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Homomorphisms between mapping class groups

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Suppose that X and Y are surfaces of finite topological type, where X has genus $g \geq 6$ and Y has genus at most $2g-1$; in addition, suppose that Y is not closed if it has genus $2g-1$. Our main result asserts that every nontrivial homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is induced by an *embedding*, ie a combination of forgetting punctures, deleting boundary components and subsurface embeddings. In particular, if X has no boundary then every nontrivial endomorphism $\mathrm{Map}(X) \to \mathrm{Map}(X)$ is in fact an isomorphism.

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1 Introduction

Throughout this article we will restrict our attention to connected orientable surfaces of finite topological type, meaning of finite genus and with finitely many boundary components and/or cusps; we will feel free to think about cusps as marked points, punctures or topological ends. The mapping class group $\mathrm{Map}(X)$ of such surface X is the group of isotopy classes of orientation preserving homeomorphisms of X which fix pointwise the union of the boundary and the set of punctures. In the terminology of Farb and Margalit [13], $\mathrm{Map}(X)$ is the *pure* mapping class group.

Mapping class groups are often compared with arithmetic lattices in higher-rank semisimple algebraic groups. This analogy, albeit limited (see Andersen [1], Bestvina and Fujiwara [7] and Ivanov [23]), has motivated many, possibly most, advances in the understanding of mapping class groups. For example, Grossman [15] proved that $\mathrm{Map}(X)$ is residually finite; Birman, Lubotzky and McCarthy [9] proved that the Tits alternative holds for subgroups of $\mathrm{Map}(X)$; the Thurston classification [44] of elements in $\mathrm{Map}(X)$ mimics the classification of elements in an algebraic group; Harvey [18] introduced the curve complex in analogy with the rational Tits building; Harer's [17] computation of the virtual cohomological dimension of $\mathrm{Map}(X)$ follows the outline of Borel and Serre's argument [10] for arithmetic groups, etc…

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In this spirit, it is natural to ask to what extent there is an analog of *Margulis's superrigidity* in the context of mapping class groups. This question, in various guises, has been addressed by a number of authors. For instance, Farb and Masur [14] proved that every homomorphism from an irreducible lattice in a higher-rank Lie group to a mapping class group has finite image. On the other hand, mapping class groups admit nontrivial homomorphisms into higher-rank lattices; see Looijenga [32].

There has also been work on what is perhaps a more natural analog of superrigidity, namely understanding homomorphisms between mapping class groups; see Aramayona, Leininger and Souto [2], Bell and Margalit [4], Berrick and Matthey [5], Castel [12], Harvey and Korkmaz [19], Ivanov [22], Ivanov and McCarthy [26], Korkmaz [28] and McCarthy [38]. Quoting Maryam Mirzakhani, the ultimate goal would be to prove that *every homomorphism between mapping class groups of sufficiently high genus has either finite image or is induced by some manipulation of surfaces.* The aim of this paper is to prove that this is indeed the case as long as the involved surfaces satisfy suitable genus bounds.

Before stating our main result we need a definition:

Definition Let X and Y be surfaces of finite topological type, and consider their cusps to be marked points. Denote by $|X|$ and $|Y|$ the compact surfaces obtained from X and Y by forgetting all their marked points. By an *embedding*

$$\iota\colon X \to Y$$

we will understand a continuous injective map $\iota\colon |X| \to |Y|$ with the property that whenever $y \in \iota(|X|) \subset |Y|$ is a marked point of Y in the image of ι, then $\iota^{-1}(y)$ is also a marked point of X.

Every embedding $\iota\colon X \to Y$ induces a homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$; see Section 3. Our main result asserts that, as long as the genus of Y is less than twice that of X, the converse is also true:

Theorem 1.1 *Suppose that X and Y are surfaces of finite topological type, of genus $g \geq 6$ and $g' \leq 2g-1$ respectively; if Y has genus $2g-1$, suppose also that it is not closed. Then every nontrivial homomorphism*

$$\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$$

is induced by an embedding $X \to Y$.

Remark As we will prove below, the conclusion of Theorem 1.1 also applies to homomorphisms $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ when both X and Y have the same genus $g \in \{4, 5\}$.

Before going any further we give some examples that highlight the necessity for the genus bounds in Theorem 1.1.

Example 1 Let X be a surface of genus $g \le 1$; if $g = 0$ then assume that X has at least four marked points or boundary components. The mapping class group $\mathrm{Map}(X)$ surjects onto $\mathrm{PSL}_2\,\mathbb{Z} \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$. In particular, any two elements $\alpha, \beta \in \mathrm{Map}(Y)$ with orders two and three, respectively, determine a homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$; notice that such elements exist if Y is closed, for example. Choosing α and β appropriately, one can in fact obtain infinitely many conjugacy classes of homomorphisms $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ with infinite image and with the property that every element in the image is either pseudo-Anosov or has finite order.

Example 1 shows that some lower bound on the genus of X is necessary in the statement of Theorem 1.1. Furthermore, since $\mathrm{Map}(X)$ has nontrivial abelianization if X has genus 2, there exist homomorphisms from $\mathrm{Map}(X)$ into mapping class groups of arbitrary closed surfaces Y that are not induced by embeddings. Other examples demonstrating the failure of Theorem 1.1 for surfaces X of genus 2 may be constructed using that the mapping class group of such a surface contains a finite index subgroup which surjects onto the free group $\mathbb{F}_2$. On the other hand, we expect Theorem 1.1 to be true for surfaces of genus $g \in \{3, 4, 5\}$.

Remark Recall that the mapping class group of a punctured disk is a finite index subgroup of the appropriate braid group (this subgroup is commonly known as the *pure braid group*). In particular, Example 1 should be compared with the rigidity results for homomorphisms between braid groups, and from braid groups into mapping class groups, due to Bell and Margalit [4] and Castel [12].

Next, an upper bound on the genus of the target surface is also necessary in the statement of Theorem 1.1 since, for instance, the mapping class group of every closed surface injects into the mapping class group of some nontrivial connected cover; see Aramayona, Leininger and Souto [2]. Moreover, the following example shows that the bound in Theorem 1.1 is in fact optimal:

Example 2 Suppose that X has nonempty connected boundary and let Y be the double of X. Let X_1, X_2 be the two copies of X inside Y, and for $x \in X$ denote by x_i the corresponding point in X_i. Given a homeomorphism $f\colon X \to X$ fixing pointwise the boundary and the cusps define

$$\hat{f}\colon Y \to Y, \quad \hat{f}(x_i) = (f(x))_i \quad \forall x_i \in X_i.$$

The map $f \to \hat{f}$ induces a homomorphism

$$\mathrm{Map}(X) \to \mathrm{Map}(Y)$$

which is not induced by any embedding.

In the body of the paper we will construct other examples of homomorphisms with more or less undesirable properties. It goes without saying that all these examples arise from manipulations of surfaces.

Applications After having established that in Theorem 1.1 a lower bound for the genus of X is necessary and that the upper bound for the genus of Y is optimal, we discuss some consequences of our main result.

By Proposition 3.1 below, every embedding is a combination of forgetting punctures, deleting boundary components, and subsurface embeddings. In particular, if X is closed then any embedding $\iota\colon X \to Y$ is a homeomorphism. Hence we obtain:

Corollary 1.2 *Suppose that X and Y satisfy the hypotheses of Theorem 1.1 and that X is closed. Then every nontrivial homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is induced by a homeomorphism $X \to Y$; in particular ϕ is an isomorphism.* □

Corollary 1.2 above settles Berrick–Matthey [5, Conjecture 4.5] in the affirmative. We remark that Castel [12] had previously obtained Corollary 1.2 in the special case when Y is a closed surface of genus $g' = g + 1$. Observe also that if there are no restrictions on the genus of Y then Corollary 1.2 is far from true. Indeed, Aramayona, Leininger and Souto [2, Theorem 1] show that for every closed surface X there exist a closed surface $Y \neq X$ and an injective homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$.

Moving away from the closed case, if X is allowed to have marked points and/or boundary then there are numerous nontrivial embeddings of X into other surfaces. That said, if X has no boundary then any embedding $X \to Y$ which induces an injective homomorphism at the level of mapping class groups is actually a homeomorphism. Thus we get:

Corollary 1.3 *Suppose that X and Y satisfy the hypotheses of Theorem 1.1 and that X has empty boundary. Then any injective homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is induced by a homeomorphism $X \to Y$; in particular ϕ is an isomorphism.* □

Again, if there are no restrictions on the genus of Y then Corollary 1.3 is not true; see Aramayona, Leininger and Souto [2] and Ivanov and McCarthy [26].

Still assuming $\partial X = \varnothing$, note that any embedding $\iota\colon X \to X$ is a homeomorphism. Since Theorem 1.1 applies for homomorphisms between surfaces of the same genus $g \geq 4$ (see the remark following the statement of the theorem) we deduce:

Corollary 1.4 *Let X be a surface of finite topological type, of genus $g \geq 4$ and with empty boundary. Then any nontrivial endomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(X)$ is induced by a homeomorphism $X \to X$; in particular ϕ is an isomorphism.* □

The analogous statement of Corollary 1.4 for injective endomorphisms was known to be true by the work of Ivanov [24], McCarthy [38] and Ivanov–McCarthy [26]. Castel [12] has obtained Corollary 1.4 independently for X closed.

Corollary 1.4 may fail if X has boundary. However, any embedding $\iota\colon X \to X$ such that the induced homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(X)$ is injective is isotopic to a homeomorphism. We hence recover the following result due to Ivanov and McCarthy [26] (see Ivanov [22] and McCarthy [38] for related earlier results):

Corollary 1.5 (Ivanov–McCarthy) *Let X be a surface of finite topological type, of genus $g \geq 4$. Then any injective homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(X)$ is induced by a homeomorphism $X \to X$; in particular ϕ is an isomorphism.* □

The corollaries above are group-theoretic consequences of Theorem 1.1. However, in a separate paper [3] we use the main result of this paper to classify all nonconstant holomorphic maps $\mathcal{M}(X) \to \mathcal{M}(Y)$ between moduli spaces of Riemann surfaces X and Y of finite type satisfying the same genus bounds as in Theorem 1.1.

Strategy of the proof of Theorem 1.1 Suppose that X, Y and ϕ are as in the statement of the theorem. The bulk of the proof of Theorem 1.1 is to show that ϕ maps Dehn twists along nonseparating curves to Dehn twists along nonseparating curves. Denoting by δ_γ the Dehn twist about a nonseparating curve $\gamma \subset X$, we obtain a map ϕ_* from the set of nonseparating curves on X to the set of nonseparating curves in Y which satisfies $\phi(\delta_\gamma) = \delta_{\phi_*(\gamma)}$. We will prove that ϕ_* preserves disjointness and intersection number 1. In particular, ϕ_* maps chains in X to chains in Y. In the closed case, it follows easily that there is a unique embedding $X \to Y$ which induces the same map on curves as ϕ_*; this is the embedding provided by Theorem 1.1. In the presence of boundary and/or cusps the argument is more involved, essentially because one needs to determine which cusps and boundary components are to be filled in.

Hoping that the reader is now convinced that Theorem 1.1 follows after a moderate amount of work once we know that ϕ maps Dehn twists along nonseparating curves to Dehn twists along nonseparating curves, we sketch the proof of this fact. The starting point is a result of Bridson [11], which asserts that ϕ maps Dehn twists to roots of multitwists. We prove that, under our assumptions, $\phi(\delta_\gamma)$ has infinite order for $\gamma \subset X$ nonseparating. We can thus associate to γ the multicurve $\phi_*(\gamma)$ supporting the multitwist powers of $\phi(\delta_\gamma)$. In principle, and also in practice if Y has sufficiently large genus, $\phi(\delta_\gamma)$ could permute the components of $\phi_*(\gamma)$. However, under the genus bounds in Theorem 1.1, we deduce from a result of Paris [41] that this is not the case. Once we know that $\phi(\delta_\gamma)$ preserves each component of $\phi_*(\gamma)$, a simple counting argument yields that $\phi_*(\gamma)$ is actually a single curve. This implies that $\phi(\delta_\gamma)$ is a root of some power of the Dehn twist along $\phi_*(\gamma)$. We then deduce that $\phi(\delta_\gamma)$ is a power of a Dehn twist from a simple computation using the Riemann–Hurwitz formula and an extension — independently due to Castel [12] — of a theorem of Harvey–Korkmaz [19] asserting that if the genus of X is larger than the genus of Y then there are no nontrivial homomorphisms $\mathrm{Map}(X) \to \mathrm{Map}(Y)$. Once we know that $\phi(\delta_\gamma)$ is a power of a Dehn twist, it follows from the braid relation that this power has to be ± 1, as we needed to prove. This finishes the sketch of the proof of Theorem 1.1.

Tournant dangereux The reader would be justified to think that, from the point of view of lattice superrigidity, it would be more natural to investigate all homomorphisms between *finite index subgroups* of mapping class groups instead of insisting on the homomorphisms to be defined on the whole group. We agree. However, it should be noticed that, so long as it is unknown whether finite index subgroups $\Gamma \subset \mathrm{Map}(X)$ have finite abelianization (see Ivanov [25]), a classification of all homomorphisms $\Gamma \to \mathrm{Map}(Y)$ is beyond reach.

Similarly, the reader could be unconvinced by the reason given above to justify the need for an upper bound on the genus of Y. Possibly we would agree: we just asserted that the given bound is optimal for the theorem to hold as stated, but Breuillard and Mangahas [34] proved that if $\Gamma \subset \mathrm{Map}(X)$ has finite index and $\phi\colon \Gamma \to \mathrm{Map}(Y)$ is a homomorphism, where $\partial Y \neq \varnothing$, then there is a surface $\widehat{Y}$ containing Y and a homomorphism $\widehat{\phi}\colon \mathrm{Map}(X) \to \mathrm{Map}(\widehat{Y})$ extending ϕ. This implies that in the absence of upper bounds for the genus there is no real difference between studying homomorphisms defined on the whole mapping class group and on finite index subgroups. We again face the possibility that there is $\Gamma \subset \mathrm{Map}(X)$ of finite index with infinite abelianization.

This possibility is the enemy from the beginning to the end of this paper. Recall for example that at some point we have to show that under the assumptions of Theorem 1.1

the image under ϕ of a Dehn twist along a nonseparating curve is infinite. That this is actually one of the key points of the proof of our main theorem might come as a surprise to the reader. However, if there is $\Gamma \subset \mathrm{Map}(X)$ of finite index with infinite abelianization then there is a surface Y and a homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ with infinite image such that the image of every Dehn twist has finite order; see the remark at the end of Section 5.

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A nuestras madres, cada uno a la suya.

2 Generalities

In this section we discuss a few well-known facts on mapping class groups. See Farb–Margalit [13] or Ivanov [24] for details.

Throughout this article, all surfaces under consideration are orientable and have finite topological type, meaning that they have finite genus, finitely many boundary components and finitely many punctures. We will feel free to consider cusps as marked points, punctures, or ends homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. For instance, if X is a surface with, say, 10 boundary components and no cusps, by deleting every boundary component we obtain a surface X' with 10 cusps and no boundary components.

A simple closed curve on a surface is said to be *essential* if it does not bound a disk containing at most one puncture; we stress that we consider boundary-parallel curves to be essential. From now on, by a *curve* we will mean an essential simple closed curve. Also, we will often abuse terminology and not distinguish between curves and their isotopy classes.

We now introduce some notation that will be used throughout the paper. Let X be a surface and let γ be an essential curve not parallel to the boundary of X. We will denote by X_γ the complement in X of the interior of a closed regular neighborhood of

γ; we will refer to the two boundary components of X_γ which appear in the boundary of the regular neighborhood of γ as the *new boundary components* of X_γ. We will denote by X'_γ the surface obtained from X_γ by deleting the new boundary components of X_γ; equivalently, $X'_\gamma = X \setminus \gamma$.

A *multicurve* is the union of a, necessarily finite, collection of pairwise disjoint, non-parallel curves. Given two multicurves γ, γ' we denote their geometric intersection number by $i(\gamma, \gamma')$.

A *cut system* is a multicurve whose complement is a connected surface of genus 0. Two cut systems are said to be related by an *elementary move* if they share all curves but one, and the remaining two curves intersect exactly once. The *cut system complex* of a surface X is the simplicial graph whose vertices are cut systems on X and where two cut systems are adjacent if the corresponding cut systems are related by an elementary move.

2.1 Mapping class group

The mapping class group $\mathrm{Map}(X)$ of a surface X is the group of isotopy classes of orientation preserving homeomorphisms $X \to X$ which fix the boundary pointwise and map every cusp to itself; here, we also require that the isotopies fix the boundary pointwise. We will also denote by $\mathrm{Map}^*(X)$ the group of isotopy classes of all orientation preserving homeomorphisms of X. Observe that $\mathrm{Map}(X)$ is a subgroup of $\mathrm{Map}^*(X)$ only in the absence of boundary; in this case $\mathrm{Map}(X)$ has finite index in $\mathrm{Map}^*(X)$.

While every element of the mapping class group is an isotopy class of homeomorphisms, it is well-known that the mapping class group cannot be realized by a group of diffeomorphisms [39], or even homeomorphisms [37]. In spite of this, in order to keep notation under control we will usually make no distinction between mapping classes and their representatives.

2.2 Dehn twists

Given a curve γ on X, we denote by δ_γ the (right) Dehn twist along γ. It is important to remember that δ_γ is solely determined by the curve γ and the orientation of X. In other words, it is independent of any chosen orientation of γ.

The following well-known result will play an important role in our arguments:

Theorem 2.1 (Dehn–Lickorish) *If X has genus at least 2, then $\mathrm{Map}(X)$ is generated by Dehn twists along nonseparating curves.*

There are a number of concrete sets of Dehn twists that generate the mapping class group; see [13] for a description of several such sets. We will consider the generating set depicted in Figure 1; we remark that, in the case of closed surfaces, these generators are the ones first identified by Humphries [21].

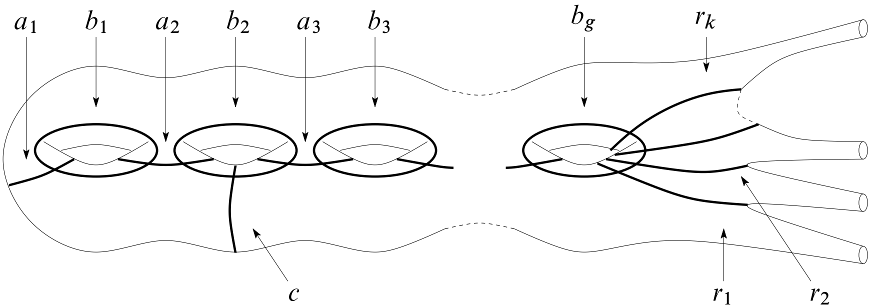


Figure 1: Dehn twists along the curves a_i, b_i, c and r_i generate $\mathrm{Map}(X)$

Algebraic relations among Dehn twists are often given by particular configurations of curves. We now discuss several of these relations; see Farb and Margalit [13], Hamidi-Tehrani [16], Margalit [35] and the references therein for proofs and details.

Conjugate Dehn twists For any curve $\gamma \subset X$ and any $f \in \mathrm{Map}(X)$ we have

$$\delta_{f(\gamma)} = f\delta_\gamma f^{-1}.$$

Hence, Dehn twists along any two nonseparating curves are conjugate in $\mathrm{Map}(X)$. Conversely, if the Dehn twist along γ is conjugate in $\mathrm{Map}(X)$ to a Dehn twist along a nonseparating curve, then γ is nonseparating. Observe that Theorem 2.1 and the fact that Dehn twists along any two nonseparating curves are conjugate immediately imply the following useful fact:

Lemma 2.2 *Let X be a surface of genus at least 3 and let $\phi\colon \mathrm{Map}(X) \to G$ be a homomorphism of groups. If $\delta_\gamma \in \mathrm{Ker}(\phi)$ for some $\gamma \subset X$ nonseparating, then ϕ is trivial.*

Disjoint curves Suppose γ, γ' are disjoint curves, meaning $i(\gamma, \gamma') = 0$. Then δ_γ and $\delta_{\gamma'}$ commute.

Curves intersecting once Suppose that $i(\gamma, \gamma') = 1$. Then

$$\delta_\gamma\delta_{\gamma'}\delta_\gamma = \delta_{\gamma'}\delta_\gamma\delta_{\gamma'}.$$

This is the so-called *braid relation*; we say that δ_γ and $\delta_{\gamma'}$ *braid*.

It is known [16] that if γ and γ' are two curves in X and $k \in \mathbb{Z}$ is such that $|k \cdot i(\gamma, \gamma')| \geq 2$, then δ_γ^k and $\delta_{\gamma'}^k$ generate a free group $\mathbb{F}_2$ of rank 2. In particular we have:

Lemma 2.3 *Suppose that $k \in \mathbb{Z} \setminus \{0\}$ and that γ and γ' are curves such that δ_γ^k and $\delta_{\gamma'}^k$ satisfy the braid relation. Then either $\gamma = \gamma'$ or $k = \pm 1$ and $i(\gamma, \gamma') = 1$.*

Chains Recall that a *chain* in X is a finite sequence of curves $\gamma_1, \ldots, \gamma_k$ such that $i(\gamma_i, \gamma_j) = 1$ if $|i - j| = 1$ and $i(\gamma_i, \gamma_j) = 0$ otherwise. Let $\gamma_1, \ldots, \gamma_k$ be a chain in X and suppose first that k is even. Then the boundary ∂Z of a regular neighborhood Z of $\bigcup \gamma_i$ is connected and we have

$$(\delta_{\gamma_1} \delta_{\gamma_2} \ldots \delta_{\gamma_k})^{2k+2} = \delta_{\partial Z}.$$

If k is odd then ∂Z consists of two components $\partial_1 Z$ and $\partial_2 Z$ and the appropriate relation is

$$(\delta_{\gamma_1} \delta_{\gamma_2} \ldots \delta_{\gamma_k})^{k+1} = \delta_{\partial Z_1} \delta_{\partial Z_2} = \delta_{\partial Z_2} \delta_{\partial Z_1}.$$

These two relations are said to be *the chain relations*.

Lanterns A *lantern* is a configuration in of seven curves a, b, c, d, x, y and z in X as represented in Figure 2.

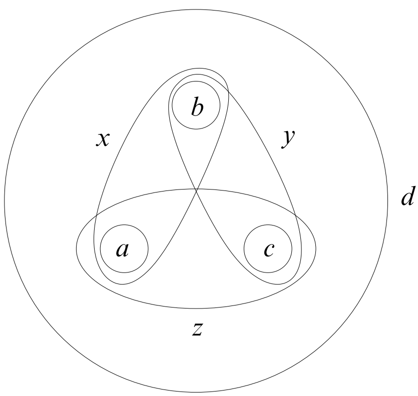


Figure 2: A lantern

If seven curves a, b, c, d, x, y and z in X form a lantern then the corresponding Dehn twists satisfy the so-called *lantern relation*:

$$\delta_a \delta_b \delta_c \delta_d = \delta_x \delta_y \delta_z.$$

Conversely, it is due to Hamidi-Tehrani [16] and Margalit [35] that, under mild hypotheses, any seven curves whose associated Dehn twists satisfy the lantern relation form a lantern. More concretely:

Theorem 2.4 (Hamidi-Tehrani, Margalit) *Let a, b, c, d, x, y, z be essential curves whose associated Dehn twists satisfy the lantern relation*

$$\delta_a\delta_b\delta_c\delta_d = \delta_x\delta_y\delta_z.$$

If the curves a, b, c, d, x are pairwise distinct and pairwise disjoint, then a, b, c, d, x, y and z form a lantern.

In the course of this paper we will continuously discriminate against separating curves. By a *nonseparating* lantern we understand a lantern with the property that all the involved curves are nonseparating. We remark that X contains a nonseparating lantern if X has genus at least 3; in particular we deduce that, as long as X has genus $g \geq 3$, every nonseparating curve belongs to a nonseparating lantern.

2.3 Centralizers of Dehn twists

Observe that the relation $f\delta_\gamma f^{-1} = \delta_{f(\gamma)}$, for $f \in \mathrm{Map}(X)$ and $\gamma \subset X$ a curve, implies that

$$\mathcal{Z}(\delta_\gamma) = \{f \in \mathrm{Map}(X) \mid f(\gamma) = \gamma\},$$

where $\mathcal{Z}(\delta_\gamma)$ denotes the centralizer of δ_γ in $\mathrm{Map}(X)$. Notice that $\mathcal{Z}(\delta_\gamma)$ is also equal to the normalizer $\mathcal{N}(\langle\delta_\gamma\rangle)$ of the subgroup of $\mathrm{Map}(X)$ generated by δ_γ.

An element in $\mathrm{Map}(X)$ which preserves γ may either switch the sides of γ or may preserve them. We denote by $\mathcal{Z}_0(\delta_\gamma)$ the group of those elements which preserve sides; observe that $\mathcal{Z}_0(\delta_\gamma)$ has index at most 2 in $\mathcal{Z}(\delta_\gamma)$.

The group $\mathcal{Z}_0(\delta_\gamma)$ is closely related to two different mapping class groups. First, let X_γ be the surface obtained by removing the interior of a closed regular neighborhood $\gamma \times [0, 1]$ of γ from X. Every homeomorphism of X_γ fixing pointwise the boundary and the punctures extends to a homeomorphism $X \to X$ which is the identity on $X \setminus X_\gamma$. This induces a homomorphism $\mathrm{Map}(X_\gamma) \to \mathrm{Map}(X)$. The sequence

$$0 \to \mathbb{Z} \to \mathrm{Map}(X_\gamma) \to \mathcal{Z}_0(\delta_\gamma) \to 1 \tag{2-1}$$

is exact unless X is a torus without boundary and/or marked points. Here, the group $\mathbb{Z}$ is generated by the difference $\delta_{\eta_1}\delta_{\eta_2}^{-1}$ of the Dehn twists along η_1 and η_2, the new boundary curves of X_γ.

Instead of deleting a regular neighborhood of γ we could also delete γ from X. Equivalently, let X'_γ be the surface obtained from X_γ by deleting the new boundary curves of X_γ. Every homeomorphism of X fixing γ induces a homeomorphism of X'_γ. This yields a second exact sequence

$$0 \to \langle \delta_\gamma \rangle \to \mathcal{Z}_0(\delta_\gamma) \to \mathrm{Map}(X'_\gamma) \to 1. \tag{2-2}$$

2.4 Multitwists

To a multicurve $\eta \subset X$ we associate the group

$$\mathbb{T}_\eta = \langle \{\delta_\gamma,\ \gamma \subset \eta\} \rangle \subset \mathrm{Map}(X)$$

generated by the Dehn twists along the components of η. We refer to the elements in $\mathbb{T}_\eta$ as *multitwists* along η. Observe that $\mathbb{T}_\eta$ is abelian; more concretely, $\mathbb{T}_\eta$ is isomorphic to the free abelian group with rank equal to the number of components of η (see Farb–Margalit [13, Lemma 3.17] for a detailed proof of the latter fact).

Let $\eta \subset X$ be a multicurve. An element $f \in \mathbb{T}_\eta$ which does not belong to any $\mathbb{T}_{\eta'}$, for some η' properly contained in η, is said to be a *generic multitwist along* η. Conversely, if $f \in \mathrm{Map}(X)$ is a multitwist, then the *support* of f is the smallest multicurve η such that f is a generic multitwist along η.

Much of what we just said about Dehn twists extends easily to multitwists. For instance, if $\eta \subset X$ is a multicurve, then we have

$$\mathbb{T}_{f(\eta)} = f\mathbb{T}_\eta f^{-1}$$

for all $f \in \mathrm{Map}(X)$. In particular, the normalizer $\mathcal{N}(\mathbb{T}_\eta)$ of $\mathbb{T}_\eta$ in $\mathrm{Map}(X)$ is equal to

$$\mathcal{N}(\mathbb{T}_\eta) = \{f \in \mathrm{Map}(X) \mid f(\eta) = \eta\}.$$

On the other hand, the centralizer $\mathcal{Z}(\mathbb{T}_\eta)$ of $\mathbb{T}_\eta$ is the intersection of the centralizers of its generators; hence

$$\mathcal{Z}(\mathbb{T}_\eta) = \{f \in \mathrm{Map}(X) \mid f(\gamma) = \gamma \text{ for every component } \gamma \subset \eta\}.$$

Notice that $\mathcal{N}(\mathbb{T}_\eta)/\mathcal{Z}(\mathbb{T}_\eta)$ acts by permutations on the set of components of η. For later use we remark that if the multicurve η happens to be a cut system, then $\mathcal{N}(\mathbb{T}_\eta)/\mathcal{Z}(\mathbb{T}_\eta)$ is in fact isomorphic to the group of permutations of the components of η.

Denote by $\mathcal{Z}_0(\mathbb{T}_\eta)$ the subgroup of $\mathcal{Z}(\mathbb{T}_\eta)$ fixing not only the components but also the sides of each component. Notice that $\mathcal{Z}(\mathbb{T}_\eta)/\mathcal{Z}_0(\mathbb{T}_\eta)$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^{|\eta|}$ and hence is abelian.

Observe that it follows from the definition of the mapping class group and from the relation $\delta_{f(\gamma)} = f\delta_\gamma f^{-1}$ that every Dehn twist along a boundary component of X is central in $\mathrm{Map}(X)$. In fact, as long as X has at least genus 3, such Dehn twists generate the center of $\mathrm{Map}(X)$:

Theorem 2.5 *If X has genus at least 3 then the group $\mathbb{T}_{\partial X}$ generated by Dehn twists along the boundary components of X is the center of $\mathrm{Map}(X)$. Moreover, we have*

$$1 \to \mathbb{T}_{\partial X} \to \mathrm{Map}(X) \to \mathrm{Map}(X') \to 1,$$

where X' is the surface obtained from X by deleting the boundary.

If X is a surface of genus $g \in \{1,2\}$, with empty boundary and no marked points, then the center of $\mathrm{Map}(X)$ is generated by the hyperelliptic involution.

2.5 Roots

It is a rather surprising, and annoying, fact that such simple elements in $\mathrm{Map}(X)$ as Dehn twists have nontrivial *roots* [36]. Recall that a root of $f \in \mathrm{Map}(X)$ is an element $g \in \mathrm{Map}(X)$ for which there is $k \in \mathbb{Z}$ with $f = g^k$. Being forced to live with roots, we state the following simple but important observation:

Lemma 2.6 *Suppose that $\delta_\eta \in \mathrm{Map}(X)$ is a Dehn twist along an essential curve η. For $f \in \mathcal{Z}_0(\delta_\eta)$ the following are equivalent:*

- *f is a root of a power of δ_η.*
- *The image of f in $\mathrm{Map}(X'_\eta)$ under the third arrow in (2-2) has finite order.*

Moreover, f is itself a power of δ_η if and only if the image of f in $\mathrm{Map}(X'_\eta)$ is trivial.

2.6 Torsion

The key to understanding torsion in mapping class groups is the resolution by Kerckhoff [27] of the Nielsen realization problem: the study of finite subgroups of the mapping class group reduces to the study of groups of automorphisms of Riemann surfaces. For instance, it follows from the classical Hurwitz theorem that the order of such a group is bounded from above solely in terms of the genus of the underlying surface. Below we will need the following bound, due to Maclachlan [33] and Nakajima [40], for the order of finite abelian subgroups of $\mathrm{Map}(X)$.

Theorem 2.7 *Suppose that X has genus $g \geq 2$. Then $\mathrm{Map}(X)$ does not contain finite abelian groups with more than $4g+4$ elements.*

We remark that if $g \le 5$ then all finite subgroups, abelian or not, of $\mathrm{Map}(X)$ have been listed; see Kuribayashi and Kuribayashi [30; 31]. In the sequel we will make use of this list in the case that $g = 3, 4$.

Finally, a finite order diffeomorphism which is isotopic to the identity is in fact the identity. This implies, for instance, that if $\overline{X}$ is obtained from X by filling in punctures, and $\tau\colon X \to X$ is a finite order diffeomorphism representing a nontrivial element in $\mathrm{Map}(X)$, then the induced mapping class of $\overline{X}$ is nontrivial as well.

2.7 Centralizers of finite order elements

By (2-1) and (2-2), centralizers of Dehn twists are closely related to other mapping class groups. Essentially the same is true for centralizers of other mapping classes. We now discuss the case of torsion elements. The following result follows directly from the work of Birman–Hilden [8]:

Theorem 2.8 (Birman–Hilden) *Suppose that $[\tau] \in \mathrm{Map}(X)$ is an element of finite order and let $\tau\colon X \to X$ be a finite order diffeomorphism representing $[\tau]$. Consider the orbifold $\mathcal{O} = X/\langle\tau\rangle$ and let $\mathcal{O}^*$ be the surface obtained from $\mathcal{O}$ by removing the singular points. Then we have an exact sequence*

$$1 \to \langle[\tau]\rangle \to \mathcal{Z}([\tau]) \to \mathrm{Map}^*(\mathcal{O}^*),$$

where $\mathrm{Map}^*(\mathcal{O}^*)$ is the group of isotopy classes of all homeomorphisms $\mathcal{O}^* \to \mathcal{O}^*$.

Hidden in Theorem 2.8 we have the following useful fact: *If X has negative Euler characteristic, then two finite order diffeomorphisms $\tau, \tau'\colon X \to X$ which are isotopic are actually conjugate as diffeomorphisms* (see the remark in [6, page 10]). Hence, it follows that the surface $\mathcal{O}^*$ in Theorem 2.8 depends only on the mapping class $[\tau]$. Abusing notation, in the sequel we will speak about the fixed-point set of a finite order element in $\mathrm{Map}(X)$.

3 Homomorphisms induced by embeddings

In this section we define what is meant by an embedding $\iota\colon X \to Y$ between surfaces. As we will observe, any embedding induces a homomorphism between the corresponding mapping class groups. We will discuss several standard examples of such homomorphisms, notably the so-called Birman exact sequences. We will conclude the section with a few observations that will be needed later on. Besides the possible differences of terminology, all the facts that we will state are either well known or simple observations in 2–dimensional topology. A reader who is reasonably acquainted with Farb–Margalit [13] or Ivanov [24] will have no difficulty filling in the details.

3.1 Embeddings

Let X and Y be surfaces of finite topological type, and consider their cusps to be marked points. Denote by $|X|$ and $|Y|$ the compact surfaces obtained from X and Y, respectively, by forgetting all the marked points, and let $P_X \subset |X|$ and $P_Y \subset |Y|$ be the sets of marked points of X and Y.

We now recall the definition of *embedding*; note that the definition below is equivalent to the one given in the introduction.

Definition An *embedding* $\iota\colon X \to Y$ is a continuous injective map $\iota\colon |X| \to |Y|$ such that $\iota^{-1}(P_Y) \subset P_X$. An embedding is said to be a *homeomorphism* if it has an inverse which is also an embedding.

We will say that two embeddings $\iota, \iota'\colon X \to Y$ are *equivalent* or *isotopic* if there is a continuous map

$$[0,1] \times |X| \to |Y|, \quad (t,x) \mapsto f_t(x),$$

with $f_0 = \iota$, $f_1 = \iota'$ and such that f_t is an embedding for all t.

Given an embedding $\iota\colon X \to Y$ and a homeomorphism $f\colon X \to X$ which pointwise fixes the boundary and the marked points of X, we consider the homeomorphism

$$\iota(f)\colon Y \to Y$$

given by $\iota(f)(x) = (\iota \circ f \circ \iota^{-1})(x)$ if $x \in \iota(X)$ and $\iota(f)(x) = x$ otherwise. Clearly, $\iota(f)$ is a homeomorphism which pointwise fixes the boundary and the marked points of Y. In particular $\iota(f)$ represents an element $\iota_\#(f)$ in $\mathrm{Map}(Y)$. We thus obtain a well-defined group homomorphism

$$\iota_\#\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$$

characterized by the following property: for any curve $\gamma \subset X$ we have $\iota_\#(\delta_\gamma) = \delta_{\iota(\gamma)}$. Notice that this characterization immediately implies that if ι and ι' are isotopic, then $\iota_\# = \iota'_\#$.

3.2 Birman exact sequences

As we mentioned above, notable examples of homomorphisms induced by embeddings are the so-called Birman exact sequences, which we now describe.

Let X and Y be surfaces of finite topological type. We will say that Y is obtained from X by *filling in a puncture* if there is an embedding $\iota\colon X \to Y$ and a marked point $p \in P_X$, such that the underlying map $\iota\colon |X| \to |Y|$ is a homeomorphism, and

$\iota^{-1}(P_Y) = P_X \setminus \{p\}$. If Y is obtained from X by filling in a puncture we have the following exact sequence:

$$1 \longrightarrow \pi_1(|Y| \setminus P_Y, \iota(p)) \longrightarrow \mathrm{Map}(X) \xrightarrow{\iota_\#} \mathrm{Map}(Y) \longrightarrow 1 \tag{3-1}$$

The second arrow in (3-1) can be described concretely. For instance, if γ is a simple loop in $|Y| \setminus P_Y$ based at $\iota(p)$, then the image of the element $[\gamma] \in \pi_1(|Y| \setminus P_Y, \iota(p))$ in $\mathrm{Map}(X)$ is the difference of the two Dehn twists along the curves forming the boundary of a regular neighborhood of $\iota^{-1}(\gamma)$.

Similarly, we will say that Y is obtained from X by *filling in a boundary component* if there is an embedding $\iota\colon X \to Y$, with $\iota^{-1}(P_Y) = P_X$, and such that the complement in $|Y|$ of the image of the underlying map $|X| \to |Y|$ is a disk which does not contain any marked point of Y. If Y is obtained from X by filling in a boundary component then we have the following exact sequence:

$$1 \longrightarrow \pi_1(T^1(|Y| \setminus P_Y)) \longrightarrow \mathrm{Map}(X) \longrightarrow \mathrm{Map}(Y) \longrightarrow 1 \tag{3-2}$$

Here $T^1(|Y| \setminus P_Y)$ is the unit-tangent bundle of the surface $|Y| \setminus P_Y$.

We refer to the sequences (3-1) and (3-2) as the Birman exact sequences. It follows from the work of Ivanov–McCarthy [26] that the Birman exact sequences do not split if the involved surfaces have genus at least 2.

3.3 Other building blocks

Continuing with the same notation as above, we will say that Y is obtained from X by *deleting a boundary component* if there is an embedding $\iota\colon X \to Y$ with $\iota(P_X) \subset P_Y$ and such that the complement of the image of the underlying map $|X| \to |Y|$ is a disk containing exactly one point in P_Y.

If X is not homeomorphic to a closed disk and Y is obtained from X by deleting a boundary component then we have

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Map}(X) \longrightarrow \mathrm{Map}(Y) \longrightarrow 1, \tag{3-3}$$

where $\mathbb{Z}$ is the group generated by the Dehn twist along the forgotten boundary component.

Finally, we will say that $\iota\colon X \to Y$ is a *subsurface embedding* if $\iota(P_X) \subset P_Y$ and if no component of the complement of the image of the underlying map $|X| \to |Y|$ is a disk containing at most one marked point. If $\iota\colon X \to Y$ is a subsurface embedding then the homomorphism

$$\iota_\#\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$$

is injective if and only if ι is *anannular*, ie if no component of the complement of the image of the underlying map $|X| \to |Y|$ is an annulus without marked points; compare with (2-1) above. We refer the reader to Farb–Margalit [13] or Paris–Rolfsen [42] for a proof of this fact.

3.4 General embeddings

Clearly, the composition of two embeddings is an embedding. For instance, filling in a boundary component is isotopic to first forgetting it and then filling in a puncture. The following proposition, whose proof we leave to the reader, asserts that every embedding is isotopic to a suitable composition of the elementary building blocks we have just discussed.

Proposition 3.1 *Every embedding $\iota\colon X \to Y$ is isotopic to a composition of the following three types of embedding: filling punctures, deleting boundary components, and subsurface embeddings. In particular, the homomorphism $\iota_\#\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is injective if and only if ι is an anannular subsurface embedding.* □

We conclude this section with an observation that will be needed below. Suppose that $\iota\colon X \to Y$ is an embedding and let $\eta \subset X$ be a multicurve. The image $\iota(\eta)$ of η in Y is an embedded 1–manifold, but it need not be a multicurve. For instance, some component of $\iota(\eta)$ may not be essential in Y; also two components of $\iota(\eta)$ may be parallel in Y. If this is not the case, that is, if $\iota(\eta)$ is a multicurve in Y, then $\iota_\#$ maps the subgroup $\mathbb{T}_\eta$ of multitwists supported on η isomorphically onto $\mathbb{T}_{\iota(\eta)}$. We record this observation in the following lemma:

Lemma 3.2 *Let $\iota\colon X \to Y$ be an embedding and let $\eta \subset X$ be a multicurve. If $\iota(\eta)$ is a multicurve in Y, then the homomorphism $\iota_\#$ maps $\mathbb{T}_\eta \subset \mathrm{Map}(X)$ isomorphically to $\mathbb{T}_{\iota(\eta)} \subset \mathrm{Map}(Y)$. Moreover, the image of a generic multitwist in $\mathbb{T}_\eta$ is generic in $\mathbb{T}_{\iota(\eta)}$.* □

Notation In order to avoid notation as convoluted as $T^1(|Y| \setminus P_Y)$, most of the time we will drop any reference to the underlying surface $|Y|$ or to the set of marked point P_Y; notice that this is consistent with taking the liberty to consider punctures as marked points or as ends. For instance, the Birman exact sequences now read

$$1 \longrightarrow \pi_1(Y) \longrightarrow \mathrm{Map}(X) \longrightarrow \mathrm{Map}(Y) \longrightarrow 1$$

if Y is obtained from X by filling in a puncture, and

$$1 \longrightarrow \pi_1(T^1Y) \longrightarrow \mathrm{Map}(X) \longrightarrow \mathrm{Map}(Y) \longrightarrow 1$$

if it is obtained by filling in a boundary component.

4 Triviality theorems

In this section we remind the reader of two triviality theorems for homomorphisms from mapping class groups to abelian groups and permutation groups; these results are widely used throughout this paper. The first of these results is a direct consequence of Powell's theorem [43] on the vanishing of the integer homology of the mapping class group of surfaces of genus at least 3:

Theorem 4.1 (Powell) *If X is a surface of genus $g \geq 3$ and A is an abelian group, then every homomorphism $\mathrm{Map}(X) \to A$ is trivial.*

We refer the reader to Korkmaz [29] for a discussion of Powell's theorem and other homological properties of mapping class groups.

As a first consequence of Theorem 4.1 we derive the following useful observation:

Lemma 4.2 *Let X, Y and $\overline{Y}$ be surfaces of finite topological type, and let $\iota\colon Y \to \overline{Y}$ be an embedding. Suppose that X has genus at least 3 and that $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a homomorphism such that the composition*

$$\overline{\phi} = \iota_\# \circ \phi\colon \mathrm{Map}(X) \to \mathrm{Map}(\overline{Y})$$

is trivial. Then ϕ is trivial as well.

Proof By Proposition 3.1 the embedding $\iota\colon Y \to \overline{Y}$ is isotopic to a suitable composition of filling in punctures, deleting boundary components and subsurface embeddings. In particular, we may argue by induction and assume that ι is of one of these three types. For the sake of concreteness suppose $\iota\colon Y \to \overline{Y}$ is the embedding associated to filling in a puncture; the other cases are actually a bit easier and are left to the reader. We have the following diagram:

$$\begin{tikzcd}
 & & \mathrm{Map}(X) \arrow[d, "\phi"'] \arrow[dr, "\overline{\phi}"] & & \\
1 \arrow[r] & \pi_1(\overline{Y}) \arrow[r] & \mathrm{Map}(Y) \arrow[r, "\iota_\#"] & \mathrm{Map}(\overline{Y}) \arrow[r] & 1
\end{tikzcd}$$

The assumption that $\overline{\phi}$ is trivial amounts to supposing that the image of ϕ is contained in $\pi_1(\overline{Y})$. Since every nontrivial subgroup of the surface group $\pi_1(\overline{Y})$ has nontrivial homology, we deduce from Theorem 4.1 that ϕ is trivial, as it was to be shown. □

Before stating another consequence of Theorem 4.1 we need a definition:

Definition A homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is said to be *irreducible* if its image does not preserve any essential curve in Y; otherwise we say it is *reducible*.

Remark Recall that we consider boundary parallel curves to be essential. In particular, every homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is reducible if Y has nonempty boundary.

Let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be a reducible homomorphism, where X has genus at least 3, and let $\eta \subset Y$ be a multicurve which is componentwise invariant under $\phi(\mathrm{Map}(X))$; in other words, $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}(\mathbb{T}_\eta)$.

Moreover, Theorem 4.1 implies that $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}_0(\mathbb{T}_\eta)$, where $\mathcal{Z}_0(\mathbb{T}_\eta)$ is the subgroup of $\mathcal{Z}(\mathbb{T}_\eta)$ consisting of those elements that preserve the sides of each component of η.

Now let $Y'_\gamma = Y \setminus \eta$ be the surface obtained by deleting η from Y. Composing (2-2) as often as necessary, we obtain an exact sequence as follows:

$$1 \longrightarrow \mathbb{T}_\eta \longrightarrow \mathcal{Z}_0(\mathbb{T}_\eta) \longrightarrow \mathrm{Map}(Y'_\eta) \longrightarrow 1 \tag{4-1}$$

The same argument of the proof of Lemma 4.2 shows that ϕ is trivial if the composition of ϕ and the third homomorphism in (4-1) is trivial. Hence we have:

Lemma 4.3 *Let X, Y be surfaces of finite topological type, with X of genus at least 3. Suppose $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a nontrivial reducible homomorphism preserving the multicurve $\eta \subset Y$. Then $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}_0(\mathbb{T}_\eta)$ and the composition of ϕ with the homomorphism (4-1) is not trivial.* □

The second triviality theorem, due to Paris [41], asserts that the mapping class group of a surface of genus $g \geq 3$ does not have subgroups of index less than or equal to $4g+4$; equivalently, any homomorphism from the mapping class group into a symmetric group $\mathcal{S}_k$ is trivial if $k \leq 4g+4$:

Theorem 4.4 (Paris) *If X has genus $g \geq 3$ and $k \leq 4g+4$, then there is no nontrivial homomorphism $\mathrm{Map}(X) \to \mathcal{S}_k$ where the latter group is the group of permutations of the set with k elements.*

Before going any further we should mention that in [41], Theorem 4.4 is only stated for closed surfaces; however, the proof works as it is also for surfaces with boundary and or punctures.

As a first consequence of Theorem 4.1 and Theorem 4.4 we prove:

Proposition 4.5 *If X has genus at least 3 and Y has genus at most 2, then every homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is trivial.*

Proof Assume for concreteness that Y has genus 2; the cases of genus 0 and genus 1 are in fact easier and are left to the reader.

Notice that by Lemma 4.2, we may assume without losing generality that Y has empty boundary and no marked points. Recall that $\mathrm{Map}(Y)$ has a central element τ of order 2, namely the hyperelliptic involution. As we discussed above, we identify the finite order mapping class τ with one of its finite order representatives, which we again denote by τ. The surface underlying the orbifold $Y/\langle\tau\rangle$ is the 6–punctured sphere $\mathbb{S}_{0,6}$. By Theorem 2.8 we have the following exact sequence:

$$1 \longrightarrow \langle\tau\rangle \longrightarrow \mathrm{Map}(Y) \longrightarrow \mathrm{Map}^*(\mathbb{S}_{0,6}),$$

where $\mathrm{Map}^*(\mathbb{S}_{0,6})$ is the group of isotopy classes of all orientation preserving homeomorphisms of $\mathbb{S}_{0,6}$. Therefore, any homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ induces a homomorphism

$$\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}^*(\mathbb{S}_{0,6}).$$

By Paris's theorem, the homomorphism obtained by composing ϕ' with the obvious homomorphism $\mathrm{Map}^*(\mathbb{S}_{0,6}) \to \mathcal{S}_6$, the group of permutations of the punctures, is trivial. In other words, ϕ' takes values in $\mathrm{Map}(\mathbb{S}_{0,6})$. Since the mapping class group of the standard sphere $\mathbb{S}^2$ is trivial, Lemma 4.2 implies that ϕ' is trivial. Therefore, the image of ϕ is contained in the abelian subgroup $\langle\tau\rangle \subset \mathrm{Map}(Y)$. Finally, Theorem 4.1 implies that ϕ is trivial, as we had to show. □

5 Getting rid of the torsion

We begin this section by asking the following question:

Question 1 Suppose that $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a homomorphism between mapping class groups of surfaces of genus at least 3, with the property that the image of every Dehn twist along a nonseparating curve has finite order. Is the image of ϕ finite?

In this section we will give a positive answer to Question 1 in the case where the genus of Y is exponentially bounded by the genus of X. Namely, we will prove:

Proposition 5.1 *Suppose that X and Y are surfaces of finite topological type with genus g and g' respectively. Suppose that $g \geq 4$ and that either $g' < 2^{g-2} - 1$ or $g' = 3, 4$. Then any homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ which maps a Dehn twist along a nonseparating curve to a finite order element is trivial.*

Under the assumption that Y is not closed, we obtain in fact a complete answer to Question 1:

Theorem 5.2 *Suppose that X and Y are surfaces of finite topological type, that X has genus at least 3, and that Y is not closed. Then any homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ which maps a Dehn twist along a nonseparating curve to a finite order element is trivial.*

Since the mapping class group of a surface with nonempty boundary is torsion-free, we deduce from Lemma 2.2 that it suffices to consider the case that $\partial Y = \varnothing$. From now on, we assume that we are in this situation.

The proofs of Proposition 5.1 and Theorem 5.2 are based on Theorem 4.1, the connectivity of the cut system complex, and the following algebraic observation:

Lemma 5.3 *For $n \in \mathbb{N}$, $n \geq 2$, consider $\mathbb{Z}^n$ endowed with the standard action of the symmetric group $\mathcal{S}_n$ by permutations of the basis elements $e_1, \dots, e_n$. If V is a finite abelian group equipped with an $\mathcal{S}_n$–action, then for any $\mathcal{S}_n$–equivariant epimorphism $\phi\colon \mathbb{Z}^n \to V$ one of the following is true:*

1. *The restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ is surjective.*
2. *V has order at least 2^n and cannot be generated by fewer than n elements.*

Moreover, if (1) does not hold and $V \neq (\mathbb{Z}/2\mathbb{Z})^n$ then V has at least 2^{n+1} elements.

Proof Let d be the order of $\phi(e_1)$ in V and observe that, by $\mathcal{S}_n$–equivariance, all the elements $\phi(e_i)$ have order d also. It follows that $(d\mathbb{Z})^n \subset \mathrm{Ker}(\phi)$ and hence that ϕ descends to an epimorphism

$$\phi'\colon (\mathbb{Z}/d\mathbb{Z})^n \to V.$$

We first treat the case $d = p^a$, where p is a prime and $a \geq 1$; we argue by induction on a. Let $a = 1$. The kernel of the epimorphism

$$\phi'\colon (\mathbb{Z}/p\mathbb{Z})^n \to V$$

is an $\mathcal{S}_n$–invariant subspace. We need the following well-known observation:

Fact *Suppose that p is prime. The only $\mathcal{S}_n$–invariant subgroups W of $(\mathbb{Z}/p\mathbb{Z})^n$ are the following:*

- *The trivial subgroup $\{0\}$.*

- $(\mathbb{Z}/p\mathbb{Z})^n$ *itself.*
- $E = \{(a, a, \dots, a) \in (\mathbb{Z}/p\mathbb{Z})^n \mid a = 0, \dots, p-1\}$.
- $F = \{(a_1, \dots, a_n) \in (\mathbb{Z}/p\mathbb{Z})^n \mid a_1 + \dots + a_n = 0\}$.

Now, either ϕ' is injective and thus V contains $p^n \geq 2^n$ elements, or its kernel is one of the spaces E or F provided by the claim. Since the union of either one of them with $(\mathbb{Z}/p\mathbb{Z})^{n-1} \times \{0\}$ spans $(\mathbb{Z}/p\mathbb{Z})^n$, it follows that the restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ surjects onto V. This concludes the proof for $a = 1$.

Suppose now that we have proved the result for $a-1$. We can then consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{Z}/p^{a-1}\mathbb{Z})^n \arrow[r] \arrow[d] & (\mathbb{Z}/p^a\mathbb{Z})^n \arrow[r] \arrow[d, "\phi'"] & (\mathbb{Z}/p\mathbb{Z})^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \phi'((\mathbb{Z}/p^{a-1}\mathbb{Z})^n) \arrow[r] & V \arrow[r] & V/\phi'((\mathbb{Z}/p^{a-1}\mathbb{Z})^n) \arrow[r] & 0
\end{tikzcd}$$

Observe that if one of the groups to the left and right of V on the bottom row has at least 2^n elements, then so does V. So, if this is not the case we may assume by induction that the restriction of the left and right vertical arrows to $(\mathbb{Z}/p^{a-1}\mathbb{Z})^{n-1} \times \{0\}$ and $(\mathbb{Z}/p\mathbb{Z})^{n-1} \times \{0\}$ are epimorphisms; in particular the restriction of the morphism ϕ' to $(\mathbb{Z}/p^a\mathbb{Z})^{n-1} \times \{0\}$ is also an epimorphism. Thus, either V has at least 2^n elements or the restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ is surjective, as desired.

We now explain how to restrict to the case that d is not a power of a prime. Consider the prime decomposition $d = \prod_j p_j^{a_j}$ of d, where $p_i \neq p_j$ and $a_i \in \mathbb{N}$. By the Chinese remainder theorem we have

$$\mathbb{Z}/d\mathbb{Z} = \prod_j (\mathbb{Z}/p_j^{a_j}\mathbb{Z}).$$

Hence, there is an $\mathcal{S}_n$–equivariant isomorphism

$$(\mathbb{Z}/d\mathbb{Z})^n = \prod_j ((\mathbb{Z}/p_j^{a_j}\mathbb{Z})^n).$$

Consider the projection

$$\pi_j \colon \mathbb{Z}^n \to (\mathbb{Z}/p_j^{a_j}\mathbb{Z})^n,$$

noting that if the restriction to $\mathbb{Z}^{n-1} \times \{0\}$ of $\phi' \circ \pi_j$ surjects onto $\phi'((\mathbb{Z}/p_j^{a_j}\mathbb{Z})^n)$ for all j, then $\phi(\mathbb{Z}^{n-1} \times \{0\}) = V$. If that is not the case, then $\phi((\mathbb{Z}/p_j^{a_j}\mathbb{Z})^n) \subset V$ has order at least 2^n, by the above.

Both the equality case and the claim on the minimal number of elements needed to generate V are left to the reader. □

We are now ready to prove:

Lemma 5.4 *Given $n \geq 4$, suppose that $g > 0$ is such that $2^{n-2} - 1 > g$ or $g \in \{3, 4\}$.*

If Y is surface of genus $g \geq 3$, $V \subset \mathrm{Map}(Y)$ is a finite abelian group endowed with an action of $\mathcal{S}_n$, and $\phi\colon \mathbb{Z}^n \to V$ is an $\mathcal{S}_n$–equivariant epimomorphism, then the restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ is surjective.

Proof Suppose, for contradiction, that the restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ is not surjective. Recall that by the resolution of the Nielsen realization problem [27] there is a conformal structure on Y such that V can be represented by a group of automorphisms.

Suppose first that $2^{n-2} - 1 > g$. Since we are assuming that the restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ is not surjective, Lemma 5.3 implies that V has at least 2^n elements. Then:

$$2^n = 4(2^{n-2} - 1) + 4 > 4g + 4,$$

which is impossible since Theorem 2.7 asserts that $\mathrm{Map}(Y)$ does not contain finite abelian groups with more than $4g + 4$ elements.

Suppose now that $g = 4$. If $n \geq 5$ we obtain a contradiction using the same argument as above. Thus assume that $n = 4$. Since $2^{4+1} = 32 > 20 = 4 \cdot 4 + 4$, it follows from the equality statement in Lemma 5.3 that V is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Luckily for us, Kuribayashi and Kuribayashi have classified all groups of automorphisms of Riemann surfaces of genus 3 and 4. From their list, more concretely [31, Proposition 2.2(c)], we obtain that $(\mathbb{Z}/2\mathbb{Z})^4$ cannot be realized as a subgroup of the group of automorphisms of a surface of genus 4, and thus we obtain the desired contradiction.

Finally, suppose that $g = 3$. As before, this case boils down to ruling out the possibility of having $(\mathbb{Z}/2\mathbb{Z})^4$ acting by automorphisms on a Riemann surface of genus 3. This is established in [31, Proposition 1.2(c)]. This concludes the case $g = 3$ and thus the proof of the lemma. □

Remark One could wonder if in Lemma 5.4 the condition $n \geq 4$ is necessary. Indeed it is, because the mapping class group of a surface of genus 3 contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, namely the group $H(8, 8)$ on the list in [31].

We are finally ready to prove Proposition 5.1:

Proof of Proposition 5.1 Recall that a cut system in X is a maximal multicurve whose complement in X is connected; observe that every cut system consists of g curves and that every nonseparating curve is contained in some cut system.

Given a cut system η consider the group $\mathbb{T}_\eta$ generated by the Dehn twists along the components of η, and recall that $\mathbb{T}_\eta \simeq \mathbb{Z}^g$. Any permutation of the components of η can be realized by a homeomorphism of X. Consider the normalizer $\mathcal{N}(\mathbb{T}_\eta)$ and centralizer $\mathcal{Z}(\mathbb{T}_\eta)$ of $\mathbb{T}_\eta$ in $\mathrm{Map}(X)$. As mentioned in Section 2.4, we have the following exact sequence:

$$1 \longrightarrow \mathcal{Z}(\mathbb{T}_\eta) \longrightarrow \mathcal{N}(\mathbb{T}_\eta) \longrightarrow \mathcal{S}_g \longrightarrow 1,$$

where $\mathcal{S}_g$ denotes the symmetric group of permutations of the components of η. Observe that the action by conjugation of $\mathcal{N}(\mathbb{T}_\eta)$ on $\mathbb{T}_\eta$ induces an action $\mathcal{S}_g = \mathcal{N}(\mathbb{T}_\eta)/\mathcal{Z}(\mathbb{T}_\eta) \curvearrowright \mathbb{T}_\eta$ which is conjugate to the standard action of $\mathcal{S}_g \curvearrowright \mathbb{Z}^g$. Clearly, this action descends to an action $\mathcal{S}_g \curvearrowright \phi(\mathbb{T}_\eta)$.

Seeking a contradiction, suppose that the image under ϕ of a Dehn twist δ_γ along a nonseparating curve has finite order. Since all the Dehn twists along the components of η are conjugate to δ_γ we deduce that all their images have finite order; hence $\phi(\mathbb{T}_\eta)$ is generated by finite order elements. On the other hand, $\phi(\mathbb{T}_\eta)$ is abelian because it is the image of an abelian group. Being abelian and generated by finite order elements, $\phi(\mathbb{T}_\eta)$ is finite.

It thus follows from Lemma 5.4 that the subgroup of $\mathbb{T}_\eta$ generated by Dehn twists along $g-1$ components of η surjects under ϕ onto $\phi(\mathbb{T}_\eta)$. This implies that

$$\phi(\mathbb{T}_\eta) = \phi(\mathbb{T}_{\eta'})$$

whenever η and η' are cut systems which differ by exactly one component. Now, since the cut system complex is connected [20], we deduce that $\phi(\delta_\alpha) \in \phi(\mathbb{T}_\eta)$ for every nonseparating curve α. Therefore the image of $\mathrm{Map}(X)$ is the abelian group $\phi(\mathbb{T}_\eta)$, as $\mathrm{Map}(X)$ is generated by Dehn twists along nonseparating curves. By Theorem 4.1, any homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ with abelian image is trivial, and thus we obtain the desired contradiction. □

Before moving on, we discuss briefly the proof of Theorem 5.2:

Proof of Theorem 5.2 Suppose that Y is not closed, in which case every finite subgroup of $\mathrm{Map}(Y)$ is cyclic. In particular, the bound on the number of generators in Lemma 5.3 implies that *if $V \subset \mathrm{Map}(Y)$ is a finite abelian group endowed with an action of $\mathcal{S}_n$ and $\phi\colon \mathbb{Z}^n \to V$ is an $\mathcal{S}_n$–equivariant epimomorphism then the restriction of ϕ to $\mathbb{Z}^{n-1} \times \{0\}$ is surjective.* Once this has been established, Theorem 5.2 follows with the same proof, word for word, as Proposition 5.1. □

Remark Let X and Y be surfaces, where Y has a single boundary component and no cusps. Let G be a finite index subgroup of $\mathrm{Map}(X)$ and let $\phi\colon G \to \mathrm{Map}(Y)$ be a homomorphism. A simple modification of a construction due to Breuillard–Mangahas [34] yields a closed surface Y' containing Y and a homomorphism

$$\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}(Y')$$

such that for all $g \in G$ we have, up to isotopy, $\phi'(g)(Y) = Y$ and $\phi'(g)|_Y = \phi(g)$.

Suppose now that G could be chosen so that there is an epimorphism $G \to \mathbb{Z}$. Assume further that $\phi\colon G \to \mathrm{Map}(Y)$ factors through this epimorphism and that the image of ϕ is purely pseudo-Anosov. Then every element in the image of the extension $\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ either has finite order or is a partial pseudo-Anosov. A result of Bridson [11], stated as Theorem 6.1 below, implies that every Dehn twist in $\mathrm{Map}(X)$ is mapped to a finite order element in $\mathrm{Map}(Y)$. Hence, the extension homomorphism ϕ' produces a negative answer to Question 1.

We have hence proved that *a positive answer to Question 1 implies that every finite index subgroup of* $\mathrm{Map}(X)$ *has finite abelianization.*

6 The map ϕ_*

In addition to the triviality results given in Theorems 4.1 and 4.4, the third key ingredient in the proof of Theorem 1.1 is the following result due to Bridson [11]:

Theorem 6.1 (Bridson) *Suppose that X, Y are surfaces of finite type and that X has genus at least 3. Any homomorphism $\phi : \mathrm{Map}(X) \to \mathrm{Map}(Y)$ maps roots of multitwists to roots of multitwists.*

A remark on the proof of Theorem 6.1 In [11], Theorem 6.1 is proved for surfaces without boundary only. However, Bridson's argument remains valid if we allow X to have boundary. That the result can also be extended to the case that Y has nonempty boundary needs a minimal argument, which we now give. Denote by Y' the surface obtained from Y by deleting all boundary components and consider the homomorphism $\pi\colon \mathrm{Map}(Y) \to \mathrm{Map}(Y')$ provided by Theorem 2.5. By Bridson's theorem, the image under $\pi \circ \phi$ of a Dehn twist δ_γ is a root of a multitwist. Since the kernel of π is the group of multitwists along the boundary of Y, it follows that $\phi(\delta_\gamma)$ is also a root of a multitwist, as claimed. □

A significant part of the sequel is devoted to proving that under suitable assumptions the image of a Dehn twist is in fact a Dehn twist. It is worth stressing that, without any restrictions on genus, there exist homomorphisms between mapping class group that map Dehn twists to nontrivial roots of multitiwists, as the next example shows:

Example 3 Suppose that X has a single boundary component and at least two punctures. By [15], the mapping class group $\mathrm{Map}(X)$ is residually finite. Fix a finite group G and an epimorphism $\pi\colon \mathrm{Map}(X) \to G$. Let Y be a connected surface on which G acts and which contains $|G|$ disjoint copies X_g $(g \in G)$ of X with $gX_h = X_{gh}$ for all $g, h \in G$; for example, such surface Y may be constructed by considering the connected sum of $|G|$ copies of X, where the connected sums are performed according to the edges of a chosen Cayley graph of G.

Given $x \in X$, denote the corresponding element in X_g by x_g. If $f\colon X \to X$ is a homeomorphism fixing pointwise the boundary and punctures, we define

$$\hat{f}\colon Y \to Y$$

with $\hat{f}(x_g) = (f(x))_{\pi([f])g}$ for $x_g \in X_g$ and $\hat{f}(y) = \pi([f])(y)$ for $y \notin \bigcup_{g\in G} X_g$; here $[f]$ is the element in $\mathrm{Map}(X)$ represented by f.

Notice that $\hat{f}$ does not fix the marked points of Y; in order to by-pass this difficulty, consider $\overline{Y}$ the surface obtained from Y by forgetting all marked points, and consider $\hat{f}$ to be a self-homeomorphism of $\overline{Y}$. The map $f \mapsto \hat{f}$ induces a homomorphism

$$\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(\overline{Y})$$

with some curious properties, namely:

- If $\gamma \subset X$ is a simple closed curve which bounds a disk with at least two punctures then the image $\phi(\delta_\gamma)$ of the Dehn twist δ_γ along γ has finite order. Moreover, $\delta_\gamma \in \mathrm{Ker}(\phi)$ if and only if $\delta_\gamma \in \mathrm{Ker}(\pi)$.
- If $\gamma \subset X$ is a nonseparating simple closed curve then $\phi(\delta_\gamma)$ has infinite order. Moreover, $\phi(\delta_\gamma)$ is a multitwist if $\delta_\gamma \in \mathrm{Ker}(\pi)$; otherwise, $\phi(\delta_\gamma)$ is a nontrivial root of a multitwist. Observe that in the latter case, $\phi(\delta_\gamma)$ induces a nontrivial permutation of the components of the multicurve supporting any of its multitwist powers.

This concludes the discussion of Example 3.

While a finite order element is by definition a root of a multitwist, Proposition 5.1 ensures that, under suitable bounds on the genus of the surfaces involved, any nontrivial homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ maps Dehn twists to infinite order elements. From now on we assume that we are in the following situation:

$(*)$ X and Y are orientable surfaces of finite topological type, of genus g and g' respectively, and such that one of the following holds:
- $g \geq 4$ and $g' \leq g$.
- $g \geq 6$ and $g' \leq 2g-1$.

Remark It is worth noticing that the reason for the genus bound $g \geq 6$ in Theorem 1.1 is that $2^{g-2}-1 < 2g-1$ if $g < 6$.

Assuming $(*)$, it follows from Proposition 5.1 that any nontrivial homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ maps Dehn twists δ_γ along nonseparating curves γ to infinite order elements in $\mathrm{Map}(Y)$. Furthermore, it follows from Theorem 6.1 that there is N such that $\phi(\delta_\gamma^N)$ is a nontrivial multitwist. We denote by $\phi_*(\gamma)$ the multicurve in Y supporting $\phi(\delta_\gamma^N)$; observe that $\phi_*(\gamma)$ is independent of N, for if two multitwists have a common root then the supporting multicurves must be equal. Notice that two multitwists commute if and only if their supports do not intersect; hence, ϕ_* preserves the property of having zero intersection number. Moreover, the uniqueness of $\phi_*(\gamma)$ implies that for any $f \in \mathrm{Map}(X)$ we have $\phi_*(f(\gamma)) = \phi(f)(\phi_*(\gamma))$. Summing up we have:

Corollary 6.2 *Suppose that X and Y are as in $(*)$ and let*

$$\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$$

be a nontrivial homomorphism. For every nonseparating curve $\gamma \subset X$, there is a uniquely determined multicurve $\phi_*(\gamma) \subset Y$ with the property that $\phi(\delta_\gamma)$ is a root of a generic multitwist in $\mathbb{T}_{\phi_*(\gamma)}$. Moreover the following holds:
- *$i(\phi_*(\gamma), \phi_*(\gamma')) = 0$ for any two disjoint nonseparating curves γ and γ'.*
- *$\phi_*(f(\gamma)) = \phi(f)(\phi_*(\gamma))$ for all $f \in \mathrm{Map}(X)$. In particular, the multicurve $\phi_*(\gamma)$ is invariant under $\phi(\mathcal{Z}(\delta_\gamma))$.* □

The remainder of this section is devoted to give a proof of the following result:

Proposition 6.3 *Suppose that X and Y are as in $(*)$; further, assume that Y is not closed if it has genus $2g-1$. Let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be an irreducible homomorphism. Then for every nonseparating curve $\gamma \subset X$ the multicurve $\phi_*(\gamma)$ is a nonseparating curve.*

Recall that a homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is irreducible if its image does not preserve any essential curve in Y, and that we consider boundary-parallel curves to be essential.

Before launching the proof of Proposition 6.3 we will establish a few useful facts.

Lemma 6.4 *Suppose X and Y satisfy (∗) and that $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is an irreducible homomorphism. Let $\overline{Y}$ be obtained from Y by filling in some, possibly all, punctures of Y, and let $\overline{\phi} = \iota_\# \circ \phi\colon \mathrm{Map}(X) \to \mathrm{Map}(\overline{Y})$ be the composition of ϕ with the homomorphism $\iota_\#$ induced by the embedding $\iota\colon Y \to \overline{Y}$. For every nonseparating curve $\gamma \subset X$ we have:*

- *$\iota(\phi_*(\gamma))$ is a multicurve.*
- *$\overline{\phi}_*(\gamma) = \iota(\phi_*(\gamma))$.*

In particular, ι yields a bijection between the components of $\phi_*(\gamma)$ and $\overline{\phi}_*(\gamma)$.

Proof First, arguing by induction, we may assume that $\overline{Y}$ is obtained from Y by filling in a single cusp. We suppose from now on that this is the case; it follows from Lemma 4.2 that $\overline{\phi}$ is not trivial. Notice also that since Y and $\overline{Y}$ have the same genus, $\overline{\phi}_*(\gamma)$ is well-defined by Corollary 6.2.

By definition of ϕ_* and $\overline{\phi}_*$, we can choose $N \in \mathbb{N}$ such that $\phi(\delta_\gamma^N)$ and $\overline{\phi}(\delta_\gamma^N)$ are generic multitwists in $\mathbb{T}_{\phi_*(\gamma)}$ and $\mathbb{T}_{\overline{\phi}_*(\gamma)}$. In particular, it follows from Lemma 3.2 that in order to prove Lemma 6.4 it suffices to show that $\iota(\phi_*(\gamma))$ does not contain (1) inessential components, or (2) parallel components.

Claim 1 *$\iota(\phi_*(\gamma))$ does not contain inessential components.*

Proof of Claim 1 Seeking a contradiction, suppose that a component η of $\phi_*(\gamma)$ is inessential in $\overline{Y}$. Since $\overline{Y}$ is obtained from Y by filling in a single cusp, it follows that η bounds a disk in Y with exactly two punctures. Observe that this implies that for any element $F \in \mathrm{Map}(Y)$ we have either $F(\eta) = \eta$ or $i(F(\eta), \eta) > 0$. On the other hand, if $f \in \mathrm{Map}(X)$ is such that $i(f(\gamma), \gamma) = 0$ then we have

$$i(\phi(f)(\eta), \eta) \leq i(\phi(f)(\phi_*(\gamma)), \phi_*(\gamma)) = i(\phi_*(f(\gamma)), \phi_*(\gamma)) = 0.$$

We deduce that $\eta = \phi(f)(\eta) \subset \phi_*(f(\gamma))$ for any such f. Since any two nonseparating curves in X are related by an element of $\mathrm{Map}(X)$ we obtain:

(⋆) *If γ' is a nonseparating curve in X with $i(\gamma, \gamma') = 0$ then $\eta = \phi(\delta_{\gamma'})(\eta)$ and $\eta \subset \phi_*(\gamma')$.*

Choose $\gamma' \subset X$ so that $X \setminus (\gamma \cup \gamma')$ is connected. It follows from (⋆) that if γ'' is any other nonseparating curve which is disjoint from at least one of γ or γ', then $\phi(\delta_{\gamma''})(\eta) = \eta$. Since $\mathrm{Map}(X)$ is generated by Dehn twists along such curves, we deduce that every element in $\phi(\mathrm{Map}(X))$ preserves η, contradicting the assumption that ϕ is irreducible. This concludes the proof of Claim 1. □

We use a similar argument to prove that $\iota(\phi_*(\gamma))$ does not contain parallel components.

Claim 2 $\iota(\phi_*(\gamma))$ *does not contain parallel components.*

Proof of Claim 2 Seeking again a contradiction suppose that there are $\eta \neq \eta' \subset \phi_*(\gamma)$ whose images in $\overline{Y}$ are parallel. Hence, $\eta \cup \eta'$ bounds an annulus which contains a single cusp. As above, it follows that for any element $f \in \mathrm{Map}(Y)$ we have either $f(\eta \cup \eta') = \eta \cup \eta'$ or $i(f(\eta), \eta) > 0$. By the same argument as before, we obtain that $\phi(\mathrm{Map}(X))$ preserves $\eta \cup \eta'$. Now, it follows from either Theorem 4.1 or Theorem 4.4 that $\phi(\mathrm{Map}(X))$ cannot permute η and η'. Hence $\phi(\mathrm{Map}(X))$ preserves η, contradicting the assumption that ϕ is irreducible. □

As we mentioned above, Lemma 6.4 follows from Claims 1 and 2 and Lemma 3.2. □

Continuing with the preliminary considerations to prove Proposition 6.3, recall that the final claim in Corollary 6.2 implies that $\phi(\delta_\gamma)$ preserves the multicurve $\phi_*(\gamma)$. Our next goal is to show that, as long as ϕ is irreducible, the element $\phi(\delta_\gamma)$ preserves every component of $\phi_*(\gamma)$.

Lemma 6.5 *Suppose that X and Y are as in $(*)$ and let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be an irreducible homomorphism. If $\gamma \subset X$ is a nonseparating simple closed curve, then $\phi(\mathcal{Z}_0(\delta_\gamma))$ preserves every component of $\phi_*(\gamma)$. Hence, $\phi(\mathcal{Z}_0(\delta_\gamma)) \subseteq \mathcal{Z}_0(\mathbb{T}_{\phi_*(\gamma)})$.*

Recall that $\mathcal{Z}_0(\delta_\gamma)$ is the subgroup of $\mathrm{Map}(X)$ preserving not only γ but also the two sides of γ and that it has at most index 2 in the centralizer $\mathcal{Z}(\delta_\gamma)$ of the Dehn twist δ_γ.

Proof We first prove Lemma 6.5 in the case that Y is closed. As in Section 2, we denote by X_γ the surface obtained by deleting the interior of a closed regular neighborhood of γ from X. Recall that by (2-1) there is a surjective homomorphism

$$\mathrm{Map}(X_\gamma) \to \mathcal{Z}_0(\delta_\gamma).$$

Consider the composition of this homomorphism with ϕ and, abusing notation, denote its image by $\phi(\mathrm{Map}(X_\gamma)) = \phi(\mathcal{Z}_0(\delta_\gamma))$.

By Corollary 6.2, the subgroup $\phi(\mathrm{Map}(X_\gamma))$ of $\mathrm{Map}(Y)$ acts on the set of components of $\phi_*(\gamma)$ and hence on $Y \setminus \phi_*(\gamma)$. Since Y is assumed to be closed and of at most genus $2g-1$ we deduce that $Y \setminus \phi_*(\gamma)$ has at most $|\chi(Y)| = 2g'-2 \leq 4g-4$ components. Since the surface X_γ has genus $g-1 \geq 3$, we deduce from Theorem 4.4 that $\phi(\mathrm{Map}(X_\gamma))$ preserves each component of $Y \setminus \phi_*(\gamma)$.

Suppose now that Z is a component of $Y \setminus \phi_*(\gamma)$ and let η be the set of components of $\phi_*(\gamma)$ contained in the closure of Z. Noticing that

$$4-4g \leq \chi(Y) \leq \chi(Z) \leq -|\eta|+2$$

we obtain that η consists of at most $4g-2$ components. Since $\phi(\mathrm{Map}(X_\gamma))$ preserves Z, it acts on the set of components of η. Again by Theorem 4.4, it follows that this action is trivial, meaning that every component of $\phi_*(\gamma)$ contained in the closure of Z is preserved. Since Z was arbitrary, we deduce that $\phi(\mathrm{Map}(X_\gamma))$ preserves every component of $\phi_*(\gamma)$ as claimed. Now, Theorem 4.1 implies that $\phi(\mathcal{Z}_0(\delta_\gamma)) = \phi(\mathrm{Map}(X_\gamma)) \subset \mathcal{Z}_0(\mathbb{T}_{\phi_*(\gamma)})$. This concludes the proof of Lemma 6.5 in the case that Y is closed.

We now turn our attention to the general case. Recall that the assumption that ϕ is irreducible implies that $\partial Y = \varnothing$. Let $\overline{Y}$ be the surface obtained from Y by closing up all the cusps and denote by $\overline{\phi}\colon \mathrm{Map}(X) \to \mathrm{Map}(\overline{Y})$ the composition of ϕ with the homomorphism $\iota_\#\colon \mathrm{Map}(Y) \to \mathrm{Map}(\overline{Y})$ induced by the embedding $\iota\colon Y \to \overline{Y}$. By the above, Lemma 6.5 holds true for $\overline{\phi}$. On the other hand, Lemma 6.4 shows that for any $\gamma \subset X$ nonseparating there is a bijection between $\phi_*(\gamma)$ and $\overline{\phi}_*(\gamma)$. Thus the lemma follows. □

Note that Lemma 6.5 yields the following sufficient condition for a homomorphism between mapping class groups to be reducible:

Corollary 6.6 *Suppose that X and Y are as in $(*)$ and let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be a nontrivial homomorphism. Let γ and γ' be disjoint curves on X such that $X \setminus (\gamma \cup \gamma')$ is connected. If the multicurves $\phi_*(\gamma)$ and $\phi_*(\gamma')$ share a component, then ϕ is reducible.*

Proof First, $\mathrm{Map}(X)$ is generated by Dehn twists along curves α which are disjoint from γ or γ'. For any such α we have $\delta_\alpha \in \mathcal{Z}_0(\delta_\gamma) \cup \mathcal{Z}_0(\delta_{\gamma'})$. In particular, it follows from Lemma 6.5 that $\phi(\mathrm{Map}(X))$ fixes every component of $\phi_*(\gamma) \cap \phi_*(\gamma')$. □

We are now ready to prove Proposition 6.3:

Proof of Proposition 6.3 Let γ be a nonseparating curve on X. Extend γ to a multicurve $\eta \subset X$ with $3g-3$ components $\gamma_1, \dots, \gamma_{3g-3}$, and such that the surface $X \setminus (\gamma_i \cup \gamma_j)$ is connected for all i, j. Since δ_{γ_i} and δ_{γ_j} are conjugate in $\mathrm{Map}(X)$ we deduce that $\phi_*(\gamma_i)$ and $\phi_*(\gamma_j)$ have the same number K of components for all i, j. Since ϕ is irreducible, Corollary 6.6 implies that $\phi_*(\gamma_i)$ and $\phi_*(\gamma_j)$ do not share any components for $i \neq j$. This shows that $\bigcup_i \phi_*(\gamma_i)$ is the union of $(3g-3)K$

distinct curves. Furthermore, since δ_{γ_i} and δ_{γ_j} commute, we deduce that $\bigcup_i \phi_*(\gamma_i)$ is a multicurve in Y.

Suppose first that Y has genus $g' \leq 2g-2$. In light of Lemma 6.4, it suffices to consider the case that Y is closed. Now, the multicurve $\bigcup_i \phi_*(\gamma_i)$ has at most $3g'-3 \leq 3(2g-2)-3 < 6g-6$ components. Hence:

$$K < \frac{6g-6}{3g-3} = 2,$$

and thus the multicurve $\phi_*(\gamma)$ consists of $K=1$ components; in other words, it is a curve. If $\phi_*(\gamma)$ were separating, then the multicurve $\bigcup_i \phi_*(\gamma_i)$ would consist of $3g-3$ separating curves; however, a closed surface of genus $g' \leq 2g-2$ contains at most $g' \leq 2g-2$ disjoint separating curves that are equivalent under the action of the mapping class group. This concludes the proof of the proposition in the case that Y has genus at most $2g-2$.

It remains to consider the case that Y has genus $g' = 2g-1$ and at least one puncture. Again by Lemma 6.4, we can assume that Y has a single puncture, which we consider as a marked point. In this case, the multicurve $\bigcup_i \phi_*(\gamma_i)$ consists of at most $3g'-2 = 6g-5$ curves. Since we know that $\bigcup_i \phi_*(\gamma_i)$ is the union of $(3g-3)K$ distinct curves, we deduce that $K \leq 2$. In the case that $\bigcup_i \phi_*(\gamma_i)$ has fewer than $6g-6$ components, we proceed as before. Therefore, it remains to rule out the possibility of having exactly $6g-6$ components.

Suppose, for contradiction, that $\bigcup_i \phi_*(\gamma_i)$ has $6g-6$ components. Since Y has genus $2g-1$ and exactly one marked point, the complement of $\bigcup_i \phi_*(\gamma_i)$ in Y is a disjoint union of pairs of pants, where one of them, call it P, contains the marked point of Y. Now, the boundary components of P are contained in the image under ϕ_* of curves $a_1, a_2, a_3 \in \{\gamma_1, \dots, \gamma_{3g-3}\}$. Assume, for the sake of concreteness, that $a_i \neq a_j$ whenever $i \neq j$; the remaining case is dealt with using minor modifications of the argument we give here.

Suppose first that the multicurve $\alpha = a_1 \cup a_2 \cup a_3$ does not disconnect X and let $\alpha' \neq \alpha$ be another multicurve with three components satisfying:

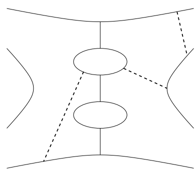
(1) $X \setminus \alpha'$ is connected,

(2) $i(\alpha, \alpha') = 0$, and

(3) $X \setminus (\gamma \cup \gamma')$ is connected for all $\gamma, \gamma' \in \alpha \cup \alpha'$.

Note that $X \setminus \alpha$ and $X \setminus \alpha'$ are homeomorphic, and thus there is $f \in \mathrm{Map}(X)$ with $f(\alpha) = \alpha'$. Now, $P' = \phi(f)(P)$ is a pair of pants which contains the marked point of Y. Taking into account that $\partial P \subset \phi_*(\alpha)$ and $\partial P' \subset \phi_*(\alpha')$ we deduce from (2) that $i(\partial P, \partial P') = \varnothing$ and hence that $P = P'$. Since $\alpha' \neq \alpha$ we may assume, up to renaming, that $a_1 \not\subset \alpha'$. Since $\phi(f)(\partial P) = \partial P'$ and $\partial P \cap \phi_*(a_1) \neq \varnothing$, we deduce that there is i such that $\phi_*(a_i) \cap \phi_*(f(a_1))$ contains a boundary curve of P. In light of (3), it follows from Corollary 6.6 that ϕ is reducible; this contradiction shows that $X \setminus \alpha$ cannot be connected.

If $X \setminus \alpha$ is not connected, then it has two components, as $X \setminus (a_1 \cup a_2)$ is connected. Suppose first that neither of the two components Z_1, Z_2 of $X \setminus \alpha$ is a (possibly punctured) pair of pants; in particular, Z_1 and Z_2 both have positive genus. Let $P_1 \subset Z_1$ be an unpunctured pair of pants with boundary $\partial P_1 = a_1 \cup a_2 \cup a_3'$ and let $P_2 \subset Z_2$ be second unpunctured a pair of pants with $Z_2 \setminus P_2$ connected and with boundary $\partial P_2 = a_3 \cup a_1' \cup a_2'$ where a_1' and a_2' are not boundary parallel in Z_2; compare with the figure below. Notice that $Z_1' = (Z_1 \cup P_2) \setminus P_1$ is homeomorphic



to Z_1. Similarly, $Z_2' = (Z_2 \cup P_1) \setminus P_2$ is homeomorphic to Z_2. Finally notice also that Z_i' contains the same punctures as Z_i for $i = 1, 2$. It follows that there is $f \in \mathrm{Map}(X)$ with $f(Z_1) = Z_1'$ and $f(Z_2) = Z_2'$. In particular, $f(\alpha) = \alpha'$ where $\alpha' = a_1' \cup a_2' \cup a_3'$. We highlight a few facts:

(1) There is $f \in \mathrm{Map}(X)$ with $f(\alpha) = \alpha'$,

(2) $i(\alpha, \alpha') = 0$, and

(3) $X \setminus (\gamma \cup \gamma')$ is connected for all $\gamma, \gamma' \in \{a_1, a_2, a_1', a_2'\}$.

As above, we deduce that $\phi(f)(\partial P_1) = \partial P_2$ and that for all $i = 1, 2, 3$ there is j such that $\phi_*(a_i) \cap \phi_*(f(a_j))$ contains a boundary curve of P. In light of (3), it follows again from Corollary 6.6 that ϕ is reducible. We have reduced to the case that one of the components of $X \setminus \alpha$, say Z_1, is a (possibly punctured) pair of pants.

We now explain how to reduce to the case that Z_1 is a pair of pants without punctures. Let $a_3' \subset Z_1$ be a curve which, together with a_3, bounds an annulus $A \subset Z_1$ such that $Z_1 \setminus A$ does not contain any marked points. Note that the multicurve $\gamma_1 \cup \cdots \cup \gamma_{3g-3}$ does not intersect a_3'. It follows that $i(\phi_*(a_3'), \bigcup \phi_*(\gamma_i)) = 0$. Now, a pants decomposition of Y consists of $3(2g-1) - 3 + 1 = 6g - 5$ curves. Since $\phi_*(a_3')$ has two components and $\bigcup \phi_*(\gamma_i)$ has $6g - 6$ components, we deduce that there exists i such that $\phi_*(a'_3)$ and $\phi_*(\gamma_i)$ share a component. If $i \neq 3$, property (3) and Corollary 6.6 imply that ϕ is reducible, since $a'_3 \bigcup \gamma_i$ does not separate X. It thus follows that $\phi_*(a'_3)$ and $\phi_*(a_3)$ share a component, and so $\partial P \subset \phi_*(a_1 \cup a_2 \cup a'_3)$.

Summing up, it remains to rule out the possibility that Z_1 is a pair of pants without punctures. Choose $\alpha' \subset X$, with $\alpha' \neq \alpha$, satisfying:

(1) α' bounds a pair of pants in X,

(2) $i(\alpha, \alpha') = 0$, and

(3) $X \setminus (\gamma \cup \gamma')$ is connected for all $\gamma, \gamma' \in \alpha \cup \alpha'$.

Now there is $f \in \mathrm{Map}(X)$ with $f(\alpha) = \alpha'$ and we can repeat word by word the argument given in the case that $X \setminus \alpha$ was connected.

After having ruled out all possibilities, we deduce that $\bigcup_i \phi_*(\gamma_i)$ cannot have $6g-6$ components. This concludes the proof of Proposition 6.3. □

7 When the genus decreases

In this section we show that every homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is trivial if the genus of X is larger than that of Y. As a consequence we obtain that, under suitable genus bounds, the centralizer of the image of a nontrivial homomorphism between mapping class groups is torsion-free.

Proposition 7.1 *Suppose that X and Y are orientable surfaces of finite topological type. If the genus of X is at least 3 and larger than that of Y, then every homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is trivial.*

Castel [12] has obtained an independent proof of the above result. For closed surfaces, Proposition 7.1 is due to Harvey–Korkmaz [19].

Proof We will proceed by induction on the genus of X. First, Proposition 4.5 establishes the base case of the induction. Observe that, by Lemma 4.2, we may assume that Y is has empty boundary and no cusps.

Suppose now that X has genus $g \geq 4$ and that we have proved Proposition 7.1 for surfaces of genus $g-1$. Our first step is to prove the following:

Claim *Under the hypotheses above, every homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is reducible.*

Proof of the claim Seeking a contradiction, suppose that there is an irreducible homomorphism ϕ: $\mathrm{Map}(X) \to \mathrm{Map}(Y)$, where Y has smaller genus than X. Let $\gamma \subset X$ be a nonseparating curve. Observing that X and Y satisfy $(*)$, we deduce that $\phi_*(\gamma)$ is a nonseparating curve by Proposition 6.3 and that $\phi(\mathcal{Z}_0(\delta_\gamma)) \subset \mathcal{Z}_0(\delta_{\phi_*(\gamma)})$ by Lemma 6.5. By (2-2), $\mathcal{Z}_0(\delta_{\phi_*(\gamma)})$ surjects onto $\mathrm{Map}(Y'_{\phi_*(\gamma)})$, where $Y'_{\phi_*(\gamma)} = Y \setminus \phi_*(\gamma)$. On the other hand, we have by (2-1) that $\mathcal{Z}_0(\delta_\gamma)$ is an image of the group $\mathrm{Map}(X_\gamma)$, where X_γ is obtained from X by deleting the interior of a closed regular neighborhood of γ.

Since $\phi_*(\gamma)$ is nonseparating, the genus of $Y'_{\phi_*(\gamma)}$ and X_γ is one less than that of Y and X, respectively. The induction assumption implies that the induced homomorphism

$$\mathrm{Map}(X_\gamma) \to \mathrm{Map}(Y'_{\phi_*(\gamma)})$$

is trivial. Lemma 4.3 proves that the homomorphism

$$\mathrm{Map}(X_\gamma) \to \mathcal{Z}_0(\delta_{\phi_*(\gamma)}) \subset \mathrm{Map}(Y)$$

is also trivial, and so $\mathcal{Z}_0(\delta_\gamma) \subset \mathrm{Ker}(\phi)$. Since $Z_0(\delta_\gamma)$ contains a Dehn twist along a nonseparating curve, we deduce that ϕ is trivial from Lemma 2.2. This contradiction concludes the proof of the claim. □

Continuing with the proof of the induction step in Proposition 7.1, suppose there exists a nontrivial homomorphism ϕ: $\mathrm{Map}(X) \to \mathrm{Map}(Y)$. By the above claim, ϕ is reducible. Let $\eta \subset Y$ be a maximal multicurve in Y which is componentwise preserved by $\phi(\mathrm{Map}(X))$, and notice that $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}_0(\mathbb{T}_\eta)$ by Lemma 4.3. Consider

$$\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}(Y'_\eta),$$

the composition of ϕ with the homomorphism (4-1). The maximality of the multicurve η implies that ϕ' is irreducible. Since the genus of Y'_η is at most equal to that of Y, we deduce from the claim above that ϕ' is trivial. Lemma 4.3 implies hence that ϕ is trivial as well. This establishes Proposition 7.1 □

As we mentioned before, a consequence of Proposition 7.1 is that, under suitable assumptions, the centralizer of the image of a homomorphism between mapping class groups is torsion-free. Namely, we have:

Lemma 7.2 *Let X and Y be surfaces of finite topological type, where X has genus $g \geq 3$ and Y has genus $g' \leq 2g$. Suppose that Y has at least one (respectively three) marked points if $g' = 2g-1$ (respectively $g' = 2g$). If ϕ: $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a nontrivial homomorphism, then the centralizer of $\phi(\mathrm{Map}(X))$ in $\mathrm{Map}(Y)$ is torsion-free.*

The proof of Lemma 7.2 relies on Proposition 7.1 and the following consequence of the Riemann–Hurwitz formula:

Lemma 7.3 *Let Y be a surface of genus $g' \geq 0$ and let $\tau\colon Y \to Y$ be a nontrivial diffeomorphism of prime order, representing an element in $\mathrm{Map}(Y)$. Then τ has $F \leq 2g'+2$ fixed-points and the underlying surface of the orbifold $Y/\langle\tau\rangle$ has genus at most $\overline{g} = (2g'+2-F)/4$.*

Proof Consider the orbifold $Y/\langle\tau\rangle$ and let F be the number of its singular points, which is also equal to the number of fixed points of τ since τ has prime order p. Denote by $|Y/\langle\tau\rangle|$ the underlying surface of the orbifold $Y/\langle\tau\rangle$. The Riemann–Hurwitz formula shows that

$$2-2g' = \chi(Y) = p\cdot\chi(|Y/\langle\tau\rangle|) - (p-1)\cdot F. \tag{7-1}$$

After some manipulations, (7-1) shows that

$$F = \frac{2g'-2+p\cdot(2-2\overline{g})}{p-1},$$

where $\overline{g}$ is the genus of $|Y/\langle\tau\rangle|$. Clearly, the quantity on the right is maximal if $\overline{g}=0$ and $p=2$. This implies that $F \leq 2g'+2$, as claimed.

Rearranging (7-1), we obtain

$$\overline{g} = \frac{2g'+(2-F)(p-1)}{2p}.$$

Again this is maximal if p is as small as possible, ie $p=2$. Hence $\overline{g} \leq (2g'+2-F)/4$. $\square$

We are now ready to prove Lemma 7.2.

Proof of Lemma 7.2 First, if Y has nonempty boundary there is nothing to prove, for in this case $\mathrm{Map}(Y)$ is torsion-free. Therefore, assume that $\partial Y = \varnothing$. Suppose, for contradiction, that there exists $[\tau] \in \mathrm{Map}(Y)$ nontrivial, of finite order, and such that $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}([\tau])$.

Let $\tau\colon Y \to Y$ be a finite order diffeomorphism representing $[\tau]$. Passing to a suitable power, we may assume that the order of τ is prime. Consider the orbifold $Y/\langle\tau\rangle$ as a surface with the singular points marked, and recall that by Theorem 2.8 we have the following exact sequence:

$$1 \longrightarrow \langle[\tau]\rangle \longrightarrow \mathcal{Z}([\tau]) \xrightarrow{\beta} \mathrm{Map}^*(Y/\langle\tau\rangle).$$

On the other hand, we have by definition

$$1 \longrightarrow \mathrm{Map}(Y/\langle\tau\rangle) \longrightarrow \mathrm{Map}^*(Y/\langle\tau\rangle) \longrightarrow \mathcal{S}_F \longrightarrow 1,$$

where F is the number of punctures of $Y/\langle\tau\rangle$. Again, F is equal to the number of fixed points of τ since τ has prime order.

Lemma 7.3 gives that $F \leq 2g' + 2 \leq 4g + 2$; hence, it follows from Theorem 4.4 that the composition

$$\mathrm{Map}(X) \xrightarrow{\phi} \mathcal{Z}([\tau]) \xrightarrow{\beta} \mathrm{Map}^*(Y/\langle\tau\rangle) \longrightarrow \mathcal{S}_F$$

is trivial; in other words,

$$(\beta \circ \phi)(\mathrm{Map}(X)) \subset \mathrm{Map}(Y/\langle\tau\rangle).$$

Our assumptions on the genus and the marked points of Y imply, by the genus bound in Lemma 7.3, that $Y/\langle\tau\rangle$ has genus less than g. Hence, the homomorphism

$$\beta \circ \phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y/\langle\tau\rangle)$$

is trivial by Proposition 7.1. This implies that the image of ϕ is contained in the abelian group $\langle[\tau]\rangle$. Theorem 4.1 shows hence that ϕ is trivial, contradicting our assumption. This concludes the proof of Lemma 7.2 $\square$

The following example shows that Lemma 7.2 is no longer true if Y is allowed to have genus $2g$ and fewer than 3 punctures.

Example 4 Let X be a surface with no punctures and such that $\partial X = \mathbb{S}^1$. Let Z be a surface of the same genus as X, with $\partial Z = \varnothing$ but with two punctures. Regard X as a subsurface of Z and consider the two-fold branched cover $Y \to Z$ corresponding to an arc in $Z \setminus X$ joining the two punctures of Z. Every homomorphism $X \to X$ fixing pointwise the boundary extends to a homeomorphism of Z fixing the punctures and which lifts to a unique homeomorphism $Y \to Y$ which preserves the two components of the preimage of X under the covering $Y \to Z$. The image of the induced homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is centralized by the involution τ associated to the two-to-one cover $Y \to Z$. Moreover, if X has genus g then Y has genus $2g$ and 2 punctures.

8 Dehn twists to Dehn twists

We are now ready to prove that under suitable genus bounds, homomorphisms between mapping class groups map Dehn twists to Dehn twists. Namely:

Proposition 8.1 *Suppose that X and Y are surfaces of finite topological type, of genus $g \geq 6$ and $g' \leq 2g-1$ respectively; if Y has genus $2g-1$, suppose also that it is not closed. Every nontrivial homomorphism*

$$\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$$

maps (right) Dehn twists along nonseparating curves to (possibly left) Dehn twists along nonseparating curves.

Remark The proof of Proposition 8.1 will apply, word for word, to homomorphisms between mapping class groups of surfaces of the same genus $g \in \{4, 5\}$.

We will first prove Proposition 8.1 under the assumption that ϕ is irreducible and then we will deduce the general case from there.

Proof of Proposition 8.1 for irreducible ϕ Suppose that ϕ is irreducible and recall that this implies that $\partial Y = \varnothing$. Let $\gamma \subset X$ be a nonseparating curve. Thus $\phi_*(\gamma)$ is also a nonseparating curve, by Proposition 6.3. We first show that $\phi(\delta_\gamma)$ is a power of $\delta_{\phi_*(\gamma)}$.

Let X_γ be the complement in X of the interior of a closed regular neighborhood of γ and $Y'_{\phi_*(\gamma)} = Y \setminus \phi_*(\gamma)$ the connected surface obtained from Y by removing $\phi_*(\gamma)$. We have that:

($\star$) X_γ and $Y'_{\phi_*(\gamma)}$ have genus $g-1 \geq 3$ and $g'-1 \leq 2g-2$ respectively. Moreover, $Y'_{\phi_*(\gamma)}$ has two more punctures than Y; in particular, $Y'_{\phi_*(\gamma)}$ has at least 3 punctures if it has genus $2g-2$.

By (2-1) and (2-2) we have epimorphisms

$$\mathrm{Map}(X_\gamma) \to \mathcal{Z}_0(\delta_\gamma) \quad \text{and} \quad \mathcal{Z}_0(\delta_{\phi_*(\gamma)}) \to \mathrm{Map}(Y'_{\phi_*(\gamma)}).$$

In addition, we know that $\phi(\mathcal{Z}_0(\delta_\gamma)) \subset \mathcal{Z}_0(\delta_{\phi_*(\gamma)})$ by Lemma 6.5. Composing all these homomorphisms we get a homomorphism

$$\phi'\colon \mathrm{Map}(X_\gamma) \to \mathrm{Map}(Y'_{\phi_*(\gamma)}).$$

It follows from Lemma 2.2 that the restriction of ϕ to $\mathcal{Z}_0(\delta_\gamma)$ is not trivial because the latter contains a Dehn twist along a nonseparating curve; Lemma 4.3 implies that ϕ' is not trivial either.

Since δ_γ centralizes $\mathcal{Z}_0(\delta_\gamma)$, it follows that $\phi'(\delta_\gamma) \in \mathrm{Map}(Y'_{\phi_*(\gamma)})$ centralizes the image of ϕ'. Now, the definition of $\phi_*(\gamma)$ implies that some power of $\phi(\delta_\gamma)$ is a power of the Dehn twist $\delta_{\phi_*(\gamma)}$. Hence, the first claim of Lemma 2.6 yields that $\phi'(\delta_\gamma)$

has finite order, and thus $\phi'(\delta_\gamma) \in \mathrm{Map}(Y'_{\phi_*(\gamma)})$ is a finite order element centralizing $\phi(\mathrm{Map}(X_\gamma))$. By $(\star)$, Lemma 7.2 applies and shows that $\phi'(\delta_\gamma)$ is in fact trivial. The final claim of Lemma 2.6 now shows that $\phi(\delta_\gamma)$ is a power of $\delta_{\phi_*(\gamma)}$; in other words, there exists $N \in \mathbb{Z} \setminus \{0\}$ such that $\phi(\delta_\gamma) = \delta^N_{\phi_*(\gamma)}$.

Note that N does not depend on the particular nonseparating curve γ since any two Dehn twists along nonseparating curves are conjugate. It remains to prove that $N = \pm 1$.

Given simple closed curves $\gamma_1, \gamma_2 \subset X$ with $i(\gamma_1, \gamma_2) = 1$, choose curves $\gamma_3, \dots, \gamma_n \subset X$ with $i(\gamma_1, \gamma_i) = 0$ for all $i \geq 3$ and such that the Dehn twists $\delta_{\gamma_1}, \dots, \delta_{\gamma_n}$ generate $\mathrm{Map}(X)$ (compare with Figure 1). Note that $i(\phi_*(\gamma_1), \phi_*(\gamma_i)) = 0$ for $i \geq 3$ and that the elements $\delta^N_{\phi_*(\gamma_1)}, \dots, \delta^N_{\phi_*(\gamma_n)}$ generate $\phi(\mathrm{Map}(X))$. Observe that $i(\phi_*(\gamma_1), \phi_*(\gamma_2)) \neq 0$, for otherwise the curve $\phi_*(\gamma_1)$ would be $\phi(\mathrm{Map}(X))$–invariant, contradicting the assumption that ϕ is irreducible. Since $i(\gamma_1, \gamma_2) = 1$, the Dehn twists δ_{γ_1} and δ_{γ_2} braid. Thus, the N–th powers $\delta^N_{\phi_*(\gamma_1)} = \phi(\delta_{\gamma_1})$ and $\delta^N_{\phi_*(\gamma_2)} = \phi(\delta_{\gamma_2})$ of the Dehn twists along $\phi_*(\gamma_1)$ and $\phi_*(\gamma_2)$ also braid. Since $i(\phi_*(\gamma_1), \phi_*(\gamma_2)) \geq 1$, Lemma 2.3 shows that $i(\phi_*(\gamma_1), \phi_*(\gamma_2)) = 1$ and $N = \pm 1$, as desired. □

Before moving on, we remark that in the final argument of the proof of the irreducible case of Proposition 8.1 we have proved the first claim of the following lemma:

Lemma 8.2 *Suppose that X, Y are as in the statement of Proposition 8.1, and let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be an irreducible homomorphism. Then the following holds:*

- *$i(\phi_*(\gamma), \phi_*(\gamma')) = 1$ for all curves $\gamma, \gamma' \subset X$ with $i(\gamma, \gamma') = 1$.*
- *If a, b, c, d, x, y and z form a lantern with the property that no two curves chosen among a, b, c, d and x separate X, then $\phi_*(a), \phi_*(b), \phi_*(c), \phi_*(d), \phi_*(x)$, $\phi_*(y)$ and $\phi_*(z)$ form a lantern in Y.*

Proof We prove the second claim. By the irreducible case of Proposition 8.1 we know that if γ is any component of the lantern in question, then $\phi_*(\gamma)$ is a single curve and $\phi(\delta_\gamma) = \delta_{\phi_*(\gamma)}$. In particular, the Dehn twists along $\phi_*(a), \phi_*(b), \phi_*(c), \phi_*(d)$, $\phi_*(x), \phi_*(y)$ and $\phi_*(z)$ satisfy the lantern relation. Since a, b, c, d, x are pairwise disjoint, Corollary 6.2 yields that the curves $\phi_*(a), \phi_*(b), \phi_*(c), \phi_*(d), \phi_*(x)$ are also pairwise disjoint. Moreover, the irreducibility of ϕ, the assumption that no two curves chosen among a, b, c, d and x separate X, and Corollary 6.6 imply that the curves $\phi_*(a), \phi_*(b), \phi_*(c), \phi_*(d)$ and $\phi_*(x)$ are pairwise distinct. Thus, the claim follows from Theorem 2.4. □

We are now ready to treat the reducible case of Proposition 8.1.

Proof of Proposition 8.1 for reducible ϕ Let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be a nontrivial reducible homomorphism, and let η be the maximal multicurve in Y which is componentwise preserved by $\phi(\mathrm{Map}(X))$. Recall the exact sequence (4-1):

$$1 \longrightarrow \mathbb{T}_\eta \longrightarrow \mathcal{Z}_0(\mathbb{T}_\eta) \longrightarrow \mathrm{Map}(Y'_\eta) \longrightarrow 0$$

Lemma 4.3 shows that $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}_0(\mathbb{T}_\eta)$ and that the composition

$$\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}(Y'_\eta)$$

of ϕ and the homomorphism $\mathcal{Z}_0(\mathbb{T}_\eta) \to \mathrm{Map}(Y'_\eta)$ is not trivial. Observe that ϕ' is irreducible because η was chosen to be maximal.

The surface Y'_η may well be disconnected; if this is the case, $\mathrm{Map}(Y'_\eta)$ is by definition the direct product of the mapping class groups of the connected components of Y'_η. Noticing that the sum of the genera of the components of Y'_η is bounded above by the genus of Y, it follows from the bound $g' \leq 2g-1$ and from Proposition 7.1 that Y'_η contains a single component Y''_η on which $\phi(\mathrm{Map}(X))$ acts nontrivially. Hence, we can apply the irreducible case of Proposition 8.1 and deduce that $\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}(Y''_\eta)$ maps Dehn twists to possibly left Dehn twists. Conjugating ϕ by an outer automorphism of $\mathrm{Map}(X)$ we may assume without loss of generality that ϕ' maps Dehn twists to Dehn twists.

Suppose now that a, b, c, d, x, y and z form a lantern in X as in Lemma 8.2; such a lantern exists because X has genus at least 3. By Lemma 8.2 we obtain that the images of these curves under ϕ'_* also form a lantern. In other words, if $S \subset X$ is the four-holed sphere with boundary $a \cup b \cup c \cup d$ then there is an embedding $\iota\colon S \to Y''_\eta \subset Y'_\eta$ such that for any $\gamma \in \{a, \dots, z\}$ we have

$$\phi'(\delta_\gamma) = \delta_{\iota(\gamma)}.$$

Identifying Y''_η with a connected component of $Y'_\eta = Y \setminus \eta$ we obtain an embedding $\hat{\iota}\colon S \to Y$. We claim that for any γ in the lantern a, b, c, d, x, y, z we have $\phi(\delta_\gamma) = \delta_{\hat{\iota}(\gamma)}$.

A priori we only have that, for any such γ, both $\phi(\delta_\gamma)$ and $\delta_{\hat{\iota}(\gamma)}$ project to the same element $\delta_{\iota(\gamma)}$ under the homomorphism $\mathcal{Z}_0(\mathbb{T}_\eta) \to \mathrm{Map}(Y'_\eta)$. In other words, there is $\tau_\gamma \in \mathbb{T}_\eta$ with $\phi(\delta_\gamma) = \delta_{\hat{\iota}(\gamma)}\tau_\gamma$. Observe that since any two curves γ, γ' in the lantern a, b, c, d, x, y, z are nonseparating, the Dehn twists δ_γ and $\delta_{\gamma'}$ are conjugate in $\mathrm{Map}(X)$. Therefore, their images under ϕ are also conjugate in $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}_0(\mathbb{T}_\eta)$. Since $\mathbb{T}_\eta$ is central in $\mathcal{Z}_0(\mathbb{T}_\eta)$, it follows that in fact $\tau_\gamma = \tau_{\gamma'}$ for any two curves γ and γ' in the lantern. Denote by τ the element of $\mathbb{T}_\eta$ so obtained.

On the other hand, both $\delta_a, \dots, \delta_z$ and $\delta_{\widehat{\iota}(a)}, \dots, \delta_{\widehat{\iota}(z)}$ satisfy the lantern relation and, moreover, τ commutes with everything. Hence

$$\begin{aligned}1 &= \phi(\delta_a)\phi(\delta_b)\phi(\delta_c)\phi(\delta_d)\phi(\delta_z)^{-1}\phi(\delta_y)^{-1}\phi(\delta_x)^{-1}\\ &= \delta_{\widehat{\iota}(a)}\tau\delta_{\widehat{\iota}(b)}\tau\delta_{\widehat{\iota}(c)}\tau\delta_{\widehat{\iota}(d)}\tau\tau^{-1}\delta_{\widehat{\iota}(z)}^{-1}\tau^{-1}\delta_{\widehat{\iota}(y)}^{-1}\tau^{-1}\delta_{\widehat{\iota}(x)}^{-1}\\ &= \delta_{\widehat{\iota}(a)}\delta_{\widehat{\iota}(b)}\delta_{\widehat{\iota}(c)}\delta_{\widehat{\iota}(d)}\delta_{\widehat{\iota}(z)}^{-1}\delta_{\widehat{\iota}(y)}^{-1}\delta_{\widehat{\iota}(x)}^{-1}\tau = \tau\end{aligned}$$

and thus

$$\phi(\delta_a) = \delta_{\widehat{\iota}(a)}\tau = \delta_{\widehat{\iota}(a)}.$$

In other words, the image under ϕ of the Dehn twist along some, and hence every, nonseparating curve is a Dehn twist. □

9 Reducing to the irreducible

In this section we explain how to reduce the proof of Theorem 1.1 to the case of irreducible homomorphisms between mapping class groups of surfaces without boundary.

9.1 Weak embeddings

Observe there are no embeddings $X \to Y$ if X has no boundary but Y does. We are going to relax the definition of embedding to allow for this possibility. For this purpose, it is convenient to regard X and Y as possibly noncompact surfaces without marked points; recall that we declared ourselves to be free to switch between cusps, marked points and ends.

Definition Let X and Y be possibly noncompact surfaces of finite topological type without marked points. A *weak embedding* $\iota\colon X \to Y$ is a topological embedding of X into Y.

Given two surfaces X and Y without marked points there are two, essentially unique, compact surfaces $\widehat{X}$ and $\widehat{Y}$ with sets $P_{\widehat{X}}$ and $P_{\widehat{Y}}$ of marked points and with $X = \widehat{X} \setminus P_{\widehat{X}}$ and $Y = \widehat{Y} \setminus P_{\widehat{Y}}$. We will say that a weak embedding $\iota\colon X \to Y$ is *induced by an embedding* $\widehat{\iota}\colon (\widehat{X}, P_{\widehat{X}}) \to (\widehat{Y}, P_{\widehat{Y}})$ if there is a homeomorphism $f\colon Y \to Y$ which is isotopic to the identity relative to $P_{\widehat{Y}}$, and $\widehat{\iota}|_X = f \circ \iota$.

Observe that a weak embedding $\iota\colon X \to Y$ is induced by an embedding if and only if the image $\iota(\gamma)$ of every curve $\gamma \subset X$ which bounds a disk in $\widehat{X}$ containing at most one marked point bounds a disk in $\widehat{Y}$ which again contains at most one marked point. Since $\iota(\gamma)$ bounds a disk without punctures if γ does, we can reformulate this equivalence in terms of mapping classes:

Lemma 9.1 *A weak embedding $\iota\colon X \to Y$ is induced by an embedding if and only if $\delta_{\iota(\gamma)}$ is trivial in $\mathrm{Map}(Y)$ for every, a fortiori nonessential, curve $\gamma \subset X$ which bounds a disk with a puncture.* □

Notice that in general a weak embedding $X \to Y$ does not induce a homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$. On the other hand, the following proposition asserts that if a homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ is, as far as it goes, induced by a weak embedding, then it is induced by an actual embedding.

Proposition 9.2 *Let X and Y be surfaces of finite type and genus at least 3. Suppose that $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a homomorphism such that there is a weak embedding $\iota\colon X \to Y$ with the property that for every nonseparating curve $\gamma \subset X$ we have $\phi(\delta_\gamma) = \delta_{\iota(\gamma)}$. Then ϕ is induced by an embedding $X \to Y$.*

We thank the referee for suggesting a simplification of the original proof.

Proof Let Z be a closed regular neighborhood of the union of the curves on X shown in Figure 1, and recall that the Dehn twists about such curves generate $\mathrm{Map}(X)$. The inclusions of Z into X, and of $\iota(Z)$ into Y, together with the homeomorphism $\iota|_Z\colon Z \to \iota(Z)$ induce homomorphisms between the corresponding mapping class groups, so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Map}(Z) & \longrightarrow & \mathrm{Map}(X) \\ \big\downarrow & & \big\downarrow \phi \\ \mathrm{Map}(\iota(Z)) & \longrightarrow & \mathrm{Map}(Y) \end{array}$$

By construction, the left vertical arrow is an isomorphism and so the claim now follows from Lemma 9.1 □

9.2 Down to the irreducible case

Armed with Proposition 9.2, we now prove that it suffices to establish Theorem 1.1 for irreducible homomorphisms. Namely, we have:

Lemma 9.3 *Suppose that Theorem 1.1 holds for irreducible homomorphisms. Then it also holds for reducible ones.*

Proof Let X and Y be surfaces as in the statement of Theorem 1.1 and suppose that $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a nontrivial reducible homomorphism. Let η be a maximal multicurve in Y whose every component is invariant under $\phi(\mathrm{Map}(X))$; by

Lemma 4.3, $\phi(\mathrm{Map}(X)) \subset \mathcal{Z}_0(\mathbb{T}_\eta)$. Consider, as in the proof of Proposition 8.1, the composition

$$\phi'\colon \mathrm{Map}(X) \to \mathrm{Map}(Y'_\eta)$$

of ϕ and the third homomorphism in (4-1). Theorem 4.1 and Lemma 4.2 show that ϕ' is nontrivial; moreover, it is irreducible by the maximality of η. Now, Proposition 8.1 implies that for any γ nonseparating both $\phi(\delta_\gamma) = \delta_{\phi_*(\gamma)}$ and $\phi'(\delta_\gamma) = \delta_{\phi'_*(\gamma)}$ are Dehn twists. As in the proof of the reducible case of Proposition 8.1 we can consider $Y'_\eta = Y \setminus \eta$ as a subsurface of Y. Clearly, $\phi_*(\gamma) = \phi'_*(\gamma)$ after this identification.

Assume that Theorem 1.1 holds for irreducible homomorphisms. Since ϕ' is irreducible, we obtain an embedding

$$\iota\colon X \to Y'_\eta$$

inducing ϕ'. Consider the embedding $\iota\colon X \to Y'_\eta$ as a weak embedding $\hat{\iota}\colon X \to Y$. By the above, $\phi(\delta_\gamma) = \delta_{\hat{\iota}(\gamma)}$, for every $\gamma \subset X$ nonseparating. Finally, Proposition 9.2 implies that ϕ is induced by an embedding. $\square$

9.3 No factors

Let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be a homomorphism as in the statement of Theorem 1.1. We will say that ϕ *factors* if there is a surface $\overline{X}$, an embedding $\bar{\iota}\colon X \to \overline{X}$, and a homomorphism $\overline{\phi}\colon \mathrm{Map}(\overline{X}) \to \mathrm{Map}(Y)$ such that the following diagram commutes:

(9-1)
$$\begin{tikzcd} \mathrm{Map}(X) \arrow[d, "\bar{\iota}_\#"'] \arrow[dr, "\phi"] & \\ \mathrm{Map}(\overline{X}) \arrow[r, "\overline{\phi}"'] & \mathrm{Map}(Y) \end{tikzcd}$$

Since the composition of two embeddings is an embedding, we deduce that ϕ is induced by an embedding if $\overline{\phi}$ is. Since a homomorphism $\mathrm{Map}(X) \to \mathrm{Map}(Y)$ may factor only finitely many times, we obtain:

Lemma 9.4 *If Theorem 1.1 holds for homomorphisms $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ which do not factor, then it holds in full generality.* $\square$

Our next step is to prove that any irreducible homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ factors if X has boundary. We need to establish the following result first:

Lemma 9.5 *Suppose that X and Y are as in the statement of Theorem 1.1 and let $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ be an irreducible homomorphism. Then the centralizer of $\phi(\mathrm{Map}(X))$ in $\mathrm{Map}(Y)$ is trivial.*

Proof Suppose that there is a nontrivial element f in $\mathcal{Z}(\phi(\mathrm{Map}(X)))$; we will show that ϕ is reducible, which is a contradiction. Noticing that the genus bounds in Theorem 1.1 are more generous than those in Lemma 7.2, we deduce from the latter that f has infinite order. Let $\gamma \subset X$ be a nonseparating curve and recall that $\phi(\delta_\gamma)$ is a Dehn twist by Proposition 8.1. Since f commutes with $\phi(\delta_\gamma)$ it follows that f is reducible; let η be the *canonical reducing multicurve* associated to f (see [9]). Since $\phi(\mathrm{Map}(X))$ commutes with f we deduce that $\phi(\mathrm{Map}(X))$ preserves η. We will prove that $\phi(\mathrm{Map}(X))$ preserves some component of η, hence obtaining a contradiction to the assumption that ϕ is irreducible. The arguments are very similar to the arguments in the proof of Lemma 6.4 and Lemma 6.5.

First, the same arguments as the ones used to prove Lemma 6.4 imply that some component of η is preserved if some component of $Y \setminus \eta$ is a disk or an annulus. Suppose that this is not the case. Then $Y \setminus \eta$ has at most $2g'-2 \le 4g-4$ components. Hence Theorem 4.4 implies that $\phi(\mathrm{Map}(X))$ preserves every component of $Y \setminus \eta$. Using again that no component of $Y \setminus \eta$ is a disk or an annulus we deduce that every such component C has at most $2g'+2 \le 4g-2$ boundary components. Hence Theorem 4.4 implies that $\phi(\mathrm{Map}(X))$ preserves every component of $\partial C \subset \eta$. We have proved that some component of η is preserved by $\phi(\mathrm{Map}(X))$ and hence that ϕ is reducible, as desired. □

We can now prove:

Corollary 9.6 *Suppose that X and Y are as in Theorem 1.1 and that $\partial X \neq \varnothing$. Then every irreducible homomorphism $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ factors.*

Proof Let $X' = X \setminus \partial X$ be the surface obtained from X by deleting the boundary and consider the associated embedding $\iota\colon X \to X'$. By Theorem 2.5, the homomorphism $\iota_\#\colon \mathrm{Map}(X) \to \mathrm{Map}(X')$ fits in the exact sequence

$$1 \longrightarrow \mathbb{T}_{\partial X} \longrightarrow \mathrm{Map}(X) \longrightarrow \mathrm{Map}(X') \longrightarrow 1,$$

where $\mathbb{T}_{\partial X}$ is the center of $\mathrm{Map}(X)$. It follows from Lemma 9.5 that if ϕ is irreducible, then $\mathbb{T}_{\partial X} \subset \mathrm{Ker}(\phi)$. We have proved that ϕ descends to $\phi'\colon \mathrm{Map}(X') \to \mathrm{Map}(Y)$ and hence that ϕ factors as we needed to show. □

Combining Lemma 9.3, Lemma 9.4 and Corollary 9.6 we deduce:

Proposition 9.7 *Suppose that Theorem 1.1 holds if*

- *X and Y have no boundary, and*
- *$\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is irreducible and does not factor.*

Then Theorem 1.1 holds in full generality. □

10 Proof of Theorem 1.1

In this section we prove the main result of this paper, whose statement we now recall:

Theorem 1.1 *Suppose that X and Y are surfaces of finite topological type, of genus $g \geq 6$ and $g' \leq 2g-1$ respectively; if Y has genus $2g-1$, suppose also that it is not closed. Then every nontrivial homomorphism*

$$\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$$

is induced by an embedding $X \to Y$.

Remark As mentioned in the introduction, the same conclusion as in Theorem 1.1 applies for homomorphisms $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ if both X and Y have the same genus $g \in \{4, 5\}$. This will be shown in the course of the proof.

By Proposition 9.7 we may assume that X and Y have no boundary, that ϕ is irreducible and that it does not factor. Moreover, by Proposition 8.1, the image of a Dehn twist δ_γ along a nonseparating curve is either the right or the left Dehn twist along the nonseparating curve $\phi_*(\gamma)$. Notice that, up to composing ϕ with an outer automorphism of $\mathrm{Map}(Y)$ induced by an orientation reversing homeomorphism of Y, we may actually assume that $\phi(\delta_\gamma)$ is actually a right Dehn twist for some, and hence every, nonseparating curve $\gamma \subset X$. In light of this, we can assume that we are in the following situation:

Standing assumption X and Y have no boundary; ϕ is irreducible and does not factor; $\phi(\delta_\gamma) = \delta_{\phi_*(\gamma)}$ for all $\gamma \subset X$ nonseparating.

Under these assumptions, we will prove that ϕ is induced by an orientation preserving homeomorphism. We will make extensive use of the concrete set of generators of $\mathrm{Map}(X)$ given in Figure 1, which we include here as Figure 4 for convenience. The reader should have Figure 4 constantly in mind throughout the rest of this section.

The sequence $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ in Figure 4 forms a chain; we will refer to it as the *$a_i b_i$–chain*. We will refer to the multicurve $r_1 \cup \cdots \cup r_k$ as the *r_i–fan*, and to the curve with label c simply as *the curve c*. Since all these curves are nonseparating, it follows from Proposition 6.3 that $\phi_*(\gamma)$ is a nonseparating curve for any curve γ in the collection a_i, b_i, r_i, c.

Before moving on, we remark that since the Dehn twists along the curves $\phi_*(a_i), \phi_*(b_i)$, $\phi_*(r_i), \phi_*(c)$ generate $\phi(\mathrm{Map}(X))$, and since we are assuming that ϕ is irreducible, we immediately obtain:

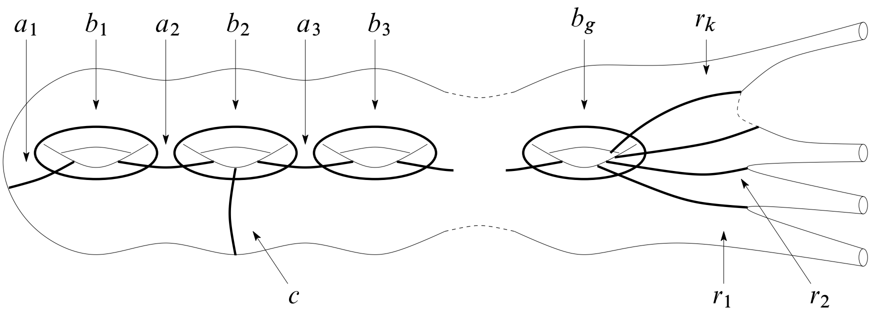


Figure 4: Dehn twists along the curves a_i, b_i, c and r_i generate $\mathrm{Map}(X)$

Lemma 10.1 *The image under ϕ_* of the a_ib_i–chain, the r_i–fan and the c–curve fill Y.* □

Suppose that γ, γ' are two distinct elements of the collection a_i, b_i, r_i, c. We now summarize several of the already established facts about the relative positions of the curves $\phi_*(\gamma), \phi_*(\gamma')$:

(1) If $i(\gamma, \gamma') = 0$ then $i(\phi_*(\gamma), \phi_*(\gamma')) = 0$ by Corollary 6.2.

(2) If γ, γ' are distinct and disjoint, and $X \setminus (\gamma \cup \gamma')$ is connected, then $\phi_*(\gamma) \neq \phi_*(\gamma')$ by Corollary 6.6.

(3) If $i(\gamma, \gamma') = 1$ then $i(\phi_*(\gamma), \phi_*(\gamma')) = 1$ by Lemma 8.2.

Note that these properties do not ensure that $\phi_*(r_i) \neq \phi_*(r_j)$ if $i \neq j$. We denote by $\mathcal{R} \subset Y$ the maximal multicurve with the property that each one of its components is homotopic to one of the curves $\phi_*(r_i)$. Note that $\mathcal{R} = \varnothing$ if and only if X has at most a puncture and that in any case $\mathcal{R}$ has at most as many components as curves has the r_i–fan. The next lemma follows easily from (1), (2) and (3) above:

Lemma 10.2 *With the notation of Figure 4 the following holds:*

- *The image under ϕ_* of the a_ib_i–chain is a chain of the same length in Y.*
- *Every component of the multicurve $\mathcal{R}$ intersects $\phi_*(b_g)$ exactly once, and is disjoint from the images of the other curves in the a_ib_i–chain.*
- *The curve $\phi_*(c)$ is disjoint from every curve in $\mathcal{R}$, intersects $\phi_*(b_2)$ exactly once, and is disjoint from the images of the other curves in the a_ib_i–chain.*

At first glance, Lemma 10.2 yields the desired embedding without any further work, but this is far from true. We sketch, for the convenience of the reader, what is left of the proof of Theorem 1.1. First, we will clarify the relative positions of the images of the $a_i b_i$–chain and the curve c under ϕ_*. This will allow us to prove Theorem 1.1 if X has at most one puncture. For the general case we will start by proving that X and Y have the same genus, and that Y has at most as many punctures as X. At this point, the main problem left will be to understand the relative positions of the curves in the multicurve $\mathcal{R}$. A few results describing the ϕ_*–images of pairs of curves bounding annuli in X will allow us to prove that ϕ_* is a bijection, preserving a certain order, from the r_i–fan to $\mathcal{R}$. Having established this, Theorem 1.1 will quickly follow.

As we just announced, we first clarify the position of $\phi_*(c)$:

Lemma 10.3 *Let Z be a regular neighborhood of the $a_i b_i$–chain with $c \subset Z$. Then there is an orientation-preserving embedding $F\colon Z \to Y$ such that $\phi_*(\gamma) = F(\gamma)$ for $\gamma = a_i, b_i, c$ and $i = 1, \ldots, g$.*

Proof The image under ϕ_* of the $a_i b_i$–chain is a chain of the same length, by Lemma 10.2. Let Z' be a regular neighborhood of the ϕ_*–image of the $a_i b_i$–chain. Since regular neighborhoods of any two chains of the same length are homeomorphic in an orientation-preserving manner, there is an orientation-preserving embedding

$$F\colon Z \to Z'$$

with $F(a_i) = \phi_*(a_i)$ and $F(b_i) = \phi_*(b_i)$ for all i. It remains to prove that F can be chosen so that $F(c) = \phi_*(c)$.

Let $Z_0 \subset Z$ be the subsurface of X filled by a_1, b_1, a_2, b_2 and observe that, up to isotopy, $c \subset Z_0$. The boundary of Z_0 is connected and, by the chain relation (see Section 2) we can write the Dehn twist along ∂Z_0 as

$$\delta_{\partial Z_0} = (\delta_{a_1}\delta_{b_2}\delta_{a_2}\delta_{b_2})^{10}.$$

Hence we have

$$\begin{aligned}
\phi(\delta_{\partial Z_0}) &= (\phi(\delta_{a_1})\phi(\delta_{b_2})\phi(\delta_{a_2})\phi(\delta_{b_2}))^{10} \\
&= (\delta_{\phi_*(a_1)}\delta_{\phi_*(b_2)}\delta_{\phi_*(a_2)}\delta_{\phi_*(b_2)})^{10} \\
&= (\delta_{F(a_1)}\delta_{F(b_2)}\delta_{F(a_2)}\delta_{F(b_2)})^{10} \\
&= \delta_{F(\partial Z_0)},
\end{aligned}$$

where the last equality follows again from the chain relation.

Since c is disjoint from ∂Z_0 we have that δ_c and $\delta_{\partial Z_0}$ commute, and hence the same is true for $\delta_{\phi_*(c)} = \phi(\delta_c)$ and $\delta_{F(\partial Z_0)} = \phi(\delta_{\partial Z_0})$. Since $\delta_{F(\partial Z_0)}$ is a nontrivial mapping class, it follows that $\phi_*(c)$ does not intersect $F(\partial Z_0)$. On the other hand, $\phi_*(c)$ intersects $\phi_*(b_2) \subset F(Z_0)$, and hence $\phi_*(c) \subset F(Z_0) \subset F(Z)$.

Observe now that $F(Z) \setminus (\bigcup \phi_*(a_i) \cup \phi_*(b_i)) \simeq Z \setminus (\bigcup a_i \cup b_i)$ is homeomorphic to an annulus A. It follows from Lemma 10.2 that the intersection of $\phi_*(c)$ with A is an embedded arc whose endpoints are in the subsegments of ∂A corresponding to $\phi_*(b_2)$. Up to isotopy, there are two choices for such an arc. However, there is an involution $\tau\colon F(Z) \to F(Z)$ with $\tau(\phi_*(a_i)) = \phi_*(a_i)$ and $\tau(\phi_*(b_i)) = \phi_*(b_i)$ and which interchanges these two arcs. It follows that, up to possibly replacing F by $\tau \circ F$, we have $F(c) = \phi_*(c)$, as we needed to prove. □

At this point we are ready to prove the first cases of Theorem 1.1.

Proof of Theorem 1.1 (X is closed or has one puncture) Let Z and $F\colon Z \to Y$ be as in Lemma 10.3. If X has one puncture, then there is a weak embedding $X \to Z \subset X$ which is homotopic to the identity $X \to X$. Denote by $f\colon X \to Y$ the weak embedding obtained by composing the weak embedding $X \to Z$ with the embedding $F\colon Z \to Y$. By construction we have $\phi(\delta_\gamma) = \delta_{f(\gamma)}$ for all γ in a_i, b_i, c. Since the elements $\delta_{a_i}, \delta_{b_i}$ and δ_c generate $\mathrm{Map}(X)$, it follows that $\phi(\delta_\gamma) = \delta_{f(\gamma)}$ for all $\gamma \subset X$. Proposition 9.2 implies that ϕ is induced by an embedding, as we needed to prove.

We now treat the case that X is closed. Since we are assuming that ϕ is irreducible, and since a collection of curves in $F(Z)$ fills Y, we obtain that $Y \setminus F(Z)$ is a disk containing at most one puncture. If $Y \setminus F(Z)$ is a disk without punctures, then we can extend the map $F\colon Z \to Y$ to a continuous injective map $X \to Y$. Since any continuous injective map between closed connected surfaces is a homeomorphism, F is an embedding and we are done in this case.

It remains to rule out the possibility that X is closed and Y has one puncture. Suppose that this is the case and let $\overline{Y}$ be the surface obtained from Y by filling in the puncture. We can now apply the above argument to the induced homomorphism

$$\overline{\phi}\colon \mathrm{Map}(X) \to \mathrm{Map}(\overline{Y}),$$

obtaining that $\overline{\phi}$ is induced by an embedding $X \to \overline{Y}$. Since any embedding from a closed surface is a homeomorphism we deduce that $\overline{\phi}$ is an isomorphism. Composing $\phi \circ \overline{\phi}^{-1}\colon \mathrm{Map}(\overline{Y}) \to \mathrm{Map}(Y)$ with the filling-in homomorphism $\mathrm{Map}(Y) \to \mathrm{Map}(\overline{Y})$ we obtain the identity. Hence, $\phi \circ \overline{\phi}^{-1}$ is a splitting of the Birman exact sequence

$$1 \longrightarrow \pi_1(\overline{Y}) \longrightarrow \mathrm{Map}(Y) \longrightarrow \mathrm{Map}(\overline{Y}) \longrightarrow 1,$$

which is impossible [26]. It follows that Y cannot have a puncture, as we needed to prove. □

From now we will assume:

Standing assumption X has at least 2 punctures.

Next, we prove that Y has the same genus as X.

Lemma 10.4 *Both surfaces X and Y have the same genus g.*

Proof With the same notation as in Lemma 10.3 we need to prove that $S = Y \setminus F(Z)$ is a surface of genus 0. By Lemma 10.1 the arcs $\rho_i = \phi_*(r_i) \cap S$ fill S. Denote by $\overline{S}$ the surface obtained by attaching a disk to S along $F(\partial Z) \subset \partial S$, noting that the arcs ρ_i can be extended to a collection of disjoint curves $\bar{\rho}_i$ on $\overline{S}$. Moreover, every curve in $\overline{S}$ either agrees with, or else intersects one of the curves $\bar{\rho}_i$ more than once. This is impossible if S has genus at least 1; this proves Lemma 10.4. □

Note that if $\eta \subset X$ is separating, all we know about $\phi(\delta_\eta)$ is that it is a root of a multitwist by Theorem 6.1; in particular, $\phi(\delta_\eta)$ may be trivial or have finite order. If this is not the case, we denote by $\phi_*(\eta)$ the multicurve supporting any multitwist power of $\phi(\delta_\eta)$. If η bounds a disk with punctures then, up to replacing the $a_i b_i$–chain by another such chain, we may assume that $i(\eta, a_i) = i(\eta, b_i) = 0$ for all i. In particular, $\phi_*(\eta)$ does not intersect any of the curves $\phi_*(a_i)$ and $\phi_*(b_i)$. It follows that every component of $\phi_*(\eta)$ is separating. We record our conclusions:

Lemma 10.5 *Suppose that $\eta \subset X$ bounds a disk with punctures and that $\phi(\delta_\eta)$ has infinite order. Then every component of the multicurve $\phi_*(\eta)$ separates Y.* □

Our next goal is to bound the number of cusps of Y:

Lemma 10.6 *Every connected component of $Y \setminus \left(\bigcup_i \phi_*(a_i) \cup \mathcal{R}\right)$ contains at most a single puncture. In particular Y has at most as many punctures as X.*

Recall that $\mathcal{R} \subset Y$ is the maximal multicurve with the property that each one of its components is homotopic to one of the curves $\phi_*(r_i)$.

Proof Observe that Lemma 10.1 and Lemma 10.3 imply that the union of $\mathcal{R}$ and the image under ϕ_* of the $a_i b_i$–chain fill Y. In particular, every component of the complement in Y of the union of $\mathcal{R}$ and all the curves $\phi_*(a_i)$ and $\phi_*(b_i)$ contains at most one puncture of Y. By Lemma 10.2, the multicurve $\bigcup \phi_*(b_i)$ does not separate any of the components of the complement of $(\bigcup \phi_*(a_i)) \cup \mathcal{R}$ in Y; we have proved the first claim.

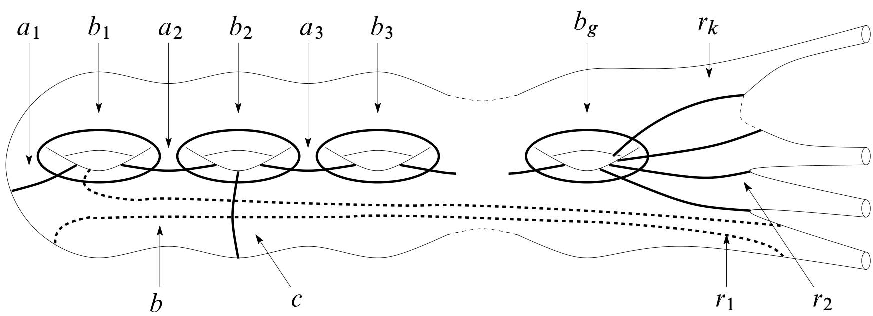
It follows again from Lemma 10.2 that the multicurve $\bigcup \phi_*(a_i) \cup \mathcal{R}$ separates Y into at most k components where $k \geq 2$ is the number of punctures of X. Thus, Y has at most as many punctures as X. □

So far, we do not know much about the relative positions of the curves in $\mathcal{R}$; this will change once we have established the next three lemmas.

Lemma 10.7 *Suppose that $a, b \subset X$ are nonseparating curves that bound an annulus A. Then $\phi_*(a)$ and $\phi_*(b)$ bound an annulus A' in Y; moreover, if A contains exactly one puncture and $\phi_*(a) \neq \phi_*(b)$, then A' also contains exactly one puncture.*

Proof Note that A is disjoint from a chain of length $2g-1$. Since ϕ_* maps chains to chains (Lemma 10.2), preserves disjointness (Corollary 6.2) and since Y has the same genus as X (Lemma 10.4), we deduce that $\phi_*(\partial A)$ consists of nonseparating curves which are contained in an annulus in Y. The first claim follows.

Suppose that $\phi_*(a) \neq \phi_*(b)$; up to translating by a mapping class, we may assume that $a = a_1$ and that b is a curve disjoint from $(\bigcup a_i) \cup (\bigcup r_i)$ and with $i(b, b_1) = 1$ and $i(b, b_i) = 0$ for $i = 2, \dots, g$ (compare with the dashed curve in the figure below).



Since ϕ_* preserves disjointness and intersection number one (Lemma 8.2), it follows that the annulus A' bounded by $\phi_*(a) = \phi_*(a_1)$ and $\phi_*(b)$ is contained in one of the two connected components of $Y \setminus \big(\bigcup_i \phi_*(a_i) \cup \bigcup_i \phi_*(r_i)\big)$ adjacent to $\phi_*(a_1)$. By Lemma 10.6, each one of these components contains at most a puncture, and thus the claim follows. □

Lemma 10.8 *Let $\gamma, \gamma' \subset X$ be nonseparating curves bounding an annulus with one puncture. Then $\phi_*(\gamma) \neq \phi_*(\gamma')$.*

Proof We will prove that if $\phi_*(\gamma) = \phi_*(\gamma')$, then ϕ factors in the sense of (9-1), contradicting our standing assumptions.

Suppose $\phi_*(\gamma) = \phi_*(\gamma')$, noting that Proposition 8.1 implies that $\phi(\delta_\gamma) = \phi(\delta_{\gamma'})$. Let p be the puncture in the annulus bounded by γ and γ'. Consider the surface $\overline{X}$ obtained from X by filling in the puncture p and the Birman exact sequence (3-1),

$$1 \longrightarrow \pi_1(\overline{X}, p) \longrightarrow \mathrm{Map}(X) \longrightarrow \mathrm{Map}(\overline{X}) \longrightarrow 1,$$

associated to the embedding $X \to \overline{X}$. Let $\alpha \in \pi_1(\overline{X}, p)$ be the unique essential simply loop contained in the annulus bounded by $\gamma \cup \gamma'$. The image of α under the left arrow of the Birman exact sequence is $\delta_\gamma \delta_{\gamma'}^{-1}$. Hence, α belongs to the kernel of ϕ. Since $\pi_1(\overline{X}, p)$ has a set of generators consisting of translates of α by $\mathrm{Map}(X)$ we deduce that $\pi_1(\overline{X}, p) \subset \mathrm{Ker}(\phi)$. This shows that $\phi\colon \mathrm{Map}(X) \to \mathrm{Map}(Y)$ factors through $\mathrm{Map}(\overline{X})$ and concludes the proof of Lemma 10.8. □

Lemma 10.9 *Let $a, b \subset X$ be nonseparating curves which bound an annulus with exactly two punctures. Then $\phi_*(a)$ and $\phi_*(b)$ bound an annulus $A' \subset Y$ with exactly two punctures. Moreover, if $x \subset A$ is any nonseparating curve in X which separates the two punctures of A, then $\phi_*(x) \subset A'$ and separates the two punctures of A'.*

Proof Let $x \subset A$ be a curve as in the statement. Suppose first that $\phi_*(a) \neq \phi_*(b)$. Consider the annuli A', A'_1 and A'_2 in Y with boundaries

$$\partial A' = \phi_*(a) \cup \phi_*(b), \quad \partial A'_1 = \phi_*(a) \cup \phi_*(x), \quad \partial A'_2 = \phi_*(x) \cup \phi_*(b).$$

By Lemmas 10.7 and 10.8, the annuli A'_1 and A'_2 contain exactly one puncture. Finally, since $\phi_*(x)$ does not intersect $\phi_*(a) \cup \phi_*(b)$, it follows that $A' = A'_1 \cup A'_2$ and the claim follows.

It remains to rule out the possibility that $\phi_*(a) = \phi_*(b)$. Seeking a contradiction, suppose that this is the case. Consider curves c, d, y, z as in Figure 6.

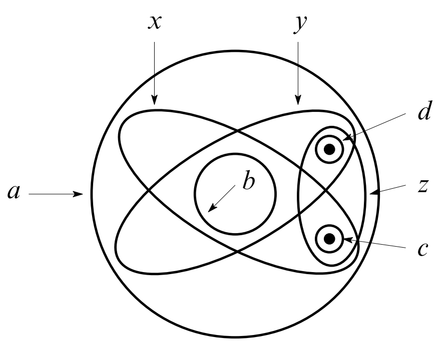


Figure 6: The black dots represent cusps

The curves a, b, c, d, x, y, z form a lantern, where c, d are not essential. In particular, the lantern relation reduces to $\delta_a \delta_b = \delta_x \delta_y \delta_z$. Applying ϕ we obtain

$$\delta^2_{\phi_*(a)} = \delta_{\phi_*(x)} \delta_{\phi_*(y)} \phi(\delta_z). \tag{10-1}$$

Observe also that each of $a \cup x$ and $a \cup y$ bound an annulus in X containing exactly one cusp. We deduce from Lemma 10.7 and Lemma 10.8 that there are annuli A_1', A_2' in Y, each containing one cusp, with

$$\partial A_1' = \phi_*(a) \cup \phi_*(x), \quad \partial A_2' = \phi_*(a) \cup \phi_*(y).$$

Noting that $i(\phi_*(x), \phi_*(y))$ is even, it remains to rule out the following three possibilities:

Case 1: $i(\phi_*(x), \phi_*(y)) > 2$ By [16, Theorem 3.10], the restriction of $\delta_{\phi_*(x)}\delta_{\phi_*(y)}$ to the subsurface of Y filled by $\phi_*(x) \cup \phi_*(y)$ is pseudo-Anosov. On the other hand,

$$\delta_{\phi_*(x)}\delta_{\phi_*(y)} = \delta_{\phi_*(a)}^2 \phi(\delta_z)^{-1}$$

is a root of a multitwist because, by Theorem 6.1, $\phi(\delta_z)^{-1}$ is a root of a multitwist that commutes with $\delta_{\phi_*(a)} = \phi(\delta_a)$; this yields a contradiction.

Case 2: $i(\phi_*(x), \phi_*(y)) = 2$ We are in the situation of Figure 7, meaning that we can

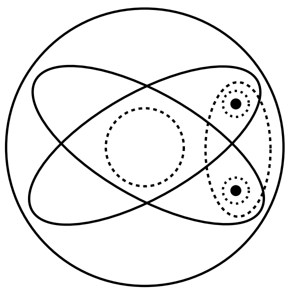


Figure 7: The solid lines are $\phi_*(a)$, $\phi_*(x)$ and $\phi_*(y)$ and the black dots are cusps

extend the collection $\phi_*(a), \phi_*(x), \phi_*(y)$ to a lantern $\phi_*(a), \widehat{b}$, $\widehat{c}$, $\widehat{d}$, $\phi_*(x), \phi_*(y)$ and $\widehat{z}$, with $\widehat{c}, \widehat{d}$ nonessential and $\widehat{b}$ nonseparating. From the lantern relation we obtain

$$\delta_{\phi_*(y)}^{-1}\delta_{\phi_*(x)}^{-1}\delta_{\phi_*(a)} = \delta_{\widehat{z}}\delta_{\widehat{b}}^{-1}. \tag{10-2}$$

From (10-1) and (10-2) we get that $\phi(\delta_z) = \delta_{\widehat{z}}\delta_{\widehat{b}}^{-1}\delta_{\phi_*(a)}$. In particular, $\phi(\delta_z)$ is a multitwist whose support contains nonseparating components, contradicting Lemma 10.5 because z bounds a disk in X.

Case 3: $i(\phi_*(x), \phi_*(y)) = 0$ Rewriting (10-1) we obtain

$$\delta_{\phi_*(x)}^{-1}\delta_{\phi_*(y)}^{-1}\delta_{\phi_*(a)}^2 = \phi(\delta_z).$$

As x and y are disjoint from a, $\phi(\delta_z)$ is a multitwist supported on a multicurve contained in $\phi_*(a) \cup \phi_*(x) \cup \phi_*(y)$. Since these three curves are nonseparating, Lemma 10.5 implies that $\phi(\delta_z) = \mathrm{Id}$, and hence $\phi_*(a) = \phi_*(x) = \phi_*(y)$. Since a and x bound an annulus which exactly one puncture, we obtain a contradiction to Lemma 10.8.

Having ruled out these three cases, we deduce that $\phi_*(a) \neq \phi_*(b)$; this concludes the proof of Lemma 10.9 □

We are now ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1 Continuing with the same notation and standing assumptions, we now introduce orderings on the r_i–fan and the multicurve $\mathcal{R} \subset Y$. In order to do so, observe that the union of the multicurve $\bigcup a_i$ and any of the curves in the r_i–fan separates X. Similarly, by Lemma 10.2 the union of the multicurve $\bigcup \phi_*(a_i)$ and any of the components of $\mathcal{R}$ is a multicurve consisting of $g+1$ nonseparating curves. Since Y has genus g, by Lemma 10.4, we deduce that the union of the multicurve $\bigcup \phi_*(a_i)$ and any of the components of $\mathcal{R}$ separates Y. We now define our orderings:

- Given two curves r_i, r_j in the r_i–fan we say that $r_i \leq r_j$ if r_i and c are in the same connected component of $X \setminus (a_1 \cup \cdots \cup a_g \cup r_j)$. Notice that the labeling in Figure 4 is such that $r_i \leq r_j$ for $i \leq j$.
- Similarly, given two curves $r, r' \in \mathcal{R}$ we say that $r \leq r'$ if r and $\phi_*(c)$ are in the same connected component of $X \setminus (\phi_*(a_1) \cup \cdots \cup \phi_*(a_g) \cup r')$.

The minimal element of the r_i–fan, the curve r_1 in Figure 4, is called the *initial curve* of the r_i–fan; we define the initial curve of the multicurve $\mathcal{R}$ in an analogous way. We claim that the image of r_1 under ϕ_* is the initial curve of $\mathcal{R}$:

Claim *$\phi_*(r_1)$ is the initial curve of $\mathcal{R}$.*

Proof of the claim Suppose, for contradiction, that $\phi_*(r_1)$ is not the initial curve in $\mathcal{R}$. Consider, besides the curves in Figure 4, a curve c' as in Figure 8. In words, c and c' bound an annulus with exactly two punctures and

$$i(c', r_i) = 0 \quad \forall i \geq 2 \quad \text{and} \quad i(c', a_i) = 0 \quad \forall i. \tag{10-3}$$

Notice that by Lemma 10.9, $\phi_*(c)$ and $\phi_*(c')$ bound an annulus A which contains exactly two punctures.

Since $\phi(r_1)$ is not the initial curve, then $i(\phi_*(c'), \bigcup \phi_*(r_j)) = 0$ for all j, as $i(c', r_j) = 0$ for all $j > 1$. Also, by disjointness $i(\phi_*(c'), \phi_*(a_i)) = 0$ for all i. Since the boundary $\partial A = \phi_*(c) \cup \phi_*(c')$ of the annulus A is disjoint from $\bigcup \phi_*(a_i) \cup \phi_*(r_i)$, it is contained

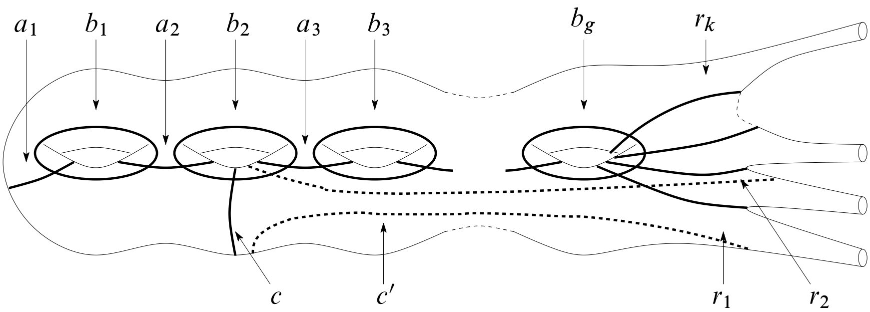


Figure 8: The dotted curve c', and c, together bound an annulus with two punctures

in one of the connected components of $X \setminus (\bigcup \phi_*(a_i) \cup \phi_*(r_i))$. However, each one of these components contains at most one puncture, by Lemma 10.6. This contradicts Lemma 10.9, and thus we have established the claim. □

We are now ready to prove that ϕ_* induces an order-preserving bijection between the r_i–fan and the multicurve $\mathcal{R}$. Denote the curves in $\mathcal{R}$ by r_i', labeled in such a way that $r_i' \leq r_j'$ if $i \leq j$. By the previous claim, $\phi_*(r_1) = r_1'$. Next, consider the curve r_2, noting that r_1 and r_2 bound an annulus with exactly one puncture. Hence, Lemma 10.8 yields that $\phi_*(r_1) = r_1'$ and $\phi_*(r_2)$ also bound an annulus with exactly one puncture. In particular, $\phi_*(r_2)$ cannot be separated from r_1' by any component of $\mathcal{R}$. This proves that $\phi_*(r_2) = r_2'$. We now consider the curve r_3. The argument just used for r_2 implies that either $\phi_*(r_3) = r_3'$ or $\phi_*(r_3) = r_1'$. The latter is impossible, as the curves r_1 and r_3 bound an annulus with exactly two punctures and hence so do $\phi_*(r_1) = r_1'$ and $\phi_*(r_3)$ by Lemma 10.9. Thus $\phi_*(r_3) = r_3'$. Repeating this argument as often as necessary we obtain that the map ϕ_* induces an injective, order-preserving map from the r_i–fan to $\mathcal{R}$. Since by definition $\mathcal{R}$ has at most as many components as the r_i–fan, we have proved that this map is in fact an order-preserving bijection.

Let $Z \subset X$ be a regular neighborhood of the $a_i b_i$–chain, and recall that $c \subset Z$. By Lemma 10.3 there is an orientation-preserving embedding $F\colon Z \to Y$ such that $\phi_*(\gamma) = F(\gamma)$ for $\gamma = a_i, b_i, c$ $(i = 1, \dots, g)$. We choose Z so that it intersects every curve in the r_i–fan in a segment. Lemma 10.2 implies that F can be isotoped so that

$$F\big(Z \cap \big(\bigcup r_i\big)\big) = F(Z) \cap \mathcal{R}.$$

The orderings of the r_i–fan and of $\mathcal{R}$ induce orderings of $Z \cap (\bigcup r_i)$ and $F(Z) \cap \mathcal{R}$. Since the map ϕ_* preserves both orderings we deduce that F preserves the induced orderings of $Z \cap (\bigcup r_i)$ and $F(Z) \cap \mathcal{R}$.

Let k be the number of curves in the r_i–fan, and thus in $\mathcal{R}$. We successively attach k annuli along the boundary ∂Z of Z, as indicated in Figure 9. In this way we get a

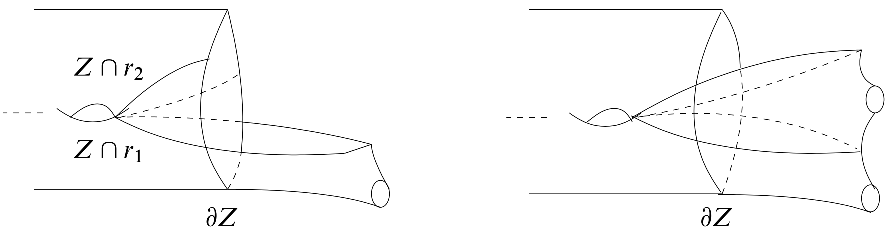


Figure 9: Attaching the first (left) and second (right) annuli along ∂Z

surface Z_1 naturally homeomorphic to X. We perform the analogous operation on $\partial F(Z)$, thus obtaining a surface Z_2 which is naturally homeomorphic to Y. Since the map ϕ_* is preserves the orderings of the r_i–fan and of R, we get that the homeomorphism $F\colon Z \to F(Z)$ extends to a homeomorphism

$$\bar{F}\colon X \to Y$$

such that

$$\bar{F}(\gamma) = \phi_*(\gamma)$$

for every curve γ in the collection a_i, b_i, c, r_i. It follows that the homomorphisms ϕ and $\bar{F}_\#$ both map the Dehn twist along γ to the Dehn twist along $\phi_*(\gamma)$ and, in particular, to the same element in $\mathrm{Map}(Y)$. Since the Dehn twists along the curves a_i, b_i, c, r_i generate $\mathrm{Map}(X)$, we deduce $\phi = \bar{F}_\#$. This finishes the proof of Theorem 1.1. □

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