 10. One then defines $\mathcal{A}$ in a fashion similar to the second special case.

Theorem 1.2 can be applied to $F^q$ for each $q \geq 2$. This gives a countable collection $\mathcal{A}(q)$ of pairwise disjoint open $F^q$–invariant annuli that (see Proposition 15.3) refines $\mathcal{A}$ in the sense that each $V_j \in \mathcal{A}(q)$ is contained in some $U_i \in \mathcal{A}$. This *renormalization process* can be iterated with $\mathcal{A}(q)$ playing the role of $\mathcal{A}$ and so on. The $V_j$ may be essential or inessential in $U_i$. In the limit, the former lead to twist-map-like behavior and the latter to solenoid-like behavior when they are nested infinitely often. It is important to note that replacing $F$ with $F^q$ changes the set of fixed points and hence changes $\mathcal{M}$ and changes the free disk recurrent points of $\mathcal{M}$.

We are interested in partitioning $\mathrm{cl}(U)$ into sets analogous to the periodic orbits in the case of the time one map of a flow. In particular we would like the rotation number to be constant on these sets. The two components of the frontier of $U$ can be somewhat problematic since such a component could be a single point or could be a complicated fractal. To deal with this issue we introduce the *annular compactification* $f_c\colon U_c \to U_c$ of $f\colon U \to U$; see Notation 2.7 and the paragraph preceding it. The compactification of an end described there is either the prime end compactification or the compactification obtained by blowing up a fixed point, whichever is appropriate.

We are now prepared to state the second of our main results. It describes the finer structure of the dynamics of $f$ on one of the annuli in $\mathcal{A}$. The proof is based on renormalization and the details are in Section 15.

**Theorem 1.4** *Suppose $F \in \mathrm{Diff}_\mu(S^2, P)$ has entropy zero, has infinite order and at least three periodic points. Let $f = F|_{\mathcal{M}}$ where $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$ and let $\mathcal{A}$ be as in Theorem 1.2. For $U \in \mathcal{A}$, let $f_c\colon U_c \to U_c$ be the annular compactification of $f|_U\colon U \to U$. Then:*

(1) *The rotation number $\rho_{f_c}(x)$ is defined and continuous at every $x \in U_c$.*

(2) *If $\mathrm{Fix}(F)$ contains at least three points then $\rho_{f_c}$ is nonconstant.*

(3) *If $C$ is a component of a level set of $\rho_{f_c}$ then $C$ is $F$–invariant. If $C$ does not contain a component of $\partial U_c$ then it is essential in $U$, meaning that $U_c \setminus C$ has two components each containing a component of $\partial U_c$.*

The components $C$ of the level sets of $\rho_{f_c}$ in Theorem 1.4 are the generalizations of the closed orbits foliating $U$ in the special case that $F$ is the time one map of a flow. Of course in the general case $C$ can be considerably more complicated. The main example constructed in Handel [14] shows $C$ can be a pseudocircle. It is also possible for $C$ to have nonempty interior.

A heuristic picture of one possibility in the case that $\rho_f|_C$ is rational is an essential "necklace" in $U$ consisting of a periodic orbit of saddle periodic points each joined to the next by a stable manifold (which is the unstable manifold of the next one) and by an unstable manifold (which is the stable manifold of the next one). This pair, stable and unstable, bound a "bead", an open disk. The diffeomorphism $f$ permutes the beads and has a periodic orbit with one point in each bead. The set $C$ containing any $x$ in one of the beads will be the entire necklace. For such a $C$ there is an $n$ such that $f^n$ will fix each bead and each saddle point joining them.

Our first application concerns area preserving diffeomorphisms of the closed annulus $A$. For expected future applications, we state our theorem in a more general context and then state the annulus result as a corollary.

Suppose that $P$ has two preferred elements $p_1, p_2$ and that $P' = P \setminus \{p_1, p_2\}$. If $H\colon A \to A$ is the homeomorphism of the closed annulus obtained from some $F \in \mathrm{Diff}_\mu(S^2, P)$ by blowing up $p_1$ and $p_2$ then we write $H \in \mathrm{Diff}_\mu(A, P')$. Note that if $P = \{p_1, p_2\}$ then $\mathrm{Diff}_\mu(A, P')$ is the group of area preserving $C^\infty$ diffeomorphism of the closed annulus $A$.

**Theorem 1.5** *For each $H \in \mathrm{Diff}_\mu(A, P')$ with entropy zero, the rotation number $\rho_H(x)$ is defined and continuous at each $x \in A$.*

**Corollary 1.6** *Suppose that $H\colon A \to A$ is an area preserving $C^\infty$ diffeomorphism of the closed annulus $A$. If $H$ has entropy zero then the rotation number $\rho_H(x)$ is defined and continuous at each $x \in A$.*

For our next application, recall that a group $G$ is *indicable* if there exists a nontrivial homomorphism $G \to \mathbb{Z}$. For finitely generated groups this is equivalent to $H^1(G, \mathbb{Z}) \neq 0$ and equivalent to the abelianization of $G$ being infinite. If a finite index subgroup of $G$ is indicable then we say that $G$ is *virtually indicable.*

For $F \in \mathrm{Diff}_\mu(S^2)$, denote the centralizer of $F$ in $\mathrm{Diff}_\mu(S^2)$ by $Z(F)$. As an application of Theorem 1.2 and Theorem 1.4 we prove:

**Theorem 1.7** *If $F \in \mathrm{Diff}_\mu(S^2)$ has infinite order then each finitely generated infinite subgroup $H$ of $Z(F)$ is virtually indicable.*

One might expect that Theorem 1.7 is proved by first proving the existence of a finite index subgroup $H_0$ of $H$ with global fixed points and then applying the Thurston stability theorem (Thurston [27]; see also Franks [10, Theorem 3.4]) to produce a nontrivial homomorphism from $H_0$ to $\mathbb{Z}$. This is easy to do (see Proposition 17.1) in the case that $F$ has positive entropy but fails when $F$ has zero entropy. Indeed, there are examples (see Examples 17.2) for which no finite index subgroup of $Z(F)$ has a global fixed point. We prove Theorem 1.7 by analyzing the possible ways in which the existence of global fixed points can fail and by showing that each allows one to define a nontrivial homomorphism to $\mathbb{Z}$.

As an application of Theorem 1.7 we have the following result about mapping class groups.

**Corollary 1.8** *If $\Sigma_g$ is the closed orientable surface of genus $g \geq 2$ then at least one of the following holds:*

(1) *No finite index subgroup of $\mathrm{MCG}(\Sigma_g)$ acts faithfully on $S^2$ by area preserving diffeomorphisms.*
(2) *For all $1 \leq k \leq g-1$, there is an indicable finite index subgroup $\Gamma$ of the bounded mapping class group $\mathrm{MCG}(S_k, \partial S_k)$ where $S_k$ is the surface with genus $k$ and connected nonempty boundary.*

Corollary 1.8 relates to the following well-known questions about mapping class groups.

**Question 1.9** *Does $\mathrm{MCG}(\Sigma_g)$, or any of its finite index subgroups, act faithfully on a closed surface $S$ by diffeomorphisms or by area preserving diffeomorphisms?*

**Question 1.10** *Does every finite index subgroup $\Gamma$ of $\mathrm{MCG}(\Sigma_g)$ satisfy $H^1(\Gamma,\mathbb{R}) = 0$?*

Question 1.9 is motivated in part by the sections problem (see Problem 6.5 and Question 6.7) of Farb's survey/problem list [4] on the mapping class group: which subgroups of $\mathrm{MCG}(\Sigma_g)$ lift to $\mathrm{Diff}(\Sigma_g)$? It is also motivated by the analogy between mapping class groups and higher rank lattices and the fact (see the authors' [11; 12] and Polterovich [25]) that every action of a nonuniform irreducible higher rank lattice on $\Sigma_g$ by area preserving diffeomorphisms factors through a finite group; see Question 12.4 of Fisher's survey article [6] on the Zimmer program.

Question 1.10 is Kirby [21, Problem 2.11]; see also Ivanov [19] and Korkmaz [22]. Corollary 1.8(1) is a negative answer to the area preserving, $S = S^2$ case of Question 1.9. The answer to Question 1.10 is no for genus 2 (see McCarthy [24]) but is unknown for genus at least three. Presumably a positive answer to Question 1.10 for genus greater than 3 would imply that Corollary 1.8(2) does not hold and so imply that Corollary 1.8(1) does hold.

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## 2 Area preserving annulus maps

We will make use of a number of results on area preserving homeomorphisms and diffeomorphisms of the annulus which we cite here.

If $A = S^1 \times [0,1]$ is the annulus, its universal covering space is $\widetilde{A} = \mathbb{R} \times [0,1]$. We will denote by $p_1$ the projection, $p_1\colon \mathbb{R} \times [0,1] \to \mathbb{R}$, of $\widetilde{A}$ onto its first factor.

**Definition 2.1** If $f\colon A \to A$ is an orientation preserving homeomorphism isotopic to the identity and $\tilde{f}$ is a lift to $\widetilde{A}$ then the *forward translation interval* $\mathcal{T}^+{}_{\tilde{f}}(\tilde{x})$ of $\tilde{x} \in \widetilde{A}$ is defined to be $[a,b]$, where

$$a = \liminf_{n\to\infty} \frac{p_1\big(\tilde{f}^n(\tilde{x})\big) - p_1(\tilde{x})}{n} \quad \text{and} \quad b = \limsup_{n\to\infty} \frac{p_1\big(\tilde{f}^n(\tilde{x})\big) - p_1(\tilde{x})}{n}.$$

If $a = b$ then $\tau^+{}_{\tilde{f}}(\tilde{x}) = a$ is called the *forward translation number* of $\tilde{x} \in \widetilde{A}$ and

$$\tau^+_{\tilde{f}}(\tilde{x}) = \lim_{n\to\infty} \frac{p_1\big(\tilde{f}^n(\tilde{x})\big) - p_1(\tilde{x})}{n}.$$

The backward translation interval and number $\mathcal{T}^-{}_{\tilde f}(\tilde x)$ and $\tau^-{}_{\tilde f}(\tilde x)$ are defined analogously. If $\tau^+{}_{\tilde f}(\tilde x)$ and $\tau^-{}_{\tilde f}(\tilde x)$ are both defined and if

$$\tau^-{}_{\tilde f}(\tilde x) = -\tau^+{}_{\tilde f}(\tilde x)$$

then we say that $\tau^+{}_{\tilde f}(\tilde x)$ is the *translation number of* $\tilde x$ and denote this number by $\tau_{\tilde f}(\tilde x)$. All of these definitions are independent of the choice of lift $\tilde x$ of $x$ and so may be viewed as functions of $x$.

The *forward rotation interval* $\mathcal{R}^+{}_f(x)$ and *forward rotation number* $\rho^+{}_f(x)$ of $x \in A$ are defined to be the projection of $\mathcal{T}^+{}_{\tilde f}(x)$ and $\tau^+{}_{\tilde f}(x)$ respectively in $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. As the notation suggests, they are independent of the choice of lift $\tilde f$ of $f$. Backward rotation interval, backward rotation number and rotation number are defined and denoted similarly.

**Lemma 2.2** *Suppose that $f\colon A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and that $\tilde f\colon \tilde A \to \tilde A$ is a lift to its universal covering space. Then $\tau_{\tilde f}(\tilde x)$ exists for almost all $\tilde x \in \tilde A$.*

**Proof** This is a standard consequence of the Birkhoff ergodic theorem applied to the function $\phi(x) = p_1(\tilde f(\tilde x)) - p_1(\tilde x)$. □

The closed interval $\mathcal{T}(\tilde f)$ of the following lemma is called the *translation interval of* $\tilde f$. Its projected image $\mathcal{R}(f)$ in $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ is called the *rotation interval of* $f$.

**Theorem 2.3** *Suppose that $f\colon A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and that $\tilde f\colon \tilde A \to \tilde A$ is a lift to its universal covering space. Then there is a closed interval $\mathcal{T}(\tilde f)$ with the following properties:*

1. *For each $r \in \mathcal{T}(\tilde f)$ there exists $\tilde x \in \tilde A$ such that $\tau_{\tilde f}(\tilde x) = r$; if $r = p/q$ is a rational number in lowest terms then one may choose $\tilde x$ to be a lift of a periodic point with period $q$.*
2. *For all $\tilde x \in \tilde A$, $\mathcal{T}^+{}_{\tilde f}(\tilde x) \subset \mathcal{T}(\tilde f)$ and $-\mathcal{T}^-_{\tilde f}(\tilde x) \subset \mathcal{T}(\tilde f)$.*

**Proof** Define $\mathcal{T}(\tilde f)$ to be the set of $r \in \mathbb{R}$ for which there exists $\tilde x \in \tilde A$ with $\tau_{\tilde f}(\tilde x) = r$. Handel [15, Theorem 0.1] implies that $\mathcal{T}(\tilde f)$ is closed. Suppose that $r_i \in \mathcal{T}^+{}_{\tilde f}(\tilde x_i)$ for $i = 1, 2$ and some $\tilde x_i \in \tilde A$. Franks [8, Corollary 2.4] implies that for any rational in lowest terms $p/q \in [r_1, r_2]$ there is a periodic point $x$ for $f$ with period $q$ and a lift $\tilde x \in \tilde A$ such that $\tau_{\tilde f}(\tilde z) = p/q$. Item (1) and the $\mathcal{T}^+{}_{\tilde f}(\tilde x) \subset \mathcal{T}(\tilde f)$ part of (2) follow immediately. The symmetric argument with $\tilde f$ replaced by $\tilde f^{-1}$ proves the $\mathcal{T}^-_{\tilde f}(\tilde x) \subset \mathcal{T}(\tilde f)$ part of (2). □

**Proposition 2.4** *Suppose $f\colon A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity. If there is a subset $Y \subset A$ with Lebesgue measure $\mu(Y) > 0$ and such that $\rho_f^+(x) = 0$ for almost all $x \in Y$ then $f$ has a fixed point in the interior of $A$.*

**Proof** By the Birkhoff ergodic theorem $\rho_f^+(x) = \rho_f^-(x)$ for almost all points of $A$, hence we may assume $\rho_f(x) = 0$ for almost all $x \in Y$. Since $\mu(Y) > 0$ there is a small open disk $D$ whose closure is in the interior of $A$ with $\mu(Y \cap D) > 0$. If $f$ has no fixed point in $D$ then by making $D$ smaller we may assume it is a free disk. We let $X = Y \cap D$. Let $r\colon X \to X$ be the first return map so $r(x) = f^n(x)$, where $n$ is the smallest positive integer such that $f^n(x) \in X$. The function $r$ is well-defined for almost all $x \in X$, so deleting a set of measure $0$ from $X$ we may assume it defined for all $x \in X$.

Let $\tilde{D}$ be a lift of $D$. If $\tilde{X}$ is the set of lifts to $\tilde{D}$ of points in $X$ then there is a positive measure subset $\tilde{X}_0 \subset \tilde{X}$ and a lift $\tilde{f}$ of $f$ such that $\tau_{\tilde{f}}(x) = 0$ for all $x \in X_0$.

Suppose the first return time for $x$ is $n$, so $r(x) = f^n(x)$. Then $\tilde{f}^n(\tilde{x}) \in T^k(\tilde{D})$ for a unique integer $k$. We define $h(x, \tilde{f})$, the *homological displacement* of $x$, to be $k$. It depends on $\tilde{f}$ but not on the choice of lift $\tilde{D}$ of $D$.

It suffices to prove that $h(x, \tilde{f}) = 0$ for some $x \in X_0$ because then $\tilde{x}$ is contained in a periodic disk chain (see [7, Proposition 1.3]) and $\tilde{f}$ has a fixed point. We note that if there are $x, y \in X_0$ such that $h(x, \tilde{f}) > 0$ and $h(y, \tilde{f}) < 0$ then $\tilde{f}$ has a fixed point. This is a consequence of [7, Theorem 2.1] since there are both positive and negative recurring disk chains for $f$. Hence we may assume $h(x, \tilde{f})$ has a constant sign.

[9, Proposition 3.2] shows that if

$$B = \bigcup_{n \in \mathbb{Z}} f^n(X_0)$$

then

$$\int_{X_0} h(x, \tilde{f})\, d\mu = \int_B \tau_{\tilde{f}}(x)\, d\mu.$$

Since $\tau_{\tilde{f}}(x) = 0$ for all $x \in X_0$ we conclude that $\int_{X_0} h(x, \tilde{f})\, d\mu = 0$. Since $h$ has constant sign it follows that $h(x, \tilde{f}) = 0$ for almost all $x \in X_0$. □

**Definition 2.5** Suppose $f\colon A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and let $\tilde{f}\colon \tilde{A} \to \tilde{A}$ be a lift to its universal covering space. Then the *mean translation number* $\tau_\mu(\tilde{f})$ is

$$\int_X \tau_{\tilde{f}}(x)\, d\mu,$$

where $X \subset \widetilde{A}$ is a fundamental domain for the universal cover. The *mean rotation number* $\rho_\mu(\widetilde{f})$ is the coset of $\tau_\mu(\widetilde{f})$ in $\mathbb{R}/\mathbb{Z}$.

**Proposition 2.6** *Suppose $f\colon A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity. If $\rho_\mu(f) = 0$ then $f$ has a fixed point in the interior of $A$.*

**Proof** Let $\widetilde{f}\colon \widetilde{A} \to \widetilde{A}$ be the lift of $f$ such that $\tau_\mu(\widetilde{f}) = 0$. If $\tau_{\widetilde{f}}$ vanishes on a set of positive measure then Proposition 2.4 gives the result. Otherwise there is a set $Y^+$ (respectively $Y^-$) with positive measure on which $\tau_{\widetilde{f}}$ is positive (respectively negative). It follows that there is a birecurrent point $x^+ \in \mathrm{int}(A)$ (respectively $x^- \in \mathrm{int}(A)$) with a positive (respectively negative) translation number. A small free disk $D^+$ containing $x^+$ will be a positively recurring disk and similarly there is a negatively recurring free disk $D^-$ containing $x^-$. [7, Theorem 2.1] then implies the existence of a fixed point for $f$ in the interior of $A$. □

**Notation 2.7** Suppose $U \subset S^2$ is an open $f$–invariant annulus. We would like to compactify $U$ to a closed annulus for which $f$ has a natural extension. The annulus $U$ has two ends which we compactify separately in a way depending on the nature of the end. We say that an end of $U$ is *singular* if the component of the complement of $U$ in $S^2$ that it determines is a single, necessarily fixed, point $x \in S^2$. In this case we compactify that end by blowing up $x$ to obtain a circle on which $f$ acts by the projectivization of $Df_x$. If the end is not singular we will take the prime end compactification (see Mather [23] for properties). In either case we obtain a closed annulus $U_c$ whose interior is naturally identified with $U$ in such a way that $f|_U$ extends to a homeomorphism $f_c\colon U_c \to U_c$.

We will call $U_c$ the *annular compactification* of $U$ and $f_c\colon U_c \to U_c$ the annular compactification of $f|_U$. If there is no ambiguity about the choice of $f$ we will denote the rotation interval $\mathcal{R}(f_c)$ by $\rho(U)$ and the two rotation numbers of the restriction of $f_c$ to its boundary circles by $\rho(\partial U_c)$.

**Lemma 2.8** *Let $f$ be an area preserving diffeomorphism of a compact surface. Suppose $U$ is an open $f$–invariant annulus and $f_c\colon U_c \to U_c$ is the extension of $f$ to its annular compactification.*

(1) *If there is a point $x \in U_c$ with $\rho_{f_c}(x) = 0$ then* $\mathrm{Fix}(f_c) \neq \emptyset$.

*If $\overline{X}$ is the component of the frontier of $U$ corresponding to a component $X$ of $\partial U_c$ then:*

(2) *If* $\mathrm{Fix}(f_c|_X) \neq \emptyset$ *then* $\mathrm{Fix}(f|_{\overline{X}}) \neq \emptyset$.

(3) *If $\overline{X} \subset \mathrm{Fix}(f)$ and $\overline{X}$ contains more than one point then $X \subset \mathrm{Fix}(f_c)$.*

**Proof** (1) follows from Theorem 2.3.

For (2), suppose that $\mathrm{Fix}(f_c|_X) \neq \varnothing$ and note that $\overline{X}$ is $f$–invariant. If $\overline{X}$ is a single point then (2) is obvious so we may assume that $\overline{X}$ has more than one point. Thus $X$ is the prime end compactification and each prime end $x \in X$ is defined by a sequence of "cross-cuts" $\{\gamma_n\}$ where each $\gamma_n$ is a Jordan arc whose interior is in $U$ and whose endpoints are in the frontier of $U$. They satisfy:

(a) $\lim_{n\to\infty} \mathrm{diam}(\gamma_n) = 0$.

(b) Each $\gamma_n$ has two complementary components in $U$, one of which is an annulus and the other of which is an open disk which we will denote $D_n$.

(c) The disk $D_{n+1}$ is a subset of $D_n$ and $\bigcap_n D_n = \varnothing$.

Two such sequences of cross-cuts $\{\gamma_n\}$ and $\{\gamma'_m\}$ determine the same prime end if for each $n$ there is an $m$ with $D'_m \subset D_n$ and for each $m$ there is an $n$ with $D_n \subset D'_m$.

Let $\{\gamma_n\}$ determine a prime end in $X$ which is fixed by $f_c$. Then from the fact that $f$ preserves area it follows that $f(\gamma_n) \cap \gamma_n \neq \varnothing$. For $n \geq 1$ choose $x_n \in \mathrm{int}(\gamma_n)$. From property (a) above it follows that any point in the limit set of the sequence $\{x_n\}$ is a fixed point of $f$. It is clearly in $\overline{X}$. This completes the proof of (2).

For (3) suppose that $\overline{X} \subset \mathrm{Fix}(f)$. By [16, Lemma 4.1] there is an isotopy rel Fix$(f)$ from $f$ to a diffeomorphism $f'$ that is the identity on a neighborhood of $\mathrm{Fix}(f)$. By [23, Theorem 18], $f_c|_X = f'_c|_X$, which is obviously the identity. □

**Corollary 2.9** *Let $\mathcal{G}$ be a group of area preserving diffeomorphisms of $S^2$ or the closed disk $D^2$. Suppose $U$ is an open $\mathcal{G}$–invariant annulus and $\mathcal{G}_c$ is the group of homeomorphisms $g_c\colon U_c \to U_c$ that are annular compactifications of the elements $g \in \mathcal{G}$. If there is a point $x \in \mathrm{Fix}(\mathcal{G}_c)$ then $\mathrm{cl}(U)$ contains a point $\bar{x}$ of $\mathrm{Fix}(\mathcal{G})$. If $x$ lies in the component $X$ of $\partial U_c$ corresponding to a component $\overline{X}$ of the frontier of $U$ then $\bar{x} \in \overline{X}$.*

**Proof** If $x$ is a point of $\mathrm{Fix}(\mathcal{G}_c)$ and $x \in \mathrm{int}(U_c) = U$ we are done. So we may assume it is in a boundary component $X$ of $U_c$. If $X$ corresponds to a singular end of $U$ then the point corresponding to that end is in $\mathrm{cl}(U) \cap \mathrm{Fix}(\mathcal{G})$. Otherwise $X$ is the prime end compactification of an end of $U$. Let $\{\gamma_n\}$ be a sequence of cross-cuts that determine a prime end in $X$ which is in $\mathrm{Fix}(\mathcal{G}_c)$. Then, as in the previous lemma, the fact that each $g \in \mathcal{G}$ preserves area implies that $g(\gamma_n) \cap \gamma_n \neq \varnothing$. Also as in the previous lemma we may choose $\gamma_n$ so that $\lim_{n\to\infty} \mathrm{diam}(\gamma_n) = 0$. For $n \geq 1$ let $x_n \in \mathrm{int}(\gamma_n)$.

It follows that any point in the limit set of the sequence $\{x_n\}$ is in $\mathrm{Fix}(g)$. Since this is independent of the choice of $g \in \mathcal{G}$ it follows that any point in the limit set is in $\mathrm{Fix}(\mathcal{G})$. □

**Proposition 2.10** *Suppose $f\colon A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and suppose every point of $A$ has the same forward rotation number. Let $U = \mathrm{int}(A)$. Then either $f$ has a fixed point in $U$ or every point of $U$ is free disk recurrent for $f|_U$.*

**Proof** If the forward rotation number of all points of $A$ is $0$, then Proposition 2.4 implies that $f$ has a fixed point in $U$. Hence we may assume the common rotation number of the points of $A$ is nonzero and consequently $\mathrm{Fix}(f) = \varnothing$. Suppose $x \in U$ and $z \in \omega(x) \subset A$. If $z \in U$, then any free disk containing $z$ intersects $\mathrm{orb}(x)$ in infinitely many points. If $z \in \partial A$ let $V$ be a free half disk neighborhood of $z$ in $A$ and let $V_0 = V \cap U$. Then $\mathrm{orb}(x)$ intersects $V_0$ infinitely often. □

The rotation number or rotation interval of a point $x$ in an open annulus may not be well-defined as in principle it can depend on the compactification of the annulus as well as the point. The following lemma addresses issue in the case that the orbit of $x$ lies in a compact (but not necessarily invariant) subannulus.

**Lemma 2.11** *Suppose $f_i\colon A_i \to A_i$, $i = 1, 2$ are homeomorphisms of closed annuli which are isotopic to the identity. Suppose further that $J_i\colon A_0 \to A_i$ is an essential embedding of a closed annulus in $A_i$ which is not necessarily $f_i$–invariant and for some $x \in A_0$ and all $n \in \mathbb{Z}$ we have $J_1^{-1}(f_1^n(J_1(x))) = J_2^{-1}(f_2^n(J_2(x)))$. Then the rotation interval of $J_1(x)$ with respect to $f_1$ equals the rotation interval of $J_2(x)$ with respect to $f_2$.*

**Proof** Identify $A_0$ with $S^1 \times [0,1]$ and let $p\colon A_0 \to S^1$ be projection onto the first coordinate. For $i = 1, 2$, extend $pJ_i^{-1}\colon J_i(A_0) \to S^1$ continuously to $p_i\colon A_i \to S^1$. Rotation intervals for $J_i(x)$ with respect to $f_i$ can be computed using $p_i$. The lemma therefore follows from the fact that $p_1 f_1^n(J_1(x)) = p_2 f_2^n(J_2(x))$. □

## 3 Planar topology

In this section we record and prove two useful elementary results.

Recall that by the Riemann mapping theorem, every open, unbounded, connected, simply connected subset of $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$. A closed set $X \subset \mathbb{R}^2$ is said to *separate* two subsets $A$ and $B$ of $\mathbb{R}^2$ provided $A$ and $B$ are contained in different components of $\mathbb{R}^2 \setminus X$.

**Lemma 3.1** *If $A$ and $B$ are disjoint closed connected subsets of $\mathbb{R}^2$ then they are separated by a simple closed curve or a properly embedded line.*

**Proof** Choose a smooth function $\phi\colon \mathbb{R}^2 \to [0,1]$ such that $\phi(A) = 0$ and $\phi(B) = 1$ and a regular value $c \in (0,1)$. Then $\phi^{-1}(c)$ is a countable union of properly embedded lines and simple closed curves. Each component of $\phi^{-1}(c)$ has a collar neighborhood which is disjoint from the other components.

Let $U$ denote the component of the complement of $\phi^{-1}(c)$ which contains $B$ and let $X$ denote the frontier of $U$. Then $X$ separates $A$ and $B$ and $X$ consists of a countable subcollection of the components of $\phi^{-1}(c)$, each of which is also a component of $X$. The set $U$ is the component of $\mathbb{R}^2 \setminus X$ which contains $B$. Each component $L$ of $X$ separates $\mathbb{R}^2$ into two open sets, one of which contains $B$ and $X \setminus L$ and the other of which is disjoint from $X$ and $B$.

Consider a curve $\gamma$ running from a point of $A$ to a point of $B$ and let $L_0$ be the first component of $X$ which $\gamma$ intersects. The component $L_0$ is independent of the choice of $\gamma$, since $L_0$ separates $A$ from all other components of $X$. It follows that $A$ and $B$ are in different components of the complement of $L_0$ since otherwise they could be joined by a $\gamma$ which does not intersect $L_0$. □

**Lemma 3.2** *If $U \subset \mathbb{R}^2$ is open and connected then each component $Z$ of the complement of $U$ has connected frontier and connected complement.*

**Proof** The complement of $Z$ is the union of $U$ with some of its complementary components and is therefore connected. If the frontier $W$ of $Z$ is not connected then by Lemma 3.1 there is a separation of $W$ by a set $Y \subset \mathbb{R}^2$ that is either a simple closed curve or a properly embedded line. Since each component of $\mathbb{R}^2 \setminus Y$ intersects the frontier of $Z$, each component must intersect both the interior of $Z$ and $\mathbb{R}^2 \setminus Z$. Since $Y$ is disjoint from the frontier $W$ of $Z$, it is contained in either the interior of $Z$ or in $\mathbb{R}^2 \setminus Z$. In the latter case $Y$ separates $Z$ and in the former case $Y$ separates $\mathbb{R}^2 \setminus Z$. This contradicts the fact that $Z$ and $\mathbb{R}^2 \setminus Z$ are connected and so proves that the frontier of $Z$ is connected. □

# 4 Normal form

Let $N$ be the genus zero surface obtained from $S^2$ by blowing up each element of $P$ to a boundary circle and let $\pi_P\colon N \to S^2$ be the "inverse" map that collapses each boundary component to a point in $P$. Given $F \in \mathrm{Diff}_\mu(S^2, P)$ there exists a diffeomorphism $F'\colon N \to N$ such that $F\pi_P = \pi_P F'$. Identify $N$ with a smooth subsurface of $S^2$ and extend $F'\colon N \to N$ to a diffeomorphism $G\colon S^2 \to S^2$ (This is possible by the isotopy extension theorem; see Hirsch's book [18].)

[11, Theorem 1.2] states that a diffeomorphism $G$ of a closed surface is isotopic relative to its fixed point set to a homeomorphism with certain nice properties. The special case that $G$ is isotopic to the identity is considered in Lemma 6.3 of that paper. Together, this theorem and lemma imply that for any diffeomorphism $G\colon S^2 \to S^2$ there is a (possibly empty) finite set $\mathcal{R}_G$ of disjoint simple closed curves in $S^2 \setminus \mathrm{Fix}(G)$ and a homeomorphism $G_1\colon S^2 \to S^2$ that is isotopic to $G$ rel $\mathrm{Fix}(G)$ such that:

- $(1_G)$ There are disjoint open $G_1$–invariant annulus neighborhoods $A_j \subset S^2 \setminus \mathrm{Fix}(G)$ of the elements $\gamma_j \in \mathcal{R}_G$.
- $(2_G)$ Each component $C_i$ of $S^2 \setminus \bigcup A_j$ is $G_1$–invariant. Moreover, if $G_1|_{C_i} \neq$ identity then $C_i \cap \mathrm{Fix}(G)$ is finite and $G_1|_{C_i}$ is pseudo-Anosov relative to $C_i \cap \mathrm{Fix}(G)$.

After removing extraneous elements of $\mathcal{R}_G$ if necessary we may also assume:

- $(3_G)$ The elements of $\mathcal{R}_G$ are essential, nonperipheral and nonparallel in $S^2 \setminus \mathrm{Fix}(G)$. For each $A_l$, if the restriction of $G_1$ to each component of $S^2 \setminus \bigcup A_j$ that is adjacent to $A_l$ is the identity, then $G|_{A_l}$ is a nontrivial Dehn twist.

Any simple closed curve in $S^2$ that is fixed up to isotopy rel $\mathrm{Fix}(G)$ is isotopic rel $\mathrm{Fix}(G)$ into one of the $A_j$ or into one of the $C_i$ on which $G$ restricts to the identity. Applying this to the components of $\partial N$, there is a diffeomorphism $H\colon S^2 \to S^2$ that is isotopic to the identity rel $\mathrm{Fix}(G)$ and satisfies $H(\mathcal{R}_G) \cap \partial N = \varnothing$. After replacing $G_1$ with $HG_1H^{-1}$, we may assume that each component of $\partial N$ is contained in an $A_j$ or in a $C_i$ on which $G_1$ restricts to the identity. After an isotopy of $G_1$ we may assume that $G_1$ restricts to the identity on $\partial N$ and hence that items (1) and (2) above hold when $\mathcal{R}_G$ is replaced by $\mathcal{R}_G \cup \partial N$. Let $F'_1 = G_1|_N$ and let $\mathcal{R}_{F'}$ be the set of simple closed curves in $N$ obtained from $\mathcal{R}_G \cap N$ by removing all peripheral elements. Then:

- $(1_{F'})$ There are disjoint open $F'_1$–invariant annulus neighborhoods
$$A_j \subset N \setminus (\partial N \cup \mathrm{Fix}(F'))$$
of the elements $\gamma_j \in \mathcal{R}_{F'}$.
- $(2_{F'})$ Each component $C_i$ of $N \setminus \bigcup A_j$ is $F'_1$–invariant. Moreover, if $F'_1|_{C_i} \neq$ identity then $C_i \cap \mathrm{Fix}(F')$ is finite and $F'_1|_{C_i}$ is pseudo-Anosov relative to $C_i \cap \mathrm{Fix}(F')$.
- $(3_{F'})$ The elements of $\mathcal{R}_{F'}$ are essential, nonperipheral and nonparallel in the space $N \setminus (\partial N \cup \mathrm{Fix}(F'))$. For each $A_l$, if the restriction of $F'_1$ to each component of $N \setminus \bigcup A_j$ that is adjacent to $A_l$ is the identity, then $F'_1|_{A_l}$ is a nontrivial Dehn twist.

Let $X = \bigcup(C_i \cap \mathrm{Fix}(F'))$ where the union is taken over those $C_i$ for which $F_1'|_{C_i}$ is not the identity. Blow up each element of $X$ to a boundary circle forming a new compact surface $N^*$ and let $F^*$ and $F_1^*$ be the diffeomorphisms of $N^*$ induced by $F'$ and $F_1'$ respectively. Then $F'$ and $F_1'$ are isotopic and $F_1^*$ is in Thurston canonical form because the nonidentity components are now pseudo-Anosov instead of pseudo-Anosov relative to a finite set of fixed points. If there are any pseudo-Anosov components, then the action of $F_1^*$, and hence $F^*$ on the fundamental group of $N^*$ has exponential growth (see [5, Exposé 11, Section V, 5.1]). In this case, [1, Theorem 1] implies that $F^*$, and hence $F'$, and hence $F$, has positive entropy. This contradiction implies that $F_1'|_{C_i}$ is the identity for each $C_i$. Thus $N \setminus \bigcup A_j \subset \mathrm{Fix}(F_1')$ and we may assume that $F_1'|_{A_j}$ is a nontrivial Dehn twist about $\gamma_j$ for each $\gamma_j \in \mathcal{R}_{F'}$.

Projecting via $\pi_P$ to $S^2$ we have shown that there is a finite collection $\mathcal{R}_F$ of essential, nonperipheral, nonparallel simple closed curves in $S^2 \setminus \mathrm{Fix}(F)$ such that $F$ is isotopic rel $\mathrm{Fix}(F)$ to a composition of nontrivial Dehn twists in the elements of $\mathcal{R}$.

A result of Brown and Kister [2] implies that $F$ preserves every component of $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$. Given a component $M$ of $\mathcal{M}$, let $f = F|_M\colon M \to M$ and let $\mathcal{R}$ be the subset of $\mathcal{R}_F \cap M$ consisting of elements that are nonperipheral in $M$. If $\mathcal{R} = \mathcal{R}_F \cap M$ then $F_1|_M$ is a composition of nontrivial Dehn twists along the elements of $\mathcal{R}$. If $\mathcal{R} \neq \mathcal{R}_F \cap M$ then $F_1|_M$ is isotopic to a composition of nontrivial Dehn twists along the elements of $\mathcal{R}$. In either case, $f$ is isotopic to a composition of nontrivial Dehn twists along the elements of $\mathcal{R}$. The elements of $\mathcal{R}$ are the *reducing curves* for $f\colon M \to M$; they are nonparallel and nonperipheral.

# 5 An intermediate proposition

To clarify the logic of the proof of Theorem 1.2 we introduce Proposition 5.1 which asserts the existence of a collection $\mathcal{A}$ of annuli satisfying the second, third and fourth items of Theorem 1.2 plus two additional properties. What is missing from this proposition is the fact that the elements of $\mathcal{A}$ are exactly the components of the set $\mathcal{W}$ of weakly free disk recurrent points for $f$. The proof of this missing fact requires renormalization and so comes at a later stage of the paper.

We have stated Proposition 5.1 in terms of a single component $M$ of $\mathcal{M}$ instead of all of $\mathcal{M}$ as in Theorem 1.2. This has obvious advantages and can be done without loss.

**Proposition 5.1** *Suppose that $F \in \mathrm{Diff}_\mu(S^2, P)$ has entropy zero, has infinite order and at least three periodic points. Suppose that $M$ is a component of $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$ and that $f = F|_M\colon M \to M$. Then there is a countable collection $\mathcal{A}$ of pairwise disjoint open $f$–invariant annuli such that:*

(1) *For each compact set $X \subset M$ there is a constant $K_X$ such that any $f$–orbit that is not contained in some $U \in \mathcal{A}$ intersects $X$ in at most $K_X$ points. In particular each birecurrent point is contained in some $U \in \mathcal{A}$.*

(2) *If $z \in M$ is not contained in any element of $\mathcal{A}$ then there are distinct components $F_+(z)$ and $F_-(z)$ of $\mathrm{Fix}(F)$ so that $\omega(F,z) \subset F_+(z)$ and $\alpha(F,z) \subset F_-(z)$.*

(3) *For each $U \in \mathcal{A}$ and each component $C_M$ of the frontier of $U$ in $M$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_M$.*

(4) *If $U \in \mathcal{A}$, and $f_c \colon U_c \to U_c$ is the extension to the annular compactification (Notation 2.7) of $U$, then each component of $\partial U_c$ corresponding to a nonsingular end of $U$ contains a fixed point of $f_c$.*

(5) *$\mathcal{A}$ is the set of maximal $f$–invariant open annuli in $M$.*

Note that it is not possible for $M$ to be simply connected, since the Brouwer plane translation theorem would then assert that $F|_M$ has a fixed point in $M$. In the special case that $M$ is an annulus, $\mathcal{A}$ is the single annulus $M$. Items (1)–(3) and (5) are obvious and item (4) follows from Lemma 5.1 of [11]. The constructions and analysis needed for the case that $M$ is not an annulus are carried out in sections 7 through 13. The final formal proof of Proposition 5.1 occurs at the end of Section 13.

# 6 Hyperbolic structures

In this section we establish notation and recall standard results about hyperbolic structures on surfaces. More details can be found, for example, in Casson and Bleiler [3].

Suppose that $M$ is a connected open subset of $S^2$ that has at least three ends or equivalently is not homeomorphic to either the open disk or open annulus. We say that a simple closed curve $\tau \subset M$ is *essential* if it is not freely homotopic to a point and is *inessential* otherwise. Similarly $\tau$ is *peripheral* if it is isotopic into arbitrarily small neighborhoods of an end of $M$ and is *nonperipheral* otherwise. Thus $\tau$ is essential if and only if each complementary component contains at least one puncture and is peripheral if and only if one of its complementary components contains exactly one puncture. We say that a properly embedded line in $M$ is *essential* if it is not properly isotopic into arbitrarily small neighborhoods of an end of $M$ or equivalently if each component of its complement contains at least one puncture.

If $M$ has infinitely many ends then it can be written as an increasing union of finitely punctured compact connected subsurfaces $M_i$ whose boundary components determine essential nonperipheral isotopy classes in $M$. We may assume that boundary curves in

$M_{i+1}$ are not parallel to boundary curves in $M_i$. It is straightforward (see [3]) to put compatible hyperbolic structures on the $M_i$ whose union defines a complete hyperbolic structure on $M$ in which all isolated punctures are cusps. Of course $M$ also has such a hyperbolic structure when it only has finitely many ends. In this paper, all hyperbolic structures are assumed to be complete and all isolated punctures are assumed to be cusps.

We use the Poincaré disk model for the hyperbolic plane $H$. In this model, $H$ is identified with the interior of the unit disk and geodesics are segments of Euclidean circles and straight lines that meet the boundary in right angles. A choice of hyperbolic structure on $M$ provides an identification of the universal cover $\widetilde{M}$ of $M$ with $H$. Under this identification, which we assume throughout this paper, covering translations of $\widetilde{M}$ are isometries of $H$ and geodesics in $M$ lift to geodesics in $H$. The compactification of the interior of the unit disk by the unit circle induces a compactification of $H$ by the "circle at infinity" $S_\infty$. Geodesics in $H$ have unique endpoints on $S_\infty$. Conversely, any pair of distinct points on $S_\infty$ are the endpoints of a unique geodesic.

Each covering translation $T\colon H \to H$ extends to a homeomorphism (also called) $T\colon H \cup S_\infty \to H \cup S_\infty$. The fixed point set of a nontrivial $T$ is either one or two points in $S_\infty$. We denote these point(s) by $T^+$ and $T^-$, allowing the possibility that $T^+ = T^-$. If $T^+ = T^-$, then $T$ is said to be *parabolic*; a root-free parabolic covering translation with fixed point $P$ is sometimes written $T_P$. If $T^+$ and $T^-$ are distinct, then $T$ is said to be *hyperbolic* and we assume that $T^+$ is a sink and $T^-$ is a source; the unoriented geodesic connecting $T^-$ and $T^+$ is called the *axis* of $T$. A root-free covering translation with axis $\tilde{\gamma}$ is sometimes denoted $T_{\tilde{\gamma}}$.

Each essential nonperipheral simple closed curve $\tau' \subset M$ is homotopic to a unique closed geodesic $\tau$. For each lift $\tilde{\tau}' \subset H$, the homotopy between $\tau'$ and $\tau$ lifts to a bounded homotopy between $\tilde{\tau}'$ and a lift $\tilde{\tau}$ of $\tau$ which is the axis of a hyperbolic covering translation $T$. The ends of both lines $\tilde{\tau}'$ and $\tilde{\tau}$ converge to $T^-$ and $T^+$.

Similarly, both ends of a lift $\tilde{\tau}'$ of a peripheral simple closed curve $\tau'$ converge to a point that is the unique fixed point of a parabolic covering translation; roughly speaking, this fixed point is a lift of the isolated puncture of $M$ that is encircled by $\tau$. Conversely, if $\tau$ is peripheral and $\tilde{\tau}$ is a sufficiently small horocycle based at $P$ then the image $\tau \subset M$, which we call a *horocycle in* $M$, is a peripheral simple closed curve. Each simple closed peripheral curve in $M$ is isotopic to a (nonunique) horocycle in $M$. Each essential properly embedded line in $M$ is properly isotopic to a unique properly embedded geodesic line.

Suppose now that $f\colon M \to M$ is a homeomorphism. If $f\colon M \to M$ and $g\colon M \to M$ are isotopic and $\tilde{f}\colon H \to H$ is a lift of $f\colon M \to M$, then the isotopy between $f$ and

$g$ lifts to an isotopy between $\widetilde{f}\colon H \to H$ and a lift $\widetilde{g}\colon H \to H$ of $g\colon M \to M$; we say that $\widetilde{f}$ and $\widetilde{g}$ are equivariantly isotopic. A proof of the following fundamental result of Nielsen theory appears in [17, Proposition 3.1].

**Proposition 6.1** *Every lift $\widetilde{f}\colon H \to H$ extends uniquely to a homeomorphism (also called) $\widetilde{f}\colon H \cup S_\infty \to H \cup S_\infty$. If $\widetilde{f}$ and $\widetilde{g}$ are equivariantly isotopic lifts of $f\colon M \to M$ and $g\colon M \to M$ then $\widetilde{f}|_{S_\infty} = \widetilde{g}|_{S_\infty}$.*

For any extended lift $\widetilde{f}\colon H \cup S_\infty \to H \cup S_\infty$ there is an *associated action $\widetilde{f}_\#$ on geodesics in $H$* defined by sending the geodesic with endpoints $P$ and $Q$ to the geodesic with endpoints $\widetilde{f}(P)$ and $\widetilde{f}(Q)$. The action $\widetilde{f}_\#$ projects to an *action $f_\#$ on geodesics in $M$*. Proposition 6.1 implies that $f_\#$ depends only on the isotopy class of $f$. Similarly, if $P$ is the unique fixed point of the parabolic covering translation $T$ then $\widetilde{f}(P)$ is the unique fixed point of the parabolic covering translation $\widetilde{f}T\widetilde{f}^{-1}$. There is an induced an action $f_\#$ on isotopy classes of simple closed peripheral curves in $M$ that agrees with the induced action of $f$ on isolated punctures in $M$. Note that if a geodesic or isotopy class of a simple closed peripheral curve is equipped with an orientation then its image under $f_\#$ has a well-defined induced orientation.

The following results are well-known and follow easily from the definitions.

**Lemma 6.2** (1) *If $\tau_1'$ and $\tau_2'$ are essential simple closed curves isotopic to geodesics $\tau_1$ and $\tau_2$ respectively, then $f(\tau_1')$ is isotopic to $\tau_2'$ if and only if $f_\#(\tau_1) = \tau_2$.*

(2) *If $\gamma_1'$ and $\gamma_2'$ are properly embedded lines properly isotopic to geodesics $\gamma_1$ and $\gamma_2$ respectively, then $f(\gamma_1')$ is properly isotopic to $\gamma_2'$ if and only if $f_\#(\gamma_1) = \gamma_2$.*

(3) *If $\tau_1'$ and $\tau_2'$ are simple closed peripheral curves encircling the punctures $p_1$ and $p_2$ respectively, then $f(\tau_1')$ is isotopic to $\tau_2'$ if and only if $f(p_1) = p_2$.*

**Lemma 6.3** *For any extended lift $\widetilde{f}\colon H \cup S_\infty \to H \cup S_\infty$ and extended covering translation $T\colon H \cup S_\infty \to H \cup S_\infty$, the following are equivalent:*

(1) *$\widetilde{f}$ commutes with $T$.*

(2) *$\widetilde{f}$ fixes $T^+$ or $T^-$.*

(3) *$\widetilde{f}$ fixes $T^+$ and $T^-$.*

**Proof** (3) $\Longrightarrow$ (2) is obvious. If $\widetilde{f}$ commutes with $T$ then it preserves $\mathrm{Fix}(T)$ mapping sources to sources and sinks to sinks. Thus (1) $\Longrightarrow$ (3). If $\widetilde{f}$ fixes an element of $\mathrm{Fix}(T)$ then $T$ and $\widetilde{f}T\widetilde{f}^{-1}$ are covering translations whose axes are asymptotic. Since these axes are periodic, they are equal and so $\widetilde{f}$ fixes both elements of $\mathrm{Fix}(T)$. Thus (2) $\Longrightarrow$ (3). □

We conclude with a definition and lemma about isotopy of families of lines.

Suppose that $\rho$ and $\sigma$ are essential properly embedded lines in $M$. We say that $\rho$ and $\sigma$ have *geodesic-like or minimal intersections* if they intersect transversely and if each component of $M \setminus (\rho \cup \sigma)$ whose frontier is the union of an interval $I \subset \sigma$ and an interval $J \subset \rho$ contains at least one puncture.

Note that:

- If $\rho$ and $\sigma$ are geodesics with respect to some hyperbolic structure on $M$ then $\rho$ and $\sigma$ have geodesic-like intersections.
- If $\rho$ and $\sigma$ have geodesic-like intersections and $h\colon M \to M$ is any homeomorphisms then $h(\rho)$ and $h(\sigma)$ have geodesic-like intersections.

**Lemma 6.4** (1) *If $\mathcal{E}$ is a locally finite collection of disjoint essential properly embedded lines in $M$ that determine distinct proper isotopy classes, then the elements of $\mathcal{E}$ are simultaneously isotopic to their associated geodesics; ie there is a homeomorphism $g\colon M \to M$, isotopic to the identity, such that $g(\rho)$ is geodesic for each $\rho \in \mathcal{E}$. If the elements of $\mathcal{E}$ are smoothly embedded then we may take $g$ to be a diffeomorphism.*

(2) *Suppose that $\mathcal{E}$ and $\mathcal{L}$ are locally finite collections of disjoint essential properly embedded lines that determine distinct proper isotopy classes. Suppose further that each element of $\mathcal{L}$ is geodesic and that each element of $\mathcal{E}$ has minimal intersections with each element of $\mathcal{L}$. Then there exists a diffeomorphism $g\colon M \to M$, isotopic to the identity, that preserves $\mathcal{L}$ and such that $g(\rho)$ is geodesic for each $\rho \in \mathcal{E}$.*

**Proof** The proofs of [3, Lemmas 2.5 and 2.6] can be modified in a straightforward manner to prove this lemma. The details are left to the reader. □

**Remark 6.5** We will use the first two parts of Lemma 6.4 to modify metrics so that certain given lines are geodesics in their isotopy classes. The key observation is that if $\mu$ is a hyperbolic metric on $M$ and $g\colon M \to M$ is a diffeomorphism then $\nu = g^*\mu$ is a hyperbolic metric on $M$ and a line $\ell$ is geodesic in $\nu$ if and only if $g(\ell)$ is geodesic in $\mu$.

## 7 The endpoint maps $\tilde{\alpha}$ and $\tilde{\omega}$ and annular covers

In this section we begin the proof of Proposition 5.1 in the case that $M$ has at least three ends. (The annulus case was considered following the statement of the proposition.)

Equip $M$ with a complete hyperbolic structure and identify the universal cover $\widetilde{M}$ with the hyperbolic disk $H$ as described in Section 6. Recall from Section 4 that $f$ is isotopic to a homeomorphism $\phi\colon M \to M$ that is supported on a finite union of disjoint annuli and that restricts to a nontrivial Dehn twist on each annulus. We may assume without loss that the core curves of these annuli, which make up the set $\mathcal{R}$ of reducing curves for $f\colon M \to M$, are geodesics. The full preimage in $H$ of $\mathcal{R}$ is denoted $\widetilde{\mathcal{R}}$. The closure of a component of $H \setminus \widetilde{\mathcal{R}}$ in $H$ is called a *domain*. If $\mathcal{R} = \emptyset$ then $H$ is the unique domain but otherwise there are infinitely many domains. The frontier of a domain is a union of elements of $\widetilde{\mathcal{R}}$. If $\mathcal{R} \neq \emptyset$ then the closure of a domain in $H \cup S_\infty$ intersects $S_\infty$ in a Cantor set. The image of the interior of a domain under projection to $M$ is a component of $M \setminus \mathcal{R}$.

For each domain $\widetilde{C}$ let $\widetilde{\phi}_{\widetilde{C}}$ be the lift of $\phi$ whose restriction to $\widetilde{C}$ is the identity outside of a product neighborhood of the frontier. If $\widetilde{C}_1$ and $\widetilde{C}_2$ are adjacent domains that intersect in a common frontier component $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ then $\widetilde{\phi}_{\widetilde{C}_1} = T_{\widetilde{\sigma}}^d \widetilde{\phi}_{\widetilde{C}_2}$ where $T_{\widetilde{\sigma}}$ is a root-free covering translation with axis $\widetilde{\sigma}$ and $|d| > 0$ is the degree of the Dehn twist of $\phi$ around $\sigma$. It is well-known, and straightforward to check, that a point $P \in S_\infty$ is fixed by $\widetilde{\phi}_{\widetilde{C}}$ if and only if it is contained in the closure of $\widetilde{C}$. Thus $\mathrm{Fix}(\widetilde{\phi}_{\widetilde{C}}|_{S_\infty})$ is a Cantor set if $\mathcal{R} \neq \emptyset$ and is all of $S_\infty$ if $\mathcal{R} = \emptyset$.

Lifting an isotopy between $f$ and $\phi$ induces a bijection between the set of lifts $\widetilde{f}$ of $f$ and the set of lifts $\widetilde{\phi}$ of $\phi$. Thus $\widetilde{f} \leftrightarrow \widetilde{\phi}$ if and only if $\widetilde{f}$ is equivariantly isotopic to $\widetilde{\phi}$. For each domain $\widetilde{C}$ let $\widetilde{f}_{\widetilde{C}}$ be the lift of $f$ corresponding to $\widetilde{\phi}_{\widetilde{C}}$. By Proposition 6.1, $\widetilde{f}_{\widetilde{C}}|_{S_\infty} = \widetilde{\phi}_{\widetilde{C}}|_{S_\infty}$ and so $\mathrm{Fix}(\widetilde{f}_{\widetilde{C}}|_{S_\infty})$ is equal to the intersection of the closure of $\widetilde{C}$ with $S_\infty$.

The subgroup of covering translations that preserves a domain $\widetilde{C}$ is denoted $\mathrm{Stab}(\widetilde{C})$ and called the *stabilizer* of $\widetilde{C}$. A covering translation $T$ is contained in $\mathrm{Stab}(\widetilde{C})$ if and only $\{T^\pm\}$ is contained in the closure of $\widetilde{C}$ (which is also equivalent to the axis of $T$ being contained in $\widetilde{C}$). Lemma 6.3 implies that $T \in \mathrm{Stab}(\widetilde{C})$ if and only if $T$ commutes with $\widetilde{f}_C$.

**Lemma 7.1** *For each lift $\widetilde{f}$ of $f$ and each $\widetilde{x} \in H$, $\alpha(\widetilde{f}, \widetilde{x})$ and $\omega(\widetilde{f}, \widetilde{x})$ are single points in $S_\infty \cap \mathrm{Fix}(\widetilde{f})$.*

**Proof** The Brouwer translation theorem implies that $\omega(\widetilde{f}, \widetilde{x}) \subset S_\infty$. We assume that $\omega(\widetilde{f}, \widetilde{x})$ is not a single point and argue to a contradiction. It must be the case that $\omega(\widetilde{f}, \widetilde{x}) \subset S_\infty \cap \mathrm{Fix}(\widetilde{f})$. If not, a nonfixed point $z \in \omega(\widetilde{f}, \widetilde{x})$ would have a free neighborhood whose intersection with $H$ would be a free disk visited by the orbit of $\widetilde{x}$ more than once (indeed infinitely often). According to [7, Proposition 1.3] this implies

$\widetilde{f}$ has a fixed point in $H$; a contradiction. Since $\omega(\widetilde{f},\widetilde{x})$ consists of fixed points it is straightforward to see that it is also connected.

If $\mathrm{Fix}(\widetilde{f})$ does not contain an interval we are done. Otherwise, Lemma 6.3 implies that every covering translation with one endpoint in this interval commutes with $\widetilde{f}$ and so preserves $\mathrm{Fix}(\widetilde{f})$. It follows that $\mathrm{Fix}(\widetilde{f}) = S_\infty$ and so $f$ is isotopic to the identity. A proof of the lemma in this special case is given in [11, Proposition 9.1]. □

In addition to lifts of $f$ to the universal cover $H$ we will also use lifts of $f$ to infinite cyclic covers.

**Definitions 7.2** Suppose that $\sigma$ is a closed geodesic that is either equal to an element of $\mathcal{R}$ or disjoint from every element of $\mathcal{R}$. For each lift $\widetilde{\sigma}$, let $T_{\widetilde{\sigma}}$ be a root free covering translation with axis $\widetilde{\sigma}$. Choose a domain $\widetilde{C}$ that contains $\widetilde{\sigma}$. (If $\sigma \in \mathcal{R}$ then there are two choices but otherwise there is just one.) Since $\widetilde{f}_{\widetilde{C}}$ fixes the ends of $\widetilde{\sigma}$, it commutes with $T_{\widetilde{\sigma}}$ by Lemma 6.3. The *annular cover* $A_\sigma$ is the closed annulus that is the quotient space of $(H \cup S_\infty) \setminus T^{\pm}_{\widetilde{\sigma}}$ by the action of $T_{\widetilde{\sigma}}$ and $f_\sigma\colon A_\sigma \to A_\sigma$ is the homeomorphism induced by $\widetilde{f}_{\widetilde{C}}$. For $\widetilde{x} \in H$ a lift of $x \in M$, we denote the image of $\widetilde{x}$ in $A_\sigma$ by $\hat{x}$. If $\alpha(\widetilde{f}_{\widetilde{C}},\widetilde{x})$ is not an endpoint of $\widetilde{\sigma}$ then $\alpha(f_\sigma,\hat{x})$ is a single point in $\partial A_\sigma$ and similarly for $\omega(f_\sigma,\hat{x})$.

Similarly, if $\widetilde{\sigma}$ is a lift of an embedded horocycle $\sigma \subset M$ then both ends of $\widetilde{\sigma}$ converge to a point $P \in S_\infty$ and there is a root free covering translation $T_P$ that preserves $\widetilde{\sigma}$. Let $\widetilde{C}$ be the unique domain that contains $\widetilde{\sigma}$. In this case, the *annular cover* $A_\sigma$ is the half-open annulus that is the quotient space of $(H \cup S_\infty) \setminus P$ by the action of $T_P$ and the boundary is a single circle denoted $\partial A_\sigma$. As in the previous case, $\widetilde{f}_{\widetilde{C}}$ induces a homeomorphism $f_\sigma\colon A_\sigma \to A_\sigma$. The end of $A_\sigma$ corresponding to $P$ projects homeomorphically to the end of $M$ circumscribed by $\sigma$. We can compactify this end exactly as in Notation 2.7 to form a closed annulus $A^c_\sigma$. There is an extension of $f_\sigma$ (also called $f_\sigma$) to a homeomorphism of $A^c_\sigma$.

As the notation suggests, $f_\sigma$ is independent of the choice of $\widetilde{C}$ and, up to conjugacy, the choice of lift $\widetilde{\sigma}$. The former follows from the fact that if $\widetilde{C}_1$ and $\widetilde{C}_2$ contain $\widetilde{\sigma}$ then $\widetilde{f}_{\widetilde{C}_1}$ and $\widetilde{f}_{\widetilde{C}_2}$ differ by an iterate of $T_{\widetilde{\sigma}}$ and the latter follows from the fact that if $\widetilde{\sigma}$ is replaced with $S(\widetilde{\sigma})$ for some covering translation $S$ then $\widetilde{C}$ is replaced by $S(\widetilde{C})$ and $T_{\widetilde{\sigma}}$ is replaced by $ST_{\widetilde{\sigma}}S^{-1}$.

**Lemma 7.3** *Suppose that $\sigma$ is a horocycle or a closed geodesic that is either equal to an element of $\mathcal{R}$ or disjoint from every element of $\mathcal{R}$.*

(1) *For each closed geodesic* $\sigma$, $\mathrm{Fix}(f_\sigma|_{\partial A_\sigma})$ *intersects both components of* $\partial A_\sigma$. *If* $\sigma \in \mathcal{R}$ *then* $f_\sigma$ *is isotopic* $\mathrm{rel}\,\mathrm{Fix}(f_\sigma|_{\partial A_\sigma})$ *to a Dehn twist of the same index that* $f$ *twists around* $\sigma$. *If* $\sigma \notin \mathcal{R}$ *then* $f_\sigma$ *is isotopic* $\mathrm{rel}\,\mathrm{Fix}(f_\sigma|_{\partial A_\sigma})$ *to the identity.*

(2) *For each horocycle* $\sigma$, $\mathrm{Fix}(f_\sigma|_{\partial A_\sigma}) \neq \varnothing$.

**Proof** Suppose at first that $\tilde{\sigma}$ is a lift of the closed geodesic $\sigma$.

If $\sigma \notin \mathcal{R}$ then the closure of the domain $\tilde{C}$ that contains $\tilde{\sigma}$ intersects both components of $S_\infty \setminus \tilde{\sigma}^{\pm}$. The points in this intersection are fixed by $\tilde{f}_{\tilde{C}}$ and project to fixed points $\hat{x}, \hat{y}$ for $f_\sigma$ in different components of $\partial A_\sigma$. A geodesic $\tilde{\alpha}$ connecting $\tilde{x}$ to $\tilde{y}$ in $H$ projects to the interior of an embedded arc $\hat{\alpha}$ connecting $\hat{x}$ to $\hat{y}$ in $A_\sigma$ such that $f_\sigma(\hat{\alpha})$ is homotopic to $\hat{\alpha}$ rel endpoints. It follows that $f_\sigma$ is isotopic $\mathrm{rel}\,\mathrm{Fix}(f_\sigma|_{\partial A_\sigma})$ to the identity.

If $\sigma \in \mathcal{R}$ and $\tilde{C}_1$ and $\tilde{C}_2$ are the domains that contain $\tilde{\sigma}$ then points in the intersection of the closure of $\tilde{C}_1$ with $S_\infty$ are fixed by $\tilde{f}_{\tilde{C}_1}$ and project to fixed points for $f_\sigma$ in one component of $\partial A_\sigma$ and points in the intersection of the closure of $\tilde{C}_2$ with $S_\infty$ are fixed by $\tilde{f}_{\tilde{C}_2}$ and project to fixed points for $f_\sigma$ in the other component of $\partial A_\sigma$. If $f$ twists with degree $k$ around $\sigma$ then $\tilde{f}_{\tilde{C}_1}$ and $\tilde{f}_{\tilde{C}_2}$ differ by $T^k_{\tilde{\sigma}}$ so $f_\sigma$ is isotopic $\mathrm{rel}\,\mathrm{Fix}(f_\sigma|_{\partial A_\sigma})$ to a Dehn twist of index $k$. This completes the proof of (1).

The proof for (2) is similar. □

## 8 Reducing arcs in annular covers

In this section we recall, adapt and improve definitions and results from [11, Section 10], where the assumption is that $F$ is periodic point free and isotopic $\mathrm{rel}\,\mathrm{Fix}(F)$ to the identity as opposed to our current assumption that $F$ has zero entropy and is isotopic $\mathrm{rel}\,\mathrm{Fix}(F)$ to a composition of Dehn twists on the elements of $\mathcal{R}$. In particular, the homeomorphisms $f_\sigma\colon A_\sigma \to A_\sigma$ of Definitions 7.2 are periodic point free in [11] and are only fixed point free in our current context. Switching from periodic point free to entropy zero requires a change in the proof of Lemma 8.9 but nothing more. Allowing $\mathcal{R}$ to be nonempty requires a fair amount of work, most of which is done in later sections.

Of primary interest are the homeomorphisms $f_\sigma\colon A_\sigma \to A_\sigma$ of Definitions 7.2. We frame the discussion more generally for clarity and for possible future applications.

**Notation 8.1** We assume throughout this section that $h\colon A \to A$ is a homeomorphism of the closed annulus $A$ that is isotopic to the identity and whose restriction to the interior $A^\circ$ of $A$ is fixed point free and that $x_1, \dots, x_r$ are points in $A^\circ$ whose $\alpha$–limit sets $\alpha(h, x_i)$ are distinct single points in $\partial A$ and whose $\omega$–limit sets $\omega(h, x_i)$ are distinct single points in $\partial A$. Let $X \subset A^\circ$ be the union of the $h$–orbits of the $x_i$ and let $A^\circ_X = A^\circ \setminus X$ be equipped with a hyperbolic structure as in Section 6.

Recall that a properly embedded line $\ell \subset A^\circ_X$ is essential if it is not properly isotopic into arbitrarily small neighborhoods of some end of $A^\circ_X$ and that each essential $\ell$ is properly isotopic to a unique geodesic. The action of $h$ on isotopy classes of properly embedded lines in $A^\circ_X$ is captured by the map $h_\#$ on geodesics defined in Section 6.

Suppose that $\ell$ is a geodesic line in $A^\circ_X$ that separates $A^\circ$ into two components, $U$ and $V$. Choose an isotopy rel $X$ from $h$ to $h'$ where $h'(\ell) = h_\#(\ell)$. The sets $h'(U)$ and $h'(V)$ are independent of the choice of $h'$ and we write $h_\#(U) = h'(U)$ and $h_\#(V) = h'(V)$. Thus, $h_\#(U)$ and $h_\#(V)$ are the components of $A^\circ \setminus h_\#(\ell)$.

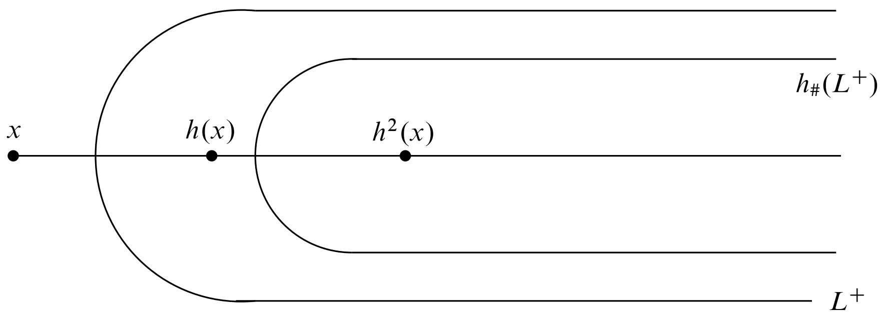
**Remark 8.2** In general, the hyperbolic metric on $A^\circ_X$ is unrelated to $\partial A$. The ends of a geodesic $\ell \subset A^\circ_X$ that is properly embedded in $A^\circ$ need not converge to single points in $\partial A$. Even if the ends of $\ell$ and $h_\#(\ell)$ converge to single points in $\partial A$, these pairs of points need not be related by $h|_{\partial A}$. We will require that our hyperbolic metrics satisfy certain extra properties (see Lemma 8.4) to guarantee some compatibility between the metric and the boundary.

If an embedded path $\beta \subset A^\circ$ has endpoints in $X$ but is otherwise disjoint from $X$ then the interior of $\beta$ determines a properly embedded line $\ell \subset A^\circ_X$. Proper isotopy of $\ell$ in $A^\circ_X$ corresponds to isotopy rel $X$ of $\beta$ in $A^\circ$. If $\ell$ is essential (respectively a geodesic) in $A^\circ_X$ then we say that $\beta$ *is essential (respectively a geodesic* rel $X$ *) in* $A^\circ$. There is an induced map $h_\#$ on geodesics rel $X$ in $A^\circ$ such that $h_\#(\beta)$ is the unique geodesic path in the isotopy class rel $X$ of $h(\beta)$.

**Definition 8.3** An arc $\beta' \subset A^\circ$ connecting $x \in X$ to $h(x)$ is called a *translation arc* for $x$ if $h(\beta') \cap \beta' = h(x)$. If $\beta'$ intersects $X$ only in its endpoints, then the geodesic rel $X$ in $A^\circ$ determined by $\beta'$ is called a *translation arc geodesic* for $x$ relative to $X$.

Assume that $\beta$ is a translation arc geodesic for $x$ relative to $X$ and let $\beta_j = h^j_\#(\beta)$, a translation arc geodesic for $h^j(x)$ relative to $X$. If $B^+ = \bigcup_{j=0}^\infty \beta_j$ is an embedded ray in $A^\circ$ that converges to $\omega(h, x)$ then we say that $\beta$ is *forward proper* with *forward homotopy streamline* $B^+$. In this case, $h_\#$ induces a self-map of $B^+$ that is conjugate to a standard translation of $[0, \infty)$ into itself.

Assume that $\beta$ is forward proper with forward homotopy streamline $B^+$ and let $L^+$ be the unique geodesic line in $A^\circ_X$ that is properly embedded in $A^\circ$ and such that one of the components, $V^+$, of $A^\circ \setminus L^+$ contains $\bigcup_{j=1}^\infty \beta_j$ and intersects $X$ exactly in $\bigcup_{j=1}^\infty h_\#^j(x)$. Topologically, $L^+$ is just the boundary of a sufficiently small regular neighborhood of $\bigcup_{j=1}^\infty \beta_j$ in $A^\circ$. Note that $h_\#(L^+)$ is the unique geodesic line in $A^\circ_X$ that is properly embedded in $A^\circ$ and such that one of the components of $A^\circ \setminus L^+$ contains $\bigcup_{j=2}^\infty \beta_j$ and intersects $X$ exactly in $\bigcup_{j=2}^\infty h_\#^j(x)$. In particular $h_\#(L^+) \subset V^+$ and $h_\#(V^+) \subset V^+$. Let $\mathrm{cl}_A(h_\#^j(V^+))$ be the closure of $h_\#^j(V^+)$ in $A$. If both ends of each $h_\#^j(L^+)$ converge to $\omega(h,x)$ and if $\bigcap_{j=0}^\infty \mathrm{cl}_A(h_\#^j(V^+)) = \omega(h,x)$ then we say that $B^+$ *has a forward translation neighborhood* and that $V^+$ is the *forward translation neighborhood* determined by $\beta$.




*Backward proper* homotopy translation arcs, *backward homotopy streamlines* $B^- = \bigcup_{j=0}^\infty h_\#^{-j}(\beta)$ and *backward translation neighborhoods* $V^-$ with boundary $L^-$ are defined similarly using $h$ instead of $h^{-1}$.

**Lemma 8.4** *Assume that $h\colon A \to A$, $x_1, \dots, x_r$, $X$ and $A^\circ_X$ are as in Notation 8.1. The hyperbolic metric on $A^\circ_X = A^\circ \setminus X$ can be chosen so that for each $1 \le i \le r$ there are translation arc geodesics $\beta_i^+$ and $\beta_i^-$ for some points, $x_i^+$ and $x_i^-$, in the orbit of $x_i$ such that:*

(1) *$\beta_i^+$ is forward proper and the forward homotopy streamline $B_i^+$ has forward translation neighborhood $V_i^+$.*

(2) *$\beta_i^-$ is backward proper and the backward homotopy streamline $B_i^-$ has backward translation neighborhood $V_i^-$.*

(3) *The $B_i^\pm$, and hence the $V_i^\pm$, are all disjoint.*

**Proof** [11, Lemma 10.6] states that there are forward proper translation arc geodesics $\beta_i^+$ and backward proper translation arc geodesics $\beta_i^-$ such that the $B_i^\pm$ are all disjoint. There are two issues that must be discussed before quoting that lemma. The first is that in the context of [11], $\mathrm{Per}(h) = \varnothing$ and $\mathcal{R} = \varnothing$. In proving [11, Lemma 10.6], the former is used only to conclude that $\mathrm{Fix}(h) = \varnothing$ and the latter is not used at all.

Since our $h$ satisfies $\mathrm{Fix}(h) = \emptyset$ by hypothesis, we are not quoting out of context. The second issue is that the role of the hyperbolic metric was not explicitly mentioned in either the statement or proof of [11, Lemma 10.6]. We attend to that now.

The accumulation set $Y \subset \partial A$ of $X$ is the union of the $\alpha$ and $\omega$ limit sets of the $x_i$. For each $y \in Y$, choose a decreasing sequence of closed half disk neighborhoods $U_i(y)$ of $y$ whose intersection is the single point $y$ and whose frontier $\partial U_i(y)$ in $A^\circ$ is disjoint from $X$. We may assume that if $y \neq y' \in Y$ then the $\partial U_i(y)$ are all disjoint from the $\partial U_j(y')$ and all these lines determine distinct proper isotopy classes in $\mathring{A}_X$. By Lemma 6.4(1), we can simultaneously isotope all of the $\partial U_i(y)$ to their associated geodesics. We may therefore modify (see Remark 6.5) the given hyperbolic metric so that all of the $\partial U_i(y)$ are geodesic. In particular, if a translation arc for an element of $X$ is contained in the interior of some $U_i(y)$ then the corresponding translation arc geodesic is also contained in the interior of $U_i(y)$.

Having chosen the metric with the above properties on translation arc geodesics, the proof of [11, Lemma 10.6] can be applied.

We must now arrange that each $B_i^+$ has forward translation neighborhoods and that each $B_i^-$ has backward translation neighborhoods. This will require a further modification of the metric. For each $J \geq 0$, choose a smooth properly embedded line $\sigma_{i,J}^+$ such that $\sigma_{i,J}^+ \cap B_i^+$ is a single point contained in $h_\#^J(\beta_i^+)$ and such that one of the two complementary components $V_J'^+$ of $\sigma_{i,J}^+$ contains $\bigcup_{j=J+1}^\infty h_\#^j(\beta_i^+)$ and intersects $X$ exactly in $\bigcup_{j=J+1}^\infty h^j(x_i)$. Assume further that both ends of each $\sigma_{i,J}^+$ converge to $\omega(h,x)$ and that $\bigcap_{J=0}^\infty \mathrm{cl}_A(V_J'^+) = \omega(h,x)$. Define $\sigma_{i,J}^-$ similarly. We may assume that all of the $\sigma_{i,J}^\pm$ are disjoint. By Lemma 6.4(2), there is an isotopy of $\mathring{A}_X$ that preserves each $B_i^\pm$ and that moves each $\sigma_{i,J}^\pm$ to the unique geodesic $L_{i,J}^\pm$ in its proper isotopy class. We may therefore change the metric so that each $\sigma_{i,J}^\pm$ is a geodesic while maintaining the property that $B_i^\pm$ is geodesic. This completes the proof of the lemma. □

Further details on the constructions in the next definition can be found in [11, Section 10].

**Definition 8.5** Assume the metric on $\mathring{A}_X$ has been chosen as in Lemma 8.4 and assume the notation of that lemma. The subsurface $W = A^\circ \setminus (X \cup (\bigcup_{i=1}^r V_i^\pm))$ is finitely punctured. We write $\partial W = \partial_+ W \cup \partial_- W$ where $\partial_\pm W = \bigcup_{i=1}^r \partial V_i^\pm$. Then $h_\#(\partial_+ W) \cap W = \emptyset$ and $\partial_- W \cap h_\#(W) = \emptyset$. We say that $W$ is the *Brouwer subsurface determined by the* $\beta_i^\pm$.

Let $RH(W, \partial_+ W)$ be the set of nontrivial relative homotopy classes $[\tau]$ determined by embedded arcs $(\tau, \partial\tau) \subset (W, \partial_+ W)$. Denote $\tau$ with its orientation reversed by $-\tau$

and $[-\tau]$ by $-[\tau]$. By a *multiset* $\mathcal{T}$ in $RH(W, \partial_+ W)$ we mean a set, each element of which is a copy of an element of $RH(W, \partial_+ W)$. The *multiplicity* of an element of $RH(W, \partial_+ W)$ in $\mathcal{T}$ is the number of copies of that element that appear in $\mathcal{T}$. An important tool in our analysis is a map that assigns to each finite multiset $\mathcal{T}$ in $RH(W, \partial_+ W)$ another finite multiset $h_\#(\mathcal{T}) \cap W$ in $RH(W, \partial_+ W)$.

Choose a homeomorphism $g\colon A^\circ \to A^\circ$ that is isotopic to $h$ rel $X$ such that $g(L) = h_\#(L)$ for each component $L$ of $\partial W$. For any arc $\tau \subset W$ with endpoints on $\partial_+ W$, $g(\tau)$ is an arc in $g(W) = h_\#(W)$ with endpoints on $h_\#(\partial_+ W)$; in particular, $g(\tau) \cap \partial_- W = \varnothing$ and $\partial g(\tau) \cap W = \varnothing$. Let $h_\#(\tau) \subset h_\#(W)$ be the geodesic arc that is isotopic rel endpoints to $g(\tau)$. The components $\tau_1, \dots, \tau_r$ of $h_\#(\tau) \cap W$ are arcs in $W$ with endpoints in $\partial_+ W$. Define $h_\#([\tau]) \cap W = \{[\tau_1], \dots, [\tau_r]\}$. It is shown in [17, pages 249–250] that $h_\#([\tau]) \cap W$ is well-defined.

More generally if $\mathcal{T}$ is a multiset in $RH(W, \partial_+ W)$ then we define $h_\#(\mathcal{T}) \cap W = \bigcup_{[\tau] \in \mathcal{T}} h_\#([\tau]) \cap W$. Note that $h_\#(\cdot) \cap W$ can be iterated. Recursively define $(h_\#)^n([\tau]) \cap W = (h_\#)^{n-1}(h_\#([\tau]) \cap W) \cap W$.

A finite multiset $\mathcal{T}$ in $RH(W, \partial_+ W)$ is a *fitted family* if:

(1) The elements of $\mathcal{T}$ are represented by disjoint simple arcs.

(2) No element of $RH(W, \partial_+ W)$ has multiplicity greater than one in $\mathcal{T}$.

(3) If $[\tau]$ has multiplicity one in $\mathcal{T}$ then $-[\tau]$ has multiplicity zero in $\mathcal{T}$.

(4) For all $n \geq 0$ and all $t \in \mathcal{T}$, each element of $h^n_\#(t) \cap W$ is, up to a change of orientation, a copy of some element of $\mathcal{T}$.

The next lemma states that one gets the same answer by either iterating the intersection operator or by first iterating $h$ and then applying the intersection operator once. Following this lemma, we will write $h^n_\#(\tau) \cap W$ for $(h_\#)^n([\tau]) \cap W = (h^n)_\#([\tau]) \cap W$.

**Lemma 8.6** *For all $\tau \in RH(W, \partial_+ W)$, $(h_\#)^n([\tau]) \cap W = (h^n)_\#([\tau]) \cap W$.*

**Proof** The statement of this lemma is the same as that of [17, Lemma 5.4]. Although the setting there is slightly different, the proof given there works here as well. □

**Notation 8.7** Assume the notation of Lemma 8.4 and Definition 8.5. Let $\mathcal{T}_i \subset RH(W, \partial_+ W)$ consist of one representative ($[\tau]$ or $-[\tau]$) of each unoriented homotopy class that is represented by a component of $h^n{}_\#(\beta_i^-) \cap W$ for some $n > 0$. The elements of $\mathcal{T}_i$ are represented by disjoint arcs. For any (not necessarily distinct) components $L_1$ and $L_2$ of $\partial W$, the number of elements of $\mathcal{T}_i$ with one endpoint on $L_1$ and the other on $L_2$ is therefore at most two plus the number of punctures in $W$. This is because a region bounded by two such elements and segments of $L_1$ and $L_2$ must contain at least one puncture. Thus $\mathcal{T}_i$ is finite. Since the fourth item in the definition of a fitted family is satisfied by construction, $\mathcal{T}_i$ is a fitted family. We say that $\mathcal{T}_i$ is the *fitted family determined by* $\beta_i^-$. The fitted family determined by $\beta_i^+$ is defined similarly.

In Section 4 we used the assumption that $F\colon S^2 \to S^2$ is smooth and has zero entropy to conclude that $f\colon M \to M$ is isotopic to a composition of Dehn twists along disjoint simple closed curves. Lemma 8.9 below (cf [17, Theorem 5.5(b)]) is the only other place in which smoothness and the entropy zero hypothesis are applied.

**Notation 8.8** We say that an element $[\tau] \in RH(W, \partial_+ W)$ *eventually doubles* if there exists $n > 0$ so that $h_\#^n([\tau]) \cap W$ contains $[\tau]$ with multiplicity at least two.

For the rest of the section we will assume that no element of $RH(W, \partial_+ W)$ eventually doubles. Before deducing implications of this assumption we show that it is satisfied by our primary examples.

Recall that $F \in \mathrm{Diff}_\mu(S^2, P)$ has entropy zero, that $M$ is a component of $S^2 \setminus \mathrm{Fix}(F)$ and that $f = F|_M$. Recall also that $F'\colon N \to N$ is a $C^\infty$ diffeomorphism of a closed genus zero surface, that $\pi_P\colon N \to S^2$ collapses components of $\partial N$ to points in $P$ and that $\pi_P F' = F \pi_P$. In particular $F'$ has zero entropy. If $\sigma$ is a horocycle or closed geodesic that is either equal to an element of $\mathcal{R}$ or disjoint from every element of $\mathcal{R}$ then $f_\sigma\colon A_\sigma \to A_\sigma$ (respectively $f_\sigma\colon A_\sigma^c \to A_\sigma^c$) is the homeomorphism of the closed annulus given in Definitions 7.2.

**Lemma 8.9** *Assume that $h = f_\sigma\colon A_\sigma \to A_\sigma$ (respectively $f_\sigma\colon A_\sigma^c \to A_\sigma^c$) is as in Definitions 7.2 and that $W$ is as in Definition 8.5. Then no element of $RH(W, \partial_+ W)$ eventually doubles.*

Before proving the lemma we state a special case of a theorem of Yomdin [28]. Suppose that $h\colon N \to N$ is a $C^\infty$ diffeomorphism of a compact surface and $\nu \subset N$ is a smooth path. Let $|\nu|_N$ be the length of $\nu$ in $N$ with respect to some smooth metric on $N$ and define the *growth rate for the length of $\nu$ with respect to $h$* to be

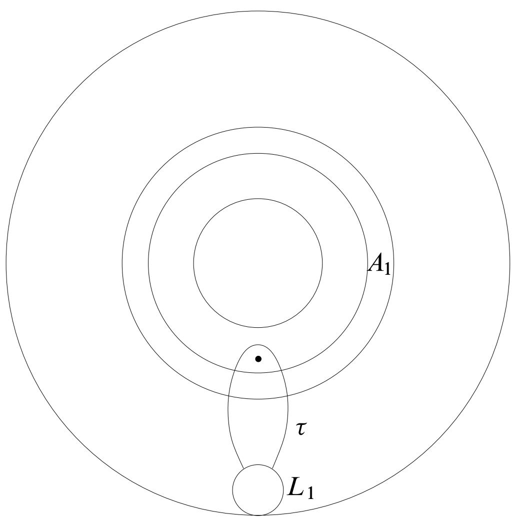
$$\mathrm{gr}(\nu, h) = \limsup_{n \to \infty} \frac{\log |h^n(\nu)|_N}{n}.$$

**Theorem 8.10** (Yomdin [28, Theorem 1.4]) *For any $C^\infty$ diffeomorphism $h\colon N \to N$ of a compact surface and for any smooth path $\nu \subset N$, $\mathrm{gr}(\nu, h) \leq$ entropy($h$).*

**Proof of Lemma 8.9** We assume to the contrary that there exist $[\tau] \in RH(W, \partial_+ W)$ and $n \geq 1$ such $[\tau]$ has multiplicity at least 2 in $(f_\sigma^n)_\#([\tau]) \cap W$ and argue to a contradiction. The obvious induction argument on $k$ implies that $[\tau]$ has multiplicity at least $2^k$ in $(f_\sigma^{kn})_\#([\tau]) \cap W$.

We denote $A_\sigma$ or $A_\sigma^c$ by $A$. By construction, the universal covering projection $H \to M$ factors through a covering projection $\pi_\sigma\colon \operatorname{int}(A) \to M$. Each smooth path $\nu \subset \operatorname{int}(A)$ projects to a smooth path $\pi_\sigma(\nu) \subset M$ whose length, with respect to the hyperbolic metric on $M$, is denoted $|\pi_\sigma(\mu)|_M$. We will prove that there is a compact set $M_0 \subset M$ and $\epsilon > 0$ so that for all $k \geq 1$ there are at least $2^k$ disjoint subpaths $\mu_j$ of $f_\sigma^{kn}(\tau)$ such that $\pi_\sigma(\mu_j) \subset M_0$ and $|\pi_\sigma(\mu_j)|_M \geq \epsilon$. The paths $\pi_\sigma(\tau), \pi_\sigma(f_\sigma^{kn}(\tau))$ and $\pi_\sigma(\mu_j)$ lift via $\pi_P$ to smooth paths $\tau'$, $F'^{kn}(\tau')$ and $\mu'_j$ respectively where the $\mu'_j$ are disjoint subpaths of $F'^{kn}(\tau')$. Since the $\mu'_j$ are contained in a compact subset of $\operatorname{int}(N)$ there exists $\epsilon' > 0$ so that each $|\mu'_j|_N \geq \epsilon'$. It follows that the growth rate for the length of $\tau'$ with respect to $F'$ is at least $\log(2)\epsilon'$ contradicting Theorem 8.10 and the assumption that $F'$ has entropy zero.

It remains to prove the existence of $M_0$ and $\epsilon$. Assume the notation of Definition 8.5. Let $L_1$ and $L_2$ be the components of $\partial_+ W$ that contain the endpoints of $\tau$. Recall that the ends of $L_i$ converges to a single point in $\partial A_\sigma$.

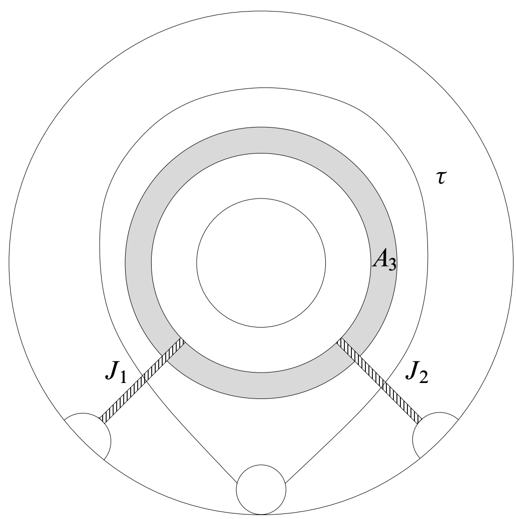



As a first case suppose that $L_1 = L_2$ and that $\tau$ and the interval in $L_1$ connecting the endpoints of $\tau$ bound a disk $D$ in $\mathring{A}_\sigma$. Choose an element $x \in X \cap D$ and a compact essential subannulus $A_1 \subset \mathring{A}_\sigma$ that separates $x$ from $L_1$. There are at least $2^k$ subpaths $\mu_j$ of $f_\sigma^{kn}(\tau)$ that cross $A_1$. In this case we let $M_0 = \pi_\sigma(A_1)$; the existence of a uniform lower bound for $|\pi_\sigma(\mu_j)|_M$ comes from the compactness of

$A_1$, which implies that there is a uniform lower bound to $|\mu_j|_A$ and that the restriction of $\pi_\sigma$ to $A_1$ is bi-Lipschitz.

In the case that the ends of $L_1$ are in one component of $\partial A_\sigma$ and the ends of $L_2$ are in the other, the same argument works with respect to a compact essential subannulus $A \subset \mathring{A}_\sigma$ that separates $L_1$ and $L_2$.

The third case is that $L_1 = L_2$ and that $\tau$ and the interval in $L_1$ connecting the endpoints of $\tau$ define a simple closed curve that is essential in $A_\sigma$. Choose a compact essential annulus $A_3 \subset \mathring{A}_\sigma$ that is disjoint from $L_1 \cup \tau$. Choose disjoint half-disks $D_1, D_2$ whose frontiers consist of intervals $I_1$ and $I_2$ in the component of $\partial A_\sigma$ that contains the endpoint of $L_1$ and half-circles $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ that project to the same simple closed curve $\rho \subset M$. Assume further that the closure of $L_1$ is disjoint from $D_1$ and $D_2$. Choose thickened arcs $J_1$ and $J_2$ connecting $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ to the far component of $\partial A_3$. Thus $J_1$ and $J_2$ overlap with $A_3$ in rectangles that cross $A_3$.




There are at least $2^k$ subpaths $\mu_j$ of $(f_\sigma^{kn})(\tau)$ such that each $\mu_j$ either crosses $J_1$, crosses $J_2$, crosses $A_3$ or is an arc with one endpoint on $\widetilde{\rho}_1$ and the other on $\widetilde{\rho}_2$. In this case we let $M_0$ be the union of $\pi_\sigma(J_1 \cup J_2 \cup A_3)$ and a compact $\delta$–neighborhood of $\rho$ where $\delta > 0$ is so small that this neighborhood is a closed annulus. The existence of a uniform lower bound for $|\pi_\sigma(\mu_j)|_M$ comes from the compactness of $J_1 \cup J_2 \cup A_3$ and the fact that if $\mu_j$ has one endpoint on $\widetilde{\rho}_1$ and the other on $\widetilde{\rho}_2$ then $|\pi_\sigma(\mu_j)|_M$ has endpoints in $\rho$ but is not contained in the $\delta$–neighborhood of $\rho$.

The fourth and final case is that $L_1 \neq L_2$ have endpoints in the same component of $\partial A_\sigma$. The obvious modification of the argument from the third case applies here. □

We now return to the more general context of this section.

Suppose that $W$ is a Brouwer subsurface and that $\tau \in RH(W, \partial_+ W)$. We say that $[\tau]$ *disappears under iteration* if $h^n_\#([\tau]) \cap W = \varnothing$ for some $n > 0$ and that $\mathcal{T}$ *disappears under iteration* if each element in $\mathcal{T}$ does. If both endpoints of $\tau$ are contained in a single component $L$ of $\partial_+ W$ and the simple closed curve that is the union of $\tau$ with an interval in $L$ does not bound a disk in $A$ then we say that $\tau$ is *essential*.

The next lemma is based on [17, Theorem 5.5(c)]. See also [11, Lemma 10.8].

**Lemma 8.11** *Suppose that $W$ is a Brouwer subsurface and $\mathcal{T}$ is a fitted family as in Definition 8.5. Suppose further that $\mathcal{T}$ does not disappear under iteration and that no element of $\mathcal{T}$ eventually doubles. Then there exists $[\tau] \in \mathcal{T}$ such that $h_\#([\tau]) \cap W$ contains $[\tau]$ with multiplicity one and does not contain $-[\tau]$; moreover, every other element of $h_\#([\tau]) \cap W$ disappears under iteration. If $\tau$ has both endpoints on the same component $L$ of $\partial_+ W$ then $[\tau]$ is essential.*

**Proof** Let $\Gamma$ be the directed graph with one vertex $v_i$ for each element $[\tau_i] \in \mathcal{T}$ and with the number of oriented edges from $v_i$ to $v_j$ equal to the sum of the multiplicities of $[\tau_j]$ and of $-[\tau_j]$ in $h_\#([\tau_i]) \cap W$. By Lemma 8.6, there is a natural bijection between the elements of $h^n_\#([\tau_i]) \cap W$ and the set of oriented paths in $\Gamma$ that have length $n$ and begin at $v_i$.

Since no $[\tau_i]$ eventually doubles, each $v_i$ is contained in at most one nonrepeating oriented closed path in $\Gamma$. The set $\mathcal{V}_0$ of vertices of $\Gamma$ that are contained in at least one oriented closed path is nonempty because $\mathcal{T}$ does not disappear under iteration. There is a partial order on the vertices of $\Gamma$ defined by $v_1 > v_2$ if there is an oriented path in $\Gamma$ from $v_1$ to $v_2$ but no oriented path from $v_2$ to $v_1$. Choose $v_p \in \mathcal{V}_0$ so that $v_p > v_q$ implies that $v_q \notin \mathcal{V}_0$. Note that if $v_p > v_q$ then $[\tau_q]$ disappears under iteration. Indeed, if it does not then there are arbitrarily long oriented paths in $\Gamma$ beginning with $v_q$ so there must be an element $v_r \in \mathcal{V}_0$ such that $v_q > v_r$; this contradicts $v_p > v_q$ and the choice of $v_p$. Note also that if there is an oriented path from $v_p$ to $v_q$ but $v_p \not> v_q$ then $v_q$ is on the oriented cycle through $v_p$ and so is uniquely determined by the length of the path from $v_p$ to $v_q$.

Let $n$ be the length of the unique oriented nonrepeating closed path $\rho$ through $v_p$. Then $h^n_\#([\tau_p]) \cap W$ contains exactly one element that does not disappear under iteration and it is $\epsilon[\tau_p]$ where $\epsilon = \pm 1$. To complete the proof of the lemma, it remains to show that $n = 1$ and $\epsilon = 1$.

Let $v_s$ be the endpoint of the unique edge in $\Gamma$ that begins at $v_p$ and is on the unique oriented closed path through $v_p$. It is possible that $v_s = v_p$. Thus, either $[\tau_s]$ or $-[\tau_s]$ is the unique element of $h_\#([\tau_p]) \cap W$ that does not disappear under iteration.

For the remainder of the proof we make use of the end of the proof of [17, Theorem 5.5, pages 253–254]. The case that $\tau_p$ has endpoints on distinct components of $\partial_+ W$ is treated in the last two paragraphs of that proof. (The labels $\tau_i^k$ in the diagram on the bottom of page 253 are incorrect; they should be $\tau_i^*$.) The argument given there applies without change in our present context so we may assume that $\tau_p$ has both endpoints on the same component, say $L$, of $\partial_+ W$.

The endpoints of $h_\#(\tau_p)$ are contained in the component of the complement of $W$ bounded by $L$. If $h_\#(\tau_p)$ intersects any other component of the complement of $W$ then at least two elements of $h_\#([\tau_p]) \cap W$ would be represented by paths with endpoints on distinct components of $\partial_+ W$. Since no such paths disappear under iteration, this can not happen and we conclude that each element of $h_\#([\tau_p]) \cap W$, and in particular $\tau_s$, has both endpoints on $L$.

Both ends of $L$ converge to the same component of $\partial A$. The argument given in [17, page 253, first and second paragraphs] (which is a proof by contradiction) carries over without change to this context and proves that $\tau_p$ is essential. By symmetry, $\tau_s$ is also essential. If $[\tau_p] \neq [\tau_s]$ then either the interval of $L$ bounded by the endpoints of $\tau_p$ contains the interval of $L$ bounded by the endpoints of $\tau_s$ or vice versa. In either case, there is a rectangle $D \subset W$ bounded by $\tau_p, \tau_s$ and intervals in $L$. It contains finitely many punctures, each of which is mapped to the complement of $W$ by all sufficiently high iterates of $h$. Thus, for all sufficiently large $k$, $h^{kn}(D)$ does not contain any punctures in $W$. It follows that either $h_\#^{kn}([\tau_p]) \cap W = h_\#^{kn}([\tau_s]) \cap W$ or $h_\#^{kn}([\tau_p]) \cap W = h_\#^{kn}([-\tau_s]) \cap W$. But $\epsilon^k[\tau_p]$ (respectively $[\epsilon^k \tau_s]$) is the unique element of $h_\#^{kn}([\tau_p]) \cap W$ (respectively $h_\#^{kn}([\tau_s]) \cap W$) that does not disappear under iteration. This contradicts the assumption that $[\tau_p] \neq [\tau_s]$. We conclude that $[\tau_p] = [\tau_s]$ and hence that $n = 1$. Since $h$ is orientation preserving and $h_\#(L)$ is parallel to $L$, it follows that $\epsilon = 1$. □

**Definition 8.12** Suppose that $W$ is a Brouwer subsurface, that $\mathcal{T}$ is a fitted family and that $[\tau] \in \mathcal{T}$. Let $L_1$ and $L_2$ be the components of $\partial_+ W$ that contain the initial and terminal endpoints $w_1$ and $w_2$ of $\tau$ respectively. We say that $[\tau]$ is *peripheral* if one (and hence all) of the following equivalent conditions are satisfied:

(1) Some component of the complement of $L_1 \cup L_2 \cup \tau$ is contractible in $A_X^\circ$.

(2) There are rays $R_1 \subset L_1$ and $R_2 \subset L_2$ whose initial points are $w_1$ and $w_2$ such that the line $R_1^{-1} \tau R_2$ can be properly isotoped rel $X$ into arbitrarily small neighborhoods of some end of $A^\circ$.

(3) If $\tilde{\tau}$ is a lift of $\tau$ to $H$ and $\tilde{L}_1$ and $\tilde{L}_2$ are the lifts of $L_1$ and $L_2$ that contain the endpoints of $\tilde{\tau}$ then $\tilde{L}_1$ and $\tilde{L}_2$ have a common endpoint.

Our next result refines Lemma 8.11. It is based on [17, Lemma 6.4].

**Lemma 8.13** *Suppose that $W$, $\mathcal{T}$ and $[\tau]$ are as in the statement of Lemma 8.11. Let $L_1 = \partial V_1$ and $L_2 = \partial V_2$ be the (possibly equal) components of $\partial_+ W$ that contain the initial and terminal endpoints $v_1$ and $v_2$ of $\tau$. Then:*

(1) $h_\#([\tau]) \cap W = \{[\tau]\}$.

(2) *If $[\tau]$ is not peripheral then there are rays $R_1 \subset L_1$ and $R_2 \subset L_2$ such that $R_1^{-1}\tau R_2$ is isotopic to an $h_\#$–invariant geodesic line $\mu$.*

**Proof** To prove (1) we assume that $(h_\#([\tau]) \cap W) \setminus \{[\tau]\} = \{s_1, \dots, s_m\}$ is not empty and argue to a contradiction. Lemma 8.11 implies that each $s_i$ disappears under iteration and so, in particular, is represented by a path with both endpoints on $L_1$ or both endpoints on $L_2$. As there is no loss in replacing $h$ by an iterate, we may assume that each $h_\#([s_i]) \cap W = \varnothing$.

We recall the alternate definition of $h_\#([\tau]) \cap W$ given in [17, page 50]. Choose a lift $\tilde{h}\colon H \cup S_\infty \to H \cup S_\infty$ to the compactified universal cover of $\mathring{A}_X$, choose a lift $\tilde{\tau}$ of $\tau$ and for $j = 1, 2$, let $\tilde{L}_j$ be the lift of $L_j$ that contains the endpoint $\tilde{v}_j$ of $\tilde{\tau}$. The lines $\tilde{h}_\#(\tilde{L}_j)$ are disjoint from the full preimage $\tilde{W}$ of $W$. There are finitely many components $\tilde{W}_l$ of $\tilde{W}$ that separate $\tilde{h}_\#(\tilde{L}_1)$ from $\tilde{h}_\#(\tilde{L}_2)$. Any geodesic path connecting $\tilde{h}_\#(\tilde{L}_1)$ to $\tilde{h}_\#(\tilde{L}_2)$ crosses through $\tilde{W}_l$ in a geodesic arc; the projection of this arc to $W$ determines a well-defined element of $RH(W, \partial_+ W)$ and the multiset of these elements, obtained by varying $l$, is exactly $h_\#([\tau]) \cap W$.

Since $[\tau] \in h_\#([\tau]) \cap W$, we may choose $\tilde{h}$ so that $\tilde{h}_\#(\tilde{\tau})$ crosses $\tilde{L}_1$ and $\tilde{L}_2$ in that order. For future reference, note that $\tilde{h}^2_\#(\tilde{\tau})$ crosses $\tilde{h}_\#(\tilde{L}_1), \tilde{L}_1, \tilde{L}_2$, and $\tilde{h}_\#(\tilde{L}_2)$ in that order.

Intersection of $\tilde{h}_\#(\tilde{\tau})$ with $\tilde{W}$ decomposes $h_\#(\tilde{\tau})$ into an alternating concatenation of subpaths $\tilde{\mu}_k$ whose projections $\mu_k \subset \mathring{A}$ represent elements of $h_\#([\tau]) \cap W$ and subpaths $\tilde{\nu}_k$ whose projections $\nu_k$ are contained in $V_j \setminus h_\#(V_j)$ for $j = 1$ or $2$. We assume without loss that some $s_i$, say $s_1$, has both endpoints in $L_1$ and hence that some $\mu_k$, say $\mu_1$, is a path in $W$ that represents $s_1$ and in particular has both endpoints in $L_1$. It follows that at least one of the $\nu_k$ is contained in $V_1 \setminus h_\#(V_1)$ and has both endpoints in $L_1$. Note that this is true not only for $\tilde{h}_\#(\tilde{\tau})$ but also for any geodesic path that connects $\tilde{h}_\#(\tilde{L}_1)$ to $\tilde{h}_\#(\tilde{L}_2)$. In particular, $h^2_\#(\tau)$ contains a subpath in $V_1 \setminus h_\#(V_1)$ with both endpoints in $L_1$.

Let $\tilde{L}_1'$ and $\tilde{L}_1''$ be the lifts of $L_1$ that contain the endpoints of $\tilde{\mu}_1$. Since $h_\#([s_1]) \cap W = \varnothing$, any geodesic path connecting $\tilde{h}_\#(\tilde{L}_1')$ to $\tilde{h}_\#(\tilde{L}_1'')$ projects to a path in $V_1 \setminus h_\#(V_1)$

with both endpoints in $h_\#(V_1)$. Since $\widetilde{h}_\#(\widetilde{L}'_1)$ and $\widetilde{h}_\#(\widetilde{L}''_1)$ separate $\widetilde{h}^2_\#(\widetilde{L}_1)$ from $\widetilde{h}^2_\#(\widetilde{L}_2)$, $h^2_\#([\tau])$ contains a subpath in $V_1 \setminus h_\#(V_1)$ with both endpoints in $h_\#(V_1)$. But $S_1 = V_1 \setminus h_\#(V_1)$ is a once punctured strip. A geodesic arc in $S_1$ with endpoints in $L_1$ can not be disjoint from a geodesic arc in $S_1$ with endpoints in $h_\#(L_1)$. Since $h^2_\#(\tau)$ is an embedded geodesic arc, we have reached the desired contradiction and so have proved (1).

Let $\widetilde{\sigma}$ be the subpath of $\widetilde{h}_\#(\widetilde{\tau})$ that connects $\widetilde{h}_\#(\widetilde{L}_1)$ to $\widetilde{L}_1$ and let $\sigma$ be its image in $A^\circ$. We now know that $\sigma$ is an arc in $S_1$ with one endpoint in $L_1$ and the other in $h_\#(L_1)$. Since $S_1$ is a once punctured strip, one of the complementary components of $\sigma$ in $S$ is unpunctured. There are rays $R' \subset L_1$ and $R'' \subset h_\#(L_1)$ so that $R'^{-1}\sigma R''$ is peripheral. Lifting this back to the universal cover, we have that $\widetilde{L}_1$ and $\widetilde{h}_\#(\widetilde{L}_1)$ are asymptotic. Their common endpoint $P_1$ is a fixed point for $\widetilde{h}|_{S_\infty}$. Symmetrically, one of the endpoints $P_2$ of $\widetilde{L}_2$ is fixed by $\widetilde{h}$ and we let $\widetilde{\mu}$ be the geodesic connecting $P_1$ to $P_2$ (which are distinct points because $[\tau]$ is not peripheral). This completes the proof of (2). □

**Definition 8.14** An embedded arc $\rho \subset A$ that is disjoint from $X$ and that has endpoints in $\mathrm{Fix}(h|_{\partial A})$ is a *reducing arc* for $h$ rel $X$ if it is $h$–invariant up to isotopy rel $X$ and rel its endpoints and is nonperipheral in the sense that it is not homotopic rel endpoints and rel $X$ into $\partial A$.

The following lemma is similar to [11, Proposition 10.10]. The conclusions of the lemma are more detailed than those of that proposition and apply to $h$ and not just some iterate of $h$. Lemma 7.3 implies that condition (b) below is satisfied in the special case that $h = f_\sigma$ for $\sigma \in \mathcal{R}$. Note also that if (b) is satisfied then all reducing arcs have their endpoints on the same component of $\partial A$.

**Lemma 8.15** *Assume that the hyperbolic metric on $A^\circ_X$ has been chosen as in Lemma 8.4 and that $\beta_i^\pm$ and $V_i^\pm$ are as in that lemma. Let $W$ be the associated Brouwer subsurface and assume that no element of $RH(W, \partial_+ W)$ eventually doubles. Let $\alpha = \bigcup_{i=1}^r \alpha(h, x_i)$ and $\omega = \bigcup_{i=1}^r \omega(h, x_i)$. Assume that for each component $\partial_l A$ of $\partial A$, $\alpha_l = \alpha \cap \partial_l A$ and $\omega_l = \omega \cap \partial_l A$ have the same cardinality $c_l$ and that if $c_l > 1$ then the elements of $\alpha_l$ and $\omega_l$ alternate around $\partial_l A$. Then:*

(1) *There is a reducing arc $\rho$ for $h$ with respect to $X$.*

*If either one of the following conditions is satisfied,*

(a) $r = 1$,

(b) *if $x$ and $y$ are fixed points in different components of $\partial A$ then $h$ is isotopic* rel$\{x, y\}$ *to a nontrivial Dehn twist,* *then:*

(2) *For each* $1 \le i \le r$, $\alpha(h, x_i)$ *and* $\omega(h, x_i)$ *belong to the same component of* $\partial A$.

(3) *For each* $1 \le i \le r$, *there is a reducing arc* $\rho_i$ *whose endpoints are* $\alpha(h, x_i)$ *and* $\omega(h, x_i)$.

(4) *For each* $1 \le i \le r$, *there is a translation arc geodesic* $\beta_i$ *for* $x_i$ *such that* $B_i = \bigcup_{j=-\infty}^{\infty} h_{\#}^j(\beta_i)$ *is a properly embedded line whose initial end converges to* $\alpha(h, x_i)$ *and whose terminal end converges to* $\omega(h, x_i)$.

(5) *The* $B_i$ *are disjoint.*

*If* $r = 1$*:*

(6) *There is a unique translation arc geodesic for* $x_1$.

**Proof** For each $i$, let $\mathcal{T}_i$ be the fitted family (Notation 8.7) determined by $\beta_i^-$.

To prove (1), it suffices to show that there is a properly embedded nonperipheral line $\ell \subset \mathring{A}_X$, whose initial and terminal ends converge to elements of $\mathrm{Fix}(h|_{\partial A})$ and such that $h(\ell)$ is properly isotopic in $\mathring{A}_X$ to $\ell$. The proper isotopy can be chosen so that it extends by the identity on $\partial A$ so we can take $\rho$ to be the closure of $\ell$ in $A$.

If $\mathcal{T}_i$ disappears under iteration then the homotopy streamline $B_i = \bigcup_{n=-\infty}^{\infty} h_{\#}^n(\beta_i^-)$ is a properly embedded $h_{\#}$–invariant line whose ends converge to $\alpha(h, x_i)$ and $\omega(h, x_i)$ and we let $\ell$ be the line obtained by pushing $B_i$ off of itself; there is always at least one direction to push that results in a nonperipheral line. If $\mathcal{T}_i$ does not disappear under iteration, let $[\tau] \in \mathcal{T}_i$ satisfy the conclusions of Lemma 8.11 and let $L_p$ and $L_q$ be the components of $\partial_+ W$ containing the initial and terminal endpoint of $\tau$ respectively. We claim that $[\tau]$ is not peripheral (Definition 8.12). This is obvious if $\omega(h, x_p)$ and $\omega(h, x_q)$ belong to distinct components of $\partial A$. If $L_p = L_q$ this follows from Lemma 8.11, which asserts that $[\tau]$ is essential, and the assumption that the component of $\partial A$ that contains $\omega(h, x_p) = \omega(h, x_q)$ intersects $\alpha$ nontrivially. In the final case, $\omega(h, x_p) \ne \omega(h, x_q)$ belong to the same component of $\partial A$ and so are separated in that boundary component by elements of $\alpha$; again $\tau$ is not peripheral. Lemma 8.13 implies that there are rays $R_p \subset L_p$ and $R_q \subset L_q$ such that $\ell = R_p^{-1} \tau R_q$, whose ends converge to $\omega(h, x_p)$ and $\omega(h, x_q)$, has the desired properties.

We now turn to the proof of (4). It suffices to show that each $\mathcal{T}_i$ disappears under iteration. We assume that some $\mathcal{T}_i$ does not disappear under iteration and, continuing with the above notation, argue to a contradiction. Note that $\omega(h, x_p)$ and $\omega(h, x_q)$ lie on the same component, say $\partial_0 A$, of $\partial A$. This is obvious for (a) and holds for (b) because $\rho$ has endpoints $\omega(h, x_p)$ and $\omega(h, x_q)$ and is isotopic to $h(\rho)$ rel endpoints. Denote the other component of $\partial A$ by $\partial_1 A$.

Let $\mu$ be the geodesic determined by $\ell$, let $Y$ be the component of $\mathring{A}_X \setminus \mu$ whose closure contains $\partial_1 A$ and let $Z$ be the other component of $\mathring{A}_X \setminus \mu$. Since $Z$ is not contractible, it contains at least one orbit of $X$.

We claim that $Y$ also intersects, and hence contains, an orbit of $X$. If $p \neq q$ this follows from the fact that the endpoints $\omega(h, x_p)$and $\omega(h, x_q)$ of $\ell$ separate $\alpha \cap \partial_0 A$. Suppose then that $p = q$. Since $\mu$ is $h_\#$–invariant and disjoint from $B_i^-$, each element of $\mathcal{T}_i$ is represented by an arc that is disjoint from $\mu$. Since $\ell$ can be isotoped to be disjoint from $\partial V_q^+$ but cannot be isotoped into $V_q^+$, $\mu$ is disjoint from $V_q^+$ and hence disjoint from $\tau \cup V_q^+$. Since $\tau$ is essential (Lemma 8.11) $Y$ contains $V_q^+$ and hence the orbit of $x_q$. This completes the proof that both $Y$ and $Z$ contain an orbit of $X$.

If $r = 1$ then we have reached the desired contradiction and so have proved (4) in this case. Arguing by induction on $r$, we may assume that $r > 1$ and that if one works relative to $X \cap Y$ or relative to $X \cap Z$ then $\mathcal{T}_i$ disappears under iteration. In other words, if $x_i \in Y$ (the argument for $x_i \in Z$ is symmetric) then for all sufficiently large $n$, $h_\#^n(\beta_i^-)$ is isotopic rel $X \cap Y$ to an arc $\gamma_{i,n} \subset V_i \subset Y$. Since $\mu$ is $h_\#$–invariant, $h_\#^n(\beta_i^-) \subset Y$. It is a standard fact that the isotopy rel $X \cap Y$ of $h_\#^n(\beta_i^-)$ to $\gamma_{i,n}$ can be taken with support in $Y$. It follows that this isotopy is rel $X$ which implies that $h_\#^n(\beta_i^-) \subset V_i$ in contradiction to the assumption that $\mathcal{T}_i$ does not disappear under iteration. This completes the proof of (4).

Items (3) and (5) follow from (4). If $r = 1$ then (2) follows from our assumption that $\alpha_l$ and $\omega_l$ have the same cardinality. If (b) is satisfied then (2) follows the fact that $\alpha(h, x_i)$ and $\omega(h, x_i)$ bound a reducing curve. Thus (2) is satisfied.

To verify (6), let $B_1$ and $\beta_1$ be as in (4) and denote $h_\#^j(\beta_1)$ by $\beta_{1,j}$. Thus $B_1 = \bigcup_{n=-\infty}^{\infty} \beta_{1,j}$ and $h_\#(\beta_{1,j}) = \beta_{i,j+1}$. We assume that there is a translation arc geodesic $\delta \neq \beta_{1,0}$ for $x$ and argue to a contradiction. Let $\eta$ be the maximum initial segment of $\delta$ whose interior is disjoint from $B_1$ and let $y$ be the terminal endpoint of $\eta$. Let $\nu$ be the maximum initial segment of $h_\#(\delta)$ whose interior is disjoint from $B_1$ and let $z$ be the terminal endpoint of $\nu$. If $y \in X$ then $z = h(y)$; otherwise $y$ is in the interior of some $\beta_{1,m}$ and $z$ is in the interior of $\beta_{1,m+1}$.

If $y \notin \beta_{1,-1} \cup \beta_{1,0}$ then the endpoints of $\eta$ and $\nu$ are linked in $B_1$ in contradiction to the fact that the interiors of $\eta$ and $\nu$ are disjoint and lie on the same side of $B_1$. We may therefore assume that $y \in \beta_{1,-1} \cup \beta_{1,0}$. In this case, the endpoints of $\eta$ and $\nu$ bound intervals $I_\eta$ and $I_\nu$ in $B_1$ that meet in at most one point. It follows that either the simple closed curve $\eta \cup I_\eta$ or the simple closed curve $\nu \cup I_\nu$ is inessential in $A$ and so bounds a disk that is disjoint from $X$ in contradiction to the fact that these simple closed curves are composed of two geodesic segments. This completes the proof that $\beta_1$ is the unique translation arc geodesic and hence the proof of (6). □

We conclude this section by applying Lemma 8.15 to the specific class of annulus homeomorphisms that concern us in this paper. Note that the statements are purely topological and so are independent of hyperbolic metrics used in their proofs.

**Corollary 8.16** *Suppose that $\sigma \in \mathcal{R}$ and that $f_\sigma\colon A_\sigma \to A_\sigma$ is as in Definitions 7.2. Let $\mathring{A}_\sigma = \mathrm{int}(A_\sigma)$. Then there do not exist $\hat{x}_1, \hat{x}_2 \in \mathring{A}_\sigma$ such that $\alpha(f_\sigma, \hat{x}_1)$ and $\omega(f_\sigma, \hat{x}_2)$ are contained in one component of $\partial A_\sigma$ and $\alpha(f_\sigma, \hat{x}_2)$ and $\omega(f_\sigma, \hat{x}_1)$ are contained in the other component of $\partial A_\sigma$.*

**Proof** Let $\hat{X} \subset \mathring{A}_\sigma$ be the union of the $f_\sigma$–orbits of $\hat{x}_1$ and $\hat{x}_2$ and assume that $\mathring{A}_\sigma \setminus \hat{X}$ is equipped with a complete hyperbolic structure as in Section 6. Let $W$ be a Brouwer subsurface as in Definition 8.5. Lemma 8.9 implies that no element of $RH(W, \partial_+ W)$ eventually doubles. If there exist $\hat{x}_1, \hat{x}_2 \in A_\sigma$ such that $\alpha(f_\sigma, \hat{x}_1)$ and $\omega(f_\sigma, \hat{x}_2)$ are contained in one component of $\partial A_\sigma$ and $\alpha(f_\sigma, \hat{x}_2)$ and $\omega(f_\sigma, \hat{x}_1)$ are contained in the other component of $\partial A_\sigma$, then the hypotheses of Lemma 8.15 are satisfied with $r = 2$ and $c_0 = c_1 = 1$. Lemma 7.3 implies that condition (b) of Lemma 8.15 is satisfied and hence by item (2) of Lemma 8.15 that for $i = 1, 2$, $\alpha(h, x_i)$ and $\omega(h, x_i)$ belong to the same component of $\partial A_\sigma$. This contradiction completes the proof. □

The next corollary states that if there is twisting across an annular cover then orbits that start and end on one boundary component can not get to close to the other boundary component.

**Corollary 8.17** *Suppose that $h\colon A \to A$ is either*

(1) *$f_\sigma\colon A_\sigma \to A_\sigma$ for some $\sigma \in \mathcal{R}$, or*

(2) *$f_\sigma\colon A^c_\sigma \to A^c_\sigma$ for some horocycle $\sigma$ corresponding to an isolated end of $M$.*

*Let $\partial_0 A$ and $\partial_1 A$ be the components of $\partial A$. In case (2) assume that $\partial_0 A$ is the unique component of $\partial A$ and that if $\mathrm{Fix}(f_\sigma|_{\partial_1 A}) \neq \emptyset$ then $f_\sigma$ is not isotopic to the identity rel $\mathrm{Fix}(f_\sigma|_{\partial A})$. Then there is a neighborhood of $\partial_1 A$ that is disjoint from the $h$–orbit of any $\hat{x} \in A$ for which both $\alpha(h, \hat{x})$ and $\omega(h, \hat{x})$ are contained in $\partial_0 A$.*

**Proof** If the corollary fails then there exist $\hat{x}_t \to \hat{P} \in \partial_1 A$ with $\alpha(h, \hat{x}_t), \omega(h, \hat{x}_t) \in \partial_0 A$. After replacing $h$ by some $h^m$ we may assume that the rotation number of $h|_{\partial_1 A}$ is less than $\frac{1}{4}$. In particular there are intervals $J_1 \subset J_2 \subset J_3$ in $\partial_1 A$ connecting $\hat{P}$ to $h(\hat{P})$, $h^2(\hat{P})$ and $h^3(\hat{P})$ respectively. These intervals will be trivial if $\hat{P}$ is fixed by $h$ and nontrivial otherwise. Additionally, after possibly increasing $m$ further, we may assume (Lemma 7.3) that if $\eta$ is a path connecting a fixed point in $\partial_0 A$ to $\hat{P}$ then $h(\eta)$ is not homotopic rel endpoints to the path obtained by concatenating $\eta$ with $J_1$.

Choose contractible neighborhoods $\hat{U}_i$ of $J_i$ in $A$ such that $\hat{U}_1 \subset \hat{U}_2 \subset \hat{U}_3$ and such that $h(\hat{U}_i) \subset \hat{U}_{i+1}$. Choose lifts $P \in \tilde{U}_1 \subset \tilde{U}_2 \subset \tilde{U}_3$ in $H \cup S_\infty$ and let $\tilde{h}$ be the lift of $h$ such that $\tilde{h}(P) \in \tilde{U}_1$. After passing to a subsequence, $\hat{x}_t \to \hat{P}$ lifts to a sequence $\tilde{x}_t \to P$ such that $\tilde{x}_t, \tilde{h}(\tilde{x}_t) \in \tilde{U}_1$ for all $t$. Recall that a translation arc for $\tilde{x}_t$ is a path from $\tilde{x}_t$ to $\tilde{h}(\tilde{x}_t)$ that intersects its $\tilde{h}$–image only in $\tilde{h}(\tilde{x}_t)$. There is a translation arc $\tilde{\delta}_t \subset \tilde{U}_2$ for $\tilde{x}_t$ by [17, Lemma 4.1]. Let $\hat{\delta}_t \subset \hat{U}_2$ be the projected image of $\tilde{\delta}_t$. Since $\tilde{h}(\tilde{\delta}_t) \cup \tilde{\delta}_t \subset \tilde{U}_3$, $h(\hat{\delta}_t) \cap \hat{\delta}_t$ is the projected image of $\tilde{h}(\tilde{\delta}_t) \cap \tilde{\delta}_t$. Thus $\hat{\delta}_t \subset \hat{U}_2$ is a translation arc for $\hat{x}_t$. We now fix such a $\hat{x}_t$ and drop the $t$ subscript.

Assume the notation of Lemma 8.15 applied with $r = 1$, $x_1 = \hat{x}$, $c_0 = 1$ and $c_1 = 0$. Lemma 8.9 implies that the hypothesis of Lemma 8.15 are satisfied. The homotopy streamline $B_1$ produced by item (4) of Lemma 8.15 can be thought of as an arc $\hat{\mu}$ with initial endpoint $\alpha(\hat{h}, \hat{x})$ and terminal endpoint $\omega(\hat{h}, \hat{x})$. Let $\hat{\mu}_0$ be the initial subpath of $\hat{\mu}$ that ends with $\hat{x}_1$ and let $\hat{\nu} \subset \hat{U}_1$ be a path connecting $\hat{x}_1$ to $\hat{P}$. The path $\hat{\eta} = \hat{\mu}_0 \hat{\nu}$ connects $\alpha(\hat{h}, \hat{x}) \in \partial_0 A$ to $\hat{P} \in \partial_1 A$. By the uniqueness part of Lemma 8.15(6), $\hat{\delta}$ is isotopic rel $\hat{X}$ to the subpath of $\hat{\mu}$ connecting $\hat{x}$ to $h(\hat{x})$. It follows that the path $\hat{\eta}^{-1} h(\hat{\eta})$ connecting $P$ to $h(P)$ is homotopic rel endpoints to $\hat{\nu}^{-1} \hat{\delta} h(\hat{\nu}) \subset \hat{U}_2$. Hence $h(\hat{\eta})$ is homotopic rel endpoints to $\hat{\eta} J_1$. This contradiction completes the proof. □

# 9 $\omega$–lifts

We assume throughout this section that $\mathcal{R} \neq \emptyset$. Recall from Section 7 that the closure of a component of $H \setminus \tilde{\mathcal{R}}$ in $H$ is called a domain. We will assign a domain or a pair of domains to each $\tilde{x} \in H$ based on its forward $\tilde{f}$–orbit. By symmetry, we can assign a domain or a pair of domains to each $\tilde{x} \in H$ based on its backward $\tilde{f}$–orbit. In the next section (Corollary 10.4) we show that these two methods give the same domain or pair of domains when $x$ is birecurrent.

Suppose that $\tilde{C}$ is a domain and that $\tilde{\sigma} \in \tilde{\mathcal{R}}$ is a frontier component of $\tilde{C}$. Let $I_{\tilde{\sigma}}$ be the component of $S_\infty \setminus \mathrm{Fix}(\tilde{f}_{\tilde{C}})$ bounded by the endpoints of $\tilde{\sigma}$. We write $\tilde{\sigma}_{\tilde{C}}$ for $\tilde{\sigma}$ equipped with the orientation which makes every point in $I_{\tilde{\sigma}}$ move away from the backward endpoint of $\tilde{\sigma}$ toward the forward endpoint of $\tilde{\sigma}$ under the action of $\tilde{f}_{\tilde{C}}$.

Equivalently, the orientation on $\widetilde{\sigma}$ is chosen so that a turn from inside $\widetilde{C}$ along $\widetilde{\sigma}$ in the direction (left or right) of the Dehn twist of $f$ across $\sigma$ has one moving toward the forward end of $\widetilde{\sigma}_{\widetilde{C}}$.

We say that a pair of disjoint oriented distinct geodesics in $H$ are *antiparallel* if either of the following conditions is satisfied:

- The four endpoints in $S_\infty$ are distinct with the pair of initial endpoints separating the pair of terminal endpoints.
- The initial endpoint of one of the geodesics equals the terminal endpoint of the other.

**Lemma 9.1** *The orientations on $\widetilde{\sigma}$ induced from the two domains that contain it are opposite.*

**Proof** This follows from the fact that left (or right) turns from the two domains containing $\widetilde{\sigma}$ result in motion in different directions along $\widetilde{\sigma}$. □

Recall from Lemma 7.1 that for all lifts $\widetilde{f}$ and all $\widetilde{x} \in H$, $\alpha(\widetilde{f}, \widetilde{x})$ and $\omega(\widetilde{f}, \widetilde{x})$ are single points in $S_\infty \cap \mathrm{Fix}(\widetilde{f})$.

**Lemma 9.2** *Suppose that $\widetilde{C}_1$ and $\widetilde{C}_2$ are domains with intersection $\widetilde{\sigma} \subset \widetilde{\mathcal{R}}$, that $\widetilde{f}_i = \widetilde{f}_{\widetilde{C}_i}$ and that $\widetilde{x} \in H$. If $\omega(\widetilde{f}_1, \widetilde{x}) \neq \widetilde{\sigma}^+{}_{\widetilde{C}_1}$ then $\omega(\widetilde{f}_2, \widetilde{x}) = \widetilde{\sigma}^+{}_{\widetilde{C}_2} = \widetilde{\sigma}^-{}_{\widetilde{C}_1}$. Symmetrically, if $\alpha(\widetilde{f}_1, \widetilde{x}) \neq \widetilde{\sigma}^-{}_{\widetilde{C}_1}$ then $\alpha(\widetilde{f}_2, \widetilde{x}) = \widetilde{\sigma}^-{}_{\widetilde{C}_2} = \widetilde{\sigma}^+{}_{\widetilde{C}_1}$.*

**Proof** Let $T_{\widetilde{\sigma}}$ be the root free covering translation with axis $\widetilde{\sigma}$ and orientation induced by $\widetilde{C}_2$. Then $\widetilde{f}_2^n = T_{\widetilde{\sigma}}^{dn} \widetilde{f}_1^n$, where $d > 0$ is the degree of Dehn twisting about $\mathcal{R}$. By hypothesis and by Lemma 9.1,

$$\omega(\widetilde{f}_1, \widetilde{x}) \neq \widetilde{\sigma}^+{}_{\widetilde{C}_1} = \widetilde{\sigma}^-{}_{\widetilde{C}_2} = T_{\widetilde{\sigma}}^-.$$

Since $\widetilde{f}_1^n(\widetilde{x})$ converges to $\omega(\widetilde{f}_1, \widetilde{x})$ it follows that $T_{\widetilde{\sigma}}^{dn} \widetilde{f}_1^n(\widetilde{x}) \to T_{\widetilde{\sigma}}^+$. This in turn proves that $\omega(\widetilde{f}_2, \widetilde{x}) = \widetilde{\sigma}^+{}_{\widetilde{C}_2}$. □

**Lemma 9.3** *There is a constant $D_1 > 0$ so that for all domains $\widetilde{C}$ and all $\widetilde{x} \in H$ such that $\mathrm{dist}(\widetilde{x}, \widetilde{C}) > D_1$, at least one of $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ and $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is an endpoint of the component $\widetilde{\sigma}$ of $\partial \widetilde{C}$ that is closest to $\widetilde{x}$.*

**Proof** Up to the action of covering translations there are only finitely many elements of $\widetilde{R}$. Thus, if the lemma is false there exists a domain $\widetilde{C}$ and a frontier component $\widetilde{\sigma}$ of $\widetilde{C}$ and a sequence $\widetilde{x}_k \in H$ such that:

- $\tilde{\sigma}$ is the component of $\partial\tilde{C}$ closest to $\tilde{x}_k$.
- Neither $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x}_k)$ nor $\omega(\tilde{f}_{\tilde{C}}, \tilde{x}_k)$ is an endpoint of $\tilde{\sigma}$.
- $\text{dist}(\tilde{x}_k, \tilde{C}) \to \infty$.

Consider the annular cover $A_\sigma$ and the induced map $f_\sigma\colon A_\sigma \to A_\sigma$. Let $\hat{x}_k$ be the image of $\tilde{x}_k$ in $A_\sigma$. By the second item,

$$\alpha(\tilde{f}_{\tilde{C}}, \tilde{x}_k), \omega(\tilde{f}_{\tilde{C}}, \tilde{x}_k) \in \text{Fix}(\tilde{f}_{\tilde{C}}) \cap (S_\infty \setminus T^{\pm}_{\tilde{\sigma}});$$

in particular $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x}_k)$ and $\omega(\tilde{f}_{\tilde{C}}, \tilde{x}_k)$ belong to the same component of $S_\infty \setminus T^{\pm}_{\tilde{\sigma}}$ because $\text{Fix}(\hat{f}_{\tilde{C}}) \cap S_\infty$ consists of ends of $\tilde{C}$ in $S_\infty$ and $\tilde{C}$ lies on one side of $\sigma$. It follows that $\alpha(f_\sigma, \hat{x}_k)$ and $\omega(f_\sigma, \hat{x}_k)$ belong to the same component of $\partial A_\sigma$. From the first and third items we conclude that every neighborhood of the other component of $\partial A_\sigma$ contains $\hat{x}_k$ for all sufficiently large $k$ in contradiction to Corollary 8.17. □

For $\tilde{C}$ a domain and $D > 0$ we let $N_D(\tilde{C})$ be the set of points in $H$ whose distance from $\tilde{C}$ is less than or equal to $D$.

**Corollary 9.4** *Suppose that $D_1$ is the constant of Lemma 9.3, that $\tilde{C}$ is a domain and that $\tilde{x} \in H$. If neither $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ nor $\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$ is an endpoint of a component of $\partial\tilde{C}$ then $\tilde{f}_{\tilde{C}}(\tilde{x}) \in N_{D_1}(\tilde{C})$.*

**Proof** This is an immediate consequence of Lemma 9.3. □

**Corollary 9.5** *For all $\tilde{x} \in H$ either:*

(1) *There is a domain $\tilde{C}$ such that $\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$ is not an endpoint of a component of $\partial\tilde{C}$.*

(2) *There is a component $\tilde{\sigma}$ of $\tilde{\mathcal{R}}$ such that both $\omega(\tilde{f}_{\tilde{C}_1}, \tilde{x})$ and $\omega(\tilde{f}_{\tilde{C}_2}, \tilde{x})$ are endpoints of $\tilde{\sigma}$, where $\tilde{C}_1$ and $\tilde{C}_2$ are the two domains that contain $\tilde{\sigma}$ in their boundaries.*

*Moreover, if (1) is satisfied then $\tilde{C}$ is unique and (2) is not satisfied and if (2) is satisfied then $\tilde{\sigma}$ is unique and (1) is not satisfied.*

**Remark 9.6** Suppose that $A$ is a closed $F$–invariant annulus in $S^2$ such that $\text{Fix}(F)$ is disjoint from the interior $A^\circ$ of $A$ but intersects both components of $\partial A$. If $F|_A$ is isotopic to the identity $\text{rel}\,\text{Fix}(F|_A)$ then the core curve $\sigma$ of $A$ is not an element of $\mathcal{R}$ and item (1) of Corollary 9.5 is satisfied for each $\tilde{x} \in H$ that projects into $A^\circ \subset M$. In the remaining case, $F|_A$ is isotopic $\text{rel}\,\text{Fix}(F|_A)$ to a nontrivial Dehn twist, $\sigma \in \mathcal{R}$ and item (2) of Corollary 9.5 is satisfied for each such $\tilde{x}$.

**Remark 9.7** In case (2), we expect (but have not proven) that $\omega(\widetilde{f}_{\widetilde{C}_1}, \widetilde{x})$ and $\omega(\widetilde{f}_{\widetilde{C}_2}, \widetilde{x})$ are distinct endpoints of $\widetilde{\sigma}$.

**Proof of Corollary 9.5** The moreover part of Corollary 9.5 follows from Lemma 9.2 and the obvious induction argument. It therefore suffices to find $\widetilde{C}$ satisfying (1) or $\widetilde{\sigma}$ satisfying (2).

Choose a domain $\widetilde{C}_1'$. If $\omega(\widetilde{f}_{\widetilde{C}_1'}, \widetilde{x})$ is not an endpoint of a component of $\partial\widetilde{C}_1'$ we are done. Otherwise, $\omega(\widetilde{f}_{\widetilde{C}_1'}, \widetilde{x})$ is an endpoint of a component $\widetilde{\sigma}_1$ of $\partial\widetilde{C}_1$ and we let $\widetilde{C}_2'$ be the domain whose intersection with $\widetilde{C}_1'$ is $\widetilde{\sigma}_1$. If $\widetilde{\omega}(\widetilde{C}_2', \widetilde{x})$ is either not the endpoint of a component of $\partial\widetilde{C}_2'$ or is an endpoint of $\widetilde{\sigma}_1$ we are done. Otherwise, let $\widetilde{C}_3'$ be the domain whose intersection with $\widetilde{C}_2'$ is the component $\widetilde{\sigma}_2$ of $\partial\widetilde{C}_2'$ whose endpoint set contains $\omega(\widetilde{f}_{\widetilde{C}_2'}, \widetilde{x})$. Iterating this procedure we either reach the desired conclusion or produce distinct domains $\widetilde{C}_k'$ such that $\omega(\widetilde{f}_{\widetilde{C}_k'}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}_k = \widetilde{C}_k' \cap \widetilde{C}_{k+1}'$. For all sufficiently large $k$, $\alpha(\widetilde{f}_{\widetilde{C}_k'}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}_{k-1}$ by Lemma 9.3.

Let $f_k\colon A_k \to A_k$ be the homeomorphism of the annular cover determined by $\widetilde{\sigma}_k$, let $\widetilde{f}_k = \widetilde{f}_{\widetilde{C}_k'}$ and let $\partial_- A_k$ and $\partial_+ A_k$ be the components of $\partial A_k$ that contain points that lift into the closure of $\widetilde{C}_k'$ and $\widetilde{C}_{k+1}'$ respectively. As usual, $\hat{x} \in A_k$ is the image of $\widetilde{x} \in H$. Then $\alpha(f_k, \hat{x}) \in \partial_- A_k$ and $\omega(f_k, \hat{x}) \in \partial_+ A_k$. The former follows from the fact that $\alpha(\widetilde{f}_k, \widetilde{x})) \in \mathrm{Fix}(\widetilde{f}_k) \cap (S_\infty \setminus T^{\pm}_{\widetilde{\sigma}_k})$ and the latter from the fact that $\omega(\widetilde{f}_{k+1}, \widetilde{x})) \in \mathrm{Fix}(\widetilde{f}_{k+1}) \cap (S_\infty \setminus T^{\pm}_{\widetilde{\sigma}_k})$.

Choose $j < l$ so that $\widetilde{\sigma}_j$ and $\widetilde{\sigma}_l$ project to the same element $\sigma \in \mathcal{R}$ but $\widetilde{\sigma}_k$ projects to a different element of $\mathcal{R}$ for all $j < k < l$. Choose an arc $\widetilde{\tau} \subset H$ with one endpoint on $\widetilde{\sigma}_j$, the other on $\widetilde{\sigma}_l$ and with interior disjoint from $\widetilde{\sigma}_j \cup \widetilde{\sigma}_l$. Then $\widetilde{\tau}$ projects to a path $\tau \subset M$ with endpoints in $\sigma$ and with interior disjoint from $\sigma$. Since $\sigma$ disconnects $S^2$, both ends of $\tau$ belong to the same component $X$ of $S^2 \setminus \mathcal{R}$. Let $Y \neq X$ be the other component of $S^2 \setminus \mathcal{R}$ that contains $\sigma$ in its closure. The interiors of the domains $\widetilde{C}_{j+1}$ and $\widetilde{C}_l$ both project to $X$ and the interiors of $\widetilde{C}_j$ and $\widetilde{C}_{l+1}$ both project to $Y$. A covering translation $T$ satisfying $T(\sigma_j) = \sigma_l$ also satisfies $T(\widetilde{C}_{j+1}) = \widetilde{C}_l$ and $T(\widetilde{C}_j) = \widetilde{C}_{l+1}$. It follows that

$$T\widetilde{f}_{j+1}T^{-1} = \widetilde{f}_l \quad \text{and} \quad T\widetilde{f}_jT^{-1} = \widetilde{f}_{l+1}.$$

Letting $\widetilde{y} = T(\widetilde{x})$, we have

$$\omega(\widetilde{f}_l, \widetilde{y}) = T\omega(\widetilde{f}_{j+1}, \widetilde{x}) \in \mathrm{Fix}(\widetilde{f}_l) \cap (S_\infty \setminus T^{\pm}_{\widetilde{\sigma}_l}),$$
$$\alpha(\widetilde{f}_{l+1}, \widetilde{y}) = T\alpha(\widetilde{f}_j, \widetilde{x}) \in \mathrm{Fix}(\widetilde{f}_{l+1}) \cap (S_\infty \setminus T^{\pm}_{\widetilde{\sigma}_l}).$$

Thus $\omega(f_l, \hat{y}) \in \partial_- A_l$ and $\alpha(f_k, \hat{x}) \in \partial_+ A_l$, which contradicts Corollary 8.16 and the fact that $\alpha(f_l, \hat{x}) \in \partial_- A_l$ and $\omega(f_l, \hat{x}) \in \partial_+ A_l$.

The process therefore terminates after finitely many steps. $\square$

**Definition 9.8** If Corollary 9.5(1) is satisfied then we say that $\widetilde{C}$ is the *$\omega$–domain for* $\widetilde{x}$ and $\widetilde{f}_{\widetilde{C}}$ the *$\omega$–lift for* $\widetilde{x}$. Otherwise, Corollary 9.5(2) is satisfied and we say that $\widetilde{C}_1$ and $\widetilde{C}_2$ are the *$\omega$–domains for* $\widetilde{x}$ and $\widetilde{f}_{\widetilde{C}_1}$ and $\widetilde{f}_{\widetilde{C}_2}$ are the *$\omega$–lifts for* $\widetilde{x}$.

**Corollary 9.9** *Let $D_1$ be the constant of Lemma 9.3.*

(1) *If $\widetilde{C}$ is the unique $\omega$–domain for $\widetilde{x}$ then $\widetilde{f}^n_{\widetilde{C}}(\widetilde{x}) \in N_{D_1}(\widetilde{C})$ for all sufficiently large $n$.*

(2) *If $\widetilde{C}_1$ and $\widetilde{C}_2$ are $\omega$–domains for $\widetilde{x}$ with intersection $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ then $\widetilde{f}^n_{\widetilde{C}_i}(\widetilde{x}) \in N_{D_1}(\widetilde{C}_1 \cup \widetilde{C}_2)$ for $i = 1, 2$ and all sufficiently large $n$.*

**Proof** If $\widetilde{C}$ is the unique $\omega$–domain for $\widetilde{x}$ and (1) fails then there exist arbitrarily large $n$ such that $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x}) \notin N_{D_1}(\widetilde{C})$. The component $\widetilde{\sigma}_n$ of $\partial\widetilde{C}$ that is closest to $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x})$ takes on infinitely many values as $n \to \infty$. By restricting to large $n$, we may assume that $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is not an endpoint of $\widetilde{\sigma}_n$. By hypothesis, $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is not an endpoint of $\widetilde{\sigma}_n$. This contradiction to Lemma 9.3 completes the proof of (1).

Suppose now that (2) fails. Since $\widetilde{f}_{\widetilde{C}_1}$ and $\widetilde{f}_{\widetilde{C}_2}$ differ by an iterate of $T_{\widetilde{\sigma}}$ and since $T_{\widetilde{\sigma}}$ preserves both $\widetilde{C}_1$ and $\widetilde{C}_2$, it follows that

$$\widetilde{f}^n_{\widetilde{C}_1}(\widetilde{x}) \notin N_{D_1}(\widetilde{C}_1 \cup \widetilde{C}_2) \quad \text{if and only if} \quad \widetilde{f}^n_{\widetilde{C}_2}(\widetilde{x}) \notin N_{D_1}(\widetilde{C}_1 \cup \widetilde{C}_2).$$

We may then assume that there exist arbitrarily large $n$ such that $\widetilde{f}^n_{\widetilde{C}_1}(\widetilde{x}) \notin N_{D_1}(\widetilde{C}_1 \cup \widetilde{C}_2)$ and such that the component $\widetilde{\sigma}_n$ of $\partial\widetilde{C}$ that is closest to $\widetilde{f}^n_{\widetilde{C}_1}(\widetilde{x})$ is not $\widetilde{\sigma}$. Since $\widetilde{f}^n_{\widetilde{C}_1}(\widetilde{x})$ converges to an endpoint of $\widetilde{\sigma}$, $\widetilde{\sigma}_n$ takes on infinitely many values as $n \to \infty$. By restricting to large $n$, we may assume that $\alpha(\widetilde{f}_{\widetilde{C}_1}, \widetilde{x})$ is not an endpoint of $\widetilde{\sigma}_n$. This contradicts Lemma 9.3 and the assumption that $\omega(\widetilde{f}_{\widetilde{C}_1}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma} \neq \widetilde{\sigma}_n$. $\square$

We record the following observation for easy reference.

**Lemma 9.10** *If $\widetilde{f}_{\widetilde{C}}{}^{k_i}(\widetilde{x}) \in N_D(\widetilde{C})$ for some $D > 0$ and some $k_i \to \infty$ then $\widetilde{C}$ is an $\omega$–domain for $\widetilde{x}$.*

**Proof** It suffices to show that if $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ and $C'$ is the other domain whose frontier contains $\widetilde{\sigma}$ then $\omega(\widetilde{f}_{\widetilde{C}'}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}$. The covering translation $T_{\widetilde{\sigma}}$ preserves $N_D(\widetilde{C})$. Since the maps $\widetilde{f}_{\widetilde{C}'}{}^{k_i}$ and $\widetilde{f}_{\widetilde{C}}{}^{k_i}$ differ by an iterate of $T_{\widetilde{\sigma}}$, it follows that $\widetilde{f}_{\widetilde{C}'}{}^{k_i}(\widetilde{x}) \in N_D(\widetilde{C})$ and hence that $\omega(\widetilde{f}_{\widetilde{C}'}, \widetilde{x})$ lies in the Cantor set of ends of $\widetilde{C}$ and in the ends of $\widetilde{C}'$. Since the ends of $\widetilde{\sigma}$ are the only points in the intersection of these Cantor sets, $\omega(\widetilde{f}_{\widetilde{C}'}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}$. $\square$

## 10 Domain covers

Let $\widetilde{C}$ be a domain and let $C$ be its image in $S$. Recall that $\mathrm{Stab}(\widetilde{C})$ is the subgroup of covering translations that preserve $\widetilde{C}$ and that elements of $\mathrm{Stab}(\widetilde{C})$ commute with $\widetilde{f}_{\widetilde{C}}$. We cannot restrict $f$ to $C$ because $C$ is not $f$–invariant and we can not replace $f$ by an isotopic map that preserves $C$ because we might lose the entropy zero property. Instead we lift to the $\pi_1(C)$ cover $\overline{C}$ of $S$. More precisely we make the following definitions.

**Definitions 10.1** Define $\overline{C}$ to be the quotient space of $H$ by the action of $\mathrm{Stab}(\widetilde{C})$ and $\overline{f}_C\colon \overline{C} \to \overline{C}$ to be the homeomorphism induced by $\widetilde{f}_{\widetilde{C}}$. Up to conjugacy, $\overline{f}_C\colon \overline{C} \to \overline{C}$ is independent of the choice of lift $\widetilde{C}$ of $C$. Define $\overline{C}_{\mathrm{core}} \subset \overline{C}$ to be the quotient space of $\widetilde{C} \subset H$ by the action of $\mathrm{Stab}(\widetilde{C})$.

**Standing notation 10.2** Our convention will be that if $\widetilde{x} \in \widetilde{C}$ then its image in $M$ is $x$ and its image in $\overline{C}$ is $\overline{x}$.

Note that $\overline{C}_{\mathrm{core}}$ is homeomorphic to $C$ and that if $\mathcal{R} \neq \varnothing$ then (topologically) $\overline{C}$ is obtained from $\overline{C}_{\mathrm{core}}$ by adding collar neighborhoods to each component of $\partial\overline{C}_{\mathrm{core}}$. Note also that $\overline{f}_C$ is isotopic to the identity.

If $\widetilde{C}$ is both an $\alpha$–domain and an $\omega$–domain for $\widetilde{x}$ then we say that $\widetilde{C}$ is a *home domain* for $\widetilde{x}$. Denote the set of birecurrent points for $f$ and $\overline{f}_C$ by $\mathcal{B}(f)$ and $\mathcal{B}(\overline{f}_C)$ respectively. Denote the full preimage in $H$ of $\mathcal{B}(f)$ by $\widetilde{\mathcal{B}}(f)$. The following proposition, whose proof is delayed until the end of the section, is the main result of this section.

**Proposition 10.3** *If $\widetilde{C}$ is an $\omega$–domain for $\widetilde{x} \in \widetilde{\mathcal{B}}(f)$ then $\overline{x} \in \mathcal{B}(\overline{f}_C)$ and $\widetilde{C}$ is a home domain for $\widetilde{x}$. Moreover if $\widetilde{\omega}(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ then $\widetilde{\alpha}(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is also an endpoint of $\widetilde{\sigma}$.*

As an immediate corollary we have:

**Corollary 10.4** *For each $\widetilde{x} \in \widetilde{\mathcal{B}}(f)$ one of the following is satisfied:*

1. *There is a unique home domain $\widetilde{C}$ for $\widetilde{x}$; neither $\widetilde{\alpha}(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ nor $\widetilde{\omega}(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is the endpoint of a component of $\partial\widetilde{C}$.*
2. *There are two home domains $\widetilde{C}_1$ and $\widetilde{C}_2$ for $\widetilde{x}$. The intersection $\widetilde{C}_1 \cap \widetilde{C}_2$ is a component $\widetilde{\sigma}$ of $\widetilde{\mathcal{R}}$ and for $i = 1, 2$, both $\widetilde{\alpha}(\widetilde{f}_{\widetilde{C}_i}, \widetilde{x})$ and $\widetilde{\omega}(\widetilde{f}_{\widetilde{C}_i}, \widetilde{x})$ are endpoints of $\widetilde{\sigma}$.*

The following definition is key to the proof of Proposition 10.3.

**Definition 10.5** A covering translation $T\colon H \to H$ is a *near-cycle of period $m$ for $\tilde{x} \in H$ with respect to $\tilde{f}_{\tilde{C}}$* if there is a free disk $U$ for $f$ and a lift $\tilde{U}$ that contains $\tilde{x}$ such that $\tilde{f}_{\tilde{C}}{}^m(\tilde{x}) \in T(\tilde{U})$. If $m$ is irrelevant then we simply say that $T$ *is a near-cycle for $\tilde{x} \in H$ with respect to $\tilde{f}_{\tilde{C}}$*.

**Remark 10.6** It is an immediate consequence of the definitions that if $T\colon H \to H$ is a near-cycle of period $m > 0$ with respect to $\tilde{f}_{\tilde{C}}$ for $\tilde{x}$ then it is also a near-cycle of period $m$ with respect to $\tilde{f}_{\tilde{C}}$ for all points in a neighborhood of $\tilde{x}$. Moreover, it is clear that by shrinking the free disk $U$ slightly to $U_0$, we may assume that $\mathrm{cl}(U_0)$ is contained in a free disk and we still have $\tilde{f}_{\tilde{C}}{}^m(\tilde{x}) \in T(\tilde{U}_0)$.

**Remark 10.7** A point $\tilde{x} \in H$ has at least one near cycle with respect to $\tilde{f}_{\tilde{C}}$ if and only if its image $x \in M$ is free disk recurrent.

**Remark 10.8** The only near-cycles for $\tilde{x} \in H$ with respect to $\tilde{f}_{\tilde{C}}$ that we make use of are those that are contained in $\mathrm{Stab}(\tilde{C})$.

The following lemma is essentially the same as [11, Lemma 10.5]. We reprove it here because our assumptions have changed.

**Lemma 10.9** *If $T \in \mathrm{Stab}(\tilde{C})$ is a near-cycle for $\tilde{x} \in H$ with respect to $\tilde{f}_{\tilde{C}}$ then $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ and $\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$ can not both lie in the same component of $S_\infty \setminus \{T^+, T^-\}$.*

**Proof** If $T$ is parabolic let $\tilde{\sigma}$ be a horocycle preserved by $T$; otherwise let $\tilde{\sigma}$ be the axis of $T$. From $T \in \mathrm{Stab}(\tilde{C})$ it follows that $\tilde{\sigma}$ is either an element of $\tilde{\mathcal{R}}$ or disjoint from $\tilde{\mathcal{R}}$. Let $f_\sigma\colon A_\sigma \to A_\sigma$ be as in Definitions 7.2. We assume the result is false and argue to a contradiction. By Lemma 8.9, we may apply Lemma 8.15 with $h = f_\sigma$, $r = 1$ and $\hat{x}_1$ the image of $\tilde{x}$ in $A_\sigma$. Assume the notation of that lemma. The lifts $\tilde{B}_1$ and $\tilde{B}_1'$ of $\hat{B}_1$ that contain $\tilde{x}$ and $T(\tilde{x})$ respectively are disjoint and $\tilde{f}$–invariant up to isotopy rel the orbits of $\tilde{x}$ and $T(\tilde{x})$. [11, Lemma 8.7(2)] implies that $\tilde{B}_1$ and $\tilde{B}_1'$ have parallel orientations. But it follows from the fact that the endpoints of $\tilde{B}_1$ are $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ and $\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$ and the endpoints of $\tilde{B}_1'$ are $T\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ and $T\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$, that these four points must occur in a configuration in $S_\infty$, which in turn implies that $\tilde{B}_1$ and $\tilde{B}_1'$ have antiparallel orientations. This contradiction completes the proof. □

**Remark 10.10** In the case that the covering translation $T$ is parabolic, Lemma 10.9 asserts that at least one of $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ and $\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$ must equal $T^{\pm}$.

**Lemma 10.11** *Suppose $\widetilde{C}$ is an $\omega$–domain for $\widetilde{x}$, that $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ and that $\bar{x}$ is $\bar{f}_C$–recurrent. Then every near cycle $T \in \mathrm{Stab}(\widetilde{C})$ for a point in the $\widetilde{f}_{\widetilde{C}}$–orbit of $\widetilde{x}$ is hyperbolic with axis $\widetilde{\sigma}$.*

**Proof** To simplify notation we write $\widetilde{f} = \widetilde{f}_{\widetilde{C}}$. There is no loss in assuming that $T$ is a near cycle for $\widetilde{x}$. Let $U$ be the free disk with respect to which $T$ is defined, let $\widetilde{U}$ be the lift of $U$ containing $\widetilde{x}$ and let $n$ satisfy $\widetilde{f}^n(\widetilde{x}) \in T(\widetilde{U})$. There is a neighborhood $x \in V \subset U$ such that $f^n(V) \subset U$. Let $\widetilde{V}$ be the lift of $V$ contained in $\widetilde{U}$. By Remark 10.6 we may assume that the diameter of $\widetilde{U}$ in the hyperbolic metric is finite.

If $\alpha(\widetilde{f}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}$ then Lemma 10.9 and the fact that $\widetilde{\sigma} \subset \partial\widetilde{C}$ complete the proof. Suppose then that $\alpha(\widetilde{f}, \widetilde{x})$ is not an endpoint of $\widetilde{\sigma}$ and in particular, $\alpha(\widetilde{f}, \widetilde{x}) \neq \omega(\widetilde{f}, \widetilde{x})$.

Since $\bar{x}$ is $\bar{f}$–recurrent, there exist $n_i \to \infty$ and $S_i \in \mathrm{Stab}(\widetilde{C})$ such that $\widetilde{f}^{n_i}(\widetilde{x}) \in S_i(\widetilde{V}) \subset S_i(\widetilde{U})$. From $\widetilde{f}^n(S_i\widetilde{V}) = S_i\widetilde{f}^n(\widetilde{V}) \subset S_iT(\widetilde{U})$ we see that $\widetilde{f}^n(\widetilde{f}^{n_i}(\widetilde{x})) \in S_iTS_i^{-1}(S_i(\widetilde{U}))$ and hence that $T_i = S_iTS_i^{-1}$ is a near cycle for $\widetilde{f}^{n_i}(\widetilde{x})$. Note also that both $\widetilde{f}^{n+n_i}(\widetilde{x})$ and $T_i\widetilde{f}^{n_i}(\widetilde{x})$ are contained in $T_i(S_i(\widetilde{U}))$ and so

$$\mathrm{dist}\big(\widetilde{f}^{n+n_i}(\widetilde{x}), T_i\widetilde{f}^{n}(\widetilde{x})\big)$$

is bounded independently of $n_i$.

If $T$, and hence each $T_i$, is parabolic then $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x}) \neq T_i^{\pm}$ because $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is an endpoint of the axis $\widetilde{\sigma}$ of a hyperbolic covering translation. Lemma 10.9 (see also Remark 10.10) therefore implies that each $T_i^{\pm} = \alpha(\widetilde{f}, \widetilde{x})$. In this case the $T_i$ are iterates of a single parabolic covering translation and there is a neighborhood of $\omega(\widetilde{f}, \widetilde{x})$ that is moved off of itself by every $T_i$. This contradicts $\lim \widetilde{f}^{n_i}(\widetilde{x}) = \omega(\widetilde{f}, \widetilde{x})$ and $\lim T_i(\widetilde{f}^{n_i}(\widetilde{x})) = \lim(\widetilde{f}^{n+n_i}(\widetilde{x})) = \omega(\widetilde{f}, \widetilde{x})$. We conclude that $T$ and each $T_i$ are hyperbolic. Let $A_T$ be the axis of $T$ and $A_i = S_i(A_T)$ the axis of $T_i$.

To complete the proof we assume that $A_T \neq \widetilde{\sigma}$ and argue to a contradiction.

We claim that $A_i \neq \widetilde{\sigma}$. This is obvious if $A_T$ is not an element of $\widetilde{\mathcal{R}}$ so we assume that $A_T$ is an element of $\widetilde{\mathcal{R}}$ and that $\widetilde{\sigma} = A_i = S_i(A_T)$ for some $S_i \in \mathrm{Stab}(\widetilde{C})$ and argue to a contradiction. Keeping in mind that $\widetilde{\sigma}$ and $A_T$ are distinct components of the frontier of $\widetilde{C}$, Lemma 10.9 implies that $\alpha(\widetilde{f}, \widetilde{x})$, which by Lemma 7.1 is a single point in the intersection of $S_\infty$ with the closure of $\widetilde{C}$, is an endpoint of $A_T$. The axis of $S_i$ is contained in $\widetilde{C}$ and is not $\widetilde{\sigma}$ or $A_T$. It follows that the axis of $S_i$ is disjoint from $A_T$ and $\widetilde{\sigma}$ and has no endpoints in common with either. Since $\widetilde{\sigma} = S_i(A_T)$, the axis of $S_i$ does not separate $A_T$ from $\widetilde{\sigma}$ and so does not separate $\alpha(\widetilde{f}, \widetilde{x})$ from $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$. This contradicts Lemma 10.9 applied to the near cycle $S_i$ and so completes the proof that $A_i \neq \widetilde{\sigma}$.

After passing to a subsequence we may assume that either the $A_i$ are all the same or all different. In the former case, $T_i$ is independent of $i$ and there is a neighborhood of $\omega(\tilde f, \tilde x)$ that is moved off of itself by each $T_i$. As above this contradicts the fact that $T_i(\tilde f^{n_i}(\tilde x)) \to \omega(\tilde f, \tilde x)$. We may therefore assume that the $A_i$ are distinct lifts of a closed curve in $M$ and hence, after passing to a subsequence, converge to some point $Q \in S_\infty$. If $Q \neq \omega(\tilde f, \tilde x)$ then there is a neighborhood of $\omega(\tilde f, \tilde x)$ that is moved off of itself by each $T_i$ and we have a contradiction. Thus $Q = \omega(\tilde f, \tilde x)$.

For sufficiently large $i$ the endpoints of $A_i$ are contained in a neighborhood of $\omega(\tilde f, \tilde x)$ that does not contain $\alpha(\tilde f, \tilde x)$ and does not contain the other endpoint of $\tilde\sigma$. Since $A_i$ is disjoint from $\tilde\sigma$, it does not separate $\alpha(\tilde f, \tilde x)$ from $\omega(\tilde f, \tilde x)$. This contradicts Lemma 10.9 applied to $T_i$ since neither $\alpha(\tilde f, \tilde x)$ nor $\omega(\tilde f, \tilde x)$ is an endpoint of $A_i$. □

**Lemma 10.12** *Suppose that $U$ is a free disk, that $x \in U$ is recurrent (birecurrent) with respect to $f$ and that the set of lifts of $U$ to $H$ that intersect $\{\tilde f_{\tilde C}^{\,k}(\tilde x) : k \ge 0\}$ is finite up to the action of* $\mathrm{Stab}(\tilde C)$*. Then $\bar x \in \bar C$ is recurrent (birecurrent) with respect to $\bar f \colon \bar C \to \bar C$.*

**Proof** The set of lifts of $U$ to $H$ that intersect $\{\tilde f_{\tilde C}^k(\tilde x) : k \ge 0\}$ is finite up to the action of $\mathrm{Stab}(\tilde C)$ if and only if the set of lifts of $U$ to $\bar C$ that intersect $\{\bar f_C^k(\bar x) : k \ge 0\}$ is finite. We may therefore replace the former with the latter in the hypotheses of this lemma.

Suppose that $x$ is recurrent. We must prove that $\bar x$ is recurrent and that if $x$ is recurrent with respect to $f^{-1}$ then $\bar x$ is recurrent with respect to $\bar f^{-1}$.

Let $\bar U_1, \dots, \bar U_m$ be the lifts of $U$ to $\bar C$ that intersect $\{\bar f_C^k(\bar x) : k \ge 0\}$ and let $\bar x_j \in \bar U_j$ be the corresponding lifts of $x$. We may assume that $\bar x_1 = \bar x$. Choose a sequence $n_i \to \infty$ such that $f^{n_i}(x) \to x$ and such that each $f^{n_i}(x) \in U$. After passing to a subsequence we may assume that $\bar f_C^{n_i}(\bar x_1) \in \bar U_s$ where $s$ is independent of $i$. Then $\bar f_C^{n_i}(\bar x_1) \to \bar x_s$ and we are done if $s = 1$. Otherwise by renumbering we may assume that $s = 2$. Since $\bar x_2$ is in the $\omega$–limit set of $\bar x_1$, each point in $\{\bar f_C^k(\bar x_2) : k \ge 0\}$ that projects to $U$ is contained in some $\bar U_j$. We may therefore apply the previous argument with $\bar x_2$ in place of $\bar x_1$. After passing to a further subsequence we may assume that $\bar f_C^{n_i}(\bar x_2) \to \bar x_t$ where $t \neq 2$ because $\bar f_C^{n_i}(\bar x_1)$ is the unique point in $\bar U_2$ that projects to $f^{n_i}(x)$. If $t = 1$ then $\bar x_1$ is in the $\omega$–limit set of $\bar x_1$ and we are done. Otherwise we may assume $t = 3$. After iterating this argument at most $m$ times, we have shown that $\bar x$ is recurrent.

From the recurrence of $\bar x$, it follows that a lift of $U$ to $\bar C$ intersects $\{\bar f_C^k(\bar x) : k \ge 0\}$ if and only if it intersects $\{\bar f_C^k(\bar x) : k \in \mathbb{Z}\}$. In particular, the set of lifts of $U$ to $\bar C$

that intersect $\{\overline{f}_C^{-k}(\overline{x}) : k \ge 0\}$ is finite. If $x$ is recurrent with respect to $f^{-1}$ then by the above argument $\overline{x} \in \overline{C}$ is recurrent with respect to $\overline{f}^{-1}\colon \overline{C} \to \overline{C}$ as desired. □

**Remark 10.13** If $\widetilde{U}$ is a lift of a disk $U$ and $T_1, T_2$ are covering translations then $T_1(\widetilde{U})$ and $T_2(\widetilde{U})$ are in the same $\mathrm{Stab}(\widetilde{C})$–orbit if and only if $T_2T_1^{-1} \in \mathrm{Stab}(\widetilde{C})$. Thus a collection of lifts $\{T_m(\widetilde{U})\}$ of $U$ is finite up to the action of $\mathrm{Stab}(\widetilde{C})$ if and only if the $T_m$ determine only finitely many right cosets of $\mathrm{Stab}(\widetilde{C})$.

**Proof of Proposition 10.3** Let $U$ be a free disk of bounded diameter that contains $x$ and let $\widetilde{U}$ be the lift that contains $\widetilde{x}$.

First suppose that $\widetilde{\omega}(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is not an endpoint of an element of $\widetilde{\mathcal{R}}$. Corollary 9.9(1) implies that for some $D$ and all $k \ge 0$, $\widetilde{f}_{\widetilde{C}}{}^k(\widetilde{x}) \in N_D(\widetilde{C})$ or equivalently, $\overline{f}_C{}^k(\overline{x}) \in N_D(\overline{C}_{\mathrm{core}})$. Since $N_D(\overline{C}_{\mathrm{core}})$ is compact, it follows that $\{\overline{f}_C{}^k(\overline{x}) \mid k \ge 0\}$ intersects only finitely many lifts of $U$. Equivalently, $\{\widetilde{f}_{\widetilde{C}}{}^k(\widetilde{x}) : k \ge 0\}$ intersects only finitely many lifts of $U$ to $H$ up to the action of the group $\mathrm{Stab}(\widetilde{C})$. Lemma 10.12 implies that $\overline{x}$, and hence $\overline{f}_C{}^k(\overline{x})$ for all $k$, is recurrent under $\overline{f}_C$. Since the forward $\overline{f}_C$–orbit of $\overline{f}_C{}^k(\overline{x})$ is eventually contained in $N_D(\overline{C}_{\mathrm{core}})$, it follows that $\overline{f}_C{}^k(\overline{x}) \in N_D(\overline{C}_{\mathrm{core}})$ for all $k$ and hence that $\widetilde{f}_{\widetilde{C}}{}^k(\widetilde{x}) \in N_D(\widetilde{C})$ for all $k$. Lemma 9.10 applied to $\widetilde{f}_{\widetilde{C}}{}^{-1}$ implies that $\widetilde{C}$ is an $\alpha$–domain for $\widetilde{x}$ and hence a home domain for $\widetilde{x}$.

We assume now that $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ is an endpoint of $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ and that $\widetilde{C}_1$ and $\widetilde{C}_2$ are the two domains that contain $\widetilde{\sigma}$ in their frontier. We will treat $\widetilde{C}_1$ and $\widetilde{C}_2$ symmetrically and prove that the proposition holds for $\widetilde{C} = \widetilde{C}_1$ and $\widetilde{C} = \widetilde{C}_2$. Denote $\widetilde{f}_{\widetilde{C}_1}$ by $\widetilde{f}_1$ and $\widetilde{f}_{\widetilde{C}_2}$ by $\widetilde{f}_2$. When near cycles are defined with respect to $\widetilde{f}_i$ we refer to them as $\widetilde{f}_i$–near cycles. Let $S$ be a root-free covering translation with axis $\widetilde{\sigma}$. Corollary 9.9(2) implies that $\widetilde{f}_1^k(\widetilde{x}), \widetilde{f}_2^k(\widetilde{x}) \in N_D(\widetilde{C}_1 \cup \widetilde{C}_2)$ for some $D$ and all $k \ge 0$. We may assume without loss that $\widetilde{U} \subset N_D(\widetilde{C}_1) \cap N_D(\widetilde{C}_2)$.

After interchanging $\widetilde{C}_1$ with $\widetilde{C}_2$ if necessary, we may assume by Lemma 9.2 that $\alpha(\widetilde{f}_1, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}$. Lemma 10.9 implies that every $\widetilde{f}_1$–near cycle $T \in \mathrm{Stab}(\widetilde{C}_1)$ for a point in the $\widetilde{f}_1$–orbit of $\widetilde{x}$ is an iterate of $S$. We will apply this as follows. If $T_1$ and $T_2$ are $\widetilde{f}_1$–near cycles for $\widetilde{x}$ and if $T_1T_2^{-1}$ (which is a near cycle for a point in the $\widetilde{f}_1$–orbit of $\widetilde{x}$) is an element of $\mathrm{Stab}(\widetilde{C}_1)$ then $T_1T_2^{-1}$ is an iterate of $S$. In particular, if $T_1$ and $T_2$ determine the same right coset of $\mathrm{Stab}(\widetilde{C}_1)$ then they also determine the same right coset of $\mathrm{Stab}(\widetilde{C}_2)$.

Let $\mathcal{U}_i$ be the set of lifts of $U$ that intersect $N_D(\widetilde{C}_i)$ and contain $\widetilde{f}_2^k(\widetilde{x})$ for some $k \ge 0$. To prove that $\overline{x}$ is $\overline{f}_2$–birecurrent it suffices by Lemma 10.12 to prove that $\mathcal{U}_1 \cup \mathcal{U}_2$ is finite up to the action of $\mathrm{Stab}(\widetilde{C}_2)$. As above, the compactness of $N_D(\overline{C}_{i_{\mathrm{core}}})$ implies that $\mathcal{U}_i$ is finite up to the action of $\mathrm{Stab}(\widetilde{C}_i)$.

Each element of $\mathcal{U}_i$ has the form $T(\widetilde{U})$ for some covering translation $T$; let $\mathcal{T}_i$ be the set of all such $T$. Each $T_m \in \mathcal{T}_1$ is an $\widetilde{f}_2$–near cycle for $\widetilde{x}$. Since $\widetilde{f}_2$ and $\widetilde{f}_1$ differ by an iterate of $S$, there exists $j_m$ such that $S^{j_m}T_m$ is an $\widetilde{f}_1$–near cycle for $\widetilde{x}$. Since $\mathcal{U}_1 = \{T_m(\widetilde{U})\}$ is finite up to the action of $\mathrm{Stab}(\widetilde{C}_1)$, Remark 10.13 implies that $\{T_m\}$, and hence $\{S^{j_m}T_m\}$, determine only finitely many right cosets of $\mathrm{Stab}(\widetilde{C}_1)$. As observed above, this implies that $\{S^{j_m}T_m\}$, and hence $\{T_m\}$, determine only finitely many right cosets of $\mathrm{Stab}(\widetilde{C}_2)$. Lemma 10.12 and a second application of Remark 10.13 complete the proof that $\bar{x}$ is $\bar{f}_2$–birecurrent.

Having established that $\bar{x}$ is recurrent for $\bar{f}_2^{-1}$, there exists $m_j \to \infty$ and $T'_j \in \mathrm{Stab}(\widetilde{C}_2)$ such that $\widetilde{f}_2^{-m_j}(\widetilde{x}) \in T'_j(\widetilde{U})$. Since $\widetilde{U}$ has bounded diameter, the distance between $\widetilde{f}_2^{-m_j}(\widetilde{x})$ and $T'_j(\widetilde{x})$ is bounded independently of $j$. It follows that $T'_j(\widetilde{x}) \to \alpha(\widetilde{f}_2, \widetilde{x})$. Lemma 10.11 implies that each $T'_j$ is an iterate of $S$. We conclude that $\alpha(\widetilde{f}_2, \widetilde{x})$ is an endpoint of the axis $\widetilde{\sigma}$ of $S$. This completes the proof for $\widetilde{C}_2$.

Now that we have established that $\alpha(\widetilde{f}_2, \widetilde{x})$ is an endpoint of $\widetilde{\sigma}$, this same argument can be applied to $\widetilde{C}_1$. □

**Lemma 10.14** (1) *If $\widetilde{C}$ is not a home domain for $\widetilde{y} \in \widetilde{\mathcal{B}}(f)$ then $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{y})$ and $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{y})$ are both endpoints of the component of $\partial\widetilde{C}$ that is closest to the home domain for $\widetilde{y}$.*

(2) *If $\widetilde{y} \in \widetilde{\mathcal{B}}(f)$, $\widetilde{C}$ is any domain and either $\alpha(\widetilde{f}_{\widetilde{C}}\widetilde{y})$ or $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{y})$ is an endpoint of a frontier component $\widetilde{\sigma}$ of $\widetilde{C}$ then both $\alpha(\widetilde{f}_{\widetilde{C}}\widetilde{y})$ and $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{y})$ are endpoints of $\widetilde{\sigma}$.*

**Proof** Item (1) follows from the existence of a home domain for $\widetilde{y}$, Lemma 9.2 and the obvious induction argument on the number of domains that separate $\widetilde{C}$ from a home domain for $\widetilde{y}$. Item (2) follows from (1) if $\widetilde{C}$ is not a home domain for $\widetilde{y}$, and from Proposition 10.3 otherwise. □

We conclude this section by strengthening Corollary 9.9.

**Corollary 10.15** *Suppose that $\widetilde{x} \in \widetilde{\mathcal{B}}(f)$ and that $D_1$ is the constant of Lemma 9.3.*

(1) *If $\widetilde{C}$ is the unique home domain for $\widetilde{x}$ then $\widetilde{f}^n_{\widetilde{C}}(\widetilde{x}) \in N_{D_1}(\widetilde{C})$ for all $n$.*

(2) *If $\widetilde{C}_1$ and $\widetilde{C}_2$ are home domains for $\widetilde{x}$ with intersection $\widetilde{\sigma} \in \widetilde{\mathcal{R}}$ then $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x}) \in N_{D_1}(\widetilde{C}_1 \cup \widetilde{C}_2)$ for all $n$.*

(3) *If $\mathrm{dist}(\widetilde{x}, \widetilde{\mathcal{R}}) > D_1$ then the domain that contains $\widetilde{x}$ is a home domain for $\widetilde{x}$.*

**Proof** Suppose that $\widetilde{C}$ is the unique home domain for $\widetilde{x}$ and that $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x}) \notin N_{D_1}(\widetilde{C})$. Choose $\epsilon$ less than the distance from $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x})$ to $N_{D_1}(\widetilde{C})$. Proposition 10.3 implies that

$\bar{x} \in \mathcal{B}(\bar{f}_C)$ and hence that there exist arbitrarily large $k$ and $S_k \in \mathrm{Stab}(\widetilde{C})$ such that the distance from $\widetilde{f}_{\widetilde{C}}{}^k(\widetilde{x})$ to $S_k \widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x})$ is less than $\epsilon$. Since $S_k$ preserves distance to $\widetilde{C}$, $\widetilde{f}_{\widetilde{C}}{}^k(\widetilde{x}) \notin N_{D_1}(\widetilde{C})$. This contradicts Corollary 9.9 and so completes the proof of (1).

Assuming the notation of (2), suppose that $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x}) \notin N_{D_1}(\widetilde{C}_1 \cup \widetilde{C}_2)$. There is no loss in assuming that $\widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x})$ is closer to $\widetilde{C}_1$ than $\widetilde{C}_2$. If $S_k \in \mathrm{Stab}(\widetilde{C}_1)$ then $S_k \widetilde{f}_{\widetilde{C}}{}^n(\widetilde{x})$ is closer to $\widetilde{C}_1$ than $\widetilde{C}_2$ and has distance greater than $D_1$ from $\widetilde{C}_1$. The argument given for (1) therefore applies in this context as well.

If $\mathrm{dist}(\widetilde{x}, \widetilde{\mathcal{R}}) > D_1$ and $\widetilde{C}$ is a domain that does not contain $\widetilde{x}$ then $\widetilde{x} \notin N_{D_1}(\widetilde{C})$. Item (3) therefore follows from items (1) and (2). □

## 11 Some results when $\mathcal{R} = \varnothing$

We say that a point $P \in S_\infty$ *projects to* a puncture $c$ in $M$ if some (and hence every) ray in $H$ that converges to $P$ projects to a ray in $M$ that converges to $c$. Note that if $P$ is the fixed point of a parabolic covering translation then $P$ projects to an isolated puncture in $M$.

**Definition 11.1** Suppose that $\widetilde{C}$ is a home domain for a lift $\widetilde{x}$ of $x \in M$ and that $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x}) = \omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x}) = P$. If there is a parabolic covering translation $T_P$ that fixes $P$ such that every near cycle $S \in \mathrm{Stab}(\widetilde{C})$ for every $\widetilde{f}_{\widetilde{C}}{}^k(\widetilde{x})$ is a positive iterate of $T_P$ then we say that $\widetilde{x}$ *tracks* $P$. If $c$ is the isolated puncture in $M$ to which $P$ projects, then we also say that $x$ *rotates about* $c$. (The latter is well-defined because (Corollary 9.5) $\widetilde{C}$ is the unique home domain for $\widetilde{x}$.)

**Definition 11.2** If $\widetilde{C}$ is a home domain for $\widetilde{x} \in H$ and $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x}) \neq \omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$, then let $\widetilde{\gamma}(\widetilde{x})$ be the oriented geodesic with endpoints $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$ and $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})$. Corollary 10.4 implies that $\widetilde{\gamma}(\widetilde{x})$ is independent of the choice of $\widetilde{C}$ in the case that $\widetilde{x}$ has two home domains. Let $\gamma(x) \subset M$ be the unoriented geodesic that is the projected image of $\widetilde{\gamma}(\widetilde{x})$. We say that $\widetilde{x}$ *tracks* $\widetilde{\gamma}(\widetilde{x})$ and that $x$ *tracks* $\gamma(x)$. Note that $\gamma(x)$ is independent of the choice of lift $\widetilde{x}$ and the choice of home domain for $\widetilde{x}$; the latter would not be true if we imposed an orientation on $\gamma(x)$.

If $f$ is isotopic to the identity then $\mathcal{R} = \varnothing$, $H$ is the only domain and there is a lift $\widetilde{f}_H$ that commutes with all covering translations and fixes every point in $S_\infty$. In the notation of Definitions 10.1, $\bar{M} = M$ and $\bar{f}\colon \bar{M} \to \bar{M}$ is just $f\colon M \to M$. We sometimes refer to $\widetilde{f}_H$ as *the preferred lift* of $f$ and sometimes drop the $H$ subscript.

In this section we import some results from [11] that apply to the case that $f$ is isotopic to the identity.

**Lemma 11.3** *Assume that $f$ is isotopic to the identity and periodic point free. Suppose that $x \in \mathcal{B}(f)$, that $\tilde{f}\colon H \to H$ is the preferred lift to the universal cover, that $\tilde{x}$ is a lift of $x$ and that $\alpha(\tilde{f},\tilde{x}) = \omega(\tilde{f},\tilde{x}) = P$. Then $P$ projects to an isolated puncture $c$ and $x$ rotates about $c$.*

**Proof** This is [11, Lemma 11.2]. □

**Lemma 11.4** *Assume that $f$ is isotopic to the identity and periodic point free. If $x \in \mathcal{B}(f)$ tracks $\gamma(x)$ then $\gamma(x)$ is a simple closed curve. If in addition $y \in \mathcal{B}(f)$ tracks $\gamma(y)$ then $\gamma(x)$ and $\gamma(y)$ are either disjoint or equal.*

**Proof** All references in this proof are to [11]. By Lemma 10.2(1) and Lemma 11.6(2), $\gamma(x)$ is simple and birecurrent. If $\gamma(x)$ is not a closed curve then by Lemma 11.6(3) there is a simple closed geodesic $\alpha$ such that $\alpha$ and $\gamma(x)$ intersect transversely and nontrivially and such that with respect to given orientations on $\alpha$ and $\gamma(x)$ all intersections have the same intersection number. This can not happen on a genus zero surface since $\alpha$ must separate. Thus $\gamma(x)$ is a simple closed curve. The second assertion of the lemma follows from Lemma 10.2(2). □

To make use of these lemmas in our present context we use the following consequence of Lemmas 8.9, 8.11 and 8.13.

**Lemma 11.5** *Assume that $h = f_\sigma\colon A_\sigma \to A_\sigma$ (respectively $f_\sigma\colon A^c_\sigma \to A^c_\sigma$) is as in Definitions 7.2 and that $W$ is as in Definition 8.5. If $\mathcal{T}$ is a fitted family that does not disappear under iteration then there exists an element $[\tau] \in \mathcal{T}$ such that $h_\#([\tau]) \cap W = \{[\tau]\}$.*

The proofs of [11, Lemmas 11.3 and 11.4] quote [11, Lemmas 10.2, 11.2 and 11.6]. The hypothesis that $f$ is periodic point free is only directly applied in the proofs of those three lemmas to prove [11, Lemma 10.8], whose conclusion is a weaker version of the conclusion of Lemma 11.5 above. Thus in each place that [11, Lemma 10.8] is applied in proving Lemmas 11.3 and 11.4 above we can replace it with Lemma 11.5. This justifies the following lemma.

**Lemma 11.6** *Lemmas 11.3 and 11.4 remain true if the hypothesis that $f$ is periodic point free is replaced by the hypothesis that the topological entropy of $F$ is zero.*

**Remark 11.7** The proof of [11, Lemma 10.8] is a pointer to the proof of [17, Theorem 5.5]. That theorem has three parts. The first two state that no element of $RH(W, \partial_+ W)$ doubles. The third uses the first two to prove the existence of $[\tau]$ as in Lemma 11.5. Thus our dividing the argument into Lemmas 8.9 and 8.11 follows the original proof.

## 12 Two compactifications

We now return to the general case, allowing the possibility that $\mathcal{R} = \emptyset$. Our goal in this section is to extend Lemma 11.6 to the case that $\mathcal{R} \neq \emptyset$. Our strategy is to apply Lemma 11.6 to $f_{\overline{C}}$ which is isotopic to the identity. Before doing so, we must address the fact that if $\mathcal{R} \neq \emptyset$ then two different compactifications of the universal cover of $\overline{C}$ are being used.

In the *extrinsic compactification*, the universal cover of $\overline{C}$ is metrically identified with the universal cover $\widetilde{M}$ of $M$, which is metrically identified with $H$ and is compactified by $S_\infty$. The covering translations of the universal cover of $\overline{C}$ are identified with the subgroup $\mathrm{Stab}(\widetilde{C})$ of covering translations of the universal cover of $M$; the closure in $S_\infty$ of the fixed points of the elements of $\mathrm{Stab}(\widetilde{C})$ is a Cantor set $K$ whose convex hull projects to $C_{\mathrm{core}} \subset \overline{C}$.

In the *intrinsic compactification*, $\overline{C}$ is viewed without regard to $M$ and is equipped with a hyperbolic structure in which the ends corresponding to the components of $\partial C$ are cusps. The universal cover of $\overline{C}$ is then metrically identified with $H$ and compactified with $S_\infty$. In this case, the set of fixed points of covering translations is dense in $S_\infty$. Topologically the intrinsic compactification of the universal cover is obtained from the extrinsic compactification by collapsing the closure of each component of $S_\infty \setminus K$ to a point.

We have defined $\overline{C}$ using the extrinsic metric so that geodesics in $\overline{C}_{\mathrm{core}}$ correspond exactly to geodesics in $C \subset M$. If one considers $\bar{f}\colon \overline{C} \to \overline{C}$ as a homeomorphism of a punctured surface without reference to $M$, as one should do when applying results from [11], then the intrinsic metric is used. To help separate the two, write $g\colon N \to N$ for $\bar{f}\colon \overline{C} \to \overline{C}$ when $\overline{C}$ has the intrinsic metric. Since $g$ is isotopic to the identity there is a preferred lift $\tilde{g}\colon \widetilde{N} \to \widetilde{N}$ to the universal cover that commutes with all covering translations. The "identity map" $p\colon \widetilde{M} \to \widetilde{N}$ conjugates $\tilde{f}_{\widetilde{C}}\colon \widetilde{M} \to \widetilde{M}$ to $\tilde{g}\colon \widetilde{N} \to \widetilde{N}$. The homeomorphism $p$, which is not an isometry, extends over the compactifying circles but not by a homeomorphism; it collapses the closure of each component of $S_\infty \setminus K$ to a point. In particular, $p|_K$ identifies a pair of points if and only if they bound a component of $\partial\widetilde{C}$.

Let $T(\widetilde{N})$ be the group of covering translations of $\widetilde{N}$ and let $B\colon \mathrm{Stab}(\widetilde{C}) \to T(\widetilde{N})$ be the bijection induced by $p$. The following properties are satisfied by $S, S' \in \mathrm{Stab}(\widetilde{C})$:

(a) If $S$ is parabolic then $B(S)$ is parabolic.

(b) If $S$ is hyperbolic then $B(S)$ is hyperbolic unless the axis of $S$ is a component of $\partial\widetilde{C}$, in which case it is parabolic.

(c) If $S^\pm = \{\alpha(\tilde f_{\tilde C}, \tilde x), \omega(\tilde f_{\tilde C}, \tilde x)\}$ then $B(S)^\pm = \{\alpha(\tilde g, \tilde x), \omega(\tilde g, \tilde x)\}$.

(d) If $S$ and $B(S)$ are hyperbolic then the axis of $S$ projects to a simple closed curve if and only if the axis of $B(S)$ projects to a simple closed curve.

(e) If $S, S', B(S)$ and $B(S')$ are hyperbolic then the axes of $S$ and $S'$ are equal or disjoint if and only if the axes of $B(S)$ and $B(S')$ are equal or disjoint.

**Lemma 12.1** *Suppose that $x \in \mathcal{B}(f)$, that $\tilde C$ is a home domain for a lift $\tilde x$ and that $\alpha(\tilde f_{\tilde C}, \tilde x) = \omega(\tilde f_{\tilde C}, \tilde x) = P$. Then $P$ projects to an isolated puncture $c$ and $x$ rotates about $c$.*

**Proof** Since $\alpha(\tilde f_{\tilde C}, \tilde x) = \omega(\tilde f_{\tilde C}, \tilde x) = P$, it follows that $\alpha(\tilde g, p(\tilde x)) = \omega(\tilde g, p(\tilde x)) = p(P)$. By Lemma 11.6, $p(P)$ projects to an isolated puncture $c'$ in $\overline{C}$ and there is a parabolic covering translation $T'$ that fixes $p(P)$ such every near cycle for every point in the orbit of $p(\tilde x)$ is a positive iterate of $T'$.

If $T \in \operatorname{Stab}(\tilde C)$ is the covering translation corresponding to $T'$ then every near cycle in $\operatorname{Stab}(\tilde C)$ for every point in the orbit of $\tilde x$ is a positive iterate of $T$. It suffices to show that $T$ is parabolic. Let $U$ be a free disk for $x$ with compact closure and let $\tilde U$ be the lift that contains $\tilde x$. Since (Proposition 10.3) $\tilde x \in \mathcal{B}(\bar f)$, there exist $n_i, a_i, m_j, b_j \to \infty$ such that $\tilde f_{\tilde C}^{\,n_i}(\tilde x) \in T^{a_i}(\tilde U)$ and $\tilde f_{\tilde C}^{\,-m_j}(\tilde x) \in T^{-b_j}(\tilde U)$. It follows that $P = \alpha(\tilde f_{\tilde C}, \tilde x) = T^-$ and $P = \omega(\tilde f_{\tilde C}, \tilde x) = T^+$. Thus $T$ is parabolic and we are done. □

**Lemma 12.2** *If $x \in \mathcal{B}(f)$ tracks $\gamma(x)$ then $\gamma(x)$ is a simple closed curve. If in addition $y \in \mathcal{B}(f)$ tracks $\gamma(y)$ then $\gamma(x)$ and $\gamma(y)$ are either disjoint or equal.*

**Proof** We may assume without loss that the axes of $\gamma(x)$ and $\gamma(y)$ are not components of $\partial C$ because such curves are simple and do not transversely intersect any other geodesics in $C$. Lemma 11.6 implies that the lemma holds with $\tilde f_{\tilde C}$ and $\tilde x$ replaced by $\tilde g$ and $p(\tilde x)$. Items (b), (d) and (e) above therefore complete the proof. □

The following corollary generalizes Lemma 10.11 which only applies when $\tilde \gamma$ is a component of $\partial \tilde C$.

**Corollary 12.3** *Suppose that $x \in \mathcal{B}(f)$, that $\tilde C$ is a home domain for a lift $\tilde x$ and that $\tilde x$ tracks $\tilde \gamma$. Then every $\tilde f_{\tilde C}$–near cycle $S \in \operatorname{Stab}(\tilde C)$ for a point in the orbit of $\tilde x$ is an iterate of $T_{\tilde \gamma}$.*

**Proof** We make use of the following consequences of Lemma 12.2 above and [11, Lemmas 8.7(2), 8.9 and 8.10]:

(1) Suppose that $\tilde{x}, \tilde{z}$ have $\tilde{C}$ as a home domain and that $\tilde{\gamma}(\tilde{x})$ and $\tilde{\gamma}(\tilde{y})$ are disjoint and antiparallel. Then $\tilde{x}$ and $\tilde{z}$ are not contained in any free disk for $\tilde{f}_{\tilde{C}}$.

(2) Suppose that $\tilde{x}, \tilde{y}, \tilde{z}$ have $\tilde{C}$ as a home domain, that $\tilde{\gamma}(\tilde{y})$ separates $\tilde{\gamma}(\tilde{x})$ and $\tilde{\gamma}(\tilde{z})$ and is antiparallel to both lines. Then $\tilde{x}$ and $\tilde{z}$ are not contained in any free disk for $\tilde{f}_{\tilde{C}}$.

We may assume without loss that $S$ is a near cycle for $\tilde{x}$. There exist $m > 0$ and a lift $\tilde{U}$ of a free disk $U \subset M$ such that $\tilde{x} \in \tilde{U}$ and $\tilde{f}_{\tilde{C}}{}^m(\tilde{x}) \in S(\tilde{U})$. Let $\tilde{z} = S(\tilde{x})$. Since $S$ is in $\mathrm{Stab}(\tilde{C})$, $S$ commutes with $\tilde{f}_{\tilde{C}}$. It follows that $\tilde{C}$ is a home domain for $\tilde{z}$ and $S(\tilde{\gamma}) = \tilde{\gamma}(\tilde{z}) \subset \tilde{C}$. Lemma 12.2 implies that $\tilde{\gamma}$ and $S(\tilde{\gamma})$ are disjoint or equal (up to perhaps a change of orientation). In the latter case we are done so we assume the former and argue to a contradiction. By (1), $\tilde{\gamma}$ and $S(\tilde{\gamma})$ are parallel. Since $M$ has genus zero there is an antiparallel translate $S'(\tilde{\gamma})$ that separates $\tilde{\gamma}$ and $S(\tilde{\gamma})$. Let $\tilde{y} = S'(\tilde{x})$. We have $S' \in \mathrm{Stab}(\tilde{C})$ because $S'(\tilde{\gamma}) \subset \tilde{C}$. Thus $S'(\tilde{\gamma}) = \tilde{\gamma}(\tilde{y})$ in contradiction to (2). □

## 13 The set of annuli $\mathcal{A}$

**Definitions 13.1** Let $\Gamma$ be the set of simple closed curves that are tracked by at least one element of $\mathcal{B}(f)$. For each lift $\tilde{\gamma}$ of $\gamma \in \Gamma$, choose a domain $\tilde{C}$ that contains $\tilde{\gamma}$ and let $\tilde{U}(\tilde{\gamma})$ be the set of points in $H$ which have a neighborhood $\tilde{V}$ such that every point in $\tilde{V} \cap \tilde{\mathcal{B}}(f)$ tracks $\tilde{\gamma}$. We say that $\tilde{C}$ is a *home domain* for $\tilde{U}(\tilde{\gamma})$, that $\tilde{\gamma}$ is the *defining parameter* of $\tilde{U}(\tilde{\gamma})$ and that $T_{\tilde{\gamma}}$ is *the covering translation associated to* $\tilde{U}(\tilde{\gamma})$.

For each $\gamma \in \Gamma$ define $U(\gamma)$ to be the projected image of $\tilde{U}(\tilde{\gamma})$ for any lift $\tilde{\gamma}$. We say that $C$ is a *home domain* for $U(\gamma)$ and that $\gamma$ is the *defining parameter* of $U(\gamma)$.

We show in Lemma 13.6 that $U(\gamma) \neq \varnothing$.

**Remark 13.2** As the notation suggests, $\tilde{U}(\tilde{\gamma})$ depends only on $\tilde{\gamma}$ and not on the choice of $\tilde{C}$. Indeed, if $\tilde{C}$ is not unique then $\tilde{\gamma} \in \tilde{\mathcal{R}}$ and (Corollary 10.4) every element of $\tilde{V} \cap \tilde{\mathcal{B}}(f)$ has exactly two home domains $\tilde{C}$ and $\tilde{C}'$ (where $\tilde{C}'$ is the other domain that contains $\tilde{\gamma}$) and both

$$\{\alpha(\tilde{f}_{\tilde{C}}, \tilde{z}), \omega(\tilde{f}_{\tilde{C}}, \tilde{z})\} \quad \text{and} \quad \{\alpha(\tilde{f}_{\tilde{C}'}, \tilde{z}), \omega(\tilde{f}_{\tilde{C}'}, \tilde{z})\}$$

are contained in $\{\tilde{\gamma}^{\pm}\}$. $U(\gamma)$ is well-defined because $\tilde{U}(S(\tilde{\gamma})) = S\tilde{U}(\tilde{\gamma})$ for any covering translation $S$.

**Definitions 13.3** Let $\mathcal{C}$ be the set of isolated punctures $c$ in $M$ for which there is at least one element of $\mathcal{B}(f)$ that rotates about $c$. For each $P \in S_\infty$ that projects to $c \in \mathcal{C}$, let $\widetilde{C}$ be the unique domain whose closure contains $P$ and let $\widetilde{U}(P)$ be the set of points in $H$ for which there is a neighborhood $\widetilde{V}$ such that every point in $\widetilde{V} \cap \widetilde{\mathcal{B}}(f)$ tracks $P$. We say that $\widetilde{C}$ is the *home domain* for $\widetilde{U}(P)$, that $P$ is the *defining parameter* of $\widetilde{U} = \widetilde{U}(P)$ and that $T_P$ is *the covering translation associated to* $\widetilde{U}(P)$.

For each $c \in \mathcal{C}$ define $U(c)$ to be the projected image of $\widetilde{U}(P)$ for any puncture $P$ that projects to $c$. We say that $C$ is the *home domain* for $U(c)$ and that $c$ is the *defining parameter* of $U(\gamma)$. As in the previous remark, $U(c)$ is well-defined. We show in Lemma 13.6 that $U(c) \neq \varnothing$.

Let $\widetilde{\mathcal{A}}$ be the set of all $\widetilde{U}(\tilde{\gamma})$ and $\widetilde{U}(P)$ and let

$$\widetilde{\mathcal{U}} = \bigcup_{\tilde{\gamma}} \widetilde{U}(\tilde{\gamma}) \cup \bigcup_{P} \widetilde{U}(P).$$

Let $\mathcal{A}$ be the set of all $U(\gamma)$ and $U(c)$ and let $\mathcal{U}$ be the projection of $\widetilde{\mathcal{U}}$ into $M$.

**Lemma 13.4** (1) *Each $\widetilde{U} \in \widetilde{\mathcal{A}}$ is open and invariant by both $T$ and $\tilde{f}_{\widetilde{C}}$ where $\widetilde{C}$ is a home domain for $\widetilde{U}$ and $T$ is the covering translation associated to $\widetilde{U}$.*

(2) *If $\widetilde{U}, \widetilde{U}' \in \widetilde{\mathcal{A}}$ have different defining parameters then $\widetilde{U} \cap \widetilde{U}' = \varnothing$.*

(3) *If $\widetilde{U} \in \widetilde{\mathcal{A}}$ and $S$ is a covering translation then $S(\widetilde{U}) \cap \widetilde{U} \neq \varnothing$ if and only if $S$ is an iterate of the covering translation associated to $\widetilde{U}$.*

(4) *Each $U \in \mathcal{A}$ is open and $f$–invariant; if $U_1$ and $U_2$ have different defining parameters then $U_1 \cap U_2 = \varnothing$.*

**Proof** (1) and (2) are immediate from the definitions. (3) follows from (2) and the fact that $S$ maps the defining parameter for $\widetilde{U}$ to the defining parameter for $S(\widetilde{U})$. (4) follows from (1)–(3). □

**Corollary 13.5** *If $h\colon M \to M$ commutes with $f$ then $h$ permutes the elements of $\mathcal{A}$.*

**Proof** Since $h_\#(\mathcal{R})$ is a reducing set for $hfh^{-1} = f$ and since reducing sets are unique, $\mathcal{R}$ is $h_\#$–invariant. It follows that both $\widetilde{\mathcal{R}}$ and the set of domains for $f$ are $\tilde{h}_\#$–invariant for any lift $\tilde{h}\colon H \to H$ of $h$.

If $\widetilde{C}$ is a home domain for $\tilde{x} \in H$ and $\tilde{x}$ tracks $\tilde{\gamma}$ (respectively $P$) under iteration by $\tilde{f}_{\widetilde{C}}$ then

$$\tilde{h}\tilde{f}_{\widetilde{C}}\tilde{h}^{-1} = \tilde{f}_{\widetilde{C}'}$$

for some domain $\widetilde{C}'$ that is a home domain for $\tilde{h}(\tilde{x})$ and $\tilde{h}(\tilde{x})$ tracks $\tilde{h}_\#(\tilde{\gamma})$ (respectively $\tilde{h}(P)$) under iteration by $\tilde{f}_{\widetilde{C}'}$. This proves that $h(U(\gamma)) = U(h_\#(\gamma))$. □

As a special case, our next lemma shows that $\mathcal{B}(f) \subset \mathcal{U}$.

**Lemma 13.6** *If either the $\alpha$–limit set $\alpha(f, y)$ or the $\omega$–limit set $\omega(f, y)$ of the $f$–orbit of $y$ is nonempty, then $y$ is contained in an element $U$ of $\mathcal{A}$. In particular, each $y \in \mathcal{B}(f)$ is contained in some $U \in \mathcal{A}$.*

**Proof** The two cases are symmetric so we may assume that $\omega(f, y) \neq \emptyset$. Choose $z \in \omega(f, y)$ and a free disk neighborhood $V$ of $z$ with compact closure. After replacing $y$ by some $f^k(y)$, we may assume that $y \in V$. Since $z \in \omega(f, y)$ there exist $m_i \to \infty$ such that $f^{m_i}(y) \to z$ and such that each $f^{m_i}(y) \in V$. Choose a lift $\widetilde{V}$ of $V$ and let $\widetilde{y}, \widetilde{z} \in \widetilde{V}$ be lifts of $y$ and $z$.

By Corollary 10.15, the distance between a point in $\widetilde{\mathcal{B}}(f)$ and a home domain for that point is uniformly bounded. It follows that there are only finitely many home domains for elements $\widetilde{x}_l \in \widetilde{\mathcal{B}}(f) \cap \widetilde{V}$ and so we may choose a sequence $\widetilde{x}_l \to \widetilde{y}$ all of which have the same home domain(s) $\widetilde{C}$ and $\widetilde{C}'$, where we allow the possibility that $\widetilde{C} = \widetilde{C}'$. By Corollary 10.15 the distance between $\widetilde{f}_{\widetilde{C}}{}^{m_i}(\widetilde{x}_l)$ and $\widetilde{C} \cup \widetilde{C}'$ is uniformly bounded. It follows that the distance between $\widetilde{f}_{\widetilde{C}}{}^{m_i}(\widetilde{y})$ and $\widetilde{C} \cup \widetilde{C}'$ is uniformly bounded. After passing to a subsequence of the $m_i$ and interchanging $\widetilde{C}$ and $\widetilde{C}'$ if necessary, we may assume that the distance between $\widetilde{f}_{\widetilde{C}}{}^{m_i}(\widetilde{y})$ and $\widetilde{C}$ is uniformly bounded.

Let $S_i$ be the covering translation such that $\widetilde{f}_{\widetilde{C}}{}^{m_i}(\widetilde{y}) \in S_i(\widetilde{V})$ and note that the distance between $S_i(\widetilde{z})$ and $\widetilde{C}$ is uniformly bounded. Up to the action of $\mathrm{Stab}(\widetilde{C})$, the number of translates of $\widetilde{z}$ that have uniformly bounded distance from $\widetilde{C}$ is finite. We may therefore choose $k > j$ such that $S = S_k S_j{}^{-1} \in \mathrm{Stab}(\widetilde{C})$. Let $\widetilde{W} = S_j(\widetilde{V})$ and let $\widetilde{W}' \subset \widetilde{W}$ be a neighborhood of $\widetilde{f}_{\widetilde{C}}{}^{m_j}(\widetilde{y})$ such that $\widetilde{f}^{m_k - m_j}(\widetilde{W}') \subset S(\widetilde{W})$. Then $S$ is a $\widetilde{f}_{\widetilde{C}}$–near cycle for every point in $\widetilde{W}'$ and in particular for $\widetilde{f}_{\widetilde{C}}{}^{m_j}(\widetilde{x}_l)$ for all sufficiently large $l$. Choose such an $\widetilde{f}_{\widetilde{C}}{}^{m_j}(\widetilde{x}_l)$ and denote it simply by $\widetilde{x}$.

To prove that $\widetilde{f}_{\widetilde{C}}{}^{m_j}(\widetilde{y})$, and hence $\widetilde{y}$, is contained in an element of $\widetilde{\mathcal{U}}$ with home domain $\widetilde{C}$ it suffices to show that if $\widetilde{w} \in \widetilde{\mathcal{B}}(f) \cap \widetilde{W}'$ then $\widetilde{C}$ is a home domain for $\widetilde{w}$ and $\{\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x}), \omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x})\} = \{\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{w}), \omega(\widetilde{f}_{\widetilde{C}}, \widetilde{w})\}$.

We proceed with a case analysis. As a first case suppose that $\widetilde{x}$ tracks a geodesic $\widetilde{\gamma}(\widetilde{x})$. Corollary 12.3 implies that $S$ is an iterate of $T_{\widetilde{\gamma}(\widetilde{x})}$. As a first subcase suppose that $\widetilde{C}$ is a home domain for $\widetilde{w}$. Since $S \in \mathrm{Stab}(\widetilde{C})$ is a near cycle for $\widetilde{w}$, Lemma 12.1 implies that $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{w}) \neq \omega(\widetilde{f}_{\widetilde{C}}, \widetilde{w})$ and Corollary 12.3 implies that $\widetilde{w}$ tracks $\widetilde{\gamma}(\widetilde{x})$.

The remaining subcase is that $\widetilde{C}$ is not a home domain for $\widetilde{w}$. Lemma 10.14 implies that $\{\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{w}), \omega(\widetilde{f}_{\widetilde{C}}, \widetilde{w})\}$ is contained in the set of endpoints for some $\widetilde{\sigma}$ in the frontier of $\widetilde{C}$. Lemma 10.9 then implies that $\widetilde{\sigma} = \widetilde{\gamma}(\widetilde{x})$. Let $\widetilde{C}'$ be the other domain that contains $\widetilde{\gamma}(\widetilde{x})$. Since some iterate of $T_{\widetilde{\gamma}(\widetilde{x})}$ is a near cycle for $\widetilde{w}$ with respect to $\widetilde{f}_{\widetilde{C}}$, the same is true with respect to $\tilde{f}_{\tilde{C}'}$. Lemma 10.9 implies that $\{\alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}), \omega(\tilde{f}_{\tilde{C}'}, \tilde{w})\} \cap \{\tilde{\gamma}^{\pm}(\tilde{x})\} \neq \emptyset$ and Lemma 10.14 implies that both $\alpha(\tilde{f}_{\tilde{C}'}, \tilde{w})$ and $\omega(\tilde{f}_{\tilde{C}'}, \tilde{w})$ are endpoints of $\tilde{\gamma}(\tilde{x})$. This contradicts the assumption that $\tilde{C}$ is not home domain for $\tilde{w}$ and so proves that the second subcase never occurs.

By Lemma 12.1, the only remaining case is that $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x}) = \omega(\tilde{f}_{\tilde{C}}, \tilde{x}) = P$ and that $S$ is an iterate of $T_P$. Lemma 10.9 implies that $P \in \{\alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}), \omega(\tilde{f}_{\tilde{C}'}, \tilde{w})\}$, Lemma 10.14 implies that $\tilde{C}$ is a home domain for $\tilde{w}$ and Lemma 12.2 implies that $\alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}) = \omega(\tilde{f}_{\tilde{C}'}, \tilde{w}) = P$. □

**Corollary 13.7** *Each $\tilde{U} \in \tilde{\mathcal{A}}$ is the interior of its closure in $\tilde{M}$.*

**Proof** Since $\tilde{U}$ is obviously contained in the interior of its closure, it suffices to show that if $\tilde{y}$ is in the interior of the closure of $\tilde{U}$ then $\tilde{y} \in \tilde{U}$. Choose a neighborhood $\tilde{V}$ of $\tilde{y}$ that is contained in the closure of $\tilde{U}$. Since the elements of $\mathcal{A}$ are open and either disjoint or equal and since each $\tilde{z} \in \tilde{\mathcal{B}}(f) \cap \tilde{V}$ is contained in some element of $\mathcal{A}$, it follows that $\tilde{\mathcal{B}}(f) \cap \tilde{V} \subset \tilde{U}$. If $\tilde{\gamma}$ (respectively $P$) is the defining parameter for $\tilde{U}$ then each element of $\tilde{\mathcal{B}}(f) \cap \tilde{V}$ tracks $\tilde{\gamma}$ (respectively $P$). By definition, $y \in \tilde{U}$. □

**Lemma 13.8** *Let $Y = M \setminus \mathcal{U}$ and let $\tilde{Y} \subset H$ be the full preimage of $Y$.*

(1) *For each $\tilde{y} \in \tilde{Y}$ there is a domain $\tilde{C}$ that is the unique $\alpha$–domain, unique $\omega$–domain and unique home domain for $\tilde{y}$; both $\alpha(\tilde{f}_{\tilde{C}}, \tilde{y})$ and $\omega(\tilde{f}_{\tilde{C}}, \tilde{y})$ project to punctures in $M$. Moreover, $\tilde{y}$ has a neighborhood $\tilde{W}$ so that $\tilde{C}$ is a home domain for all points in $\tilde{W} \cap \tilde{\mathcal{B}}(f)$.*

(2) *If $\tilde{C}$ is the home domain for $\tilde{y} \in \tilde{Y}$ then $\tilde{y}$ has no $\tilde{f}_{\tilde{C}}$–near cycles in* $\mathrm{Stab}(\tilde{C})$.

(3) *For any compact subset $X \subset M$ there is a constant $K_X$ such that for each $y \in Y$, $f^i(y) \in X$ for at most $K_X$ values of $i$.*

(4) *There exists $\epsilon > 0$ so that if $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$ and* $\mathrm{dist}(\tilde{y}_1, \tilde{y}_2) < \epsilon$ *then $\tilde{y}_1$ and $\tilde{y}_2$ have the same home domain. As a consequence, points in the same component of $\tilde{Y}$ have the same home domain.*

**Proof** Suppose at first that $\mathcal{R} = \emptyset$ and hence that there is only one domain. Items (1) and (4) are obvious. Every neighborhood of $\tilde{y} \in \tilde{Y}$ contains points in $\tilde{\mathcal{B}}(f)$ that are contained in different elements of $\mathcal{A}$. Lemma 12.1 and Corollary 12.3 imply that such points have no common near cycles. Item (2) therefore follows from Remark 10.6. Item (3) follows from item (2) and the fact that every compact set has a finite cover by free disks.

We now assume that $\mathcal{R} \neq \emptyset$. Write $M$ as an increasing sequence of compact connected subsurfaces $M_1 \subset M_2 \subset \ldots$ such that

$$N_{D_1}\mathcal{R} \subset M_1 \quad \text{and} \quad M_i \subset \operatorname{int}(M_{i+1})$$

for all $i$ where $D_1$ is the constant of Lemma 9.3 and so that every component of $M \setminus M_i$ contains a puncture. Moreover we choose $M_i$ so that for any sequence $\{V_i\}$ of components of $M \setminus M_i$ satisfying $V_{i+1} \subset V_i$ we have $f(V_{i+1}) \subset V_i$. We also assume without loss that the frontier $\partial M_i$ of $M_i$ is a finite union of geodesics and horocycles.

Since $y \notin \mathcal{U}$, Lemma 13.6 implies that $\omega(f, y) = \emptyset$ and hence that the forward orbit of $y$ intersects each $M_i$ in a finite set. After replacing $y$ by some point in its forward orbit, we may assume that $f^j(y) \in M \setminus M_2$ for all $j \geq 0$. Let $W_1$ and $W_2'$ be, respectively, the components of $M \setminus M_1$ and $M \setminus M_2$ that contain $y$ and let $\mu \subset W_2'$ be a ray connecting $y$ to a puncture $c'$. Note that $f(\mu) \subset W_1$.

Given a lift $\tilde{y}$, let $\tilde{C}$ be the domain that contains $\tilde{y}$ and let $\tilde{W}_2' \subset \tilde{W}_1$ be the lifts that contain $\tilde{y}$. Since the distance from a point in $\tilde{W}_1$ to a domain other than $\tilde{C}$ is greater than $D_1$, Corollary 10.15 implies that $\tilde{C}$ is a home domain for every point in $\tilde{\mathcal{B}}(f) \cap \tilde{W}_1$. The lift $\tilde{\mu}$ of $\mu$ that begins at $\tilde{y}$ converges to some $Q \in S_\infty$ that belongs to the closure of $\tilde{C}$ because $\tilde{\mu}$ does not cross any element of $\tilde{\mathcal{R}}$. In particular, $\tilde{f}_{\tilde{C}}(Q) = Q$.

If $\partial W_1$ is a horocycle then $\partial \tilde{W}_1$ is a single lift of $\partial W_1$ with both endpoints at $Q$. Otherwise $\partial W_1$ is a single simple closed geodesic, $\partial \tilde{W}_1$ has countably many components and the closure of $\partial \tilde{W}_1$ intersects $S_\infty$ in a Cantor set that contains $Q$. In both cases, $\tilde{W}_1$ is the only lift of $W_1$ that contains $Q$ in its closure. It follows that $\tilde{f}_{\tilde{C}}(\tilde{\mu}) \subset \tilde{W}_1$ and in particular that $\tilde{f}_{\tilde{C}}(\tilde{y}) \in \tilde{W}_1$.

Applying this argument to $\tilde{f}^j$ for $j \geq 2$, perhaps with $W_2'$ replaced by some other component of $M \setminus M_2$ that depends on $j$, shows that $\tilde{f}_{\tilde{C}}{}^j(\tilde{y}) \in \tilde{W}_1$ for all $j \geq 0$. There exists $J_2$ so that $f^j(y) \in M \setminus M_3$ for all $j \geq J_2$. Let $\tilde{W}_2$ be the component of $H \setminus \tilde{M}_2$ that contains $\tilde{f}_{\tilde{C}}{}^{J_2}(\tilde{y})$. By the same argument, $\tilde{f}_{\tilde{C}}{}^j(\tilde{y}) \in \tilde{W}_2$ for all $j \geq J_2$. Continuing in this manner, we can choose a decreasing sequence of components $\tilde{W}_i$ of $H \setminus \tilde{M}_i$ such that for all $i$, $\tilde{f}_{\tilde{C}}{}^j(\tilde{y}) \in \tilde{W}_i$ for all sufficiently large $j$. One may therefore choose a ray $\tilde{\tau}$ that converges to $\omega(\tilde{f}_{\tilde{C}}, \tilde{y})$ so that the terminal end of the projected ray $\tau \subset M$ lies in the complement of each $M_i$. Thus $\tau$ converges to a puncture $c$ which lifts to $\omega(\tilde{f}_{\tilde{C}}, \tilde{y})$. It follows (Corollary 9.5) that $\tilde{f}_{\tilde{C}}$ is the unique $\omega$–lift for $\tilde{y}$ and $\tilde{C}$ is its unique $\omega$–domain.

By the symmetric argument applied to $f^{-1}$, there is a unique domain $\tilde{C}^*$ that is an $\alpha$–domain for $\tilde{y}$; moreover there is a neighborhood of $\tilde{y}$ such that $\tilde{C}^*$ is a home domain for every birecurrent point in this neighborhood. To complete the proof of (1) it suffices to prove that $\widetilde{C} = \widetilde{C}^*$. If $\widetilde{C} \neq \widetilde{C}^*$, then both $\widetilde{C}$ and $\widetilde{C}^*$ are home domains for every birecurrent point in a neighborhood of $\tilde{y}$. But then (Corollary 10.4) $y \in U(\sigma)$ where $\sigma = \widetilde{C} \cap \widetilde{C}^*$ contradicting the assumption that $y$ is not contained in any $U \in \mathcal{U}$. This completes the proof of (1).

Every neighborhood of $\tilde{y}$ contains points in $\widetilde{\mathcal{B}}(f)$ that are contained in different elements of $\mathcal{U}$. Lemma 12.1 and Corollary 12.3 imply that such points have no common $\tilde{f}_{\widetilde{C}}$–near cycle in $\mathrm{Stab}(\widetilde{C})$. Item (2) now follows from Remark 10.6.

Any compact $X \subset M$ has a cover by finitely many, say $D$, free disks with compact closure. Since $N_{D_1}(\overline{C}_{\mathrm{core}})$ is a compact subset of $\overline{C}$, there is a constant $L$ so that for each of these $D$ free disks $B$, there are at most $L$ disjoint lifts of $B$ to $\overline{C}$ that intersect $N_{D_1}(\overline{C}_{\mathrm{core}})$. Equivalently, there are at most $L$ $\mathrm{Stab}(\widetilde{C})$–orbits of lifts of $B$ to $H$ that intersect $N_{D_1}(\widetilde{C})$. Item (1) and Corollary 9.4 imply that $\tilde{f}_{\widetilde{C}}^{\,j}(\tilde{y}) \in N_{D_1}(\widetilde{C})$ for all $j$. Item (2) therefore implies that there are at most $K_X = DL$ values of $j$ such that $f^j(y) \in X$. This proves (3).

It remains to prove (4). Corollary 10.15(3) implies that any two elements of $\widetilde{\mathcal{B}}(f)$ in the same component of $H \setminus \widetilde{M}_1$ have the same home domain. We may therefore assume that $\tilde{y}_1, \tilde{y}_2$ project into $M_1$. Since the forward orbit of $y_1$ intersects $M \setminus M_1$, there exists $\epsilon(y_1)$ such that $\mathrm{dist}(\tilde{y}_1, \tilde{y}_2) < \epsilon(y_1)$ implies that $\tilde{y}_1$ and $\tilde{y}_2$ have the same home domain. Since $M_1$ is compact, we may choose $\epsilon(y_1)$ independently of $y_1$. This completes the proof of (4). □

**Corollary 13.9** *Suppose that $\widetilde{V}$ is a component of $\widetilde{U} \in \widetilde{\mathcal{A}}$ and that the union $\widetilde{V}'$ of $\widetilde{V}$ with all of its bounded complementary components has finite area. Then each point in the frontier $\mathrm{fr}(\widetilde{V})$ of $\widetilde{V}$ has the same home domain.*

**Proof** Choose $\epsilon > 0$ as in Lemma 13.8 (4). It suffices to show that $\mathrm{fr}(\widetilde{V})$ can not be written as a union of two nonempty sets $X_1$ and $X_2$ whose $\epsilon/2$ neighborhoods are disjoint. We assume that such $X_1$ and $X_2$ exist and argue to a contradiction.

Since $\widetilde{V}'$ is simply connected it is the union of an increasing sequence of compact disks $\{B_i, i = 1 \ldots \infty\}$. Since $\widetilde{V}'$ has finite area we may assume that each $\partial B_i \subset N_{\epsilon/2}(\mathrm{fr}(\widetilde{V}'))$ and hence that $\partial B_i \cap N_{\epsilon/2}(X_1)$ and $\partial B_i \cap N_{\epsilon/2}(X_2))$ is an open cover of $\partial B_i$. Since $\partial B_i$ is connected one of these sets must be empty. But this can only happen for all $B_i$ if one of the sets $X_1$ and $X_2$ is empty. □

Item (4) of Proposition 5.1 asserts that if $f_c \colon U_c \to U_c$ is the annular compactification (Notation 2.7) of $U \in \mathcal{A}$, then a component of $\partial U_c$ corresponding to a nonsingular end of $U$ contains fixed points for $f_c$. We will prove this by viewing $U$ as an essential subannulus of the annular cover determined by the defining parameter of $U$.

**Definition 13.10** If $\widetilde{U} = \widetilde{U}(\widetilde{\gamma})$ choose a parameterization of the annular cover $A_\gamma$ (see Definitions 7.2) as $S^1 \times [0,1]$ with $S^1$ having circumference one. Lift this to a parameterization of $(H \cup S_\infty) \setminus \widetilde{\gamma}^{\pm}$ as $\mathbb{R} \times [0,1]$ and let $\pi\colon (H \cup S_\infty) \setminus \widetilde{\gamma}^{\pm} \to \mathbb{R}$ be projection onto the $\mathbb{R}$ factor. (Alternately, one can define this directly as orthogonal projection onto $\widetilde{\gamma}$ parameterized as $\mathbb{R}$ and with fundamental domain having length one.) If $\widetilde{U} = \widetilde{U}(P)$ where $P$ projects to an isolated end $M$ with horocycle $\tau$ define $\pi\colon (H \cup S_\infty) \setminus P \to \mathbb{R}$ as above using the compactified annular cover $A^c_P = A^c_\tau$. In both case we say that the $\pi$ is the *projection associated to the defining parameter of* $\widetilde{U}$.

**Corollary 13.11** *Suppose that $T$ is the covering translation associated to $\widetilde{U} \in \widetilde{\mathcal{A}}$, that $\pi$ is the projection associated to the defining parameter of $\widetilde{U}$, and that $\widetilde{C}$ is a home domain for $\widetilde{U}$. Given $p, q > 0$ set $\widetilde{g} = T^{-p} \widetilde{f}_{\widetilde{C}}{}^q$. Then there exists $r > 0$ so that $\pi(\widetilde{g}^r(\widetilde{y})) < \pi(\widetilde{y}) - 1$ for all $\widetilde{y} \in \mathrm{fr}(\widetilde{U})$ for which $\widetilde{C}$ is a home domain.*

**Proof** To simplify notation slightly, we let $h = f^q$ and $\widetilde{h} = \widetilde{f}_{\widetilde{C}}{}^q$. Increasing $p$ makes the desired inequality easier to satisfy so we may assume that $p = 1$ and $\widetilde{g} = T^{-1}\widetilde{h}$. The goal is to prove the existence of $r$ such that

$$\pi(\widetilde{h}^r(\widetilde{y})) < \pi(\widetilde{y}) + r - 1 \tag{13-1}$$

for all $\widetilde{y}$.

Choose compact subsurfaces $M_1 \subset M_2 \subset M$ such that

$$N_{D_1}\mathcal{R} \subset M_1 \quad \text{and} \quad M_1 \subset \mathrm{int}(M_2)$$

and so that the following hold for each component $W_1$ of $M \setminus M_1$ and each component $W_2$ of $M \setminus M_2$:

(1) $W_i$ contains at least one puncture.

(2) $\partial W_i$ is connected and is either a geodesic or a horocyle.

(3) $W_2 \subset W_1 \Longrightarrow h(W_2) \subset W_1$.

The existence of $r$ is independent of the exact choice of projection $\pi$ so we may assume:

(4) If $\widetilde{U} = \widetilde{U}(\widetilde{\gamma})$ then $\pi$ is orthogonal projection onto $\widetilde{\gamma}$; if $\widetilde{U} = \widetilde{U}(P)$ then there is a horocycle $\widetilde{\nu}$ whose ends converge to $P$ such that the restriction of $\pi$ to the component of $H \setminus \widetilde{\nu}$ whose closure contains $S_\infty \setminus P$ is orthogonal projection onto $\widetilde{\nu}$.

We will eventually add one more property satisfied by $M_1$, namely:

(5) For any lift $\widetilde{W}_1$ of a component of $M \setminus M_1$, any $\widetilde{y} \in \mathrm{fr}(\widetilde{U})$ and for all $J_1 < J_2$,

$$\widetilde{h}^j(\widetilde{y}) \in \widetilde{W}_1 \quad \text{for all } J_1 \le j \le J_2 \\ \implies \pi\big(\widetilde{h}^{J_2}(\widetilde{y})\big) - \pi\big(\widetilde{h}^{J_1}(\widetilde{y})\big) \le 1 + \tfrac{1}{10}(J_2 - J_1).$$

Assuming (5) for now, we complete the proof of the corollary.

Suppose that $h^j(y) \notin M_2$ for some $J_1 < J_2$ and all $J_1 \le j \le J_2$. Let $W_1$ be the component of $M \setminus M_1$ that contains $h^{J_1}(y)$ and let $\widetilde{W}_1$ be the lift of $W_1$ that contains $\widetilde{h}^{J_1}(\widetilde{y})$. Arguing exactly as in the proof of Lemma 13.8, we conclude that $\widetilde{h}^j(\widetilde{y}) \in \widetilde{W}_1$ for all $J_1 \le j \le J_2$. By (5)

$$\pi(\widetilde{h}^j(\widetilde{y})) \le \pi(\widetilde{h}^{J_1}(\widetilde{y})) + 1 + (j - J_1)/10$$

for all $J_1 \le j \le J_2$.

By Lemma 13.8(3) and the assumption that $\widetilde{y} \in \mathrm{fr}(\widetilde{U})$, there is a constant $K$ such that there are at most $K$ values of $j$ with $f^j(y) \in M_2$. There is a constant $B$ so that $\pi(\widetilde{h}(\widetilde{y})) < \pi(\widetilde{y}) + B$ for all $\widetilde{y} \in H$. Thus

$$\pi(\widetilde{h}^r(\widetilde{y})) < \pi(\widetilde{y}) + KB + (K+1) + r/10$$

for all $r$. A straightforward calculation shows that inequality (13-1) therefore holds for

$$r > \frac{10(KB + (K+1) + 1)}{9}.$$

It remains to verify (5). If $\widetilde{U} = \widetilde{U}(\widetilde{\gamma})$ then by enlarging $M_1$ we may assume that $\gamma \subset \mathrm{int}(M_1)$. Each component of $\partial \widetilde{W}_1$ is disjoint from $\widetilde{\gamma}$. There is a component $\widetilde{\delta}$ of $\partial \widetilde{W}_1$ that separates $\widetilde{\gamma}$ from all other components of $\partial \widetilde{W}_1$. Since $\delta$ is a simple geodesic or horocycle, $\widetilde{\delta} \cap T_{\widetilde{\gamma}}(\widetilde{\delta}) = \varnothing$. It follows that $\widetilde{W}_1 \cap T_{\widetilde{\gamma}}(\widetilde{W}_1) = \varnothing$ and hence that the diameter of $\pi(\widetilde{W}_1)$ is less than one. (Recall that we have normalized the projection so that a fundamental domain of $\widetilde{\gamma}$ has length one.) This completes the proof of (5) in the $\widetilde{U} = \widetilde{U}(\widetilde{\gamma})$ case.

Suppose then that $\widetilde{U} = \widetilde{U}(P)$ and that $\widetilde{\nu}$ is as in (4). Assuming without loss that $\widetilde{\nu}$ projects to a simple closed curve $\nu \subset M_1$, the previous argument applies to all lifts of $W_1$ except the one $\widetilde{W}_1$ whose closure contains $P$. It therefore suffices to verify (5) for this one lift $\widetilde{W}_1$ and for this we are allowed to enlarge $M_1$ if necessary.

Let $c$ be the puncture that lifts to $P$. If $U$ contains a neighborhood of $c$ then we may assume that $W_1 \subset U$ in which case (5) is vacuously true. We may therefore assume that $U$ does not contain a neighborhood of $c$ and hence that there exist $\widetilde{z}_i \in \widetilde{\mathcal{B}}(f)$ such that $\tilde{z}_i \to P$ and $\tilde{z}_i \notin \tilde{U}$. By Lemma 13.6, $\tilde{z}_i$ belongs to some element of $\tilde{\mathcal{A}}$ and so $\alpha(\tilde{f}_{\tilde{C}}, \tilde{z}_i)$ and $\omega(\tilde{f}_{\tilde{C}}, \tilde{z}_i)$ are both unequal to $P$.

Let $f_\nu\colon A^c_\nu \to A^c_\nu$ be the homeomorphism of the compactified annular cover $A^c_\nu$ (see Definitions 7.2), let $\partial_1 A^c_\nu$ be the component of $\partial A^c_\nu$ that corresponds to $c$ and let $\partial_0 A^c_\nu$ be the other component of $\partial A^c_\nu$. The projected images $\hat{z}_i \in A^c_\nu$ of $\tilde{z}_i$ satisfy $\alpha(f_\nu, \hat{z}_i), \omega(f_\nu, \hat{z}_i) \in \partial_0 A^c_\nu$ and any given neighborhood of $\partial_1 A^c_\nu$ contains $\hat{z}_i$ for all sufficiently large $i$. Corollary 8.17 therefore implies that $\mathrm{Fix}(f_\nu|_{\partial A^c_\nu})$ intersects both components of $\partial A^c_\nu$ and that $f_\nu$ is isotopic to the identity relative to $\mathrm{Fix}(f_\nu|_{\partial A^c_\nu})$.

Let $\tilde{f}_\nu\colon \tilde{A}^c_\nu \to \tilde{A}^c_\nu$ be the lift to the universal cover that fixes points in both components of $\partial \tilde{A}^c_\nu$. Then $\tilde{f}_\nu|_{\mathrm{int}(\tilde{A}^c_\nu)}$ is naturally identified with $\tilde{f}_{\tilde{C}}$ by construction and so $\tilde{h}$ is naturally identified with $\tilde{f}^q_\nu|_{\mathrm{int}(\tilde{A}^c_\nu)}$. Since $\tilde{f}_\nu|_{\partial_1 \tilde{A}^c_\nu}$ has translation number zero, we can enlarge $M_1$ to arrange that (5) is satisfied. □

**Lemma 13.12** *Suppose that $U \in \mathcal{A}$.*

(1) *$U$ is an open annulus that is essential in $M$.*

(2) *If $U = U(\gamma)$ then each simple closed curve in $U$ that is essential in $U$ is isotopic to $\gamma$. If $U = U(P)$ then each simple closed curve in $U$ that is essential in $U$ is isotopic to a horocycle surrounding the isolated end of $M$ corresponding to $P$.*

(3) *If $U = U(P)$ and $\mathcal{C}$ is the component of $\mathrm{Fix}(F)$ whose corresponding puncture in $M$ lifts to $P$ then $\mathcal{C}$ contains a component of the frontier of $U$ in $S^2$. In other words, $U$ contains a deleted neighborhood of $\mathcal{C}$.*

(4) *Each component of $\partial U_c$ corresponding to a nonsingular end of $U$ has a fixed point for $f_c$.*

**Proof** Choose $\tilde{U} \in \tilde{\mathcal{A}}$ projecting to $U$ and let $T$ be the covering translation associated to $\tilde{U}$. We will prove that $\tilde{U}$ is connected and simply connected. The first and third items of Lemma 13.4 then imply that $U$ is an open annulus and that (2) is satisfied. Since (2) implies that $U$ is essential in $M$, (1) is also proved.

As part of our proof that $\tilde{U}$ is simply connected we will show that each component $\tilde{V}$ of $\tilde{U}$ is:

(a) unbounded

(b) simply connected

(c) $T$–invariant

We verify (a) by assuming that $\widetilde{V}$ is bounded and arguing to a contradiction. Let $\tilde{f} = \tilde{f}_{\widetilde{C}}$ where (Corollary 13.9) $\widetilde{C}$ is a home domain for each point in the frontier of $\widetilde{V}$. Since $f$ preserves area there exists $q > 0$ and a covering translation $S$ so that $\tilde{f}^q(\widetilde{V}) \cap S(\widetilde{V}) \neq \emptyset$. Lemma 13.4(3) implies that $S = T^p$ for some $p \in \mathbb{Z}$. After replacing $T$ with $T^{-1}$ if necessary we may assume that $p \geq 0$. From the fact that $\tilde{f}^q(\widetilde{V})$ and $S(\widetilde{V})$ are both components of $\widetilde{U}$, it follows that $\tilde{f}^q(\widetilde{V}) = S(\widetilde{V}) = T^p(\widetilde{V})$. Thus $\widetilde{V}$ is $\tilde{g}$–invariant where $\tilde{g} = T^{-p}\tilde{f}^q$. If $p = 0$ then $\tilde{f}$ has bounded orbits (since we are assuming $\widetilde{V}$ is bounded) and hence fixed points by the Brouwer plane translation theorem. Since $\tilde{f}$ is fixed point free, $p \neq 0$. This contradicts Corollary 13.11 and so completes the proof of (a).

If (b) fails then some component of the complement of $\widetilde{V}$ is bounded. Thus there is a closed disk $D$ that is not contained in $\widetilde{U}$ but whose boundary is contained in $\widetilde{U}$. By the definition of $\widetilde{U}$ there exist $\tilde{z} \in \widetilde{\mathcal{B}}(f) \cap D$ such that $\tilde{z} \notin \widetilde{U}$. By Lemma 13.6 there is $U' \in \mathcal{A}$ such that $\tilde{z} \in \widetilde{U}'$. But then the component of $\widetilde{U}'$ containing $\tilde{z}$ is bounded in contradiction to (a). This proves (b).

We next assume that (c) fails and argue to a contradiction. A closed curve homotopic to an iterate of $\gamma$ contains a closed curve homotopic to $\gamma$. Thus $T^p(\widetilde{V}) \neq \widetilde{V}$ for all $p \neq 0$. Lemma 13.4(3) implies that $\widetilde{V}$ is moved off itself by every covering translation. In particular, $\widetilde{V}$ has finite area because the covering projection into $M$ is injective on $\widetilde{V}$. Define $\tilde{f} = \tilde{f}_{\widetilde{C}}$ where $\widetilde{C}$ is a home domain for each point in the frontier of $\widetilde{V}$. As in the previous argument, there exists an integer $p$ and a positive integer $q$ so that $\tilde{f}^q(\widetilde{V}) = T^p(\widetilde{V})$. If $p = 0$, then $\widetilde{V}$ has recurrent points, and hence fixed points for $\tilde{f}$, which is impossible. Thus $p \neq 0$ and we assume without loss that $p > 0$.

Let $\pi$ be the projection associated to the defining parameter of $\widetilde{U}$ and let $\tilde{g} = T^{-p}\tilde{f}^q$. Then $\tilde{g}(\widetilde{V}) = \widetilde{V}$ and by Corollary 13.11, there is an $r > 0$ such that $\pi(\tilde{g}^r(\tilde{y})) < \pi(\tilde{y}) - 1$ for every $\tilde{y}$ in $\partial\widetilde{V}$. The function $\pi\tilde{g}^r - \pi$ is defined on the universal cover of a compact annulus (either $A_\gamma$ or $A^c_P$ in the notation of Definition 13.10) and is invariant under the cyclic group of covering translations of that covering space. It follows that $\pi\tilde{g}^r - \pi$ is uniformly continuous. Consequently, there is $\delta > 0$ such that every $\tilde{x} \in \widetilde{V}$ which is within $\delta$ of $\partial\widetilde{V}$ satisfies $\pi(\tilde{g}^r(\tilde{x})) < \pi(\tilde{x}) - 1$.

Let $\widetilde{V}_n = \{\tilde{x} \in \widetilde{V} \mid \pi(\tilde{x}) < -n\}$. Then $\{\widetilde{V}_n\}_{n\geq 0}$ is a nested family whose intersection is empty. Moreover, each $\widetilde{V}_n$ is nonempty because $\widetilde{V}$ is $\tilde{g}$–invariant and $\lim_{n\to\infty} \pi\tilde{g}^{nr}(\tilde{y}) = -\infty$ for all $\tilde{y} \in \partial\widetilde{V}$. Since $\widetilde{V}$ has finite area there exists $N > 0$ such that $\widetilde{V}_N$ contains no ball of diameter $\delta$, and hence every point of $\widetilde{V}_N$ must be within $\delta$ of $\partial\widetilde{V}$. We conclude the $\tilde{g}^r(\widetilde{V}_N) \subset \widetilde{V}_{N+1} \subset V_N$. But then $\tilde{g}^r(\widetilde{V}_N)$ is a proper open subset of $V_N$ with the same finite area as $V_N$. This contradiction completes the proof of (c).

We have now proved that each component of $U$ contains a simple closed curve that is essential in $M$ and that all such simple closed curves in $U$ are in the same isotopy class. Moreover if $U' \in \mathcal{A}$ and $U \neq U'$ then $U$ and $U'$ do not contain isotopic simple closed curves. If $U$ has more than one component then there is an unpunctured annulus $A$ whose boundary curves are in $U$ and whose interior intersects a component of $\mathrm{fr}(U)$ and hence intersects the interior of some $U' \neq U$. It follows that $A$ contains a component of $U'$ and hence contains an essential simple closed curve not isotopic to the components of $\partial A$. This contradiction implies that $U$ and hence $\widetilde{U}$ is connected. Item (b) therefore implies that $\widetilde{U}$ is simply connected. This completes the proof of (1) and (2).

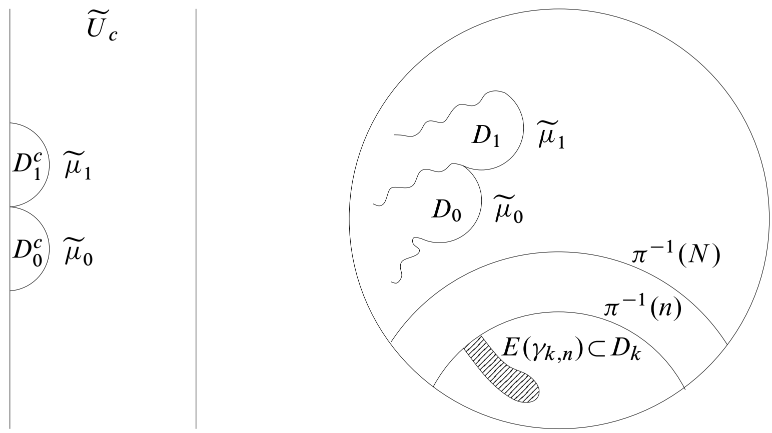
A similar argument proves (3): If $U$ does not contain a neighborhood of the puncture $c$ corresponding to $\mathcal{C}$ then the once punctured disk neighborhood of $c$ determined by a core curve $\tau$ of $U$ contains some $U' \neq U$ and hence contains an essential simple closed curve that is not isotopic to $\tau$. This contradiction proves (3).

We now consider (4). Suppose that $\partial_0 U_c$ is a component of $\partial U_c$ corresponding to a nonsingular end of $U$, meaning that the corresponding component $Z$ of the frontier of $U$ in $S^2$ is not a single point. The compactification of this end of $U$ is by prime ends. By Lemma 2.8(3) we may assume that $Z \not\subset \mathrm{Fix}(F)$ or equivalently that $M \cap Z \neq \varnothing$. Let $\widetilde{f}_c \colon \widetilde{U}_c \to \widetilde{U}_c$ be the lift to the universal cover such that $\widetilde{f}_c|_{\widetilde{U}} = \widetilde{f}_{\widetilde{C}}|_{\widetilde{U}}$. We will prove that $f_c|_{\partial_0 U_c}$ has a fixed point by showing that the translation number $\tau$ for $\widetilde{f}_c|_{\partial_0 \widetilde{U}_c}$ (see Definition 2.1) is zero. By symmetry, it suffices to assume that $\tau > 0$ and argue to a contradiction.

Choose a degree one closed path $\mu$ with embedded interior in $U$ and with both endpoints at $z \in M \cap Z$. Let $\widetilde{\mu}_0$ be a lift of the interior of $\mu$ to $\widetilde{U}$. Since $\mu$ has degree one, the ends of $\widetilde{\mu}_0$ converge to lifts $\widetilde{z}$ and $T(\widetilde{z})$ of $z$ in the frontier of $\widetilde{U}$ in $H$. Denote the bounded area component of $\widetilde{U} \setminus \widetilde{\mu}_0$ by $D_0$. For each $k$, let $\widetilde{\mu}_k = T^k(\widetilde{\mu}_0)$ and $D_k = T^k(D_0)$.

From the point of view of $\widetilde{U}_c$, $D_0$ is the interior of a half-disk $D_0^c$ whose frontier is the union of $\widetilde{\mu}_0$ and an interval $I_0 \subset \partial_0 \widetilde{U}_c$ that is a fundamental domain for the action on $\partial_0 \widetilde{U}_c$ of the covering translation $T_c \colon \widetilde{U}_c \to \widetilde{U}_c$ corresponding to $T$. Let $D_k^c = T_c^k(D_0^c)$. Choose $0 < p/q < \tau$, let $\widetilde{g} = T^{-p} \widetilde{f}_{\widetilde{C}}{}^q$ and let $\widetilde{g}_c = T_c^{-p} \widetilde{f}_{\widetilde{C}}{}^q$. Identify $\partial_0 \widetilde{U}_c$ with $\mathbb{R}$. Under the action of $\widetilde{g}_c$, points in $\partial_0 \widetilde{U}_c$ move in the positive direction at an average rate of $\tau - p/q > 0$. In particular, given any $\overline{z}$ in the interior of $I_0$ and any $L > 0$, there exists $j > 0$ so that for any sufficiently small half disk neighborhood $B$ of $\overline{z}$ in $\widetilde{U}_c$, we have $\widetilde{g}_c^j(B) \subset D_l^c$ for some $l \geq L$.

From the point of view of $\pi$, $D_0$ is not so small. The image under $\pi$ of $\widetilde{\mu}_0$ is bounded so the image under $\pi$ of $\widetilde{\mu}_l$ goes to infinity with $l$. The frontier of the set $B$ from the previous paragraph is the union of an interval in $\partial_0 \widetilde{U}_c$ with an open embedded path $\widetilde{\nu}_c \subset \mathrm{int}(\widetilde{U}_c)$. We may choose $B$ so that, under the identification of $\mathrm{int}(\widetilde{U}_c)$ with $\widetilde{U}$, $\widetilde{\nu}_c$ corresponds to the interior of a path $\widetilde{\nu}$ with endpoints in $M \cap Z$. Corollary 13.11 implies that the $\pi$–image of the endpoints of $\widetilde{g}^j(\nu)$ decrease linearly in $j$. Since $L$ can be arbitrarily large, this proves that there is no uniform bound to the diameter of the image under $\pi$ of $D_l$. Since $T(D_l) = D_{l+1}$, this diameter is independent of $l$ and we conclude that each $\pi(D_l)$ is not bounded below.




Choose a positive integer $N$ so that $\pi(\widetilde{\mu}_0) > -N$. For every fixed $n > N$ and $k > 0$ consider all cross cuts $\gamma_{k,n} \subset D_k$ such that $\pi(\gamma_{k,n}) = -n$. (In other words, $\gamma_{k,n}$ is a nontrivial component of the intersection of $D_k$ with the properly embedded line $\pi^{-1}(-n)$.) Let $E(\gamma_{k,n})$ be the complementary component of $\gamma_{k,n}$ that is contained in $D_k$ and let $d_{k,n}$ be the maximum area of all such $E(\gamma_{k,n})$. To see that this maximum is achieved, it suffices to show that any ascending chain

$$E(\gamma^1_{k,n}) \subset E(\gamma^2_{k,n}) \subset E(\gamma^3_{k,n}) \subset \dots$$

is finite. Suppose not. Let $E$ be the union of an infinite ascending chain. Choose $\widetilde{w} \in E(\gamma^1_{k,n})$, choose $\widetilde{w}' \in \widetilde{U} \setminus E$ and choose a path $\widetilde{\rho} \subset \widetilde{U}$ connecting $\widetilde{w}$ to $\widetilde{w}'$. Then $\widetilde{\rho}$ intersects $\gamma^i_{k,n}$ for all $i$. Choose a point $\widetilde{v}_i \in \widetilde{\rho} \cap \gamma^i_{k,n}$ for each $i$ and a limit point $\widetilde{v}$ of some subsequence of the $\widetilde{v}_i$. Then $\widetilde{v} \in \widetilde{U}$ because $\widetilde{\rho} \subset \widetilde{U}$ is compact. However, this is impossible because $\pi^{-1}(-n)$ is a properly embedded line so the $\widetilde{v}_i$ converge to $\widetilde{v}$ in this line and $\widetilde{v}$ is in one component of the open subset $\pi^{-1}(-n) \cap \widetilde{U}$ of this line while each $\widetilde{v}_i$ is in a different component. This contradiction shows that $d_{k,n}$ is well-defined.

We have

$$d_{k,n} = d_{k+1,n-1} > d_{k+1,n}.$$

The equality follows from the fact that $\gamma_{k+1,n-1} = T(\gamma_{k,n}) \subset D_{k+1}$ is a cross cut with $\pi(\gamma_{k+1,n-1}) = -n+1$. The inequality follows from the fact that each $E(\gamma_{k,n})$ is contained in some $E(\gamma_{k,n-1})$.

Fix $k$ and choose $\gamma_{k,n}$ so that $d_{k,n} = E(\gamma_{k,n})$. Since $D_k$ has finite area, we have $\lim_{n\to\infty} d_{k,n} = 0$. Arguing as in the proof of (c), there exits $r > 0$ and $N' > N$ such that $\pi(\tilde{g}^r(\gamma_{k,n})) < -n-1$ for all $n > N'$. Our choice of $N$ guarantees that $\tilde{g}^r(\gamma_{k,n}) \cap \tilde{\mu}_l = \emptyset$ for $l \geq k$. Since the endpoints of $\gamma_{k,n}$ move upward under the action of $\tilde{g}_c^r$, it follows that $\tilde{g}^r(E(\gamma_{k,n}))$ is contained in $D_l$ for some $l \geq k$ and hence that $\tilde{g}^r(E(\gamma_{k,n}))$ is contained in some $E(\gamma_{l,n+1})$. This contradicts the fact that $d_{l,n+1} < d_{k,n}$ for all $l \geq k$. □

# 14 Proof of Proposition 5.1

**Lemma 14.1** *$\mathcal{A}$ is the set of maximal $f$–invariant open annuli in $M$.*

**Proof** By Lemma 13.12, the elements of $\mathcal{A}$ are disjoint $f$–invariant open annuli. It therefore suffices to show that for every $f$–invariant open annulus $V$ there exists $U \in \mathcal{A}$ such that $V \subset U$.

If $V$ is inessential in $M$ then the union of $V$ with one of its complementary components in $M$ is an $f$–invariant open disk. Since $f$ preserves area, the Brouwer plane translation theorem implies that this open disk contains a fixed point which is impossible because $M$ is fixed point free. We conclude that $V$ is essential in $M$.

Let $\alpha$ be an essential simple closed curve in $V$ and let $\gamma$ be either a simple closed geodesic or a horocycle in $M$ that is isotopic to $\alpha$. Since $V$ is $f$–invariant, $\gamma$ is isotopic to $f(\gamma)$ and so does not cross any reducing curves.

Choose a lift $\tilde{\gamma} \subset H$ of $\gamma$ and let $T$ be a root free covering translation that preserves $\tilde{\gamma}$. The ends of $\tilde{\gamma}$ converge to the (possibly equal) endpoints $T^{\pm}$ of $T$. If $\gamma$ is not a reducing curve then $\tilde{\gamma}$ lies in a unique domain $\tilde{C}$. The lift $\tilde{f}_1 = \tilde{f}_{\tilde{C}}$ of $f$ fixes $T^{\pm}$ and so commutes with $T$ by Lemma 6.3. If $\gamma$ is a reducing curve then $\tilde{\gamma}$ is the common frontier of two domains $\tilde{C}_1$ and $\tilde{C}_2$. Let $\tilde{f}_j$, $j = 1, 2$ be the lift which fixes the ends of $\tilde{C}_j$. In this case too $\tilde{f}_j$ fixes $\tilde{\gamma}^{\pm}$ and commutes with $T$.

The components of the full preimage of $V$ are copies of the universal cover of $V$; we refer to each component as a lift of $V$. There is a compactly supported homotopy from $\gamma$ to $\alpha$ which lifts to a homotopy between $\tilde{\gamma}$ and a lift $\tilde{\alpha}$ of $\alpha$. Let $\tilde{V}$ be the lift of $V$ that contains $\tilde{\alpha}$. Since the lifted homotopy moves points a uniformly bounded distance, the ends of $\tilde{\alpha}$ converge to $T^{\pm}$. Since this uniquely determines $\tilde{\alpha}$ and since the ends of

$T(\tilde{\alpha})$ converge to $T^{\pm}$, it follows that $T(\tilde{\alpha}) = \tilde{\alpha}$ and hence that $T(\tilde{V}) = \tilde{V}$. For the same reason, $\tilde{f}_j(\tilde{\alpha})$ is the unique lift of $f(\alpha)$ whose ends converge to $T^{\pm}$. Since there is such a lift of $f(\alpha)$ in $\tilde{V}$, it follows that $\tilde{V}$ and $\tilde{f}_j(\tilde{V})$ have nontrivial intersection and so, being lifts of $V$, are equal.

Given $\tilde{x} \in \tilde{\mathcal{B}}(f) \cap \tilde{V}$ projecting to $x \in \mathcal{B}(f) \cap V$, let $W \subset V$ be a free disk neighborhood of $x$ with compact closure and let $\tilde{W} \subset \tilde{V}$ be the lift of $W$ that contains $\tilde{x}$. There exist $k_i \to \infty$ such that $f^{k_i}(x) \in W$ and covering translations $S_i$ satisfying $\tilde{f}_j^{k_i}(\tilde{x}) \in S_i(\tilde{W})$. Since $\tilde{f}_j^{k_i}(\tilde{x}) \in \tilde{V}$, $S_i$ preserves $\tilde{V}$ and so must be an iterate of $T$. After passing to a subsequence and reversing the orientation of $T$ if necessary, we may assume that $S_i = T^{m_i}$ for $m_i \to \infty$. In particular, the distance between $\tilde{f}_j^{k_i}(\tilde{x})$ and $\tilde{C}$ is uniformly bounded. Lemma 9.10 implies that $\tilde{C}$ is an $\omega$ domain and hence (Proposition 10.3) a home domain for $\tilde{x}$. Lemma 13.6 implies that $\tilde{x}$ is contained in some $\tilde{U} \in \mathcal{A}$; Lemma 12.1 and Corollary 12.3 imply that $T$ is the covering translation associated to $\tilde{U}$. Since $T$ is independent of the choice of $\tilde{x}$, $\mathcal{B}(f) \cap \tilde{V} \subset \tilde{U}$. The interior of the closure of $\mathcal{B}(f) \cap \tilde{V}$ contains $\tilde{V}$ so $\tilde{V} \subset \tilde{U}$ by Corollary 13.7. This completes the proof. □

Recall (see Notation 2.7) that for any open $f$–invariant annulus $V \subset M$ there is a natural annular compactification of $V$ denoted $V_c$ and an extension of $f$ to the closed annulus $f_c\colon V_c \to V_c$. See Definition 2.1 for the definition of translation number, translation interval, rotation number and rotation interval.

**Lemma 14.2** *Suppose that $U \in \mathcal{A}$ and that $X$ is a component of $\partial U_c$ corresponding to a nonsingular end. Then the translation number $\tau(\tilde{f}_c|_{\tilde{X}})$ of any lift of $f_c$ restricted to the universal covering space $\tilde{X}$ is an integer $p$. Moreover the translation interval $\mathcal{T}(\tilde{f}_c)$ is a nontrivial interval containing $p$ as an endpoint and having length at most 1.*

**Proof** No integer can be in the interior of the translation interval $\mathcal{T}(\tilde{f}_c)$. To see this we suppose to the contrary that an integer (which without loss we assume is 0) is in the interior of $\mathcal{T}(\tilde{f}_c)$ and show this leads to a contradiction. In this case by Theorem 2.3 there would be periodic points in $U$ with both positive and negative rotation numbers. [7, Theorem 2.1] then implies that $f$ has a fixed point in the open annulus $U$, which is a contradiction.

By part (4) of Lemma 13.12, $f_c$ has a fixed point in $X$. It follows that the translation number of the lift $\tilde{f}_c|_{\tilde{X}}\colon \tilde{X} \to \tilde{X}$ is an integer, say $p$. Hence $p \in \mathcal{T}(\tilde{f}_c)$. There is a point in the interior of $U$ with a well-defined noninteger translation number. This is because almost all points of $U$ have a well-defined translation number by Lemma 2.2 and if these were all integers then Proposition 2.4 would imply $U$ contains a fixed point – a contradiction. Since $p \in \mathcal{T}(\widetilde{f}_c)$ and no integer can be in its interior, it follows that $\mathcal{T}(\widetilde{f}_c)$ is nontrivial, $p$ is one endpoint and it must be contained in either $[p, p+1]$ or $[p-1, p]$. □

Suppose that $\mu_1$ and $\mu_2$ are disjoint nonhomotopic essential oriented simple closed curves in $M$ and that $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are lifts to $H$. The initial and terminal ends of $\widetilde{\mu}_i$ converge to the fixed points $T_i^-, T_i^+ \in S_\infty$ respectively of some covering translation $T_i$. If $\mu_1$ and $\mu_2$ are nonperipheral then $T_1$ and $T_2$ are hyperbolic and the four endpoints are distinct. Moreover, $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are antiparallel if $\{T_1^-, T_2^-\}$ links $\{T_1^+, T_2^+\}$ and parallel otherwise. If either $\mu_1$ or $\mu_2$ is peripheral then it requires more care to decide if $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are antiparallel.

**Definition 14.3** Suppose that $T$ is the covering translation associated to $\widetilde{U} \in \widetilde{\mathcal{A}}$ and that $\widetilde{C}$ is a home domain for $\widetilde{U}$. Let $\widetilde{f} = \widetilde{f}_{\widetilde{C}}$. Identify the annular compactification $U_c$ with $S^1 \times [0,1]$ and so the universal cover of $U_c$ with $\mathbb{R} \times [0,1]$. Let $p_1 \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be projection onto the first coordinate.

Since there are no fixed points for $f_c$ in $U$, Proposition 2.4 implies that the set of points in $U_c$ with zero rotation number has measure zero. Thus there is a full measure set $\mathcal{P} \subset U$ consisting of points in $B(f)$ which have a well-defined nonzero rotation number for $f_c \colon U_c \to U_c$. Each lift $\widetilde{x} \in \widetilde{U}$ of each $x \in \mathcal{P}$ has a well-defined nonzero translation number with respect to $\widetilde{f}_c$. These translation numbers must either all be positive or all be negative since the existence of a point with positive translation number and a point with negative translation number would imply the existence of positively and negatively recurring free disks in $U$ and then [7, Theorem 2.1] implies the existence of a fixed point.

Let $\mu \subset U$ be an essential simple closed curve and let $\widetilde{\mu}$ be its lift to $\widetilde{U}$. If all $\widetilde{x}$ as above have positive translation number then we orient $\widetilde{\mu}$ so that the $p_1$–image of its initial end converges to $-\infty$ and the $p_1$–image of its terminal end converges to $+\infty$. Otherwise, all $\widetilde{x}$ as above have negative translation number and we orient $\widetilde{\mu}$ so that the $p_1$–image of its initial end converges to $+\infty$ and the $p_1$–image of its terminal end converges to $-\infty$. We say that $\widetilde{\mu}$ has the *orientation determined by* $\widetilde{C}$. (If $\widetilde{U}$ has two home domains then the orientations that they induce on $\widetilde{\mu}$ are opposite from each other.)

For any pair of disjoint properly embedded oriented lines $\ell_1, \ell_2$ in $\mathbb{R}^2$ there is an ambient isotopy that moves $\ell_1$ and $\ell_2$ to a pair of oriented horizontal lines. If the horizontal lines are both oriented to the right or both oriented to the left then we say that $\ell_1$ and $\ell_2$ are parallel. Otherwise, we say that $\ell_1$ and $\ell_2$ are antiparallel. It is easy to check that this is well-defined.

**Lemma 14.4** *Suppose that* $\widetilde{U}_1$ *and* $\widetilde{U}_2$ *are distinct elements of* $\widetilde{\mathcal{A}}$ *that have a common home domain* $\widetilde{C}$. *Suppose further that both* $\widetilde{U}_1$ *and* $\widetilde{U}_2$ *intersect a lift* $\widetilde{D} \subset H$ *of some free disk* $D \subset M$. *For* $i = 1, 2$, *let* $\mu_i$ *be an essential simple closed curve in* $U_i$ *and let* $\widetilde{\mu}_i \subset \widetilde{U}_i$ *be its lift endowed with the orientation determined by* $\widetilde{C}$. *Then* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *are parallel.*

**Proof** Following Definition 14.3, we let $\mathcal{P}_i$ be the full measure subset of $U_i$ consisting of points with well-defined nonzero rotation number for $f_c\colon U_{ic} \to U_{ic}$. Since $U_i \cap D$ is an open set we may choose $x_i \in \mathcal{P}_i$ and lifts $\widetilde{x}_i \in \widetilde{D}$.

Let $\widetilde{f} = \widetilde{f}_{\widetilde{C}}$. By [17, Theorem 2.6] there exists an oriented properly embedded line $L_i$ with the following properties:

(1) $L_i$ contains the $\widetilde{f}_{\widetilde{C}}$–orbit of $\widetilde{x}_i$.

(2) The initial and terminal ends of $L_i$ converge to $\alpha(\widetilde{f}_{\widetilde{C}}, \widetilde{x}_i)$ and $\omega(\widetilde{f}_{\widetilde{C}}, \widetilde{x}_i)$ respectively.

(3) If $i < j$ then $\widetilde{f}^i(\widetilde{x}_1) < \widetilde{f}^j(\widetilde{x}_1)$ in the ordering induced on $\widetilde{L}$ by its orientation.

(4) $L_i$ is $\widetilde{f}$–invariant, up to isotopy rel the orbit of $\widetilde{x}_i$.

As $L_1$ is only defined up to isotopy rel the orbit of $\widetilde{x}_1$, we may assume that $L_1 \subset \widetilde{U}_1$. The lines $L_1$ and $\widetilde{f}(L_1)$ are isotopic rel the orbit of $\widetilde{x}_1$. Since $L_1$ and $\widetilde{f}(L_1)$ are both contained in $\widetilde{U}_1$ and the orbit of $\widetilde{x}_2$ is disjoint from $\widetilde{U}_1$, $L_1$ and $\widetilde{f}(L_1)$ are isotopic rel the orbits of $\widetilde{x}_1$ and $\widetilde{x}_2$. By symmetry we may assume that $L_2 \subset \widetilde{U}_2$ is $\widetilde{f}$–invariant up to isotopy rel the orbits of $\widetilde{x}_1$ and $\widetilde{x}_2$. Since $\widetilde{x}_1, \widetilde{x}_2$ are contained in a free disk for $\widetilde{f}$, [11, Lemma 8.7(2)] implies that $L_1$ and $L_2$ are parallel. Items (2) and (3) imply that the orientation on $L_i$ is consistent with one on $\widetilde{\mu}_i$ determined by $\widetilde{C}$ and we conclude that $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are parallel. □

**Lemma 14.5** *Each* $U \in \mathcal{A}$ *is the interior of its closure in* $M$.

**Proof** It is obvious that $U \subset \operatorname{int}(\operatorname{cl}(U))$ so it suffices to show that if $x \in \operatorname{fr}(U)$ then every neighborhood of $x$ intersects some element $U' \neq U$ of $\mathcal{A}$.

Choose $\widetilde{U} \in \widetilde{\mathcal{A}}$ projecting to $U$ and a lift $\widetilde{x} \in \operatorname{fr}(\widetilde{U})$. By Lemma 13.8(1) there is a free disk neighborhood $D$ of $x$ lifting to a neighborhood $\widetilde{D}$ of $\widetilde{x}$ and there is a domain $\widetilde{C}$ that is a home domain for each point in $\widetilde{D} \cap \widetilde{\mathcal{B}}(f)$ and hence a home domain for every element of $\widetilde{A}$ that intersects $\widetilde{D}$. Let $\widetilde{f} = \widetilde{f}_{\widetilde{C}}\colon H \to H$. Since $\widetilde{x}$ is in the frontier of $\widetilde{U}$, $\widetilde{D}$ intersects at least one element $\widetilde{U}' \neq \widetilde{U}$ of $\widetilde{\mathcal{A}}$. We must show that for any $\widetilde{D}$ there is such a $\widetilde{U}'$ whose projection $U'$ in $\mathcal{A}$ is not equal to $U$. Let $\mathcal{S}$ be the set of covering translations $S$ such that $\widetilde{U}' = S(\widetilde{U})$ intersects $\widetilde{D}$ but is not equal to $\widetilde{U}$. It suffices to show that $\mathcal{S} = \varnothing$, and we do this by assuming that $\mathcal{S}$ contains at least one element $S$ and arguing to a contradiction.

If $\widetilde{C} \neq S(\widetilde{C})$ then they must have a common frontier component $\widetilde{\sigma}$ because there are points for which they are both home domains. But $\widetilde{\sigma}$ projects to a simple closed curve $\sigma$ that separates $M$ with the interior of $\widetilde{C}$ projecting into one side and the interior of $S(\widetilde{C})$ projecting to the other in contradiction to the fact that $S$ is a covering translation. We conclude that $\widetilde{C} = S(\widetilde{C})$. Thus $S \in \mathrm{Stab}(\widetilde{C})$ and $S$ commutes with $\widetilde{f}$. It follows that if $\widetilde{\mu} \subset \widetilde{U}$ and $\widetilde{\mu}' \subset S(\widetilde{U})$ are lifts of a simple closed curve $\mu \subset U$ equipped with the orientation determined by $\widetilde{C}$ as in Definition 14.3 then $S$ maps $\widetilde{\mu}$ to $\widetilde{\mu}'$ and preserves orientations.

If $\mu$ and $\mu'$ are antiparallel we have contradicted Lemma 14.4. If $\mu$ and $\mu'$ are parallel then there is a covering translation $S'$ such that $S'(\widetilde{\mu})$ separates $\widetilde{\mu}$ and $\widetilde{\mu}'$ and such that the orientation on $S'(\widetilde{\mu})$ is antiparallel to that of $\widetilde{\mu}$; the existence of $S'$ follows from the fact that $M$ has genus zero. We are now reduced to the previous case and so are done. □

We are now able to prove Proposition 5.1.

**Proposition 5.1** *Suppose that $F \in \mathrm{Diff}_\mu(S^2, P)$ has entropy zero, has infinite order and at least three periodic points. Suppose that $M$ is a component of $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$ and that $f = F|_M \colon M \to M$. Then $\mathcal{A}$ (see Definitions 13.3) is a countable collection of pairwise disjoint essential open $f$–invariant annuli in $M$ such that:*

1. *For each compact set $X \subset M$ there is a constant $K_X$ such that any $f$–orbit that is not contained in some $U \in \mathcal{A}$ intersects $X$ in at most $K_X$ points. In particular each birecurrent point is contained in some $U \in \mathcal{A}$.*
2. *If $z \in M$ is not contained in any element of $\mathcal{A}$ then there are components $F_+(z)$ and $F_-(z)$ of $\mathrm{Fix}(F)$ so that $\omega(F,z) \subset F_+(z)$ and $\alpha(F,z) \subset F_-(z)$.*
3. *For each $U \in \mathcal{A}$ and each component $C_M$ of the frontier of $U$ in $M$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_M$.*
4. *If $U \in \mathcal{A}$, and $f_c \colon U_c \to U_c$ is the extension to the annular compactification (Notation 2.7) of $U$, then each component of $\partial U_c$ corresponding to a nonsingular end of $U$ contains a fixed point of $f_c$.*
5. *$\mathcal{A}$ is the set of maximal $f$–invariant open annuli in $M$*

**Proof of Proposition 5.1** The case in which $M$ has less than three ends is proved in Section 5 so we may assume that $M$ has at least three ends.

The elements of $\mathcal{A}$ are essential open annuli by Lemma 13.12(1) and are disjoint and $f$–invariant by Lemma 13.4(4). Lemma 13.8(3) implies (1) which implies (2). Item (4) follows from Lemma 13.12 (4). Item (5) is Lemma 14.1.

We now turn to (3). Let $Z$ be a component of the frontier of $U$ in $M$, let $\tilde{U}$ be a component of the full preimage of $U$ and let $\tilde{Z}$ be a component of the frontier of $\tilde{U}$ that projects onto $Z$.

Given $\tilde{z} \in \tilde{Z}$, let $\tilde{C}$ be the unique (Lemma 13.8) home domain for $\tilde{z}$, let $\tilde{f} = \tilde{f}_{\tilde{C}}$ and let $\tilde{D}$ be a neighborhood of $\tilde{z}$ that projects to a free disk for $f$ and is disjoint from a lift $\tilde{\mu}$ of a simple closed curve $\mu \subset U$ that is essential in $U$. By Lemma 13.8(1), we may assume that $\tilde{C}$ is a home domain for each element of $\tilde{\mathcal{B}}(f) \cap \tilde{D}$ and hence a home domain for each element of $\tilde{A}$ that intersects $\tilde{D}$. We claim that $\tilde{D}$ intersects exactly one component $V_{\tilde{z}}$ of $H \setminus \mathrm{cl}(\tilde{U})$. If the claim is false then there exist $\tilde{U}', \tilde{U}'' \in \mathcal{A}$ that intersect $\tilde{D}$ and that are contained in distinct components of $H \setminus \mathrm{cl}(\tilde{U})$. Let $\tilde{\mu}' \subset \tilde{U}'$ and $\tilde{\mu}'' \subset \tilde{U}''$ be lifts of essential simple closed curves $\mu' \subset U'$ and $\mu'' \subset U''$. Equip $\tilde{\mu}, \tilde{\mu}'$ and $\tilde{\mu}''$ with the orientation determined by $\tilde{C}$. Since $\tilde{\mu}'$ and $\tilde{\mu}''$ are contained in distinct components of $H \setminus \mathrm{cl}(\tilde{U})$ and are contained in the same component of the complement of $\tilde{\mu}$, no one of these three lines separates the other two. It follows that two of these lines are antiparallel in contradiction to Lemma 14.4. This completes the proof of the claim. We conclude that each $\tilde{z} \in \tilde{Z}$ has a neighborhood that intersects exactly one component $V_{\tilde{z}}$ of $H \setminus \mathrm{cl}(\tilde{U})$.

The next step in the proof of (3) is to show that the intersection $B$ of $S_\infty$ with the closure of $\tilde{Z}$ cannot have more than two components. The open set $V_{\tilde{z}} = V_{\tilde{Z}}$ depends only on $\tilde{Z}$ and not on $\tilde{z}$. In particular, $\tilde{Z}$ is contained in the frontier of $V_{\tilde{Z}}$. Let $W$ be the component of the complement of $\tilde{U}$ that contains $V_{\tilde{Z}}$ and so contains $\tilde{Z}$. Lemma 3.2 implies that the frontier of $W$ is connected and hence is contained in a component of the frontier of $\tilde{U}$. Thus $\tilde{Z}$ is the frontier of $W$. The argument in the preceding paragraph shows that $W \setminus \tilde{Z}$ is connected. Lemma 3.2 also implies that the complement of $W$ is connected and hence that the complement of $\tilde{Z}$ in $H$ has exactly two components. If $B$ has more than two components there would be two components of $S_\infty \setminus B$ with neighborhoods contained in the same component of $H \setminus \tilde{Z}$ and so there would be a line in $H \setminus \tilde{Z}$ that separates $\tilde{Z}$. This contradiction completes the second step.

The third step is to prove that each component of $B$ is a single point. If $\mathcal{R} \neq \emptyset$ then this follows from the fact (Corollary 10.15) that $\tilde{U}$, and hence $\tilde{Z}$, is contained in a uniformly bounded neighborhood of either one or two domains. Similarly, we are done if there is an essential nonperipheral simple closed curve $\tau$ in $M$ that is contained in an element of $\mathcal{A}$, for in this case each interval in $S_\infty$ contains the endpoints of a lift of $\tau$ that is disjoint from $\tilde{U}$.

We are now reduced to the case that $f$ is isotopic to the identity and that $U$ is peripheral. Choose compact subsurfaces $M_1 \subset M_2 \subset M$ so that $M \setminus M_1$ has at least three components and so that the following hold for each component $W$ of $M \setminus M_1$ and each component $V$ of $M \setminus M_2$:

- $\partial V$ (respectively $\partial W$) is connected and is either a geodesic or a horocyle.
- $V \subset W \Longrightarrow h(V) \subset W$.

By Lemma 13.12(3), $U$ contains a deleted neighborhood of some puncture $p$. We may assume without loss that the component $W_p$ of $M \setminus M_1$ that contains $p$ contains no other puncture and is contained in $U$. Let $\widetilde{W}_P$ be the component of the full preimage of $W_p$ that is contained in $\widetilde{U}$ and let $P \in S_\infty$ be the fixed point of the covering translation $T_P$ corresponding to $\widetilde{U}$. (Thus $P$ projects to $p$.)

For $\widetilde{\rho}$ a path in $H$, define $d(\widetilde{\rho})$ to be the total length of the maximal subpaths of $\widetilde{\rho}$ that are contained in the full preimage $\widetilde{M}_1$ of $M_1$. Equivalently, project $\widetilde{\rho}$ to a path $\rho \subset M$ and take the total length of $\rho \cap M_1$. For all $\widetilde{x} \in H$, let $\widetilde{\rho}_{\widetilde{x}}$ be a path connecting $\widetilde{x}$ to $\partial \widetilde{W}_P$ such that $d(\widetilde{x}) := d(\widetilde{\rho}_{\widetilde{x}})$ is minimal among all such paths. Since there is a lower bound to the distance between components of $M \setminus M_1$, $\widetilde{\rho}_{\widetilde{x}}$ decomposes as a finite alternating concatenation of subpaths in $M \setminus \widetilde{M}_1$ and subpaths in $M_1$ with all intersections with $\partial M_1$ being orthogonal. Note also that $|d(\widetilde{y}_1) - d(\widetilde{y}_2)| \le \mathrm{dist}(\widetilde{y}_1, \widetilde{y}_2)$. If $\tau$ is an essential closed curve in $M_1$ that is nonperipheral in $M_1$ then an endpoint in $S_\infty$ of any lift of $\tau$ is the limit of points $\widetilde{y}_j$ with $d(\widetilde{y}_j) \to \infty$. The third step will therefore be completed once we show that there is a uniform bound to $d(\widetilde{x})$ for birecurrent $\widetilde{x} \in \widetilde{U}$ and hence for all $\widetilde{x} \in \widetilde{U}$.

Let $Q \in S_\infty$ be a translate of $P$, let $\widetilde{W}_Q$ be the horodisk neighborhood of $P$ that is a lift of $W_p$, let $\widetilde{\mu}$ be the geodesic connecting $P$ to $Q$ and let $\widetilde{F}$ be the fundamental domain for the action of $T_P$ on $H$ that is bounded by $\partial \widetilde{F} = \widetilde{\mu} \cup T(\widetilde{\mu})$. We may assume without loss that $\widetilde{x} \in \widetilde{F}$. Since $f$ is isotopic to the identity, there exists a constant $C_0$ so that $\mathrm{dist}(\widetilde{y}, \widetilde{f}(\widetilde{y})) < C_0$ for all $\widetilde{y}$. It follows that if $\widetilde{y} \in \widetilde{F}$ and $\mathrm{dist}(\widetilde{y}, \partial \widetilde{F}) > C_0$ then $\widetilde{f}(\widetilde{y}) \in \widetilde{F}$. Applying this to the orbit of $\widetilde{x}$ we conclude that there exists $m \ge 0$ so that $\widetilde{f}^j(\widetilde{x}) \in \widetilde{F}$ for all $0 \le j \le m$ and so that $\mathrm{dist}(\widetilde{f}^m(\widetilde{x}), \partial \widetilde{F}) \le C_0$.

Letting $C_1$ be the length of the finite arc $\widetilde{\mu} \cap \widetilde{M}_1$, we have

$$d(\widetilde{f}^j(\widetilde{x})) \le C_1 + \mathrm{dist}(\widetilde{f}^j(\widetilde{x}), \partial \widetilde{F})$$

for all $j$. Since $M_2$ is compact, it is covered by finitely many, say $K$, free disks. If the orbit of $\widetilde{x}$ contains more than $K$ points in $\widetilde{F} \cap \widetilde{M}_2$ then there would be a near cycle $S$ for some points in the orbit of $\widetilde{x}$ that was not an iterate of $T_P$ in contradiction to Lemma 11.3. Thus the orbit of $\widetilde{x}$ intersects $\widetilde{F} \cap \widetilde{M}_2$ in at most $K$ points. Suppose that

$d(\tilde{x}) > (2K+1)C_0 + C_1$. If both $\tilde{f}^j(\tilde{x})$ and $\tilde{f}^{j+1}(\tilde{x})$ are contained in $H \setminus \tilde{M}_2$ then both $\tilde{f}^j(\tilde{x})$ and $\tilde{f}^{j+1}(\tilde{x})$ are contained in the same component of $H \setminus \tilde{M}_1$ and so $d(\tilde{f}^j(\tilde{x})) = d(\tilde{f}^{j+1}(\tilde{x}))$. It follows that $d(\tilde{f}^m(\tilde{x})) \geq d(\tilde{f}(\tilde{x})) - 2KC_0 > C_0 + C_1$ and hence that $\operatorname{dist}(\tilde{f}^m(\tilde{x}), \partial\tilde{F}) > C_0$. This contradiction shows that $d(\tilde{x})$ is bounded above and so completes the proof of step 3.

If $B$ is a single point $P$, then $P$ is also the intersection of $S_\infty$ with the closure of one of the complementary components of $\tilde{Z}$. It follows that $\alpha(\tilde{f}, \tilde{y}) = \omega(\tilde{f}, \tilde{y}) = P$ for each $\tilde{y} \in \mathcal{B}(f)$ contained in this component and hence that this component is $\tilde{U}(P)$. Projecting to $M$, we have by Lemma 13.12(3), that $Z$ is disjoint from a neighborhood of the puncture to which $P$ projects. This contradicts (2) and the fact that $\alpha(\tilde{f}, \tilde{z}) = \omega(\tilde{f}, \tilde{z}) = P$ for all $\tilde{z} \in \tilde{Z}$. We conclude that the limit set of $\tilde{Z}$ in $S_\infty$ is a pair of points, say $a$ and $b$.

The final step in the proof of (3) is to show that either $\alpha(\tilde{f}, \tilde{z}) = a$ and $\omega(\tilde{f}, \tilde{z}) = b$ for all $\tilde{z} \in \tilde{Z}$ or $\alpha(\tilde{f}, \tilde{z}) = b$ and $\omega(\tilde{f}, \tilde{z}) = a$ for all $\tilde{z} \in \tilde{Z}$. Since $\tilde{Z}$ is connected, it suffices to verify this for all $\tilde{z} \in \tilde{D}$.

Choose $\tilde{z} \in \tilde{D}$. For $i = 1, 2$, choose $\tilde{y}_i \in \tilde{D} \cap \tilde{\mathcal{B}}(f)$ and $\tilde{U}_i \in \tilde{A}$ such that $\tilde{y} \in \tilde{U}_i$ and such that $\tilde{U}_1$ and $\tilde{U}_2$ are in different components of $\tilde{M} \setminus \tilde{Z}$. As shown in the proof of Lemma 14.4 there are oriented lines $L_1$ and $L_2$ with the following properties:

(a) $L_i \subset U_i$ contains the $\tilde{f}$–orbit of $\tilde{y}_i$.

(b) The initial and terminal ends of $L_i$ converge to $\alpha(\tilde{f}, \tilde{y}_i)$ and $\omega(\tilde{f}, \tilde{y}_i)$ respectively.

(c) $L_i$ is $\tilde{f}$–invariant, up to isotopy rel the orbits of $\tilde{y}_1$ and $\tilde{y}_2$.

The isotopy of (c) between $L_i$ and $\tilde{f}(L_i)$ can be taken with compact support in $\tilde{U}_1 \cup \tilde{U}_2$. We may therefore assume:

(d) the isotopy of (c) is rel the orbits of $\tilde{y}_1, \tilde{y}_2$ and $\tilde{z}$.

By [17, Theorem 2.2] there is an oriented line $L_3$ satisfying:

(e) $L_3$ contains the $\tilde{f}$–orbit of $\tilde{z}$.

(f) The initial and terminal ends of $L_3$ converge to $\alpha(\tilde{f}, \tilde{z})$ and $\omega(\tilde{f}, \tilde{z})$ respectively.

(g) $L_3$ is $\tilde{f}$–invariant, up to isotopy rel the orbit of $\tilde{z}$.

As $L_3$ is only defined rel the orbit of $\tilde{z}$ we may assume that $L_3$ is disjoint from $L_1$ and $L_2$. By (d), $f(L_3)$ is isotopic rel the orbits of $\tilde{y}_1, \tilde{y}_2$ and $\tilde{z}$ to a line $L_3'$ that is disjoint from $L_1$ and $L_2$. Item (g) implies that $L_3$ is isotopic to $L_3'$ rel the orbit of $\tilde{z}$. This isotopy can be chosen to leave $L_1$ and $L_2$ invariant so $L_3$ is isotopic to $L_3'$ rel the orbits of $\tilde{y}_1, \tilde{y}_2$ and $\tilde{z}$. In other words $L_3$ is $\tilde{f}$–invariant rel the orbits of $\tilde{y}_1, \tilde{y}_2$ and $\tilde{z}$.

Lemma 8.7(2) of [11] implies that $L_3$ is parallel to $L_1$ and $L_2$. It follows that the ends of $L_3$ converge to distinct points and that the orientation on $L_3$ is independent of $\tilde{z} \in \tilde{D}$. □

# 15 Renormalization

In this section we study the finer structure of $f|_U$, the restriction of $f$ to one of the annuli $U \in \mathcal{A}$.

For each $q \geq 1$ let $\mathcal{M}_q = S^2 \setminus \mathrm{Fix}(F^q) \subset S^2 \setminus \mathrm{Fix}(F) = \mathcal{M}$. Recall that by the main theorem of [2], each component $M$ of $\mathcal{M}$ is $F$–invariant and similarly each component $M_q$ of $\mathcal{M}_q$ is $F^q$–invariant. Let $\mathcal{A}(q)$ be the family of open $F^q$–invariant annuli obtained by applying Definitions 13.3 to the restriction of $F^q$ to a component $M_q$ of $\mathcal{M}_q$ that is contained in the component $M$ of $\mathcal{M}$. See Proposition 5.1 for several useful properties of $\mathcal{A}(q)$.

**Lemma 15.1** *$f$ permutes the elements of $\mathcal{A}(q)$.*

**Proof** As $f$ commutes with $f^q$ this follows from Corollary 13.5 applied to $\mathcal{A}(q)$. □

**Lemma 15.2** *If $V \in \mathcal{A}(q)$ is essential in $M$ then $V$ is $f$–invariant.*

**Proof** Lemma 15.1 implies that $f(V)$ is an element of $\mathcal{A}(q)$ and hence that $f(V)$ is either equal to or disjoint from $V$. Since $V$ is essential in $M$, $f(V)$ is essential in $M$. If $f(V)$ is disjoint from $V$ then $f$ maps one component of the complement of $V$ to a proper subset of itself because every component of the complement of $V$ in $S^2$ contains fixed points of $F$. This contradicts the fact that $f$ preserves area. □

The following proposition shows that elements of the family $\mathcal{A}(q)$ refine the elements of $\mathcal{A}$.

**Proposition 15.3** *Each $V \in \mathcal{A}(q)$ is a subset of some $U \in \mathcal{A}$.*

**Proof** The case that $V$ is essential follows from Lemma 15.2 and Lemma 14.1 so we may assume that $V$ is inessential in $M$. An essential closed curve in $V$ bounds a closed disk in $M$ and we let $W$ be the open disk that is the union of $V$ and this disk. Since $f$ preserves area and $W$ is open and invariant under $f^q$ there is a periodic point $p \in W \cap \mathrm{Fix}(f^q)$ by the Brouwer plane translation theorem.

Let $\tilde{p} \in \tilde{W} \subset H$ be lifts of $p \in W$, let $\tilde{C}$ be a home domain for $\tilde{p}$ and let $\tilde{U}$ be the element of $\tilde{\mathcal{A}}$ that contains $\tilde{p}$. We will show that $\tilde{W} \cap \tilde{\mathcal{B}}(f^q) \subset \tilde{U}$. Corollary 13.7 then implies that $\tilde{W} \subset \tilde{U}$ and hence that $V \subset W \subset U$.

Given $z \in W \cap \mathcal{B}(f^q)$, choose $k_i \to \infty$ such that each $f^{qk_i}(z)$ is connected to $z$ by a path in $W$ of length less than $1$. Also, choose $d$ so that $z$ is connected to $p$ by a path in $W$ of length less than $d$. If $\tilde{z}$ is the lift of $z$ into $\tilde{W}$ then $\mathrm{dist}(\tilde{f}_{\tilde{C}}^{qk_i}(\tilde{z}), \tilde{f}_{\tilde{C}}^{qk_i}(\tilde{p})) < d+1$. It follows that $\omega(\tilde{f}_{\tilde{C}}, \tilde{z}) = \omega(\tilde{f}_{\tilde{C}}, \tilde{p})$. If $\tilde{C}$ is not the unique home domain for $\tilde{p}$ then the equality of $\omega$ limit sets holds for the other home domain as well. Thus $\tilde{C}$ is an $\omega$ domain, and hence a home domain (Proposition 10.3) for $\tilde{z}$. Since the element of $\tilde{\mathcal{A}}$ that contains $\tilde{z}$ is determined by $\omega(\tilde{f}_{\tilde{C}}, \tilde{z}) = \omega(\tilde{f}_{\tilde{C}}, \tilde{p})$, we have $\tilde{z} \in \tilde{U}$ as desired. □

**Remark 15.4** $V \in \mathcal{A}(q)$ is essential in $\mathcal{M}$ if and only if it is essential in the unique $U \in \mathcal{A}$ containing it, since Proposition 5.1 asserts $U$ is essential in $M$. In this case we will simply say that $V$ is essential.

The next lemmas provide information about the translation and rotation intervals of the extension $f_c \colon U_c \to U_c$ of $f$ to the annular compactification of $U$.

**Lemma 15.5** *Suppose that $q > 1$ and that $x \in U$ is not contained in any $V \in \mathcal{A}(q)$. Then $\omega(f_c, x)$ is contained in a component of $\partial U_c \cup \mathrm{Fix}(f_c^q)$. Moreover:*

1. *The forward rotation number, $\rho^+_{f_c}(x)$, with respect to $f_c$ is well-defined.*
2. *If $\omega(f_c, x)$ contains a point of $U$ then $\rho^+_{f_c}(x) = p/q$ for some $0 < p < q$.*
3. *If $\omega(f_c, x)$ is contained in a component of $\partial U_c$ corresponding to a nonsingular end of $U$ then $\rho^+_{f_c}(x) = 0$.*
4. *If $\omega(f_c, x)$ is contained in a component $B$ of $\partial U_c$ corresponding to a singular end of $U$ then $\rho^+_{f_c}(x) = \rho(B)$.*

*An analogous statement holds for backward rotation number.*

**Proof** Lemma 13.6 implies that $\omega(f^q, x) \cap \mathcal{M}_q = \emptyset$. Thus $\omega(f^q, x) \subset \mathrm{Fix}(f^q)$ and $\omega(f_c^q, x) \subset \mathrm{Fix}(f_c^q) \cup \partial U_c$. Since each component of $\mathrm{Fix}(f_c^q) \cup \partial U_c$ is $f_c^q$–invariant, $\omega(f_c^q, x)$ is contained in a component $K$ of $\mathrm{Fix}(f_c^q) \cup \partial U_c$.

The rotation number $\rho_{f_c}$ is well-defined and constant on each component of $\partial U_c$. It is also well-defined and locally constant on $\mathrm{Fix}(f_c^q)$. Since both sets are closed, $\rho_{f_c}$ is locally constant on their union and hence constant on $K$, say $\rho(f_c|_K) = \rho_K$. It follows that $\rho(f_c^q|_K) = q\rho_K$.

In fact, more is true. There is a lift $\widetilde{f}_c\colon \widetilde{U}_c \to \widetilde{U}_c$ of $f$ such that $\tau_{\widetilde{f}_c}(\widetilde{y}) = \rho_K$ for each $\widetilde{y}$ that projects into $K$. Let $p_1\colon \widetilde{U}_c \to \mathbb{R}$ be the projection used to define $\tau_{\widetilde{f}_c}$. Then for any $k \in \mathbb{Z}$

$$\left| p_1 \widetilde{f}^{kq}(\widetilde{y}) - p_1\widetilde{y} - kq\rho_K \right| < 1$$

for any $\widetilde{y}$ that projects into $K$. For any fixed $k$, this inequality holds for any point $\widetilde{z}$ that projects into a neighborhood, say $W_k$, of $K$. Suppose that $z$ and the forward $f^q$–orbit of $z$ is contained in $W_k$. Then by applying the above inequality with $\widetilde{y}$ equal, in order, to $\widetilde{z}$, $\widetilde{f}_c^{kq}(\widetilde{z})$, $\widetilde{f}_c^{2kq}(\widetilde{z}), \dots, \widetilde{f}_c^{(j-1)kq}(\widetilde{z})$ and summing, we obtain

$$\left| p_1 \widetilde{f}^{qjk}(\widetilde{z}) - p_1\widetilde{z} - qjk\rho_K \right| < j$$

for all $j$. Setting $n = jk$ and dividing by $n$ we obtain

$$\left| \frac{p_1 \widetilde{f}^{nq}(\widetilde{z}) - p_1\widetilde{z}}{n} - q\rho_K \right| < \frac{1}{k}$$

for all $n$ which are multiples of $k$. An easy computation for $n$ which are not multiples of $k$ proves that

$$q\rho_K - \frac{1}{k} \le \liminf_{n\to\infty} \frac{p_1(\widetilde{f}^{nq}(\widetilde{z})) - p_1(\widetilde{z})}{n} \le \limsup_{n\to\infty} \frac{p_1(\widetilde{f}^{nq}(\widetilde{z})) - p_1(\widetilde{z})}{n} \le q\rho_K + \frac{1}{k}$$

for all $z$ with $\omega(f_c^q, z) \subset W_k$. Since $\omega(f_c^q, x) \subset W_k$ for all $k$ it follows that $\rho^+{}_{f_c^q}(x) = q\rho_K$ and hence that $\rho^+{}_{f_c}(x) = \rho_K$. This completes the proof of (1).

If $K$ contains a point $y$ in the interior of $U_c$ then $y \in \mathrm{Fix}(f_c^q)$ and $\rho(f_c|_K) = p/q$ for some $0 \le p < q$. If $p = 0$ then there is a lift $\widetilde{f}\colon \widetilde{U} \to \widetilde{U}$ and a lift $\widetilde{y} \in \mathrm{Fix}(\widetilde{f}^q)$ in contradiction to the Brouwer translation theorem applied to $\widetilde{f}$ and the fact that $\mathrm{Fix}(\widetilde{f}) = \varnothing$. Thus $0 < p < q$.

If $K$ is a component of $\partial U_c$ corresponding to a nonsingular end then $\rho_K = 0$ by Proposition 5.1(4). This proves (3) and (4) is clear. The analogous result for backward rotation numbers comes from considering $f^{-1}$. □

**Corollary 15.6** *Suppose $q > 1$ and $V \in \mathcal{A}(q)$ is essential in $U \in \mathcal{A}$ and $U$ has a nonsingular end. If $f_c\colon V_c \to V_c$ is the extension to its annular compactification then for any lift $\widetilde{f}_c$ to the universal covering space $\widetilde{V}_c$, there is $p \in \mathbb{Z}$ such that the translation interval $\mathcal{T}(\widetilde{f}_c)$ is a nontrivial subinterval of $[p/q, (p+1)/q]$ and contains at least one of its endpoints.*

**Proof** Lemma 15.2 says $V$ is $f$–invariant. The fact that $U$ has a nonsingular end implies that $V$ does also. The result now follows from Lemma 14.2 since

$$\mathcal{T}(\tilde{f}_c) = \frac{\mathcal{T}(\tilde{f}_c^q)}{q}. \qquad \square$$

**Lemma 15.7** *If $V \in \mathcal{A}(q)$ is a proper subset of $U \in \mathcal{A}$, then there is a full measure subset $W \subset V$, containing $V \cap \mathrm{Per}(f)$, with the following properties:*

(1) *If $V$ is essential in $U$, then every $x \in W$ has the same well-defined rotation number in $V_c$ as in $U_c$.*

(2) *The annulus $V$ is inessential in $U$ if and only if there is $p \in \mathbb{Z}$ with $0 < p < q$, such that every $x \in W$ has rotation number $p/q$ in $U_c$.*

**Proof** Let $W$ be the full measure subset of $V$ which consists of birecurrent points which have a well-defined rotation numbers in both $U_c$ and $V_c$. Suppose that $x \in W$.

If $V$ is essential in $U$ then it is $f$–invariant by Lemma 15.2. The inclusion of $V$ in $U$ induces an isomorphism on the fundamental group. The rotation numbers in the two annuli can be computed along a subsequence of iterates which recurs. More precisely we may join $f^{n_i}(x)$ to $x$ by arcs $\alpha_i$ in $V$ whose lengths are bounded uniformly in $i$. If $\tilde{f}$ is a lift to the universal cover $\tilde{V}$ then we can join a lift $\tilde{x}$ of $x$ to $\tilde{f}^{n_i}(\tilde{x})$ by an arc $\tilde{\beta}_i$ in $\tilde{V}$. The projection $\beta_i$ of $\tilde{\beta}_i$ in $V$ concatenated with $\alpha_i$ forms a closed loop in $V$ and the rotation number is the limit of $1/n_i$ times the homology class of this loop as $n_i$ goes to infinity. This is easily seen to be independent of the choices of $\tilde{f}$ and $\tilde{x}$. This homology class is the same in $U$ and $V$ so (1) follows.

If $V$ is inessential in $U$ there is a component $X$ of its complement in $S^2$ contained in $U$. The set $Q = V \cup X$ is an open disk invariant under $f^q$. Since $f$ is area preserving and $f^q(Q) = Q$, by the Brouwer plane translation theorem there is a point $x_0 \in Q \cap \mathrm{Fix}(f^q)$. Since $x_0 \in \mathrm{Fix}(f^q)$ it is not in any $V \in \mathcal{A}(q)$, so by Lemma 15.5(2) the rotation number of $x_0$ in $U$ is $p/q$ for some $0 < p < q$. Let $N$ be a compact disk neighborhood of $x$ in $V$ and let $\{n_i = k_i q\}$ be a sequence such that $f^{n_i}(x) \in N$. Choose a path $\sigma \subset Q$ connecting $x_0$ to $x$. Since $Q$ is contractible, $f^{n_i}(\sigma)$ is homotopic rel endpoints to the concatenation of $\sigma$ with a path in $N$. Choose a lift $\tilde{f}_c \colon \tilde{U}_c \to \tilde{U}_c$ and a lift $\tilde{N}$ of $N$. Let $\tilde{x}, \tilde{x}_0 \in \tilde{N}$ be lifts of $x, x_0$ and let $p_1 \colon \tilde{U}_c \to \mathbb{R}$ be the projection used to define $\tau_{\tilde{f}_c}$. The $p_1$–image of $\tilde{N}$ is a bounded subset of $\mathbb{R}$. It follows that

$$p_1\big(\tilde{f}_c^{n_i}(\tilde{x})\big) - p_1\big(\tilde{f}_c^{n_i}(\tilde{x}_0)\big)$$

is bounded uniformly in $i$ and hence that $x$ and $x_0$ have the same rotation number in $U$, namely $p/q$.

This proves that if $V$ is inessential in $U$ then all the points of $W \cap V$ have the same rotation number $p/q$. To show the converse observe that since $V$ is a proper subset of $U$ it has a nonsingular end. If $V$ is essential then Lemma 14.2 applied to $f^q|_V$ asserts that $f|_V$ has a nontrivial rotation interval and, in particular, by Theorem 2.3, there are periodic points in $V$ with infinitely many distinct rotation numbers. □

**Corollary 15.8** *If $x \in U$ and $\omega(f_c, x)$ is not contained in $\partial U_c$, then there is a positive integer $r = r(x)$ so that if $q > r$ and either $r = 1$ or $q$ is relatively prime to $r$, then there is $V \in \mathcal{A}(q)$ which is essential in $U$ and contains $x$.*

**Proof** If there is a positive integer $n$ such that $\omega(f_c, x) \subset \mathrm{Fix}(f_c^n) \cup \partial U_c$, let $r_0 = r_0(x)$ be the smallest such $n$ and note that $r_0 > 1$ since $\mathrm{Fix}(f) \cap U = \varnothing$ and $\omega(f_c, x)$ is not contained in $\partial U_c$. If there is no such $n$, let $r_0 = 1$. Note that if $k \geq 1$ and $\omega(f_c, x) \subset \mathrm{Fix}(f_c^k) \cup \partial U_c$ then $r_0 > 1$ and $k$ is a multiple of $r_0$.

Suppose now that $r = r_1 r_0$ for some positive integer $r_1$ (to be chosen below), that $q > r$ and that either $r = 1$ or $q$ is relatively prime to $r$. By the above observation, $\omega(f_c, x) \not\subset \mathrm{Fix}(f_c^q) \cup \partial U_c$ so Lemma 15.5 implies that $x$ is contained in some $V_q \in \mathcal{A}_q$. It remains to show that if $r_1$ is properly chosen then $V_q$ is essential.

If there is a positive integer $m$ such that $x$ is contained in an inessential element of $\mathcal{A}(m)$, note that $m \geq 2$ since $U$ is essential and let $r_1$ be the smallest such $m$. If there is no such $m$, let $r_1 = 1$. In this case we are done so suppose that $r_1 > 1$ and hence that there is an inessential $V_{r_1} \in \mathcal{A}(r_1)$ with $x \in V_{r_1}$. We complete the proof by assuming that $V_q$ is inessential and arguing to a contradiction. By Lemma 15.7(2), a full measure subset of the nonempty open set $V_q \cap V_{r_1}$ consists of points with the same rotation number in $U$. Moreover by the same result this number must have the form $p/q$ and $p'/r_1$ with $p, p' \neq 0$. Since $q$ is relatively prime to $r_1$ this is impossible and we have reached the desired contradiction. □

Recall (see Notation 2.7) that to simplify notation we denote the rotation interval $\mathcal{R}(f_c)$ by $\rho(U)$ when there is no ambiguity about the choice of diffeomorphism $f$ but various annuli $U$ are under consideration.

**Lemma 15.9** *Suppose $U \in \mathcal{A}$ has a nontrivial rotation interval $\rho(U)$. For any sufficiently large prime $q$ there is $V \in \mathcal{A}(q)$ which is essential in $U$ and satisfies* $\mathrm{cl}(V) \subset U$.

**Proof** Since $\rho(U)$ is nontrivial, by Theorem 2.3 we may choose three periodic points $\{x_i\}$, $i = 1, 2, 3$ in $U$ whose rotation numbers $\{p_i/q_i\}$ have distinct denominators and are contained in the interior of $\rho(U)$. Choose a prime $q$, larger than each $q_i$ and sufficiently large that any three intervals of length $1/q$ containing the three numbers $\{p_i/q_i\}$, must be pairwise disjoint and must lie in the interior of $\rho(U)$. The points $\{x_i\}$ lie in elements of $\mathcal{A}(q)$ by Lemma 15.5 and these elements must be distinct by Corollary 15.6. They are essential in $U$ by Lemma 15.7. At least one of these annuli, say $V$, must be separated by the other two from the components of the complement of $U$. Hence $\mathrm{cl}(V) \subset U$. □

We will write $|\rho(V)|$ for the length of the interval $\rho(V)$.

**Lemma 15.10** *Suppose that $Y$ is a component of the frontier of $U$ in $S^2$ and that $\{V_i\}$ is an infinite sequence of distinct essential elements of $\mathcal{A}(q)$ such that $V_{i+1}$ separates $V_i$ from $Y$. Then*

$$\lim_{i \to \infty} |\rho(V_i)| = 0.$$

**Proof** Let $W_i$ be the open annulus that is the union of $V_1, V_i$ and a closed annulus bounded by an essential curve in $V_1$ and an essential curve in $V_i$. The complementary components of $W_i$ in $S^2$ are the component of $S^2 \setminus V_i$ that contains $Y$ and the component of $S^2 \setminus V_1$ that contains the other component of the frontier of $U$. In particular, these complementary components are compact and connected. Let $W \subset U$ be the union $\bigcup_i W_i$. Since the nested intersection of compact connected sets is compact and connected, $W$ is open and has two complementary components in $S^2$ so it must be an open annulus.

Let $B$ be the boundary component of $\partial W_c$ corresponding to the end of $W$ that is disjoint from $V_1$. Every neighborhood of $B$ contains $V_i$ for all sufficiently large $i$. It follows that if $x_i \in V_i$ is periodic, then the rotation number of $x_i$ with respect to $f$ converges to the rotation number $a$ of the restriction of $f_c$ to $B$. Theorem 2.3 therefore implies that the interval $\rho(V_i)$ converges to the point $a$ and so has length tending to zero. □

**Lemma 15.11** *Suppose $\rho(U)$ is nontrivial and $\partial_0 U_c$ is a component of $\partial U_c$. Let $\rho_0$ be the rotation number of $f_c$ on $\partial_0 U_c$. There exists $Q > 0$ such that:*

1. *If $q$ is any product of primes, each bigger than $Q$, then $\partial_0 U_c$ is a frontier component of a (necessarily unique) essential $V_0(q) \in \mathcal{A}(q)$ with $V_0(q) \subset U$.*
2. *The rotation number of the homeomorphism induced by $f$ on $\partial_0 V_0(q)_c$ is $\rho_0$. In particular $\rho_0 \in \rho(V_0(q))$.*
3. *If $\rho_0 \neq p/q$ for some $0 < p < q$ then $\mathrm{cl}_U(V_0(q')) \subset V_0(q)$ for all sufficiently large $q'$.*

**Proof** The first step in the proof of (1) is to prove that for sufficiently large $Q$ and for $q$ as in the hypothesis there exists a (necessarily unique) essential $V_0(q) \in \mathcal{A}(q)$ that is not separated from $\partial_0 U_c$ by any other essential element in $\mathcal{A}(q)$.

By Lemma 15.9 we may assume that there exists an essential $V_1 \in \mathcal{A}(q)$ whose closure is contained in $U$. Let $\rho(\partial U_c) = \{\rho_0, \rho_1\}$. We may assume that $Q$ is so large that neither $\rho_0$ nor $\rho_1$ has the form $p/q$ with $0 < p < q$. Choose $\delta$ such that $\delta < |\rho_0 - p/q|, |\rho_1 - p/q|$ for all $0 < p < q$.

The proof is by contradiction: assuming that no such $V_0(q)$ exists we will inductively define an infinite sequence $\{V_i\}$ of distinct essential elements of $\mathcal{A}(q)$ such that $V_{i+1}$ separates $V_i$ from $\partial_0 U_c$ and such that $|\rho(V_i)| > \delta/2$ in contradiction to Lemma 15.10. It suffices to assume that $V_1, \dots V_{i-1}$ have been defined for $i \geq 2$, and define $V_i$. By the assumption we wish to contradict, any element of $\mathcal{A}(q))$ is separated from $\partial_0 U_c$ by another element of $\mathcal{A}(q)$. In particular $V_{i-1}$ is separated from $\partial_0 U_c$, say by $V_i^*$. Since $V_i^*$ is also separated from $\partial_0 U_c$ by (yet another) element of $\mathcal{A}(q)$, it is contained between two open essential annuli in $U$. Hence each component of its frontier is contained in the interior of $U$. Lemma 15.5 implies that the rotation number of the restriction of $f_c$ to a component of $\partial {V_i^*}_c$ has the form $p/q$ with $0 < p < q$.

Choose an essential closed curve $\alpha$ in $V_i^*$ and let $W_i$ be the union of $V_i^*$ with the component of $U \setminus \alpha$ that does not contain $V_{i-1}$. Then $W_i$ is an open annulus whose frontier components are $\partial_0 U_c$ and a component of the frontier of $V_i^*$. Theorem 2.3 implies that $W_i$ contains a periodic point $z$ whose rotation number has distance less than $\delta/2$ from $\rho_0$ and so is not of the form $p/q$.

In particular, $z \in \mathcal{M}_q$ and it is also contained in some $V_i \in \mathcal{A}(q)$ by Lemma 15.5. Lemma 15.7(2) implies that $V_i$ is essential and hence separates $\partial_0 U_c$ from $V_{i-1}$. The rotation number of the restriction of $f_c$ to a component of $\partial V_{ic}$ has the form $p/q$ with $0 < p < q$ for the same reason that components of $\partial {V_i^*}_c$ satisfy this property. Since $z \in V_i$, it follows that $|\rho(V_i)| > \delta/2$. This completes the induction step and hence shows the existence an infinite family $\{V_i\}$ contradicting Lemma 15.10. We conclude there is a unique essential $V_0(q) \in \mathcal{A}(q)$ that is not separated from $\partial_0 U_c$ by any essential element of $\mathcal{A}(q)$.

Since there exists $V_2 \in \mathcal{A}(q)$ whose closure is contained in $U$, the component $B(q)$ of $\mathrm{fr}(V_0(q))$ which is separated from $\partial_0 U_c$ by $V_0(q)$ is contained in $U$. Lemma 15.5 implies that if $x \in B(q)$ then $\omega(f_c, x)$ is contained in a component of $\mathrm{Fix}(f^q)$ that is disjoint from $\mathrm{Fix}(f)$. It follows that if $q'$ is a product of primes all greater than $Q$ and with $q$ and $q'$ relatively prime, then $B(q) \cap B(q') = \emptyset$. Let $W(q)$ be the open subannulus of $U_c$ bounded by $\partial_0 U_c$ and $B(q)$. Note that either $B(q)$ separates $B(q')$

from $\partial_0 U_c$ or $B(q')$ separates $B(q)$ from $\partial_0 U_c$. Hence either $\mathrm{cl}_U(W(q)) \subset W(q')$ or $\mathrm{cl}_U(W(q')) \subset W(q)$.

Theorem 2.3 implies that $W(q)$ contains a periodic point $w$ whose rotation number is arbitrarily close to $\rho_0$, but not equal to $\rho_0$, and in particular not of the form $p/q$. Lemma 15.5 and Lemma 15.7(2) imply that $w$ is contained in some element of $\mathcal{A}(q)$ that is essential and hence this element must be $V_0(q)$. Theorem 2.3 implies that $\rho_0 \in \rho(V_0(q))$. Now choose $q'$ sufficiently large that $1/q' < |\rho_0 - \rho_{f_c}(w)|$. Since $\rho_0 \in \rho_{f_c}(V_0(q'))$, by the same argument used for $V_0(q)$, we conclude that $\rho_{f_c}(w) \notin \rho(V_0(q'))$ and hence $w \notin V_0(q')$. It follows that $\mathrm{cl}_U(W(q')) \subset W(q)$.

Items (1) and (3) will follow once we prove that $W(q) = V_0(q)$ (which is equivalent to showing that $\partial_0 U_c$ is the boundary component $B'(q)$ of $V_0(q)$ which is not $B(q)$). In particular this will show that there are no inessential elements of $\mathcal{A}(q)$ contained in $W$. In order to show this we first prove the following.

**Claim** *If $q'$ is sufficiently large then for any open set $P$ in $W(q) \setminus V_0(q)$ we have $P \cap W(q') = \varnothing$.*

We choose $q'$ so that, in addition to its properties above, it is large enough that $\rho(V_0(q'))$ does not contain a point of the form $p/q$ with $0 < p < q$. Assuming that there is an open $P \subset W(q) \setminus V_0(q)$ with $P \cap W(q') \neq \varnothing$, we will argue to a contradiction, thus proving the claim. The open set $P \cap W(q')$ (like any open subset of $U_c$) contains a positive measure subset $P_0$ of points which are birecurrent and have well-defined rotation numbers in $U_c$. By Lemma 13.6, the fact that $\rho(V_0(q'))$ does not contain a point of the form $p/q$ implies that $V_0(q') \subset \mathcal{M}_q$ and hence there is a positive measure subset $P_1 \subset P_0$ contained in some $V \in \mathcal{A}(q)$. This $V$ is necessarily inessential since otherwise it would separate $\partial_0 U_c$ and $V_0(q)$. Lemma 15.7(2) therefore implies that points in a full measure subset of $P_1$ have the same rotation number which is of the form $p/q$ with $0 < p < q$.

It follows that there is a positive measure subset $P_2 \subset P_1$ which is not contained in $V_0(q')$ but is contained in an essential element of $\mathcal{A}(q')$, (by Lemmas 15.5 and 15.7 (2) again). This contradicts the assumption that $P_0 \subset W(q')$ and so verifies the claim.

One consequence of the claim is that $\partial_0 U_c \subset B'(q)$. This is because if $x \in \partial_0 U_c \setminus B'(q)$ then $x$ has a neighborhood which is disjoint from $V_0(q)$ but intersects $W(q')$ in an open set contradicting the claim.

We want now to prove $B'(q) \subset \partial_0 U_c$ and hence that $B'(q) = \partial_0 U_c$. We note that $\partial_0 U_c$ has neighborhoods which are disjoint from $\mathrm{Fix}(f_c^q) \setminus \mathrm{Fix}(f_c)$, since otherwise points in $\partial_0 U_c$ would have a rotation number of the form $p/q$, $p \neq 0$. We let $W_0$

denote such a neighborhood which is chosen sufficiently small that it is a subset of $W(q') \cup \partial_0 U_c$ and let $x$ be a point of $B'(q) \cap W_0$. Any point on the frontier (in $U_c$) of an element of $\mathcal{A}(q)$ is either in $\text{Fix}(f^q)$ or $\partial U_c$ or has arbitrarily small neighborhoods meeting more than one element of $\mathcal{A}(q)$. This is because each element of $\mathcal{A}(q)$ is the interior of its closure by Corollary 13.7 and the union of all elements of $\mathcal{A}(q)$ and $\text{Fix}(f^q)$ is dense in $S^2$. But $x$ has a neighborhood which intersects no element of $\mathcal{A}(q)$ other than $V_0(q)$, because otherwise there would be an open $P \subset W(q')$ which is disjoint from $V_0(q)$ contradicting the claim above.

We conclude that $x \in \partial_0 U_c \cup \text{Fix}(f^q)$. Since $x \in W_0$ implies $x \notin \text{Fix}(f_c^q) \setminus \text{Fix}(f_c)$ and $\text{Fix}(f_c) \subset \partial U_c$, we conclude that $x \in \partial_0 U_c$. We have shown that $B'(q) \cap W_0 = \partial_0 U_c$, but $B'(q)$ is connected, so in fact $B'(q) = \partial_0 U_c$. This completes the proof of (1) and (3).

Finally to prove (2) we observe that one component of the complement of $U$ in $S^2$ coincides with a component of the complement of $V_0(q)$, namely the component corresponding to $\partial_0 U_c$. Indeed $V_0(q)$ is a neighborhood of the corresponding end of $U$. It follows that $\partial_0 U_c$ and $\partial_0 V_0(q)_c$ (in the annular compactifications $U_c$ and $V_0(q)_c$ respectively) can be naturally identified. Hence the rotation number of the map induced by $f$ on $\partial_0 V_0(q)_c$ is $\rho_0$. □

**Notation 15.12** For each $U \in \mathcal{A}$ there are two components of its frontier in $S^2$. Associated to each component and each $q$ satisfying the hypothesis of Lemma 15.11, there is an element of $\mathcal{A}(q)$ as described in this lemma. We will refer to these as the *end elements* of $\mathcal{A}(q)$ and denote them $V_0(q)$ and $V_1(q)$. They are neighborhoods of the ends of $U$. We label them $V_0$ and $V_1$ consistent with a transverse orientation; ie for any $q, q'$, we have $V_0(q) \cap V_0(q') \neq \varnothing$. Any element of $\mathcal{A}$ which is not an end element will be called an *interior element*.

**Lemma 15.13** *Suppose $x \in U \in \mathcal{A}$. If $\{q_n\}$ is a sequence of primes tending to infinity and $x \in V_0(q_n)$ for all $n$ then $\rho_{f_c}(x)$ is well-defined and equal to the rotation number $\rho_0$ of the component of $\partial U_c$ corresponding to the end elements $V_0(q_n)$.*

**Proof** For $n$ sufficiently large, $\rho_0 \neq p/q_n$ for $0 < p < q_n$. Hence by Lemma 15.11(3), by choosing a subsequence we may assume $\text{cl}_U(V_0(q_{n+1})) \subset V_0(q_n)$.

We now apply Lemma 2.11 letting $A_0$ be a closed annulus in the annular compactification of $V_0(q_n)$ which has $\partial_0 U_c$ as one boundary component and the other an essential closed curve in $V_0(q_n) \setminus \text{cl}_U(V_0(q_{n+1}))$. We are identifying $\partial_0 U_c$ as a component of the boundary of both $U_c$ and the annular compactification of $V_0(q_n)$. Lemma 2.11 implies the rotation interval of $x$ is the same in $U_c$ as it is in the compactification of

$V_0(q_n)$. Since this holds for all $q_n$ and since $\rho(V_0(q_n))$ contains $\rho_0$ and has length $\le 1/q_n$, $\rho_{f_c}(x_0) = \rho_0$. □

**Lemma 15.14** *Suppose $U \in \mathcal{A}$. If $V \in \mathcal{A}(q)$ is essential in $U$ and $\rho(V)$ is disjoint from $\rho(\partial U_c)$, then $\mathrm{cl}(V) \subset U$. Moreover, if $x \in U$ and the rotation interval for $x$ in $U_c$ is disjoint from $\rho(\partial U_c)$, then for every sufficiently large prime $q$ there exists an essential $V \in \mathcal{A}(q)$ such that $x \in V$, $\mathrm{cl}(V) \subset U$.*

**Proof** By Lemma 15.11 the fact that $\rho(V)$ is disjoint from $\rho(\partial U_c)$ means that $V$ is neither $V_0(q)$ nor $V_1(q)$, the two end elements whose frontiers contain the components of the boundary of $U_c$. It follows that $V$ is separated by the essential annuli $V_0(q)$ and $V_1(q)$ from the boundary of $U_c$ and hence $\mathrm{cl}(V) \subset U$.

To see the moreover part we observe that if the rotation interval for $x$ in $U_c$ is disjoint from $\rho(\partial U_c)$, then for sufficiently large $q$, the rotation interval of $x$ will be disjoint from $\rho(V_0(q))$ and $\rho(V_1(q))$. If the rotation interval of $x$ is a single rational point choose $q$ larger than its denominator; otherwise choose any $q > 1$. By Lemma 15.5 this will guarantee that $x$ lies in some $V \in \mathcal{A}(q)$. Corollary 15.8 implies that if $q$ is sufficiently large this $V$ will be essential. This $V$ is disjoint $V_0(q)$ and $V_1(q)$ and hence will satisfy $\mathrm{cl}(V) \subset U$. □

In principle a point $x \in V \in \mathcal{A}(q)$ might have a different rotation interval when viewed in $V$ than when viewed in $U$. The following proposition shows this does not happen, and as a consequence every point of $U$ has a well-defined rotation number.

**Proposition 15.15** *Suppose $x \in U \in \mathcal{A}$. Then the rotation number $\rho_{f_c}(x)$ of $x$ with respect to $f_c \colon U_c \to U_c$ exists. Moreover, if $x$ is in an essential $V \in \mathcal{A}(q)$ and $\rho_{f_c}(x) \notin \rho(\partial V_c)$, then $\rho_{f_c}(x) = \rho_{h_c}(x)$ where $h = f|_V$.*

**Proof** We will first show that $\rho_{f_c}(x)$ exists. Given $\epsilon > 0$ it suffices to show that the rotation interval of $x$ in $U_c$ has length $< \epsilon$. Choose an integer $Q$ such that $1/Q < \epsilon$ and such that $Q$ is greater than the number $r(x)$ from Corollary 15.8. Hence if $q'$ is any product of primes each of which is $> Q$ then $q'$ is relatively prime to $r(x)$ so $x$ is contained in an essential element of $\mathcal{A}(q)$.

Choose three primes $q_i$, $i = 1, 2, 3$ all greater than $Q$. Let $V^i$ be the essential element of $\mathcal{A}(q_i)$ which contains $x$. We may assume that each of the $V^i$ are interior in $U_c$, as otherwise for every sufficiently large prime $q$ the $V'$ in $\mathcal{A}(q)$ containing $x$ is an end element and it follows from Lemma 15.13 that $x$ has a well-defined rotation number which equals the rotation number of one boundary component of $U_c$.

Let $V$ be the essential annulus in $\mathcal{A}(q_1q_2q_3)$ which contains $x$. Note that $V \subset V^i$ and if we define $h_i$ to be the restriction of $f^{q_i}$ to $V^i$ then $V$ can be considered as an element of $\mathcal{A}(q_2q_3, h_1)$, $\mathcal{A}(q_1q_3, h_2)$, and $\mathcal{A}(q_1q_2, h_3)$.

Suppose for one choice of $i$, say $i = 1$, the element $V$ is an interior element of $\mathcal{A}(q_2q_3, h_1)$, ie $\mathrm{cl}(V) \subset V^1$. Then there exists an annulus $A_0$ whose boundary consists of two essential simple closed curves, one in each component of $V^1 \setminus \mathrm{cl}(V)$. The orbit of $x$ lies in $A_0$ and $A_0$ is an essential closed annulus embedded in both $U_c$ and $V_c^1$. Lemma 2.11 implies that the rotation interval of $x$ is the same when calculated in $V_c^1$ as when calculated in $U_c$ and since $|\rho(V^1)| < 1/Q < \epsilon$, the fact that $\epsilon$ is arbitrary proves that the rotation number of $x$ in $U_c$ is well-defined.

We are left with the possibility that each of $\mathcal{A}(q_2q_3, h_1)$, $\mathcal{A}(q_1q_3, h_2)$, and $\mathcal{A}(q_1q_2, h_3)$ has $V$ as an end element. We will show this leads to a contradiction. We chose three primes $q_i$ in order to guarantee that $V$ corresponds to the same end (ie a $V_0$ or a $V_1$) for two of the $h_i$: $V^i \to V^i$. Hence we may assume without loss of generality that $V = V_0(q_2q_3) \in \mathcal{A}(q_2q_3, h_1)$ and $V = V_0(q_1q_3) \in \mathcal{A}(q_1q_3, h_2)$. But Lemma 15.11(2) applied to $f^{q_1}$ implies that the rotation numbers of the maps induced by $f$ on $\partial_0 V_c^1$ and $\partial_0 V_c$ coincide. Likewise, so do the rotation numbers on $\partial_0 V_c^2$ and $\partial_0 V_c$. Since $V^i$ is interior in $U_c$ both its ends are nonsingular. But it follows from Lemma 14.2 applied to $h_i$ that the rotation number of the map induced by $f$ on $\partial_0 V_c^i$ has the form $p_i/q_i$ with $0 < p_i < q_i$ and hence it is not possible for two of these rotation numbers to coincide. This contradiction completes the proof that $\rho_{f_c}(x)$ exists.

To prove the second assertion of the proposition we note that $\rho_{h_c}(x)$ exists by the first part applied to $f^q|_V$. The fact that $\rho_{f_c}(x) \notin \rho(\partial V_c)$, and Lemma 15.14 imply that there is a prime $q'$ and an essential $V' \in \mathcal{A}(q')$ containing $x$ and such that $\mathrm{cl}(V') \subset V$. Hence we may choose a closed annulus $A_0$ containing $V'$ and contained in $V$. Applying Lemma 2.11 we conclude that $\rho_{f_c}(x) = \rho_{h_c}(x)$ where $h = f|_V$. □

**Remark 15.16** In the following definition we assume that $\rho(U)$ is nontrivial. If we are willing to pass to a power of $F$ this is a consequence of our standing hypothesis that $F$: $S^2 \to S^2$ has at least three periodic points, because then $F^q$ will have three fixed points and any $U \in \mathcal{A}(q)$ will have a nontrivial end which is sufficient by Lemma 14.2 to imply $\rho(U)$ is nontrivial.

**Definitions 15.17** Suppose that $\rho(U)$ is nontrivial, that $\partial_0 U_c$ and $\partial_1 U_c$ are the frontier components of $U \in \mathcal{A}$ and that $a_i$ is the rotation number of $f$ on $\partial_i U_c$. Choose $Q = Q(U)$ so that:

- If $a_i = p/q$ for some $0 < p < q$ then $q < Q$.

- For every prime $q > Q$ there is an essential element $V \in \mathcal{A}(q)$ such that $\mathrm{cl}(V) \subset U$. (See Lemma 15.14.)
- For every prime $q > Q$ there are distinct elements $V_0(q), V_1(q) \in \mathcal{A}(q)$ (as in Lemma 15.11) that are contained in $U$ such that $\mathrm{fr}(V_i(q)) \subset U_c$ is $\partial_i U_c \bigcup B_i(q)$ for $i = 0, 1$ where $B_0(q) \cap B_1(q) = \varnothing$.

Define

$$\hat{Y}_i = \bigcap_{q>Q} \mathrm{cl}_{U_c}(V_i(q)),$$

where the intersection is taken over all primes $q > Q$ and define

$$\check{U} = U_c \setminus (\hat{Y}_0 \cup \hat{Y}_1) \subset U.$$

Recall from Definition 1.1 that $\mathcal{W}_0$ is the set of free disk recurrent points.

**Lemma 15.18** *Assume notation as above and assume that $\rho(U)$ is nontrivial. The following hold for $i = 0$ and $i = 1$:*

(1) *$\hat{Y}_i$ is well-defined, ie independent of the choice of $Q$.*
(2) *$\check{U}$ and $\check{U} \cup (\hat{Y}_i \cap U)$ are essential open $f$–invariant annuli.*
(3) *If $a_i = 0$ then $\hat{Y}_i \cap U$ has measure $0$.*
(4) *If $a_i \neq 0$ then $\hat{Y}_i \cap U \subset \mathcal{W}_0$.*
(5) *$\rho(y) = a_i$ for each $y \in \hat{Y}_i \cap U$.*

**Remark 15.19** If $x \in \check{U}$ then it has nonzero rotation number. To see this observe that if $x \in \check{U}$, the $\omega$–limit set $\omega(f_c, x)$ is separated from $\partial U_c$ because the orbit of $x$ is separated from $\partial U_c$ by $V_0(q) \cup V_1(q)$ for some $q$. Then Corollary 15.8 implies that for every sufficiently large prime $q$ the point $x$ must lie in some essential $V \in \mathcal{A}(q)$. For large $q$ this $V$ must be interior, ie separated from $\partial U_c$.

Let $h = f|_V$ and consider $h_c\colon V_c \to V_c$. Observe that $0 \notin \rho_{h_c}(V_c)$, because if it were Theorem 2.3 would imply $h_c$ has a fixed point in $V_c$. But then Lemma 2.8 implies there is a fixed point for $F$ in $\mathrm{cl}(V) \subset U$ which is a contradiction.

Suppose now $\rho_{f_c}(x) = 0$ in $U_c$. Since $0 \notin \rho_{h_c}(\partial V_c)$, Proposition 15.15 implies that $x$ also has rotation number $0$ for $h_c\colon V_c \to V_c$, but as noted above this is a contradiction.

If $a_0 = a_1 = 0$, then item (5) of Lemma 15.18 implies that $U \setminus \check{U}$ consists of points with rotation number $0$. Hence in this case $\check{U}$ is precisely the subset of $U$ whose points have nonzero rotation number.

**Proof** By part (3) of Lemma 15.11 there is a sequence of primes $\{q_j\}$ tending to infinity such that

$$\widehat{Y}_i = \bigcap_{q_j} \mathrm{cl}_{U_c}(V_i(q_j)) \quad \text{and} \quad V_i(q_{j+1}) \cup B_i(q_{j+1})) \subset V_i(q_j)$$

for all $j$, where $B_i(q)$ denotes the component of the frontier of $V_i(q)$ which lies in $U$. This proves that $\widehat{Y}_i$ does not depend on the choice of $Q$ in its definition and so is well-defined.

Let $Z_i(q)$ be the component of $S^2 \setminus B_i(q)$ that contains $Y_i$. Then $\mathrm{cl}_{S^2}(Z_i(q_{j+1})) = Z_i(q_{j+1}) \cup B_i(q_{j+1}) \subset Z_i(q_j)$ for the sequence of primes $\{q_j\}$ chosen above. Define

$$\widehat{Z}_i = \bigcap_{q_j} \mathrm{cl}_{S^2}\, Z_i(q_j)) = \bigcap_{q>Q} \mathrm{cl}_{S^2}(Z_i(q))$$

Then $\widehat{Z}_0$ and $\widehat{Z}_1$ are disjoint, compact, connected sets and $\check{U} = U_c \setminus (\widehat{Y}_0 \cup \widehat{Y}_1) = S^2 \setminus (\widehat{Z}_0 \cup \widehat{Z}_1)$. Thus $\check{U}$ is an open subsurface of $S^2$ with two ends and hence an annulus. The set $\check{U}$ separates $\widehat{Z}_0$ and $\widehat{Z}_1$ each of which contains a point of $\mathrm{Fix}(F)$. Hence $\check{U}$ is essential in $\mathcal{M}$ and therefore in $U$. The same argument applies to $\check{U} \cup (\widehat{Y}_i \cap U)$: Its complement in $S^2$ has two components. For example, if $i = 0$ then one of the components is $\widehat{Z}_1$ and the other is the component of the complement of $U$ that intersects $\widehat{Z}_0$. Each of these complementary components contains a point of $\mathrm{Fix}(F)$ and hence $\check{U} \cup (\widehat{Y}_i \cap U)$ is an annulus which is essential in $\mathcal{M}$ and therefore in $U$.

To show $\check{U}$ is $f$–invariant it suffices to show $\widehat{Y}_i$ is $f_c$–invariant, but this follows from the definition of $\widehat{Y}_i$ and the fact that $V_i(q)$ is $f_c$–invariant. Having verified that $\check{U}$ is $f$–invariant, the same is true for $\check{U} \cup (\widehat{Y}_i \cap U)$. This completes the proof of (2).

Item (5) follows from Lemma 15.13. Item (3) then follows from Proposition 2.4 and that fact that $\mathrm{Fix}(f) = \varnothing$.

For (4) suppose that $a_i \neq 0$ and that $x \in \widehat{Y}_i$. If the $\omega$–limit set of $x$ contains a point in $U$ then $x \in \mathcal{W}_0$. Otherwise there is a nonfixed point $z$ in $\omega(x, f_c)$. If $D_0$ is a disk neighborhood of $z$ then $D_0 \cap U$ is a free disk that the orbit of $x$ intersects more than once and again $x \in \mathcal{W}_0$. □

We are now prepared to complete the proof of Theorem 1.2. For the definition of free disk recurrent and weakly free disk recurrent see Definition 1.1.

**Theorem 1.2** *Suppose $F \in \mathrm{Diff}_\mu(S^2, P)$ has entropy zero, infinite order and at least three periodic points. Let $f = F|_{\mathcal{M}}$ where $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$. Then there is a countable collection $\mathcal{A}$ of pairwise disjoint open $f$–invariant annuli such that:*

(1) *$\mathcal{U} = \bigcup_{U \in \mathcal{A}} U$ is the set $\mathcal{W}$ of weakly free disk recurrent points for $f$.*

(2) *$\mathcal{A}$ is the set of maximal $f$–invariant open annuli in $\mathcal{M}$.*

(3) *If $z \notin \mathcal{U}$, there are components $F_+(z)$ and $F_-(z)$ of $\text{Fix}(F)$ so that $\omega(F, z) \subset F_+(z)$ and $\alpha(F, z) \subset F_-(z)$.*

(4) *For each $U \in \mathcal{A}$ and each component $C_{\mathcal{M}}$ of the frontier of $U$ in $\mathcal{M}$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_{\mathcal{M}}$.*

**Proof** It suffices to verify items (1)–(4) for one component $M$ of $\mathcal{M}$ at a time. Items (2), (3) and (4) follow from the second, third and fifth items of Proposition 5.1 so it suffices to prove (1).

If $x \in \mathcal{W}_0$ then there is a free disk $D$ and $n > 0$ such that $x, f^n(x) \in D$. Choose lifts $\tilde{x} \in \tilde{D}$ to $H$, let $\tilde{C}$ be a home domain for $\tilde{x}$ and let $T$ be a covering translation such that $\tilde{f}_{\tilde{C}}{}^n(x) \in T(D)$. Thus $T$ is an $\tilde{f}_{\tilde{C}}$–near cycle for $\tilde{x}$. Since $\tilde{x}$ and $\tilde{f}_{\tilde{C}}{}^n(\tilde{x})$ are both contained in $\tilde{U}$, $T$ preserves $\tilde{U}$ and so preserves $\tilde{C}$. Lemma 13.8(2) implies that $x \in \mathcal{U}$ thereby proving that $\mathcal{W}_0 \subset \mathcal{U}$. Lemma 14.5 therefore implies that $\mathcal{W} \subset \mathcal{U}$.

To prove the converse note that the $\omega$–limit set of any point in $\check{U}$ lies in $U$ and hence contains points that are not fixed by $f$. It follows that $\check{U} \subset \mathcal{W}_0$. If both $a_0$ and $a_1$ are nonzero then $U = \check{U} \cup (U \cap (\hat{Y}_0 \cup \hat{Y}_1)) \subset \mathcal{W}_0 \subset \mathcal{W}$ by item (4) of Lemma 15.18. If both $a_0$ and $a_1$ are zero then $\check{U}$ is dense in $U$ by item (3) of Lemma 15.18. Thus $U \subset \text{int}_M \, \text{cl}_M(\check{U}) \subset \mathcal{W}$ since $\check{U}$ is a connected (item (2) of Lemma 15.18) subset of $\mathcal{W}_0$. For the remaining case we may assume that $a_0 = 0$ and $a_1 \neq 0$. Then $U \subset \text{int}_M \, \text{cl}_M(\check{U} \cup (\hat{Y}_1 \cap U)) \subset \mathcal{W}$ because $\check{U} \cup (\hat{Y}_1 \cap U)$ is a connected subset of $\mathcal{W}_0$. □

**Theorem 1.4** *Suppose $F \in \text{Diff}_\mu(S^2, P)$ has entropy zero, has infinite order and at least three periodic points. Let $f = F|_{\mathcal{M}}$ where $\mathcal{M} = S^2 \setminus \text{Fix}(F)$ and let $\mathcal{A}$ be as in Theorem 1.2. For $U \in \mathcal{A}$, let $f_c\colon U_c \to U_c$ be the annular compactification of $f|_U\colon U \to U$. Then:*

(1) *The rotation number $\rho_{f_c}(x)$ is defined and continuous at every $x \in U_c$.*

(2) *If $\text{Fix}(F)$ contains at least three points then $\rho_{f_c}$ is nonconstant.*

(3) *If $C$ is a component of a level set of $\rho_{f_c}$ then $C$ is $F$–invariant. If $C$ does not contain a component of $\partial U_c$ then it is essential in $U$, meaning that $U_c \setminus C$ has two components each containing a component of $\partial U_c$.*

**Proof** For notational simplicity we write $\rho(x) = \rho_{f_c}(x)$. Proposition 15.15 says that $\rho(x)$ is defined for all $x \in U_c$ so to prove (1) we must verify continuity of $\rho$.

Assume notation as in Definitions 15.17. Recall that $a_i$ is the rotation number of $f_c$ on $\partial_i U_c$. By item (5) of Lemma 15.18, $\rho(y) = a_i$ for all $y \in \widehat{Y}_i$. By construction, there is a sequence of primes $\{q_k\}$ tending to infinity such that $N_k = V_i(q_k) \cup \widehat{Y}_i$ is a nested sequence of $f_c$–invariant neighborhoods of $\widehat{Y}_i$ whose frontiers have rotation number $p/q_k$ with $0 < p < q_k$. For sufficiently large $k$, $p/q_k \neq a_0$ and we conclude that $\widehat{Y}_i$ is a level set for $\rho_{f_c}$. Also $\partial_i U_c$ can be identified with $\partial_i V_i(q_k)_c$ since $N_k$ is a neighborhood of $Y_i$ in both $\mathrm{cl}(U)$ and $\mathrm{cl}(V_i(q_k))$. It follows that $a_i \in \rho(V_i(q_k))$ for all $k$. Proposition 15.15 implies that if $r \in \rho(V_i(q_k))$ then $r$ is in the rotation interval for the induced action of $f$ on $V_i(q_k)_c$. Since the length of this interval tends to zero as $k \to \infty$ (Corollary 15.6), the same is true for the length of $\rho(V_i(q_k))$. This proves continuity of $\rho$ at points of $Y_i$.

The level sets $C(x)$ for $x \in \check{U}$ are defined similarly. We specify $Q = Q(x)$ by a series of largeness conditions. By Lemma 15.11(3) and the assumption that $x \in \check{U}$, we may assume that $x \notin V_0(q) \cup V_1(q)$ for $q \geq Q$. In particular, $\omega(x) \subset U$. We may also assume that $\rho(x) \neq p/q$ for $q \geq Q$ and $0 < p < q$. Lemma 15.5 therefore implies that $x$ is contained in some $V(q,x) \in \mathcal{A}(q)$ which is essential by Lemma 15.7. Since $x \notin V_0(q) \cup V_1(q)$, we have $\mathrm{cl}(V(q,x)) \subset U$.

Define

$$C(x) = \bigcap_{q>Q} \mathrm{cl}(V(q,x)),$$

where the intersection is over all primes $> Q$.

Given $q$ let $\delta$ be the minimum value of $|p/q - \rho(x)|$ for $0 < p < q$. If $q' > 1/\delta$ then $\rho(V(q',x))$ does not contain $p/q$ for $0 < p < q$ and so does not contain any points in the frontier of $V(q,x)$. It follows that $\mathrm{cl}(V(q',x)) \subset V(q,x)$. We may therefore choose a sequence of primes $\{q_j\}$ tending to infinity such that

$$C(x) = \bigcap_{q_j} \mathrm{cl}(V(q_j,x)) \quad \text{and} \quad \mathrm{cl}(V(q_{j+1},x)) \subset V(q_j,x)$$

for all $j$. This proves that $C(x)$ is nonempty and does not depend on the choice of $Q$ in its definition and so is well-defined.

Item (1) follows from the fact that $|\rho(V(q,x))| \leq 1/q$.

If $y \notin C(x)$ then there exists $q > Q$ such that $y \notin \mathrm{cl}(V(q,x))$. The frontier of $V(q,x)$ separates $C(x)$ and $y$ and $q$ may be chosen so that this frontier consists of points whose rotation number is not equal to $\rho(x)$ so $y$ is not in the same connected component as $x$ of the level set of $\rho$. It follows that $C(x)$ is a connected component of this level set. Since it is clear from the construction that $C(x)$ is $F$–invariant and essential, we have proved (3).

If $\mathrm{Fix}(F)$ contains at least three points then $U$ has at least one nonsingular end. It then follows from Lemma 14.2 that $\rho(U)$ is nontrivial so $\rho_{f_c}$ is nonconstant. This verifies (2) and completes the proof. □

## 16 Proof of Theorem 1.5

The proof of Theorem 1.5 is given at the end of this section following the statement and proofs of some preliminary lemmas. Recall that $A^\circ = \mathrm{int}(A)$ and that if $H \in \mathrm{Diff}_\mu(A, P')$ then the homeomorphism $F\colon S^2 \to S^2$ obtained from $H$ by collapsing each component of $\partial A$ to a point satisfies $F \in \mathrm{Diff}_\mu(S^2, P)$. As throughout the paper, $\mathcal{M} = S^2 \setminus \mathrm{Fix}(F)$ and $f = F|_{\mathcal{M}}$. We identify $\mathcal{M}$ with $A \setminus (\mathrm{Fix}(H) \cup \partial A)$ and $H|_{\mathcal{M}}$ with $f$.

**Theorem 1.5** *For each $H \in \mathrm{Diff}_\mu(A, P')$ with entropy zero, the rotation number $\rho_H(x)$ is defined and continuous at each $x \in A$.*

We assume without loss that $H$, and hence $F$, has infinite order.

**Lemma 16.1** *Suppose that $H \in \mathrm{Diff}_\mu(A, P')$ has entropy zero and that $\mathrm{Fix}(H)$ contains at least one point in $A^\circ$. Let $\mathcal{A}$ be as in Theorem 1.4 applied to the element $F \in \mathrm{Diff}_\mu(S^2, P)$ corresponding to $H$.*

1. *If $U \in \mathcal{A}$ is essential in $A$ then $\rho_H(x) = \rho_{f_c}(x)$ for all $x \in U$. If a component $\partial_0 A$ of $\partial A$ is a frontier component of $U$, then $\rho_H$ is defined and continuous on a neighborhood of $\partial_0 A$.*
2. *If $U \in \mathcal{A}$ is inessential in $A$ then $\rho_H(x) = 0$ for all $x \in U$.*

**Proof** We first consider the case that $U \in \mathcal{A}$ is essential. Theorem 1.4(2) and Theorem 2.3 imply that the image of $\rho_{f_c}$ is a nontrivial interval. Suppose that $B$ is a component of a level set of $\rho_{f_c}$. If $B$ is disjoint from $\partial U_c$, then $B$ is $H$–invariant and essential by Theorem 1.4(3). Since any such $B$ is contained in a closed annulus in $U$ that is essential in both $U$ and $A$, Lemma 2.11 implies that $\rho_{f_c}|_B = \rho_H|_B$. It therefore suffices to prove the lemma for $B$ containing a component of $\partial U_c$.

Since $U$ is essential, $B$ corresponds to a singular end of $U$ if and only if it corresponds to an end of $A^\circ$ determined by a component, say $\partial_0 A$, of $\partial A$. In this case, Lemma 13.12 (3) implies that $U \cup \partial_0 A$ is a neighborhood of $\partial_0 A$ in $A$. Since $\rho_{f_c}$ is not constant there is a core curve in $U$ which separates $B$ from the end of $U$ which does not correspond to $B$. Let $N$ be this curve together with the component of its complement in $U$ which contains the end corresponding to $B$. Note that $N$ is a neighborhood of an end of $U$ and also a neighborhood of an end of $A^{\circ}$. The compactification of this end can be done in $N$, so it is the same in $U_c$ and in $A$. Hence $\partial_0 A$ can be thought of as a component of $\partial U_c$ and $B$ can be thought of as a subset of $A$. Since $N \cup \partial_0 A$ is a closed annulus in $U_c$ containing $B$ which is disjoint from the other component of $\partial U_c$, Lemma 2.11 implies that $\rho_H|_B = \rho_{f_c}|_B$. Since $\rho_{f_c}$ is continuous, $\rho_H$ is defined and continuous on a neighborhood of $\partial_0 A$.

We may therefore assume that $B$ corresponds to a nonsingular end of $U$. Let $X_A$ be the component of the frontier of $U$ in $A$ determined by this nonsingular end and let $B_A = X_A \cup (B \cap U)$. In other words, $B_A \subset A$ is obtained from $B \subset U_c$ by replacing a component of the frontier of $U$ in $U_c$ with a component of the frontier of $U$ in $A$. Lemma 13.12(4) implies that $\rho_{f_c}|_B = 0$ so it suffices to show that $\rho_H|_{B_A} = 0$. For reference below, we note that the remainder of the proof of (1) makes no use of the fact that $U$ is essential.

Let $\widetilde{U}_A \subset \widetilde{A}$ be the lift of $U$ to the universal (cyclic) cover $\widetilde{A}$ of $A$ and let $\widetilde{X}_A$ be the component of the frontier of $\widetilde{U}_A$ in $\widetilde{A}$ that projects to $X_A$. Denote the full preimage of $\mathrm{Fix}(H)$ by $\widetilde{\mathrm{Fix}}(H)$. We claim that there is a lift $\widetilde{H}\colon \widetilde{A} \to \widetilde{A}$ that fixes each point in $\widetilde{X}_A \cap \widetilde{\mathrm{Fix}}(H)$. Up to isotopy rel $\mathrm{Fix}(H)$, $H$ is isotopic to a composition of Dehn twists along a finite set $\Sigma$ of disjoint simple closed curves in $A \setminus \mathrm{Fix}(H)$. (See Section 4.) To prove the claim, it suffices to show that no two points in $\mathrm{Fix}(H) \cap X_A$ are separated from each other by an element $\sigma$ of $\Sigma$.

Let $M$ be the component of $A^{\circ} \setminus \mathrm{Fix}(H)$ that contains $U$. If $M$ is an annulus then $M = U$ and $X_A \subset \mathrm{Fix}(H) \cup \partial A$. In this case, $X_A \cap \Sigma = \varnothing$ so the claim is clear. We may therefore assume that $M$ has at least three ends. Equip $M$ with a complete hyperbolic metric in which all isolated punctures are cusps, in which the core curve $\tau$ of $U$ is a geodesic and in which the elements of $\Sigma$ are disjoint simple closed geodesics that have no transverse intersections with $\tau$. If $\tau$ is not an element of $\Sigma$, let $C$ be the component of $M \setminus \Sigma$ that contains $\tau$. Corollary 9.9(1) implies that $U$ is contained in a bounded (as measured in the hyperbolic metric) neighborhood of $C$. In particular, all points in the intersection of $\mathrm{Fix}(H)$ with the closure of $U$ in $A$ are contained in the component of $A \setminus \Sigma$ that contains $C$. Since any two such points can be connected by an arc in $A \setminus \Sigma$, the claim is proved. If $\tau$ is an element of $\Sigma$, let $C_1$ and $C_2$ be the components of $M \setminus \Sigma$ on either side of $\tau$. Corollary 9.9(2) implies that $U$ is contained in a bounded (as measured in the hyperbolic metric) neighborhood of $C_1 \cup C_2$. Since $X_A$ is disjoint from $\tau$, we may assume that $X_A$ is contained in the closure of $C_1$ and the proof of the claim concludes as in the previous case.

Identify $\widetilde{A}$ with $\mathbb{R} \times [0,1]$ and let $p_1\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be projection onto the first coordinate. Let $\widetilde{B}_A$ be the preimage of $B_A$ and choose $\tilde{y} \in \widetilde{B}_A$. For $\delta > 0$, we say that

*$k$ is $\delta$–good* if $p_1(\widetilde{H}^k(\widetilde{y})) - p_1(\widetilde{y}) < k\delta$. We complete the proof of (1) by choosing $\epsilon > 0$ arbitrarily and showing that all sufficiently large $k$ are $\epsilon$–good.

Lemma 13.8(3) implies that for any neighborhood $W$ of $\mathrm{Fix}(H)$ there exists a positive integer $M$ so that for all $x \in X_A$, we have $H^k(x) \in W$ for all but at most $M$ values of $k$. We may therefore choose $K_1$ so that

$$p_1(\widetilde{H}^{K_1}(\widetilde{x})) - p_1(\widetilde{x}) < K_1\epsilon/2$$

for all $\widetilde{x} \in \widetilde{X}_A$ and hence for all $\widetilde{x}$ in a neighborhood $\widetilde{V}$ of $\widetilde{X}_A$. Note that this inequality can be concatenated. Thus, if $\widetilde{H}^{jK_1}(\widetilde{x}) \in \widetilde{V}$ for all $0 \le j \le J$ then $p_1(\widetilde{H}^{JK_1}(\widetilde{x})) - p_1(\widetilde{x}) < JK_1\epsilon/2$. In particular, if the forward orbit of $\widetilde{y}$ is eventually contained in $\widetilde{V}$ then all sufficiently large $k$ are $\epsilon$ good. We may therefore assume that there exists arbitrarily large $k$ with $\widetilde{H}^k(\widetilde{y}) \notin \widetilde{V}$.

There is a compact essential subannulus of $U$ whose lift to $\widetilde{A}$ contains $\widetilde{B}_A \cap (\widetilde{A} \setminus \widetilde{V})$. We may therefore assume that $p_1$ and the projection $p_1'\colon \widetilde{U}_C \to \mathbb{R}$ used to define $\rho_{f_c}$ agree on $\widetilde{B}_A \cap (\widetilde{A} \setminus \widetilde{V})$. Since $\rho_{f_c}|B = 0$, we may assume after reducing the size of $\widetilde{V}$ if necessary, that there exists $K_2$ so that $k$ is $\epsilon/2$–good whenever $k \ge K_2$ and $\widetilde{H}^k(\widetilde{y}) \notin \widetilde{V}$.

Given arbitrary $k > K_2$, let $k'$ be the largest value between $k$ and $K_2$ such that $\widetilde{H}^k(\widetilde{y}) \notin \widetilde{V}$ and let $m$ be the largest integer such that $l := K_2 + mK_1 < k$. Then $k' + mK_1$ is $\epsilon/2$–good, and $k - l$ is bounded by $K_1$. It follows that $k$ is $\epsilon$–good for all sufficiently large $k$. This completes the proof of (1)

If $U$ is inessential then the union of $U$ with one of the components of $A \setminus U$ is an open $H$–invariant disk. If $B$ is a level set of $\rho_{f_c}$ that does not contain a component of $\partial U_c$ then $B$ is contained in an $H$–invariant open disk whose closure does not separate the boundary components of $A$. It follows that the complete lift of this disk to $\widetilde{A}$ has bounded components and hence all points in the disk have 0 rotation number. The lemma therefore holds for all such $B$ and for the component of the frontier of $U$ in $A$ that is contained in the $H$–invariant disk. We are therefore reduced to considering the level set $B$ corresponding to the other, necessarily nonsingular, end of $U$. The proof given above applies without change. □

**Lemma 16.2** *Suppose that $H \in \mathrm{Diff}_\mu(A, P')$ has entropy zero and that $\mathrm{Fix}(H)$ contains at least one point in $A^\circ$. Let $\mathcal{A}$ be as in Theorem 1.4 applied to the element $F \in \mathrm{Diff}_\mu(S^2, P)$ corresponding to $H$. If $\partial_0 A$ is a component of $\partial A$ and $\rho_H|_{\partial_0 A} \neq 0$ then $\partial_0 A$ is a frontier component of some essential $U \in \mathcal{A}$.*

**Proof** Since $\mathrm{Fix}(H) \cap \partial_0 A = \varnothing$, there is a component $M$ of $\mathcal{M}$ that contains a deleted neighborhood $V$ of $\partial_0 A$. If $M$ is an annulus, then it is an element of $\mathcal{A}$ and we are done. We may therefore assume that $M$ has at least three ends. Choose a component $\widetilde{V}$ of the full preimage of $V$ in the universal cover of $M$, let $T$ be the parabolic covering translation that preserves $\widetilde{V}$ and let $P \in S_\infty$ be the unique fixed point of $T$. After shrinking $V$ if necessary, we may assume that $V$ is covered by a finite collection of free disks, say $k$ free disks. If $x \in V$ is sufficiently close to $\partial_0 A$ then $f^j(x) \in V$ for all $0 \le j \le k$ and so there exists $0 \le j_1 \le j_2 \le k$ such that $f^{j_1}(x)$ and $f^{j_2}(x)$ belong to the same free disk. It follows that some iterate of $T$ is a near cycle for all points in $\widetilde{V}$ that are sufficiently close to $P$. The domain containing $P$ in its closure is a home domain for all such points by Corollary 9.9 and is obviously $T$–invariant. Lemma 12.1 therefore implies that $T$ is the covering translation associated to some $\widetilde{U} \in \widetilde{A}$ and Lemma 13.12(3) implies that $U$ contains a deleted neighborhood of $\partial_0 A$. □

**Lemma 16.3** *Suppose that $H\colon A \to A$ is a homeomorphism of the closed annulus and that $U_1, U_2, \dots$ is an infinite sequence of disjoint invariant open essential annuli in $A$. Let $H_i\colon U_i^c \to U_i^c$ be the annular compactification (see Notation 2.7) of $H|_{U_i}$ and let $L_i$ be the length of the forward translation interval (see Definition 2.1) of some (any) lift of $H_i$. Then $L_i \to 0$.*

**Proof** If the lemma fails then, after passing to a subsequence, there exists $\epsilon > 0$ such that $L_i > \epsilon$ for all $i$. After replacing $H$ with an iterate, we may assume that $L_i > 2$ for all $i$. Since each $U_i$ contains points with rotation number $\frac{1}{2}$, we may choose $y_i \in U_i$ such that $H(y_i)$ is antipodal to $y_i$ in the $S^1 \times [0,1]$ structure of $A$. After passing to a further subsequence we may assume that $y_i \to x$ for some nonfixed $x \in A$ and that $U_{i+1}$ separates $U_i$ from $x$ for all $i$. Choose a free disc neighborhood of $x$ and an arc $\nu$ in this free disk that begins at $x$ and ends at a point in some $\mathrm{fr}(U_{i_0})$. For all $i > i_0$, there is a subarc $\nu_i$ with interior in $U_i$ and with endpoints on both components of $\mathrm{fr}(U_i)$. Transporting this to $H_i\colon U_i^c \to U_i^c$, there is an arc $\nu_i$ with interior in $U_i$, with endpoints on distinct components of $\partial U_i^c$ and satisfying $H(\nu_i) \cap \nu_i = \emptyset$.

We fix such an $i$ and drop the $i$ subscript from the notation, renaming $H_i\colon U_i^c \to U_i^c$ by $h\colon A \to A$ and the arc $\nu_i$ by $\nu$. Identify $\widetilde{A}$ with $\mathbb{R} \times [0,1]$ and let $p_1\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be projection onto the first coordinate. Let $T$ be the covering translation $T(r,s) = (r+1,s)$, let $\widetilde{\nu}$ be a lift of $\nu$ and let $\widetilde{h}$ be the lift of $H$ such that $\widetilde{H}(\widetilde{\nu})$ is contained in the interior of the region bounded by $\widetilde{\nu}$ and $T(\widetilde{\nu})$. Then $\widetilde{H}^2(\widetilde{\nu})$ is contained in the interior of the region bounded by $\widetilde{h}(\widetilde{\nu})$ and $\widetilde{h}(T(\widetilde{\nu})) = T(\widetilde{h}(\widetilde{\nu}))$ and so is also contained in the interior of the region bounded by $\widetilde{\nu}$ and $T^2(\widetilde{\nu})$. Continuing in this manner we conclude that $\widetilde{h}^k(\widetilde{\nu})$ is contained in the interior of the region bounded by $\widetilde{\nu}$ and $T^k(\widetilde{\nu})$, which implies that forward translation interval for $\widetilde{h}$ has length at most one. This contradiction completes the proof. □

**Proof of Theorem 1.5** Let $F \in \mathrm{Diff}_\mu(S^2, P)$ be the element $F \in \mathrm{Diff}_\mu(S^2, P)$ corresponding to $H$. If $H|_{A^\circ}$ has no interior periodic points then every point in $A$ has the same irrational rotation number by Theorem 2.3. We may therefore assume that $F$ satisfies the hypotheses of Theorems 1.2 and 1.4. Let $\mathcal{A}$ be the set of annuli produced by those theorems, let $\mathcal{U} = \bigcup_{U \in \mathcal{A}} U$, let $\mathcal{U}_e$ be the union of all essential elements of $\mathcal{A}$ and let $\mathcal{U}_e'$ be the union of $\mathcal{U}_e$ with any components of $\partial A$ for which $\rho_F$ is nonzero. After replacing $H$ with an iterate, we may assume that $H$ has at least one interior fixed point. Lemma 16.2 implies that $\mathcal{U}_e'$ is an open subset of $A$.

Each $x \in A \setminus \mathcal{U}_e'$ satisfies one of the following:

- $x$ is contained in a components of $\partial A$ with zero rotation number.
- $x \in \mathrm{Fix}(H)$.
- $x \in A^\circ \setminus \mathcal{U}$.
- $x$ is contained in an inessential element of $\mathcal{A}$.

In all of these cases $\rho_H(x) = 0$. This is obvious for the first two and follows from Theorem 1.2(3) for the third and Lemma 16.1(2) for the fourth.

To complete the proof of Theorem 1.5, it suffices to show that $\rho_H$ is defined and continuous on the closure of $\mathcal{U}_e'$. By Theorem 1.4(1), $\rho_H$ is defined and continuous on $\mathcal{U}_e$. It is obvious that $\rho_H$ is defined on $\partial A$. Lemma 16.1(1) implies that $\rho_H$ is continuous at points in a component of $\partial A$ with nonzero rotation number. It remains only to show that if $x_i \to x$ where $x_i \in \mathcal{U}_e$ and $x \in \mathrm{fr}(\mathcal{U}_e)$ does not belong to a boundary component with nonzero rotation number, then $\rho_H(x_i) \to 0$. By Lemma 16.3 we may assume that the $x_i$ belong to a single essential $U \in A$. Since $\rho_H(x_i) = \rho_{f_c}(x_i)$ and $\rho_{f_c}$ is continuous, it suffices to show that $\rho_{f_c}|_{\partial_0 U_c} = 0$ where $\partial_0 U_c$ is the component of $\partial U_c$ to which the $x_i$ converge. This follows from Lemma 13.12(4) if $x$ is not contained in $\partial A$ and is obvious if $x \in \partial A$ because we have excluded components of $\partial A$ with nonzero rotation number. □

## 17 The proof of Theorem 1.7

Recall that a group $G$ is called *indicable* if there is a nontrivial homomorphism $\phi\colon G \to \mathbb{Z}$. We say $G$ is *virtually indicable* if it has a finite index subgroup which is indicable.

**Proposition 17.1** *Suppose that $S$ is a surface and $F\colon S \to S$ is $C^{1+\epsilon}$ and has positive topological entropy. Then every finitely generated infinite subgroup $H$ of the centralizer $Z(F)$ of $F$ is virtually indicable and has a finite index subgroup that has a global fixed point.*

**Proof** A result of Katok [20] asserts that $F^q$ has a hyperbolic saddle fixed point $p$ for some $q \geq 1$. The orbit of $p$ under $H$ consists of hyperbolic fixed points of $F^q$ at which the derivative of $DF^q$ has the same eigenvalues as $DF^q_p$. If the $H$ orbit of $p$ were infinite, continuity of the derivative would imply that at any limit point of this orbit $DF^q$ would have the same eigenvalues and in particular would be hyperbolic. But this is impossible since hyperbolic fixed points are isolated. We conclude the orbit of $p$ under $H$ is finite and hence that the subgroup $H_0$ of $H$ that fixes $p$ has finite index.

After passing to a further finite index subgroup we may assume that $Dh_p$ has positive eigenvalues and the same eigenspaces as $DF_p$ for each $h \in H_0$. For each eigenspace the function which assigns to $h$ the log of the eigenvalue of $Dh_p$ on that eigenspace is a homomorphism from $H_0$ to $\mathbb{R}$. If this is nontrivial we are done. Otherwise both eigenvalues are 1 for each $Dh_p$. Hence in the appropriate basis

$$Dh_p = \begin{pmatrix} 1 & r_h \\ 0 & 1 \end{pmatrix}$$

for some $r_h \in \mathbb{R}$. The function $h \mapsto r_h$ defines a homomorphism from $H_0$ to $\mathbb{R}$, so we are done unless $r_h = 0$ for all $h \in H_0$. But in this latter case $Dh_p = I$ for all $h \in H_0$ so we may apply the Thurston stability theorem ([27]; see also [10, Theorem 3.4]) to conclude there is a nontrivial homomorphism from $H_0$ to $\mathbb{R}$. □

**Examples 17.2** Let $S = S^2$ be the unit sphere in $\mathbb{R}^3$. Let $F\colon S \to S$ be a diffeomorphism whose restriction to each of the level sets $z = c$ is a rotation of that circle and with the property that $F = \mathrm{id}$ for all points $(x, y, z)$ with $|z| \geq \frac{3}{4}$. We assume that $F$ is not the identity on the equator $z = 0$. Let $g\colon S \to S$ be a rotation about the $z$–axis by an angle which is an irrational multiple of $\pi$. Let $h\colon S \to S$ be a diffeomorphism supported in the interior of the disks $|z| > \frac{3}{4}$ with the property that $h$ preserves area and the $h$–orbits of $(0, 0, 1)$ and $(0, 0, -1)$ are infinite. Let $G$ be the group of all rotations about the $z$–axis through angles which are rational multiples of $\pi$.

(1) The group $H$ generated by $g$ and $h$ lies in the centralizer $Z(F)$ of $F$ but has no finite index subgroup with a global fixed point.

(2) The group $G$ is a subgroup of $Z(g)$. Every element of $G$ has finite order so there are no nontrivial homomorphisms from any subgroup of $G$ to $\mathbb{R}$ and hence $G$ is not virtually indicable.

The first example above shows that we cannot generalize Proposition 17.1 to the centralizer of a diffeomorphism $F$ with zero entropy, even in the group of area preserving diffeomorphisms. The second example shows the necessity of the hypothesis of finitely generated in the following.

**Theorem 1.7** *If $F \in \mathrm{Diff}_\mu(S^2)$ has infinite order then each finitely generated infinite subgroup $H$ of $Z(F)$ is virtually indicable.*

**Proof** The case that $F$ has positive entropy is covered by Proposition 17.1 so we need only consider the case when $F$ has entropy zero. We assume that every finite index subgroup of $H$ admits only the trivial homomorphism to $\mathbb{R}$ and show this leads to a contradiction.

Assuming for now that $\mathrm{Per}(F)$ contains at least three points, we may apply Theorem 1.2 to $F$ and its iterates obtaining the families $\mathcal{A}(q)$ of $F^q$–invariant annuli guaranteed by that theorem. Since there is no loss in replacing $F$ by an iterate, we may assume that $F$ has at least three fixed points. Choose once and for all an element $U \in \mathcal{A} = \mathcal{A}(1)$. Item (2) of Theorem 1.4 implies that $f_c\colon U_c \to U_c$ has a nontrivial rotation interval so by Theorem 2.3 we may choose $x \in U$ such that $\rho_{f_c}(x) = \lambda$ is irrational and not equal to the rotation number of either component of $\partial U_c$.

By Corollary 13.5, each $h \in H$ permutes the elements of $\mathcal{A}$. In particular, for any $h \in H$ the open annuli $U$ and $h(U)$ must be disjoint or equal. Since elements of $H$ preserve area the $H$ orbit of the open set $U$ must be finite. We let $H'$ be the finite index subgroup of $H$ which leaves $U$ invariant.

Let $C(x)$ be the component of the level set of $\rho_{f_c}$ that contains $x$. Since $h \in H'$ preserves level sets of $\rho_{f_c}$, $h(C(x))$ is either equal to or disjoint from $C(x)$. By Theorem 1.4 (3), $C(x)$ is essential in $U$. Since $h$ preserves area, it cannot move $C(x)$ off of itself and we conclude that $C(x)$ is $h$–invariant.

Choose a sequence of primes $\{q_n\}$ tending to infinity. By Corollary 15.8, for $n$ sufficiently large, $x \in V_n$ for some essential $V_n \in \mathcal{A}(q_n)$. Lemma 15.5 implies that $C(x)$ is disjoint from the frontier of $V_n$ and hence contained in $V_n$. Since $h$ preserves $C(x)$ and permutes the elements of $\mathcal{A}(q_n)$, it follows that $V_n$ is $h$–invariant.

Choose one component, $V_n^+$, of the complement of $C(x)$ in $V_n$ in such a way that $V_{n+1}^+ \subset V_n^+$, ie always choose the component on the same side of $C(x)$. Let $\overline{A}_n$ denote $V_n^+$ with its ends compactified by the prime end compactification. Let $\partial^+ A$ denote the circle of prime ends added to the end corresponding to $C(x)$. The natural identification of these circles for different $n$ is reflected in the notation which is independent of $n$. Let $A_n = V_n^+ \cup \partial^+ A$, ie $V_n^+$ with only one end (the one corresponding to $C(x)$)

compactified. Then $A_{n+1} \subset A_n$ and

$$\bigcap_{n>0} A_n = \partial^+ A.$$

Let $\bar{f}\colon \bar{A}_n \to \bar{A}_n$ and $\bar{h}\colon \bar{A}_n \to \bar{A}_n$ denote the natural extensions of $F$ and $h \in H'$ to $\bar{A}_n$.

The rotation number $\rho(\bar{f}|_{\partial^+ A})$ of the restriction of $\bar{f}$ to $\partial^+ A$ must be $\lambda$. This is because if it were not and $p/q$ is between $\rho(\bar{f}|_{\partial^+ A})$ and $\lambda$ then by Theorem 2.3 applied to $\bar{A}_n$ there would be periodic points in the interior of $\bar{A}_n \subset V_n$ with rotation number $p/q$ for all $n$, a contradiction.

For each $n$ there is a homomorphism $\phi_n\colon H' \to S^1 = \mathbb{R}/\mathbb{Z}$ given by $h \mapsto \rho_\mu(h|_{\bar{A}_n})$ where $\rho_\mu(h|_{\bar{A}_n})$ denotes the mean rotation number of $h$ on the annulus $\bar{A}_n$ (see Definition 2.5). Let $H''$ denote the subgroup of $H'$ which is the kernel of the canonical homomorphism from $H'$ to its abelianization. Then $\rho_\mu(h|_{\bar{A}_n}) = 0$ for all $h \in H''$. Also the abelianization of $H'$ must be finite since this is one of the equivalent conditions for $H'$ not to be indicable. Therefore $H''$ has finite index in $H'$ (and hence in $H$).

Since $\rho_\mu(\bar{h}|_{\bar{A}_n}) = 0$ for each $n$ and each $h \in H''$ we conclude from Proposition 2.6 that each $\bar{h}$ has a fixed point $x_n$ in the interior of $\bar{A}_n$, ie in $A_n^+$, for all $n$. Let $B$ be the closed disk which is the union of $\partial^+ A$ and the component of the complement of $C(x)$ in $S^2$ which contains $V_n^+$. Then $\mathrm{cl}_B(V_n^+)$ contains a fixed point $x_n$ of $\bar{h}$.

Taking the limit of a subsequence we note that for each $h \in H''$ there is a fixed point of $\bar{h}$ in $\partial^+ A$. But the rotation number of $\bar{f}$ on $\partial^+ A$ is irrational so $\bar{f}$ has a unique invariant minimal set which is the omega limit set $\omega(x, \bar{f})$ for each $x \in \partial^+ A$. Since $\bar{f}$ preserves $\mathrm{Fix}(\bar{h})$ we conclude this minimal set is in $\mathrm{Fix}(\bar{h})$. Since the minimal set depends only on $\bar{f}$ and not $\bar{h}$ we conclude that the this minimal set is in $\mathrm{Fix}(\bar{h})$ for every $h \in H''$.

We have found a prime end (in fact infinitely many ) in $\partial^+ A$ which is fixed by $\bar{h}$ for every $h \in H''$. It follows from Corollary 2.9 that there is a point of $\mathrm{Fix}(H'')$ in $\mathrm{cl}(V_n^+)$ for each $n$. Taking the limit of a subsequence again we find a point of $\mathrm{Fix}(H'')$ which lies in $\bigcap_n \mathrm{cl}(V_n^+) = C(x)$.

Choosing an infinite collection $\{\lambda_i\}$ of distinct irrationals in the rotation interval of $F|_U$ and repeating the construction we obtain an infinite collection of global fixed points for $H''$ with distinct rotation numbers for $F|_U$. They must possess a limit point in $\mathrm{Fix}(H'')$.

[13, Proposition 3.1] asserts that if there is an accumulation point of $\mathrm{Fix}(H'')$ then there is a homomorphism from $H''$ to $\mathbb{R}$. So $H''$ is indicable.

We are left with addressing the special case that $\mathrm{Per}(F)$ contains only two points. Since $F$ cannot have an empty fixed point set we conclude $\mathrm{Per}(F) = \mathrm{Fix}(F)$ and this set contains two points. If $H = Z(F)$ is the centralizer of $F$ then it has an index two subgroup $H'$ which fixes both points and hence the annulus $U = S^2 \setminus \mathrm{Per}(F)$ is invariant under $H'$ with each element isotopic to the identity. Let $f = F|_U$. Then $\rho_f(U)$ consists of a single point, by Theorem 2.3. By Proposition 2.4 (applied to iterates of $f$) it cannot consist of a single rational in $\mathbb{R}/\mathbb{Z}$. We conclude that $\rho_f(U)$ contains a single irrational number $\lambda$.

Blowing up the two fixed points of $F$ we obtain the annular compactification homeomorphism $f_c\colon U_c \to U_c$. The restriction to the boundary component corresponding to the fixed point $x$ is conjugate to the projectivization of $DF_x$. It must have rotation number $\lambda$ since otherwise there would be additional periodic points in $U$ by Theorem 2.3.

This map on the boundary circle is the projectivization of an element of $\mathrm{SL}(2,\mathbb{R})$, ie a fractional linear transformation. Since its rotation number is irrational it is an irrational rotation in appropriately chosen coordinates. It follows that the restrictions of blow-ups of elements of $H'$ to this circle are rotations, since the centralizer of an irrational rotation consists of rotations. Therefore this group of restrictions is abelian. It is finitely generated because it is the image under a homomorphism of a finitely generated group. Since it admits no nontrivial homomorphisms to $\mathbb{R}$ and is finitely generated it must be finite. We conclude there is a finite index subgroup $H''$ of $H'$ whose restrictions to the boundary circle are all the identity. In other words, the projectivization of $Dh_x$ is the identity for all $h \in H''$. Since there are no nontrivial homomorphisms from $H''$ to $\mathbb{R}$, $\det(Dh_x) = 1$. The Thurston stability theorem [27] therefore produces a nontrivial homomorphisms from $H''$ to $\mathbb{R}$ and we have arrived at the desired contradiction. □

We now provide the proof of Corollary 1.8.

**Corollary 1.8** *If $\Sigma_g$ is the closed orientable surface of genus $g \geq 2$ then at least one of the following holds:*

(1) *No finite index subgroup of* $\mathrm{MCG}(\Sigma_g)$ *acts faithfully on $S^2$ by area preserving diffeomorphisms.*

(2) *For all $1 \leq k \leq g-1$, there is an indicable finite index subgroup $\Gamma$ of the bounded mapping class group* $\mathrm{MCG}(S_k, \partial S_k)$ *where $S_k$ is the surface with genus $k$ and connected nonempty boundary.*

**Proof** We assume that (1) fails, ie that there is a finite index subgroup $G$ of $\mathrm{MCG}(\Sigma_g)$ which acts faithfully on $S^2$ by area preserving diffeomorphisms, and show that this implies (2).

Suppose $1 \le k \le g-1$ and $S$ is the compact surface with genus $k$ and a connected nonempty boundary, $\partial S$. We assume $S$ is embedded in $\Sigma_g$ with $\partial S$ a separating closed curve and let $S'$ be the closure of the complement of $S$, a surface with genus $g-k$ and boundary $\partial S$. There is a natural embedding of $\mathrm{MCG}(S, \partial S)$ into $\mathrm{MCG}(\Sigma_g)$ obtained by extending a representative of an element of $\mathrm{MCG}(S, \partial S)$ to all of $\Sigma_g$ by letting it be the identity on the complement of $S$. Similarly there is a natural embedding of $\mathrm{MCG}(S', \partial S')$ into $\mathrm{MCG}(\Sigma_g)$. If $\Gamma_0$ and $\Gamma_0'$ are the images of these two embeddings it is clear that every element of $\Gamma_0$ commutes with every element of $\Gamma_0'$ since they have representatives in $\mathrm{Diff}(\Sigma_g)$ which commute.

We let $\Gamma_1 = \Gamma_0 \cap G$ and $\Gamma_1' = \Gamma_0' \cap G$. Since $\Gamma_1'$ has finite index in $\mathrm{MCG}(S', \partial S)$ it contains an element $\gamma$ of infinite order. Suppose $\phi\colon G \to \mathrm{Diff}_\mu(S^2)$ is the injective homomorphism defining the action of $G$. Let $F = \phi(\gamma)$. Then $\phi(\Gamma_1)$ is in the centralizer $Z(F)$. According to Theorem 1.7, $\Gamma_1$ is virtually indicable. Therefore there is an indicable $\Gamma$ of finite index in $\Gamma_1$. □

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