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A nonboundary nef divisor on $\overline{M}_{0,12}$

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We describe a nef divisor $D_{\mathcal{P}}$ on $\overline{M}_{0,12}$ that is not numerically equivalent to an effective sum of boundary divisors.

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0 Introduction

The moduli space of stable curves of genus 0 with n marked points, $\overline{M}_{0,n}$, has a stratification where the codimension k strata are the irreducible components of the locus of curves with at least k nodes. Fulton, motivated by analogy with toric varieties, asked whether any effective cycle is numerically equivalent to an effective sum of these strata (see Keel and McKernan [7, Question 1.1]). In the case of divisors, this is known to be false; Keel and Vermeire [9] constructed effective divisors on $\overline{M}_{0,6}$ that are not effective sums of boundary divisors (the codimension 1 strata).

For curves, however, the question is still open. The 1–dimensional strata are called *F–curves* and the conjecture in this case (known as the F–conjecture) is that the cone of effective curves is generated by the F–curves. It is often stated in the following equivalent form:

Conjecture 1 (The F–conjecture) *A divisor on $\overline{M}_{0,n}$ is nef if and only if it has nonnegative intersection with every F–curve.*

Keel and McKernan [7] proved this conjecture for $n \leq 7$, and for $n \geq 8$ the conjecture is still open. We usually say that a divisor that intersects every F–curve nonnegatively is *F–nef*, so the F–conjecture states that a divisor on $\overline{M}_{0,n}$ is nef if and only if it is F–nef.

The following related conjecture will be the central subject of this paper. It is sometimes called "Fulton's Conjecture," though it seems to have first been stated by Gibney, Keel, and Morrison [5].

Conjecture 2 [5, Question 0.13] *Every F–nef divisor on $\overline{M}_{0,n}$ is numerically equivalent to an effective sum of boundary divisors.*

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This conjecture has also been proven for $n \leq 7$ (by Larsen [8]), and it has long been understood that it would imply the F–conjecture by a straightforward inductive argument, if true.

Gibney showed that this conjecture can be weakened slightly and still imply the F–conjecture:

Conjecture 3 [4, Conjecture 1] *Every F–nef divisor on $\overline{M}_{0,n}$ is of the form $cK_{\overline{M}_{0,n}} + E$, where $c \geq 0$ and E is an effective sum of boundary classes.*

However, we provide a counterexample to Conjectures 2 and 3 when $n = 12$.

Proposition 1 *The divisor $D_{\mathcal{P}}$ on $\overline{M}_{0,12}$ is F–nef but is not numerically equivalent to a nonnegative linear combination of boundary divisors and the canonical divisor $K_{\overline{M}_{0,12}}$.*

We currently have no geometric explanation for this counterexample, but it is related to a highly symmetric combinatorial arrangement of subsets, the $(11,5,2)$ biplane (see Section 1.1).

We do not know whether $n = 12$ is the minimal n for which Conjecture 2 is false. We do give a simple argument (Lemma 2) allowing us to shift the counterexample to any larger n, though, so there are only four values of n for which the status of Conjecture 2 is still unknown.

Question Does Conjecture 2 hold for $\overline{M}_{0,n}$ with $8 \leq n \leq 11$?

We begin by reviewing notation in Section 1. In Section 2, we describe the counterexample divisor $D_{\mathcal{P}}$ appearing in Proposition 1 and show that its pullbacks under forgetful maps also provide counterexamples. In Section 3, we list a few additional properties of this divisor calculated by computer; in particular, it is basepoint free, and hence nef.

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1 Notation

1.1 The $(11,5,2)$ biplane

The divisor we construct is based on the combinatorial data of the $(11,5,2)$ biplane, a collection $\mathcal{P}$ of 11 five-element subsets of $[11] := \{1,2,\dots,11\}$ with the properties:

(1) Any two elements of $[11]$ are in precisely two elements of $\mathcal{P}$.

(2) Any two elements of $\mathcal{P}$ have intersection of size two.

It turns out that $\mathcal{P}$ is unique up to renaming the elements of $[11]$; one way of constructing such a biplane is to take the set $\{1,3,4,5,9\}$ (the nonzero quadratic residues mod 11) together with its translates (taken mod 11).

The $(11,5,2)$ biplane has symmetry group of order 660, isomorphic to $\mathrm{PSL}_2(11)$. The divisor $D_{\mathcal{P}}$ that we construct shares this symmetry group.

1.2 Divisors on $\overline{M}_{0,n}$

The simplest divisors on $\overline{M}_{0,n}$ are the boundary divisors $\Delta_{S,T} = \Delta_S = \Delta_T$, where $\{S,T\}$ is a partition of $\{1,\dots,n\}$ with $|S|,|T| \geq 2$.

It is also convenient to define $\Delta_{\{i\}} = -\psi_i$, the negative of the cotangent line at the i^{th} marked point.

These two types of divisors suffice to generate $\mathrm{Pic}(\overline{M}_{0,n})$ (in fact, the boundary divisors alone do), and the space of relations between these generators has basis consisting of the relations

$$\sum_{i\in S,\, j\notin S} \Delta_S = 0 \tag{1}$$

for $1 \leq i < j \leq n$.

1.3 F–curves

The dual graph to an F–curve has exactly one vertex of degree 4. The class of the F–curve just depends on the partition of the markings into 4 subsets determined by this vertex. If $\{A_1,A_2,A_3,A_4\}$ is a partition of $\{1,\dots,n\}$ into four nonempty subsets, we write $C_{A_1A_2A_3A_4}$ for the corresponding F–curve class.

The intersection number of any F–curve with any divisor $\Delta_{S,T}$ has a simple formula:

$$\Delta_{S,T}\cdot C_{A_1A_2A_3A_4} = \begin{cases} 1 & \text{if } S = A_i \cup A_j \text{ for some } i \neq j, \\ -1 & \text{if } S \text{ or } T = A_i \text{ for some } i, \\ 0 & \text{else.} \end{cases} \tag{2}$$

2 The construction

We begin by describing a curve class $C_{\mathcal{P}}$ on $\overline{M}_{0,12}$ that directly corresponds to the biplane configuration $\mathcal{P}$. It is uniquely determined by its pairings with the boundary divisors $\Delta_{S,T}$, since they generate the Picard group:

$$C_{\mathcal{P}} \cdot \Delta_{S,T} = \begin{cases} 1 & \text{if } S \text{ or } T \in \mathcal{P}, \\ 0 & \text{else.} \end{cases}$$

Of course, for such a divisor to exist, these pairings have to respect the relations between the divisors $\Delta_{S,T}$. The fact that they do turns out to be equivalent to every pair of elements of [11] being contained in the same number of sets in the biplane $\mathcal{P}$. This is actually the only biplane property that we will directly use. Of course, there are many similarly balanced configurations of subsets that we could consider using instead of $\mathcal{P}$ to construct different interesting curve classes.

Since $C_{\mathcal{P}}$ intersects every boundary divisor nonnegatively, any divisor that intersects it negatively cannot be an effective sum of boundary. Thus we view $C_{\mathcal{P}}$ as a witness to the nonboundary nature of certain divisors.

We can also compute that

$$C_{\mathcal{P}} \cdot \Delta_{\{i\}} = \begin{cases} -2 & \text{if } i = 12, \\ -3 & \text{else.} \end{cases}$$

We now describe the divisor itself. Unlike with $C_{\mathcal{P}}$, the definition of $D_{\mathcal{P}}$ does not seem to have an obvious generalization to other combinatorial configurations of subsets.

The notation $E_S = \Delta_{S \cup \{n\}}$ will be convenient here; these are the exceptional divisors in the Kapranov model of $\overline{M}_{0,n}$ with respect to the n^{th} marked point (see [6]). Then we define the divisor $D_{\mathcal{P}} \in \operatorname{Pic}(\overline{M}_{0,12})$ by

$$D_{\mathcal{P}} = -5E_{\varnothing} - 4\sum E_i - 3\sum E_{ij} - 2\sum E_{ijk} - \sum E_{ijkl} - \sum E_S,$$

where the first four sums are over all subsets of [11] of sizes $1, 2, 3, 4$ and the last sum runs over the subsets of [11] of size 5 or 6 that are either equal to or disjoint from one of the eleven five-element subsets in the chosen biplane $\mathcal{P}$.

We can now check that we have a counterexample to Conjectures 2 and 3.

Proof of Proposition 1 This is just a matter of checking four things:

(a) $D_{\mathcal{P}} \cdot C_{A_1A_2A_3A_4} \geq 0$ for any F–curve class $C_{A_1A_2A_3A_4}$.

(b) $\Delta_{S,T} \cdot C_{\mathcal{P}} \geq 0$ for $|S|, |T| \geq 2$.

(c) $K_{\overline{M}_{0,12}} \cdot C_{\mathcal{P}} \geq 0$.

(d) $D_{\mathcal{P}} \cdot C_{\mathcal{P}} < 0$.

Of these, (b), (c) and (d) all follow immediately from the definition of $C_{\mathcal{P}}$, combined with the identity

$$K_{\overline{M}_{0,12}} = -\sum_{i=1}^{12} \Delta_{\{i\}} - 2 \sum_{|S|,|T|\geq 2} \Delta_{S,T}$$

in the case of (c).

This leaves (a), which is just a matter of computing the intersection of $D_{\mathcal{P}}$ with each of the finitely many F–curve classes on $\overline{M}_{0,12}$. There are 611501 such classes, so this check can be easily completed with a computer using (2). Alternatively, we can write $D_{\mathcal{P}} = D_0 - D'_{\mathcal{P}}$, where

$$D_0 = -5E_{\varnothing} - 4\sum E_i - 3\sum E_{ij} - 2\sum E_{ijk} - \sum E_{ijkl},$$
$$D'_{\mathcal{P}} = \sum_{S\in\mathcal{P}} \Big(\Delta_S + \sum_{i\notin S} \Delta_{S\cup\{i\}}\Big).$$

The divisor D_0 here is very special; it is fully S_{12}–symmetric and can either be interpreted as the pullback of the distinguished polarization of the (symmetrically linearized) GIT quotient $(\mathbb{P}^1)^{12} /\!\!/ \mathrm{SL}_2$ (see [1]) or as a conformal block divisor (see [3, Theorem 4.5]). It is nef, and its degree on an F–curve $C_{A_1A_2A_3A_4}$ has a simple formula:

$$D_0 \cdot C_{A_1A_2A_3A_4} = \begin{cases} 0 & \text{if } \max(|A_i|) \geq 6, \\ \min(\min(|A_i|), 6 - \max(|A_i|)) & \text{else.} \end{cases}$$

Using this formula along with (2), it is straightforward to check that

$$D_0 \cdot C_{A_1A_2A_3A_4} \geq D'_{\mathcal{P}} \cdot C_{A_1A_2A_3A_4}$$

for any F–curve $C_{A_1A_2A_3A_4}$. For example, if $D_0 \cdot C_{A_1A_2A_3A_4} = 0$ then $|A_1| \geq 6$ (without loss of generality), and then the only possibility of $D'_{\mathcal{P}} \cdot C_{A_1A_2A_3A_4}$ being positive is if $A_2 \cup A_3 \in \mathcal{P}$. But then $A_2 \cup A_4$ and $A_3 \cup A_4$ cannot be elements of $\mathcal{P}$ (since the union of any two sets in $\mathcal{P}$ has cardinality $8 > 12 - 6$), and $A_2 \cup A_3 \cup A_4$ is an element of $\mathcal{P}$ with a single point added, so $D'_{\mathcal{P}} \cdot C_{A_1A_2A_3A_4} = 1 - 1 = 0$. The other cases are similar. □

Although we expect that both Conjecture 2 and Conjecture 3 remain false for all $n \geq 12$, we are only able to prove this for Conjecture 2 at the moment.

Lemma 2 *Let $n \geq 3$ and let $\pi\colon \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be the map given by forgetting the final marking. If D is an F–nef divisor on $\overline{M}_{0,n}$ that is not numerically equivalent to an effective boundary divisor, then π^*D also is F–nef and not numerically equivalent to an effective boundary divisor.*

Proof First, π^*D being F–nef follows immediately from the fact that for any F–curve class C, π_*C is either 0 or another F–curve class.

Now suppose for contradiction that π^*D is numerically equivalent to

$$\sum a_{S,T}\Delta_{S,T}$$

with $a_{S,T} \geq 0$, where the sum runs over all partitions of $\{1,\dots,n+1\}$ into two sets S and T of cardinality at least 2.

The F–curve $C_{A,B,C,\{n+1\}}$ is contracted by π, so we have

$$\begin{aligned}(3)\quad 0 &= \pi^*D\cdot C_{A,B,C,\{n+1\}}\\ &= (a_{A\cup\{n+1\},B\cup C} - a_{A,B\cup C\cup\{n+1\}}) + (a_{B\cup\{n+1\},A\cup C} - a_{B,A\cup C\cup\{n+1\}})\\ &\qquad - (a_{A\cup B\cup\{n+1\},C} - a_{A\cup B,C\cup\{n+1\}}).\end{aligned}$$

Let $f(S) = a_{S\cup\{n+1\},S^c} - a_{S,S^c\cup\{n+1\}}$ for $\varnothing \neq S \subset \{1,\dots,n\}$, where S^c is the complement of S inside $\{1,\dots,n\}$. Then (3) becomes

$$f(A\cup B) = f(A) + f(B),$$

where A and B are any two disjoint nonempty subsets of $\{1,\dots,n\}$ such that $A\cup B$ is a proper subset of $\{1,\dots,n\}$.

We also have $f(A^c) = -f(A)$ by the definition of f. Thus

$$f(\{1\}) + \dots + f(\{n\}) = 0.$$

However, for each k we have $f(\{k\}) = a_{\{k,n+1\},\{k\}^c} \geq 0$. Thus $f(\{k\}) = 0$ for all k, and hence $f(S) = 0$ for all S. In other words, $a_{S\cup\{n+1\},S^c} = a_{S,S^c\cup\{n+1\}}$.

Note now that $\pi^*\Delta_{S,S^c} = \Delta_{S\cup\{n+1\},S^c} + \Delta_{S,S^c\cup\{n+1\}}$. Then π^*D is numerically equivalent to π^*B for

$$B = \sum_{\{S,S_c\}} a_{S\cup\{n+1\},S^c}\Delta_{S,S^c}.$$

But $\pi^*\colon \mathrm{Pic}(\overline{M}_{0,n}) \to \mathrm{Pic}(\overline{M}_{0,n+1})$ in injective (because π has a section), so this implies that D is numerically equivalent to B, which is effective boundary. This is a contradiction. □

3 Computations

We have checked a few basic properties of the divisor $D_{\mathcal{P}}$ by computer. First, $D_{\mathcal{P}}$ generates an extremal ray in the cone of F–nef divisors on $\overline{M}_{0,12}$. We do not know whether there are any extremal rays on $\overline{M}_{0,12}$ other than the S_{12}–orbit containing $D_{\mathcal{P}}$ that contradict Conjecture 2.

Also, $D_{\mathcal{P}}$ is effective, with a 66–dimensional space of sections. By straightforward computation of the intersection of these sections using the Kapranov model for $\overline{M}_{0,12}$ as an iterated blow-up of $\mathbb{P}^9$, we were able to check that this linear system is basepoint free. Thus $D_{\mathcal{P}}$ is nef, which can be interpreted as very weak evidence for the F–conjecture.

Since $D_{\mathcal{P}}$ is nef and extremal F–nef, it is also an extremal nef divisor. Unsurprisingly, it is not extremal in the cone of effective divisors. Castravet and Tevelev have computed that $D_{\mathcal{P}}$ is in fact an effective sum of boundary divisors and a pullback from $\overline{M}_{0,8}$ of the unique (up to symmetries) "hypertree" divisor there, so it is consistent with their conjecture that the cone of effective divisors is generated by the boundary divisors and the hypertree divisors (defined in [2]).

Finally, we can give a somewhat more geometric description of $D_{\mathcal{P}}$, again in terms of the Kapranov morphism $\kappa_{12}\colon \overline{M}_{0,12} \to \mathbb{P}^9$. If Q is a generic quintic hypersurface in $\mathbb{P}^9$ containing the eleven $\mathbb{P}^4$ spanned by the five-tuples in $\mathcal{P}$ along with the eleven complementary $\mathbb{P}^5$, then $D_{\mathcal{P}}$ is the class of the proper transform of Q with respect to κ_{12}.

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