Addendum to “Commensurations and subgroups of finite index of Thompson’s group $F$”

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We show that the abstract commensurator of $F$ is composed of four building blocks: two isomorphism types of simple groups, the multiplicative group of the positive rationals and a cyclic group of order two. The main result establishes the simplicity of a certain group of piecewise linear homeomorphisms of the real line.

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The purpose of this note is to extend our earlier work [2], where we described the commensurator group of Thompson’s group $F$. We prove that an interesting subgroup of $\text{Com}(F)$ is simple and describe the algebraic structure of $\text{Com}(F)$ in terms of short exact sequences of simple groups and the multiplicative group of the positive rationals. For all the details and notation, see the paper [2].

1 The group of eventually periodic maps

Previously [2] we described the commensurator group of $F$ as the group of the eventually integrally periodically affine maps in $P$, which is defined in [2, Section 1]. These elements may preserve or reverse the orientation of the real line. We also showed that the index two subgroup $\text{Com}^+(F)$ of orientation-preserving maps fits into the short exact sequence

$$1 \longrightarrow K \longrightarrow \text{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1,$$

whose kernel $K$ is exactly those elements $f$ of $P_+$ for which there exists $M > 0$ and two positive integers $p, p'$ such that

$$f(t + p) = f(t) + p \quad \text{for } t \geq M,$$
$$f(t + p') = f(t) + p' \quad \text{for } t \leq -M.$$

Now we can associate to each element $f \in K$ two integrally periodically affine maps $f_+$ and $f_-$, which coincide with $f$ near $\infty$ and $-\infty$, respectively. This property leads to the following definitions.
For \( p \in \mathbb{N} \), we denote by \( H_p \) the subgroup of \( P_+ \) of \( p \)-periodically affine maps, that is
\[
H_p = \{ f \in P_+ \mid f(t+p) = f(t) + p \quad \text{for all } t \in \mathbb{R} \}.
\]
Clearly, if \( p \mid q \), then \( H_p \subset H_q \), whence we define the subgroup \( H \) as a direct limit under inclusion by
\[
H = \bigcup_{p=1}^{\infty} H_p.
\]
The maps \( f_+ \) and \( f_- \) now give rise to a homomorphism
\[
\rho: K \to H \times H,
\]
given by \( \rho(f) = (f_-, f_+) \). The kernel consists of the eventually trivial elements, and therefore equals \( F' \), the commutator subgroup of \( F \) (see [1] or [2]). In other words, we get the short exact sequence
\[
1 \to F' \to K \to H \times H \to 1.
\]
Brin [1] showed that \( \text{Aut}^+(F) = \rho^{-1}(H_1 \times H_1) \) and established the short exact sequence
\[
1 \to F \to \text{Aut}^+(F) \to T \times T \to 1,
\]
where \( T \) is Thompson’s group \( T \) (see Cannon, Floyd and Parry [3]). Since we clearly have a map \( H_1 \to T \), due to the fact that a map which is \( 1 \)-periodically affine can be viewed as a map on the circle \( S^1 \) given by \( \mathbb{R}/\mathbb{Z} \), an alternative version of this sequence is
\[
1 \to F' \to \text{Aut}^+(F) \to H_1 \times H_1 \to 1.
\]
These two sequences are related by the short exact sequence
\[
1 \to A_1 \to H_1 \to T \to 1,
\]
whose kernel \( A_1 \) is the maps \( t \mapsto t + k \) for integers \( k \). Clearly \( A_1 \) is isomorphic to \( \mathbb{Z} \).
It is straightforward to verify that any element \( \alpha \) of \( \text{Com}^+(F) \) which satisfies \( \alpha(t+1) = \alpha(t) + p \) for all \( t \in \mathbb{R} \) conjugates \( H_1 \) to \( H_p \) and \( A_1 \) to \( A_p \), the group of maps of the form \( t \mapsto t + kp \) with \( k \in \mathbb{Z} \). So we clearly have a short exact sequence
\[
1 \to A_p \to H_p \to T \to 1.
\]
We note that this extension is, in fact, central, and that one may view this copy of \( T \) as acting on the circle of length \( p \) given by \( \mathbb{R}/p\mathbb{Z} \). We summarise this discussion as follows.

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**Theorem 1** The structure of the group $\text{Com}(F)$ and its index two subgroup $\text{Com}^+(F)$ is given by the following short exact sequences and equalities:

\[
\begin{align*}
1 & \longrightarrow \text{Com}^+(F) \longrightarrow \text{Com}(F) \longrightarrow C_2 \longrightarrow 1, \\
1 & \longrightarrow K \longrightarrow \text{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1, \\
1 & \longrightarrow F' \longrightarrow K \longrightarrow H \times H \longrightarrow 1, \\
1 & \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1,
\end{align*}
\]

where $H = \bigcup_{p=1}^\infty H_p$ and $A_p \cong \mathbb{Z}$ is central in $H_p$.

**2 Simplicity of the group $H$**

Here we exploit the well-known fact that $T$ is simple [3] to prove our main result.

**Theorem 2** The group $H = \{ f \in P_+ \mid f(t + p) = f(t) + p \text{ for some } p \in \mathbb{N} \}$ is simple.

Note that for $p, q \in \mathbb{N}$ with $p \mid q$, we have $H_p \subseteq H_q$ and $A_p \supseteq A_q$. So the theorem says that in the union $H$ the groups $A_p$ cease to be normal. This is due to the following.

**Lemma 3** A normal subgroup of $H_p$ is either $H_p$ or it is contained in $A_p$.

**Proof** In the light of the isomorphism between $H_p$ and $H_1$ which carries $A_p$ to $A_1$, it suffices to consider the case $p = 1$. Let $N$ be a normal subgroup of $H_1$ and consider its image in $T$. Since $T$ is simple, the image of $N$ is either $\{1\}$ or the whole of $T$. If the image is $\{1\}$, then $N \subseteq A_1$. So we assume that the image is $T$, which yields the exact sequence

\[
1 \longrightarrow B \longrightarrow N \longrightarrow T \longrightarrow 1,
\]

with kernel $B = N \cap A_1 \subseteq A_1$. It follows that $B = A_r$ for some $r$, and we find that

$H_1/N \cong A_1/A_r \cong \mathbb{Z}/r\mathbb{Z}$.

In particular $H_1/N$ is abelian. The proof will be complete once we show that $H_1$ is equal to its commutator subgroup, because then $N = H_1$. In order to establish this, we recall from [3] that $T$ is generated by three elements $x_0$, $x_1$ and $c$ subject to the relators

\[
\begin{align*}
[x_0x_1^{-1}, x_0^{-1}x_1x_0], & \quad [x_0x_1^{-1}, x_0^{-2}x_1x_0^2], & \quad x_1x_0^{-1}cx_1c^{-1}, \\
(x_0^{-1}cx_1)^2x_0^{-1}c^{-1}, & \quad x_1x_0^{-2}cx_1^2x_0^{-1}x_1^{-1}x_0x_1^{-1}c^{-1}x_0, & \quad c^3.
\end{align*}
\]
This easily gives rise to a finite presentation for $H_1$ with three generators $x_0$, $x_1$ and $c$ subject to the same relators, except for $c^3$ which has to be replaced by the two relators $[c^3, x_0]$ and $[c^3, x_1]$. Here $x_0$, $x_1$ and $c$ are the preimages of the corresponding generators for $T$, as defined in [3], with $x_0(0) = x_1(0) = 0$ and $c(0) = -\frac{1}{4}$; composition is then to be read from right to left, as in [3]. In this case $c^3$ is the map $t \mapsto t - 1$, which generates $A_1$. Modulo the commutator subgroup of $H_1$, the third, fourth and fifth relators yield the relators $x_0^{-1}x_1^2$, $x_0^{-3}x_1^2c$ and $x_0^{-1}x_1$, respectively, which in turn imply $x_0 = x_1 = c = 1$. This proves that $[H_1, H_1] = H_1$. 

**Proof of Theorem 2** Let $N$ be a nontrivial normal subgroup of $H$. According to Lemma 3, for each $p$, we have that $N \cap H_p$ is either $H_p$ or it is contained in $A_p$.

We claim that if $N \cap H_p = H_p$ for some $p$, then this happens for all $p \in \mathbb{N}$. We take $q \in \mathbb{N}$. Then $N \cap H_{pq}$ is a normal subgroup of $H_{pq}$, and

$$N \cap H_{pq} \supset N \cap H_p = H_p \supset A_p \supset A_{pq},$$

which shows that $N \cap H_{pq} = H_{pq}$ by the lemma. Thus, in this case $N$ contains all $H_q$, and hence $N = H$.

The only case left now is that $N \cap H_p \subset A_p$ for all $p$. Since $N$ is nontrivial and the $A_p$ are infinite cyclic, there exists a $p$ with $N \cap H_p = A_{rp}$ for some $r \geq 1$. But then

$$A_{2pr} \supset N \cap H_{2pr} \supset N \cap H_p = A_{pr} \supset A_{2pr},$$

which is a contradiction. Thus the only normal subgroups of $H$ are $H$ and the identity as claimed. 

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**References**


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