Geometry & Topology 17 (2013) 1199–1203

Addendum to "Commensurations and subgroups of finite index of Thompson's group F"

José Burillo
Sean Cleary
Claas E Röver

We show that the abstract commensurator of F is composed of four building blocks: two isomorphism types of simple groups, the multiplicative group of the positive rationals and a cyclic group of order two. The main result establishes the simplicity of a certain group of piecewise linear homeomorphisms of the real line.

20E32, 20E34; 20F65, 20E36

The purpose of this note is to extend our earlier work [2], where we described the commensurator group of Thompson's group F. We prove that an interesting subgroup of $\mathrm{Com}(F)$ is simple and describe the algebraic structure of $\mathrm{Com}(F)$ in terms of short exact sequences of simple groups and the multiplicative group of the positive rationals. For all the details and notation, see the paper [2].

1 The group of eventually periodic maps

Previously [2] we described the commensurator group of F as the group of the eventually integrally periodically affine maps in P, which is defined in [2, Section 1]. These elements may preserve or reverse the orientation of the real line. We also showed that the index two subgroup $\mathrm{Com}^+(F)$ of orientation-preserving maps fits into the short exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1,$$

whose kernel K is exactly those elements f of P_+ for which there exists $M > 0$ and two positive integers p, p' such that

$$f(t+p) = f(t) + p \quad \text{for } t \geq M,$$
$$f(t+p') = f(t) + p' \quad \text{for } t \leq -M.$$

Now we can associate to each element $f \in K$ two integrally periodically affine maps f_+ and f_-, which coincide with f near ∞ and $-\infty$, respectively. This property leads to the following definitions.

Published: 29 May 2013 DOI: 10.2140/gt.2013.17.1199

For $p \in \mathbb{N}$, we denote by H_p the subgroup of P_+ of p–periodically affine maps, that is

$$H_p = \{f \in P_+ \mid f(t+p) = f(t) + p \quad \text{for all } t \in \mathbb{R}\}.$$

Clearly, if $p \mid q$, then $H_p \subset H_q$, whence we define the subgroup H as a direct limit under inclusion by

$$H = \bigcup_{p=1}^{\infty} H_p.$$

The maps f_+ and f_- now give rise to a homomorphism

$$\rho\colon K \longrightarrow H \times H,$$

given by $\rho(f) = (f_-, f_+)$. The kernel consists of the eventually trivial elements, and therefore equals F', the commutator subgroup of F (see [1] or [2]). In other words, we get the short exact sequence

$$1 \longrightarrow F' \longrightarrow K \longrightarrow H \times H \longrightarrow 1.$$

Brin [1] showed that $\mathrm{Aut}^+(F) = \rho^{-1}(H_1 \times H_1)$ and established the short exact sequence

$$1 \longrightarrow F \longrightarrow \mathrm{Aut}^+(F) \longrightarrow T \times T \longrightarrow 1,$$

where T is Thompson's group T (see Cannon, Floyd and Parry [3]). Since we clearly have a map $H_1 \to T$, due to the fact that a map which is 1–periodically affine can be viewed as a map on the circle S^1 given by $\mathbb{R}/\mathbb{Z}$, an alternative version of this sequence is

$$1 \longrightarrow F' \longrightarrow \mathrm{Aut}^+(F) \longrightarrow H_1 \times H_1 \longrightarrow 1.$$

These two sequences are related by the short exact sequence

$$1 \longrightarrow A_1 \longrightarrow H_1 \longrightarrow T \longrightarrow 1,$$

whose kernel A_1 is the maps $t \mapsto t+k$ for integers k. Clearly A_1 is isomorphic to $\mathbb{Z}$.

It is straightforward to verify that any element α of $\mathrm{Com}^+(F)$ which satisfies $\alpha(t+1) = \alpha(t) + p$ for all $t \in \mathbb{R}$ conjugates H_1 to H_p and A_1 to A_p, the group of maps of the form $t \mapsto t + kp$ with $k \in \mathbb{Z}$. So we clearly have a short exact sequence

$$1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1.$$

We note that this extension is, in fact, central, and that one may view this copy of T as acting on the circle of length p given by $\mathbb{R}/p\mathbb{Z}$. We summarise this discussion as follows.

Theorem 1 *The structure of the group* $\mathrm{Com}(F)$ *and its index two subgroup* $\mathrm{Com}^{+}(F)$ *is given by the following short exact sequences and equalities:*

$$1 \longrightarrow \mathrm{Com}^{+}(F) \longrightarrow \mathrm{Com}(F) \longrightarrow C_2 \longrightarrow 1,$$
$$1 \longrightarrow K \longrightarrow \mathrm{Com}^{+}(F) \longrightarrow \mathbb{Q}^{*}\times\mathbb{Q}^{*} \longrightarrow 1,$$
$$1 \longrightarrow F' \longrightarrow K \longrightarrow H\times H \longrightarrow 1,$$
$$1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1,$$

where $H=\bigcup_{p=1}^{\infty} H_p$ *and* $A_p\cong\mathbb{Z}$ *is central in* H_p.

2 Simplicity of the group H

Here we exploit the well-known fact that T is simple [3] to prove our main result.

Theorem 2 *The group* $H=\{f\in P_+ \mid f(t+p)=f(t)+p \text{ for some } p\in\mathbb{N}\}$ *is simple.*

Note that for $p,q\in\mathbb{N}$ with $p\mid q$, we have $H_p\subset H_q$ and $A_p\supset A_q$. So the theorem says that in the union H the groups A_p cease to be normal. This is due to the following.

Lemma 3 *A normal subgroup of* H_p *is either* H_p *or it is contained in* A_p.

Proof In the light of the isomorphism between H_p and H_1 which carries A_p to A_1, it suffices to consider the case $p=1$. Let N be a normal subgroup of H_1 and consider its image in T. Since T is simple, the image of N is either $\{1\}$ or the whole of T. If the image is $\{1\}$, then $N\subset A_1$. So we assume that the image is T, which yields the exact sequence

$$1\longrightarrow B\longrightarrow N\longrightarrow T\longrightarrow 1,$$

with kernel $B=N\cap A_1\subset A_1$. It follows that $B=A_r$ for some r, and we find that

$$H_1/N\cong A_1/A_r\cong\mathbb{Z}/r\mathbb{Z}.$$

In particular H_1/N is abelian. The proof will be complete once we show that H_1 is equal to its commutator subgroup, because then $N=H_1$. In order to establish this, we recall from [3] that T is generated by three elements x_0, x_1 and c subject to the relators

$$[x_0x_1^{-1},x_0^{-1}x_1x_0],\qquad [x_0x_1^{-1},x_0^{-2}x_1x_0^{2}],\qquad x_1x_0^{-1}cx_1c^{-1},$$
$$(x_0^{-1}cx_1)^2x_0^{-1}c^{-1},\qquad x_1x_0^{-2}cx_1^{2}x_0^{-1}x_1^{-1}x_0x_1^{-1}c^{-1}x_0,\qquad c^3.$$

This easily gives rise to a finite presentation for H_1 with three generators x_0, x_1 and c subject to the same relators, except for c^3 which has to be replaced by the two relators $[c^3, x_0]$ and $[c^3, x_1]$. Here x_0, x_1 and c are the preimages of the corresponding generators for T, as defined in [3], with $x_0(0) = x_1(0) = 0$ and $c(0) = -\frac{1}{4}$; composition is then to be read from right to left, as in [3]. In this case c^3 is the map $t \mapsto t - 1$, which generates A_1. Modulo the commutator subgroup of H_1, the third, fourth and fifth relators yield the relators $x_0^{-1}x_1^2$, $x_0^{-3}x_1^2c$ and $x_0^{-1}x_1$, respectively, which in turn imply $x_0 = x_1 = c = 1$. This proves that $[H_1, H_1] = H_1$. □

Proof of Theorem 2 Let N be a nontrivial normal subgroup of H. According to Lemma 3, for each p, we have that $N \cap H_p$ is either H_p or it is contained in A_p.

We claim that if $N \cap H_p = H_p$ for some p, then this happens for all $p \in \mathbb{N}$. We take $q \in \mathbb{N}$. Then $N \cap H_{pq}$ is a normal subgroup of H_{pq}, and

$$N \cap H_{pq} \supset N \cap H_p = H_p \supsetneq A_p \supset A_{pq},$$

which shows that $N \cap H_{pq} = H_{pq}$ by the lemma. Thus, in this case N contains all H_q, and hence $N = H$.

The only case left now is that $N \cap H_p \subset A_p$ for all p. Since N is nontrivial and the A_p are infinite cyclic, there exists a p with $N \cap H_p = A_{rp}$ for some $r \geq 1$. But then

$$A_{2pr} \supset N \cap H_{2pr} \supset N \cap H_p = A_{pr} \supsetneq A_{2pr},$$

which is a contradiction. Thus the only normal subgroups of H are H and the identity as claimed. □

Acknowledgements We would like to thank the de Brún Centre at NUI Galway for its generous support. The first author acknowledges support from MEC grant MTM2011-25955 and the second author is grateful for funding from NSF grant number 0811002 and Simons Foundation grant number 234548.

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Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya
Escola Politècnica Superior de Castelldefels, Esteve Terrades, 5, 08860 Castelldefels, Spain

Department of Mathematics R8133, The City College of New York
160 Convent Ave, New York, NY 10031, USA

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, University Road, Galway, Ireland

burillo@ma4.upc.edu, cleary@sci.ccny.cuny.edu,
claas.roever@nuigalway.ie

http://www-ma4.upc.edu/~burillo/, http://www.sci.ccny.cuny.edu/~cleary, http://www.maths.nuigalway.ie/~chew/

Proposed: Walter Neumann
Seconded: Benson Farb, Ronald Stern

Received: 24 September 2012
Revised: 20 December 2012