

Deriving Deligne–Mumford stacks with perfect obstruction theories

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We show that every n -connective quasi-coherent obstruction theory on a Deligne–Mumford stack comes from the structure of a connective spectral Deligne–Mumford stack on the underlying topos. Working over a base ring containing the rationals, we obtain the corresponding result for derived Deligne–Mumford stacks.

14A20, 18G55; 55P43

1 Introduction

Some moduli spaces playing an important role in enumerative geometry carry an additional structure. Apart from the cotangent complex, which controls deformations and obstructions of the objects parametrized by the moduli space, there sometimes exist another complex doing the same job. If the moduli space in question is very singular, the cotangent complex will have cohomology in arbitrary many degrees. In many cases, the replacement complex has much better finiteness properties, being locally isomorphic to a finite complex of vector bundles. If the replacement complex is perfect and of Tor-amplitude at most 1 it gives rise to the virtual fundamental class of Li and Tian [3], and Behrend and Fantechi [1], which is the key to actually producing numbers.

Ever since this phenomenon was observed by Kontsevich [2], it was suspected that the replacement complex is a shadow of a derived structure on the moduli space. In the meantime, the foundations of derived algebraic geometry have been firmly laid out by Toën and Vezzosi [11], and Lurie [4]. Using these theories, in many examples derived moduli spaces having the ‘correct’ cotangent complex have been found (for examples see Toën’s overview [9]). These derived enhancements have the same underlying topological space as their classical counterpart, the derived structure just being a nilpotent thickening of the structure sheaf. In [9, Section 4.4.3], Toën observed that such a derived enhancement automatically induces a replacement complex for the cotangent complex of the classical moduli space. The replacement is simply the cotangent complex of the derived enhancement, which might well be very different from the cotangent complex of the classical part and enjoy much better finiteness properties.

Using the approach of Toën mentioned above, we can regard the replacement complex as the cotangent complex of some possible derived enhancement. We thus already know quite a lot about a possible derived enhancement inducing the replacement complex: we know its underlying space and cotangent complex. The problem then is to lift this information on the tangent level to an actual derived structure sheaf on the space.

This formulation makes the problem tractable to using obstruction calculus to find a possible derived structure on the moduli space in question inducing the replacement complex. In a certain sense, this defeats the purpose of derived algebraic geometry, as part of the motivation for derived algebraic geometry was precisely avoiding such calculations and simply writing down the functor the lifted moduli problem should represent.¹

Behrend and Fantechi in [1] axiomatized the phenomenon of a replacement complex for the cotangent complex of a moduli space to the notion of an *obstruction theory*. The main theorem of this paper is that any obstruction theory of arbitrary length on a Deligne–Mumford stack comes from the structure of a spectral Deligne–Mumford stack on the same underlying topos (Theorem 3.6). Over a base ring containing the rationals the theory of spectral Deligne–Mumford stacks and derived Deligne–Mumford stacks coincide. Adding this extra assumption, we obtain that any obstruction theory comes from the structure of a derived Deligne–Mumford stack on the same underlying topos.

Recall that an obstruction theory is called *perfect* if the replacement complex is in fact a perfect complex. If we assume the given obstruction theory to be perfect it follows that any spectral Deligne–Mumford stack inducing the perfect obstruction theory is locally of finite presentation (Lemma 3.8). Again adding the assumption that our base ring contains the rationals, the main theorem thus implies that any perfect obstruction theory of any length on a Deligne–Mumford stack comes from a derived Deligne–Mumford stack locally of finite presentation with the same underlying topos. In particular, every 1–perfect obstruction theory comes from a structure of quasi-smooth Deligne–Mumford stack (Corollary 3.9) on the same underlying topos. The structure inducing the obstruction theory is by no means expected to be unique.

The method of proof uses obstruction calculus for nilpotent thickenings of derived rings. The basic observation underlying the whole work is easily described. An obstruction theory for a commutative ring A is given by a morphism $\phi: E \rightarrow L_A$, where E is a complex of A –modules and L_A is the cotangent complex. Such a morphism can always be completed to a cofiber sequence

$$E \xrightarrow{\phi} L_A \xrightarrow{\eta} K.$$

¹ See the overviews by Toën and Vezzosi [10, Section 6], and Lurie [5, page 9]

Now the datum of a morphism $\eta: L_A \rightarrow K$ defines a square-zero extension $A^\eta \rightarrow A$. The cotangent complex of A^η is already an excellent approximation of E . There exists a comparison map $E \rightarrow L_{A^\eta}$ which is an equivalence in low degrees. The remaining work then is to find further square-zero extensions of A^η that successively correct the difference in higher degrees. This process will allow us to lift the structure sheaf of the classical part step by step to a structure sheaf of derived rings which has the right cotangent complex. The main advantage of this approach is that it is global from the start, thus avoiding all gluing issues. The practical value of such a result is small though. Given a moduli problem equipped with an obstruction theory it is far better to find the appropriate derived formulation of the moduli problem, as the true derived moduli space contains much more information than just the induced perfect obstruction theory on the truncation. As a derived extension of a moduli space is highly non-unique, the construction presented here will seldom give the correct derived moduli space.

Conventions

- Given a stable ∞ -category \mathcal{C} equipped with a t -structure in the sense of Lurie’s treatise on higher algebra [8], an object X of \mathcal{C} is said to be n -connective if $X \in \mathcal{C}_{\geq n}$. A morphism $f: X \rightarrow Y$ is n -connective if its fiber $\text{fib}(f)$ is n -connective.
- Given a commutative ring A we will denote by Mod_A the ∞ -category of A -module spectra. This category contains the category of ordinary A -modules as heart of a t -structure. Roughly, objects of Mod_A consist of possibly unbounded chain complexes of ordinary A -modules.

Acknowledgments

I would like to thank Manfred Lehn, Marc Nieper-Wißkirchen and Gabriele Vezzosi for countless helpful discussions on the subject, Moritz Groth, Parker Lowrey and David Carchedi for many explanations about ∞ -categories, and the referee for his/her suggestions. Finally I thank Barbara Fantechi for emphasizing the benefits of global constructions. The idea to put on the derived structure globally step by step is by her. The dependence on Lurie’s volumes is obvious from the number of citations.

This work was supported by the SFB/TR 45 ‘Periods, Moduli Spaces and Arithmetic of Algebraic Varieties’ of the DFG (German Research Foundation).

2 The algebraic case

In this section we first treat the problem of finding derived structures inducing an obstruction theory in an abstract setting. The abstract setting will be given by a stable

symmetric monoidal ∞ -category \mathcal{C} equipped with a t -structure satisfying some assumptions (Assumption 2.1). After reviewing some results on the cotangent complex of a commutative algebra object in such a category, we then define the notion of an n -connective obstruction theory on a commutative algebra object $A \in \text{CAlg}(\mathcal{C})$ (Definition 2.4). We then define how a morphism $f: B \rightarrow A$ can induce a given obstruction theory. As main result we prove that for any given commutative algebra object A with fixed obstruction theory there always exists a morphism $f: B \rightarrow A$ inducing the obstruction theory.

The main example for \mathcal{C} we have in mind is the ∞ -category of k -module spectra where k is a commutative ring containing \mathbb{Q} (Example 2.19). Concrete models for connective commutative algebra objects in this category are given by simplicial k -algebras (Example 2.20) or by connective commutative differential graded algebras over k (Example 2.21). In this example we also show a finiteness result for any commutative algebra object B inducing an obstruction theory on a finitely presented discrete commutative k -algebra in case the n -connective obstruction theory is perfect.

2.1 Background

Throughout this paper, the following assumption will be made with regard to the ∞ -category \mathcal{C} in question.

Assumption 2.1 Let \mathcal{C} be a symmetric monoidal stable ∞ -category equipped with a t -structure satisfying the following assumptions [8, Construction 8.4.3.9]:

- (i) The ∞ -category \mathcal{C} is presentable.
- (ii) The tensor product $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ preserves small colimits separately in each variable.
- (iii) The full subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ contains the unit object and is closed under tensor products.

Denote by $\text{CAlg}(\mathcal{C})$ the category of commutative algebra objects in \mathcal{C} . We will make constant use of several fundamental facts proven in [8]. The first concerns the existence of a cotangent complex in such a situation. Lurie proves that in this generality for every commutative algebra object $A \in \text{CAlg}(\mathcal{C})$ there exists a cotangent complex L_A which is an A -module [8, Theorem 8.3.4.18]. Note that in the case where A is an \mathbb{E}_∞ -ring, the homotopy groups of L_A are the topological André–Quillen homology groups of A , and in characteristic different from zero these do not have to coincide with classical André–Quillen homology groups. The second concerns the question what the cotangent

complex classifies. It turns out that maps from the cotangent complex L_A of an object $A \in \text{CAlg}(\mathcal{C})$ to an A -module $M[1]$ correspond to square-zero extensions of A with fiber M . To make a precise statement we recall the following definitions from [8, Section 8.4].

The ∞ -category of *derivations* in $\text{CAlg}(\mathcal{C})$ consists of pairs $(A, \eta: L_A \rightarrow M[1])$ where L_A is the cotangent complex of A and M is an A -module. This category will be denoted by $\text{Der}(\text{CAlg}(\mathcal{C}))$. We can now impose connectivity assumptions on A and the module M . Let $\text{Der}_{n\text{-con}}(\text{CAlg}(\mathcal{C}))$ be the full subcategory of n -connective derivations, defined by the conditions that $A \in \mathcal{C}_{\geq 0}$ and $M \in \mathcal{C}_{\geq n}$. Imposing even stricter conditions, let $\text{Der}_{n\text{-sm}}(\text{CAlg}(\mathcal{C}))$ be the full subcategory of n -small derivations of $\text{Der}_{n\text{-con}}(\text{CAlg}(\mathcal{C}))$ spanned by those objects such that $M \in \mathcal{C}_{\leq 2n}$.

To each derivation we can associate a *square-zero extension*. This associates to a derivation $(A, \eta: L_A \rightarrow M[1])$ a morphism $A^\eta \rightarrow A$ of objects in $\text{CAlg}(\mathcal{C})$. The fiber of $A^\eta \rightarrow A$ can be identified as an A^η -module with M . More generally, we say that a morphism $\tilde{A} \rightarrow A$ is a square-zero extension if there exists a derivation $(A, L_A \rightarrow M[1])$ and an equivalence $\tilde{A} \simeq A^\eta$. As above, we can impose connectivity assumptions on square-zero extensions. A morphism $f: A \rightarrow B$ in $\text{CAlg}(\mathcal{C})$ is an n -connective extension if $A \in \mathcal{C}_{\geq 0}$ and $\text{fib}(f) \in \mathcal{C}_{\geq n}$. Again imposing further connectivity assumptions, we call an extension n -small if $\text{fib}(f) \in \mathcal{C}_{\leq 2n}$ and the multiplication map $\text{fib}(f) \otimes_A \text{fib}(f) \rightarrow \text{fib}(f)$ is nullhomotopic. Denote by $\text{Fun}_{n\text{-con}}(\Delta^1, \text{CAlg}(\mathcal{C}))$ the full subcategory of the category of morphisms $\text{Fun}(\Delta^1, \text{CAlg}(\mathcal{C}))$ spanned by the n -connective extensions, and by $\text{Fun}_{n\text{-sm}}(\Delta^1, \text{CAlg}(\mathcal{C}))$ the full subcategory spanned by the n -small extensions.

The process described above in fact defines a functor of ∞ -categories

$$\Phi: \text{Der}(\text{CAlg}(\mathcal{C})) \longrightarrow \text{Fun}(\Delta^1, \text{CAlg}(\mathcal{C}))$$

given on objects by $(A, \eta: L_A \rightarrow M[1]) \mapsto (A^\eta \rightarrow A)$. This functor has a left adjoint

$$\Psi: \text{Fun}(\Delta^1, \text{CAlg}(\mathcal{C})) \longrightarrow \text{Der}(\text{CAlg}(\mathcal{C}))$$

given on objects by $(\tilde{A} \rightarrow A) \mapsto (A, d: L_A \rightarrow L_{A/\tilde{A}})$.

Lurie proves that this adjunction restricts to subcategories with the appropriate connectivity assumptions and gives an equivalence of categories.

Theorem 2.2 [8, Theorem 8.4.1.26] *Let \mathcal{C} be as above. Then*

$$\Phi_{n\text{-sm}}: \text{Der}_{n\text{-sm}} \longrightarrow \text{Fun}_{n\text{-sm}}(\Delta^1, \text{CAlg}(\mathcal{C}))$$

is an equivalence of ∞ -categories.

The third fundamental fact we will use concerns the connectivity of the cotangent complex. In short, the cotangent complex of a highly connected morphism is again highly connected. The precise statement is the following:

Theorem 2.3 [8, Theorem 8.4.3.11] *Let \mathcal{C} be an ∞ -category as above. Let $f: A \rightarrow B$ be a morphism of objects of $\mathrm{CAlg}(\mathcal{C})$ such that both A and $B \in \mathcal{C}_{\geq 0}$. Assume that $\mathrm{cofib}(f) \in \mathcal{C}_{\geq n}$. Then there exists a canonical morphism $\epsilon_f: B \otimes_A \mathrm{cofib}(f) \rightarrow L_{B/A}$, and furthermore $\mathrm{fib}(\epsilon_f) \in \mathcal{C}_{\geq 2n}$.*

In the special case of square-zero extension $f: A^\eta \rightarrow A$ with cofiber $M[1]$, the map ϵ_f allows us to compare $M[1]$ with the relative cotangent complex L_{A/A^η} .

2.2 The construction

We begin by giving the definition of an obstruction theory in this abstract setting.

Definition 2.4 Let $A \in \mathcal{C}_{\geq 0}$ be a commutative algebra object. An n -connective obstruction theory for A is a morphism

$$\phi: E \longrightarrow L_A$$

of connective A -modules such that $\mathrm{cofib}(\phi) \in \mathcal{C}_{\geq n+1}$.

Remark 2.5 Let A be a discrete object of $\mathrm{CAlg}(\mathcal{C})$ equipped with a 1-connective obstruction theory $\phi: E \rightarrow L_A$. The condition $\mathrm{cofib}(\phi) \in \mathcal{C}_{\geq 2}$ is equivalent to $\pi_0\phi$ being an isomorphism and $\pi_1\phi$ being surjective, thus recovering the definition of [1].

Remark 2.6 The datum of an n -connective obstruction theory for connective $A \in \mathrm{CAlg}(\mathcal{C})$ is equivalent to giving an n -connective derivation of A . To see this, simply complete $\phi: E \rightarrow L_A$ to a cofiber sequence

$$E \xrightarrow{\phi} L_A \xrightarrow{\eta} K.$$

By definition, $\eta: L_A \rightarrow K$ is an n -connective derivation.

Definition 2.7 Let $n \geq 1$ and let $(A, \phi: E \rightarrow L_A)$ be an n -connective obstruction theory, and $(A, \eta: L_A \rightarrow K)$ the associated n -connective derivation. We say that a pair

$$(f: B \rightarrow A, \tilde{\delta}: K \rightarrow L_{A/B})$$

induces the obstruction theory if

- (i) $\tau_{\leq n-1} f: \tau_{\leq n-1} B \rightarrow \tau_{\leq n-1} A$ is an equivalence,
- (ii) the diagram

$$\begin{array}{ccc}
 L_A & \xrightarrow{\phi} & K \\
 d \downarrow & \searrow \tilde{\delta} & \\
 L_{A/B} & &
 \end{array}$$

commutes and $\tilde{\delta}$ is an equivalence.

Remark 2.8 Let $(A, \phi: E \rightarrow L_A)$ be an n -connective obstruction theory, and assume that $(f: B \rightarrow A, \tilde{\delta}: K \rightarrow L_{A/B})$ induces the obstruction theory. This induces an equivalence

$$\tilde{\phi}: E \rightarrow A \otimes_B L_B.$$

Example 2.9 Let $A \in \text{CAlg}(\mathcal{C})$. Then $\pi_0 A$ can be equipped with a canonical 1-connective obstruction theory. The obstruction theory is given by

$$\phi: \pi_0 A \otimes_A L_A \longrightarrow L_{\pi_0 A}.$$

It immediately follows from Theorem 2.3 that the cofiber of ϕ is in $\mathcal{C}_{\geq 2}$. This obstruction theory is trivially induced by $(A \rightarrow \pi_0 A, \text{id}: L_{\pi_0 A/A} \rightarrow L_{\pi_0 A/A})$. More generally, A induces an $n + 1$ -connective obstruction theory on all truncations $A \rightarrow \tau_{\leq n} A$.

Starting from the data of an n -connective obstruction theory $(A, \phi: E \rightarrow L_A)$, or equivalently, an n -connective derivation $(A, \eta: L_A \rightarrow K)$, we now want to begin with explicitly constructing a pair $(f: B \rightarrow A, \tilde{\delta}: K \rightarrow L_{B/A})$ inducing the n -connective derivation. We will construct B as an increasingly connective tower of square-zero extensions of A . We begin with a simple result on the connectivity of square-zero extensions.

Lemma 2.10 *Let $(A, \eta: L_A \rightarrow M[1])$ be an n -connective derivation. Then the square-zero extension $A^\eta \rightarrow A$ is n -connective.*

Proof We have a fiber sequence of A^η -modules $M \rightarrow A^\eta \rightarrow A$. Since $M \in \mathcal{C}_{\geq n}$ by assumption, the claim follows. □

We now introduce the key technical tool. We have seen that given an n -connective derivation $(A, \eta: L_A \rightarrow M[1])$ there exists an associated n -connective square-zero extension $f: A^\eta \rightarrow A$. We now want to study how the relative cotangent complex

L_{A/A^n} compares to the module $M[1]$. In the following we will construct a morphism δ_f that compares the two. This δ_f -map is a slight refinement of the map ϵ_f of Theorem 2.3, and it can also be directly deduced from ϵ_f .

Recall that we have an adjunction $\Phi \rightleftarrows \Psi$ between the categories of extensions and derivations. Let v be the co-unit of this adjunction. By definition, on a derivation $(A, \eta: L_A \rightarrow M[1])$ with corresponding extension $f: A^\eta \rightarrow A$ the co-unit v is given by

$$(A, \eta: L_A \rightarrow M[1]) \mapsto (A, d: L_A \rightarrow L_{A/A^n}).$$

In particular, we obtain the following diagram in the category Mod_A :

$$(1) \quad \begin{array}{ccc} L_A & \xrightarrow{\eta} & M[1] \\ & \searrow d & \downarrow \delta_f \\ & & L_{A/A^n} \end{array}$$

Definition 2.11 Let $(A, \eta: L_A \rightarrow M[1])$ be a derivation. Let δ_f be the morphism defined by the co-unit v of the adjunction $\Phi \rightleftarrows \Psi$ as in (1).

Given an object $A \in \text{CAlg}(C)$ with obstruction theory $\phi: E \rightarrow L_A$ and associated derivation $\eta: L_A \rightarrow K$ the morphism δ_f fits into the fundamental diagram of cofiber sequences:

$$\begin{array}{ccccc} E & \xrightarrow{\phi'} & A \otimes_{A^n} L_{A^n} & \xrightarrow{\eta'} & \text{fib}(\delta_f) \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{\phi} & L_A & \xrightarrow{\eta} & K \\ \downarrow & & \downarrow d & & \downarrow \delta_f \\ 0 & \longrightarrow & L_{A/A^n} & \longrightarrow & L_{A/A^n} \end{array}$$

We next prove a connectivity estimate for δ_f analogous to the connectivity estimate of Theorem 2.3 for ϵ_f . This result again could also be easily deduced from the result for ϵ_f .

Proposition 2.12 Let $(A, \eta: L_A \rightarrow M[1])$ be an n -connective derivation, and let $f: A^\eta \rightarrow A$ be the corresponding square-zero extension. Then $\text{fib}(\delta_f) \in \mathcal{C}_{\geq 2n+2}$, where $\delta_f: M[1] \rightarrow L_{A/A^n}$ is the canonical morphism.

Proof We have to show that $\tau_{\leq 2n+1} \delta_f$ is an equivalence. But $\tau_{\leq 2n+1} \delta_f$ is the co-unit of the adjunction $\Phi_{n-\text{sm}} \rightleftarrows \Psi_{n-\text{sm}}$, which is an equivalence. \square

Corollary 2.13 *Let $(A, \eta: L_A \rightarrow M[1])$ be an n –connective obstruction theory, and let $f: A^\eta \rightarrow A$ be the associated square-zero extension. Then there exists a canonical $2n + 1$ –connective derivation on A^η .*

Proof Define $\theta: L_{A^\eta} \rightarrow \text{fib}(\delta_f)$ to be the adjoint map to

$$A \otimes_{A^\eta} L_{A^\eta} \rightarrow \text{fib}(\delta_f). \quad \square$$

Remark 2.14 In the same vein, assume again that $(A, \eta: L_A \rightarrow M[1])$ is an n –connective derivation and $f: B \rightarrow A \in \text{CAlg}(\mathcal{C})$ such that there is a $2n + 2$ –connective map

$$\delta: M[1] \rightarrow L_{A/B}.$$

Then there is a canonical square-zero extension of $B^\eta \rightarrow B$ defined by the $2n + 1$ –connective derivation

$$(B, \eta: L_B \rightarrow \text{fib}(\delta)).$$

Starting from an n –connective derivation $(A, \eta: L_A \rightarrow M[1])$, using Corollary 2.13 we have a series of square-zero extensions

$$A^\theta \longrightarrow A^\eta \longrightarrow A.$$

We again want to compare the relative cotangent complex L_{A/A^θ} to the module $M[1]$ we began with. For later applications we will be studying the slightly more general situation of Remark 2.14.

Lemma 2.15 *Let $n \geq 1$, and let $(A, \eta: L_A \rightarrow M[1])$ be an n –connective derivation. Assume further we have an n –connective square-zero extension $f: B \rightarrow A \in \text{CAlg}(\mathcal{C})$ and a $2(n + 1)$ –connective map $\delta: M[1] \rightarrow L_{A/B}$. Let $g: B^\eta \rightarrow B$ be the associated square extension as in Remark 2.14. Then there is a canonical $4(n + 1)$ –connective map*

$$\delta_{gf}: M[1] \rightarrow L_{A/B^\eta}.$$

Proof We have the cofiber sequences

$$M[1] \xrightarrow{\delta} L_{A/B} \rightarrow \text{cofib}(\delta)$$

and, coming from the composition $B^\eta \xrightarrow{g} B \xrightarrow{f} A$,

$$L_{A/B^\eta} \rightarrow L_{A/B} \rightarrow A \otimes_B L_{B/B^\eta}[1].$$

Since $f: B \rightarrow A$ is a square-zero extension and $\text{fib}(\delta)$ has the structure of an A -module, the map $\text{id}_A \otimes_B \delta_g: A \otimes_B \text{fib}(\delta) \rightarrow A \otimes_B L_{B/B^n}$ factors as

$$A \otimes_B \text{fib}(\delta) \xrightarrow{\alpha} \text{fib}(\delta) \xrightarrow{\delta'} A \otimes_B L_{B/B^n}$$

in Mod_B , where α is obtained by adjunction. Using $n \geq 1$ and since α is $n \cdot (2n + 1) + 1$ -connective and $\text{id}_A \otimes_B \delta_g$ is $4(n + 1)$ -connective, δ' is $4(n + 1)$ -connective. Using $\delta'[1]$, it follows that $\delta: M[1] \rightarrow L_{A/B}$ factors over $L_{A/B}$. Let δ_{gf} be the induced map. The connectivity statement follows by identifying $\text{cofib}(\delta_{gf})$ with $\text{fib}(\delta'[1])$. \square

Finally we need a result allowing us to compute the cotangent complex of an increasingly connected tower of square-zero extensions.

Lemma 2.16 *Let*

$$A_0 \xleftarrow{f_1} A_1 \xleftarrow{f_2} A_2 \xleftarrow{f_3} \dots$$

be a sequence of square-zero extensions where f_n is n -connective. Let B be the inverse limit $\lim\{A_n\}$. Then

$$L_B \simeq \lim\{L_{A_n}\}.$$

Proof Passing to the Postnikov decomposition of B we have a sequence of equivalences:

$$\begin{array}{ccccccc} \tau_{\leq 0} B & \longleftarrow & \tau_{\leq 1} B & \longleftarrow & \tau_{\leq 2} B & \longleftarrow & \dots \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ \tau_{\leq 0} A_0 & \longleftarrow & \tau_{\leq 1} A_1 & \longleftarrow & \tau_{\leq 2} A_2 & \longleftarrow & \dots \end{array}$$

This induces equivalences on the Postnikov decomposition of L_B

$$\begin{array}{ccccccc} \tau_{\leq 0} L_B & \longleftarrow & \tau_{\leq 1} L_B & \longleftarrow & \tau_{\leq 2} L_B & \longleftarrow & \dots \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ \tau_{\leq 0} L_{A_0} & \longleftarrow & \tau_{\leq 1} L_{A_1} & \longleftarrow & \tau_{\leq 2} L_{A_2} & \longleftarrow & \dots \end{array}$$

and the claim follows. \square

We now have all tools to prove the main result. We will use the previous results to give a tower of increasingly connected square-zero extensions $A_{m+1} \rightarrow A_m$. In every step we will measure the difference between L_{A/A_m} and K using the maps defined in Lemma 2.15. Using Remark 2.14 we can construct a square-zero extension $A_{m+1} \rightarrow A_m$ correcting the defect in some degrees. In every step the degrees in which a defect still exists will be pushed up by a factor of 2.

Theorem 2.17 *Let \mathcal{C} be an ∞ –category as in Assumption 2.1, and let $A \in \text{CAlg}(\mathcal{C})$ be a connective commutative algebra object. Assume that $(A, \phi: E \rightarrow L_A)$ is an n –connective obstruction theory with $n \geq 1$, and let $\text{cofib}(\phi) = K$. Then there exists a pair*

$$(f: B \rightarrow A, \tilde{\delta}: K \rightarrow L_{A/B})$$

inducing the obstruction theory.

Proof Let $A = A_0$, and let $\eta_0: L_{A_0} \rightarrow K$ be the n –connective derivation associated to the obstruction theory. We inductively define a tower of increasingly connected square-zero extensions $f_{m+1,m}: A_{m+1} \rightarrow A_m$, where $f_{m+1,m}$ is $2^m(n+1) - 1$ –connective, along with $2^m(n+1)$ –connective maps $\delta_m: K \rightarrow L_{A_0/A_{m+1}}$. The first step of the induction is given by setting $f_{1,0}: A_1 \rightarrow A_0$ to be the square-zero extension $A^\eta \rightarrow A$. By Lemma 2.10, the connectivity assumption on $f_{1,0}$ follows. Finally, define $\delta_1 = \delta_{f_{1,0}}$.

Now assume that $f_{m,m-1}: A_m \rightarrow A_{m-1}$ and $\delta_m: K \rightarrow L_{A_0/A_m}$ have already been constructed. Denote by f_m the composition $f_{1,0} \circ \dots \circ f_{m,m-1}$. By applying Remark 2.14 to

$$f_m: A_m \rightarrow A, \quad \delta_m: K \rightarrow L_{A_0/A_m}$$

we obtain a square zero extension $f_{m+1,m}: A_{m+1} \rightarrow A_m$ which is $2 \cdot 2^m(n+1) - 1$ –connective. Finally, define δ_{m+1} to be the $2 \cdot 2^m(n+1)$ –connective map $\delta_{f_{m+1,m}, f_m}$ of Lemma 2.15.

Now define B to be the inverse limit $\lim\{A_m\}$. Using the maps δ_n we have a series of maps

$$\begin{array}{ccccccc}
 & & & & & & K \\
 & & & & & & \downarrow \delta_1 \\
 & & & & & \delta_3 & \swarrow \delta_2 \\
 \dots & \longrightarrow & L_{A_0/A_3} & \longrightarrow & L_{A_0/A_2} & \longrightarrow & L_{A_0/A_1}
 \end{array}$$

where δ_m is $2^m(n+1)$ –connective. Passing to the limit and using Lemma 2.16, we have an equivalence $\tilde{\delta}: K \rightarrow \lim L_{A/A_n} \simeq L_{A/B}$. □

We can now apply this result in some examples.

Example 2.18 Let $\mathcal{C} = \text{Sp}$ be the ∞ –category of spectra. An object of $\text{CAlg}(\text{Sp})$ then is an \mathbb{E}_∞ –ring. Discrete objects of $\text{CAlg}(\text{Sp})$ can be identified with ordinary commutative rings. Applying the above theorem, it follows that every 1–connective obstruction theory for a commutative ring A is induced by some \mathbb{E}_∞ –ring B with $\pi_0 B = A$.

Example 2.19 Let k be a connective \mathbb{E}_∞ -ring and let \mathcal{C} denote the category $\text{Mod}_k(\text{Sp})$ of k -module spectra. Define CAlg_k to be $\text{CAlg}(\text{Mod}_k(\text{Sp}))$. A discrete object of CAlg_k is an ordinary commutative algebra over $\pi_0 k$. By the above theorem every 1-connective obstruction theory for a commutative $\pi_0 k$ -algebra A is induced by an \mathbb{E}_∞ -algebra B over k such that $\pi_0 B = A$.²

Example 2.20 In the previous example, let k be an ordinary commutative ring containing the rationals \mathbb{Q} viewed as a discrete \mathbb{E}_∞ -ring. Then we can identify connective objects of CAlg_k with the nerve of the category of simplicial commutative k -algebras [8, Proposition 8.1.4.20]. Applying the above theorem, every 1-connective obstruction theory for a commutative k -algebra is induced by a simplicial commutative k -algebra.

Example 2.21 Taking k as in the previous example, a further explicit model for the ∞ -category CAlg_k is given by commutative differential graded algebras over k [8, Proposition 8.1.4.11]. The above theorem thus shows that every 1-connective obstruction theory on a commutative k -algebra is induced by a connective commutative differential graded algebra.

Remark 2.22 An important case not covered by Theorem 2.17 is simplicial algebras over k where k is an ordinary commutative ring not necessarily containing \mathbb{Q} . This case is important since it is a homotopical algebra context in the sense of [11] and leads to derived algebraic geometry over any base ring. This case is not covered by Theorem 2.17 since simplicial algebras over k no longer provide a model for \mathbb{E}_∞ -algebras over k if k does not contain \mathbb{Q} . Thus it is not directly possible to apply the formalism developed in [8] to this case.

Nevertheless, it should be possible to extend all results to this case. The main tool we have used is comparing the target module $M[1]$ in a derivation $(A, \eta: L_A \rightarrow M[1])$ to the relative cotangent complex L_{A/A^n} of the corresponding square-zero extension $f: A^n \rightarrow A$ via the map δ_f . This map was defined via the adjunction between derivations and square-zero extensions, and its connectivity properties were deduced from Theorem 2.2. Proving the analogous results in the context of simplicial algebras over any base k would provide all necessary tools to carry out the proof. Several results in this direction can be found in [11]. There the relative cotangent complex L_{A/A^n} is explicitly computed [11, Lemma 1.4.3.7] and is shown to have the same homotopy groups as $M[1]$ up to degree $n + 1$ for an n -connective module M [11, Lemma 2.2.2.7]. The main difficulty lies in extending the connectivity estimate to the estimate of Proposition 2.12.

²The category CAlg_k can be identified with the under-category $\text{CAlg}(\text{Sp})_{k/}$. Under this identification the cotangent complex associated to the category CAlg_k can be identified with the relative cotangent complex $L_{A/k}$.

In Example 2.19 one will typically impose some finiteness property on both the $\pi_0 k$ -algebra A and the obstruction theory $\phi: E \rightarrow L_{A/k}$. We recall the relevant definition.

Definition 2.23 Let k be a connective \mathbb{E}_∞ -ring, and let A be a finitely presented discrete commutative $\pi_0 k$ -algebra equipped with a 1-connective obstruction theory $\phi: E \rightarrow L_A$. Let $m \geq 0$.

- (i) The obstruction theory is an m -obstruction theory if $E \in (\text{Mod}_A)_{\leq m}$.
- (ii) The obstruction theory is perfect if E is a perfect A -module.

We next want to ensure that if we start with a finitely presented commutative $\pi_0 k$ -algebra equipped with a 1-connective perfect m -obstruction theory, any k -algebra inducing the obstruction theory satisfies a strong finiteness property. We briefly recall the relevant finiteness property following [8, Definition 8.2.5.26].

Definition 2.24 Let k be a connective \mathbb{E}_∞ -ring, and let $\text{CAlg}_k = \text{CAlg}(\text{Mod}_k(\text{Sp}))$. A commutative k -algebra B is *locally of finite presentation over k* if B is a compact object of CAlg_k .

Note that a finitely presented discrete commutative $\pi_0 k$ -algebra viewed as an object in CAlg_k will usually not satisfy the above finiteness property. As an example one can take any finitely presented discrete commutative $\pi_0 k$ -algebra with non-perfect cotangent complex, as in light of [8, Theorem 8.4.3.17] perfectness of the cotangent complex is necessary for being locally of finite presentation in CAlg_k . Nevertheless, we will later see that given a finitely presented discrete commutative $\pi_0 k$ -algebra A equipped with an n -connective perfect m -obstruction theory, any commutative k -algebra B inducing the obstruction theory does satisfy the above finiteness property, although A itself will usually not.

Remark 2.25 Note that in [11, Definition 1.2.3.1] slightly different terminology is used. There a compact object of CAlg_k is called *finitely presented*, whereas Lurie reserves finitely presented for algebras which lie in the smallest full subcategory which contains finitely generated free algebras and is stable under retracts and finite colimits.

Before we begin we need a series of lemmas. These lemmas will allow us to deduce properties of an A -module M from the respective properties of the $\pi_0 A$ -module $\pi_0 A \otimes_A M$. Recall the following definitions for a connective \mathbb{E}_∞ -ring A and an A -module M :

(i) M is of Tor–amplitude at most n if for any discrete A –module N we have

$$\pi_i(N \otimes_A M) = 0 \quad \text{for } i > n.$$

(ii) M is perfect to order n if $\tau_{\leq n} M$ is a compact object of $(\text{Mod}_A)_{\leq n}$.

(iii) M is almost perfect if it is perfect to order n for all integers n .

Lemma 2.26 *Let A be a connective \mathbb{E}_∞ –ring and M a connective A –module. If $\pi_0 A \otimes_A M$ has Tor–amplitude at most n as $\pi_0 A$ –module, then M has Tor–amplitude at most n as A –module.*

Proof Let N be a discrete A –module. In particular, N is a $\pi_0 A$ –module. The claim then follows from

$$\pi_i(N \otimes_A M) = \pi_i(N \otimes_{\pi_0 A} \pi_0 A \otimes_A M) = \text{Tor}_i^{\pi_0 A}(N, \pi_0 A \otimes_A M). \quad \square$$

Lemma 2.27 *Let A be a connective \mathbb{E}_∞ –ring, M a connective A –module, and $n \geq 0$. If $\pi_0 A \otimes_A M$ is perfect to order n as $\pi_0 A$ –module, then M is perfect to order n as A –module.*

Proof We prove the claim by induction over n . For the case of $n = 0$ recall that M is perfect to order 0 as A –module if and only if $\pi_0 M$ is finitely generated as a module over $\pi_0 A$. The claim then follows from

$$\pi_0 M = \pi_0 A \otimes_{\pi_0 A} \pi_0 M = \pi_0(\pi_0 A \otimes_A M).$$

Now let $n > 0$. Recall that given a map of A –modules $\phi: A^k \rightarrow M$ which induces a surjection $\pi_0 A^k \rightarrow \pi_0 M$, then M is perfect to order n if and only if $\text{fib}(\phi)$ is perfect to order $n - 1$ [6, Proposition 2.6.12]. The argument now follows [6, Proposition 2.6.13]. As $\pi_0 M$ is finitely generated, we can choose a fiber sequence of connective A –modules

$$M' \longrightarrow A^k \longrightarrow M.$$

Tensoring with $\pi_0 A$, we obtain a fiber sequence of $\pi_0 A$ –modules

$$\pi_0 A \otimes_A M' \longrightarrow \pi_0 A^k \longrightarrow \pi_0 A \otimes_A M.$$

By assumption, $\pi_0 A \otimes_A M$ is perfect to order n as $\pi_0 A$ –module, so $\pi_0 A \otimes_A M'$ is perfect to order $n - 1$ as $\pi_0 A$ –module. By the inductive hypothesis M' is perfect to order $n - 1$ as A –module, and thus M is perfect to order n as A –module. \square

Corollary 2.28 *Let A be a connective \mathbb{E}_∞ –ring and M a connective A –module. If $\pi_0 A \otimes_A M$ is almost perfect and of finite Tor–amplitude as $\pi_0 A$ –module, then M is perfect as A –module.*

Proof By Lemma 2.26, M is of finite Tor–amplitude. By Lemma 2.27, M is perfect to order n for all n and thus almost perfect. Now M being almost perfect and of finite Tor–amplitude imply that M is perfect. \square

We can now prove the finiteness result alluded to above.

Proposition 2.29 *Let k be a connective \mathbb{E}_∞ –ring, and let $\text{CAlg}_k = \text{CAlg}(\text{Mod}_k(\text{Sp}))$. Let A be a discrete object of CAlg_k such that A is finitely presented as $\pi_0 k$ –algebra. Assume that A is equipped with an n –connective perfect m –obstruction theory $\phi: E \rightarrow L_{A/k}$. Then in any pair $(f: B \rightarrow A, \tilde{\delta}: L_{A/B} \rightarrow K)$ inducing the obstruction theory, the object B of CAlg_k is locally of finite presentation and $L_{B/k}$ is of Tor–amplitude at most m .*

Proof Using the equivalence $\tilde{\phi}: E \rightarrow A \otimes_B L_{B/k} = \pi_0 B \otimes_B L_{B/k}$ obtained from $\tilde{\delta}$ (see Remark 2.8) it follows that $\pi_0 B \otimes_B L_{B/k}$ is perfect and of Tor–amplitude at most m . In particular, $\pi_0 B \otimes_B L_{B/k}$ is almost perfect and of finite Tor–amplitude at most m . By the previous corollary, $L_{A/k}$ is perfect and of Tor–amplitude at most m . Now $\pi_0 B$ being of finite presentation over $\pi_0 k$ and $L_{B/k}$ being perfect imply that B is locally of finite presentation over k [8, Theorem 8.4.3.17]. \square

3 The geometric case

In this section we want to apply the above results in the setting of spectral Deligne–Mumford stacks. We first recall some of the definitions we will use.

3.1 Background

In [7] Lurie defines a ∞ –category $\text{Sch}(\mathcal{G}_{\text{ét}}^{\text{Sp}})$ of connective spectral Deligne–Mumford stacks. An object \mathfrak{X} of this category is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ consisting of an ∞ –topos \mathcal{X} and a sheaf of \mathbb{E}_∞ –rings satisfying further conditions. There also is a relative version $\text{Sch}(\mathcal{G}_{\text{ét}}^{\text{Sp}}(k))$ of connective spectral Deligne–Mumford stacks over a connective \mathbb{E}_∞ –ring k . Here the structure sheaf takes its values in the category CAlg_k of \mathbb{E}_∞ –rings over k . This category can in fact be identified with the category of $\mathcal{G}_{\text{ét}}^{\text{Sp}}$ –schemes equipped with a morphism to $\text{Spec}(k)$.³

A key property of this category is that ordinary Deligne–Mumford stacks over the discrete commutative ring $\pi_0 k$ sit inside $\text{Sch}(\mathcal{G}_{\text{ét}}^{\text{Sp}}(k))$ as the full subcategory spanned by the 0–truncated and 1–localic $\mathcal{G}_{\text{ét}}^{\text{Sp}}(k)$ –schemes.

³In Lurie’s notation, $\text{Spec}(k)$ would be $\text{Spec}^{\text{ét}}(k)$. As we will only encounter this Spec –functor omitting the superscript hopefully does not lead to confusion.

The theory of spectral connective Deligne–Mumford stacks over k is compatible with n –truncations in the sense that for every such stack \mathfrak{X} its n –truncation $\tau_{\leq n}\mathfrak{X} = (\mathcal{X}, \tau_{\leq n}\mathcal{O}_{\mathcal{X}})$ is again a spectral connective Deligne–Mumford stack over k . In particular, given a 1–localic connective spectral Deligne–Mumford stack, its 0–truncation $\tau_{\leq 0}\mathfrak{X} = (\mathcal{X}, \pi_0\mathcal{O}_{\mathcal{X}})$ is an ordinary Deligne–Mumford stack over π_0k with the same underlying ∞ –topos as \mathfrak{X} . Furthermore, we have a canonical morphism $\tau_{\leq 0}\mathfrak{X} \rightarrow \mathfrak{X}$.

Finally, recall that given a connective spectral Deligne–Mumford stack \mathfrak{X} over $\text{Spec } k$ we say that \mathfrak{X} is *locally of finite presentation over $\text{Spec } k$* if it is possible to choose a covering by affine schemes $\text{Spec}(A_\alpha)$ such that each A_α is locally of finite presentation over k , ie, a compact object of CAlg_k . If a connective spectral Deligne–Mumford stack \mathfrak{X} is locally of finite presentation over k and has a cotangent complex of Tor–amplitude at most 1 we say that \mathfrak{X} is *quasi-smooth*.

In the following we will make use of the following identification. Let \mathcal{X} be an ∞ –topos, and take \mathcal{C} to be the ∞ –category of sheaves of spectra $\text{Shv}_{\text{Sp}}(\mathcal{X})$ on \mathcal{X} . This is a symmetric monoidal ∞ –category equipped with a t –structure satisfying Assumption 2.1. Using the equivalence

$$(2) \quad \text{Shv}_{\text{CAlg}}(\mathcal{X}) \simeq \text{CAlg}(\text{Shv}_{\text{Sp}}(\mathcal{X}))$$

we can identify the structure sheaf $\mathcal{O}_{\mathcal{X}}$ of a connective spectral Deligne–Mumford stack with a commutative algebra object in $\text{Shv}_{\text{Sp}}(\mathcal{X})$.

3.2 The construction

We first give the definition of an n –connective obstruction theory in the geometric setting. The only difference to the algebraic case is that we want to assume the module defining the obstruction theory to be quasi-coherent.

Definition 3.1 Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be connective spectral Deligne–Mumford stack, and let $n \geq 1$. An n –connective quasi-coherent obstruction theory for \mathfrak{X} is a morphism

$$\phi: \mathcal{E} \longrightarrow L_{\mathcal{O}_{\mathcal{X}}}$$

of connective quasi-coherent $\mathcal{O}_{\mathcal{X}}$ –modules such that $\text{cofib}(\phi) \in \text{QCoh}(\mathfrak{X})_{\geq n+1}$.

We also have the analogous definition in the relative setting over $\text{Spec}(k)$ using the relative cotangent complex.

Definition 3.2 Let k be a connective \mathbb{E}_∞ –ring, and let \mathfrak{X} be a connective spectral Deligne–Mumford stack over $\text{Spec}(k)$. Let $n \geq 1$.

- (i) An n –connective quasi-coherent obstruction theory for \mathfrak{X} over $\text{Spec}(k)$ is a morphism

$$\phi: \mathcal{E} \longrightarrow L_{\mathcal{O}_{\mathfrak{X}}/k}$$

of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$ –modules such that $\text{cofib}(\phi) \in \text{QCoh}(\mathfrak{X})_{\geq n+1}$.

- (ii) Let \mathfrak{X} be locally of finite presentation over $\text{Spec}(k)$, and let $\phi: \mathcal{E} \rightarrow L_{\mathcal{O}_{\mathfrak{X}}/k}$ be an n –connective obstruction theory. We say that the obstruction theory is *perfect* if \mathcal{E} is perfect. The obstruction theory is an m –obstruction theory if $\mathcal{E} \in \text{QCoh}(\mathfrak{X})_{\geq 0} \cap \text{QCoh}(\mathfrak{X})_{\leq m}$.

In complete analogy to Definition 2.7 we have the notion of an object inducing the obstruction theory.

Definition 3.3 Let $\mathfrak{X}_0 = (\mathcal{X}, \mathcal{O}_{\mathcal{X}_0})$ be a spectral connective Deligne–Mumford stack equipped with an n –connective quasi-coherent obstruction theory $\phi: \mathcal{E} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}}$ with $n \geq 1$. Let $\text{cofib}(\phi) = \mathcal{K}$, and let $\delta: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}}$ be the induced morphism. We say that the pair

$$(i: \mathfrak{X}_0 \rightarrow \mathfrak{X}, \tilde{\delta}: \mathcal{K} \rightarrow L_{\mathfrak{X}_0/\mathfrak{X}})$$

induces the obstruction theory if

- (i) $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a connective spectral Deligne–Mumford stack with the same underlying ∞ –topos as \mathfrak{X}_0 ,
- (ii) $\tau_{\leq n} \mathfrak{X} = \tau_{\leq n} \mathfrak{X}_0$,
- (iii) $\tilde{\delta}: \mathcal{K} \rightarrow L_{\mathfrak{X}_0/\mathfrak{X}}$ is an equivalence of quasi-coherent $\mathcal{O}_{\mathcal{X}_0}$ –modules such that

$$\begin{array}{ccc} L_{\mathcal{O}_{\mathcal{X}_0}} & \xrightarrow{\delta} & \mathcal{K} \\ d \downarrow & \swarrow \tilde{\delta} & \\ L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}} & & \end{array}$$

commutes in $\text{QCoh}(\mathfrak{X}_0)$.

We can now begin to prove geometric versions of our main theorem. Since we want the objects inducing the obstruction theories to be of geometric nature, we have to make sure that in every step of the construction we obtain geometric objects. The key to this is verifying that a square-zero extension of a connective spectral Deligne–Mumford stack by a quasi-coherent sheaf is again a spectral connective Deligne–Mumford stack.

Lemma 3.4 *Let $n \geq 1$, and let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a connective spectral Deligne–Mumford stack. Furthermore, let $\eta: L_{\mathcal{O}_{\mathcal{X}}} \rightarrow \mathcal{M}[1]$ be an n –connective derivation with \mathcal{M} a quasi-coherent $\mathcal{O}_{\mathcal{X}}$ –module. Let $\mathcal{O}_{\mathcal{X}}^{\eta} \rightarrow \mathcal{O}_{\mathcal{X}}$ be the corresponding extension under Theorem 2.2. Then $\mathfrak{X}' = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\eta})$ is a connective spectral Deligne–Mumford stack.*

Proof This is immediate by verifying the conditions of [7, Theorem 8.42]. To verify the first condition, note that by the assumption $n \geq 1$ the 0–truncations of \mathfrak{X}' and \mathfrak{X} are equivalent. □

Remark 3.5 Note that due to the connectivity assumption on \mathcal{M} in the above lemma we have never changed the underlying ∞ –topos, but only have altered the structure sheaf.

Using the algebraic construction theorem proven above, we can now prove a geometric version.

Theorem 3.6 *Let $\mathfrak{X}_0 = (\mathcal{X}, \mathcal{O}_{\mathcal{X}_0})$ be a spectral connective Deligne–Mumford stack equipped with an n –connective quasi-coherent obstruction theory $\phi: \mathcal{E} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}}$ with $n \geq 1$. Let $\mathcal{K} = \text{cofib}(\phi)$. Then there exists a pair*

$$(i: \mathfrak{X}_0 \rightarrow \mathfrak{X}, \tilde{\delta}: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}})$$

inducing the obstruction theory.

Proof Let $\mathcal{C} = \text{Shv}_{\text{Sp}}(\mathcal{X})$, and using (2) identify $\mathcal{O}_{\mathcal{X}_0}$ with an object of $\text{CAlg}(\mathcal{C})$. Applying Theorem 2.17, we obtain a morphism $\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}_0}$ in $\text{CAlg}(\mathcal{C})$ and an equivalence $\tilde{\delta}: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}}$ in $\text{Mod}_{\mathcal{O}_{\mathcal{X}_0}}$. As every step of the construction given in the proof of Theorem 2.17 is a square-zero extension, the pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is indeed a spectral connective Deligne–Mumford stack by Lemma 3.4. In particular, $L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}}$ is quasi-coherent. As $\text{QCoh}(\mathfrak{X}_0)$ is a full subcategory of $\text{Mod}_{\mathcal{O}_{\mathcal{X}_0}}$, the claim follows. □

We now proceed to prove a relative version of Theorem 3.6 over $\text{Spec}(k)$. So let $p: \mathfrak{X}_0 = (\mathcal{X}, \mathcal{O}_{\mathcal{X}_0}) \rightarrow \text{Spec}(k)$ be a morphism of spectral connective Deligne–Mumford stacks. Pulling back the structure sheaf of $\text{Spec}(k)$, we obtain a morphism $p^* \mathcal{O}_{\text{Spec}(k)} \rightarrow \mathcal{O}_{\mathcal{X}_0}$ of connective commutative algebra objects in $\text{Shv}_{\text{Sp}}(\mathcal{X})$. In particular we can view $\mathcal{O}_{\mathcal{X}_0}$ as a connective object of $\text{CAlg}(\text{Mod}_{p^* \mathcal{O}_{\text{Spec}(k)}})$.

Proposition 3.7 *Let k be a connective \mathbb{E}_∞ -ring, and let $\mathfrak{X}_0 = (\mathcal{X}, \mathcal{O}_{\mathcal{X}_0})$ be a spectral connective Deligne–Mumford stack over $\mathrm{Spec}(k)$ equipped with an n -connective quasi-coherent obstruction theory $\phi: \mathcal{E} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/k}$ with $n \geq 1$. Then there exists a pair*

$$(i: \mathfrak{X}_0 \rightarrow \mathfrak{X}, \tilde{\delta}: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}})$$

inducing the obstruction theory, where \mathfrak{X} is a spectral connective Deligne–Mumford stack over $\mathrm{Spec}(k)$.

Proof By applying Theorem 2.17 to the category $\mathrm{CAlg}(\mathrm{Mod}_{p^* \mathcal{O}_{\mathrm{Spec}(k)}})$ we obtain a morphism of commutative algebra objects $\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}_0}$ in $\mathrm{CAlg}(\mathrm{Mod}_{p^* \mathcal{O}_{\mathrm{Spec}(k)}})$ and an equivalence $\tilde{\delta}: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}}$ of $\mathcal{O}_{\mathcal{X}_0}$ -modules. Define $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The remainder of the proof is analogous to the proof of Theorem 3.6. \square

As in Proposition 2.29 we want to ensure certain finiteness properties if we start with an ordinary Deligne–Mumford stack locally of finite presentation over $\pi_0 k$ equipped with an n -connective perfect m -obstruction theory.

Lemma 3.8 *Let \mathfrak{X}_0 be an ordinary Deligne–Mumford stack locally of finite presentation over $\mathrm{Spec}(\pi_0 k)$ equipped with an n -connective perfect m -obstruction theory $\phi: \mathcal{E} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/k}$ with $n \geq 1$. Then in any pair $(i: \mathfrak{X}_0 \rightarrow \mathfrak{X}, \tilde{\delta}: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}})$ inducing the obstruction theory, the connective spectral Deligne–Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is locally of finite presentation over $\mathrm{Spec}(k)$ and the cotangent complex $L_{\mathcal{O}_{\mathcal{X}}/k}$ is of Tor-amplitude at most m .*

Proof As both assertions are local this follows from Proposition 2.29. \square

Corollary 3.9 *If \mathfrak{X}_0 is locally of finite presentation over $\mathrm{Spec}(\pi_0 k)$ equipped with an n -connective perfect 1-obstruction theory for $n \geq 1$, there exists a pair $(i: \mathfrak{X}_0 \rightarrow \mathfrak{X}, \tilde{\delta}: \mathcal{K} \rightarrow L_{\mathcal{O}_{\mathcal{X}_0}/\mathcal{O}_{\mathcal{X}}})$ inducing the obstruction theory where \mathfrak{X} is quasi-smooth.*

Remark 3.10 Assume that we are working over a discrete commutative ring k containing the rationals \mathbb{Q} . Using the equivalence of spectral algebraic geometry and derived algebraic geometry over k [7, Corollary 9.28] the results of Proposition 3.7 and Corollary 3.9 hold in derived algebraic geometry over k .

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Received: 22 March 2012
 Revised: 7 June 2012