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Saturated fusion systems as idempotents in the double Burnside ring

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We give a new characterization of saturated fusion systems on a p–group S in terms of idempotents in the p–local double Burnside ring of S that satisfy a Frobenius reciprocity relation. Interpreting our results in stable homotopy, we characterize the stable summands of the classifying space of a finite p–group that have the homotopy type of the classifying spectrum of a saturated fusion system, and prove an invariant theorem for double Burnside modules analogous to the Adams–Wilkerson criterion for rings of invariants in the cohomology of an elementary abelian p–group. This work is partly motivated by a conjecture of Haynes Miller that proposes p–tract groups as a purely homotopy-theoretical model for p–local finite groups. We show that a p–tract group gives rise to a p–local finite group when two technical assumptions are made, thus reducing the conjecture to proving those two assumptions.

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1 Introduction

Fusion systems are an abstract model for the p–local structure of a finite group. To a finite group G with Sylow p–subgroup S one associates the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are the group homomorphisms induced by conjugation in G and inclusion. Alperin and Broué showed in [4] that a similar structure arises when one considers the G–conjugation among Brauer pairs in a block of defect S in the group algebra of G in characteristic p, and this prompted Puig to give an axiomatic definition for an abstract fusion system. More precisely, a fusion system on a finite p–group S is a category $\mathcal{F}$ whose objects are the subgroups of S, and whose morphism sets model a system of conjugations among subgroups of S induced by the inclusion of S in an ambient group, without reference to the ambient group. (In particular every morphism is a group monomorphism and $\mathcal{F}$ contains all morphisms induced by conjugation in S.)

Among fusion systems, the important ones are the *saturated* fusion systems. Informally, a saturated fusion system on S models a "Sylow inclusion" of S in an ambient object.

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Formally, a fusion system is saturated if its morphism sets satisfy two axioms that mimic the Sylow theorems. One is a "prime to p" axiom, corresponding to the index of a Sylow subgroup being prime to p, and the other is a "maximal extension" axiom, replacing the result that a Sylow subgroup contains all p–subgroups up to conjugacy. Saturated fusion systems are now widely studied. In modular representation theory they are considered a helpful venue in which to reformulate and approach the Alperin weight conjecture; see Linckelmann [26]. Building on work of Martino and Priddy, Broto, Levi and Oliver popularized saturated fusion systems among homotopy theorists as a model for studying the p–completed classifying space of a finite group. Lately saturated fusion system have been embraced by group theorists as a possible framework for one of the masterpieces of modern mathematics: the classification of finite simple groups. Recent work of Aschbacher in [5], [8], [6] and [7], and Aschbacher and Chermak in [9] transports deep group theoretic tools into the world of fusion systems.

Saturated fusion systems were first defined by Puig, who originally called them full Frobenius systems. His definition was simplified by Broto, Levi and Oliver in [16], and we follow their conventions and terminology. Further simplifications of the saturation axioms were made by the second author and Kessar in [24], and by Roberts and Shpectorov in [37]. These simplified (but equivalent) definitions are all of a similar nature: each consists of a prime to p axiom and an extension axiom on the morphism sets, with one or both axioms being weaker than in the definition by Broto, Levi and Oliver. In this paper we give a substantially different characterization of saturated fusion systems. Instead of axioms on morphism sets, we formulate the saturation property for a fusion system on S in the double Burnside ring $A(S,S)$ of left-free (S,S)–bisets.

The idea of relating fusion systems to bisets originates from Linckelmann and Webb. Looking at the $\mathbb{F}_p$–cohomology of fusion systems, they realized that in the case of a Sylow inclusion $S \leq G$, the (S,S)–biset G plays a special role. Cohomology is a Mackey functor, and so an (S,S)–biset induces a map $H^*(BS;\mathbb{F}_p) \to H^*(BS;\mathbb{F}_p)$. The map induced by the (S,S)–biset G is idempotent up to scalar with image isomorphic to $H^*(BG;\mathbb{F}_p)$. Linckelmann and Webb synthesized the properties of the (S,S)–biset G and found the appropriate replacement Ω for an abstract fusion system $\mathcal{F}$ on S that allows one to think of the cohomology of $\mathcal{F}$ as the image of the map $H^*(BS;\mathbb{F}_p) \to H^*(BS;\mathbb{F}_p)$ induced by Ω. To this end they defined a *characteristic biset* Ω for $\mathcal{F}$ to be an (S,S)–biset with augmentation $|S\backslash\Omega|$ prime to p that is $\mathcal{F}$–stable—in the sense that restricting either S–action to a subgroup P via a morphism in $\mathcal{F}$ yields a biset isomorphic to restriction along the inclusion $P \hookrightarrow S$—and also satisfies an additional condition relating the irreducible components of Ω to $\mathcal{F}$ (see Section 4 for a precise definition). More generally a *characteristic element* for $\mathcal{F}$ is an element in the p–localized double Burnside ring $A(S,S)_{(p)}$ that has the Linckelmann–Webb properties.

The existence of characteristic elements for saturated fusion systems was established by Broto, Levi and Oliver in [16, Proposition 5.5]. Characteristic elements are by no means unique. Indeed, a saturated fusion system $\mathcal{F}$ has infinitely many of them. However, the first author showed in [34, Proposition 5.6] that among characteristic elements there is exactly one that is idempotent in the p–local double Burnside ring, and we refer to this as the *characteristic idempotent* of $\mathcal{F}$. A consequence of [34, Proposition 5.2] is that a fusion system can be reconstructed as the stabilizer fusion system of any characteristic element Ω, meaning the largest fusion system $\mathcal{F}$ with respect to which Ω is $\mathcal{F}$–stable. Thus, a characteristic element contains exactly the same information as its fusion system. It is then natural to ask whether only saturated fusion systems admit characteristic elements, and to seek an intrinsic criterion for characteristic elements.

For (S,S)–bisets X and Y, let $(X\times Y)_\Delta$ be the biset with $(S\times S)$ acting coordinatewise on the left, and S acting on the right via the diagonal. We say that X satisfies *Frobenius reciprocity* if there is an isomorphism of bisets

$$(X\times X)_\Delta \cong (X\times 1)_\Delta \circ X, \tag{1}$$

where $-\circ- = -\times_S-$ is the multiplication in the double Burnside ring, and $1=S$ is the unit. More generally, an element in $A(S,S)_{(p)}$ satisfies Frobenius reciprocity if it satisfies (1). Frobenius reciprocity is discussed in more detail in Section 7, and the connection to classical Frobenius reciprocity in group cohomology is explained in Section 9.2.

Our first main result shows that Frobenius reciprocity is equivalent to saturation. To state the theorem we need some technical conditions. An (S,S)–biset X is *bifree* if both the left and right S–actions are free, and a general element in $A(S,S)_{(p)}$ is bifree if it is in the subring generated by bifree bisets. Characteristic elements are always bifree. A *right-characteristic element* for a fusion system $\mathcal{F}$ is one that is stable under restricting the right S–action along morphisms in $\mathcal{F}$, but not necessarily the left action. The *right-stabilizer fusion system* $\mathrm{RSt}(X)$ of $X\in A(S,S)_{(p)}$ is the largest fusion system with respect to which X is right-stable. Furthermore, $\mathrm{RSt}(X)=\mathcal{F}$ when X is right-characteristic for a fusion system $\mathcal{F}$, again by [34, Proposition 5.2].

Theorem A *Let S be a finite p-group. If Ω is a bifree element in $A(S,S)_{(p)}$ with augmentation not divisible by p that satisfies Frobenius reciprocity, then $\mathrm{RSt}(\Omega)$ is a saturated fusion system, and Ω is a right characteristic element for $\mathrm{RSt}(\Omega)$.*

When Ω is idempotent, the bifreeness condition can be replaced by a dominance condition (in the sense of Nishida [30]), and this is done in Corollary 8.9. Although Corollary 8.9 is a little more technical to state, it is often more useful in practice.

Since characteristic elements always satisfy Frobenius reciprocity, this implies in particular that a fusion system with a characteristic element must always be saturated, a result first proved by Puig for characteristic bisets in [33, Proposition 21.9]. In Corollary 6.7 we also give an independent, direct proof of that fact as a consequence of our work in Section 6.

An element in $A(S,S)_{(p)}$ is *symmetric* if it is unchanged under transposing the two S–actions (see Section 3.8). A symmetric right-characteristic element is a bifree characteristic element. Using this observation along with the uniqueness of characteristic idempotents, Theorem A yields a bijection among objects that a priori seem unrelated.

Theorem B *For a finite p–group S, there is a bijective correspondence between saturated fusion systems on S and symmetric idempotents in $A(S,S)_{(p)}$ of augmentation 1 that satisfy Frobenius reciprocity. The bijection sends a saturated fusion system to its characteristic idempotent, and an idempotent to its stabilizer fusion system.*

A fascinating extension of this bijection has now been given by Boltje and Danz in [13, Theorem 7.15], giving a bijection between (not necessarily saturated) fusion systems on S and symmetric idempotents in $\mathbb{Q} \otimes A(S,S)$ of augmentation 1 that satisfy Frobenius reciprocity. In that context, Theorem B says that the saturated fusion systems are precisely the ones whose corresponding idempotents live in $\mathbb{Z}_{(p)} \otimes A(S,S) = A(S,S)_{(p)}$.

Theorems A and B are proved in Section 7. This result gives us a new way to think of saturated fusion systems. Rather than looking at a category of subgroups of S with Sylow-like axioms on the morphism sets, we can encode saturation in the one-line Frobenius reciprocity relation (3). In light of Theorem B it is interesting to see to what extent results on fusion systems can be reformulated in terms of idempotents. The authors have some results in this direction that will appear separately to keep the current paper at a reasonable length. (These were previously contained in a section of a preprint version of the current paper.) Instead we focus on applying Theorems A and B to obtain new results on the stable homotopy theory of classifying spaces of finite groups. Those results are rather technical to state here. We provide highlights of these results in the outline below.

Outline

The paper has three main parts. In the first part, we recall background material and establish notational conventions that will be used throughout the paper. The theory of fusion systems is recalled in Section 2, bisets, the double Burnside ring and fixed-point homomorphisms are covered in Section 3, and Section 4 contains a discussion of characteristic bisets and idempotents.

The second part of the paper deals with the new results on fusion systems. In Section 5 we introduce the notions of stabilizer, fixed-point, and orbit-type fusion systems, list their basic properties and reformulate the Linckelmann–Webb properties in this context. In Section 6 we set up congruences for the fixed points of characteristic bisets, and use these to tease out the saturation axioms. Section 7 is the focal point of the paper, in which we introduce Frobenius reciprocity in the double Burnside ring and, using tools from Section 6, show that it implies saturation, proving Theorems A and B. In Section 8 we prepare for the applications to stable homotopy theory by showing in Theorem 8.8 that the bifreeness condition (which has no reasonable interpretation in stable homotopy) can be replaced by a dominance condition (in the sense of Nishida [30])

The third part of the paper covers applications to algebraic topology. In Section 9 we review the Segal conjecture and the construction of classifying spectra for saturated fusion systems, which allows us to make the transition from algebra to stable homotopy theory. An immediate application to stable splittings of classifying spaces is then given in Corollary 9.5. The motivation for this paper comes in part from a conjecture of Haynes Miller, who suggested that p–local finite groups could be modelled by p–tract groups, which consist of a map $BS \to X$ (with suitable technical conditions) that admits a stable retract satisfying a Frobenius reciprocity relation. In Section 10 we discuss this conjecture, and reduce it to proving two technical assumptions about p–tract groups. Finally, in Section 11, we prove an invariant theorem for double Burnside modules (Theorem 11.3) and explain its analogy to the Adams–Wilkerson criterion for characterizing rings of invariants in the cohomology of an elementary abelian p–group [3, Theorem 1.2].

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2 Fusion systems

Fusion systems and their saturation axioms were introduced by Puig in [32] in an effort to axiomatize the p–local structure of a finite group and, more generally, of a block of a group algebra. Broto, Levi and Oliver developed this axiomatic approach in [16], and gave a different set of saturation axioms, which they prove to be equivalent to Puig's definition. In this section we present Broto, Levi and Oliver's axiomatic system, that is used throughout our paper. We include in this section some basic properties of fusion systems and we introduce the concept of pre-fusion system, which is a structure designed to keep track of a set of homomorphisms used to generate a fusion systems.

2.1 Basic notation and definitions

For subgroups H and K of a finite group G, denote the *transporter* from H to K in G by $N_G(H,K) := \{g \in G : c_g(H) \leq K\}$ where $c_g(x) := gxg^{-1}$ is the *conjugation homomorphism*. For $g \in G$ we write ${}^gH$ for gHg^{-1}, and H^g for $g^{-1}Hg$. We say that H and K are G–conjugate if ${}^gH = K$ for some $g \in G$, and denote the G–conjugacy class of H by $[H]_G$. Also, as usual, the *normalizer* of P is denoted by $N_S(P) := N_S(P,P)$ and the *centralizer* of P is $C_S(P) := \{y \in N_S(P) : c_y|_P = \mathrm{id}_P\}$. Other useful notation: $\mathrm{Hom}_S(P,Q) := N_S(P,Q)/C_S(P)$, $\mathrm{Aut}_S(P) := \mathrm{Hom}_S(P,P)$ and $\mathrm{Inn}(P) := \mathrm{Aut}_P(P)$.

Definition 2.1 A *fusion system $\mathcal{F}$ on a finite p–group S* is a category whose objects are the subgroups of S, and whose set of morphisms between the subgroups P and Q of S is a set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ of injective group homomorphisms from P to Q, with the following properties:

(1) $\mathrm{Hom}_S(P,Q) \subset \mathrm{Hom}_{\mathcal{F}}(P,Q)$,

(2) for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ the induced isomorphism $P \simeq \varphi(P)$ and its inverse are morphisms in $\mathcal{F}$,

(3) the composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Let $\mathcal{F}$ be a fusion system on S. We say that two subgroups P and Q of S are *$\mathcal{F}$–conjugate* if there exist an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$. The $\mathcal{F}$–conjugacy class of P is denoted by $[P]_{\mathcal{F}}$.

Definition 2.2 Let $\mathcal{F}$ be a fusion system on a finite p–group S, and let $P \leq S$.

(a) P is *fully $\mathcal{F}$–centralized* if $|\mathbb{C}SP| \geq |\mathbb{C}SQ|$ for all $Q \in [P]_{\mathcal{F}}$,

(b) P is *fully $\mathcal{F}$–normalized* if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in [P]_{\mathcal{F}}$,

(c) P is *$\mathcal{F}$–centric* if $C_S(Q) = Z(Q)$ for all $Q \in [P]_{\mathcal{F}}$.

We define the *$\mathcal{F}$–representations* from P to Q as the quotient

$$\mathrm{Rep}_{\mathcal{F}}(P, Q) := \mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathcal{F}}(P, Q),$$

and the *outer automorphisms* in $\mathcal{F}$ as

$$\mathrm{Out}_{\mathcal{F}}(P) := \mathrm{Rep}_{\mathcal{F}}(P, Q) = \mathrm{Inn}(P)\backslash\mathrm{Aut}_{\mathcal{F}}(P).$$

2.2 Morphisms of fusion systems and fusion-preserving homomorphisms

Let $\mathcal{F}$ be a fusion system on S and $\mathcal{F}'$ be a fusion system on S'.

Definition 2.3 A *morphism* from $\mathcal{F}$ to $\mathcal{F}'$ is a pair (α, α_0), where $\alpha\colon \mathcal{F} \to \mathcal{F}'$ is a covariant functor and $\alpha_0\colon S \to S'$ is a group homomorphism satisfying $\alpha(P) = \alpha_0(P)$ and $\alpha(\phi) \circ \alpha_0(u) = \alpha_0 \circ \phi(u)$ for all $u \in P \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.

Definition 2.4 We say that a group homomorphism $\beta\colon S \to S'$ is *$(\mathcal{F}, \mathcal{F}')$–fusion preserving* if

$$\beta|_Q \circ \mathrm{Hom}_{\mathcal{F}}(P, Q) \subset \mathrm{Hom}_{\mathcal{F}'}(\beta(P), \beta(Q)) \circ \beta|_P\,.$$

If $(\alpha, \alpha_0)\colon \mathcal{F} \to \mathcal{F}'$ is a morphism of fusion systems then α_0 is $(\mathcal{F}, \mathcal{F}')$–fusion preserving. Conversely, if $\alpha_0\colon S \to S'$ is an $(\mathcal{F}, \mathcal{F}')$–fusion-preserving homomorphism then α_0 induces a unique functor $\alpha\colon \mathcal{F} \to \mathcal{F}$ such that the pair (α, α_0) is a morphism of fusion systems.

2.3 Saturation axioms

Fusion systems provide a model for the conjugation action on S by an ambient object, but this model is far too general to be interesting in practice. Classically, the interesting fusion systems are the ones coming from the p–local structure of a finite group or of a block of the group algebra of a finite group in characteristic p. In both cases, the fusion systems satisfy certain axioms that correspond to the Sylow theorems, and this generalizes to the following definition, originally due to Puig [32] but presented here in the form developed by Broto, Levi and Oliver [16].

Definition 2.5 A fusion system $\mathcal{F}$ on a finite p–group S is *saturated* if it satisfies the following axioms:

(I) If $P \leq S$ is fully $\mathcal{F}$–normalized then P is fully $\mathcal{F}$–centralized and $\mathrm{Aut}_S(P)$ is a Sylow p–subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(II) If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ is a homomorphism such that $\varphi(P)$ is fully $\mathcal{F}$–centralized, then φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$, where

$$N_\varphi = \{x \in N_S(P) \mid \exists y \in N_S(\varphi(P)), \varphi({}^x u) = {}^y\varphi(u), \forall u \in P\}.$$

Notice that if $R \leq N_S(P)$ is a subgroup containing P, to which φ can be extended in $\mathcal{F}$, then $R \leq N_\varphi$. More generally, if φ extends in $\mathcal{F}$ to a group R with $P < R \leq S$, then $P < N_R(P) = N_\varphi \cap R$. In particular, if $N_\varphi = P$, then φ cannot be extended in $\mathcal{F}$.

2.4 Pre-fusion systems

A fusion system on a given finite p–group S is determined by its morphism sets. Thus one can construct a fusion system $\mathcal{F}$ by specifying a set of morphisms it should contain and then taking $\mathcal{F}$ to be the fusion system generated by those morphisms. This approach will be taken in Section 4 when we construct fusion systems from bisets. To capture this construction we introduce the notion of pre-fusion systems.

Definition 2.6 A *pre-fusion system* on a finite p–group S is a collection

$$\mathcal{P} = \{\mathrm{Hom}_{\mathcal{P}}(P, Q) \mid P, Q \leq S\},$$

satisfying the following conditions:

(1) $\mathrm{Hom}_{\mathcal{P}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for each pair of subgroups $P, Q \leq S$.

(2) If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P, Q)$ and $\varphi(P) \leq R \leq S$, then the composite $P \xrightarrow{\varphi} \varphi(P) \hookrightarrow R$ is in $\mathcal{P}$.

Note that a pre-fusion system need not be a category as we require neither that the composite of two morphisms in $\mathcal{P}$ is again in $\mathcal{P}$, nor that the identity morphism of a subgroup in S is in $\mathcal{P}$. The second condition says we can restrict or extend morphisms in the target, and it follows that a pre-fusion system $\mathcal{P}$ on S is determined by the sets $\mathrm{Hom}_{\mathcal{P}}(P, S)$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two pre-fusion systems on a finite p–group S, let $\mathcal{P}_1 \cap \mathcal{P}_2$ be the pre-fusion system with morphism sets $\mathrm{Hom}_{\mathcal{P}_1}(P, Q) \cap \mathrm{Hom}_{\mathcal{P}_2}(P, Q)$. It is easy to see that the intersection of two fusion systems is a fusion system, and this allows us to make the following definition.

Definition 2.7 The *closure of* $\mathcal{P}$, denoted $\overline{\mathcal{P}}$, is the smallest fusion system on S such that $\mathrm{Hom}_{\mathcal{P}}(P,Q) \subseteq \mathrm{Hom}_{\overline{\mathcal{P}}}(P,Q)$ for each pair of subgroups $P,Q \le S$. We say that $\mathcal{P}$ is *closed* if $\overline{\mathcal{P}} = \mathcal{P}$.

On occasion we will consider pre-fusion systems with a weaker form of closure.

Definition 2.8 A pre-fusion system $\mathcal{P}$ on a finite p–group S is *level-wise closed* if the following holds for all $P,Q,R \le S$.

(1) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{P}}(P,Q)$.

(2) If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,Q)$ is a group isomorphism, then $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{P}}(Q,P)$.

(3) If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{P}}(Q,R)$ are group isomorphisms, then $\psi \circ \varphi \in \mathrm{Hom}_{\mathcal{P}}(P,R)$.

It is easy to show that a level-wise-closed pre-fusion system that is closed under restriction is closed. In a level-wise-closed pre-fusion system $\mathcal{P}$ the morphism sets $\mathrm{Hom}_{\mathcal{P}}(P,P)$ are groups of automorphisms, and we denote them by $\mathrm{Aut}_{\mathcal{P}}(P)$. Furthermore, the notions of $\mathcal{P}$–conjugacy, fully $\mathcal{P}$–centralized and fully $\mathcal{P}$–normalized subgroups extend to this context. Hence we can consider the following local saturation conditions.

Definition 2.9 Let $\mathcal{P}$ be a level-wise-closed pre-fusion system on a finite p–group S. For a subgroup $P \le S$, we say that $\mathcal{P}$ is *saturated at* P if the following two conditions hold.

(I_P) If $Q \in [P]_{\mathcal{P}}$ is fully $\mathcal{P}$–normalized, then Q is fully $\mathcal{P}$–centralized and $\mathrm{Aut}_S(Q)$ is a Sylow p–subgroup of $\mathrm{Aut}_{\mathcal{P}}(Q)$.

(II_P) If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ is a homomorphism such that $\varphi(P)$ is fully $\mathcal{P}$–centralized, then φ extends to a homomorphism $\overline{\varphi} \in \mathrm{Hom}_{\overline{\mathcal{P}}}(N_\varphi, S)$.

Clearly, a fusion system on S is saturated if and only if it is saturated at every subgroup P of S. As the notational distinction is slight, let us explicitly point out that in condition II_P we require only that the extension $\overline{\varphi}$ be in the closure $\overline{\mathcal{P}}$, and not necessarily in $\mathcal{P}$ itself.

3 Bisets, the Burnside category and Mackey functors

In this section we recall the structure and main properties of the double Burnside ring $A(G,G)$ of finite, left-free (G,G)–bisets for a finite group G. In fact we work more generally, studying the modules $A(G,H)$ of finite, left-free (G,H)–bisets for finite groups G and H. These modules form the morphism sets in the Burnside category $\mathbf{A}$, and we think of Mackey functors as functors defined on this category. Throughout this section G, H and K will denote finite groups.

3.1 The (single) Burnside ring and fixed points

We start by considering sets with a single group action as many properties of bisets follow from this case. The material in this subsection is described in detail in Sections 2 and 3 of Bouc [14].

Definition 3.1 The Burnside ring of a finite group G is the Grothendieck group $A(G)$ of isomorphism classes of finite, left G–sets, with addition induced by disjoint union, and multiplication induced by Cartesian product.

$A(G)$ is a free $\mathbb{Z}$–module with basis given by isomorphism classes of the G–sets G/H, where H ranges over conjugacy classes of subgroups in G. Multiplication is described by the double coset formula

$$[G/H]\cdot[G/K] = \prod_{x\in H\backslash G/K} [G/(H\cap K^x)],$$

where the square brackets denote isomorphism classes.

A *family* of subgroups in G is a set $\mathcal{H}$ of subgroups of G that is closed under conjugacy and taking subgroups. For a family $\mathcal{H}$, let $A_{\mathcal{H}}(G)$ be the submodule generated by G–sets of the form G/H with $H\in\mathcal{H}$. By the double coset formula, this is a subring of $A(G)$.

For a G–set X and a subgroup $H\le G$, we let X^H denote the set of H–fixed points in G. The cardinality $|X^H|$ is additive in X and does not change when we replace H by a conjugate subgroup in G. There results a *fixed-point homomorphism*

$$\Phi_{[H]}\colon A(G)\to\mathbb{Z},$$

where $[H]$ denotes the conjugacy class of H in G. A straightforward computation shows that

$$\Phi_{[H]}([G/K]) = |N_G(H,K)|/|K|. \tag{2}$$

In particular, $\Phi_{[H]}([G/K])=0$ unless H is subconjugate to K.

Proposition 3.2 *For a finite group G and a family $\mathcal{H}$ of subgroups in G, the map*

$$\Phi_{\mathcal{H}}\colon A_{\mathcal{H}}(G)\xrightarrow{\prod_{[H]}\Phi_{[H]}}\prod_{[H]}\mathbb{Z},$$

where the product runs over conjugacy classes of groups in $\mathcal{H}$, is injective.

Proof This is proved in [14] in the case where $\mathcal{H}$ is the family of all subgroups. The main point of the proof is that (2) implies that $\Phi_{\mathcal{H}}$ can be represented as an upper triangular matrix with nonzero coefficients on the diagonal. This remains true when $\mathcal{H}$ is a proper subfamily, and the result follows in the same way. □

The morphism $\Phi_{\mathcal{H}}$ is commonly called the *table of marks* for $A_{\mathcal{H}}(G)$. The upper triangular property of the table of marks also proves the following.

Lemma 3.3 *Let G be a finite group and let $\mathcal{H}$ be a family of subgroups in G. An element $X \in A(G)$ belongs to $A_{\mathcal{H}}(G)$ if and only if $\Phi_{[K]}(X) = 0$ for each subgroup $K \leq G$ that does not belong to $\mathcal{H}$.*

3.2 Bisets and the double Burnside module

A (G, H)*–biset* is a set equipped with a right G–action and a left H–action, such that the two actions commute. A biset is *left-free* if the H–action is free, and *right-free* if the G–action is free. A biset is *bifree* if it is both left- and right-free.

Definition 3.4 Let $B(G, H)$ denote the Grothendieck group of isomorphism classes of finite (G, H)–bisets, with addition induced by disjoint union. The *Burnside module* of G and H is the subgroup $A(G, H)$ of $B(G, H)$ generated by left-free (G, H)–sets. The *free Burnside module* of G and H is the subgroup $A_{\mathrm{fr}}(G, H)$ generated by bifree (G, H)–sets.

Given a (G, H)–biset X one obtains a $(H \times G)$–set $\widehat{X}$, with the same underlying set, and $(H \times G)$–action given by $(h, g)x := hxg^{-1}$. This gives a bijective correspondence between (G, H)–bisets and $(H \times G)$–sets, and it is often convenient to characterize a (G, H)–biset by the corresponding $(H \times G)$–set. The assignment $X \mapsto \widehat{X}$ induces an isomorphism

$$\Psi\colon B(G, H) \xrightarrow{\cong} A(H \times G).$$

3.3 (G, H)–pairs and (sub)conjugacy

A (G, H)*–pair* is a pair (K, φ), consisting of a subgroup $K \leq G$ and a homomorphism $\varphi\colon K \to H$. We say that (K, φ) is *injective* if φ is injective. The *graph* of (K, φ) is the subgroup

$$\Delta(K, \varphi) = \{(\varphi(k), k) \mid k \in K\} \leq (H \times G).$$

Let (K, φ) and (L, ψ) be (G, H)–pairs. We say that (K, φ) is *conjugate* to (L, ψ), and write $(K, \varphi) \sim (L, \psi)$, if $\Delta(K, \varphi)$ is conjugate to $\Delta(L, \psi)$ in $H \times G$. We refer to (G,H)–conjugacy classes of (G,H)–pairs as (G,H)–classes, and denote the (G,H)–class of (K,φ) by $\langle K,\varphi\rangle$.

Similarly we say that (K,φ) is *subconjugate* to (L,ψ), and write $(K,\varphi)\precsim(L,\psi)$, if $\Delta(K,\varphi)$ is subconjugate to $\Delta(L,\psi)$. In this case we also say that the (G,H)–class $\langle K,\varphi\rangle$ is *subconjugate* to $\langle L,\psi\rangle$, and write $\langle K,\varphi\rangle\precsim\langle L,\psi\rangle$. Notice that the subconjugacy relation $\langle K,\varphi\rangle\precsim\langle L,\psi\rangle$ implies that every representative of the (G,H)–class $\langle K,\varphi\rangle$ is subconjugate to every representative of the (G,H)–class $\langle L,\psi\rangle$.

The following characterization of subconjugacy is sometimes useful. The proof is left to the reader.

Lemma 3.5 *Let (K,φ) and (L,ψ) be (G,H)–pairs. Then (K,φ) is subconjugate to (L,ψ) if and only if there exist $x\in N_G(K,L)$ and $y\in N_H(\varphi(K),\psi(L))$ such that $c_y\circ\varphi=\psi\circ c_x$. Conjugacy holds if and only if the additional condition $L={}^xK$ is satisfied.*

3.4 The standard basis of $A(G,H)$

The Burnside module $A(G,H)$ is a free abelian $\mathbb{Z}$–module with basis the isomorphism classes of indecomposable left-free (G,H)–sets. Using the isomorphism Ψ we can conveniently describe and parametrize this basis.

We consider two families of subgroups of $H\times G$, defined as follows:

$$\mathcal{B}=\{\Delta(K,\varphi)\mid(K,\varphi)\text{ is a }(G,H)\text{–pair}\}$$
$$\mathcal{B}_{\mathrm{fr}}=\{\Delta(K,\varphi)\mid(K,\varphi)\text{ is an injective }(G,H)\text{–pair}\}$$

Lemma 3.6 *The isomorphism $\Psi\colon B(G,H)\to A(H\times G)$ restricts to isomorphisms*

$$A(G,H)\xrightarrow{\cong}A_{\mathcal{B}}(H\times G)$$

and

$$A_{\mathrm{fr}}(G,H)\xrightarrow{\cong}A_{\mathcal{B}_{\mathrm{fr}}}(H\times G).$$

Proof One proves the first isomorphism by showing that a transitive $(H\times G)$–set $(H\times G)/K$ has free G–action if and only if K is the graph of a (G,H)–pair. The second isomorphism is then obtained by showing that $(H\times G)/K$ has free H–action if and only if K is the graph of an injective (G,H)–pair. The details are left to the reader. □

Lemma 3.6 allows us to pull back the bases for $A_{\mathcal{B}}(H\times G)$ and $A_{\mathcal{B}_{\mathrm{fr}}}(H\times G)$ to obtain bases for $A(G,H)$ and $A_{\mathrm{fr}}(G,H)$, respectively, and we proceed to describe these bases. From a (G,H)–pair (K,φ), one obtains a left-free (G,H)–set

$$H\times_{(K,\varphi)} G := (H\times G)/\sim,$$

where $\sim$ is the relation

$$(x,ky)\sim(x\varphi(k),y), \text{ for all } x\in H, y\in G, k\in K,$$

and G and H act in the obvious way. The corresponding $(H\times G)$–set is isomorphic to

$$(H\times G)/\Delta(K,\varphi)$$

under the map $(h,g)\mapsto(h,g^{-1})$.

Definition 3.7 For a (G,H)–pair (K,φ), let $[K,\varphi]_G^H\in A(G,H)$ denote the isomorphism class of $H\times_{(K,\varphi)} G$.

We often write $[K,\varphi]$ instead of $[K,\varphi]_G^H$ when there is no danger of confusion.

Lemma 3.8 *The Burnside module $A(G,H)$ is a free $\mathbb{Z}$–module with one basis element $[K,\varphi]_G^H$ for each (G,H)–class $\langle K,\varphi\rangle$. A basis for $A_{\mathrm{fr}}(G,H)$ is obtained by restricting to injective (G,H)–classes.*

Proof This follow from Lemma 3.6 and the fact that

$$\Psi([K,\varphi]_G^H)=[(H\times G)/\Delta(K,\varphi)]. \qquad \square$$

The basis described in Lemma 3.8 will be used throughout the paper, and we refer to it as the *standard basis* of $A(G,H)$ (or $A_{\mathrm{fr}}(G,H)$ when appropriate). Notice that $A(G,H)$ is in particular a finitely generated $\mathbb{Z}$–module. Therefore the p–localization $A(G,H)_{(p)}$ and p–completion $A(G,H)^\wedge_p$ can be obtained by tensoring with $\mathbb{Z}_{(p)}$ and $\mathbb{Z}^\wedge_p$, respectively. Hence Lemma 3.8 holds after p–localization or p–completion.

Definition 3.9 For each (G,H)–class $\langle K,\varphi\rangle$, let $c_{\langle K,\varphi\rangle}\colon A(G,H)\to\mathbb{Z}$ be the homomorphism sending $X\in A(G,H)$ to the coefficient at $[K,\varphi]$ in the standard basis decomposition of X.

The homomorphisms $c_{\langle K,\varphi\rangle}$ are equivalently defined by requiring that

$$X=\sum_{\langle K,\varphi\rangle} c_{\langle K,\varphi\rangle}(X)[K,\varphi]$$

for every $X \in A(G,H)$. At times we will break this up into a double sum

$$X = \sum_{[K]_G} \Bigg(\sum_{[\varphi]\in \mathrm{Rep}(K,H)} c_{\langle K,\varphi\rangle}(X)\,[K,\varphi] \Bigg),$$

where the outer sum runs over G–conjugacy classes of subgroups, and the inner sum runs over H–conjugacy classes of morphisms.

We also denote the p–localization or p–completion of $c_{\langle K,\varphi\rangle}$ by $c_{\langle K,\varphi\rangle}$.

3.5 Fixed points of bisets

For a (G,H)–biset X, and a (G,H)–pair (K,φ), set

$$X^{(K,\varphi)} := \{x \in X \mid \forall k \in K : xk = \varphi(k)x\}.$$

Notice that $X^{(K,\varphi)} = \widehat{X}^{\Delta(K,\varphi)}$ (as sets). As the number $|\widehat{X}^{\Delta(K,\varphi)}|$ does not change when we conjugate $\Delta(K,\varphi)$, the same is true for the number $|X^{(K,\varphi)}|$, and we can make the following definition.

Definition 3.10 For a (G,H)–class $\langle K,\varphi\rangle$, let $\Phi_{\langle K,\varphi\rangle}\colon A(G,H) \to \mathbb{Z}$ be the $\mathbb{Z}$–module homomorphism defined by setting

$$\Phi_{\langle K,\varphi\rangle}(X) = \left|X^{(K,\varphi)}\right|$$

for (G,H)–set X, and extending linearly.

The homomorphisms $\Phi_{\langle K,\varphi\rangle}$ form a *table of marks* for $A(G,H)$.

Proposition 3.11 *For finite groups G and H, the morphism*

$$\prod_{\langle K,\varphi\rangle} \Phi_{\langle K,\varphi\rangle}\colon \Phi\colon A(G,H) \to \prod_{\langle K,\varphi\rangle} \mathbb{Z},$$

where the products run over all (G,H)–classes $\langle K,\varphi\rangle$, is injective.

Proof Since $\Phi(X) = \Phi_{\mathcal{B}} \circ \Psi(X)$ for $X \in A(G,H)$, this follows from Proposition 3.2. □

Again, this proposition holds with $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p^\wedge$ coefficients, and we also use $\Phi_{\langle K,\varphi\rangle}$ to denote the p–localization or p–completion of $\Phi_{\langle K,\varphi\rangle}$.

The following lemma describes the relationship between the standard basis and fixed-point methods of bookkeeping for (G,H)–bisets.

Lemma 3.12 *Let G and H be finite groups, and let $\langle K,\varphi\rangle$ and $\langle L,\psi\rangle$ be (G,H)–classes. Then*

$$\Phi_{\langle K,\varphi\rangle}([L,\psi]) = \frac{|N_{\varphi,\psi}|}{|L|}\cdot|C_H(\varphi(K))|\,,$$

where

$$N_{\varphi,\psi} = \{x\in N_G(K,L)\mid \exists y\in H : c_y\circ\varphi = \psi\circ c_x\}\,.$$

In particular, $\Phi_{\langle K,\varphi\rangle}([L,\psi]) = 0$ unless $\langle K,\varphi\rangle$ is subconjugate to $\langle L,\psi\rangle$.

Proof We count the pairs $(x,y)\in G\times H$ whose class in $H\times_{(L,\psi)} G$ is preserved by the action of every element in $\Delta(K,\varphi)$. They are such that for every $k\in K$ there exists $l\in L$ with $(\varphi(k)y, xk^{-1}) = (y\psi(l), l^{-1}x)$. This gives $x\in N_{\varphi,\psi}$. Once $x\in N_{\varphi,\psi}$ is fixed, y is determined up to an element in $C_H(\varphi(K))$. Thus there are $|N_{\varphi,\psi}|\cdot|C_H(\varphi(K))|$ pairs $(x,y)\in G\times H$ whose class in $H\times_{(L,\psi)} G$ is preserved by the action of every element in $\Delta(K,\varphi)$. Now, $\Delta(L,\psi)$ acts freely on these pairs by $(\psi(l),l)\cdot(y,x) = (y\psi(l), l^{-1}x)$ and any orbit of the action is an equivalence class in $H\times_{(L,\psi)} G$. The result follows. □

We use the notation $N_\varphi := N_{\varphi,\varphi}$. Note that, when φ is a morphism in a fusion system, the notation N_φ here has the same meaning as in Definition 2.5. We shall need the following observation later.

Lemma 3.13 *Let G and H be finite groups, and let $\langle K,\varphi\rangle$ and $\langle L,\psi\rangle$ be (G,H)–classes. Then $N_{\varphi,\psi}$ is a bifree (N_φ,N_ψ)–biset with action given by the multiplication in G.*

Proof Take $x\in N_{\varphi,\psi}$ and $u\in N_\psi$. By definition, there exist $y\in H$ and $v\in N_H(\psi(L))$ such that $c_y\circ\varphi = \psi\circ c_x$ and $c_v\circ\psi = \psi\circ c_u$. But then

$$c_{vy}\circ\varphi = c_v\circ c_y\circ\varphi = c_v\circ\psi\circ c_x = \psi\circ c_u\circ c_x = \psi\circ c_{ux}$$

and, hence, $ux\in N_{\varphi,\psi}$. One similarly shows that, for $x\in N_{\varphi,\psi}$ and $u\in N_\varphi$, we have $xu\in N_{\varphi,\psi}$. Thus $N_{\varphi,\psi}$ is a (N_φ,N_ψ)–subset of G. Bifreeness of (N_φ,N_ψ) now follows from the bifreeness of G. □

3.6 The Burnside category and Mackey functors

There is a composition pairing

$$A(H,K)\times A(G,H)\to A(G,K)$$

induced on isomorphism classes of bisets by

$$[X]_H^K \circ [Y]_G^H = [X \times_H Y]_G^K.$$

This can be described on basis elements via the double coset formula:

$$[A,\varphi]_H^K \circ [B,\psi]_G^H = \sum_{x \in A\backslash H/\psi(B)} \big[\psi^{-1}\big(\psi(B) \cap A^x\big), \varphi \circ c_x \circ \psi\big]_G^K$$

The composition pairing is associative and bilinear, prompting us to make the following definition.

Definition 3.14 The *Burnside category* $\mathbf{A}$ is the category whose objects are the finite groups, and whose morphism sets are given by

$$\mathrm{Mor}_{\mathbf{A}}(G,H) = A(G,H),$$

with composition given by the composition pairing $\circ$.

The Burnside category is a $\mathbb{Z}$–linear category, in the sense that morphism sets are $\mathbb{Z}$–modules and composition is bilinear. Given a commutative ring R, we obtain an R–linear category $\mathbf{A}R$ by tensoring every morphism set in $\mathbf{A}$ with R. We are primarily interested in the cases $\mathbf{A}\mathbb{Z}_{(p)}$ and $\mathbf{A}\mathbb{Z}_p^\wedge$, which we call the *$p$–local* and *$p$–complete* Burnside categories, respectively.

Definition 3.15 Let R be a commutative ring. A *globally defined, R–linear Mackey functor* is a functor $M\colon \mathbf{A} \to R\text{-}\mathrm{mod}$, where $R\text{-}\mathrm{mod}$ is the category of R–modules. The functor can be either covariant or contravariant.

Notice that an R–linear Mackey functor M extends uniquely to a functor $\mathbf{A}R \to R\text{-}\mathrm{mod}$, which we also denote by M. When R is a p–local ring, we say that an R–linear Mackey functor is *p–local*. In this case a functor $M\colon \mathbf{A} \to R\text{-}\mathrm{mod}$ extends uniquely to a functor $M\colon \mathbf{A}\mathbb{Z}_{(p)} \to R\text{-}\mathrm{mod}$, and we often think of a p–local Mackey functor as a functor defined on $\mathbf{A}\mathbb{Z}_{(p)}$. Similarly, when R is a p–complete ring, we say that an R–linear Mackey functor is *p–complete*, and a p–complete Mackey functor is equivalent to a functor defined on $\mathbf{A}\mathbb{Z}_p^\wedge$. We will also have the occasion to consider more restrictive Mackey functors.

Definition 3.16 Let $\mathbf{A}_p$ be the full subcategory of $\mathbf{A}$ whose objects are the finite p–groups. For a commutative ring R, a *p–defined, R–linear Mackey functor* is a functor $M\colon \mathbf{A}_p \to R\text{-}\mathrm{mod}$.

The "classical notion" of a Mackey functor for a group G is a pair of functors (M_*, M^*), one covariant the other contravariant, that are defined on the category of finite G–sets and satisfy $M_*(X) = M^*(X)$. The functors are required to be additive with respect to disjoint union and to satisfy a "pullback condition" that corresponds to the double coset formula. Such a functor can be seen to be equivalent to a functor M defined on the category $\mathbf{A}_G$ whose objects are the subgroups of G, and with morphism sets $A_G(H, K)$ the submodules of $A(H, K)$ generated by basis elements of the form $[L, c_g]_H^K$ for $g \in N_G(H, K)$. In other words, M only allows restriction along conjugation in G. The analogous construction for fusion systems is the following.

Definition 3.17 Let $\mathcal{F}$ be a fusion system on a finite p–group S. For subgroups $P, Q \leq S$, let $A_{\mathcal{F}}(P, Q)$ be the submodule of $A(P, Q)$ generated by basis elements of the form $[T, \varphi]$ with $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. An element X in $A(P, Q)$ is *$\mathcal{F}$–generated* if $X \in A_{\mathcal{F}}(P, Q)$. Let $\mathbf{A}_{\mathcal{F}}$ be the subcategory whose objects are the subgroups of S, and whose morphism sets are the modules $A_{\mathcal{F}}(P, Q)$.

To make sense in the context of a fusion system $\mathcal{F}$, a Mackey functor should at least be defined on the category $\mathbf{A}_{\mathcal{F}}$.

Definition 3.18 Let $\mathcal{F}$ be a fusion system on a finite p–group S. For a commutative ring R, an *$\mathcal{F}$–defined* Mackey functor is a functor $\mathbf{A}' \to R\text{-}\mathrm{mod}$ defined on a subcategory $\mathbf{A}'$ of $\mathbf{A}$ containing $\mathbf{A}_{\mathcal{F}}$.

3.7 Augmentation

A particular Mackey functor that will be used throughout the paper is the *augmentation functor* ϵ, which we describe here.

Definition 3.19 The *augmentation* $\epsilon\colon A(G, H) \to \mathbb{Z}$ is the homomorphism defined on the isomorphism class of a biset X by

$$\epsilon([X]) = |H\backslash X| = |X|/|H|.$$

The augmentation homomorphism admits a convenient description on basis elements.

Lemma 3.20 *The augmentation of a basis element $[K, \varphi]$ of $A(G, H)$ is given by*

$$\epsilon([K, \varphi]) = |G|/|K|.$$

Proof $\epsilon([K, \varphi]) = |(H \times_{K,\varphi} G)|/|H| = |(H \times G)/\Delta(K, \varphi)|/|H| = |G|/|K|$ □

It is not hard to show that the augmentation homomorphisms satisfy

$$\epsilon(X \circ Y) = \epsilon(X) \cdot \epsilon(Y)$$

when X and Y are composable. Hence, we can regard them as the components of a Mackey functor

$$\epsilon\colon \mathbf{A} \to \mathbb{Z},$$

where $\mathbb{Z}$ is regarded as a $\mathbb{Z}$–linear category with a single object and morphism set $\mathbb{Z}$, composition being given by multiplication. We also use the symbol ϵ to denote the p–localization or p–completion of ϵ.

3.8 Opposite sets

Given a (G,H)–biset X, one obtains an (H,G)–biset X^{op} with the same underlying set by reversing the G and H actions. That is, if $x \in X$, and x^{op} is the same element regarded as an element of X^{op}, then $gx^{\mathrm{op}}h := h^{-1}xg^{-1}$. Notice that $\widehat{X^{\mathrm{op}}}$ is obtained by regarding the $(H \times G)$–set $\widehat{X}$ as a $(G \times H)$–set in the obvious way.

Taking opposite sets induces an isomorphism of $\mathbb{Z}$–modules

$$\mathrm{op}\colon B(G,H) \xrightarrow{\cong} B(H,G),$$

that restricts to an isomorphism

$$\mathrm{op}\colon A_{\mathrm{fr}}(G,H) \xrightarrow{\cong} A_{\mathrm{fr}}(H,G),$$

as the opposite of a bifree set is again bifree. However, op does not map $A(G,H)$ into $A(H,G)$.

Definition 3.21 An element $X \in A_{\mathrm{fr}}(G,G)$ is *symmetric* if $X^{\mathrm{op}} = X$.

We conclude this section by recording some basic properties of the opposite homomorphism.

Lemma 3.22 *For an injective (G,H)–pair (K,φ),*

$$\left([K,\varphi]_G^H\right)^{\mathrm{op}} = [\varphi(K),\varphi^{-1}]_H^G.$$

Lemma 3.23 *For $X \in A_{\mathrm{fr}}(G,H)$ and an injective (G,H)–pair (K,φ),*

$$\Phi_{\langle K,\varphi\rangle}(X) = \Phi_{\langle \varphi(K),\varphi^{-1}\rangle}(X^{\mathrm{op}}).$$

Lemma 3.24 *If $X \in A_{\mathrm{fr}}(G,H)$ and $Y \in A_{\mathrm{fr}}(H,K)$, then*

$$(Y \circ X)^{\mathrm{op}} = X^{\mathrm{op}} \circ Y^{\mathrm{op}}.$$

Corollary 3.25 *If $X \in A_{\mathrm{fr}}(G,H)$, then $X^{\mathrm{op}} \circ X \in A_{\mathrm{fr}}(G,G)$ and $X \circ X^{\mathrm{op}} \in A_{\mathrm{fr}}(H,H)$ are symmetric.*

The entire discussion of opposite sets carries over to the p–local setting.

4 Characteristic bisets and idempotents

In this section we recall the notion of characteristic elements for fusion systems, which play a central role in this paper, and list some of their important properties. Characteristic elements were introduced by Linckelmann and Webb in order to produce a transfer theory for the cohomology of fusion systems, and subsequently to construct classifying spectra for fusion systems. Their definition was motivated by the special role that the (S,S)–biset G plays in the $\mathbb{F}_p$–cohomology of a Sylow inclusion $S \le G$. (G acts on $H^*(BS;\mathbb{F}_p)$ as cohomology is a Mackey functor.) Linckelmann and Webb distilled the important properties of the biset G in this context, and generalized them to fusion systems. A characteristic biset for a fusion system $\mathcal{F}$ on S is a (S,S)–biset with certain properties that mimic the properties of the biset G in the group case, and a characteristic element is the generalization to arbitrary elements in $A(S,S)$. We begin this section by an account of the motivating example, and then move on to discuss the generalization to fusion systems.

4.1 Motivation

Given a finite group G with Sylow p–subgroup S, the restriction map

$$H^*(BG;\mathbb{F}_p) \xrightarrow{i^*} H^*(BS;\mathbb{F}_p)$$

is a monomorphism with image the G–stable elements in S. That is, the elements $x \in H^*(BS;\mathbb{F}_p)$ such that for every subgroup P of S, restricting x to $H^*(BP;\mathbb{F}_p)$ along a conjugation in G has the same effect as restricting along the inclusion. The transfer map

$$H^*(BS;\mathbb{F}_p) \xrightarrow{\mathrm{tr}} H^*(BG;\mathbb{F}_p)$$

provides a right inverse to i^*, up to scalar. More precisely, the composite $\mathrm{tr} \circ i^*$ is multiplication by the index $|G:S|$ on $H^*(BG;\mathbb{F}_p)$, in particular, an automorphism. This implies that the composite $i^* \circ \mathrm{tr}$ is idempotent up to scalar on $H^*(BS;\mathbb{F}_p)$, with image isomorphic to $H^*(BG;\mathbb{F}_p)$.

Now, $H^*(B(\,\cdot\,);\mathbb{F}_p)$ is a globally defined Mackey functor and so $H^*(BS;\mathbb{F}_p)$ admits an $A(S,S)$–action. Under this action the class of the (S,S)–biset G acts by $i^* \circ \mathrm{tr}$.

We can now identify $H^*(BG;\mathbb{F}_p)$ with the image of $H^*([G])\colon H^*(BS;\mathbb{F}_p)\to H^*(BS;\mathbb{F}_p)$, which consists of the G–stable elements in $H^*(BS;\mathbb{F}_p)$. Under this identification, the restriction map $H^*(BG;\mathbb{F}_p)\to H^*(BS;\mathbb{F}_p)$ corresponds to the inclusion of $\mathrm{Im}(H^*([G]))$ in $H^*(BS;\mathbb{F}_p)$, and the transfer $H^*(BS;\mathbb{F}_p)\to H^*(BG;\mathbb{F}_p)$ corresponds to the map

$$H^*([G])\colon H^*(BS;\mathbb{F}_p)\to \mathrm{Im}(H^*([G])).$$

The key point here is that one can approach the $\mathbb{F}_p$–cohomology of BG without knowing G itself. The isomorphism class of $H^*(BG;\mathbb{F}_p)$ is determined by G–stability, which depends on the fusion system, and the transfer theory can be recovered from $[G]$. The notion of G–stability generalizes readily to fusion systems, so one can apply this approach to the cohomology of fusion systems if one has an appropriate replacement for $[G]$ in the fusion setting. Linckelmann and Webb determined the properties that such a replacement should have, leading to the definition of characteristic elements.

4.2 Characteristic elements

To state the definition of a characteristic element, we need to define the notion of $\mathcal{F}$–stability in the double Burnside ring.

Definition 4.1 Let $\mathcal{F}$ be a fusion system on a finite p–group S. We say that an element X in $A(S,S)_{(p)}$ is *right $\mathcal{F}$–stable* if for every $P\le S$ and every $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$, the following equation holds in $A(P,S)_{(p)}$:

$$X\circ[P,\varphi]_P^S = X\circ[P,\mathrm{incl}]_P^S.$$

Similarly, X is *left $\mathcal{F}$–stable* if for every $P\le S$ and every $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$, the following equation holds in $A(S,P)_{(p)}$:

$$[\varphi(P),\varphi^{-1}]_S^P\circ X = [P,\mathrm{id}_P]_S^P\circ X.$$

We say that X is *fully $\mathcal{F}$–stable* if it is both left and right $\mathcal{F}$–stable.

Definition 4.2 Let $\mathcal{F}$ be a fusion system on a finite p–group S. We say that an element Ω in $A(S,S)_{(p)}$ is a *right (resp. left, fully) characteristic element* for $\mathcal{F}$ if it satisfies the following three conditions.

(a) Ω is $\mathcal{F}$–generated (see Definition 3.17).

(b) Ω is right (resp. left, fully) $\mathcal{F}$–stable.

(c) $\epsilon(\Omega)$ is prime to p.

We refer to conditions (a), (b) and (c) as the *Linckelmann–Webb* properties.

Observe that an element is fully characteristic for $\mathcal{F}$ if and only if it is both left and right characteristic. Furthermore, an element Ω is right characteristic if and only if Ω^{op} is left characteristic. We will usually drop the prefix "fully" and take "characteristic element" to mean "fully characteristic element". In practice there is usually no loss of generality in considering only fully characteristic elements, because of the following lemma.

Lemma 4.3 *Let $\mathcal{F}$ be a fusion system on a finite p–group S and let $\Omega \in A(S,S)_{(p)}$. Then Ω is a right characteristic element for $\mathcal{F}$ if and only if Ω^{op} is a left characteristic element for $\mathcal{F}$. In that case $\Omega^{\mathrm{op}} \circ \Omega$ is a symmetric, fully characteristic element for $\mathcal{F}$.*

Proof The first claim is obvious. In the second claim the symmetry, augmentation and stability conditions are clear, and $\mathcal{F}$–generation is an easy consequence of the double coset formula. □

The existence of characteristic elements is far from obvious. It was established by Broto, Levi and Oliver in [16].

Theorem 4.4 [16, Proposition 5.5] *Every saturated fusion system has a characteristic biset, in particular a characteristic element.*

4.3 Characteristic idempotents

Linckelmann and Webb showed that a right characteristic element for $\mathcal{F}$ induces a self-map of $H^*(BS;\mathbb{F}_p)$ that is idempotent up to scalar with image the $\mathcal{F}$–stable elements of $H^*(BS;\mathbb{F}_p)$. (A proof can be found in [16, Proposition 5.5].) Hence, characteristic elements are appropriate for defining transfer in the $\mathbb{F}_p$–cohomology of fusion systems. However, if one tries to replace $\mathbb{F}_p$–cohomology with another Mackey functor, this is not so simple because a characteristic element will not act by an idempotent in general. Moreover, a given saturated fusion system has infinitely many characteristic elements, which can give rise to different transfer constructions. Both of these problems were circumvented in Ragnarsson [34] by introducing characteristic idempotents.

Definition 4.5 Let $\mathcal{F}$ be a fusion system on a finite p–group S. A *characteristic idempotent* for $\mathcal{F}$ is a characteristic element for $\mathcal{F}$ that is idempotent.

Theorem 4.6 [34] *A saturated fusion system $\mathcal{F}$ on a finite p–group S has a unique characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$. Furthermore, $\omega_{\mathcal{F}}$ is symmetric.*

Proof The existence and uniqueness of a characteristic idempotent was first shown in [34, Propositions 4.9 and 5.6]. A cleaner and more direct argument is given by Reeh in [36, Theorem 2.4.11]. Since ${\omega_{\mathcal{F}}}^{\mathrm{op}}$ is also a characteristic idempotent for $\mathcal{F}$, uniqueness implies that ${\omega_{\mathcal{F}}}^{\mathrm{op}} = \omega_{\mathcal{F}}$. □

Remark 4.7 In the proof [34, Proposition 5.6], a consequence of left stability is incorrectly attributed to right stability, and as a result, the first author incorrectly concluded that one-sided idempotents are unique in [34, Remark 5.7]. This mistake was pointed out in [36, Observation 2.4.9] by Reeh, who provides a counter-example in [36, Section 2.5].

4.4 Fixed points of characteristic element

We will prove our main theorems by carefully analyzing and keeping track of fixed points of characteristic elements. Therefore it is important to reformulate the properties of characteristic elements in terms of fixed points, as we do in the following lemma.

Lemma 4.8 *Let $\mathcal{F}$ be a fusion system on a finite p–group S, and let X be an element in $A(S,S)_{(p)}$.*

(a) *X is $\mathcal{F}$–generated if and only if $\Phi_{\langle Q,\psi\rangle}(X) = 0$ for all (S,S)–classes $\langle Q,\psi\rangle$ where ψ is not in $\mathcal{F}$.*

(b) *X is right $\mathcal{F}$–stable if and only if for every (S,S)–class $\langle Q,\psi\rangle$, and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$,*
$$\Phi_{\langle Q,\psi\rangle}(X) = \Phi_{\langle \varphi(Q),\psi\circ\varphi^{-1}\rangle}(X).$$

(c) *X is left $\mathcal{F}$–stable if and only if for every (S,S)–class $\langle Q,\psi\rangle$, and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\psi(Q),S)$,*
$$\Phi_{\langle Q,\psi\rangle}(X) = \Phi_{\langle Q,\varphi\circ\psi\rangle}(X).$$

Proof

(a) Take $\mathcal{H}$ to be the family of subgroups of $S\times S$ of the form $\Delta(P,\varphi)$, with $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. The submodule of $\mathcal{F}$–generated elements in $A(S,S)_{(p)}$ is then the image of $A_{\mathcal{H}}$ under the isomorphism Ψ described in Section 3.2. The result now follows from Lemma 3.3.

(b) It suffices to consider the case where X is a biset; the general case follows by linearity. For any $Q \le P \le S$, $\varphi \in \mathrm{Inj}(P,S)$ and $\psi \in \mathrm{Inj}(Q,S)$ there are canonical bijections between the fixed-point sets
$$(X\circ[P,\varphi]_P^S)^{(Q,\psi)} \quad \text{and} \quad X^{(\varphi(Q),\psi\circ\varphi^{-1})}$$

sending (x,u,v) to x, and also between

$$X^{(Q,\psi)} \quad \text{and} \quad (X\circ[P,\mathrm{incl}]_P^S)^{(Q,\psi)}$$

sending x to $(x,1,1)$. This results in equations of fixed-point homomorphisms,

$$\Phi_{\langle Q,\psi\rangle}(X\circ[P,\varphi]) = \Phi_{\langle \varphi(Q),\psi\circ\varphi^{-1}\rangle}(X),$$
$$\Phi_{\langle Q,\psi\rangle}(X\circ[P,\mathrm{incl}]) = \Phi_{\langle Q,\psi\rangle}(X),$$

valid for any $X \in A(S,S)_{(p)}$. Consequently, if X is right $\mathcal{F}$–stable we get an equality $\Phi_{\langle Q,\psi\rangle}(X) = \Phi_{\langle \varphi(Q),\psi\circ\varphi^{-1}\rangle}(X)$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$ and $\psi \in \mathrm{Inj}(Q,S)$.
Conversely, if for a fixed $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ and all pairs $Q \le P$ and $\psi \in \mathrm{Inj}(Q,S)$ we have $\Phi_{\langle Q,\psi\rangle}(X) = \Phi_{\langle \varphi(Q),\psi\circ\varphi^{-1}\rangle}(X)$, then the fixed-points map

$$\Phi = \prod_{\langle Q,\psi\rangle} \Phi_{\langle Q,\psi\rangle}\colon A(P,S)_{(p)} \to \prod_{\langle Q,\psi\rangle} \mathbb{Z}_{(p)},$$

where the product runs over all (P,S)–classes $\langle Q,\psi\rangle$, have the same image at $X\circ[P,\varphi]$ and $X\circ[P,\mathrm{incl}]$. Hence, by Proposition 3.11, $X\circ[P,\varphi] = X\circ[P,\mathrm{incl}]$.

(c) Analogous to (b). □

Although we do not use it here, it is worth remarking that the table of marks for a characteristic idempotent was computed by Reeh in [36, Theorem 2.4.11], and also in [13, Equation (35)].

4.5 The universal stable element property

We have previously discussed $\mathcal{F}$–stability in $\mathbb{F}_p$–cohomology and in the double Burnside ring. The notion generalizes readily to any Mackey functor, and we will show below that characteristic idempotents characterize $\mathcal{F}$–stable elements.

Definition 4.9 Let $\mathcal{F}$ be a fusion system on a finite p–group S, and let M be a (covariant or contravariant) $\mathcal{F}$–defined Mackey functor. An element $x \in M(S)$ is *$\mathcal{F}$–stable* if for every $P \le S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ we have

$$M([P,\varphi]_P^S)(x) = M([P,\mathrm{incl}]_P^S)(x) \in M(P)$$

in the contravariant case, or

$$M([\varphi(P),\varphi^{-1}]_S^P)(x) = M([P,\mathrm{id}]_S^P)(x) \in M(P),$$

in the covariant case. In either case we denote by $M(\mathcal{F})$ the set of $\mathcal{F}$–stable elements in $M(S)$.

A key property of characteristic idempotents is the following theorem, which is implicit in [34]. The theorem can be interpreted as saying that the characteristic idempotent of a saturated fusion system $\mathcal{F}$ is a universal $\mathcal{F}$–stable element for Mackey functors.

Theorem 4.10 (Universal stable element theorem) *Let $\mathcal{F}$ be a saturated fusion system on a finite p–group S, and let M be a (covariant or contravariant) p–local, $\mathcal{F}$–defined Mackey functor. Then an element $x \in M(S)$ is $\mathcal{F}$–stable if and only if $M(\omega_{\mathcal{F}})(x) = x$.*

Proof One direction is easy: If $M(\omega_{\mathcal{F}})(x) = x$, then $\mathcal{F}$–stability of $\omega_{\mathcal{F}}$ implies the $\mathcal{F}$–stability of x. We prove the converse only for contravariant M, the covariant case being analogous. Now, if x is $\mathcal{F}$–stable, then we have

$$\begin{aligned} M(\omega_{\mathcal{F}})(x) &= \sum_{[P]_S} \Bigg(\sum_{[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,S)} c_{\langle P,\varphi\rangle}(\omega_{\mathcal{F}})\, M([P,\varphi])(x) \Bigg) \\ &= \sum_{[P]_S} \Bigg(\sum_{[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,S)} c_{\langle P,\varphi\rangle}(\omega_{\mathcal{F}}) \Bigg) M([P,\mathrm{incl}])(x) \\ &= \sum_{[P]_S} m_P(\omega_{\mathcal{F}})\, M([P,\mathrm{incl}])(x). \end{aligned}$$

where

$$m_P(X) = \sum_{[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,S)} c_{\langle P,\varphi\rangle}(X)$$

and the latter sum runs over representatives of conjugacy classes of group homomorphisms. Running through the formula in the proof of [34, Lemma 5.5], one easily proves the converse of the statement for a characteristic element Ω for $\mathcal{F}$ there, ie, that Ω is idempotent if and only if $m_S(\Omega) = 1$, and $m_P(\Omega) = 0$ for $P < S$ (this is done explicitly in [36, Lemma 2.4.5]). Hence, we have $m_S = 1$, and $m_P = 0$ for $P < S$, so

$$M(\omega_{\mathcal{F}})(x) = M([S,\mathrm{id}])(x) = x. \qquad \square$$

The same argument proves a universal stable element theorem for (left or right) $A(S,S)_{(p)}$–modules, where one defines $\mathcal{F}$–stability for $x \in M$ by demanding that for all $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, one has $[P,\varphi]_S^S \cdot x = [P,\mathrm{incl}]_S^S \cdot x$ (left modules) or $x \cdot [P,\varphi]_S^S = x \cdot [P,\mathrm{incl}]_S^S$ (right modules), as appropriate.

Theorem 4.11 *Let $\mathcal{F}$ be a saturated fusion systems on a finite p–group S, and let M be a left (resp. right) $A(S,S)_{(p)}$–module. Then, an element $x \in M$ is $\mathcal{F}$–stable if and only if $\omega_{\mathcal{F}} \cdot x = x$ (resp. $x \cdot \omega_{\mathcal{F}} = x$).*

5 Fusion systems induced by bisets

In this section we introduce certain fusion systems on S induced by (S,S)–bisets and, more generally, elements in $A(S,S)_{(p)}$. These fusion systems are the stabilizer fusion system, fixed-point fusion system, and orbit-type fusion system. The latter two are initially defined at the level of pre-fusion systems, and their fusion closures are always equal (although the pre-fusion systems generally are not). When applied to a characteristic element for a fusion system $\mathcal{F}$, each of the three fusion systems constructions recover $\mathcal{F}$. Thus characteristic elements contain exactly the same information as their fusion systems, encoded in the double Burnside ring. Furthermore, equality of the three fusion systems holds only for characteristic elements, giving us a criterion to characterize characteristic elements from the fusion systems they induce. By [33, Proposition 21.9], the existence of a characteristic element implies saturation for a fusion system, so this criterion can also be thought of as a saturation criterion.

5.1 Stabilizer fusion systems

The stabilizer fusion system of an element is the largest fusion system that stabilizes the element. Stabilizer fusion systems come in three flavors, depending on whether one is looking at right stability, left stability, or both. Formally, they are defined as follows.

Definition 5.1 Let S be a finite p–group, and let X be an element in $A(S,S)_{(p)}$.

(a) The *right stabilizer fusion system* of X is the fusion system $\mathsf{RSt}(X)$ with morphism sets

$$\mathrm{Hom}_{\mathsf{RSt}(X)}(P,Q)=\{\varphi\in\mathrm{Inj}(P,Q)\mid X\circ[P,\varphi]_P^S=X\circ[P,\mathrm{incl}]_P^S\}.$$

(b) The *left stabilizer fusion system* of X is the fusion system $\mathsf{LSt}(X)$ with morphism sets

$$\mathrm{Hom}_{\mathsf{LSt}(X)}(P,Q)=\{\varphi\in\mathrm{Inj}(P,Q)\mid [\varphi(P),\varphi^{-1}]_S^P\circ X=[P,\mathrm{id}_P]_S^P\circ X\}.$$

(c) The *full stabilizer fusion system* of X is the intersection

$$\mathsf{St}(X)=\mathsf{RSt}(X)\cap\mathsf{LSt}(X).$$

We leave it to the reader to verify that the three stabilizer fusion systems are indeed fusion systems. As the following lemma shows, the three stabilizer fusion systems are related, so in practice it usually suffices to prove results for left or right stabilizer systems.

Lemma 5.2 *Let S be a finite p–group. For an element X in $A_{\mathrm{fr}}(S,S)_{(p)}$, we have*

$$\mathtt{RSt}(X) = \mathtt{LSt}(X^{\mathrm{op}}) \quad \textit{and} \quad \mathtt{LSt}(X) = \mathtt{RSt}(X^{\mathrm{op}}).$$

In particular, if X is symmetric, then

$$\mathtt{RSt}(X) = \mathtt{LSt}(X) = \mathtt{St}(X)\,.$$

Proof Lemma 3.22 implies that $\mathtt{RSt}(X) = \mathtt{LSt}(X^{\mathrm{op}})$, from which the other claims follow. □

As most of our arguments in the next two sections are in terms of fixed-point homomorphisms, it is helpful to record how morphisms stabilizing an element can be recognized in that context.

Lemma 5.3 *Let S be a finite p–group and let X be an element in $A(S,S)_{(p)}$. For every subgroup $P \leq S$, the following hold:*

(a) *If $\psi \in \mathrm{Hom}(P,S)$ and $\varphi \in \mathrm{Hom}_{\mathtt{RSt}(X)}(P,S)$, then*

$$\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle \varphi(P),\psi\circ\varphi^{-1}\rangle}(X)\,.$$

(b) *If $\psi \in \mathrm{Hom}(P,S)$ and $\varphi \in \mathrm{Hom}_{\mathtt{LSt}(X)}(\psi(P),S)$, then*

$$\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle P,\varphi\circ\psi\rangle}(X).$$

Proof As X is right $\mathtt{RSt}(X)$–stable and left $\mathtt{LSt}(X)$–stable, this follows from Lemma 4.8. □

5.2 Fixed-point and orbit-type fusion systems

Definition 5.4 Let S be a finite p–group, and let X be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$.

(a) The *orbit-type pre-fusion system* of X is the pre-fusion system $\mathtt{Pre\text{-}Orb}(X)$ with morphism sets

$$\mathrm{Hom}_{\mathtt{Pre\text{-}Orb}(X)}(P,Q) = \{\varphi \in \mathrm{Inj}(P,Q) \mid c_{\langle P,\varphi\rangle}(X) \neq 0\}.$$

The *orbit-type fusion system* of X, denoted $\mathtt{Orb}(X)$, is the closure of $\mathtt{Pre\text{-}Orb}(X)$.

(b) The *fixed-point pre-fusion system* of X is the pre-fusion system $\mathtt{Pre\text{-}Fix}(X)$ with morphism sets

$$\mathrm{Hom}_{\mathtt{Pre\text{-}Fix}(X)}(P,Q) = \{\varphi \in \mathrm{Inj}(P,Q) \mid \Phi_{\langle P,\varphi\rangle}(X) \neq 0\}.$$

The *fixed-point fusion system* of X, denoted $\mathtt{Fix}(X)$, is the closure of $\mathtt{Pre\text{-}Fix}(X)$.

Although the fixed-point and orbit-type pre-fusion systems are different in general, their closures are the same, as shown in the following lemma.

Lemma 5.5 *Let S be a finite p–group. For any $X \in A_{\mathrm{fr}}(S,S)_{(p)}$,*

$$\mathtt{Orb}(X) = \mathtt{Fix}(X).$$

Proof It suffices to show that $\mathtt{Pre\text{-}Orb}(X) \subseteq \mathtt{Fix}(X)$, and $\mathtt{Pre\text{-}Fix}(X) \subseteq \mathtt{Orb}(X)$.

As both $\mathtt{Orb}(X)$ and $\mathtt{Fix}(X)$ are fusion systems they are closed under restriction and conjugation by elements of S. Thus $\psi \in \mathrm{Hom}_{\mathtt{Orb}(X)}(P,S)$ implies that, for every $\langle Q,\varphi\rangle \precsim \langle P,\psi\rangle$, we have $\varphi \in \mathrm{Hom}_{\mathtt{Orb}(X)}(Q,S)$. The analogous statement for $\mathtt{Fix}(X)$ is also true.

Suppose that $\varphi \in \mathrm{Hom}_{\mathtt{Pre\text{-}Orb}(X)}(Q,S)$, so $c_{\langle Q,\varphi\rangle}(X) \neq 0$. Let $\langle P,\psi\rangle$ be a maximal (with respect to subconjugacy) (S,S)–class so that $\langle Q,\varphi\rangle$ is subconjugate to $\langle P,\psi\rangle$, and $c_{\langle P,\psi\rangle}(X) \neq 0$. By Lemma 3.12, maximality implies

$$\Phi_{\langle P,\psi\rangle}(X) = \frac{|N_\psi|}{|P|}|C_S(\psi(P)| \cdot c_{\langle P,\psi\rangle}(X) \neq 0,$$

so $\psi \in \mathtt{Fix}(X)$. Using the remark in the previous paragraph we get $\varphi \in \mathtt{Fix}(X)$.

Suppose now that $\varphi \in \mathrm{Hom}_{\mathtt{Pre\text{-}Fix}(X)}(Q,S)$. Then, using Lemma 3.12 again, there exists an (S,S)–class $\langle P,\psi\rangle$ to which $\langle Q,\varphi\rangle$ is subconjugate and such that $c_{\langle P,\psi\rangle}(X) \neq 0$. Using the remark in the first paragraph we get $\varphi \in \mathtt{Orb}(X)$. □

Lemma 5.6 *Let S be a finite p–group and let X be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$. For every $P \leq S$ and every monomorphism $\varphi\colon P \to S$, we have*

$$\varphi \in \mathrm{Hom}_{\mathtt{Pre\text{-}Fix}(X)}(P,S) \quad \textit{if and only if } \varphi^{-1} \in \mathrm{Hom}_{\mathtt{Pre\text{-}Fix}(X^{\mathrm{op}})}(\varphi(P),S).$$

In particular,

$$\mathtt{Fix}(X) = \mathtt{Fix}(X^{\mathrm{op}}).$$

Proof The first claim is an immediate consequence of Lemma 3.23, and the second claim follows since $\mathtt{Fix}(X)$ is closed under inverses. □

Given that the only extra property of $\mathtt{Fix}(X)$ over $\mathtt{Pre\text{-}Fix}(X)$ that was used in the proof of Lemma 5.6 is the closure under inverses, we can give a more precise statement.

Lemma 5.7 *Let S be a finite p–group and let X be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$. If $\mathtt{Pre\text{-}Fix}(X)$ is level-wise closed, then $\mathtt{Pre\text{-}Fix}(X^{\mathrm{op}}) = \mathtt{Pre\text{-}Fix}(X)$.*

5.3 Fusion systems induced by characteristic elements

The Linckelmann–Webb properties can be rephrased in terms of fusion systems induced by characteristic elements.

Lemma 5.8 *Let $\mathcal{F}$ be a fusion system on a finite p–group S. An element Ω in $A_{\mathrm{fr}}(S,S)_{(p)}$ is right characteristic for $\mathcal{F}$ if and only if $\epsilon(\Omega)$ is prime to p and*

$$\mathtt{Pre\text{-}Orb}(\Omega) \subseteq \mathcal{F} \subseteq \mathtt{RSt}(\Omega).$$

The analogous statement holds for left and fully characteristic elements.

Proof The inclusion $\mathtt{Pre\text{-}Orb}(\Omega) \subseteq \mathcal{F}$ is equivalent to (a) in Definition 4.2, while the inclusion $\mathcal{F} \subseteq \mathtt{RSt}(\Omega)$ is equivalent to condition (b). □

With significant extra work, the inclusion $\mathcal{F} \subseteq \mathtt{RSt}(\Omega)$ in Lemma 5.8 can in fact be strengthened to an equality.

Theorem 5.9 [34] *Let $\mathcal{F}$ be a fusion system on a finite p–group S. If Ω is a right characteristic element for $\mathcal{F}$, then $\mathtt{RSt}(\Omega) = \mathcal{F}$. The analogous result holds for left characteristic and fully characteristic elements for $\mathcal{F}$.*

Proof This is a consequence of [34, Proposition 5.2] in the case where Ω is a characteristic idempotent, and the same argument works for left, right or fully characteristic elements. □

6 Characteristic elements and saturation

In this section we present some key results needed to prove Theorem A in the next section. In Section 6.1 we establish some congruences for fixed-point homomorphisms that we will use repeatedly. In Section 6.2 we use these congruences to prove local saturation axioms.

6.1 Congruence relations for fixed-point homomorphisms

The key to extracting information about the stabilizer and fixed-point fusion systems induced by an element in the double Burnside ring are the congruence relations in the following lemma.

Lemma 6.1 (Broto, Castellana, Grodal, Levi and Oliver [15]) *Let S be a finite p–group, let $X \in A_{\mathrm{fr}}(S,S)_{(p)}$, and put $\mathcal{P} := \mathtt{Pre\text{-}Fix}(X)$.*

(a) *For each* $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$, *the number* $\Phi_{\langle P,\varphi\rangle}(X)$ *is divisible by* $|C_S(\varphi(P))|$ *in* $\mathbb{Z}_{(p)}$. *Furthermore,*

$$\sum_{[\varphi]\in \mathrm{Rep}_{\mathcal{P}}(P,S)} \frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_S(\varphi(P))|} \equiv \epsilon(X) \mod p.$$

(b) *For each* $Q \in [P]_{\mathcal{P}}$, *the number*

$$\sum_{\varphi\in \mathrm{Hom}_{\mathcal{P}}(P,Q)} \Phi_{\langle P,\varphi\rangle}(X)$$

is divisible by $|N_S(\varphi(P))|$ *in* $\mathbb{Z}_{(p)}$. *Furthermore,*

$$\sum_{[Q]\in [P]_{\mathcal{P}}} \frac{\sum_{\varphi\in \mathrm{Hom}_{\mathcal{P}}(P,Q)} \Phi_{\langle P,\varphi\rangle}(X)}{|N_S(Q)|} \equiv \epsilon(X) \mod p,$$

where the sum runs over S*–conjugacy classes of subgroups* $Q \le S$ *that are* $\mathcal{P}$*–images of* P.

Proof This follows by adapting and expanding on an argument used in the proof of Proposition 1.16 in [15]. We outline that argument here, referring the reader to [15] for details, and emphasize the parts that need to be adapted.

We treat here the case where X is an (S,S)–biset. For the sake of clarity, we shall distinguish between the left and right S–actions by considering X as an (S_1,S_2)–set, with the understanding that $S_1 = S_2 = S$. Then $S_2\backslash X$ is a right S_1–set, and we let $X_0 \subseteq X$ be the pre-image of $(S_2\backslash X)^P$ under the projection $X \to S_2\backslash X$, where P acts on the right via the inclusion $P \le S_1$. As explained in [15], this means that for every $x \in X_0$, there is a group monomorphism $\theta(x)\colon P \to S_2$ such that, for all $g \in P$, we have $\theta(x)(g)x = xg$. Thus we get a map $\theta\colon X_0 \to \mathrm{Inj}(P,S_2)$ such that $\theta^{-1}(\varphi) = X^{(P,\varphi)}$, and we have

$$|X_0| = \sum_{\varphi\in \mathrm{Hom}(P,S_2)} |\theta^{-1}(\varphi)| = \sum_{\varphi\in \mathrm{Inj}(P,S_2)} \Phi_{\langle P,\varphi\rangle}(X).$$

Furthermore, $\theta(ax) = c_a \circ \theta(x)$ for $a \in S_2$ and $x \in X_0$, so we get an induced map $\overline{\theta}\colon S_2\backslash X_0 \to \mathrm{InjRep}(P,S_2) := S_2\backslash \mathrm{Inj}(P,S_2)$.

Letting $I(P)$ be the set of subgroups of S_2 that are isomorphic to P, we get a map Image: $\mathrm{Inj}(P,S_2) \to I(P)$ sending a monomorphism to its image. This induces a map $\overline{\text{Image}}$: $\mathrm{Rep}_I(P,S_2) \to \overline{I}(P)$, where $\overline{I}(P)$ is the set of S_2–conjugacy classes in $I(P)$.

These maps all fit into a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\theta"] \arrow[d, "q"] & \mathrm{Inj}(P, S_2) \arrow[r, "\mathrm{Image}"] \arrow[d, "q"] & I(P) \arrow[d, "q"] \\
S_2\backslash X_0 \arrow[r, "\bar{\theta}"] & \mathrm{Rep}_I(P, S_2) \arrow[r, "\overline{\mathrm{Image}}"] & \bar{I}(P)
\end{tikzcd}$$

where the vertical maps are the canonical projections onto S_2–orbits.

For each $\varphi \in \mathrm{Inj}(P, S_2)$, the conjugacy class $[\varphi] \in \overline{\mathrm{Inj}}(P, S_2)$ contains $|S|/|C_{S_2}(\varphi(P)|$ distinct monomorphisms φ', each of which is conjugate to φ, so $\Phi_{\langle P,\varphi'\rangle}(X) = \Phi_{\langle P,\varphi\rangle}(X)$. Therefore,

$$|(q \circ \theta)^{-1}([\varphi])| = \frac{|S|}{|C_{S_2}(\varphi(P))|}\, \Phi_{\langle P,\varphi\rangle}(X).$$

Since the S_2–action on X_0 is free, we obtain

$$\bar{\theta}^{-1}([\varphi]) = \frac{|(q \circ \theta)^{-1}([\varphi])|}{|S_2|} = \frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_{S_2}(\varphi(P))|},$$

and in particular, $\Phi_{\langle P,\varphi\rangle}(X)$ is divisible by $|C_{S_2}(\varphi(P))|$. We have

$$|S_2\backslash X_0| = |(S_2\backslash X)^P| \equiv |S_2\backslash X| = \epsilon(X) \mod p,$$

where the congruence in the middle comes from the fact each non-trivial orbit in a P–set must have cardinality a power of p. Combining this with the above, we obtain

$$\sum_{[\varphi]\in\overline{\mathrm{Inj}}(P,S_2)} \frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_S(\varphi(P))|} = \sum_{[\varphi]\in\overline{\mathrm{Inj}}(P,S_2)} |\bar{\theta}^{-1}([\varphi])| = |S_2\backslash X_0| \equiv \epsilon(X) \mod p.$$

This congruence, and the divisibility property, extend to general $X \in A_{\mathrm{fr}}(S_1, S_2)_{(p)}$ by linearity of the morphisms ϵ and $\Phi_{\langle P,\varphi\rangle}$. Part (a) now follows by observing that we need only sum over $[\varphi] \in \mathrm{Rep}_{\mathcal{P}}(P, S)$, as $\Phi_{\langle P,\varphi\rangle}(X_0) = 0$ when $\varphi \notin \mathrm{Hom}_{\mathcal{P}}(P, S)$.

Part (b) is proved similarly by first showing that, when X is a biset,

$$|(\overline{\mathrm{Image}} \circ \bar{\theta})^{-1}([Q])| = \frac{\sum_{\varphi\in\mathrm{Hom}_{\mathcal{P}}(P,Q)} \Phi_{\langle P,\varphi\rangle}(X)}{|N_S(Q)|}$$

for each $[Q] \in \bar{I}(P)$, then summing over $[Q] \in \bar{I}(P)$ and extending the resulting congruence to general $X \in A_{\mathrm{fr}}(S_1, S_2)_{(p)}$ by linearity. □

6.2 Levelwise saturation results

Adding a stability condition, we get the following characterization of fully centralized and fully normalized subgroups.

Lemma 6.2 *Let S be a finite p–group, and let X be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$ with $\epsilon(X)$ not divisible by p, such that $\mathcal{P} := \texttt{Pre-Fix}(X)$ is level-wise closed. Let $P \le S$ and assume that for all $\varphi,\psi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ we have $\Phi_{\langle P,\varphi\rangle}(X) = \Phi_{\langle P,\psi\rangle}(X)$. Then the following hold:*

(a) *For $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$, the image $\varphi(P)$ is fully $\mathcal{P}$–centralized if and only if*
$$\frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_S(\varphi(P))|} \not\equiv 0 \mod p.$$

(b) *For $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ we have*
$$\sum_{\psi \in \mathrm{Hom}_{\mathcal{P}}(P,\varphi(P))} \Phi_{\langle P,\psi\rangle}(X) = |\mathrm{Aut}_{\mathcal{P}}(\varphi(P))| \cdot \Phi_{\langle P,\varphi\rangle}(X),$$
and $\varphi(P)$ is fully $\mathcal{P}$–normalized if and only if
$$\frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi(P))| \cdot \Phi_{\langle P,\varphi\rangle}(X)}{|N_S(\varphi(P))|} \not\equiv 0 \mod p.$$

Proof By assumption, there is a constant $k \in \mathbb{Z}_{(p)}$ such that $\Phi_{\langle P,\varphi\rangle} = k$ for every $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$. By Lemma 6.1, k is divisible by $|C_S(\varphi(P))|$ for each $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$, and we have the congruence
$$\sum_{[\varphi] \in \mathrm{Rep}_{\mathcal{P}}(P,S)} \frac{k}{|C_S(\varphi(P))|} \equiv \epsilon(X) \not\equiv 0 \mod p.$$
Hence, there is some $[\psi] \in \mathrm{Rep}_{\mathcal{P}}(P,S)$ such that $k/|C_S(\psi(P))| \not\equiv 0 \bmod p$. Since $|C_S(\psi(P))|$ is the highest power of p that divides k, $\psi(P)$ must be fully $\mathcal{P}$–centralized. It follows that, for $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$, $\varphi(P)$ is fully $\mathcal{P}$–centralized if and only
$$k/|C_S(\varphi(P))| \not\equiv 0 \mod p,$$
proving part (a).

The equation in part (b) follows from the facts that $\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle P,\varphi\rangle}(X)$ for all $\psi \in \mathrm{Hom}_{\mathcal{P}}(P,\varphi(P))$, and that $|\mathrm{Hom}_{\mathcal{P}}(P,\varphi(P))| = |\mathrm{Aut}_{\mathcal{P}}(P)|$. The criterion for when $\varphi(P)$ is fully $\mathcal{P}$–normalized is now proved in a similar way to part (a). □

We are now ready to identify a set of conditions that guarantee that the fixed-point pre-fusion system of an element in the double Burnside ring is locally saturated, in the sense of Definition 2.9.

Proposition 6.3 *Let S be a finite p–group, and let X be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$ with $\epsilon(X)$ not divisible by p, such that $\mathcal{P} := \mathtt{Pre\text{-}Fix}(X)$ is level-wise closed. Let $P \le S$ and assume that for all $\varphi, \psi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ we have $\Phi_{\langle P,\varphi\rangle}(X) = \Phi_{\langle P,\psi\rangle}(X)$. Then $\mathcal{P}$ is saturated at P.*

Proof Let $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ be such that $\varphi(P)$ is fully $\mathcal{P}$–normalized. Then, by Lemma 6.2, we have

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi(P))| \cdot \Phi_{\langle P,\varphi\rangle}(X)}{|N_S(\varphi(P))|} \not\equiv 0 \mod p\,.$$

As $|N_S(\varphi(P))| = |\mathrm{Aut}_S(\varphi(P))| \cdot |C_S(\varphi(P))|$, and $|C_S(\varphi(P))|$ divides $\Phi_{\langle P,\varphi\rangle}(X)$ by Lemma 6.1, while $|\mathrm{Aut}_S(\varphi(P))|$ divides $|\mathrm{Aut}_{\mathcal{P}}(\varphi(P))|$ since $\mathrm{Aut}_S(\varphi(P))$ is a subgroup of $\mathrm{Aut}_{\mathcal{P}}(\varphi(P))$, this implies that

$$\frac{|\mathrm{Aut}_{\mathcal{P}}(\varphi(P))|}{|\mathrm{Aut}_S(\varphi(P))|} \not\equiv 0 \mod p \quad \text{and} \quad \frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_S(\varphi(P))|} \not\equiv 0 \mod p.$$

The former incongruence implies that $\mathrm{Aut}_S(\varphi(P))$ is a Sylow subgroup of $\mathrm{Aut}_{\mathcal{P}}(\varphi(P))$, and by Lemma 6.2; the latter implies that $\varphi(P)$ is fully $\mathcal{P}$–centralized. This proves (I_P).

To prove (II_P), let $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ be such that $\varphi(P)$ is fully $\mathcal{P}$–centralized. By Lemma 3.12,

$$\begin{aligned}\Phi_{\langle P,\varphi\rangle}(X) &= \sum_{\langle Q,\psi\rangle} c_{\langle Q,\psi\rangle}(X) \cdot \Phi_{\langle P,\varphi\rangle}([Q,\psi]) \\ &= \sum_{\langle Q,\psi\rangle} c_{\langle Q,\psi\rangle}(X) \cdot \frac{|N_{\varphi,\psi}|}{|Q|} \cdot |C_S(\varphi(P))|,\end{aligned}$$

so the incongruence $\Phi_{\langle P,\varphi\rangle}(X)/|C_S(\varphi(P))| \not\equiv 0 \mod p$ from Lemma 6.2 implies that there exists a (S,S)–pair (Q,ψ) with $c_{\langle Q,\psi\rangle}(X) \neq 0$ and $|N_{\varphi,\psi}|/|Q| \not\equiv 0 \mod p$.

Recall from Lemma 3.13 that $N_{\varphi,\psi}$ is a left-free (N_φ, N_ψ)–biset. In particular, as $Q \le N_\psi$, we can regard $Q\backslash N_{\varphi,\psi}$ as a right N_φ–set. As $|Q\backslash N_{\varphi,\psi}|$ is not divisible by p, and N_φ is a p–group, there must exist at least one $x \in N_{\varphi,\psi}$ such that the orbit Qx is fixed by the N_φ–action on $Q\backslash N_{\varphi,\psi}$. This means that for each $g \in N_\varphi$, there exists an $h \in Q$ such that $xg = hx$. In other words, x conjugates N_φ into Q. Recall that the condition $x \in N_{\varphi,\psi}$ implies that there exists $y \in S$ such that $c_y \circ \varphi = \psi \circ c_x$ as homomorphisms $P \to \psi(Q)$. We now have a commutative diagram

$$\begin{tikzcd}
 & Q \arrow[r, "\psi"] & \psi(Q) \arrow[r, "c_y^{-1}"] & S \\
N_\varphi \arrow[ur, "c_x"] & & & \\
 & P \arrow[ul, "\mathrm{incl}"] \arrow[uu, "c_x"'] \arrow[r, "\varphi"] & \varphi(P), \arrow[uu, "c_y"'] \arrow[uur, "\mathrm{incl}"'] &
\end{tikzcd}$$

and putting $\overline{\varphi} = c_y^{-1} \circ \psi \circ c_x \colon N_\varphi \to S$ we get an extension of φ as required. Finally, we note that as $c_{\langle Q,\psi\rangle}(X) \neq 0$, we have $\psi \in \mathsf{Pre\text{-}Orb}(X)$, and hence $\overline{\varphi} \in \mathsf{Orb}(X) = \mathsf{Fix}(X) = \overline{\mathcal{P}}$. □

In the proof of Proposition 6.3 the left multiplication action of N_ψ on $N_{\varphi,\psi}$ is free and $Q \le N_\psi$, so the fact that $|Q \backslash N_{\varphi,\psi}|$ is not divisible by p implies that $N_\psi = Q$. By the comments made after Definition 2.5, ψ, and hence $\overline{\varphi}$, cannot be extended in $\mathsf{Fix}(X)$. This argument can be used to prove the following result.

Proposition 6.4 *Let $\mathcal{F}$ be a saturated fusion system on a finite p–group S. For $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $\varphi(P)$ is fully $\mathcal{F}$–centralized, there exists a homomorphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ that extends φ and cannot be extended further in $\mathcal{F}$.*

Using the congruence relations proved in this section we get that, if Ω is a left or right characteristic element for $\mathcal{F}$, then $\mathsf{Pre\text{-}Fix}(\Omega)$ is a fusion system.

Proposition 6.5 *Let $\mathcal{F}$ be a fusion system on a finite p–group S. If Ω is a left or right characteristic element for $\mathcal{F}$, then*

$$\mathsf{Pre\text{-}Fix}(\Omega) = \mathsf{Fix}(\Omega) = \mathsf{Orb}(\Omega) = \mathcal{F}.$$

Proof We already have

$$\mathsf{Pre\text{-}Fix}(\Omega) \subseteq \mathsf{Fix}(\Omega) = \mathsf{Orb}(\Omega) \subseteq \mathcal{F},$$

where the first inclusion is immediate, the equality is by Lemma 5.5, and the last inclusion follows from $\mathsf{Pre\text{-}Orb}(\Omega) \subseteq \mathcal{F}$ upon taking closures. Thus it suffices to show that $\mathcal{F} \subseteq \mathsf{Pre\text{-}Fix}(\Omega)$. In other words we show that, for every (S,S)–pair (P,φ) with φ in $\mathcal{F}$, we have $\Phi_{\langle P,\varphi\rangle}(\Omega) \neq 0$.

Consider first the left characteristic case. By Lemma 4.8, left $\mathcal{F}$–stability of Ω implies that $\Phi_{\langle P,\psi\rangle}(\Omega) = \Phi_{\langle P,\varphi\rangle}(\Omega)$ for all $\psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Hence it is enough to consider the case where $\varphi(P)$ is fully $\mathcal{F}$–centralized. But in this case Lemma 6.2 shows that $\Phi_{\langle P,\varphi\rangle}(\Omega)/|C_S(\varphi(P))|$ is nonzero mod p. In particular, $\Phi_{\langle P,\varphi\rangle}(\Omega) \neq 0$. This proves $\mathcal{F} \subseteq \mathtt{Pre\text{-}Fix}(\Omega)$, and the string of equalities follows.

If Ω is a right characteristic element, then Ω^{op} is left characteristic, and we have $\mathtt{Pre\text{-}Fix}(\Omega^{\mathrm{op}}) = \mathcal{F}$. In particular, $\mathtt{Pre\text{-}Fix}(\Omega^{\mathrm{op}})$ is levelwise closed, so $\mathtt{Pre\text{-}Fix}(\Omega) = \mathtt{Pre\text{-}Fix}(\Omega^{\mathrm{op}})$ by Lemma 5.7, and the result follows. □

The equations $\mathtt{RSt}(\Omega) = \mathcal{F}$ and $\mathtt{Pre\text{-}Fix}(\Omega) = \mathcal{F}$ were independently proved by Puig in [33] in the case where Ω is a symmetric characteristic biset.

Note that it is generally not true that $\mathtt{Pre\text{-}Orb}(\Omega) = \mathcal{F}$ for a characteristic element Ω. For instance, the (S,S)–biset $[S,\mathrm{id}]$ is always a characteristic idempotent for the minimal fusion system on S, and $\mathtt{Pre\text{-}Orb}([S,\mathrm{id}])$ contains only one morphism: the identity of S.

These results can also be applied when the fusion system is not specified. This gives a criterion for recognizing characteristic elements, and reconstructing their fusion system.

Proposition 6.6 *Let S be a finite p–group, and let Ω be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$ such that $\epsilon(\Omega)$ is prime to p. If $\mathtt{Pre\text{-}Fix}(\Omega) \subseteq \mathtt{RSt}(\Omega)$, then $\mathtt{RSt}(\Omega)$ is saturated, Ω is a right characteristic element for $\mathtt{RSt}(\Omega)$ and*

$$\mathtt{Pre\text{-}Fix}(\Omega) = \mathtt{Fix}(\Omega) = \mathtt{Orb}(\Omega) = \mathtt{RSt}(\Omega).$$

The analogous statement holds for left and fully characteristic elements.

Proof We prove this for left characteristic elements, as that proof is easiest to write out. The result for a right characteristic element follows by taking opposite sets, and the result for a fully characteristic element follows from the two one-sided results.

Taking closures, the inclusion $\mathtt{Pre\text{-}Fix}(\Omega) \subseteq \mathtt{LSt}(\Omega)$ gives $\mathtt{Fix}(\Omega) \subseteq \mathtt{LSt}(\Omega)$, and since

$$\mathtt{Pre\text{-}Orb}(\Omega) \subseteq \mathtt{Orb}(\Omega) = \mathtt{Fix}(\Omega),$$

this implies $\mathtt{Pre\text{-}Orb}(\Omega) \subseteq \mathtt{LSt}(\Omega)$. Taking $\mathcal{F} := \mathtt{LSt}(\Omega)$ in Lemma 5.8, we deduce that Ω is a left characteristic element for $\mathtt{LSt}(\Omega)$. Theorem 5.9 and Proposition 6.5 then imply the stated equalities.

It remains to prove saturation. Lemma 5.3 implies that for each $P \leq S$, and for $\varphi,\psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ we have $\Phi_{\langle P,\varphi\rangle}(\Omega) = \Phi_{\langle P,\psi\rangle}(\Omega)$. Hence Proposition 6.3 applies, showing that $\mathcal{F} = \mathtt{Pre\text{-}Fix}(\Omega)$ is saturated at P for every $P \leq S$. Hence, $\mathcal{F}$ is saturated. □

In [33, Proposition 21.9], Puig shows that if a fusion system has a right or left characteristic biset, then it is saturated. As a consequence of Proposition 6.6, we get a slight generalization of this result.

Corollary 6.7 *If a fusion system has a right or left characteristic element, then it is saturated.*

Proof If Ω is a right characteristic element for a fusion system $\mathcal{F}$, then $\epsilon(\Omega)$ is not divisible by p, and, by Theorem 5.9 and Proposition 6.5,

$$\mathsf{Pre\text{-}Fix}(\Omega) = \mathcal{F} = \mathsf{RSt}(\Omega).$$

In particular, part (a) of Proposition 6.6 applies to show that $\mathcal{F}$ is saturated. The argument for left characteristic elements is analogous. $\square$

7 Frobenius reciprocity implies saturation

In this section we introduce a condition on elements in the double Burnside ring, called Frobenius reciprocity, and show that under mild technical assumptions, satisfying this condition implies that an element is a right characteristic element for its stabilizer fusion system. The Frobenius reciprocity is independent of fusion systems, and thus this gives an intrinsic way of recognizing characteristic elements.

7.1 Frobenius reciprocity

For finite groups G_1, G_2, H_1 and H_2, Cartesian product induces a bilinear pairing

$$A(G_1, H_1) \times A(G_2, H_2) \longrightarrow A(G_1 \times G_2, H_1 \times H_2), \quad (X, Y) \longmapsto X \times Y,$$

that passes to the p–local setting. In particular, for a finite p–group S, one obtains a bilinear pairing

$$A(S,S)_{(p)} \times A(S,S)_{(p)} \to A(S \times S, S \times S)_{(p)}.$$

Frobenius reciprocity is a condition on the behavior of an element with respect to this pairing and restricting along the diagonal map $\Delta\colon S \to S \times S$.

Definition 7.1 Let S be a finite p–group, and let X be an element in $A(S,S)_{(p)}$. We say that X *satisfies Frobenius reciprocity*, or that X is a *Frobenius reciprocity element*, if

$$(X \times X) \circ [S, \Delta] = (X \times 1) \circ [S, \Delta] \circ X \in A(S, S \times S)_{(p)}, \tag{3}$$

where $1 = [S, \mathrm{id}]$ is the identity in $A(S,S)_{(p)}$.

As the terminology suggests, the Frobenius reciprocity condition is related to classical Frobenius reciprocity in cohomology. The link is explained in Section 9.2.

Proposition 7.2 [16, Proposition 5.5] *A characteristic element of a fusion system satisfies Frobenius reciprocity.*

Proof This was proved on the level of cohomology in [16, Proposition 5.5], and the proof lifts to the double Burnside ring. □

7.2 Fixed-point homomorphisms of Frobenius reciprocity elements

We currently have three copies of S appearing in different roles. To avoid confusion, it is helpful to distinguish them notationally by putting $S_1 := S_2 := S$, and writing $A_{\mathrm{fr}}(S, S_1 \times S_2)_{(p)}$ instead of $A_{\mathrm{fr}}(S, S \times S)_{(p)}$.

Suppose X is a (S, S_1)–biset and Y is a (S, S_2)–biset, and let $[X]$ and $[Y]$ be their isomorphism classes. Then $([X] \times [Y]) \circ [S, \Delta]$ is the isomorphism class of the $(S, S_1 \times S_2)$–biset $(X \times Y) \times_{(S_1 \times S_2)} ((S_1 \times S_2) \times_{(S,\Delta)} S)$, while $([X] \times 1) \circ [S, \Delta] \circ [Y]$ is the isomorphism class of $(X \times S_2) \times_{(S_1 \times S_2)} ((S_1 \times S_2) \times_{(S,\Delta)} S) \times_S Y$. These sets admit a far more convenient description.

Lemma 7.3 *Let S be a finite p–group and write $S_1 := S_2 := S$. Let X be a (S, S_1)–biset and let Y be a (S, S_2)–biset.*

(a) *The $(S, S_1 \times S_2)$–biset $(X \times Y) \times_{(S_1 \times S_2)} ((S_1 \times S_2) \times_{(S,\Delta)} S)$ is isomorphic to $X \times Y$ endowed with the $(S, S_1 \times S_2)$–action*
$$(b_1, b_2)(x, y)a := (b_1 x a, b_2 y a),$$
for $(b_1, b_2) \in S_1 \times S_2, a \in S, x \in X, y \in Y$.

(b) *The $(S, S_1 \times S_2)$–biset $(X \times S_2) \times_{(S_1 \times S_2)} ((S_1 \times S_2) \times_{(S,\Delta)} S) \times_S Y$ is isomorphic to $X \times Y$ endowed with the $(S, S \times S)$–action*
$$(b_1, b_2)(x, y)a := (b_1 x b_2^{-1}, b_2 y a),$$
for $(b_1, b_2) \in S_1 \times S_2, a \in S, x \in X, y \in Y$.

Proof The proof is a straightforward verification. □

An $(S, S_1 \times S_2)$–pair has the form $(P, \psi \times \varphi)$, where P is a subgroup of S, and $\psi\colon P \to S_1$ and $\varphi\colon P \to S_2$ are homomorphisms.

Lemma 7.4 *Let S be a finite p–group and write $S_1 := S_2 := S$. Let $X \in A(S,S_1)_{(p)}$ and let $Y \in A(S,S_2)_{(p)}$. Then, for every $(S, S_1 \times S_2)$–pair $(P, \psi \times \varphi)$:*

(a) $\Phi_{\langle P,\psi\times\varphi\rangle}((X \times Y)\circ[S,\Delta]) = \Phi_{\langle P,\psi\rangle}(X)\cdot\Phi_{\langle P,\varphi\rangle}(Y)$.

(b) *If* $\mathrm{Ker}(\varphi) \le \mathrm{Ker}(\psi)$*, and $\rho\colon \varphi(P) \to \psi(P)$ is the unique homomorphism such that $\rho\circ\varphi = \psi$, then*

$$\Phi_{\langle P,\psi\times\varphi\rangle}((X\times 1)\circ[S,\Delta]\circ Y) = \Phi_{\langle\varphi(P),\rho\rangle}(X)\cdot\Phi_{\langle P,\varphi\rangle}(Y).$$

Proof We prove part (b), leaving the simpler part (a) to the reader. It suffices to prove this for bisets X and Y. In this case we observe that if Z is $X\times Y$ endowed with the $(S, S_1\times S_2)$–action $(b_1,b_2)(x,y)a = (b_1 x b_2^{-1}, b_2 y a)$, then the fixed-point set $Z^{(P,\psi\times\varphi)}$ consists of the pairs (x,y) such that for every $a\in P$ we have

$$(x, ya) = (\psi(a)x\varphi(a)^{-1}, \varphi(a)y).$$

The condition $ya = \varphi(a)y$ for all $a\in P$ is equivalent to $y \in Y^{(P,\varphi)}$. The condition $x = \psi(a)x\varphi(a)^{-1}$ for all $a\in P$ is equivalent to $x\varphi(a) = \psi(a)x$ for all $a\in P$. This can also be written as $xb = \rho(b)x$ for all $b\in\varphi(P)$, which is equivalent to $x\in X^{(\varphi(P),\rho)}$. We deduce that

$$\psi(a)x\varphi(a)^{-1} = X^{(\varphi(P),\rho)} \times Y^{(P,\varphi)},$$

and the result follows. □

Consequent to Lemma 7.4 we get the following lemma, which will be useful to prove closure and stability results for the fixed-point pre-fusion system of a Frobenius reciprocity element.

Lemma 7.5 *Let S be a finite group, and let X be an element in $A(S,S)_{(p)}$ that satisfies Frobenius reciprocity. Let $P \le S$, let $\psi,\varphi\colon P\to S$ be group homomorphisms such that* $\mathrm{Ker}(\varphi)\le\mathrm{Ker}(\psi)$*, and let $\rho\colon\varphi(P)\to\psi(P)$ be the unique homomorphism such that $\rho\circ\varphi=\psi$. Then*

$$\Phi_{\langle P,\varphi\rangle}(X)\cdot\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle P,\varphi\rangle}(X)\cdot\Phi_{\langle\varphi(P),\rho\rangle}(X).$$

In particular, if $\Phi_{\langle P,\varphi\rangle}(X)\neq 0$, then

$$\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle\varphi(P),\rho\rangle}(X).$$

Proof The first equation follows from Lemma 7.4, and the second follows by canceling. □

Observe that when φ is injective, the last equation in Lemma 7.5 is

$$\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle\varphi(P),\psi\circ\varphi^{-1}\rangle}(X).$$

7.3 Level-wise closure and local saturation

Using Lemma 7.5, we now show that, for a Frobenius reciprocity element X in $A(S,S)_{(p)}$, $\mathsf{Pre\text{-}Fix}(X)$ is level-wise closed and saturated at every $P \leq S$. At times it will be easier to work with X^{op} than X, but this makes no difference: once level-wise closure is established we have $\mathsf{Pre\text{-}Fix}(X) = \mathsf{Pre\text{-}Fix}(X^{\mathrm{op}})$, so either way we get the desired results.

Lemma 7.6 *Let S be a finite group, let X be a Frobenius reciprocity element in $A_{\mathrm{fr}}(S,S)_{(p)}$ with augmentation not divisible by p, and set $\mathcal{P} := \mathsf{Pre\text{-}Fix}(X^{\mathrm{op}})$. Let $P \leq S$, and let ι denote the inclusion $P \hookrightarrow S$.*

(a) *If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$, then $\Phi_{\langle P,\varphi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle P,\iota\rangle}(X^{\mathrm{op}})$.*

(b) $\mathrm{Hom}_{\mathcal{S}}(P,S) \subseteq \mathrm{Hom}_{\mathcal{P}}(P,S)$.

(c) *If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ then $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{P}}(\varphi(P),S)$.*

(d) *If $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$ and $\psi \in \mathrm{Hom}_{\mathcal{P}}(\varphi(P),S)$, then $\psi \circ \varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$.*

Proof First recall that $\Phi_{\langle P,\varphi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle \varphi(P),\varphi^{-1}\rangle}(X)$ for all (S,S)–pairs (P,φ). Hence Lemma 7.5 implies that if (P,φ) is an (S,S)–pair with $\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$, then, for any (S,S)–pair (Q,ψ) with $\psi(Q) = \varphi(P)$, we have

$$\Phi_{\langle Q,\psi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle Q,\varphi^{-1}\circ\psi\rangle}(X^{\mathrm{op}}). \tag{4}$$

Part (a) follows by taking $\psi = \varphi$ in (4).

Part (b) is equivalent to $\Phi_{\langle P,\iota\rangle}(X^{\mathrm{op}}) \neq 0$. By part (a), it suffices to show that $\mathrm{Hom}_{\mathcal{P}}(P,S)$ is nonempty. This follows from part (a) of Lemma 6.1 since $\epsilon(X^{\mathrm{op}}) = \epsilon(X)$ is not divisible by p.

For part (c), we take ψ to be the inclusion $i\colon \varphi(P) \hookrightarrow S$ in (4) and obtain

$$\Phi_{\langle \varphi(P),\varphi^{-1}\rangle}(X^{\mathrm{op}}) = \Phi_{\langle \varphi(P),i\rangle}(X^{\mathrm{op}}).$$

As $\Phi_{\langle \varphi(P),i\rangle}(X^{\mathrm{op}}) \neq 0$ by part (b), this implies that φ^{-1} is in $\mathcal{P}$.

For part (d), we first observe that by part (c), $\psi \in \mathrm{Hom}_{\mathcal{P}}(\varphi(P),S)$ implies that $\psi^{-1} \in \mathrm{Hom}_{\mathcal{P}}(\psi\circ\varphi(P),S)$. Applying (4) with $(\psi\circ\varphi(P),\psi^{-1})$ in place of (P,φ) and (P,φ) in place of (Q,ψ), we obtain $\Phi_{\langle P,\psi\circ\varphi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle P,\varphi\rangle}(X^{\mathrm{op}}) \neq 0$, so $\psi\circ\varphi \in \mathrm{Hom}_{\mathcal{P}}(P,S)$. □

Corollary 7.7 *Let S be a finite group, and let X be a Frobenius reciprocity element in $A_{\mathrm{fr}}(S,S)_{(p)}$ with augmentation not divisible by p. Then $\mathsf{Pre\text{-}Fix}(X) = \mathsf{Pre\text{-}Fix}(X^{\mathrm{op}})$ and both are level-wise closed. Furthermore, the following hold.*

(a) *If* $\varphi\colon P \to S$ *and* $\psi\colon Q \to S$ *are morphisms in* $\mathtt{Pre\text{-}Fix}(X)$ *with* $\varphi(P) = \psi(Q)$, *then* $\Phi_{\langle P,\varphi\rangle}(X) = \Phi_{\langle Q,\psi\rangle}(X)$.

(b) *If* $\varphi, \psi\colon P \to S$ *are morphisms in* $\mathtt{Pre\text{-}Fix}(X)$, *then*

$$\Phi_{\langle P,\varphi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle P,\psi\rangle}(X^{\mathrm{op}}).$$

(c) $\mathtt{Pre\text{-}Fix}(X)$ *is saturated at* P *for every* $P \le S$.

Proof The level-wise closure of $\mathtt{Pre\text{-}Fix}(X^{\mathrm{op}})$ follows from parts (b)–(d) of Lemma 7.6, and consequently Lemma 5.6 implies that $\mathtt{Pre\text{-}Fix}(X) = \mathtt{Pre\text{-}Fix}(X^{\mathrm{op}})$. The claim in (b) follows from part (a) of Lemma 7.6, and (a) is just the reformulation for $\mathtt{Pre\text{-}Fix}(X)$. By (a) the conditions in Proposition 6.3 are satisfied, so we apply it to get (c). □

7.4 Closure under restriction

Now that we have shown that the fixed-point pre-fusion system of a Frobenius reciprocity element X is level-wise closed and saturated at every subgroup, all that remains to prove saturation is to show that $\mathtt{Pre\text{-}Fix}(X)$ is closed under restriction. This is easy when X is a biset but difficult for general elements of the double Burnside ring as the relations between fixed points don't behave well with respect to the restriction. The good news is that we can use level-wise saturation and Frobenius reciprocity to prove closure of $\mathtt{Pre\text{-}Fix}(X)$, as we show below. The first step in this direction is to get some control over the possible ways to extend a given morphism in $\mathtt{Pre\text{-}Fix}(X)$.

Lemma 7.8 *Let* S *be a finite group and let* X *be an element in* $A_{\mathrm{fr}}(S,S)_{(p)}$. *For subgroups* $P < Q$ *of* S *such that* P *has index* p *in* Q, *and a monomorphism* $\varphi\colon P \to S$ *that can be extended to a homomorphism* $\overline{\varphi}\colon Q \to S$, *we have*

$$\frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_S(\varphi(P))|} \equiv \sum_{[\psi]\in E(P,\varphi;Q)} \frac{\Phi_{\langle Q,\psi\rangle}(X)}{|C_S(\psi(Q))|} \mod p,$$

where

$$E(P,\varphi;Q) = \{[\psi] \in \mathrm{Rep}(Q,S) \mid [\psi|_P] = [\varphi]\}.$$

Proof It is enough to prove this when X is an (S,S)–biset. For clarity, we make a notational distinction between the two copies of S by regarding X as an (S_1,S_2)–biset, with the understanding that $S_1 = S_2 = S$.

Now, consider the subset

$$Y := S_2 X^{(P,\varphi)} \subseteq X,$$

consisting of elements of the form ax where $a \in S_2$ and $x \in X^{(P,\varphi)}$. Although Y is not necessarily a (S_1, S_2)–subset of X, we show that Y is closed under the (Q, S_2)–action obtained by restriction. To prove this, it is enough to show that if $x \in X^{(P,\varphi)}$ and $b \in Q$, then $xb \in Y$. As $xb = \overline{\varphi}(b)(\overline{\varphi}(b)^{-1}xb)$, it suffices to show that $\overline{\varphi}(b)^{-1}xb \in X^{(P,\varphi)}$. To do this, we first note that P must be normal in Q because of the index, and for $g \in P$ we have $\varphi(bgb^{-1}) = \overline{\varphi}(b)\varphi(g)\overline{\varphi}(b)^{-1}$. Hence, for all $g \in P$ we have

$$\begin{aligned}(\overline{\varphi}(b)^{-1}xb)g &= \overline{\varphi}(b)^{-1}x(bgb^{-1})b = \overline{\varphi}(b)^{-1}\varphi(bgb^{-1})xb\\ &= \overline{\varphi}(b)^{-1}(\overline{\varphi}(b)\varphi(g)\overline{\varphi}(b)^{-1})xb = \varphi(g)(\overline{\varphi}(b)^{-1}xb),\end{aligned}$$

so $\overline{\varphi}(b)^{-1}xb \in X^{(P,\varphi)}$.

Next, we consider the induced right Q–subset $S_2\backslash Y$. We have a congruence

$$|S_2\backslash Y| \equiv |(S_2\backslash Y)^Q| \mod p,$$

and the result follows once we show that

$$|S_2\backslash Y| = \frac{\Phi_{\langle P,\varphi\rangle}(X)}{|C_S(\varphi(P))|}, \tag{5}$$

and

$$|(S_2\backslash Y)^Q| = \sum_{[\psi]\in E(P,\varphi;Q)} \frac{\Phi_{\langle Q,\psi\rangle}(X)}{|C_S(\psi(Q))|}. \tag{6}$$

Let $x \in X^{(P,\varphi)}$ and $a \in S_2$. For $g \in P$, we have

$$axg = a\varphi(g)x = \varphi(g)(\varphi(g)^{-1}a\varphi(g))x.$$

Since X is left-free, this implies that $ax \in X^{(P,\varphi)}$ if and only if $a \in C_S(\varphi(P))$. Equation (5) follows.

To prove (6), let $Y_0 \subseteq Y$ be the pre-image of $(S_2\backslash Y)^Q$ under the projection $Y \to S_2\backslash Y$. Just as in the proof of Lemma 6.1, we obtain a map $\theta\colon Y_0 \to \mathrm{Inj}(Q, S_2)$ such that for all $g \in Q$ we have $yg = \theta(y)(g)y$, and, since $\theta(ax) = c_a \circ \theta(x)$, an induced map $\overline{\theta}\colon S_2\backslash Y_0 \to \overline{\mathrm{Inj}}(Q, S_2) = S_2\backslash \mathrm{Inj}(Q, S_2)$, fitting into a commutative diagram

$$\begin{tikzcd} Y_0 \arrow[r, "\theta"] \arrow[d, "q"] & \mathrm{Inj}(Q, S_2) \arrow[d, "q"] \\ S_2\backslash Y_0 \arrow[r, "\overline{\theta}"] & \overline{\mathrm{Inj}}(Q, S_2) \end{tikzcd}$$

where the vertical maps are the canonical projection onto S_2–orbits. Again, as in the proof of Lemma 6.1, we deduce that

$$|(S_2\backslash Y)^Q| = |S_2\backslash Y_0| = \sum_{[\psi]\in\overline{\mathrm{Inj}}(Q,S_2)} \overline{\theta}^{-1}([\psi]) = \sum_{[\psi]\in\overline{\mathrm{Inj}}(Q,S_2)} \frac{\Phi_{\langle Q,\psi\rangle}(Y)}{|C_S(\psi(Q))|}.$$

The proof is completed by showing that

$$\Phi_{\langle Q,\psi\rangle}(Y) = \begin{cases}\Phi_{\langle Q,\psi\rangle}(X) & \text{if } \psi\in E(P,\varphi;Q),\\ 0 & \text{otherwise.}\end{cases}$$

If $y\in Y^{(Q,\psi)}$, then $yb=\psi(b)y$ for all $b\in Q$, and in particular for $b\in P$. We can also write $y=ax$ with $x\in X^{(P,\varphi)}$ and $a\in S_2$. Therefore, for $b\in P$, we have

$$yb = axb = a\varphi(b)x = c_a\circ\varphi(b)ax = c_a\circ\varphi(b)y,$$

so $\psi(b)y = c_a\circ\varphi(b)y$, and by left-freeness $\psi(b)=c_a\circ\varphi(b)$. We deduce that if $Y^{(Q,\psi)}$ is nonempty, then $\psi\in E(P,\varphi;Q)$, so $\Phi_{\langle Q,\psi\rangle}(Y)=0$ when $\psi\notin E(P,\varphi;Q)$.

Next we show that $Y^{(Q,\psi)} = X^{(Q,\psi)}$ when $[\psi]\in E(P,\varphi;Q)$. We certainly have $Y^{(Q,\psi)}\subseteq X^{(Q,\psi)}$ since $Y\subseteq X$. Now, $[\psi]\in E(P,\varphi;Q)$ implies that $\psi|_P = c_a\circ\varphi$ for some $a\in S_2$. If $x\in X^{(Q,\psi)}$, then, for all $b\in P$,

$$a^{-1}xb = a^{-1}\psi(b)x = a^{-1}a\varphi(b)a^{-1}x = \varphi(b)a^{-1}x,$$

so $a^{-1}x\in X^{(P,\varphi)}$, and $x\in Y = SX^{(P,\varphi)}$. Thus $X^{(Q,\psi)} = Y\cap X^{(Q,\psi)} = Y^{(Q,\psi)}$. □

Lemma 7.9 *Let S be a finite group, and let X be a Frobenius reciprocity element in $A_{\mathrm{fr}}(S,S)_{(p)}$ with augmentation not divisible by p. Then $\mathsf{Pre\text{-}Fix}(X)$ is closed.*

Proof Write $\mathcal{P}$ for $\mathsf{Pre\text{-}Fix}(X)$ to simplify notation. By Corollary 7.7, $\mathcal{P}$ is level-wise closed and saturated at P for every $P\le S$.

We prove that $\mathcal{P}$ is closed by showing that

$$\mathrm{Hom}_{\mathcal{P}}(Q,S) = \mathrm{Hom}_{\overline{\mathcal{P}}}(Q,S)$$

for all $Q\le S$ by downward induction on conjugacy classes of subgroups of S.

For the base case, $Q=S$, level-wise closure of $\mathcal{P}$ implies that $\mathrm{Aut}_{\mathcal{P}}(S) = \mathrm{Aut}_{\overline{\mathcal{P}}}(S)$.

For the inductive step, let $\mathcal{H}$ be a family of subgroups of S that is closed under $\mathcal{P}$–conjugacy and taking supergroups, and assume that for all $Q\in\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{P}}(Q,S) = \mathrm{Hom}_{\overline{\mathcal{P}}}(Q,S).$$

Let P be maximal among subgroups of S not in $\mathcal{H}$, and set $\mathcal{H}' := \mathcal{H} \cup [P]_{\mathcal{P}}$. To show that the induction hypothesis holds for $\mathcal{H}'$, it suffices, since P was chosen arbitrarily from its $\mathcal{P}$–conjugacy class, to show that $\mathrm{Hom}_{\mathcal{P}}(P, S) = \mathrm{Hom}_{\overline{\mathcal{P}}}(P, S)$. Furthermore, as $\mathcal{P}$ is level-wise closed, it suffices to show that $\mathcal{P}$ is closed under restricting morphisms to P. That is, we need to show that if $P < Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{P}}(Q, S)$, then the restriction $\varphi|_P$ is in $\mathcal{P}$. By the induction hypothesis, $\mathcal{P}$ is closed under restriction of morphisms to groups in $\mathcal{H}$, so it is enough to consider the case where $P \lhd Q$ is an extension of index p.

Using Lemma 7.8 we have

$$\frac{\Phi_{\langle P,\varphi|_P\rangle}(X)}{|C_S(\varphi(P))|} \equiv \sum_{[\psi]\in E(P,\varphi|_P;Q)} \frac{\Phi_{\langle Q,\psi\rangle}(X)}{|C_S(\psi(Q))|} \mod p.$$

Similarly, for the inclusion $i\colon \varphi(P) \hookrightarrow S$, we have

$$\frac{\Phi_{\langle \varphi(P),i\rangle}(X)}{|C_S(\varphi(P))|} \equiv \sum_{[\rho]\in E(\varphi(P),i;\varphi(Q))} \frac{\Phi_{\langle \varphi(Q),\rho\rangle}(X)}{|C_S(\rho(\varphi(Q)))|} \mod p .$$

There is a bijection $E(P,\varphi; Q) \to E(\varphi(P), i; \varphi(Q))$ sending $[\psi]$ to $[\psi \circ \varphi^{-1}]$. Moreover, since $\varphi \in \mathcal{P}$, Lemma 7.5 implies that for $\psi \in E(P, \varphi|_P; Q)$,

$$\Phi_{\langle Q,\psi\rangle}(X) = \Phi_{\langle \varphi(Q),\psi\circ\varphi^{-1}\rangle}(X).$$

Thus the sums on the right sides of the two congruences above actually agree term by term, and we deduce that

$$\frac{\Phi_{\langle P,\varphi|_P\rangle}(X)}{|C_S(\varphi(P))|} \equiv \frac{\Phi_{\langle \varphi(P),i\rangle}(X)}{|C_S(\varphi(P))|} \mod p.$$

Now, by Lemma 5.6, we have

$$\Phi_{\langle \varphi(P),i\rangle}(X) = \Phi_{\langle \varphi(P),i\rangle}(X^{\mathrm{op}}).$$

First consider the case where $\varphi(P)$ is fully centralized in $\mathcal{P}$. Then, Lemma 6.2 and Corollary 7.7 give

$$\frac{\Phi_{\langle \varphi(P),i\rangle}(X^{\mathrm{op}})}{|C_S(\varphi(P))|} \not\equiv 0 \mod p.$$

We conclude that $\Phi_{\langle P,\varphi|_P\rangle}(X) \neq 0$, and hence $\varphi|_P \in \mathrm{Hom}_{\mathcal{P}}(P, S)$.

Second, consider the general case, no longer assuming that $\varphi(P)$ is fully $\mathcal{P}$–centralized. We shall apply an argument analogous to the proof of [16, Proposition A.2] to obtain a homomorphism $\alpha \in \mathrm{Hom}_{\mathcal{P}}(\varphi(Q), S)$ such that $\alpha(\varphi(P))$ is fully $\mathcal{P}$–centralized. The previous argument then implies that the restrictions $\alpha|_{\varphi(P)}$ and $(\alpha \circ \varphi)|_P$ are in $\mathcal{P}$, and as $\mathcal{P}$ is level-wise closed, this implies that $\varphi|_P$ is in $\mathcal{P}$.

To obtain α, let $\gamma\colon \varphi(P) \xrightarrow{\cong} P'$ be an isomorphism in $\mathcal{P}$ such that P' is fully $\mathcal{P}$–normalized. As $\mathcal{P}$ has Property (I_P), this implies that P' is fully $\mathcal{P}$–centralized, and that $\mathrm{Aut}_S(P')$ is a Sylow subgroup of $\mathrm{Aut}_{\mathcal{P}}(P')$. The latter implies that $\gamma^{-1}\mathrm{Aut}_S(P')\gamma$ is a Sylow subgroup of $\mathrm{Aut}_{\mathcal{P}}(\varphi(P))$, and hence there exists $\chi \in \mathrm{Aut}_{\mathcal{P}}(\varphi(P))$ such that

$$\mathrm{Aut}_S(\varphi(P)) \leq \chi^{-1} \circ \gamma^{-1}\mathrm{Aut}_S(P')\gamma \circ \chi.$$

This in turn implies that $N_{\gamma\circ\chi} = N_S(\varphi(P))$, and as $\gamma \circ \chi(\varphi(P)) = P'$ is fully $\mathcal{P}$–centralized, Property ($\mathrm{II}_{\varphi(P)}$) implies that there exists a homomorphism $\overline{\alpha} \in \mathrm{Hom}_{\overline{\mathcal{P}}}(N_S(P), S)$ such that $\overline{\alpha}|_{\varphi(P)} = \gamma \circ \chi$, and in particular $\overline{\alpha}(\varphi(P)) = P'$. The desired α is obtained by restricting $\overline{\alpha}$ to $\varphi(Q)$. (Recall that P is normal in Q, so $\varphi(Q) \leq N_S(\varphi(P))$. By the induction hypothesis, $\mathrm{Hom}_{\overline{\mathcal{P}}}(\varphi(Q), S) = \mathrm{Hom}_{\mathcal{P}}(\varphi(Q), S)$, so α is in $\mathcal{P}$.

This completes the induction, and hence the proof that $\mathcal{P}$ is closed. □

7.5 Proof of Theorems A and B

Collecting our results from this section, we deduce that Frobenius reciprocity implies saturation. Theorem A is a consequence of the following result.

Theorem 7.10 *Let S be a finite p–group and let X be an element in $A_{\mathrm{fr}}(S,S)_{(p)}$. If $\epsilon(X)$ is not divisible by p and X satisfies Frobenius reciprocity, then*

$$\mathtt{Pre\text{-}Fix}(X) = \mathtt{Fix}(X) = \mathtt{Orb}(X) = \mathtt{RSt}(X),$$

and $\mathtt{RSt}(X)$ is saturated with right characteristic element X.

Proof Corollary 7.7 and Lemma 7.9 combine to show that $\mathcal{F} := \mathtt{Pre\text{-}Fix}(X)$ is a saturated fusion system. Corollary 7.7 and Lemma 4.8 show that X is right $\mathcal{F}$–stable, so $\mathcal{F} \subseteq \mathtt{RSt}(X)$. The rest follows as in Proposition 6.6. □

Theorem 7.10 (combined with Proposition 7.2) gives an intrinsic criterion for recognizing characteristic elements without mentioning fusion systems: A bifree element in $A(S,S)_{(p)}$ is a characteristic element for a fusion system on S if and only if it has augmentation prime to p and satisfies Frobenius reciprocity. The fusion system, which must be saturated, can be recovered via a stabilizer, fixed-point or orbit-type construction.

Using the correspondence between saturated fusion systems and their characteristic idempotents, we obtain a new characterization of saturated fusion systems, which is Theorem B in the introduction.

Theorem 7.11 *For a finite p–group S, there is a bijective correspondence between saturated fusion systems on S and symmetric idempotents in $A(S,S)_{(p)}$ of augmentation 1 that satisfy Frobenius reciprocity. The bijection sends a saturated fusion system to its characteristic idempotent, and an idempotent to its stabilizer fusion system.*

Proof By Theorems 5.9 and 4.6, the maps described in the statement give a bijection between saturated fusion systems on S and their characteristic idempotents. Also, the characteristic idempotent of a saturated fusion system is symmetric, has augmentation 1, and satisfies Frobenius reciprocity. Conversely, suppose we have a symmetric idempotent in $A(S,S)_{(p)}$ of augmentation 1 that satisfies Frobenius reciprocity. By symmetry, left-freeness of ω implies that ω is bifree. Theorem 7.10 then implies that $\mathtt{RSt}(\omega)$ is saturated, and ω is a right characteristic idempotent. Finally, symmetry implies that $\mathtt{St}(\omega) = \mathtt{RSt}(\omega)$, and ω is a full characteristic idempotent. □

8 Relaxing the right freeness condition

The right freeness condition in Theorem A is prohibitively restrictive for some applications, and we now examine to which extent it can be relaxed. Although the material in this section is purely algebraic, the motivation comes from stable homotopy: in the remainder of the paper we focus on interpreting Theorem A in the context of the stable homotopy theory of classifying spaces via the Segal conjecture, and the right freeness condition has no reasonable interpretation in that context. Note that the right freeness assumption in Theorem 7.10 cannot just be removed: If S is a finite group, and ψ is a non-injective, idempotent endomorphism of S, then $[S,\psi]$ is an idempotent in $A(S,S)^\wedge_p$ that satisfies Frobenius reciprocity and has augmentation 1, but is certainly not the characteristic idempotent of any fusion system. Instead we show that the right freeness condition is automatically satisfied by an element of $A(S,S)^\wedge_p$ that satisfies Frobenius reciprocity if we assume that it is not generated by maps that factor through proper subgroups of S. This is a familiar condition in stable homotopy theory, first considered by Nishida in [30].

We begin by formulating the freeness condition in terms of fixed points.

Lemma 8.1 *Let S be a finite p–group.*

(a) *An element* $X \in A(S,S)^\wedge_p$ *is in* $A_{\mathrm{fr}}(S,S)^\wedge_p$ *if and only if for every* (S,S)*–pair* (P,ψ) *where* ψ *is not injective,*

$$\Phi_{\langle P,\psi\rangle}(X) = 0.$$

(b) *If* $X \in A(S,S)^\wedge_p$ *and* (P,ψ) *is an* (S,S)*–pair where* ψ *is not injective, then*

$$\Phi_{\langle P,\psi\rangle}(X^{\mathrm{op}}) = 0.$$

Proof Part (a) follows from Lemmas 3.6 and 3.3. Part (b) is easy to prove for bisets and the general result follows by linearity. □

Our goal is to identify suitable conditions, under which an element that satisfies Frobenius reciprocity has $\Phi_{\langle P,\psi\rangle}(X) = 0$ for every (S,S)–pair (P,ψ) where ψ is not injective. To this end we note the following consequences of Frobenius reciprocity.

Lemma 8.2 *Let* S *be a finite* p*–group, and assume that* $X \in A(S,S)^\wedge_p$ *satisfies Frobenius reciprocity. If* (P,ψ) *is an* (S,S)*–pair such that* $\Phi_{\langle P,\psi\rangle}(X) \neq 0$*, then*

(a) $\Phi_{\langle \psi(P),\mathrm{incl}\rangle}(X) = \Phi_{\langle P,\psi\rangle}(X)$*, and*

(b) $\Phi_{\langle P,\psi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle P,\mathrm{incl}\rangle}(X)$*.*

Proof Part (a) is a special case of Lemma 7.5. Part (b) is proved similarly by first showing that Frobenius reciprocity implies

$$\Phi_{\langle P,\mathrm{incl}\rangle}(X)\cdot\Phi_{\langle P,\psi\rangle}(X) = \Phi_{\langle P,\psi\rangle}(X^{\mathrm{op}})\cdot\Phi_{\langle P,\psi\rangle}(X).$$

The left side of this equation is equal to $\Phi_{\langle P,\mathrm{incl}\times\psi\rangle}((X\times X)\circ[S,\Delta])$, so it is enough to prove

$$\Phi_{\langle P,\mathrm{incl}\times\psi\rangle}((X\times 1)\circ[S,\Delta]\circ X) = \Phi_{\langle P,\psi\rangle}(X^{\mathrm{op}})\cdot\Phi_{\langle P,\psi\rangle}(X).$$

It suffices to consider the case where X is a biset, in which case we can look at actual fixed-point sets. By Lemma 7.3, $(X\times 1)\circ[S,\Delta]\circ X$ is isomorphic to $Z := X\times X$, with $(S, S\times S)$ action given by $(b_1,b_2)(x,y)a = (b_1 x b_2^{-1}, b_2 y a)$. The fixed-point set $Z^{(P,\mathrm{incl}\times\psi)}$ consists of pairs $(x,y)\in X\times X$ such that for all $a\in P$ we have

$$a x \psi(a)^{-1} = x \quad\text{and}\quad ya = \psi(a)y.$$

The latter condition is equivalent to $y \in X^{(P,\psi)}$, and rewriting the former condition as $ax = x\psi(a)$, we see that it is equivalent to $x\in (X^{\mathrm{op}})^{(P,\psi)}$. Hence $Z^{(P,\mathrm{incl}\times\psi)} = (X^{\mathrm{op}})^{(P,\psi)}\times X^{(P,\psi)}$, and the result follows. □

We establish the nonzero-condition in Lemma 8.2 by a counting argument, dualizing Lemma 6.1. For this we need a dual of the augmentation.

Definition 8.3 For a finite group S, let $\epsilon^R\colon A(S,S)^\wedge_p \to \mathbb{Z}^\wedge_p$ be the $\mathbb{Z}^\wedge_p$–linear map defined on bisets by $\epsilon^R(X) = |X/S|$.

Notice that for a generator $[P,\psi]$ of $A(S,S)$ we have $\epsilon([P,\psi]) = |S|/|P|$, while $\epsilon^R([P,\psi]) = |S|/|\psi(P)|$. Hence, for $X \in A_{\mathrm{fr}}(S,S)^\wedge_p$ we have $\epsilon(X) = \epsilon^R(X)$, but this is not true for general $X \in A(S,S)^\wedge_p$.

Lemma 8.4 *Let S be a finite group and let $X \in A(S,S)^\wedge_p$. For $P \le S$, let* $\mathrm{Sur}(P)$ *be the set of (S,S)–pairs (Q,ψ) such that $\psi(Q) = P$, and let* $\mathrm{SurRep}(P)$ *be the set of conjugacy classes under the conjugacy relation $(Q,\psi) \sim (Q^x, \psi \circ c_x)$ for $x \in S$. Then*

$$\sum_{(Q,\psi)\in \mathrm{SurRep}(P)} \frac{\Phi_{\langle Q,\psi\rangle}(X)}{|C_S(Q)|} \equiv \epsilon^R(X) \mod p$$

where the sum runs over representatives of conjugacy classes in $\mathrm{SurRep}(P)$.

Proof This follows from an argument similar to that used in the proof of part (a) of Lemma 6.1. As in that proof, it suffices to consider the case where X is a biset. This time we let $X_0 \subseteq X$ be the preimage of $(X/S)^P$ under the projection $X \to X/S$, where P now acts on the left. There results a map $\theta\colon X_0 \to \mathrm{Sur}(P)$ with $\theta^{-1}(Q,\psi) = \Phi_{\langle Q,\psi\rangle}(X)$. This descends to a map $\overline{\theta}\colon X_0/S \to \mathrm{SurRep}(P)$. The desired congruence now follows as in Lemma 6.1. □

We now have all the ingredients to prove the first result relaxing the right freeness condition in Theorem 7.10.

Proposition 8.5 *Let S be a finite group, and let X be an element in $A(S,S)^\wedge_p$. If X satisfies Frobenius reciprocity and $\epsilon^R(X)$ is not divisible by p, then $X \in A_{\mathrm{fr}}(S,S)^\wedge_p$. In particular $\epsilon(X) = \epsilon^R(X)$, and Theorem 7.10 applies to X.*

Proof We first show that for every $P \le S$, we have $\Phi_{\langle P,\mathrm{incl}\rangle}(X) \ne 0$. Indeed, the condition on $\epsilon^R(X)$ and Lemma 8.4 imply that there exists an (S,S)–pair (Q,ψ) with $\psi(Q) = P$ such that $\Phi_{\langle Q,\psi\rangle}(X) \ne 0$. Part (a) of Lemma 8.2 then implies $\Phi_{\langle P,\mathrm{incl}\rangle}(X) \ne 0$.

Now suppose (P,ψ) is a (S,S)–pair such that $\Phi_{\langle P,\psi\rangle}(X) \ne 0$. By part (b) of Lemma 8.2 and the condition $\Phi_{\langle P,\mathrm{incl}\rangle}(X) \ne 0$ proved above, we have $\Phi_{\langle P,\psi\rangle}(X^{\mathrm{op}}) \ne 0$. But part (b) of Lemma 8.1 then implies that ψ is injective.

We have shown that $\Phi_{\langle P,\psi\rangle}(X) = 0$ whenever ψ is not injective, and by part (a) of Lemma 8.1 this implies that $X \in A_{\mathrm{fr}}(S,S)^\wedge_p$. □

While Proposition 8.5 does relax the right freeness conditions, it is replaced by the condition on $\epsilon(R)$, which is equally problematic for the intended applications in later sections. However, with further work we obtain a result that is better suited to our needs. For this we recall the following definition.

Definition 8.6 [30] Let S be a finite group. The *Nishida ideal* $J(S) \subset A(S,S)$ is the $\mathbb{Z}$–submodule generated by elements $[P,\psi]$, where $\psi(P) < S$. An element $X \in A(S,S)^\wedge_p$ is *dominant* if $X \notin J(S)^\wedge_p$.

Strictly speaking, this is an extension of Nishida's definition, as Nishida applied the term "dominant" only to indecomposable idempotents. The double coset formula readily shows that $J(S)$ is a two-sided ideal of $A(S,S)$.

Lemma 8.7 *Let S be a finite p–group, and let X be an element of $A(S,S)^\wedge_p$. If X is dominant and satisfies Frobenius reciprocity, then, for every non-injective homomorphism $\psi\colon S \to S$, we have $c_{\langle S,\psi\rangle}(X) = 0$. In particular $\epsilon^R(X) \equiv \epsilon(X)$ mod p.*

Proof First note that for a homomorphism $\psi\colon S \to S$, Lemma 3.12 says that for an (S,S)–class $\langle P,\varphi\rangle$, we have

$$\Phi_{\langle S,\psi\rangle}([P,X]) = \begin{cases} Z(\psi(P)) & \text{if } \langle P,\varphi\rangle = \langle S,\psi\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$\Phi_{\langle S,\psi\rangle}(X) = |Z(S)| \cdot c_{\langle S,\psi\rangle}(X), \tag{7}$$

and it follows that $c_{\langle S,\psi\rangle}(X) \neq 0$ if and only if $\Phi_{\langle S,\psi\rangle}(X) \neq 0$.

Now, since X is dominant, there exists an automorphism $\varphi \in \mathrm{Aut}(S)$ such that $c_{\langle S,\varphi\rangle}(X) \neq 0$, and hence $\Phi_{\langle S,\varphi\rangle}(X) \neq 0$. Lemma 8.2 then implies that $\Phi_{\langle S,\mathrm{incl}\rangle}(X) \neq 0$. If $\psi\colon S \to S$ is a group homomorphism such that $\Phi_{\langle S,\psi\rangle}(X) \neq 0$, then Lemma 8.2 implies that $\Phi_{\langle S,\psi\rangle}(X^{\mathrm{op}}) = \Phi_{\langle S,\mathrm{incl}\rangle}(X) \neq 0$, and by Lemma 8.1 this means that ψ must be injective. Thus we conclude that for non-injective homomorphisms $\psi\colon S \to S$ we have $c_{\langle S,\psi\rangle}(X) = 0$.

For an (S,S)–pair (P,ψ) we have $\epsilon([P,\psi]) = |S|/|P|$ and $\epsilon^R([P,\psi]) = |S|/|\psi(P)|$. It follows that $\epsilon(X)$ is congruent mod p to the sum of coefficients $c_{\langle S,\psi\rangle}(X)$ where ψ runs over conjugacy classes of homomorphisms $S \to S$, while $\epsilon^R(X)$ is congruent mod p to the sum of coefficients $c_{\langle S,\varphi\rangle}(X)$ where φ runs over conjugacy classes of automorphisms of S. Since $c_{\langle S,\psi\rangle}(X) = 0$ for non-injective ψ, these sums are the same and we have $\epsilon^R(X) \equiv \epsilon(X)$ mod p. □

We can now replace the right-freeness condition in Theorem A by a dominance condition.

Theorem 8.8 *Let S be a finite p–group, and let X be a dominant element of $A(S,S)^\wedge_p$ that satisfies Frobenius reciprocity. If $\epsilon(X)$ is not divisible by p, then $\mathtt{RSt}(X)$ is a saturated fusion system and X is a right-characteristic element for $\mathtt{RSt}(X)$.*

Proof By Lemma 8.7, $\epsilon^R(X)$ is not divisible by p. The result now follows from Proposition 8.5 and Theorem 7.10. □

When working with dominant idempotents we can even remove the augmentation condition.

Corollary 8.9 *Let S be a finite p–group, and let ω be a dominant idempotent in $A(S,S)^\wedge_p$. If ω satisfies Frobenius reciprocity, then ω is a right-characteristic idempotent for $\mathtt{RSt}(\omega)$, which is saturated.*

Proof The result follows from Theorem 8.8 if we can show that ω has augmentation not divisible by p. We have

$$\epsilon(\omega) \equiv \sum_{\psi \in W} c_{\langle S,\psi\rangle}(\omega) \mod p,$$

where W is the set of conjugacy classes of homomorphisms $\psi\colon S \to S$ with $c_{\langle S,\psi\rangle}(\omega) \neq 0$. By Lemma 8.7 we have $W \subseteq \mathrm{Out}(S)$. Using (7), Lemma 8.2 then implies that for all $\varphi \in W$, we have $c_{\langle S,\varphi\rangle}(\omega) = c_{\langle S,\mathrm{incl}\rangle}(\omega)$, and hence

$$\epsilon(\omega) \equiv |W| \cdot c \mod p$$

where $c = c_{\langle S,\mathrm{incl}\rangle}(\omega)$ is a nonzero constant.

Now consider the projection

$$\pi\colon A(S,S)^\wedge_p \twoheadrightarrow A(S,S)^\wedge_p / J(S)^\wedge_p \cong \mathbb{Z}^\wedge_p \mathrm{Out}(S).$$

This is a homomorphism of $\mathbb{Z}^\wedge_p$–algebras, so $\pi(\omega) = \sum_{\varphi\in W} c \cdot \varphi$ is an idempotent in $\mathbb{Z}^\wedge_p\mathrm{Out}(S)$ with augmentation $|W|\cdot c$. Hence $|W|\cdot c$ equals 0 or 1. The former would imply that $|W|$ or c is zero, contradicting dominance, and hence $|W|\cdot c = 1$. Consequently $\epsilon(\omega)$ is not divisible by p, and the result follows from Theorem 8.8. (In fact $\epsilon(\omega) = 1$ since ω is idempotent.) □

9 Relation to stable homotopy of classifying spaces

The correspondence between saturated fusion systems and Frobenius idempotents provides a tool to study the role of fusion in the stable homotopy theory of classifying spaces. By the Segal conjecture, stable self-maps of the classifying space of a finite p–group S correspond to elements in the p–completion of the double Burnside ring, $A(S,S)^\wedge_p$. The characteristic idempotent of a saturated fusion system $\mathcal{F}$ on S can thus be used to split off a stable summand of BS, and we call this summand a *classifying spectrum* $\mathbb{B}\mathcal{F}$ for $\mathcal{F}$. This extends what happens for finite groups, as $\mathbb{B}\mathcal{F}_S(G) \simeq \Sigma^\infty BG^\wedge_{p+}$ when G is a finite group with S as Sylow p–subgroup.

In this section we provide the background needed to pass to the stable world, and present an immediate application of Theorem A (more specifically, Corollary 8.9) by characterizing the stable summands of the classifying space of a finite p–group that have the homotopy type of the classifying spectrum of a saturated fusion system. The reader is assumed to have some familiarity with stable homotopy theory and spectra; this is an extensive subject and providing the necessary background cannot be reasonably done in this paper. For the results presented here one requires only a category of spectra with minimal structure, such as the homotopy category of spectra developed in Adams [1]. The reader is also referred to Adams [2] for background on stable transfer maps arising from finite covering maps, such as maps of classifying spaces induced by subgroup inclusions.

9.1 The Segal conjecture

The stable homotopy of classifying spaces of finite groups is linked to the Burnside category via the Segal conjecture, and we briefly describe this link here. First, recall that for finite groups G and H, there is a natural map

$$\alpha\colon A(G,H) \longrightarrow \{BG_+, BH_+\}, \quad [K,\varphi] \longmapsto \Sigma^\infty B\varphi \circ \mathrm{tr}_K,$$

where the subscript $+$ denotes an added disjoint basepoint, $\{BG_+, BH_+\}$ is the group of homotopy classes of stable maps, $\mathrm{tr}_K\colon \Sigma^\infty BG_+ \to \Sigma^\infty BK_+$ is the stable transfer associated to the subgroup inclusion $K \le G$, and $\Sigma^\infty B\varphi\colon \Sigma^\infty BK_+ \to \Sigma^\infty BH_+$ is the obvious map. The (single) Burnside ring of finite G–sets, $A(G)$, acts on $A(G,H)$ by Cartesian product, and the Segal conjecture deals with completion at the augmentation ideal $I(G) \subseteq A(G)$.

Theorem 9.1 (Carlsson [19], Lewis, May and McClure [25]) *For finite groups G and H, the natural map $\alpha\colon A(G,H) \to \{BG_+, BH_+\}$ is an $I(G)$–adic completion map.*

May and McClure showed in [28] that when S is a p–group, the I–adic completion of $A(S,H)$ is "essentially p–completion". This is helpful as the p–completion of $A(S,H)$ admits a convenient description: Since $A(S,H)$ is a finitely generated $\mathbb{Z}$–module, [10, Proposition 10.13] implies that $A(S,H)^\wedge_p \cong \mathbb{Z}^\wedge_p \otimes A(S,H)$. Hence $A(S,H)^\wedge_p$ is a free $\mathbb{Z}^\wedge_p$–module on the standard basis of $A(S,H)$ (cf. Lemma 3.8). We offer the following formulation of the result of May and McClure.

Proposition 9.2 [28] *For a finite p–group P and any finite group G, the $I(P)$–adic topology on $A(P,G)$ is finer than the p–adic topology, and the resulting completion map $A(P,G)^\wedge_I \to A(P,G)^\wedge_p$ is an injection whose image is the submodule of elements with augmentation in $\mathbb{Z}$.*

Proof Write $I = I(P)$. May and McClure showed in [28] that if $|P| = p^n$, then $I^{n+1} \subseteq pI$, proving the first claim. They also showed that if K is the kernel of the restriction map $r\colon A(P,G) \to A(1,G)$, then the $I(P)$–adic topology on K coincides with the p–adic topology. Observe also that r is a map of $A(P)$–modules, where $A(P)$ acts on $A(1,G) \cong \mathbb{Z}$ by $X \cdot n = |X| \cdot n$. Since I acts on $A(1,G)$ by the zero map, $A(1,G)^\wedge_I \cong A(1,G)$. By [10, Proposition 10.12], I–adic and p–adic completions are both exact on finitely generated modules, so the short exact sequence $K \to A(P,G) \to A(1,G)$ gives rise to a commutative diagram

$$\begin{tikzcd}
K^\wedge_p \arrow[r] \arrow[d, equal] & A(P,G)^\wedge_I \arrow[r] \arrow[d] & A(1,G) \arrow[d] \\
K^\wedge_p \arrow[r] & A(P,G)^\wedge_p \arrow[r] & A(1,G)^\wedge_p
\end{tikzcd}$$

with exact rows that arise from I–adic and p–adic completions, respectively. The vertical maps are the canonical maps from I–adic to p–adic completion, coming from the fact that I–adic topology is finer. In particular, the right-hand map can be identified with the canonical map $\mathbb{Z} \to \mathbb{Z}^\wedge_p$, and is thus injective. Injectivity of the map $A(P,G)^\wedge_I \to A(P,G)^\wedge_p$ follows by a simple diagram chase. Observing that one can identify the restriction $A(P,G) \to A(1,G)$ with the augmentation $\epsilon\colon A(P,G) \to \mathbb{Z}$ completes the proof. □

In particular, Proposition 9.2 allows us to regard $A(P,G)^\wedge_I$ as a submodule of $A(P,G)^\wedge_p$, and so it makes sense to talk about the element in $A(P,G)^\wedge_p$ corresponding to a stable map $\Sigma^\infty BS_+ \to \Sigma^\infty BG_+$, bypassing the I–adic completion. Note that we can also regard $A(S,G)_{(p)}$ as a submodule of $A(S,G)^\wedge_p$ in the usual way. One can extend the notion of characteristic element to include elements in the p–completed or I–adically completed double Burnside ring, and all the results obtained thus far for the p–localized double Burnside ring carry over to this setting.

9.2 Frobenius reciprocity and the push-pull formula

For a finite group S, the Frobenius reciprocity condition

$$(X \times X) \circ [S, \Delta_S] = (X \times 1) \circ [S, \Delta_S] \circ X \tag{8}$$

on an element $X \in A(S,S)$ readily translates to a familiar condition in stable homotopy, which allows us to explain the relationship to the classical Frobenius reciprocity property in cohomology. Applying α turns (8) into a homotopy

$$(\alpha(X) \wedge \alpha(X)) \circ \Sigma^\infty B\Delta_S \simeq (\alpha(X) \wedge \mathrm{id}_{\Sigma^\infty BS_+}) \circ \Sigma^\infty B\Delta_S \circ \alpha(X) \tag{9}$$

of stable maps from $\Sigma^\infty BS_+$ to $\Sigma^\infty BS_+ \wedge \Sigma^\infty BS_+$.

When G is a finite group with Sylow subgroup S, let $[G]$ be G regarded as an (S,S)–biset. Then $[G]$ is a characteristic biset for $\mathcal{F}_S(G)$, and $\alpha([G])$ factors as

$$\alpha([G])\colon \Sigma^\infty BS_+ \xrightarrow{\Sigma^\infty Bi} \Sigma^\infty BG_+ \xrightarrow{\mathrm{tr}_S} \Sigma^\infty BS_+,$$

where i is the inclusion $S \le G$, and tr_S is the associated transfer. The *push-pull formula* (see [2]) expresses the naturality of transfers with respect to Cartesian products as the homotopy

$$(\mathrm{id}_{\Sigma^\infty BG_+} \wedge \mathrm{tr}_S) \circ \Sigma^\infty B\Delta_G \simeq (\Sigma^\infty Bi \wedge \mathrm{id}_{\Sigma^\infty BS_+}) \circ \Sigma^\infty B\Delta_S \circ \mathrm{tr}_S \tag{10}$$

of stable maps from $\Sigma^\infty BG_+$ to $\Sigma^\infty BG_+ \wedge \Sigma^\infty BS_+$. Applying the cohomology functor, the diagonal maps $B\Delta_S$ and $B\Delta_G$ induce multiplication maps μ_G and μ_S, respectively, and we obtain the commutative diagram:

$$\begin{tikzcd}
H^*(BG) \otimes H^*(BS) \arrow[r, "\mathrm{Res} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mathrm{Tr}"] & H^*(BS) \otimes H^*(BS) \arrow[r, "\mu_S"] & H^*(BS) \arrow[d, "\mathrm{Tr}"] \\
H^*(BG) \otimes H^*(BG) \arrow[rr, "\mu_G"] & & H^*(BG)
\end{tikzcd}$$

This diagram expresses the familiar Frobenius reciprocity relation in cohomology, namely that for all $x \in H^*(BG)$ and $y \in H^*(BS)$ one has

$$\mathrm{Tr}(\mathrm{Res}(x)y) = x\,\mathrm{Tr}(y).$$

Composing with $(\mathrm{tr}_S \wedge \mathrm{id}_{\Sigma^\infty BS_+})$ on the left and $\Sigma^\infty Bi$ on the right of both sides of (10), and using $\Sigma^\infty B\Delta_G \circ \Sigma^\infty Bi \simeq (\Sigma^\infty Bi \wedge \Sigma^\infty Bi) \circ \Sigma^\infty B\Delta_S$, yields the homotopy

$$(\alpha([G]) \wedge \alpha([G])) \circ \Sigma^\infty B\Delta_S \simeq (\alpha([G]) \wedge \mathrm{id}_{\Sigma^\infty BS_+}) \circ \Sigma^\infty B\Delta_S \circ \alpha([G]),$$

which is equivalent to the Frobenius reciprocity condition in Definition 7.1 for the biset $[G]$.

9.3 Pointed classifying spectra of saturated fusion systems

A functorial assignment of classifying spectra to saturated fusion systems was given in [34, Section 7], based on results from [16] and building on ideas by Linckelmann and Webb. We recall the construction and some basic properties of that assignment in this subsection. We use the opportunity to remedy an unfortunate choice made by the first author in [34] by framing the current account in terms of "pointed" classifying spectra.

We start by observing that if $\mathcal{F}$ is a saturated fusion system on a finite p–group S, then the characteristic idempotent $\omega_{\mathcal{F}}$ can, by Proposition 9.2, be regarded as an element of $A(S,S)^\wedge_I$, since it has augmentation 1. Hence there is a corresponding stable map $\widetilde{\omega}_{\mathcal{F}} := \alpha(\omega_{\mathcal{F}})\colon \Sigma^\infty BS_+ \to \Sigma^\infty BS_+$, which we call the *stable characteristic idempotent* of $\mathcal{F}$. The *classifying spectrum* of $\mathcal{F}$ is defined as the stable summand carved out of BS_+ by $\widetilde{\omega}_{\mathcal{F}}$ via the standard mapping telescope construction

$$\mathbb{B}\mathcal{F}_+ := \mathrm{Tel}(\widetilde{\omega}_{\mathcal{F}}) := \mathrm{HoColim}\Big(\Sigma^\infty BS_+ \xrightarrow{\widetilde{\omega}_{\mathcal{F}}} \Sigma^\infty BS_+ \xrightarrow{\widetilde{\omega}_{\mathcal{F}}} \cdots\Big).$$

We denote the structure map of the homotopy colimit by $\sigma_{\mathcal{F}}\colon \Sigma^\infty BS_+ \to \mathbb{B}\mathcal{F}_+$ and refer to it as the *structure map* of the classifying spectrum. There is a unique (up to homotopy) map $t_{\mathcal{F}}\colon \mathbb{B}\mathcal{F}_+ \to \Sigma^\infty BS_+$ such that $\sigma_{\mathcal{F}} \circ t_{\mathcal{F}} \simeq \mathrm{id}_{\mathbb{B}\mathcal{F}_+}$ and $t_{\mathcal{F}} \circ \sigma_{\mathcal{F}} \simeq \widetilde{\omega}_{\mathcal{F}}$, to which we refer as a *transfer map*. Classifying spectra are functorial with respect to fusion-preserving homomorphisms. A fusion-preserving monomorphism also gives rise to a transfer between classifying spectra, and this transfer construction is functorial.

Restricting the diagonal map Δ_S of $\Sigma^\infty BS_+$ to $\mathbb{B}\mathcal{F}_+$, one obtains a map

$$\Delta_{\mathcal{F}} := (\sigma_{\mathcal{F}} \wedge \sigma_{\mathcal{F}}) \circ \Delta_S \circ t_{\mathcal{F}}\colon \mathbb{B}\mathcal{F}_+ \to \mathbb{B}\mathcal{F}_+ \wedge \mathbb{B}\mathcal{F}_+ .$$

The Frobenius reciprocity relation for $\omega_{\mathcal{F}}$ implies that $\Delta_{\mathcal{F}}$ is coassociative up to homotopy, so we can think of $\Delta_{\mathcal{F}}$ as a homotopy diagonal map of $\mathbb{B}\mathcal{F}_+$. Frobenius reciprocity for $\omega_{\mathcal{F}}$ also implies

$$\Delta_{\mathcal{F}} \circ \sigma_{\mathcal{F}} \simeq (\sigma_{\mathcal{F}} \wedge \sigma_{\mathcal{F}}) \circ \Delta_S,$$

and the Frobenius reciprocity relation

$$(\mathrm{id}_{\mathbb{B}\mathcal{F}_+} \wedge t_{\mathcal{F}}) \circ \Delta_{\mathcal{F}} \simeq (\sigma_{\mathcal{F}} \wedge \mathrm{id}_{\Sigma^\infty BS_+}) \circ \Delta_S \circ t_{\mathcal{F}}.$$

The homotopy classes of maps to or from classifying spectra of saturated fusion systems admit a nice stable elements description.

Proposition 9.3 [34, Remark 9.3] *Let $\mathcal{F}$ be a saturated fusion system on a finite p–group S, and let E be any spectrum. The maps*

$$E^*(\sigma_{\mathcal{F}})\colon E^*(\mathbb{B}\mathcal{F}_+)\longrightarrow E^*(\Sigma^\infty BS_+) \quad\text{and}\quad E_*(t_{\mathcal{F}})\colon E_*(\mathbb{B}\mathcal{F}_+)\longrightarrow E_*(\Sigma^\infty BS_+)$$

are split injections with image the $\mathcal{F}$–stable elements in $E^*(\Sigma^\infty BS_+)$ and $E_*(\Sigma^\infty BS_+)$, respectively. Furthermore, if E^* is a ring spectrum, then $E^*(\sigma_{\mathcal{F}})$ is a map of algebras, and $E^*(t_{\mathcal{F}})$ is a map of $E^*(\mathbb{B}\mathcal{F}_+)$–modules.

In particular, applying Proposition 9.3 twice, one obtains a description of the group of homotopy classes of stable maps between classifying spectra.

Corollary 9.4 *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are saturated fusion systems on finite p–groups S_1 and S_2, respectively, then the map*

$$[\mathbb{B}\mathcal{F}_{1+},\mathbb{B}\mathcal{F}_{2+}]\longrightarrow[\Sigma^\infty BS_{1+},\Sigma^\infty BS_{2+}],\quad f\longmapsto t_{\mathcal{F}_2}\circ f\circ\sigma_{\mathcal{F}_1},$$

is a split injection with image the $(\mathcal{F}_1,\mathcal{F}_2)$–stable maps in $[\Sigma^\infty BS_{1+},\Sigma^\infty BS_{2+}]$.

Corollary 9.4 can be regarded as a generalization of the Segal conjecture to fusion systems. This analogy is strengthened by Diaz and Libman in [21, Theorem B] where they reformulate the result as a completion theorem in the case where $S_2=1$, so $[\mathbb{B}\mathcal{F}_{1+},\mathbb{B}\mathcal{F}_{2+}]=[\mathbb{B}\mathcal{F}_{1+},\mathbb{S}^0]$ is the zeroth cohomotopy group of $\mathbb{B}\mathcal{F}_{1+}$, which matches Segal's original formulation. It is not hard to extend this reformulation to the general case but that would take us too far afield.

9.4 Applications to stable splittings

The stable splitting of p–completed classifying spaces has been studied intensively by many authors, most notably by Martino and Priddy in [27], and Benson and Feshbach in [12]. A good overview of the subject was given by Benson in [11]. Let G be a finite group with Sylow subgroup S. By a simple transfer argument, $BG_+{}^\wedge_p$ is a stable summand of BS_+. Thus the stable splitting of $BG_+{}^\wedge_p$ can be described by first determining the complete stable splitting of BS_+, and then determining how many copies of each stable summand of BS_+ can be found in a stable splitting of $BG_+{}^\wedge_p$. This is done in [27] and [12].

Given an idempotent in $e\in\{BS_+,BS_+\}$, the mapping telescope $\mathrm{Tel}(e)$ is a stable summand of BS_+. This gives a correspondence between (homotopy types of) stable summands of BS_+ and (conjugacy classes of) idempotents in $\{BS_+, BS_+\}$, which, by the Segal conjecture, correspond to (conjugacy classes of) idempotents in $A(S,S)^\wedge_p$. Under this correspondence, an indecomposable summand corresponds to an irreducible idempotent. Thus a complete stable splitting of BS corresponds to a decomposition of the identity in $A(S,S)^\wedge_p$ as an orthogonal sum of irreducible idempotents. In particular, since the double Burnside ring satisfies the Krull–Schmidt Theorem (see [11, Section 2]), one obtains a uniqueness result for the complete stable splitting of BS_+.

Stewart Priddy has asked what characterizes stable summands of BS_+ that have the stable homotopy type of $BG_+{}^\wedge_p$ for some finite group G with Sylow subgroup S. An equivalent question is when an idempotent in $A(S,S)^\wedge_p$ splits off a summand of BS that has the stable homotopy type of $BG_+{}^\wedge_p$. If we redefine the question, and instead ask for a characterization of the more general class of stable summands that have the homotopy type of the classifying spectrum of a saturated fusion system, we can go some way toward providing an answer.

Corollary 9.5 *Let S be finite p–group, and let $e \in A(S,S)^\wedge_p$ be a dominant idempotent. The stable summand* $\mathrm{Tel}(e)$ *of BS has the homotopy type of the classifying spectrum of a saturated fusion system if and only if e is conjugate to an idempotent in $A(S,S)^\wedge_p$ that is dominant and satisfies Frobenius reciprocity.*

Proof This follows directly from Corollary 8.9. □

It should be noted that Priddy's question goes beyond Corollary 9.5, as he was interested in a description of the collection of indecomposable summands that together make up a summand of the form $BG_+{}^\wedge_p$. This question remains open, but Corollary 9.5 raises hopes that one can answer this by analyzing, given an indecomposable idempotent e, the idempotents "detecting" $(e \times e) \circ [S, \Delta]$ and $(e \times 1) \circ [S, \Delta] \circ e$.

10 Miller's conjecture on the homotopy characterization of p–local finite groups

p–local finite groups were introduced by Broto, Levi and Oliver in [16] as a model for the classifying space of a fusion system. Haynes Miller proposed an alternative model for p–local finite groups, which provided a starting point for the work in this paper and motivated the investigation of the Frobenius reciprocity condition. We discuss his conjecture in this section and apply our main results to reduce the conjecture to proving two technical conditions for the spaces he proposed.

10.1 p–local finite groups

A p–local finite group, as defined by Broto, Levi and Oliver in [16], is a model for the classifying space of a saturated fusion system. Their definition is quite technical in nature, and we recount only the basic facts needed for the discussion that follows, referring the reader to [16] for details. A *p–local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite p–group, $\mathcal{F}$ is a saturated fusion system on S and $\mathcal{L}$ is a *centric linking system* associated to $\mathcal{F}$. The latter is a category whose objects are the $\mathcal{F}$–centric subgroups of S, and whose morphism sets $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ are free $Z(P)$–sets with $\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \cong \mathrm{Hom}_{\mathcal{F}}(P, Q)$. The geometric realization $|\mathcal{L}|^\wedge_p$ is called the *classifying space* of the p–local finite group, and it comes equipped with a map $\theta\colon BS \to |\mathcal{L}|^\wedge_p$, which we think of as an inclusion map.

The driving question in the subject of p–local finite groups has been the existence and uniqueness of centric linking systems (and hence classifying spaces) associated to saturated fusion systems. This question was settled in the affirmative (on both counts) by Chermak in [20] (see also Oliver [31]) while this article was under review.

The homotopical properties of classifying spaces of p–local finite groups closely resemble those of p–completed classifying spaces of finite groups. We recall only one important property here, and refer the interested reader to [16] for further information.

Given a finite p–group S, a space X and a map $f\colon BS \to X$, define a fusion system $\mathcal{F}_{S,f}(X)$ by setting

$$\mathrm{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q) := \{\varphi \in \mathrm{Inj}(P, Q) \,:\, f|_{BP} \simeq f|_{BQ} \circ B\varphi\}$$

for groups $P, Q \le S$. Here $f|_{BP}$ is the composite

$$BP \xrightarrow{B\,\mathrm{incl}} BS \xrightarrow{f} X,$$

and $\simeq$ means non-basepoint-preserving homotopy. In general one should not expect $\mathcal{F}_{S,f}(X)$ to be a saturated fusion system, although this is true when X is the classifying space of a p–local finite group (see Theorem 10.1 below). One can also define a category $\mathcal{L}_{S,f}(X)$ whose objects are the $\mathcal{F}_{S,f}(X)$–centric subgroups of S, with morphism sets

$$\mathrm{Mor}_{\mathcal{L}_{S,f}(X)}(P, Q) = \{(\varphi, [H])\},$$

where $[H]$ is a homotopy class of homotopies between $f|_{BP}$ and $f|_{BQ} \circ B\varphi$. As the following result shows, a p–local finite group is determined by its classifying space. We refer the reader to [16, Section 7] for the precise meaning of the isomorphism of linking systems in the statement.

Theorem 10.1 [16, Proposition 7.3] *For a p–local finite group $(S, \mathcal{F}, \mathcal{L})$ we have*

(1) $\mathcal{F}_{S,\theta}(|\mathcal{L}|^{\wedge}_{p}) = \mathcal{F}$, *and*

(2) $\mathcal{L}_{S,\theta}(|\mathcal{L}|^{\wedge}_{p}) \cong \mathcal{L}$.

Applying the same construction in stable homotopy, one has the following result.

Theorem 10.2 [34, Theorem 7.3] *For a saturated fusion system $\mathcal{F}$ on a finite p–group S we have*

$$\mathcal{F}_{S,\sigma_{\mathcal{F}}}(\mathbb{B}\mathcal{F}) = \mathcal{F}.$$

By [34, Proposition 10.1] one can identify $\Sigma^\infty\theta\colon \Sigma^\infty BS_+ \to \Sigma^\infty|\mathcal{L}|^{\wedge}_{p+}$ with the structure map $\sigma_{\mathcal{F}}\colon \Sigma^\infty BS_+ \to \mathbb{B}\mathcal{F}_+$. Thus Theorem 10.2 says that part (a) of Theorem 10.1 remains true after passing to stable homotopy.

10.2 p–tract groups

The definition of p–local finite groups includes elements of group theory and category theory, and it would be highly desirable to have a purely homotopy-theoretic model for the p–local homotopy theory of classifying spaces of finite groups. Such a model was suggested by Haynes Miller, defined as follows.

Definition 10.3 A p–tract group is a triple (S, f, X), where

- S is a finite p–group,
- X is a connected, p–complete space with finite fundamental group,
- $f\colon BS \to X$ is a homotopy monomorphism that admits a transfer retract.

Here *homotopy monomorphism* means that f induces a finite extension $H^*(X;\mathbb{F}_p) \to H^*(BS;\mathbb{F}_p)$. A *transfer retract* is a map $\Sigma^\infty X_+ \to \Sigma^\infty BS_+$ that satisfies

$$\Sigma^\infty f_+ \circ t \simeq \mathrm{id}_{\Sigma^\infty X_+}$$

and the Frobenius reciprocity relation

$$(\mathrm{id}_{\Sigma^\infty X_+} \wedge t) \circ \Delta_X \simeq (\Sigma^\infty f_+ \wedge \mathrm{id}_{\Sigma^\infty BS_+}) \circ \Delta_{BS} \circ t,$$

where Δ_{BS} and Δ_X denote the diagonals of $\Sigma^\infty BS_+$ and $\Sigma^\infty X_+$, respectively. In this case $t \circ \Sigma^\infty f_+$ is idempotent up to homotopy, and hence corresponds to an idempotent ω in $A(S,S)^{\wedge}_{p}$. It is not hard to show that ω satisfies Frobenius reciprocity (the argument given in Section 9.2 can easily be adapted). Note that X is then the stable summand of BS corresponding to the idempotent ω. We say that the p–tract group is *dominant* if X contains a dominant summand of BS, which is equivalent to $\omega \notin J(S)^{\wedge}_{p}$.

10.3 A reduction of Miller's conjecture

Miller conjectured that there is a correspondence between p–tract groups and p–local finite groups, up to appropriate equivalence relations. A partial confirmation was obtained in the first author's thesis and published in [35]. More precisely, it was shown that if (V, f, X) is a p–tract group with V an elementary abelian p–group, then $(V, \mathcal{F}_{V,f}(X), \mathcal{L}_{V,f}(X))$ is a p–local finite group with classifying space homotopy equivalent to X. The converse direction was treated more generally, showing that if $(S, \mathcal{F}, \mathcal{L})$ is a p–local finite group on *any* finite p–group S, then there exists a transfer retract t for θ, and thus $(S, \theta, |\mathcal{L}|^{\wedge}_p)$ is a p–tract group on S.

The results in the current paper allow us to make further progress toward proving Miller's conjecture. One major obstacle to showing that a p–tract group gives rise to a p–local finite group is associating a saturated fusion system to it. This was overcome in [35], when the Sylow p–subgroup is an elementary abelian group V, by using a variant of the Adams–Wilkerson Invariant Theorem [3] (Theorem 11.1) to show that X has the homology type of the classifying space of a semi-direct product $W \ltimes V$, and then using Miller's Theorem [29] to deduce the necessary homotopical information from the homological information. The Adams–Wilkerson Theorem can be replaced by Theorem 8.8 to obtain the following result.

Theorem 10.4 *If (f, t, X) is a dominant p–tract group on a finite p–group S, then $\mathcal{F}_{S,f}(\Sigma^\infty X_+)$ is a saturated fusion system on S.*

Proof Let ω be the idempotent in $A(S, S)^{\wedge}_p$ corresponding to the homotopy idempotent $t \circ f$ of $\Sigma^\infty BS_+$. Then ω is a dominant idempotent that satisfies Frobenius reciprocity, so, by Corollary 8.9, $\mathrm{RSt}(\omega)$ is a saturated fusion system on S. As $\mathcal{F}_{S,f}(\Sigma^\infty X_+) = \mathrm{RSt}(\omega)$, this completes the proof. □

Theorem 10.4 shows that X has the stable homotopy type of the classifying spectrum of a saturated fusion system. More precisely, it shows that $(f, \Sigma^\infty X_+)$ has the homotopy type of the structured classifying spectrum of a saturated fusion system. To show that X has the homotopy type of the classifying space of a p–local finite group, it now remains to show that the necessary unstable homotopy information can be extracted from the stable homotopy information. This can be done in a fairly simple manner, using existing techniques from Broto, Levi and Oliver [17], and Broto and Møller [18], if one assumes two further technical conditions on p–tract groups.

Theorem 10.5 *Let (S, f, X) be a dominant p–tract group. Suppose that (S, f, X) satisfies the following two conditions:*

(1) *For every $P \leq S$, the map $\Sigma^\infty \colon [BP, X] \to \{BP_+, X_+\}$ is injective, and*

(2) *for every $\mathcal{F}_{S,f}(X)$–centric subgroup $P \leq S$, the induced map of mapping space components,*

$$\mathrm{Map}(BP, BS)_{B\,\mathrm{incl}} \xrightarrow{f \circ \cdot} \mathrm{Map}(BP, X)_{f|_{BP}},$$

is a homotopy equivalence.

Then $(S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}(X))$ is a p–local finite group with classifying space homotopy equivalent to X.

Proof By Theorem 10.4, $\mathcal{F}_{S,\Sigma^\infty f}(\Sigma^\infty X_+)$ is a saturated fusion system on S. In general one has an inclusion $\mathcal{F}_{S,f}(X) \subseteq \mathcal{F}_{S,\Sigma^\infty f}(\Sigma^\infty X_+)$, and Condition (1) implies that this is an equality; in particular $\mathcal{F}_{S,f}(X)$ is saturated. Condition (2) and [17, Lemma 1.8] now imply that $\mathcal{L}_{S,f}(X)$ is a centric linking system associated to $\mathcal{F}_{S,f}(X)$. Hence $(S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}(X))$ is a p–local finite group.

It remains to show that $|\mathcal{L}|^\wedge_p \simeq X$. By [18, Proposition 4.6] there is a map

$$h \colon |\mathcal{L}_{S,f}(X)| \to X$$

such that $h \circ \theta \simeq f$. Passing to cohomology, f and θ both induce injections with image the $\mathcal{F}$–stable elements in $H^*(BS; \mathbb{F}_p)$. Therefore h induces an isomorphism in cohomology with $\mathbb{F}_p$–coefficients, and consequently the p–completion $h \colon |\mathcal{L}|^\wedge_p \to X^\wedge_p \simeq X$ is a homotopy equivalence. □

At first glance, Theorem 10.5 is a statement about a quite restrictive special case of p–tract groups, but it can also be interpreted as a reduction of Miller's conjecture to showing that the two conditions in Theorem 10.5 are always true for a p–tract group. Although the two conditions are quite strong, this point of view is reasonable as the conditions are indeed satisfied when X is the classifying space of a p–local finite group: Condition (1) follows from the computation of $[BP, |\mathcal{L}|^\wedge_p]$ in [16, Theorem C] and the computation of $\{BP_+, |\mathcal{L}|^\wedge_{p+}\}$ in [34, Theorem B], while Condition (2) is proved in [16, Theorem 4.4(c)]. In joint work with Matthew Gelvin [22], the first author has developed a draft proof that, if Lannes's unpublished generalization of the Segal conjecture for elementary abelian groups holds, then the two conditions in Theorem 10.5 are indeed satisfied, and Miller's conjecture is true.

Theorem 10.5 can also be regarded as a characterization of classifying spaces of p–local finite groups, with some group-theoretic input coming from Condition (2). Such a characterization, also relying on Condition (2), was also given by Broto, Levi and Oliver in [16, Theorem 7.5]. In addition to Condition (2), they assume directly that

$\mathcal{F}_{S,f}$ is saturated, and that $X \simeq |\mathcal{L}_{S,f}(X)|^{\wedge}_{p}$, while we have the Frobenius reciprocity condition and Condition (1) in Theorem 10.5. Another interesting comparison is [17, Theorem 2.1], in which Broto, Levi and Oliver give conditions under which a map $f\colon BS \to X$ induces a p–local finite group $(S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}(X))$, without claiming that X is the classifying space. This again assumes Condition (2), and in addition a Sylow property on $f\colon BS \to X$ along with an assumption that $\mathcal{F}_{S,f}(X)$ is generated by maps between centric subgroups. The work in [17] inspired us to consider Condition (2) for formulating Theorem 10.5, specifically [17, Lemma 1.8], on which the proof relies.

11 An analogue of the Adams–Wilkerson invariant theorem

In their celebrated paper [3], Adams and Wilkerson developed and studied Galois theory for even-graded integral algebras over the mod p Steenrod algebra $\mathcal{A}_p$. Among their results is a characterization of the even-graded integral rings over the Steenrod algebra that can be realized as a ring of invariants in a polynomial ring with generators of degree 2. Following ideas of Lannes, in [23], Goerss, Smith and Zarati described how the $\mathbb{F}_p$–cohomology of the classifying space of an elementary abelian p–group is controlled by its evenly graded part, which is a polynomial ring with generators in degree 2. Thus the work of Adams and Wilkerson can be applied to the cohomology of elementary abelian p–groups, yielding the following variant of their result.

Theorem 11.1 [3; 23; 35] *Let V be a finite, elementary abelian p–group, put $H^* := H^*(BV;\mathbb{F}_p)$, regarded as an $\mathcal{A}_p$–algebra, and let $f\colon R^* \to H^*$ be the inclusion of a $\mathcal{A}_p$–subalgebra, making H^* a finite R^*–algebra. There exists a subgroup $W \le \mathrm{Aut}(V)$ of order prime to p such that $R^* = (H^*)^W$ if and only if there exists an R^*–linear map of $\mathcal{A}_p$–modules $t^*\colon H^* \to R^*$ such that $t \circ f = \mathrm{id}_{R^*}$.*

Proof The proof of [35, Proposition 3.11] shows how the R^*–linearity implies the conditions in [23, Theorem 1.3], yielding the desired result. □

In the setting of Theorem 11.1 one can identify f with the restriction map

$$H^*(BG;\mathbb{F}_p) \to H^*(BV;\mathbb{F}_p),$$

where $G = W \ltimes V$ is the semidirect product, and identify t with a normalized transfer map. The R^*–linearity condition on t is then the usual Frobenius reciprocity relation in cohomology. Given the connection between the Frobenius reciprocity relation in the double Burnside ring and characteristic idempotents of saturated fusion systems (Theorem 8.8), and the relationship between characteristic idempotents and stable elements (Theorem 4.10), it is natural to wonder whether one can make an analogous statement, replacing cohomology with the double Burnside ring. We formulate and prove such a statement in Theorem 11.3. In the absence of cup products, this statement necessarily takes a more functorial (and admittedly less elegant) form than Theorem 11.1. However, working with the double Burnside ring also has its advantages, as Theorem 11.3 holds for general finite p–groups, whereas one must restrict to elementary abelian groups in Theorem 11.1. We explain the analogy between Theorems 11.1 and 11.3 in Section 11.2 by converting Theorem 11.1 to a more functorial (but equivalent) statement.

11.1 Characterizing fusion-stable subfunctors of $A(S,\cdot)_{(p)}$

Throughout this subsection, we fix a finite p–group S and consider the functor

$$\alpha = A(S,\cdot)_{(p)}\colon \mathbf{A}_{(p)} \to \mathbb{Z}_{(p)}\text{-}\mathrm{mod}\,.$$

Let ρ be a subfunctor of α, meaning that $\rho(P)$ is a submodule of $\alpha(P)$ for each finite p–group P, and let $i\colon \alpha \to \rho$ be the natural transformation given by inclusion. Our goal is to give a criterion that characterizes when ρ is a subfunctor of elements that are stable with respect to a saturated fusion system $\mathcal{F}$ on S, in other words, when for each p–group P we have:

$$\rho(P) = \lim_{\mathcal{F}} A(\,\cdot\,, P)_{(p)}.$$

We write $\rho = \alpha^{\mathcal{F}}$ in this case. Staying true to the theme of this paper, the criterion is the existence of a retract $t\colon \alpha \to \rho$ that satisfies an appropriate form of Frobenius reciprocity, and we proceed to set up the tools needed to formulate this condition before giving a formal statement. Although our discussion is framed p–locally, we stress that everything carries over verbatim to the p–complete or I–adically complete worlds.

Definition 11.2 Let $\kappa\colon \alpha\times\alpha \to \alpha$ be the natural transformation defined by:

$$\kappa_{P,Q}\colon A(S,P)_{(p)} \times A(S,Q)_{(p)} \xrightarrow{\cdot\times\cdot} A(S\times S, P\times Q)_{(p)} \xrightarrow{\cdot\circ[S,\Delta]} A(S, P\times Q)_{(p)}.$$

We say that a subfunctor ρ of α is *κ–preserving* if κ restricts to a natural transformation $\kappa_\rho\colon \rho\times\rho \to \rho$.

Theorem 11.3 *Let S be a finite p–group, and let $i\colon \rho \to \alpha$ be the inclusion of a κ–preserving submodule such that $\rho(S)$ is not contained in the Nishida ideal $J(S)_{(p)}$. There exists a saturated fusion system $\mathcal{F}$ on S such that $\rho = \alpha^{\mathcal{F}}$ if and only if there* *exists a natural transformation $t\colon \alpha \to \rho$ that satisfies $t \circ i = \mathrm{id}_\rho$ and the Frobenius reciprocity relation*

$$\kappa_\rho \circ (\mathrm{id}_\rho \times t) = t \circ \kappa \circ (f \times \mathrm{id}_\alpha).$$

Proof To reduce confusion, we will write $x \cdot y$ for the composition in $\mathbf{A}_{(p)}$, and $g \circ h$ for the composition of functors, while we write 1 for the unit element in $A(S,S)_{(p)}$ and id for the identity transformation of α.

Suppose $\rho = \alpha^{\mathcal{F}}$ for a saturated fusion system $\mathcal{F}$ on S, and let ω be the characteristic idempotent of $\mathcal{F}$. Then, by the universal stable element theorem (Theorem 4.11), we have $\rho(P) = \alpha(P) \cdot \omega$ for each finite p–group P, and we define a homomorphism

$$t_P\colon \alpha(P) \to \rho(P), \quad X \mapsto X \cdot \omega.$$

We leave the reader to check that the maps t_P assemble into a natural transformation $t\colon \alpha \to \rho$, and that the Frobenius reciprocity relation for $\omega_{\mathcal{F}}$ implies the desired Frobenius reciprocity relation for t.

Conversely, if ρ admits a transfer retract t, put $\omega = i_S \circ t_S(1) \in A(S,S)_{(p)}$. For a finite p–group P and an element $X \in \alpha(P) = A(S,P)_{(p)}$, let f_X denote the corresponding morphism in $\mathrm{Mor}_{\mathbf{A}_{(p)}}(S,P) = A(S,P)_{(p)}$, and observe that $X = X \cdot 1 = \alpha(f_X)(1)$. Naturality of i and t now gives

$$i_P \circ t_P(x) = i_P \circ t_P(\alpha(f_X)(1)) = \alpha(f_X)(i_S \circ t_S(1)) = x \cdot \omega.$$

In particular,

$$\omega \cdot \omega = i_S \circ t_S(\omega) = i_S \circ \underbrace{t_S \circ i_S}_{\mathrm{id}} \circ t_S(1) = f \circ t(1) = \omega,$$

so ω is idempotent. We deduce that $\rho(P) = \alpha(P) \cdot \omega$ for every finite p–group P.

The Frobenius reciprocity relation for i and t implies that ω satisfies Frobenius reciprocity, and the assumption that $\rho(S)$ is not contained in $J(S)_{(p)}$ implies that ω is dominant, so ω is a right-characteristic idempotent for the saturated fusion system $\mathcal{F} = \mathrm{RSt}(\omega)$ by Corollary 8.9. In particular ω is right $\mathcal{F}$–stable, so it is clear that, for every finite p–group P, $\rho(P) = \alpha(P) \cdot \omega$ consists of right $\mathcal{F}$–stable elements. To see that $\rho(P)$ contains all the right $\mathcal{F}$–stable elements in $\alpha(P)$, we apply [36, Proposition 2.4.6], in which Reeh proves that for a right $\mathcal{F}$–stable element $X \in A(S,P)_{(p)}$, one has $X \cdot \omega = X$, so $X \in \alpha(P) \cdot \omega = \rho(P)$. $\square$

11.2 A functorial version of Theorem 11.1

Theorems 11.1 and 11.3 have quite different forms, making the similarity between them somewhat opaque. In this subsection we reformulate Theorem 11.1 to a functorial form similar to that of Theorem 11.3 in an attempt to clarify the analogy.

Let $H\mathcal{E}$ be the category whose objects are the finite elementary abelian p–groups, and with morphisms given by graded $\mathbb{F}_p$–modules of natural transformations

$$\mathrm{Mor}_{H\mathcal{E}}(E, E') = \mathrm{Nat}(H^*(\,\cdot\,, E), H^*(\,\cdot\,, E')).$$

Fix an elementary abelian group V, and consider the functor

$$\eta = H^*(BV;\,\cdot)\colon H\mathcal{E} \to \mathbb{F}_p\text{-}\mathtt{mod}\,.$$

This time we have a natural transformation $\kappa\colon \eta\otimes\eta \to \eta$ defined by

$$\kappa_{E,E'}\colon H^*(BV;E)\otimes H^*(BV;E') \xrightarrow{\cdot\times\cdot} H^*(BV\times BV;E\times E') \xrightarrow{B\Delta^*} H^*(BV;E\times E'),$$

where the first map is the cross product in cohomology. Again, a subfunctor ρ of η is *κ–preserving* if κ restricts to a natural transformation $\kappa_\rho\colon \rho\otimes\rho\to\rho$. Since $\kappa_{\mathbb{F}_p,\mathbb{F}_p}$ is the cup product in cohomology, this implies that $\rho(\mathbb{F}_p)$ is a subring of $\eta(\mathbb{F}_p)$.

Given a κ–preserving subfunctor ρ, we can use Theorem 11.1 to determine whether there exists a subgroup $W \le \mathrm{Aut}(V)$ of order prime to p such that $\rho = \eta^W$.

Corollary 11.4 *Let V be a finite elementary abelian p–group, let η be as above, and let $i\colon \rho\to\eta$ be the inclusion of a κ–preserving submodule such that $\rho(\mathbb{F}_p)\to\eta(\mathbb{F}_p)$ is a finite extension. There exists a subgroup $W\le\mathrm{Aut}(V)$ of order prime to p such that $\rho=\eta^W$ if and only if there exists a natural transformation $t\colon\eta\to\rho$ that satisfies $t\circ i = id_\rho$ and the Frobenius reciprocity relation*

$$\kappa_\rho\circ(\mathrm{id}_\rho\times t) = t\circ\kappa\circ(f\times\mathrm{id}_\alpha).$$

Proof One direction is obvious: if $\rho=\eta^W$ where W has order prime to p, then $\rho(E)=H^*(W\ltimes V;E)$ for every elementary abelian E, and one can take $t=|W|^{-1}\tau$, where τ is the usual transfer in group cohomology.

Now suppose i admits a transfer retract t. Put $H=\eta(\mathbb{F}_p)$ and $R=\rho(\mathbb{F}_p)$. Since $\mathrm{A}_p=\mathrm{Mor}_{H\mathcal{E}}(\mathbb{F}_p,\mathbb{F}_p)$, functoriality of ρ implies that R is a A_p–subalgebra. By assumption, the map $t_{\mathbb{F}_p}\colon H\to R$ is a retract of the inclusion $i_{\mathbb{F}_p}\colon R\to H$. Since κ restricts to the cup product on H and R, the Frobenius reciprocity relation for t

implies that $t_{\mathbb{F}_p}$ is R–linear. Theorem 11.1 then implies that there exists a subgroup $W \le \mathrm{Aut}(V)$ of order prime to p with $R = H^W$.

For an elementary abelian p–group E, the group W still acts on $\eta(E) = H^*(BV;E)$. Pick a basis $(e_1,\dots,e_n)$ for E (regarded as a vector space over $\mathbb{F}_p$) and let $E_j \le E$ be the subspace generated by e_j. There is an isomorphism

$$\bigoplus_{k=1}^{n} \eta(E_k) \xrightarrow{\bigoplus \eta(\iota_k)} \eta(E),$$

where $\iota_k\colon E_k \to E$ is inclusion of the k^{th} component. This isomorphism respects i, t and the W–action. Using the injectivity of i we can deduce that the homomorphism

$$\bigoplus_{k=1}^{n} \rho(E_k) \xrightarrow{\bigoplus \rho(\iota_k)} \rho(E)$$

is an injection, and using surjectivity of t we deduce it is a surjection, and hence an isomorphism. Since $\rho(E_k) = \eta(E_k)^W$ for each k, it follows that $\rho(E) = \eta(E)^W$. □

Corollary 11.4 is in fact equivalent to Theorem 11.1: Given a subring R of $H^*(BV;\mathbb{F}_p)$ one can extend to a subfunctor ρ of α by demanding that ρ be additive (as in the proof of Corollary 11.4). The conclusion of Corollary 11.4 then implies the conclusion of Theorem 11.1.

There is a strong analogy between Theorem 11.3 and Corollary 11.4 (and hence Theorem 11.1), especially if one thinks in terms of stable homotopy. In Corollary 11.4 we can think of $H\mathcal{E}$ as the category whose objects are finite elementary abelian groups, with morphisms given by

$$\mathrm{Mor}_{H\mathcal{E}}(E,E') = [HE, HE']_*$$

where HE and HE' denote the Eilenberg–Mac Lane spectra of E and E', respectively and the subscript $*$ indicates that homotopy classes of stable maps $HE \to HE'$ are graded by their degree. We can also identify α with the functor $E \mapsto [\Sigma^\infty BV_+, HE]_*$. In the I–adically complete (as opposed to p–local) version of Theorem 11.3, one is looking at a category whose objects are all finite p–groups, and with morphisms given by $[\Sigma^\infty BP_+, \Sigma^\infty BQ_+]$ (here we take only maps of degree 0) for finite p–groups P and Q, and ρ can be identified with the functor $P \mapsto [\Sigma^\infty BS_+, \Sigma^\infty BP_+]$. Thus the I–adically complete version of Theorem 11.3 is obtained by replacing $H(\cdot)$ with $\Sigma^\infty B(\cdot)_+$ everywhere, and expanding to allow general finite p–groups instead of just elementary abelian ones.

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