

Constructing derived moduli stacks

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We introduce frameworks for constructing global derived moduli stacks associated to a broad range of problems, bridging the gap between the concrete and abstract conceptions of derived moduli. Our three approaches are via differential graded Lie algebras, via cosimplicial groups, and via quasicomonoids, each more general than the last. Explicit examples of derived moduli problems addressed here are finite schemes, polarised projective schemes, torsors, coherent sheaves and finite group schemes.

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Introduction

In [25], representability was established by the author for many derived moduli problems involving schemes and quasicohherent sheaves. However, the derived stacks there were characterised as nerves of ∞ -groupoids with very many objects, making it difficult to understand the derived stacks concretely.

By contrast to the indirect approach of satisfying a representability theorem, Ciocan-Fontanine and Kapranov [6; 5] construct explicit derived Hilbert and Quot schemes as dg-schemes with the necessary properties, but give no universal family, so the derived moduli spaces lack functorial interpretations. In this paper, we will show how to reconcile these approaches, thereby giving explicit presentations for the derived moduli spaces of [25].

In fact, we go substantially beyond the problems considered in [6; 5], and give a framework valid in all characteristics (rather than just over \mathbb{Q}). This is done by working with quasicomonoid valued functors, which give a global analogue of the simplicial deformation complexes of the author [20]. In broad terms, derived moduli constructions over \mathbb{Q} tend to be based on differential graded Lie algebras (DGLAs), while quasicomonoids perform the same role in much greater generality. Since quasicomonoids arise naturally from algebraic theories, they are much more general than DGLAs, even in characteristic zero.

Beware that for the purposes of this paper, derived algebraic geometry will mean the theory of Lurie [18] based on simplicial commutative rings, or on dg algebras when working over \mathbb{Q} , rather than the more exotic HAG contexts of Toën and Vezzosi [32]. This enables us to apply Lurie's Representability Theorem in Section 1, but is also needed in later sections. The key to Section 3 is that tensoring a commutative algebra with a Lie algebra gives a Lie algebra, but similar constructions could be made with any pair of algebras for Koszul-dual operads. Likewise, the constructions of Section 4–5 adapt to give functors on any category of simplicial objects. However, they will not adapt to give functors on symmetric spectra, since they depend on the functor $A_\bullet \rightsquigarrow A_0$.

The structure of the paper is as follows. Section 1 summarises various results from the author [26] concerning representability of derived stacks, and gives a few minor generalisations. Section 2 develops some technical results on the pro-Zariski and pro-étale sites. Lemma 2.3 shows that any finitely presented sheaf is a sheaf for the associated pro site, and our main results are Lemmas 2.10 and 2.13, concerning the existence of weakly universal coverings. These are applied in later sections to deal with infinite sums of locally free sheaves, which feature when studying polarised projective varieties.

In Section 3, DGLAs are introduced, together with the Deligne groupoid $\mathcal{D}\mathcal{E}\mathcal{L}(L)$ associated to any DGLA L with a gauge action. By adapting the techniques of [5], DGLAs are used to construct derived moduli stacks for pointed finite schemes (Proposition 3.16) and for polarised projective schemes (Proposition 3.33). The resulting functors are shown (in Propositions 3.18 and 3.34, respectively) to be equivalent to the corresponding functors in [25], defined as nerves of ∞ -groupoids of derived geometric stacks.

DGLAs only tend to work in characteristic 0, and Section 4 shows how to construct derived moduli stacks using cosimplicial groups instead. For any simplicial cosimplicial group G , there is a derived Deligne groupoid $\underline{\mathcal{D}\mathcal{E}\mathcal{L}}(G)$; Proposition 4.15 shows that cosimplicial group valued functors G give rise to well-behaved derived moduli functors $\underline{\mathcal{D}\mathcal{E}\mathcal{L}}(G)$. For any DGLA L with gauge G_L , there is an associated cosimplicial group $D(\exp(L), G_L)$, and Corollary 4.27 shows that the Deligne groupoids associated to L and $D(\exp(L), G_L)$ are isomorphic. Section 4.4 defines a kind of sheafification $G^\#$ for cosimplicial group valued functors G , removing the need to sheafify $\underline{\mathcal{D}\mathcal{E}\mathcal{L}}(G)$; this gives an immediate advantage of cosimplicial groups over DGLAs. Proposition 4.38 gives a cosimplicial group governing derived moduli of torsors, a problem not easily accessible via DGLAs.

Cosimplicial groups cannot handle all moduli problems, so Section 5 begins by recalling the quasicomonoids from the author [23], and the derived Deligne groupoid $\underline{\mathcal{D}\mathcal{E}\mathcal{L}}(E)$ of a simplicial quasicomonoid E . Corollary 5.43 then shows that quasicomonoid

valued functors E give rise to well-behaved derived moduli functors $\underline{\mathcal{D}el}(E)$. In Section 5.2.1, we recall basic properties of monads, together with results from [23] showing how these give rise to quasicomonoids. Monads are ubiquitous, arising whenever there is some kind of algebraic structure. Section 5.2.2 goes further, by associating quasicomonoids to diagrams. In particular, this allows derived moduli of morphisms to be constructed for all the examples considered in Section 6. Section 5.5 then defines a kind of sheafification E^\sharp for quasicomonoid valued functors E , removing the need to sheafify $\underline{\mathcal{D}el}(E)$.

For every cosimplicial group G , there is a quasicomonoid $\mathcal{E}(G)$, and Lemma 5.12 shows that $\underline{\mathcal{D}el}(\mathcal{E}(G)) \simeq \underline{\mathcal{D}el}(G)$, ensuring consistency between the various approaches. For moduli problems based on additive categories, the associated quasicomonoid E is linear. This means that its normalisation NE is a DG associative noncommutative algebra (so *a fortiori* a DGLA), so the techniques of this section give DGLAs for abelian moduli problems. Moreover, Proposition 5.40 gives an equivalence $\underline{\mathcal{D}el}(E) \simeq \underline{\mathcal{D}el}(NE)$, so quasicomonoids and DGLAs give equivalent derived moduli.

Section 6 gives a selection of examples which can be tackled by quasicomonoids. Derived moduli of finite schemes, of polarised projective schemes, and of finite group schemes are constructed in Propositions 6.4, 6.17 and 6.28, respectively. In Propositions 6.6, 6.18 and 6.32, these are shown to be equivalent to the corresponding functors in [25], defined as nerves of ∞ -groupoids of derived geometric stacks. Proposition 6.11 constructs derived moduli of coherent sheaves, and Proposition 6.12 shows that this is equivalent to the nerve of ∞ -groupoids of hypersheaves considered in [25].

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1 Background on representability

Let \mathbb{S} be the category of simplicial sets. Denote the category of simplicial commutative rings by $s\text{Ring}$, the category of simplicial commutative R -algebras by $s\text{Alg}_R$, and the category of simplicial R -modules by $s\text{Mod}_R$. If $\mathbb{Q} \subset R$, we let $dg_+\text{Alg}_R$ be the category of differential graded-commutative R -algebras in nonnegative chain degrees, and $dg_+\text{Mod}_R$ the category of R -modules in chain complexes in nonnegative chain degrees.

Definition 1.1 Given a simplicial abelian group A_\bullet , we denote the associated normalised chain complex by $N^s A$ (or, when no confusion is likely, by NA). Recall

that this is given by $N(A)_n := \bigcap_{i>0} \ker(\partial_i: A_n \rightarrow A_{n-1})$, with differential ∂_0 . Then $H_*(NA) \cong \pi_*(A)$.

When $\mathbb{Q} \subset R$, using the Eilenberg–Zilber shuffle product (see Weibel [33, 8.5.4]), normalisation N extends to a right Quillen equivalence

$$N: s\text{Alg}_R \rightarrow dg_+\text{Alg}_R,$$

by Quillen [28, Section I.4].

Definition 1.2 Define $dg_+\mathcal{N}_R$ (resp. $s\mathcal{N}_R$) to be the full subcategory of $dg_+\text{Alg}_R$ (resp. $s\text{Alg}_R$) consisting of objects A for which the map $A \rightarrow H_0A$ (resp. $A \rightarrow \pi_0A$) has nilpotent kernel. Define $dg_+\mathcal{N}_R^{\flat}$ (resp. $s\mathcal{N}_R^{\flat}$) to be the full subcategory of $dg_+\mathcal{N}_R$ (resp. $s\mathcal{N}_R$) consisting of objects A for which $A_i = 0$ (resp. $N_iA = 0$) for all $i \gg 0$.

From now on, we will write $d\mathcal{N}^{\flat}$ (resp. $d\text{Alg}_R$, resp. $d\text{Mod}_R$) to mean either $s\mathcal{N}_R^{\flat}$ (resp. $s\text{Alg}_R$, resp. $s\text{Mod}_R$) or $dg_+\mathcal{N}_R^{\flat}$ (resp. $dg_+\text{Alg}_R$, resp. $dg_+\text{Mod}_R$), noting that we only use chain algebras in characteristic 0.

Definition 1.3 Say that a surjection $A \rightarrow B$ in $dg_+\text{Alg}_R$ (resp. $s\text{Alg}_R$) is a *tiny acyclic extension* if the kernel K satisfies $I_A \cdot K = 0$, and K (resp. NK) is of the form $\text{cone}(M)[-r]$ for some H_0A -module (resp. π_0A -module) M . In particular, $H_*K = 0$.

1.1 Formal quasismoothness and homogeneity

The following definitions are mostly taken from [25].

Definition 1.4 Say a natural transformation $\eta: F \rightarrow G$ of functors $F, G: d\mathcal{N}^{\flat} \rightarrow \mathbb{S}$ is homotopic (resp. prehomotopic) if for all tiny acyclic extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is a trivial fibration (resp. a surjective fibration). Say that F is homotopic if $F \rightarrow \bullet$ is so, where \bullet denotes the one point set.

Definition 1.5 Say a natural transformation $\eta: F \rightarrow G$ of functors $F, G: d\mathcal{N}^{\flat} \rightarrow \mathbb{S}$ is formally quasipresmooth (resp. formally presmooth) if for all square zero extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is a fibration (resp. a surjective fibration).

Say that η is formally quasismooth (resp. formally smooth) if it is formally quasipresmooth (resp. formally presmooth) and homotopic.

Definition 1.6 Say that a natural transformation $\eta: F \rightarrow G$ of functors on $d\mathcal{N}^\flat$ is formally étale if for all square zero extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is an isomorphism.

Definition 1.7 Say that a natural transformation $F \rightarrow G$ of functors on $d\mathcal{N}^\flat$ is (relatively) homogeneous if for all square zero extensions $A \rightarrow B$, the map

$$F(A \times_B C) \rightarrow G(A \times_B C) \times_{[G(A) \times_{G(B)} G(C)]} [F(A) \times_{F(B)} F(C)]$$

is an isomorphism. Say that F is homogeneous if $F \rightarrow \bullet$ is relatively homogeneous.

Proposition 1.8 Let $\alpha: F \rightarrow G$ be a formally étale morphism of the functors $F, G: d\mathcal{N}^\flat \rightarrow \text{Set}$. If G is homogeneous, then so is F . Conversely, if α is surjective and F is homogeneous, then so is G .

Proof This is [25, Proposition 2.18]. □

1.2 Tangent complexes

Given a category \mathcal{C} , write $\text{Ab}(\mathcal{C})$ for the category of abelian group objects in \mathcal{C} .

Definition 1.9 For a homogeneous functor $F: d\mathcal{N}^\flat \rightarrow \mathbb{S}$, $A \in d\mathcal{N}^\flat$ and $M \in d\text{Mod}_A$, define the tangent space by

$$T(F, M) := F(A \oplus M) \in \mathbb{S} \downarrow F(A),$$

noting that this is an abelian group object in this category. Here, $\mathbb{S} \downarrow F(A)$ denotes the category of objects over $F(A)$.

Given a natural transformation $\alpha: F \rightarrow G$ of homogeneous functors $F, G: d\mathcal{N}^\flat \rightarrow \mathbb{S}$, define the relative tangent space by

$$T(F/G, M) := \ker(T(F, M) \rightarrow T(G, M) \times_{G(A)} F(A)) \in \text{Ab}(\mathbb{S} \downarrow F(A)).$$

Given $x \in F(A)$, define $T_x(F/G, M) := T(F/G, M) \times_{F(A)} \{x\} \in \text{Ab}(\mathbb{S}) = s\text{Ab}$.

When $\alpha: F \rightarrow G$ is formally quasipresmooth, note this definition is compatible with [26, Definition 1.8], in the sense that for $x \in \pi_0 F(A)$, the space $T_x(F/G)(M)$ of [26] is the homotopy fibre of $T(F/G, M) \rightarrow F(A)$ over x , since $T(F/G, M) \rightarrow F(A)$ is a fibration.

Definition 1.10 Given a prehomotopic formally quasipresmooth transformation $\alpha: F \rightarrow G$ of homogeneous functors $F, G: d\mathcal{N}^b \rightarrow \mathbb{S}$, an object $A \in d\mathcal{N}^b$, a point $x \in F_0(A)$ and a module $M \in d\text{Mod}_A$, define $D_x^i(F/G, M)$ as follows, following [26, Definition 3.14].

For $i \leq 0$, set

$$D_x^i(F/G, M) := \pi_{-i}(T_x(F/G, M)).$$

For $i > 0$, set

$$D_x^i(F, M) := \pi_0 F(T_x(F/G, M[-i]))/\pi_0(T_x(F/G, \text{cone}(M)[1-i])).$$

Note that homogeneity of F ensures that these are abelian groups for all i , and that the multiplicative action of A on M gives them the structure of A -modules.

If $\alpha: F \rightarrow G$ is formally quasismooth, note that [26, Lemma 1.12] gives

$$D_x^{n-i}(F/G, M) = \pi_i(T_x(F/G, M[-n])).$$

The following is immediate.

Lemma 1.11 If $X, Y, Z: d\mathcal{N}^b \rightarrow \mathbb{S}$ are homogeneous, and $X \xrightarrow{\alpha} Y$ is formally quasipresmooth, with $\beta: Z \rightarrow Y$ any map, set $T := X \times_Y Z$, and observe that $T \rightarrow Z$ is quasipresmooth. There is an isomorphism

$$D_t^*(T/Z, M) \cong D_x^*(X/Y, M),$$

for $t \in T(A)$ with image $x \in X(A)$.

Proposition 1.12 Let $X, Y, Z: d\mathcal{N}^b \rightarrow \mathbb{S}$ be homogeneous functors, with $\alpha: X \rightarrow Y$ and $\beta: Y \rightarrow Z$ formally quasismooth. For $x \in X(A)$, there is then a long exact sequence

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & D_x^j(X/Y, M) & \longrightarrow & D_x^j(X/Z, M) & \longrightarrow & D_y^j(Y/Z, M) \\ & & \xrightarrow{\partial} & & D_x^{j+1}(X/Y, M) & \longrightarrow & D_x^{j+1}(X/Z, M) & \longrightarrow & \cdots \end{array}$$

where $y \in Y(A)$ is the image of x .

Proof Since $T_x(X/Y, M) = \ker(\alpha: T_x(X/Z, M) \rightarrow T_y(Y/Z, M))$, we have fibration sequences

$$\cdots \rightarrow \pi_i T_x(X/Y, M[-n]) \rightarrow \pi_i T_x(X/Z, M[-n]) \rightarrow \pi_i T_x(Y/Z, M[-n]) \rightarrow \cdots$$

for all $i, n \geq 0$ so the result follows because $\pi_i T_x(X/Y, M[-n]) = D_x^{n-i}(X/Y, M)$, and similarly for $X/Z, Y/Z$. □

Definition 1.13 Recall that a local coefficient system on $S \in \mathbb{S}$ is an object V of $\text{Ab}(\mathbb{S} \downarrow S)$ for which the maps $\partial_i: V_s \rightarrow V_{\partial_i s}$ are isomorphisms for all $s \in S_n$, where $V_s := V_n \times_{S_n} \{s\}$.

Lemma 1.14 If $\alpha: X \rightarrow Y$ is a formally quasismooth morphism between homogeneous functors, take an object $A \in d\mathcal{N}^\flat$ and $M \in d\text{Mod}_A$. Then there is a local coefficient system

$$D^*(X/Y, M)$$

on $X(A)$, whose stalk at $x \in X(A)$ is $D_x^*(X/Y, M)$. In particular, $D_x^*(X/Y, M)$ depends (up to noncanonical isomorphism) only on the image of x in $\pi_0 X(A)$.

Proof As with [26, Lemma 1.16], this follows straightforwardly from the proof of [26, Lemma 1.9]. □

1.2.1 Obstructions

Proposition 1.15 If $F, G: d\mathcal{N}^\flat \rightarrow \mathbb{S}$ are homogeneous, with G prehomotopic and $\alpha: F \rightarrow G$ formally quasismooth, then for any square zero extension $e: I \rightarrow A \rightarrow B$ (with $f: A \rightarrow B$) in $d\mathcal{N}^\flat$, there is a sequence of sets

$$\pi_0(\text{FA}) \xrightarrow{f_*} \pi_0(\text{FB} \times_{\text{GB}} \text{GA}) \xrightarrow{o_e} \Gamma(\text{FB}, D^1(F/G, I)),$$

where $\Gamma(-)$ denotes the global section functor. This is exact in the sense that the fibre of o_e over 0 is the image of f_* . Moreover, there is a group action of $D_x^0(F/G, I)$ on the fibre of $\pi_0(\text{FA}) \rightarrow \pi_0(\text{FB})$ over x , whose orbits are precisely the fibres of f_* .

For any $y \in F_0 A$, with $x = f_* y$, the fibre of $\text{FA} \rightarrow \text{FB} \times_{\text{GB}} \text{GA}$ over x is isomorphic to $T_x(F/G, I)$, and the sequence above extends to a long exact sequence

$$\begin{array}{ccccccc} \cdots & \xrightarrow{e_*} & \pi_n(\text{FA}, y) & \xrightarrow{f_*} & \pi_n(\text{FB} \times_{\text{GB}} \text{GA}, x) & \xrightarrow{o_e} & D_y^{1-n}(F/G, I) & \xrightarrow{e_*} & \pi_{n-1}(\text{FA}, y) \\ & & \xrightarrow{f_*} & \cdots & \xrightarrow{f_*} & \xrightarrow{o_e} & D_y^0(F/G, I) & \xrightarrow{-*y} & \pi_0(\text{FA}). \end{array}$$

Proof The proof of [26, Proposition 1.17] carries over to this generality. □

Corollary 1.16 If $F, G: d\mathcal{N}^\flat \rightarrow \mathbb{S}$ are homogeneous, with G prehomotopic and $\alpha: F \rightarrow G$ prehomotopic and formally quasipresmooth, then α is formally presmooth if and only if $D_x^i(F/G, M) = 0$ for all $i > 0$, all discrete rings A , all $x \in \pi_0 F(A)$ and all A -modules M .

Definition 1.17 Given a functor F on $d\mathcal{N}_R^\flat$, define the functor $\pi^0 F$ on Alg_{H_0R} by $(\pi^0 F)(A) := F(A)$.

Corollary 1.18 Take a morphism $\alpha: F \rightarrow F'$ of homogeneous formally quasismooth functors $F, F': d\mathcal{N}^\flat \rightarrow \mathbb{S}$. Then α is a weak equivalence if and only if

- (1) $\pi^0\alpha: \pi^0 F \rightarrow \pi^0 F'$ is a weak equivalence of functors $\text{Alg}_{H_0R} \rightarrow \mathbb{S}$;
- (2) the maps $D_x^i(F, M) \rightarrow D_{\alpha x}^i(F', M)$ are isomorphisms for all $A \in \text{Alg}_{H_0R}$, all A -modules M , all $x \in F(A)_0$, and all $i > 0$.

Proof For any $A \in d\mathcal{N}^\flat$, we need to show that $\alpha_A: F(A) \rightarrow F'(A)$ is a weak equivalence. By hypothesis, we know that this holds if we replace A with H_0A . Now, the map $A \rightarrow H_0A$ is a nilpotent extension; let the kernel be I_A . The maps $A/I_A^{n+1} \rightarrow A/I_A^n$ are square zero extensions, and their kernels I_A^n/I_A^{n+1} are H_0A -modules. This allows us to proceed inductively, using the long exact sequence of Proposition 1.15 to deduce that $F(A/I_A^{n+1}) \rightarrow F'(A/I_A^{n+1})$ is a weak equivalence whenever $F(A/I_A^n) \rightarrow F'(A/I_A^n)$ is so. □

1.3 Representability

For the remainder of this section, R will be a derived G -ring admitting a dualising module (in the sense of [18, Definition 3.6.1]). In particular, this is satisfied if R is a G -ring admitting a dualising complex in the sense of Hartshorne [13, Chapter V]. Examples are \mathbb{Z} , any field, or any Gorenstein local ring.

Theorem 1.19 Take a functor $F: d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$ satisfying the following conditions:

- (1) F is formally quasismooth.
- (2) For all discrete rings A , $F(A)$ is n -truncated, ie $\pi_i F(A) = 0$ for all $i > n$.
- (3) F is homogeneous.
- (4) $\pi^0 F: \text{Alg}_{H_0R} \rightarrow \mathbb{S}$ is a hypersheaf for the étale topology.
- (5) $\pi_0\pi^0 F: \text{Alg}_{H_0R} \rightarrow \text{Set}$ preserves filtered colimits.
- (6) For all $A \in \text{Alg}_{H_0R}$ and all $x \in F(A)$, the functors $\pi_i(\pi^0 F, x): \text{Alg}_A \rightarrow \text{Set}$ preserve filtered colimits for all $i > 0$.
- (7) For all finitely generated integral domains $A \in \text{Alg}_{H_0R}$, all $x \in F(A)_0$ and all étale morphisms $f: A \rightarrow A'$, the maps

$$D_x^*(F, A) \otimes_A A' \rightarrow D_{fx}^*(F, A')$$

are isomorphisms.

- (8) For all finitely generated $A \in \text{Alg}_{\mathbb{H}_0 R}$ and for all $x \in F(A)_0$, the functors $D_x^i((F/R), -): \text{Mod}_A \rightarrow \text{Ab}$ preserve filtered colimits for all $i > 0$.
- (9) For all finitely generated integral domains $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $x \in F(A)_0$, the groups $D_x^i(F/R, A)$ are all finitely generated A -modules.
- (10) For any complete discrete local Noetherian $\mathbb{H}_0 R$ -algebra A , with maximal ideal \mathfrak{m} , the map

$$\pi^0 F(A) \rightarrow \varprojlim F(A/\mathfrak{m}^r)$$

is a weak equivalence.

Then F is the restriction to $d\mathcal{N}_R^b$ of an almost finitely presented geometric derived n -stack $F': d\text{Alg}_R \rightarrow \mathbb{S}$. Moreover, F' is uniquely determined by F (up to weak equivalence).

Proof This variant of Lurie’s Representability Theorem essentially appears in [26, Theorem 2.17], which takes a homotopy preserving, homotopy homogeneous functor instead of a formally quasismooth homogeneous functor. However, every homotopic functor is homotopy preserving (by [25, Lemma 2.24]), while every formally quasipresmooth homogeneous functor is homotopy homogeneous (by [25, Lemma 2.27]). Finally, note that formal quasipresmoothness allows us to replace homotopy limits with limits. □

Remark 1.20 For the definition of hypersheaves featuring in (4) above, see [26, Definition 1.29]. For all the applications in this paper, the following observation suffices. Given a groupoid valued functor $\Gamma: \text{Alg}_{\mathbb{H}_0 R} \rightarrow \text{Gpd}$, the nerve $B\Gamma: \text{Alg}_{\pi_0 R} \rightarrow \mathbb{S}$ is a hypersheaf if and only if Γ is a stack (in the sense of Laumon and Moret-Bailly [17]).

Remark 1.21 Note that there are slight differences in terminology between [32; 18]. In the former, only disjoint unions of affine schemes are 0-representable, so arbitrary schemes are 2-geometric stacks, and Artin stacks are 1-geometric stacks if and only if they have affine diagonal. In the latter, algebraic spaces are 0-stacks. A geometric n -stack is called n -truncated in [32], and it follows easily that every n -geometric stack in [32] is n -truncated. A weak converse is that every geometric n -stack is $(n + 2)$ -geometric.

Theorem 1.19 follows the convention from [18], so “geometric derived n -stack” means “ n -truncated derived geometric stack”.

Beware, however, that condition (2) of the theorem only applies to *discrete* rings. In general, if $A \in d\mathcal{N}_R^b$ with $H_i A = 0$ for $i > m$, then a geometric derived n -stack F will have the property that $\pi_j F(A) = 0$ for all $j > m + n$.

1.4 Prerepresentability

Concrete approaches to derived moduli can naturally produce functors $F: d\mathcal{N}_R^{\flat} \rightarrow \mathbb{S}$ with the property that $\pi_i F(A) = 0$ for all $i > n$ and all A . Such functors are not geometric derived n -stacks, since they cannot be both homotopic and homogeneous. The purpose of this section is to establish weaker conditions which can be satisfied by such functors, and still allow us to associate a geometric derived n -stack \underline{F} to F . In particular, all of the examples in Sections 3–6 will work by constructing derived geometric 1-stacks \underline{F} from groupoid valued functors F .

Definition 1.22 Define a simplicial enrichment of $s\mathcal{N}_R^{\flat}$ as follows. For $A \in s\mathcal{N}_R^{\flat}$ and a finite simplicial set K , $A^K \in s\mathcal{N}_R^{\flat}$ is defined by

$$(A^K)_n := \text{Hom}_{\mathbb{S}}(K \times \Delta^n, A).$$

Spaces $\underline{\text{Hom}}(A, B) \in \mathbb{S}$ of morphisms are then given by

$$\underline{\text{Hom}}_{s\mathcal{N}_R^{\flat}}(A, B)_n := \text{Hom}_{s\mathcal{N}_R^{\flat}}(A, B^{\Delta^n}).$$

Definition 1.23 Define a simplicial enrichment of $dg_+\mathcal{N}_R^{\flat}$ as follows. First set Ω_n to be the differential graded algebra

$$\mathbb{Q}[t_0, t_1, \dots, t_n, dt_0, dt_1, \dots, dt_n] / \left(\sum t_i - 1, \sum dt_i \right)$$

of rational differential forms on the n -simplex Δ^n , where t_i is of degree 0. These fit together to form a simplicial complex Ω_{\bullet} of graded-commutative DG-algebras, and we define A^{Δ^n} as the good truncation $A^{\Delta^n} := \tau_{\geq 0}(A \otimes \Omega_n)$. (Note that this construction commutes with finite limits, so extends to define A^K for finite simplicial sets K .)

Spaces $\underline{\text{Hom}}(A, B) \in \mathbb{S}$ of morphisms are then given by

$$\underline{\text{Hom}}_{dg_+\mathcal{N}_R^{\flat}}(A, B)_n := \text{Hom}_{dg_+\mathcal{N}_R^{\flat}}(A, B^{\Delta^n}).$$

Definition 1.24 Given a functor $F: d\mathcal{N}^{\flat} \rightarrow \mathbb{S}$, we define $\underline{F}: d\mathcal{N}^{\flat} \rightarrow s\mathbb{S}$, (for $s\mathbb{S}$ the category of bisimplicial sets), by

$$\underline{F}(A)_n := F(A^{\Delta^n}).$$

Definition 1.25 Define $\overline{W}: s\mathbb{S} \rightarrow \mathbb{S}$ to be the right adjoint to Illusie’s total Dec functor given by $\text{DEC}(X)_{mn} = X_{m+n+1}$. Explicitly,

$$\overline{W}_p(X) = \left\{ (x_0, x_1, \dots, x_p) \in \prod_{i=0}^p X_{i,p-i} \mid \partial_0^v x_i = \partial_{i+1}^h x_{i+1}, \forall 0 \leq i < p \right\},$$

with operations

$$\begin{aligned} \partial_i(x_0, \dots, x_p) &= (\partial_i^v x_0, \partial_{i-1}^v x_1, \dots, \partial_1^v x_{i-1}, \partial_i^h x_{i+1}, \partial_i^h x_{i+2}, \dots, \partial_i^h x_p), \\ \sigma_i(x_0, \dots, x_p) &= (\sigma_i^v x_0, \sigma_{i-1}^v x_1, \dots, \sigma_0^v x_i, \sigma_i^h x_i, \sigma_i^h x_{i+1}, \dots, \sigma_i^h x_p). \end{aligned}$$

In [3], Cegarra and Remedios established that the canonical natural transformation

$$\text{diag} X \rightarrow \overline{W} X$$

from the diagonal is a weak equivalence for all X .

Lemma 1.26 *For a homotopic functor $F: d\mathcal{N}^b \rightarrow \mathbb{S}$, the natural transformation $F \rightarrow \overline{W} F$ is a weak equivalence.*

Proof This is [26, Lemma 3.13]. □

Proposition 1.27 *If a formally quasipresmooth homogeneous functor $F: d\mathcal{N}^b \rightarrow \mathbb{S}$ is prehomotopic, then the functor $\overline{W} F: d\mathcal{N}^b \rightarrow \mathbb{S}$ is homogeneous and formally quasismooth.*

Proof This is essentially the same as [26, Corollaries 3.10 and 3.12] (replacing weak equivalences with isomorphisms, and homotopy fibre products with fibre products), using the result from Cegarra and Remedios [4] that diagonal fibrations are \overline{W} -fibrations. □

Lemma 1.28 *Given a formally quasipresmooth prehomotopic homogeneous functor $F: d\mathcal{N}_R^b \rightarrow \mathbb{S}$, an object $A \in d\mathcal{N}^b$, a point $x \in F_0(A)$, and a module $M \in d\text{Mod}_A$, there are canonical isomorphisms*

$$D_x^i(F, M) \cong D_x^i(\overline{W} F, M).$$

Proof The proof of [26, Lemma 3.15], which deals with the case when A and M are discrete, carries over to this generality. □

Corollary 1.29 *If a formally quasipresmooth homogeneous functor $F: d\mathcal{N}^b \rightarrow \mathbb{S}$ is prehomotopic, and admits a morphism $\alpha: F \rightarrow G$ to a formally quasismooth homogeneous functor, then α induces a functorial weak equivalence*

$$\overline{W} F \simeq G$$

if and only if

- (1) $\pi^0 \alpha: \pi^0 F \rightarrow \pi^0 G$ is a weak equivalence of functors $\text{Alg}_{\mathbb{H}_0 R} \rightarrow \mathbb{S}$;
- (2) the maps $D_x^i(F, M) \rightarrow D_{\alpha x}^i(G, M)$ are isomorphisms for all $A \in \text{Alg}_{\mathbb{H}_0 R}$, all A -modules M , all $x \in F(A)_0$, and all $i > 0$.

Proof By Lemma 1.28, the map from F to \overline{WF} induces isomorphisms on D^i , so the maps

$$D_x^i(\overline{WF}, M) \rightarrow D_{\alpha x}^i(\overline{WG}, M)$$

are isomorphisms. Proposition 1.27 shows that \overline{WF} and \overline{WG} are formally quasismooth homogeneous functors. Since $F \rightsquigarrow \overline{WF}$ does not change $\pi^0 F$, Lemma 1.28 and Corollary 1.18 imply that the map

$$\overline{WF} \rightarrow \overline{WG}$$

is a weak equivalence.

Since G is also homogeneous and formally quasismooth, Corollary 1.18 gives a weak equivalence $G \rightarrow \overline{WG}$. Combining this with the weak equivalence above, we see that \overline{WF} and G are canonically weakly equivalent. \square

Remark 1.30 By replacing Proposition 1.15 with [26, Proposition 1.17], the proof of Corollary 1.29 works just as well if F is homotopy homogeneous and homotopy surjecting, while G is homotopy homogeneous and homotopy preserving. In particular, this holds if G is any presentation of a derived geometric n -stack.

Theorem 1.31 Take a functor $F: d\mathcal{N}_R^{\flat} \rightarrow \mathbb{S}$ satisfying the following conditions:

- (1) F is prehomotopic.
- (2) F is formally quasipresmooth.
- (3) For all discrete rings A , $F(A)$ is n -truncated, ie $\pi_i F(A) = 0$ for all $i > n$.
- (4) F is homogeneous.
- (5) $\pi^0 F: \text{Alg}_{\mathbb{H}_0 R} \rightarrow \mathbb{S}$ is a hypersheaf for the étale topology.
- (6) $\pi_0 \pi^0 F: \text{Alg}_{\mathbb{H}_0 R} \rightarrow \text{Set}$ preserves filtered colimits.
- (7) For all $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $x \in F(A)$, the functors $\pi_i(\pi^0 F, x): \text{Alg}_A \rightarrow \text{Set}$ preserve filtered colimits for all $i > 0$.
- (8) For all finitely generated integral domains $A \in \text{Alg}_{\mathbb{H}_0 R}$, all $x \in F(A)_0$ and all étale morphisms $f: A \rightarrow A'$, the maps

$$D_x^*(F, A) \otimes_A A' \rightarrow D_{fx}^*(F, A')$$

are isomorphisms.

- (9) For all finitely generated $A \in \text{Alg}_{\mathbb{H}_0 R}$ and for all $x \in F(A)_0$, the functors $D_x^i(F, -): \text{Mod}_A \rightarrow \text{Ab}$ preserve filtered colimits for all $i > 0$.

- (10) For all finitely generated integral domains $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $x \in F(A)_0$, the groups $D_x^i(F, A)$ are all finitely generated A -modules.
- (11) For all complete discrete local Noetherian $\mathbb{H}_0 R$ -algebras A , with maximal ideal \mathfrak{m} , the map

$$\pi^0 F(A) \rightarrow \varprojlim F(A/\mathfrak{m}^r)$$

is a weak equivalence.

Then \overline{WF} is the restriction to $d\mathcal{N}_R^b$ of an almost finitely presented geometric derived n -stack $F': s\text{Alg}_R \rightarrow \mathbb{S}$ (resp. $F': dg_+\text{Alg}_R \rightarrow \mathbb{S}$). Moreover, F' is uniquely determined by F (up to weak equivalence).

Proof This essentially appears as [26, Theorem 3.16], which takes a homotopy surjecting, homotopy homogeneous functor instead of a formally quasismooth prehomotopic homogeneous functor. However, every prehomotopic functor is automatically homotopy surjecting, while every formally quasipresmooth homogeneous functor is homotopy homogeneous (by [25, Lemma 2.27]).

Alternatively, note that Proposition 1.27 ensures that \overline{WF} is homogeneous and formally quasismooth, so we may apply Theorem 1.19. □

2 Sheaves on the pro-Zariski and pro-étale sites

Our primary motivation for this section is the following. In general, an infinite direct sum $M = \bigoplus_i M_i$ of locally free A -modules is not locally free for the étale topology, in the sense that there need not exist any faithfully flat étale morphism $A \rightarrow A'$ with $M \otimes_A A'$ free. However, for all maximal ideals \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $M \otimes_A A_{\mathfrak{m}}$ is free. Indeed, for any set S of maximal ideals, the $\prod_{\mathfrak{m} \in S} A_{\mathfrak{m}}$ -module $M \otimes_A \prod_{\mathfrak{m} \in S} A_{\mathfrak{m}}$ is free. As we will show below, this amounts to saying that M is locally free for the pro-Zariski topology, and hence for the pro-étale topology.

Definition 2.1 A presheaf $\mathcal{F}: \text{Alg}_R \rightarrow \text{Set}$ is said to be locally of finite presentation if for any filtered direct system $\{A_i\}_i$, the map

$$\varinjlim_i \mathcal{F}(A_i) \rightarrow \mathcal{F}\left(\varinjlim_i A_i\right)$$

is an isomorphism.

Definition 2.2 Given a property \mathbf{P} of morphisms of affine schemes, we say that $f: X \rightarrow Y$ is pro- \mathbf{P} if it can be expressed as the limit $X = \varprojlim_i X_i$ of a filtered inverse system $\{X_i\}_i$ of \mathbf{P} -morphisms $X_i \rightarrow Y$, in which all structure maps $X_i \rightarrow X_j$ are \mathbf{P} -morphisms. Likewise, we say that a map $A \rightarrow B$ of rings is ind- \mathbf{P} if $\text{Spec } B \rightarrow \text{Spec } A$ is pro- \mathbf{P} .

Lemma 2.3 *If $\mathcal{F}: \text{Alg}_R \rightarrow \text{Set}$ is locally of finite presentation and a sheaf for a class \mathbf{P} of covering morphisms, then \mathcal{F} is also a sheaf for the class $\text{pro}(\mathbf{P})$.*

Proof Given any finite (possibly empty) set $\{A_s\}_{s \in S}$ of objects of Alg_R , we automatically have an isomorphism

$$\mathcal{F}\left(\prod_{s \in S} A_s\right) \rightarrow \prod_{s \in S} \mathcal{F}(A_s),$$

so we need only check that for any ring homomorphism $A \rightarrow B$ in $\text{ind}(\mathbf{P})$, the diagram

$$\mathcal{F}(A) \rightarrow \mathcal{F}(B) \implies \mathcal{F}(B \otimes_A B)$$

is an equaliser diagram.

Now, we can express $A \rightarrow B$ as a direct limit $B = \varinjlim_i B_i$ of \mathbf{P} -morphisms $A \rightarrow B_i$, so

$$\mathcal{F}(B) \cong \varinjlim_i \mathcal{F}(B_i), \quad \mathcal{F}(B \otimes_A B) \cong \varinjlim_i \mathcal{F}(B_i \otimes_A B_i),$$

\mathcal{F} being locally of finite presentation. Since \mathcal{F} is a \mathbf{P} -sheaf, the diagram

$$\mathcal{F}(A) \rightarrow \mathcal{F}(B_i) \implies \mathcal{F}(B_i \otimes_A B_i)$$

is an equaliser, and the required result now follows from the observation that finite limits commute with filtered direct limits. □

We will now construct weak universal covers for the topologies which concern us.

2.1 The pro-Zariski topology

Definition 2.4 A morphism $A \rightarrow B$ of commutative rings is said to be conservative if the map

$$A^\times \rightarrow A \times_B B^\times$$

is an isomorphism, where A^\times denotes units in A . Say a morphism $\text{Spec } B \rightarrow \text{Spec } A$ of affine schemes is conservative if $A \rightarrow B$ is so.

Definition 2.5 Say that a map $A \rightarrow B$ of commutative rings is a localisation if $B \cong A[S^{-1}]$, for some subset $S \subset A$.

Note that $\text{Spec } D \rightarrow \text{Spec } C$ is an open immersion if and only if $D \cong C[S^{-1}]$ for some finite set S . Thus $A \rightarrow B$ is a localisation if and only if $\text{Spec } B \rightarrow \text{Spec } A$ is a pro-(open immersion).

Lemma 2.6 Any commutative ring homomorphism $f: A \rightarrow B$ has a unique factorisation $A \rightarrow C \rightarrow B$ as a localisation followed by a conservative map.

Proof This is from Anel [1, Proposition 52]. The factorisation is given by setting $S := \{a \in A \mid f(a) \in B^\times\}$, then letting $C := A[S^{-1}]$. \square

In order to study the Zariski topology, we wish to use local isomorphisms rather than open immersions. Likewise, for the pro-Zariski topology, we want pro-(local isomorphisms) rather than pro-(open immersions).

Definition 2.7 A morphism $A \rightarrow B$ of commutative rings is said to be strongly conservative if it is conservative, and the map $\text{id}(A) \rightarrow \text{id}(B)$ on sets of idempotent elements is an isomorphism. Say that a morphism $\text{Spec } B \rightarrow \text{Spec } A$ of affine schemes is strongly conservative if $A \rightarrow B$ is so.

Remark 2.8 The set $\text{id}(A)$ just consists of ring homomorphisms $\mathbb{Z}^2 \rightarrow A$. If A is finitely generated, then $\text{Spec } A$ has a finite set $\pi(\text{Spec } A)$ of components. Since an arbitrary ring A can be expressed as a filtered colimit $A = \varinjlim A_i$ of finitely generated rings, we can then define $\pi(A)$ to be the profinite set $\varprojlim \pi(\text{Spec } A_i)$. Thus a conservative morphism $\text{Spec } B \rightarrow \text{Spec } A$ is strongly conservative if and only if $\pi(\text{Spec } B) \rightarrow \pi(\text{Spec } A)$ is an isomorphism of profinite sets.

Lemma 2.9 Every morphism $f: X \rightarrow Y$ of affine schemes has a unique factorisation $X \rightarrow (X/Y)^{\text{loc}} \rightarrow Y$ as a strongly conservative map followed by a pro-(local isomorphism).

Proof This is remarked at the end of [1, Section 4.2], where strongly conservative maps are denoted by Conv , and pro-(local isomorphisms) by Zet . Explicitly, we first factorise f as $X \rightarrow Y \times_{\pi(Y)} \pi(X) \rightarrow Y$, and then apply Lemma 2.6 to the first map, obtaining $X \rightarrow (X/Y)^{\text{loc}} \rightarrow Y \times_{\pi(Y)} \pi(X) \rightarrow Y$. If $X = \text{Spec } B$ and $Y = \text{Spec } A$, note that

$$Y \times_{\pi(Y)} \pi(X) = \text{Spec}(A \otimes_{\mathbb{Z}.\text{id}(A)} \mathbb{Z}.\text{id}(B)).$$

Note we would get the same construction by applying Lemma 2.6 to $X \rightarrow Y \times \pi(X)$. \square

Lemma 2.10 *For any commutative ring A , the category of pro-Zariski covers of $\text{Spec } A$ has a weakly initial object $\text{Spec } C$. In other words, for any covering pro-(local isomorphism) $Y \rightarrow \text{Spec } A$, there exists a map $\text{Spec } C \rightarrow Y$ over $\text{Spec } A$, although the map need not be unique.*

Moreover, every covering pro-(local isomorphism) $Z \rightarrow \text{Spec } C$ has a section.

Proof Let S be the set of maximal ideals of A , and set $C := (A / \prod_{\mathfrak{m} \in S} (A/\mathfrak{m}))^{\text{loc}}$, as constructed in Lemma 2.9. Explicitly, we first form the subring A' of A^S consisting of functions $f: S \rightarrow A$ with finite image. To form C , we then invert any element $f \in A'$ whenever for all $s \in S$, $f(s) \notin \mathfrak{m}_s$.

Now, given any covering pro-(local isomorphism) $\text{Spec } B \rightarrow \text{Spec } A$, use the covering property to lift the closed points of A to closed points of B ; this gives us a map

$$g: B \rightarrow \prod_{\mathfrak{m}} A/\mathfrak{m}.$$

Properties of unique factorisation systems then give a unique map

$$B \rightarrow \left(A / \prod_{\mathfrak{m} \in S} (A/\mathfrak{m}) \right)^{\text{loc}}$$

compatible with g .

For the second part, take a covering pro-(local isomorphism) $Z = \text{Spec } D \rightarrow \text{Spec } C$, and choose a lift $D \rightarrow A/\mathfrak{m}$ of each canonical map $C \rightarrow A/\mathfrak{m}$. This gives a diagram $A \rightarrow D \rightarrow \prod_{\mathfrak{m} \in S} A/\mathfrak{m}$ with $h: A \rightarrow D$ opposite to a pro-(local isomorphism), so the universal property of C then gives a unique factorisation $D \rightarrow C \rightarrow \prod_{\mathfrak{m} \in S} A/\mathfrak{m}$. The composition $C \rightarrow D \rightarrow C$ must then be the identity, since $C \rightarrow \prod_{\mathfrak{m} \in S} A/\mathfrak{m}$ is strongly conservative. \square

2.2 The pro-étale topology

Definition 2.11 A morphism $f: A \rightarrow B$ is said to be Henselian if any factorisation $A \rightarrow A' \rightarrow B$, with $A \rightarrow A'$ étale, has a section $A' \rightarrow A$ over B . Say that a morphism $\text{Spec } B \rightarrow \text{Spec } A$ of affine schemes is Henselian if $A \rightarrow B$ is so.

Lemma 2.12 *Every morphism $f: X \rightarrow Y$ of affine schemes has a unique factorisation $X \rightarrow (X/Y)^{\text{hen}} \rightarrow Y$ as a Henselian map followed by a pro-étale morphism.*

Proof This is an immediate consequence of [1, Proposition 64], which shows that ind-étale morphisms and Henselian morphisms form the left and right classes of a unique

factorisation system on the category of commutative rings. Explicitly, if $Y = \text{Spec } A$ and $X = \text{Spec } B$, then

$$(A/B)^{\text{hen}} := \varinjlim A_i,$$

where A_i runs over all factorisations $A \rightarrow A_i \rightarrow B$ of $f^\#$ with $A \rightarrow A_i$ étale. Then $(X/Y)^{\text{hen}} := \text{Spec } (A/B)^{\text{hen}}$. □

Lemma 2.13 *For any commutative ring A , there is a weakly initial object $\text{Spec } C$ in the category of pro-étale coverings of $\text{Spec } A$.*

Moreover, every pro-étale covering $Z \rightarrow \text{Spec } C$ has a section.

Proof For each point x of $\text{Spec } A$, choose a geometric point \bar{x} over x , so $k(\bar{x})$ is a separably closed field, and let the set of all these points be S . Now, use Lemma 2.12 to construct the unique factorisation

$$A \rightarrow \left(A / \prod_{\bar{x} \in S} k(\bar{x}) \right)^{\text{hen}} \rightarrow \prod_{\bar{x} \in S} k(\bar{x})$$

of $A \rightarrow \prod_{\bar{x} \in S} k(\bar{x})$. The arguments used in Lemma 2.10 now adapt to show that $\text{Spec } C := [(\text{Spec } \prod_{\bar{x} \in S} k(\bar{x})) / \text{Spec } A]^{\text{hen}}$ is weakly initial in the category of pro-étale coverings of $\text{Spec } A$, and that every covering of $\text{Spec } C$ has a section. □

2.3 Sheaves on derived rings

Definition 2.14 Given a subclass \mathbf{P} of flat morphisms of commutative rings, closed under pushouts and composition, say that a morphism $f: A \rightarrow B$ in $s\text{Ring}$ is

- (1) homotopy \mathbf{P} if $\pi_0 f: \pi_0 A \rightarrow \pi_0 B$ is in \mathbf{P} , and the maps

$$\pi_n(A) \otimes_{\pi_0 A} \pi_0 B \rightarrow \pi_n B$$

are isomorphisms for all n ;

- (2) strictly \mathbf{P} if $f_0: A_0 \rightarrow B_0$ is in \mathbf{P} , and the maps

$$A_n \otimes_{A_0} B_0 \rightarrow B_n$$

are isomorphisms for all n .

Definition 2.15 Given \mathbf{P} as above, say that a morphism $f: A \rightarrow B$ in $dg_+ \text{Alg}_{\mathbb{Q}}$ is

- (1) homotopy \mathbf{P} if $H_0 f: H_0 A \rightarrow H_0 B$ is in \mathbf{P} , and the maps

$$H_n(A) \otimes_{H_0 A} H_0 B \rightarrow H_n B$$

are isomorphisms for all n ;

(2) strictly \mathbf{P} if $f_0: A_0 \rightarrow B_0$ is in \mathbf{P} , and the maps

$$A_n \otimes_{A_0} B_0 \rightarrow B_n$$

are isomorphisms for all n .

Lemma 2.16 *Every strictly \mathbf{P} morphism in $s\text{Ring}$ or $dg_+\text{Alg}_{\mathbb{Q}}$ is homotopy \mathbf{P} .*

Proof We first prove this in the simplicial case. Take a strictly \mathbf{P} morphism $f: A \rightarrow B$; taking homotopy groups gives $\pi_n(B) \cong \pi_n(A) \otimes_{A_0} B_0$, by flat base change. We then have isomorphisms

$$\begin{aligned} \pi_n(B) &\cong \pi_n(A) \otimes_{A_0} B_0 \\ &\cong \pi_n(A) \otimes_{\pi_0 A} (\pi_0 A \otimes_{A_0} B_0) \\ &\cong \pi_n(A) \otimes_{\pi_0 A} \pi_0 B, \end{aligned}$$

as required. For the chain algebra case, replace π_n with H_n . □

Definition 2.17 On $s\text{Ring}^{\text{opp}}$ and $dg_+\text{Alg}_{\mathbb{Q}}^{\text{opp}}$, we define topologies for every class \mathbf{P} as above by setting \mathbf{P}_c to be the intersection of \mathbf{P} with faithfully flat morphisms, and saying that $f: A \rightarrow B$ is a homotopy \mathbf{P} covering (resp. a strict \mathbf{P} covering) if f is homotopy \mathbf{P}_c (resp. strictly \mathbf{P}_c).

In this way, we define both homotopy and strict sites for the étale, Zariski, pro-étale and pro-Zariski topologies.

3 Moduli from DGLAs

3.1 DGLAs

Definition 3.1 A differential graded Lie algebra (DGLA) is a graded \mathbb{Q} -vector space $L = \bigoplus_{i \in \mathbb{N}_0} L^i$, equipped with operators $[-, -]: L \times L \rightarrow L$ bilinear and $d: L \rightarrow L$ linear, satisfying

- (1) $[L^i, L^j] \subset L^{i+j}$;
- (2) $[a, b] + (-1)^{\bar{a}\bar{b}}[b, a] = 0$;
- (3) $(-1)^{\bar{c}\bar{a}}[a, [b, c]] + (-1)^{\bar{a}\bar{b}}[b, [c, a]] + (-1)^{\bar{b}\bar{c}}[c, [a, b]] = 0$;
- (4) $d(L^i) \subset L^{i+1}$;
- (5) $d \circ d = 0$;
- (6) $d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$.

Here \bar{a} denotes the degree of $a \pmod 2$, for a homogeneous.

3.1.1 Maurer–Cartan

Definition 3.2 Given a DGLA L^\bullet , define the Maurer–Cartan set by

$$\text{MC}(L) := \{\omega \in L^1 \mid d\omega + \frac{1}{2}[\omega, \omega] = 0 \in L^2\}$$

Lemma 3.3 If a map $e: L \twoheadrightarrow M$ of DGLAs has kernel K , with $[K, K] = 0$, then for any $\omega \in \text{MC}(M)$, the obstruction to lifting ω to $\text{MC}(L)$ lies in

$$H^2(K, d + [\omega, -]).$$

Proof This is well-known. Given $\omega \in \text{MC}(M)$, choose a lift $\tilde{\omega} \in L^1$, and look at $u(\tilde{\omega}) := d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$. Since $[a, [a, a]] = 0$ for any $a \in L^1$, we get

$$du + [\tilde{\omega}, u(\tilde{\omega})] = [d\tilde{\omega}, \tilde{\omega}] + [\tilde{\omega}, d\tilde{\omega}] = 0,$$

so $u \in Z^2(K, d + [\omega, -])$. Another choice for $\tilde{\omega}$ is of the form $\tilde{\omega} + a$, for $a \in K^1$, and then

$$u((\tilde{\omega} + a)) = u((\tilde{\omega})) + da + [\tilde{\omega}, a],$$

so the obstruction is

$$o_e(\omega) := [u(\tilde{\omega})] \in Z^2(K, d + [\omega, -]) / (d + [\omega, -])K^1 = H^2(K, d + [\omega, -]). \quad \square$$

3.1.2 The gauge action

Definition 3.4 Given a DGLA L , we say that a group G_L is a gauge group for L if it is equipped with the extra data

- (1) group homomorphisms $\text{ad}: G_L \rightarrow \text{GL}(L^n)$ for all n ,
- (2) a map $D: G_L \rightarrow L^1$,

satisfying the following conditions for $g, h \in G_L, v, w \in L$:

- (1) $\text{ad}_g([v, w]) = [\text{ad}_g v, \text{ad}_g w]$.
- (2) $D(gh) = Dg + \text{ad}_g(Dh)$.
- (3) $d(Dg) = \frac{1}{2}[Dg, Dg]$.
- (4) $d(\text{ad}_g(v)) = [Dg, \text{ad}_g(v)] + \text{ad}_g(dv)$.

Examples 3.5 If the DGLA L is nilpotent, then a canonical choice for G_L is the group $\exp(L^0)$, with $D(g) = (dg) \cdot g^{-1}$.

When L^0 is finite-dimensional, G_L will typically be an algebraic group integrating L^0 , again with $D(g) = (dg) \cdot g^{-1}$.

Definition 3.6 Given a gauge group G_L for a DGLA L , define the *gauge action* of G_L on $\text{MC}(L)$ by

$$g \star \omega := \text{ad}_g(\omega) - Dg$$

for $g \in G_L$ and $\omega \in \text{MC}(L)$, noting that the conditions on ad_g and D ensure that this is well-defined and a group homomorphism.

Definition 3.7 Given a DGLA L with gauge group G_L , define the Deligne groupoid by $\mathcal{D}\text{el}(L) := [\text{MC}(L)/G_L]$. In other words, $\mathcal{D}\text{el}(L)$ has objects $\text{MC}(L)$, and morphisms from ω to ω' consist of $\{g \in G_L \mid g \star \omega = \omega'\}$.

Define $\text{Del}(L) \in \mathbb{S}$ to be the nerve $B\mathcal{D}\text{el}(L)$ of $\mathcal{D}\text{el}(L)$.

3.2 Moduli of pointed finite schemes

For a fixed $r \in \mathbb{N}$, we now construct a DGLA governing moduli of pointed finite schemes of rank $r + 1$. For any commutative \mathbb{Q} -algebra A , our moduli groupoid consists of nonunital commutative A -algebras B , with the A -module underlying B being locally free of rank r . Our approach is analogous to the treatment of finite subschemes in [6, Section 3].

Definition 3.8 Given a graded vector space V over \mathbb{Q} , let $\text{CL}(V)$ be the free (ind-conilpotent) graded Lie coalgebra $\bigoplus_{n \geq 1} \text{CL}_n(V)$ cogenerated by V . Note that $\text{CL}_n(V)$ is a quotient of $V^{\otimes n}$ by graded shuffle permutations.

Definition 3.9 Given a graded-commutative chain algebra A , define $\beta(A)$ to be the dg Lie coalgebra $\text{CL}(A[-1])$, with coderivation d_C given on cogenerators by

$$d_C(a_1 \otimes a_2 \cdots \otimes a_n) = \begin{cases} da_1 & n = 1, \\ a_1 a_2 & n = 2, \\ 0 & n > 2. \end{cases}$$

Definition 3.10 Define a DGLA L by

$$L^n := \text{Hom}_{\mathbb{Q}}(\text{CL}_{n+1}(\mathbb{Q}^r[-1]), \mathbb{Q}^r[-1]);$$

this can be identified with the space of degree $-n$ Lie coalgebra derivations of $\beta(\mathbb{Q}^r)$, and this latter description allows us to define differential and bracket as

$$d_L(f) = d_\beta \circ f \pm f \circ d_\beta, \quad [f, g] = f \circ g \mp g \circ f.$$

Define a gauge group for L by setting $G_L = \text{GL}(\mathbb{Q}^r) = \text{GL}_r(\mathbb{Q})$. This has a canonical action on $\beta(\mathbb{Q}^r)$, so we set $\text{ad}: G_L \rightarrow \text{GL}(L^n)$ to be the adjoint action on derivations. Finally, $D: G_L \rightarrow L^1$ is given by $D(g) = d_\beta - \beta(g) \circ d_\beta \circ \beta(g)^{-1}$.

Definition 3.11 Given a differential graded (chain) Lie coalgebra C , define the graded-commutative chain algebra $\beta^*(C)$ to be the free graded-commutative algebra on generators $C[1]$, with derivation given on generators by

$$d_{\beta^*(C)} = d_C + \Delta: C[1] \rightarrow C \oplus S^2(C[1])[-1],$$

where $\Delta: C[1] \rightarrow S^2(C[1])[-1] = \text{CL}_2(C)[1]$ is the cobracket.

Note that β^* is left adjoint to the functor β from graded-commutative chain algebras to ind-conilpotent chain Lie coalgebras.

Lemma 3.12 *If we set $G_{L \otimes A} := \text{GL}_r(A)$, then for any commutative \mathbb{Q} -algebra A , $\mathcal{D}\text{el}(L \otimes A)$ is canonically isomorphic to the groupoid of nonunital commutative A -algebra structures on the A -module A^r .*

Proof This is standard. Square-zero A -linear degree -1 derivations on $\beta(\mathbb{Q}^r) \otimes_{\mathbb{Q}} A$ are all of the form $d_{\beta(\mathbb{Q}^r)} + \omega$, for $\omega \in \text{MC}(L \otimes A)$. Given $g \in \text{GL}(A)$, the derivation $d_{\beta(\mathbb{Q}^r)} + g \star \omega$ is then $\beta(g) \circ (d_{\beta(\mathbb{Q}^r)} + \omega) \circ \beta(g)^{-1}$.

An element $\omega \in \text{MC}(L \otimes A)$ is just an associative multiplication $S^2(A^r) \rightarrow A^r$, so corresponds to a nonunital commutative A -algebra structure. \square

Definition 3.13 Given $A \in dg_+ \mathcal{N}_{\mathbb{Q}}^{\flat}$, define $L \otimes A$ to be the DGLA

$$(L \otimes A)^n := \bigoplus_i L^{n+i} \otimes A_i,$$

with differential $d_L \pm d_A$ and bracket given by $[v \otimes a, w \otimes b] = \pm[v, w] \otimes (ab)$, where signs follow the usual graded conventions.

Definition 3.14 For the DGLA L of Definition 3.10, define the groupoid valued functor $\mathcal{G}: dg_+ \mathcal{N}_{\mathbb{Q}}^{\flat} \rightarrow \text{Gpd}$ to be the stackification of the groupoid presheaf

$$A \mapsto [\text{MC}(L \otimes A) / \text{GL}_r(A_0)]$$

in the strict Zariski topology of Definition 2.17.

Explicitly, objects of $\mathcal{G}(A)$ are pairs $(\omega, g) \in \text{MC}(L \otimes A \otimes_{A_0} B \otimes_{A_0} B) \times \text{GL}_r(B \otimes_{A_0} B)$, for $A_0 \rightarrow B$ a faithfully flat local isomorphism (so $\text{Spec } B \rightarrow \text{Spec } A_0$ is an open cover), satisfying the following conditions:

- (1) $g \star (\text{pr}_1^* \omega) = \text{pr}_0^* \omega \in \text{MC}(L \otimes A \otimes_{A_0} B \otimes_{A_0} B)$
- (2) $\text{pr}_{02}^* g = (\text{pr}_{01}^* g) \cdot (\text{pr}_{12}^* g) \in \text{GL}_r(B \otimes_{A_0} B \otimes_{A_0} B)$

An isomorphism from (B, ω, g) to (C, ν, h) is a local isomorphism $B \otimes_{A_0} C \rightarrow D$ with $A_0 \rightarrow D$ faithfully flat, together with an element $\alpha \in \text{GL}_r(D)$ such that we have $\alpha \star \omega = \nu \in \text{MC}(L \otimes A \otimes_{A_0} D)$, with $(\text{pr}_0^* \alpha) \cdot g = h \cdot (\text{pr}_1^* \alpha) \in \text{GL}_r(D \otimes_{A_0} D)$.

Definition 3.15 As in Dwyer and Kan [8], given a simplicial object \mathcal{C} in the category of categories, we define the simplicial set \overline{WC} by first forming the nerve BC (a bisimplicial set), then applying the functor \overline{W} of Definition 1.25, giving

$$\overline{WC} := \overline{W}BC.$$

Explicitly,

$$(\overline{W}\Gamma)_n = \{(\underline{x}, \underline{g}) \mid \underline{x} \in \text{Ob } \Gamma_n \times \text{Ob } \Gamma_{n-1} \times \cdots \times \text{Ob } \Gamma_0, \\ \underline{g} \in \Gamma_{n-1}(\partial_0 x_n, x_{n-1}) \times \Gamma_{n-2}(\partial_0 x_{n-1}, x_{n-2}) \times \cdots \times \Gamma_0(\partial_0 x_1, x_0)\},$$

with operations giving $\partial_i(x_n, \dots, x_0; g_{n-1}, \dots, g_0)$ as

$$\begin{cases} (x_{n-1}, \dots, x_0; g_{n-2}, \dots, g_0) & i = 0, \\ (\partial_i x_n, \partial_{i-1} x_{n-1}, \dots, \partial_1 x_{n-i+1}, x_{n-i-1}, \dots, x_0; \\ \quad \partial_{i-1} g_{n-1}, \dots, \partial_1 g_{n-i+1}, (\partial_0 g_{n-i}) g_{n-i-1}, g_{n-i-2}, \dots, g_0) & 0 < i < n, \\ (\partial_n x_n, \dots, \partial_1 x_1; \partial_{n-1} g_{n-1}, \dots, \partial_1 g_1) & i = n, \end{cases}$$

and $\sigma_i(x_n, \dots, x_0; g_{n-1}, \dots, g_0)$ as

$$(\sigma_i x_n, \sigma_{i-1} x_{n-1}, \dots, \sigma_0 x_{n-i}, x_{n-i}, \dots, x_0; \\ \sigma_{i-1} g_{n-1}, \dots, \sigma_0 g_{n-i}, \text{id}_{x_{n-i}}, g_{n-i-1}, \dots, g_0).$$

Proposition 3.16 The functor $\overline{W}\underline{\mathcal{G}}: dg_+ \mathcal{N}_{\mathbb{Q}}^b \rightarrow \mathbb{S}$ is representable by an almost finitely presented derived geometric 1-stack.

Proof We verify the conditions of Theorem 1.31 for $B\mathcal{G}: dg_+ \mathcal{N}_{\mathbb{Q}}^b \rightarrow \mathbb{S}$. Homogeneity follows immediately, because both $\text{MC}(L \otimes -)$ and GL_r preserve finite limits. Lemma 3.3 implies that $\text{MC}(L \otimes -)$ is prehomotopic, since for any tiny acyclic extension $A \rightarrow B$ with kernel I , it gives the obstruction space as

$$\text{H}^2(L \otimes I, d + [\omega, -]) = \bigoplus_n \text{H}^{2+n}(L \otimes \text{H}_0 B, d + [\omega, -]) \otimes_{\text{H}_0 B} \text{H}_n(I) = 0.$$

It follows immediately that $B\mathcal{G}$ is prehomotopic, and formal quasipresmoothness is a consequence of the smoothness of GL_r .

Now, for $A \in \text{Alg}_{\mathbb{Q}}$, Lemma 3.12 implies that $\mathcal{G}(A)$ is equivalent to the groupoid of rank r commutative algebras over A . This implies that $\pi^0 \mathcal{G}$ is a stack, so $\pi^0 B\mathcal{G}$ is a

hypersheaf, and it also guarantees that the other conditions relating to \mathcal{G} hold, so we need only verify the cohomological conditions.

For an A -algebra B corresponding to an object $[B]$ of $\mathcal{G}(A)$, the results of [6, Section 2] imply that

$$D_{[B]}^i(B\mathcal{G}, M) \cong \text{Ext}_{A \oplus B}^{i+1}(\mathbb{L}_{\bullet}^{A \oplus B/A}, M \otimes_A B),$$

which has all the properties we require. Here, $\mathbb{L}^{A \oplus B/A}$ denotes the cotangent complex (in the sense of Illusie [15]) of the unital algebra $A \oplus B$ over A . This corresponds to the cotangent complex $\mathbb{L}^{B/A}$ in the category of nonunital commutative rings, defined using the formalism of Quillen [29]. \square

Remark 3.17 Alternatively, we can describe the associated derived geometric 1-stack explicitly. The functor $A \mapsto \text{MC}(L \otimes A)$ is an affine dg scheme, and $\overline{W}\underline{\mathcal{G}}$ is just the hypersheafification of the quotient $B[\text{MC}(L)/\text{GL}_r]$ in the homotopy Zariski (and indeed homotopy étale) topologies. In the terminology of the author [27], the simplicial affine dg scheme $B[\text{MC}(L)/\text{GL}_r]$ is a derived Artin 1-hypergroupoid representing $\overline{W}\underline{\mathcal{G}}$.

Proposition 3.18 For $A \in dg_+ \mathcal{N}_{\mathbb{Q}}^b$, the space $\overline{W}\underline{\mathcal{G}}(A)$ is functorially weakly equivalent to the nerve $\overline{W}\mathfrak{G}(A)$ of the ∞ -groupoid $\mathfrak{G}(A)$ of nonunital graded-commutative chain A -algebras B in nonnegative degrees for which $B \otimes_A^{\mathbb{L}} H_0 A$ is weakly equivalent to a locally free module rank r over $H_0 A$.

Proof The data $(\omega, g) \in \mathcal{G}(A)$ amount to giving a locally free A_0 -module M of rank r (defined by the descent datum g), and a closed degree -1 differential δ on the free chain Lie A -coalgebra $\text{CL}_{A_0}(M[-1]) \otimes_{A_0} A$. Note that in the notation of [6, 3.5], $\text{RCA}(\mathbb{Q}^r)$ is the dg scheme representing $\text{MC}(L \otimes -)$.

The functor β^* from Definition 3.11 maps from dg Lie A -coalgebras to nonunital graded-commutative chain A -algebras, giving us a chain algebra

$$\beta^*(\text{CL}_{A_0}(M[-1]) \otimes_{A_0} A, \delta)$$

over A . Thus we have defined a functor $\beta^*: \mathcal{G}(A) \rightarrow \mathfrak{G}(A)$, and Lemma 3.12 implies that this is a weak equivalence when $A \in \text{Alg}_{\mathbb{Q}}$, so $\pi^0 \mathcal{G} \simeq \pi^0 \mathfrak{G}(A)$.

Now, for $A \in \text{Alg}_{\mathbb{Q}}$, $\omega \in \text{MC}(L \otimes A)$ and an A -module N , a standard calculation gives

$$D_{(\omega, \text{id})}^i(B\mathcal{G}, N) \cong H^{i+1}(L \otimes N, d + [\omega,]),$$

which by [6, Section 2] is just $\text{Ext}_{A \oplus B}^{i+1}(\mathbb{L}^{A \oplus B/A}, N \otimes_A B)$, where B is the nonunital A -algebra corresponding to ω . By faithfully flat descent, we deduce that if an A -algebra B is associated to $(\omega, g) \in \mathcal{G}(A)$, then

$$D_{(\omega, \text{id})}^i(B\mathcal{G}, N) \cong \text{Ext}_{A \oplus B}^{i+1}(\mathbb{L}^{A \oplus B/A}, N \otimes_A B).$$

Adapting [25, Corollary 3.10 and Example 3.11] to nonunital algebras,

$$D_{[B]}^i(\overline{W}\mathcal{G}, N) \cong \text{Ext}_{A \oplus B}^{i+1}(\mathbb{L}^{A \oplus B/A}, N \otimes_A B),$$

so f induces isomorphisms on the cohomology groups D^i .

As [25, Example 3.11] adapts to nonunital algebras, the functor $\overline{W}\mathcal{G}$ is also representable by a derived geometric 1-stack, so the weak equivalence follows from Remark 1.30. □

3.3 Derived moduli of polarised projective schemes

Fix a numerical polynomial $h \in \mathbb{Q}[t]$, with $h(i) \geq 0$ for $i \gg 0$. We will now study the moduli of polarised projective schemes $(X, \mathcal{O}_X(1))$ over an affine base, with $\mathcal{O}_X(1)$ ample, for which $\Gamma(X, \mathcal{O}_X(n))$ is locally free of rank $h(n)$ for $n \gg 0$. As in Mumford [19, Lecture 7 Corollary 3], such a polynomial h exists for every flat projective scheme over a connected Noetherian base.

Note that a \mathbb{G}_m -representation M in A -modules is equivalent to an A -linear decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M\{n\},$$

with $\lambda \in \mathbb{G}_m(A)$ acting on $M\{n\}$ as multiplication by λ^n . The functors β^* and β of the previous section both extend naturally to \mathbb{G}_m -equivariant objects.

Definition 3.19 Given $p \gg 0$ and $q \geq p$, define a DGLA $L_{[p,q]}$ over \mathbb{Q} by

$$L_{[p,q]}^n := \text{Hom}_{\mathbb{Q}}^{\mathbb{G}_m} \left(\text{CL}_{n+1} \left(\bigoplus_{q \geq r \geq p} \mathbb{Q}^{h(r)}\{r\}[-1] \right), \bigoplus_{q \geq r \geq p} \mathbb{Q}^{h(r)}\{r\}[-1] \right);$$

this can be identified with the space of \mathbb{G}_m -equivariant degree $-n$ Lie coalgebra derivations of $\beta(\bigoplus_{q \geq r \geq p} \mathbb{Q}^{h(r)}\{r\})$, and this latter description allows us to define differential and bracket as

$$d_{L_{[p,q]}}(f) = d_\beta \circ f \pm f \circ d_\beta, \quad [f, g] = f \circ g \mp g \circ f.$$

Definition 3.20 Given $p \gg 0$, define the pro-DGLA L_p over \mathbb{Q} to be the inverse system $L_p = \{L_{[p,q]}\}_q$, so the underlying DGLA is $\varprojlim_q L_{[p,q]}$, given by

$$L_p^n = \text{Hom}_{\mathbb{Q}}^{\mathbb{G}_m} \left(\text{CL}_{n+1} \left(\bigoplus_{r \geq p} \mathbb{Q}^{h(r)} \{r\}[-1] \right), \bigoplus_{r \geq p} \mathbb{Q}^{h(r)} \{r\}[-1] \right),$$

which can be identified with the space of \mathbb{G}_m -equivariant degree $-n$ Lie coalgebra derivations of $\beta(\bigoplus_{r \geq p} \mathbb{Q}^{h(r)} \{r\})$, the latter of which is regarded as the colimit $\varinjlim_q \beta(\bigoplus_{q \geq r \geq p} \mathbb{Q}^{h(r)} \{r\})$.

Given a \mathbb{Q} -vector space V , we then define $L_p \widehat{\otimes} V$ to be the completed tensor product

$$L_p \widehat{\otimes} V := \varprojlim_q (L_{[p,q]} \otimes V),$$

so

$$(L_p \widehat{\otimes} V)^n := \text{Hom}_{\mathbb{Q}}^{\mathbb{G}_m} \left(\text{CL}_{n+1} \left(\bigoplus_{r \geq p} \mathbb{Q}^{h(r)} \{r\}[-1] \right), \bigoplus_{r \geq p} V^{h(r)} \{r\}[-1] \right).$$

Definition 3.21 Given $A \in \text{Alg}_{\mathbb{Q}}$, we define a gauge group for $L_p \widehat{\otimes} A$ by setting $G_{L_p}(A) := \prod_{r \geq p} \text{GL}_{h(r)}(A)$. This has a canonical action on $\beta(\bigoplus_{r \geq p} \mathbb{Q}^{h(r)}(r)) \otimes_{\mathbb{Q}} A$, so we set $\text{ad}: G_{L_p} \rightarrow \text{GL}(L_p^n)$ to be the adjoint action on derivations. Finally, $D: G_{L_p} \rightarrow L^1$ is given by $D(g) = d_{\beta} - \beta(g) \circ d_{\beta} \circ \beta(g)^{-1}$.

Lemma 3.22 For any commutative \mathbb{Q} -algebra A , the groupoid $[\text{MC}(L_p \widehat{\otimes} A)/G_{L_p}(A)]$ is naturally equivalent to the groupoid of \mathbb{G}_m -equivariant nonunital commutative A -algebra structures on

$$\bigoplus_{r \geq p} A^{h(r)} \{r\}.$$

Proof This is just a graded version of Lemma 3.12. □

Definition 3.23 Given $A \in dg_+ \mathcal{N}_{\mathbb{Q}}^b$, define $L_p(A)$ to be the DGLA

$$L_p(A)^n := \bigoplus_i L_p^{n+i} \widehat{\otimes} A_i$$

with differential $d_L \pm d_A$ and bracket given by $[v \otimes a, w \otimes b] = \pm[v, w] \otimes (ab)$, where signs follow the usual graded conventions.

Definition 3.24 For the DGLA L_p of Definition 3.20, define the groupoid valued functor $\mathcal{G}_p: dg_+ \mathcal{N}_{\mathbb{Q}}^b \rightarrow \text{Gpd}$ to be the stackification of the groupoid presheaf

$$A \mapsto [\text{MC}(L_p(A))/G_{L_p}(A_0)]$$

in the strict pro-Zariski topology of Definition 2.17.

If we make use of Lemma 2.10, we can describe this explicitly by first setting $A'_0 := (A_0/(\prod_{\mathfrak{m}} A_0/\mathfrak{m}))^{\text{loc}}$, where \mathfrak{m} runs over all maximal ideals of A_0 , and then by setting $A' := A'_0 \otimes_{A_0} A$. Objects of $\mathcal{G}_p(A)$ are then pairs $(\omega, g) \in \text{MC}(L_p(A')) \times G_{L_p}(A'_0 \otimes_{A_0} A')$, satisfying the following conditions:

- (1) $g \star (\text{pr}_1^* \omega) = (\text{pr}_0^* \omega) \in \text{MC}(L(A' \otimes_A A'))$
- (2) $\text{pr}_{02}^* g = (\text{pr}_{01}^* g) \cdot (\text{pr}_{12}^* g) \in G_{L_p}(A'_0 \otimes_{A_0} A'_0 \otimes_{A_0} A'_0)$

An isomorphism from (ω_1, g_1) to (ω_2, g_2) is an element $\alpha \in G_{L_p}(A'_0)$ such that $\alpha \star \omega_1 = \omega_2 \in \text{MC}(L_p(A'))$, with $(\text{pr}_0^* \alpha) \cdot g_1 = g_2 \cdot (\text{pr}_1^* \alpha) \in G_{L_p}(A' \otimes_A A')$

Lemma 3.25 For $A \in \text{Alg}_{\mathbb{Q}}$, the groupoid $\mathcal{G}_p(A)$ is canonically equivalent to the groupoid of nonunital \mathbb{G}_m -equivariant commutative A -algebras

$$B = \bigoplus_{r \geq p} B\{r\},$$

with each A -module $B\{r\}$ locally free of rank $h(r)$.

Proof If we set $A' := (A/\prod_{\mathfrak{m}} A/\mathfrak{m})^{\text{loc}}$, where \mathfrak{m} runs over all maximal ideals of A , then Lemma 2.10 shows that any Zariski cover $\text{Spec } B \rightarrow \text{Spec } A'$ must have a section. Hence locally free A' -modules are free. Lemma 3.22 then implies that $[\text{MC}(L_p(A'))/G_{L_p}(A')]$ is equivalent to the groupoid of nonunital \mathbb{G}_m -equivariant commutative A' -algebras $B' = \bigoplus_{r \geq p} B'\{r\}$, with each A' -module $B'\{r\}$ locally free of rank $h(r)$.

Given an object (ω, g) of $\mathcal{G}_p(A)$, it thus follows that ω corresponds to such an A' -algebra B' , while g is a descent datum. This determines a unique A -algebra B with $B' = B \otimes_A A'$, and isomorphisms behave as required. □

Definition 3.26 For $A \in \text{Alg}_{\mathbb{Q}}$, define $\mathcal{M}_p(A)$ to be the full subgroupoid of $\mathcal{G}_p(A)$ whose objects correspond under Lemma 3.25 to finitely generated commutative A -algebras.

Lemma 3.27 The morphism $\mathcal{M}_p \rightarrow \pi^0 \mathcal{G}_p$ of groupoid valued functors on $\text{Alg}_{\mathbb{Q}}$ is formally étale.

Proof For any square zero extension $A \twoheadrightarrow B$ of commutative \mathbb{Q} -algebras, we need to show that

$$\mathcal{M}_p(A) \rightarrow \mathcal{M}_p(B) \times_{\mathcal{G}_p(B)} \mathcal{G}_p(A)$$

is an isomorphism. This follows because any flat A -algebra C is finitely generated if and only if $C \otimes_A B$ is finitely generated as a B -algebra, since any lift of a generating set for $C \otimes_A B$ must give a generating set for C . \square

Lemma 3.28 *The functor $\mathcal{M}_p: \text{Alg}_{\mathbb{Q}} \rightarrow \text{Gpd}$ is locally of finite presentation, in the sense that for any filtered direct system $\{A_i\}_i$ of commutative \mathbb{Q} -algebras with $A = \varinjlim A_i$, the map*

$$\varinjlim_i \mathcal{M}_p(A_i) \rightarrow \mathcal{M}_p(A)$$

is an equivalence of groupoids.

Proof We first show essential surjectivity. Take an object $B \in \mathcal{M}_p(A)$. Since B is finitely generated, we can choose homogeneous generators x_j of degree d_j for $1 \leq j \leq n$, giving us a surjection

$$f: A[x_1, \dots, x_n] \twoheadrightarrow B.$$

If we let $S := \mathbb{Z}[x_1, \dots, x_n]$, then $I := \ker f$ is a graded ideal of $S \otimes A$. In the notation of Haiman and Sturmfels [12], we have a degree functor $\text{deg}: \mathbb{N}^n \rightarrow \mathbb{N}$ given by $(a_1, \dots, a_n) \mapsto \sum_i a_i d_i$, and the Hilbert polynomial is given by $h_I = h$. By [12, Corollary 1.2], there is a projective scheme H_S^h over \mathbb{Z} with $H_S^h(A)$ the set of all graded ideals of S with Hilbert function

$$h_p(i) := \begin{cases} h(i) & i \geq p, \\ 0 & i < p. \end{cases}$$

(Note that we do not use Grothendieck’s construction from [10], since that only describes A -valued points of the Hilbert scheme for A Noetherian.)

In particular, $H_S^{h_p}$ is of finite presentation, so $H_S^{h_p}(A) = \varinjlim_i H_S^{h_p}(A_i)$. Therefore there exists $B_i \in H_S^{h_p}(A_i)$ with $B \cong B_i \otimes_{A_i} A$. The forgetful functor $H_S^{h_p} \rightarrow \mathcal{M}_p$ then ensures that $B_i \in \mathcal{M}_p(A_i)$.

It only remains to show that $\varinjlim_i \mathcal{M}_p(A_i) \rightarrow \mathcal{M}_p(A)$ is full and faithful. Now, [12, Proposition 3.2] shows that the ideal I above is finitely generated, so B is finitely presented over A . Likewise, any objects $B_i, B'_i \in \mathcal{M}_p(A_i)$ will be finitely presented, which implies that

$$\text{Hom}_{\mathcal{M}_p(A)}(B_i \otimes_{A_i} A, B'_i \otimes_{A_i} A) \cong \varinjlim_j \text{Hom}_{\mathcal{M}_p(A_j)}(B_i \otimes_{A_i} A_j, B'_i \otimes_{A_i} A_j),$$

completing the proof. □

Definition 3.29 Define $\mathcal{M}: \text{Alg}_{\mathbb{Q}} \rightarrow \text{Gpd}$ by $\mathcal{M}(A) := \varinjlim_p \mathcal{M}_p(A)$. Likewise, define

$$\mathcal{G} := \varinjlim_p \mathcal{G}_p: dg_+ \mathcal{N}_{\mathbb{Q}}^b \rightarrow \text{Gpd}$$

and $\tilde{\mathcal{M}}: dg_+ \mathcal{N}_{\mathbb{Q}}^b \rightarrow \text{Gpd}$ by

$$\tilde{\mathcal{M}}(A) := \mathcal{G}(A) \times_{\mathcal{G}(\text{H}_0 A)} \mathcal{M}(\text{H}_0 A).$$

Proposition 3.30 For $A \in \text{Alg}_{\mathbb{Q}}$, $\mathcal{M}(A)$ is equivalent to the groupoid of flat polarised schemes $(X, \mathcal{O}_X(1))$ of finite type over A , with $\mathcal{O}_X(1)$ ample and the A -modules $\Gamma(X, \mathcal{O}_X(n))$ locally free of rank $h(n)$ for all $n \gg 0$.

Proof This is fairly standard—the analogue for subschemes is [12, Lemma 4.1]. Given an object $B \in \mathcal{M}(A)$, there exists p with B lifting to $B \in \mathcal{M}_p(A)$. Therefore we can define

$$(X, \mathcal{O}_X(1)) := \text{Proj}(A \oplus B).$$

Replacing B with its image in $\mathcal{M}_q(A)$ (for $q > p$) does not affect $\text{Proj}(A \oplus B)$, so we have a functor $\text{Proj}(A \oplus -)$ from $\mathcal{M}(A)$ to polarised projective schemes over A .

For the quasi-inverse functor, take a polarised scheme $(X, \mathcal{O}_X(1))$ and some p for which $\Gamma(X, \mathcal{O}_X(n))$ is locally free of rank $h(n)$ for all $n \geq p$. Then define $B \in \mathcal{M}_p(A)$ by

$$B := \bigoplus_{n \geq p} \Gamma(X, \mathcal{O}_X(n)). \quad \square$$

Remark 3.31 Note that the hypothesis $\Gamma(X, \mathcal{O}_X(n))$ be locally free for $n \gg 0$ ensures that X is flat over A . If A is Noetherian, then the proof of Hartshorne [14, III.9.9] shows that the converse holds, and indeed that if A is connected, then there exists a Hilbert polynomial h with $\Gamma(X, \mathcal{O}_X(n))$ locally free of rank $h(n)$ for all $n \gg 0$.

Proposition 3.32 If $A \in \text{Alg}_{\mathbb{Q}}$ and $X = \text{Proj}(A \oplus C)$ for $C \in \mathcal{M}_p(A)$, then for any A -module M , there are canonical isomorphisms

$$D_{[C]}^i(B\tilde{\mathcal{M}}, M) \cong \text{Ext}_X^{i+1}(\mathbb{L}^{X/BG_m \otimes A}, \mathcal{O}_X \otimes_A M),$$

where \mathbb{L} is the cotangent complex of [15].

Proof Given $C \in \mathcal{M}_{p_0}(A)$, first let $\tilde{X} := \text{Spec}(A \oplus C) - \{0\}$, where $\{0\}$ denotes the copy of $\text{Spec } A$ defined by the ideal C . \tilde{X} inherits a \mathbb{G}_m -action from C (with trivial action on A), and in fact

$$\tilde{X} = \text{Spec}_X \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n) \right),$$

with $X = \tilde{X}/\mathbb{G}_m$ and $\tilde{X} = X \times_{B\mathbb{G}_m \otimes A}^h \text{Spec } A$. Writing $\pi: \tilde{X} \rightarrow X$ for the projection, base change gives

$$\pi^* \mathbb{L}^{X/B\mathbb{G}_m \otimes A} \simeq \mathbb{L}^{\tilde{X}/A}.$$

Since $j: \tilde{X} \rightarrow \text{Spec}(A \oplus C)$ is an open immersion, it is étale, so $\mathbb{L}^{\tilde{X}/A} \simeq j^* \mathbb{L}^{(A \oplus C)/A}$.

For any C -module N , the associated quasicoherent sheaf N^\sharp on X is given by $N^\sharp = (\pi_* j^* N)^{\mathbb{G}_m}$. Now, Lemma 3.28 implies that there exists a finitely generated \mathbb{Q} -subalgebra $A^0 \subset A$ and an object $C^0 \in \mathcal{M}_{p_0}(A^0)$ with $C = C^0 \otimes_{A^0} A$. Since both D^i and $\mathbb{E}xt$ are compatible with base change (the former by [26, Lemma 1.15]), it suffices to show that

$$D_{[C^0]}^i(B\tilde{\mathcal{M}}, M) \cong \mathbb{E}xt_{X_0}^{i+1}(\mathbb{L}^{X_0/B\mathbb{G}_m \otimes A^0}, \mathcal{O}_{X_0} \otimes_{A^0} M),$$

where $X_0 = \text{Proj}(A^0 \oplus C^0)$. Replacing A and C with A^0 and C^0 , we may therefore reduce to the case where A is a finitely generated \mathbb{Q} -algebra (hence Noetherian). Because both expressions above commute with filtered colimits of the modules M , we may assume that M is a finitely generated A -module.

Since $\mathcal{M} \rightarrow \pi^0 \mathcal{G}$ is formally étale by Lemma 3.27,

$$D_{[C]}^i(B\tilde{\mathcal{M}}, M) \cong D_{[C]}^i(B\mathcal{G}, M).$$

As $A \oplus \beta^* \beta(B)$ is a cofibrant resolution of $A \oplus B$, we have

$$D_{[C]}^i(B\mathcal{G}, M) \cong \lim_{\substack{\rightarrow \\ p \geq p_0}} \text{Ext}_{A \oplus B\{\geq p\}}^{i+1}(\mathbb{L}^{(A \oplus B\{\geq p\})/A}, B\{\geq p\} \otimes_A M)^{\mathbb{G}_m}.$$

Now, the proof of Serre’s Theorem [30, Section 59] still works over any Noetherian base, so shows that for a finitely generated C -module N and any $n \in \mathbb{Z}$,

$$\text{Ext}_X^i(\mathcal{O}_X(n), N^\sharp) \cong \lim_{\substack{\rightarrow \\ p}} \text{Ext}_C^i(C(n)\{\geq p\}, N\{\geq p\})^{\mathbb{G}_m}.$$

Indeed, a spectral sequence argument shows that the same is true if we replace $C(n)$ with any finitely generated C -module L , since L will then admit a resolution by finite

sums of $C(n)$'s. In fact, another spectral sequence argument allows us to take a chain complex L whose homology groups $H_i(L)$ are finite and bounded below, giving

$$\mathbb{E}xt_X^i(L^\#, N^\#) \cong \varinjlim_p \mathbb{E}xt_C^i(L_{\{\geq p\}}, N_{\{\geq p\}})^{\mathbb{G}_m}.$$

Thus for any fixed $p \geq p_0$,

$$\begin{aligned} \mathbb{E}xt_X^i(\mathbb{L}^{\tilde{X}/B\mathbb{G}_m \otimes A}, \mathbb{O}_X \otimes_A M) \\ \cong \varinjlim_{q \geq p} \mathbb{E}xt_{A \oplus B_{\{\geq p\}}}^i((\mathbb{L}^{(A \oplus B_{\{\geq p\}})/A})_{\{\geq q\}}, B_{\{\geq p\}} \otimes_A M)^{\mathbb{G}_m}. \end{aligned}$$

We can then take the colimit over the poset of pairs (p, q) with $q \geq p \geq p_0$. Since the set of pairs (p, p) is cofinal in this poset, we get

$$\begin{aligned} \mathbb{E}xt_X^i(\mathbb{L}^{\tilde{X}/B\mathbb{G}_m \otimes A}, \mathbb{O}_X \otimes_A M) \\ \cong \varinjlim_{p \geq p_0} \mathbb{E}xt_{A \oplus B_{\{\geq p\}}}^i(\mathbb{L}^{(A \oplus B_{\{\geq p\}})/A}, B_{\{\geq p\}} \otimes_A M)^{\mathbb{G}_m}, \end{aligned}$$

as required. □

Proposition 3.33 *The functor $B\tilde{\mathcal{M}}: dg_+\mathcal{N}_{\mathbb{Q}}^b \rightarrow \mathbb{S}$ satisfies the conditions of Theorem 1.31, and therefore the associated functor $\overline{W}\tilde{\mathcal{M}}: dg_+\mathcal{N}_{\mathbb{Q}}^b \rightarrow \mathbb{S}$ is representable by an almost finitely presented derived geometric 1–stack.*

Proof We apply Theorem 1.31 to $\tilde{\mathcal{M}}$. First, note that $\pi^0\tilde{\mathcal{M}} = \mathcal{M}$, which is a stack, locally of finite presentation by Lemma 3.28. Proposition 3.30 and Grothendieck’s formal existence theorem [11, 5.4.5] ensure that for any complete local Noetherian \mathbb{Q} –algebra Λ , the map

$$\mathcal{M}(\Lambda) \rightarrow \varprojlim_n \mathcal{M}(\Lambda/\mathfrak{m}^n)$$

is surjective on objects. That the map is an isomorphism is a consequence of [11, 5.1.4].

Now, homogeneity of $B\tilde{\mathcal{M}}$ is immediate, and Lemma 3.3 gives prehomotopicity. All the remaining conditions follow from Proposition 3.32, with the same reasoning as for Proposition 3.16. □

Proposition 3.34 *For $A \in dg_+\mathcal{N}_{\mathbb{Q}}^b$, the space $\overline{W}\tilde{\mathcal{M}}(A)$ is functorially weakly equivalent to the nerve $\overline{W}\mathfrak{M}(A)$ of the ∞ –groupoid $\mathfrak{M}(A)$ of derived geometric 0–stacks \mathfrak{X} over $B\mathbb{G}_m \times \text{Spec } A$ for which $X := \mathfrak{X} \otimes_A^{\mathbb{L}} H_0 A$ is weakly equivalent to a flat projective scheme over $H_0 A$, with the polarisation $X \rightarrow B\mathbb{G}_m \otimes H_0 A$ ample with Hilbert polynomial h .*

Proof We adapt the proof of Proposition 3.18. An object of $\mathcal{G}_p(A)$ corresponds to a locally free \mathbb{G}_m -equivariant A_0 -module $N\{\geq p\}$, with $N\{r\}$ locally free of rank $h(r)$, together with a closed degree -1 differential δ on the free chain Lie coalgebra $\text{CL}_{A_0}(N[-1]) \otimes_{A_0} A$. We may therefore form the DG-scheme

$$\mathfrak{X} := \text{Proj}(A \oplus \beta^*(\text{CL}_{A_0}(N[-1]) \otimes_{A_0} A, \delta)).$$

As in [27, Section 6.4], there is a canonical derived geometric 1-stack associated to \mathfrak{X} . To give an explicit map from this to $B\mathbb{G}_m$, we first let $\tilde{\mathfrak{X}} := \text{Spec}(A \oplus \beta^*(\text{CL}_{A_0}(N[-1]) \otimes_{A_0} A, \delta)) - \{0\}$, and then form the simplicial scheme

$$\tilde{\mathfrak{X}} \times^{\mathbb{G}_m} E\mathbb{G}_m,$$

which is a simplicial resolution of X , and has a canonical map to the simplicial scheme $B\mathbb{G}_m$. Here, $E\mathbb{G}_m$ is the universal \mathbb{G}_m -space over $B\mathbb{G}_m$, given by the simplicial 0-coskeleton $E\mathbb{G}_m = \text{cosk}_0 \mathbb{G}_m$, so $(E\mathbb{G}_m)_n = (\mathbb{G}_m)^{n+1}$. For an explicit Artin hypergroupoid representation of X , we could go further and replace $\tilde{\mathfrak{X}}$ with its Čech nerve associated to any open affine cover.

Now, if our object C lies in $\tilde{\mathcal{M}}_p \subset \mathcal{G}_p(A)$, then $C \otimes_A H_0 A$ lies in $\mathcal{M}_p(H_0 A)$, so $C \otimes_A H_0 A = \beta(B)$, for a finitely generated commutative algebra structure B on N . Since the map

$$\beta^* \beta(B) \rightarrow B$$

is a quasi-isomorphism, this means that

$$\mathfrak{X} \otimes_A^L H_0 A \simeq \text{Proj}(H_0 A \oplus B),$$

which is a polarised projective scheme with Hilbert polynomial h .

Since Proj is unchanged on replacing N with $N\{\geq q\}$ for $q > p$, we have defined a functor

$$\alpha_A: \tilde{\mathcal{M}}(A) \rightarrow \mathfrak{M}(A).$$

By [25, Example 3.39], the functor $\overline{W}\mathfrak{M}$ is also representable by a derived geometric 1-stack, so we just need to check that $\overline{W}\tilde{\mathcal{M}} \rightarrow \overline{W}\mathfrak{M}$ satisfies the conditions of Remark 1.30.

If $A \in \text{Alg}_{\mathbb{Q}}$, then Proposition 3.30 implies that α_A is an equivalence of groupoids. Combining Proposition 3.32 with [25, Corollary 3.32 and Example 3.39], we have isomorphisms

$$D_{[C]}^i(B\tilde{\mathcal{M}}, M) \cong \text{Ext}_X^{i+1}(\mathbb{L}^{X/\mathbb{G}_m \otimes A}, \mathbb{O}_X \otimes_A M) \cong D_{[C]}^i(\overline{W}\tilde{\mathfrak{M}}, M),$$

so Remark 1.30 applies. □

Remark 3.35 Replacing the DGLA L_p with the finite-dimensional DGLA $L_{[p,q]}$ in the definitions above gives us a functor $\widetilde{\mathcal{M}}_{[p,q]}$. Since $L_p = \lim_{\leftarrow q} L_{[p,q]}$, we will have $\widetilde{\mathcal{M}}_p = \lim_{\leftarrow q} \widetilde{\mathcal{M}}_{[p,q]}$, and hence

$$\widetilde{\mathcal{M}} = \lim_{\rightarrow p} \lim_{\leftarrow q} \widetilde{\mathcal{M}}_{[p,q]}.$$

It is natural to seek an open substack of \mathcal{M} on which these limits stabilise. If we define $\mathcal{M}^{(k)} \subset \mathcal{M}$ to be the open substack consisting of polarised schemes $(X, \mathcal{O}_X(1))$ for which $\mathcal{O}_X(k)$ is very ample, then we may regard X as a subscheme of $\mathbb{P}^{h(k)}$, and so [5, Theorem 1.2.3(b) and Theorem 1.4.1] imply that for $q \gg p \gg 0$, the maps

$$\mathcal{M}^{(k)} \leftarrow \mathcal{M}_p^{(k)} \rightarrow \mathcal{M}_{[p,q]}^{(k)}$$

are equivalences of underived stacks.

Moreover, for fixed i , [6, Theorem 4.1.1] implies that for $q \gg p \gg 0$, the maps

$$D_{[C]}^i(B\widetilde{\mathcal{M}}^{(k)}, M) \leftarrow D_{[C]}^i(B\widetilde{\mathcal{M}}_p^{(k)}, M) \rightarrow D_{[C]}^i(B\widetilde{\mathcal{M}}_{[p,q]}^{(k)}, M)$$

are isomorphisms for all $[C]$ and M . This does not give suitable p, q for all i simultaneously.

However, if we restrict further to the open substack $\mathcal{M}^{(k), \text{LCI}} \subset \mathcal{M}^{(k)}$ of local complete intersections, then the cotangent complex $\mathbb{L}^{X/\mathbb{G}_m \otimes A}$ will be concentrated in chain degrees $[0, 1]$. Thus $D_{[C]}^i(B\widetilde{\mathcal{M}}^{(k), \text{LCI}}, M) = 0$ for $i \notin [-1, \text{deg } h]$, so for $q \gg p \gg 0$, we have weak equivalences

$$\overline{W}\widetilde{\mathcal{M}}^{(k), \text{LCI}} \leftarrow \overline{W}\widetilde{\mathcal{M}}_p^{(k), \text{LCI}} \rightarrow \overline{W}\widetilde{\mathcal{M}}_{[p,q]}^{(k), \text{LCI}}$$

of derived stacks, by applying Remark 1.30.

4 Moduli from cosimplicial groups

Since suitable DG Lie algebras can usually only be constructed in characteristic 0, we now work with cosimplicial groups, which form the first step towards a more general construction.

4.1 Cosimplicial groups

Definition 4.1 Let $c\text{Gp}$ be the category of cosimplicial groups, and $cs\text{Gp}$ the category of cosimplicial simplicial groups.

4.1.1 Maurer–Cartan

Definition 4.2 Define $\text{MC}: c\text{Gp} \rightarrow \text{Set}$ by

$$\text{MC}(G) := Z^1(G) = \{\omega \in G^1 \mid \sigma^0\omega = 1, \partial^1\omega = \partial^2\omega \cdot \partial^0\omega\}.$$

Definition 4.3 Define $\underline{\text{MC}}: c\mathcal{S}\text{Gp} \rightarrow \mathcal{S}$ by setting $\underline{\text{MC}}(G) \subset \prod_{n \geq 0} (G^{n+1})^{\Delta^n}$ to consist of elements $(\omega_n)_{n \geq 0}$ satisfying

$$\begin{aligned} \partial_i \omega_n &= \begin{cases} \partial^{i+1} \omega_{n-1} & i > 0, \\ (\partial^1 \omega_{n-1}) \cdot (\partial^0 \omega_{n-1})^{-1} & i = 0, \end{cases} \\ \sigma_i \omega_n &= \sigma^{i+1} \omega_{n+1}, \\ \sigma^0 \omega_n &= 1. \end{aligned}$$

Define $\text{MC}: c\mathcal{S}\text{Gp} \rightarrow \text{Set}$ by $\text{MC}(G) = \underline{\text{MC}}(G)_0$, noting this agrees with Definition 4.2 when $G \in c\text{Gp}$.

Remark 4.4 Note that by the author’s proof of [21, Lemma 3.3],

$$\text{MC}(G) \cong \text{Hom}_{c\mathcal{S}}(\Delta, \overline{WG}),$$

for \overline{W} as in Definition 3.15, where the cosimplicial simplicial set Δ is given by the n -simplex Δ^n in cosimplicial level n . Thus $\text{MC}(G) = \text{Tot}_0 \overline{WG}$, for $\text{Tot}: c\mathcal{S} \rightarrow \mathcal{S}$ the total space functor of Goerss and Jardine [9, Chapter VIII], originally defined in Bousfield and Kan [2, Chapter X].

In fact, we have that \overline{W} has a left adjoint G (the loop group functor), and also that $\underline{\text{MC}}(G) \cong \underline{\text{Hom}}_{c\text{Gp}}(G(\Delta), \overline{WG})$. However, \overline{W} is not simplicial right Quillen, so this does not equal $\underline{\text{Hom}}_{c\mathcal{S}}(\Delta, \overline{WG}) = \text{Tot}(\overline{WG})$.

Definition 4.5 Given a cosimplicial group G , define the n^{th} matching object $M^n G$ to be the group

$$M^n G = \{(g_0, g_1, \dots, g_{n-1}) \in (G^{n-1})^n \mid \sigma^i g_j = \sigma^{j-1} g_i \ \forall i < j\}.$$

The Reedy matching map $G^n \rightarrow M^n G$ sends g to $(\sigma^0 g, \sigma^1 g, \dots, \sigma^{n-1} g)$.

There is then a Reedy model structure on $s\text{Gp}$ (analogous to [9, Section VII.4]) in which a morphism $f: G \rightarrow H$ is a (trivial) fibration whenever the canonical maps

$$G^n \rightarrow H^n \times_{M^n H} M^n G$$

are (trivial) fibrations in $s\text{Gp}$ for all $n \geq 0$.

Definition 4.6 Given an element $G \in cGp$, we define the cosimplicial normalisation by $N_c^n G := G^n \cap \bigcap_{i=0}^{n-1} \ker \sigma^i$. If G is abelian, then we make $N_c G$ into a cochain complex by setting

$$d_c := \sum_{i=0}^n (-1)^i \partial^i: N_c^{n-1} G \rightarrow N_c^n G.$$

Lemma 4.7 A morphism $f: G \rightarrow H$ in $scGp$ is a (trivial) fibration whenever the maps

$$f^n: G^n \rightarrow H^n$$

are all (trivial) fibrations in sGp .

Proof First note that $N_c^n G = \ker(G^n \rightarrow M^n G)$. Given $(g_0, g_1, \dots, g_{n-1}) \in M^n G$, we can functorially construct a preimage.

First, set $g(1) := \partial^1 g_0$; this has $\sigma^0 g(1) = g_0$. Proceeding by induction, assume that we have constructed $g(r) \in G^n$ with $\sigma^i g(r) = g_i$ for all $i < r$. Set $g_i(r) := \sigma^i g(r)^{-1} \cdot g_i$, so $(g_0(r), g_1(r), \dots, g_{n-1}(r)) \in M^n G$, with $g_i(r) = 1$ for all $i < r$. Now let $g(r+1) := g(r) \cdot \partial^{r+1} g_r(r)$, noting that this satisfies the inductive hypothesis.

Thus we have an isomorphism $G^n \cong N^n G \times M^n G$ as simplicial sets, and $f: G \rightarrow H$ is therefore a (trivial) fibration whenever $N^n f: N^n G \rightarrow N^n H$ is a (trivial) fibration in \mathbb{S} for all n . Since $N^n f$ is a retraction of f^n , the result follows. \square

Lemma 4.8 If $f: G \rightarrow H$ is a (trivial) fibration in $scGp$, then the map

$$\underline{MC}(f): \underline{MC}(G) \rightarrow \underline{MC}(H)$$

is a (trivial) fibration in \mathbb{S} . In particular, if $f: G \rightarrow H$ is a trivial fibration, then $\underline{MC}(f)$ is surjective.

Proof In the proof of [23, Proposition 6.7], a cofibrant object Φ is constructed in $scGp$, with the property that

$$\underline{MC}(G) \cong \underline{Hom}(\Phi, G),$$

where the simplicial sets \underline{Hom} come from a simplicial model structure. Since Φ is cofibrant, $\underline{Hom}(\Phi, -)$ is right Quillen, so has the properties claimed. \square

Definition 4.9 Define the total complex functor Tot^Π from chain cochain complexes (ie bicomplexes) to chain complexes by

$$(\text{Tot}^\Pi V)_n := \prod_{a-b=n} V_a^b,$$

with differential $d := d^s + (-1)^a d_c$ on V_a^b .

Lemma 4.10 *If $A \in csGp$ is abelian, then*

$$\begin{aligned} MC(A) &\cong Z_{-1}(\text{Tot}^\Pi N^s N_c A), \\ \pi_n \underline{MC}(A) &\cong H_{n-1}(\text{Tot}^\Pi \sigma^{\geq 1} N^s N_c A), \end{aligned}$$

where $\sigma^{\geq 1}$ denotes brutal truncation in cochain degrees greater than or equal to 1.

Proof This is a fairly straightforward application of the simplicial and cosimplicial Dold–Kan correspondences. Alternatively, we could appeal to Lemma 4.23, noting that $A = \exp(DN_c A)$. □

4.1.2 The gauge action

Definition 4.11 For $G \in scGp$, there is an action of the simplicial group G^0 on the simplicial set $\underline{MC}(G)$, called the *gauge action*, and given by writing

$$(g \star \omega)_n = ((\partial^1)^{n+1}(\sigma_0)^n g) \cdot \omega_n \cdot (\partial^0(\partial^1)^n(\sigma_0)^n g^{-1}),$$

as in [21, Definition 3.8], with $(\sigma_0)^n$ denoting the canonical map $G \rightarrow G^{\Delta^n}$.

Definition 4.12 Given an element $G \in scGp$, we then define the Deligne groupoid by $\mathcal{D}el(G) := [\underline{MC}(G)/G^0]$. In other words, $\mathcal{D}el(G)$ has objects $\underline{MC}(G)$, and morphisms from ω to ω' consist of $\{g \in G^0 \mid g \star \omega = \omega'\}$.

Define the derived Deligne groupoid to be the simplicial object in groupoids given by $\underline{\mathcal{D}el}(G) := [\underline{MC}(G)/G^0]$, so $\mathcal{D}el(G) = \underline{\mathcal{D}el}(G)_0$.

Define the simplicial sets $\underline{Del}(G), \overline{Del}(G) \in \mathbb{S}$ to be the nerves $B\mathcal{D}el(G), \overline{B}\underline{\mathcal{D}el}(G)$, respectively.

Lemma 4.13 *If $A \in csGp$ is abelian, then*

$$\pi_n \underline{Del}(A) \cong H_{n-1}(\text{Tot}^\Pi N^s N_c A),$$

whereas $\pi_1 \underline{Del}(A) \cong H^0(A_0)$, with

$$\pi_0 \underline{Del}(A) \cong Z_{-1}(\text{Tot}^\Pi N^s N_c A)/d_c(A_0^0).$$

Proof This is a straightforward consequence of Lemma 4.10. □

4.2 Moduli functors from cosimplicial groups

Proposition 4.14 *If $G: \text{Alg}_R \rightarrow c\text{Gp}$ is a homogeneous functor, with each G^n formally smooth, then the functor*

$$\underline{\text{MC}}(\underline{G}): d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$$

is homogeneous and formally quasismooth, so

$$\text{MC}(\underline{G}): d\mathcal{N}_R^\flat \rightarrow \text{Set}$$

is homogeneous and prehomotopic.

Proof Homogeneity is automatic, as $\underline{\text{MC}}$ preserves arbitrary limits.

We can extend G^n to a functor $G^n: d\mathcal{N}_R^\flat \rightarrow \text{Gp}$, given by $G^n(A) := G^n(A_0)$. Then formal smoothness of G^n implies that the extended G^n is prehomotopic. It is automatically formally quasipresmooth, as all discrete morphisms are fibrations. Thus Proposition 1.27 implies that $\underline{G}^n: d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$ is formally smooth, and hence formally quasismooth.

Lemma 4.7 therefore implies that $\underline{G}(A) \rightarrow \underline{G}(B)$ is a (trivial) fibration in $sc\text{Gp}$ for all (acyclic) square zero extensions $A \rightarrow B$, and Lemma 4.8 then implies that $\underline{\text{MC}}(\underline{G})(A) \rightarrow \underline{\text{MC}}(\underline{G})(B)$ is a (trivial) fibration, as required. \square

Proposition 4.15 *If $G: \text{Alg}_R \rightarrow c\text{Gp}$ is a homogeneous functor, with each G^n formally smooth, then the functor*

$$\underline{\text{Del}}(\underline{G}): d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$$

is homogeneous and formally quasismooth, while

$$\text{Del}(\underline{G}): s\mathcal{N}_R^\flat \rightarrow \mathbb{S}$$

is homogeneous, prehomotopic and formally quasipresmooth.

Proof Homogeneity is immediate, combining Proposition 4.14 with the fact that \underline{G}^0 is homogeneous. Now, take a square zero (acyclic) extension $f: A \rightarrow B$ in $d\mathcal{N}_R^\flat$. Since $\underline{\text{MC}}(\underline{G})$ is formally quasismooth, the map $\underline{\text{MC}}(\underline{G}(A)) \rightarrow \underline{\text{MC}}(\underline{G}(B))$ is a (trivial) fibration.

Combining [9, Lemma IV.4.8] with [4], this means that

$$\overline{W}[\underline{\text{MC}}(\underline{G}(A))/\underline{G}^0(A)] \rightarrow \overline{W}[\underline{\text{MC}}(\underline{G}(B))/\underline{G}^0(A)]$$

is a (trivial) fibration in \mathbb{S} . Now, the map

$$\overline{W}[\underline{\text{MC}}(\underline{G}(B))/\underline{G}^0(A)] \rightarrow \overline{W}[\underline{\text{MC}}(\underline{G}(B))/\underline{G}^0(B)]$$

is a pullback of $\overline{W}\underline{G}^0(A) \rightarrow \overline{W}\underline{G}^0(B)$, which is a (trivial) fibration as \underline{G}^0 is formally smooth. Composing the two morphisms above, we see that

$$\underline{\text{Del}}(\underline{G})(A) \rightarrow \underline{\text{Del}}(\underline{G})(B)$$

is a (trivial) fibration, so $\underline{\text{Del}}(\underline{G})$ is formally quasismooth.

Meanwhile, prehomotopicity of $\underline{\text{Del}}(\underline{G})$ follows immediately from prehomotopicity of $\underline{\text{MC}}(\underline{G})$, while formal quasipresmoothness of $\underline{\text{Del}}(\underline{G})$ is an immediate consequence of formal smoothness of G^0 , by [9, Chapter V]. \square

4.2.1 Cohomology

Definition 4.16 Given a ring $A \in \text{Alg}_R$, an A -module M , a homogeneous, levelwise formally smooth functor $G: \text{Alg}_R \rightarrow c\text{Gp}$ and $\omega \in \text{MC}(G(A))$, define the cosimplicial module $C_\omega^\bullet(G, M)$ to be the tangent space

$$C_\omega^n(G, M) := T_1(G^n, M)$$

with operations on $a \in C_\omega^n(G, M)$ given by

$$\begin{aligned} \sigma^i a &= \sigma_G^i a, \\ \partial^i a &= \begin{cases} ((\partial_G^2)^n \omega)(\partial_G^0 a)((\partial_G^2)^n \omega^{-1}) & i = 0, \\ \partial_G^i a & i \geq 1. \end{cases} \end{aligned}$$

Definition 4.17 For G, A, M, ω as above, define

$$H_\omega^i(G, M) := H^i C_\omega^\bullet(G, M).$$

Lemma 4.18 Given $A \in \text{Alg}_R$, $M \in d\text{Mod}_A$, a homogeneous, levelwise formally smooth functor $G: \text{Alg}_R \rightarrow sc\text{Gp}$ and $\omega \in \text{MC}(G_0(A))$, then the fibre of the map $\text{MC}(G(A \oplus M)) \rightarrow \text{MC}(G(A))$ over ω is canonically isomorphic to $\text{MC}(C_\omega^\bullet(G, M))$.

Proof Given $\alpha \in \text{MC}(C_\omega^\bullet(G, M))$, the associated element $\beta \in \text{MC}(G(A \oplus M))$ is given by

$$\beta_n := \alpha_n (\partial_G^2)^n \omega \in T_{(\partial_G^2)^n \omega}(G_n^{n+1}, M). \quad \square$$

Lemma 4.19 *If $G: \text{Alg}_R \rightarrow c\text{Gp}$ is a homogeneous, levelwise formally smooth functor, with $A \in \text{Alg}_R$ and M an A -module, then*

$$D_\omega^i(\underline{\text{MC}}(\underline{G}), M) \cong D_\omega^i(\text{MC}(\underline{G}), M) \cong \begin{cases} H_\omega^{i+1}(G, M) & i > 0, \\ Z^1 C_\omega^\bullet(G, M) & i = 0. \end{cases}$$

Proof By Lemma 4.10, for any $L \in d\text{Mod}_A$,

$$D_\omega^i(\underline{\text{MC}}(\underline{G}), L) \cong H_{i-1}(\text{Tot}^\Pi \sigma^{\geq 1} N^s N_c C_\omega^\bullet(\underline{G}, L)).$$

Thus we have a spectral sequence

$$E_2^{i,-j} = H^i(\sigma^{\geq 1} \pi_j C_\omega^\bullet(\underline{G}, L)) \implies D_\omega^{i-j-1}(\underline{\text{MC}}(\underline{G}), L);$$

in the terminology of [33, Page 142], this is a second quadrant spectral sequence, so is weakly convergent.

The simplicial abelian group $C_\omega^n(\underline{G}, L)$ is given in level i by $C_\omega^n(\underline{G}, (L^{\Delta^i})_0)$. Moreover, $C^\omega(G, -)$ is an exact functor; left exactness follows from homogeneity, and right exactness from formal smoothness. Thus $\pi_j C_\omega^n(\underline{G}, L) \cong C_\omega^n(\underline{G}, \pi_j(L^{\Delta^\bullet})_0)$.

Now, when $d\mathcal{N}_R^b = s\mathcal{N}_R^b$, we have that $(L^{\Delta^n})_0 = L_n$, and so $(L^{\Delta^\bullet})_0 = L$. When $d\mathcal{N}_R^b = dg_+ \mathcal{N}_R^b$, then $N^s(L^{\Delta^\bullet})_0$ is weakly equivalent to L . In either case, we have $\pi_j(L^{\Delta^\bullet})_0 \cong H_j L$, so our spectral sequence is

$$H^i(\sigma^{\geq 1} C_\omega^\bullet(G, H_j L)) \implies D_\omega^{i-j-1}(\underline{\text{MC}}(\underline{G}), L).$$

Taking $L = M[-n]$, the spectral sequence degenerates, giving

$$H^i(\sigma^{\geq 1} C_\omega^\bullet(G, M)) \cong D_\omega^{i-n-1}(\underline{\text{MC}}(\underline{G}), M[-n]) = D_\omega^{i-1}(\underline{\text{MC}}(\underline{G}), M),$$

completing the proof for $\underline{\text{MC}}$.

Now, $T_\omega(\underline{\text{MC}}, -): d\text{Mod}_A \rightarrow \mathbb{S}$ preserves fibrations and trivial fibrations. Since $M[-j] \oplus \text{cone}(M)[1-j]$ is a path object for $M[-j]$ when $j \geq 1$ (recalling that M is a discrete A -module), this means that $T_\omega(\underline{\text{MC}}, M[-j] \oplus \text{cone}(M)[1-j])$ must be a path object for $T_\omega(\underline{\text{MC}}, M[-j])$. Therefore for $j \geq 1$,

$$\begin{aligned} \pi_0 T_\omega(\underline{\text{MC}}, M[-j]) &= T_\omega(\text{MC}, M[-j]) / T_\omega(\text{MC}, \text{cone}(M)[1-j]), \\ D_\omega^j(\underline{\text{MC}}, M) &= D_\omega^j(\text{MC}, M). \end{aligned}$$

The proof for $j = 0$ is even simpler, since we have that $M^{\Delta^n} = M$, and so we conclude that $T_\omega(\underline{\text{MC}}, M) = T_\omega(\text{MC}, M)$. □

Lemma 4.20 *If $G: \text{Alg}_R \rightarrow c\text{Gp}$ is a homogeneous, levelwise formally smooth functor, with $A \in \text{Alg}_R$ and M an A -module, then*

$$D_\omega^i(\underline{\text{Del}}(G)) \cong D_\omega^i(\text{Del}(G)) \cong H_\omega^{i+1}(G, M).$$

Proof The description of $D_\omega^i(\underline{\text{Del}}(G))$ follows with the same reasoning as Lemma 4.19, substituting Lemma 4.13 for Lemma 4.10.

Now, there is a morphism

$$\begin{array}{ccccc} \text{MC}(G) & \longrightarrow & \text{Del}(G) & \longrightarrow & BG^0 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\text{MC}}(G) & \longrightarrow & \underline{\text{Del}}(G) & \longrightarrow & \overline{WG}^0 \end{array}$$

of fibration sequences, with the outer maps inducing isomorphisms on D^i , and so Proposition 1.12 gives the required isomorphisms $D_\omega^i(\text{Del}(G)) \cong D_\omega^i(\underline{\text{Del}}(G))$. \square

4.3 Denormalisation

Definition 4.21 Given a DGLA L in nonnegative degrees, let DL be its cosimplicial denormalisation. Explicitly,

$$D^n L := \bigoplus_{\substack{m+s=n \\ 1 \leq j_1 < \dots < j_s \leq n}} \partial^{j_s} \dots \partial^{j_1} L^m,$$

for formal symbols ∂^j . We then define operations ∂^j and σ^i using the cosimplicial identities, subject to the conditions that $\sigma^i L = 0$ and $\partial^0 v = dv - \sum_{i=1}^{n+1} (-1)^i \partial^i v$ for all $v \in L^n$.

We now have to define the Lie bracket $\llbracket -, - \rrbracket$ from $D^n L \otimes D^n L$ to $D^n L$. Given a finite set I of distinct strictly positive integers, write $\partial^I = \partial^{i_s} \dots \partial^{i_1}$, for $I = \{i_1, \dots, i_s\}$, with $i_1 < \dots < i_s$. The Lie bracket is then defined on the basis by

$$\llbracket \partial^I v, \partial^J w \rrbracket := \begin{cases} \partial^{I \cap J} (-1)^{(J \setminus I, I \setminus J)} [v, w] & v \in L^{|J \setminus I|}, w \in L^{|I \setminus J|}, \\ 0 & \text{otherwise,} \end{cases}$$

where for disjoint sets S, T of integers, $(-1)^{(S, T)}$ is the sign of the shuffle permutation of $S \sqcup T$ which sends the first $|S|$ elements to S (in order), and the remaining $|T|$ elements to T (in order). Beware that this formula cannot be used to calculate $\llbracket \partial^I v, \partial^J w \rrbracket$ when $0 \in I \cup J$ (for the obvious generalisation of ∂^I to finite sets I of distinct nonnegative integers).

Of course, the denormalisation functor above extends a denormalisation functor D from nonnegatively graded cochain complexes to cosimplicial complexes. The latter D is quasi-inverse to the normalisation functor N_c of Definition 4.6.

Definition 4.22 Given a pro-nilpotent Lie algebra \mathfrak{g} , define $\widehat{U}(\mathfrak{g})$ to be the pro-unipotent completion of the universal enveloping algebra of \mathfrak{g} , regarded as a pro-object in the category of algebras. As in [28, Appendix A], this is a pro-Hopf algebra, and we define $\exp(\mathfrak{g}) \subset \widehat{U}(\mathfrak{g})$ to consist of elements g with $\varepsilon(g) = 1$ and $\Delta(g) = g \otimes g$, for $\varepsilon: \widehat{U}(\mathfrak{g}) \rightarrow k$ the augmentation (sending \mathfrak{g} to 0), and $\Delta: \widehat{U}(\mathfrak{g}) \rightarrow \widehat{U}(\mathfrak{g}) \otimes \widehat{U}(\mathfrak{g})$ the comultiplication.

Since k is assumed to have characteristic 0, exponentiation gives an isomorphism from \mathfrak{g} to $\exp(\mathfrak{g})$, so we may regard $\exp(\mathfrak{g})$ as having the same elements as \mathfrak{g} , but with multiplication given by the Campbell–Baker–Hausdorff formula.

If L is a DGLA in strictly positive degrees, observe that we can write L as the inverse limit $L = \varprojlim_n \sigma^{\leq n} L$ of nilpotent DGLAs, where $\sigma^{\leq n}$ denotes brutal truncation. We may thus regard DL as the pro-nilpotent cosimplicial Lie algebra $\varprojlim_n D(\sigma^{\leq n} L)$, so we can exponentiate to obtain $\exp(DL) := \varprojlim_n \exp(D(\sigma^{\leq n} L))$.

Lemma 4.23 *Given a simplicial DGLA L_\bullet in strictly positive cochain degrees, there is a canonical isomorphism*

$$\text{MC}(\exp(DL)) \cong \text{MC}(\text{Tot} \Pi N^s L),$$

Here, N^s is simplicial normalisation (as in Definition 1.1).

Proof This is [23, Theorem 6.23]. □

Definition 4.24 Given a DGLA L with gauge G_L , define the cosimplicial group $D(\exp(L), G_L)$ as follows:

$$D^n(\exp(L), G_L) := \exp(D^n L^{>0}) \rtimes G_L,$$

(with G_L acting on $\exp(D^n L^{>0})$ via the adjoint action ad), with operations

$$\begin{aligned} \sigma^i(a, g) &= (\sigma^i a, g), \\ \partial^i(a, g) &= \begin{cases} (\partial^i a, g) & i > 0, \\ (\partial^0 a \cdot \exp((\partial^2)^n Dg), g) & i = 0, \end{cases} \end{aligned}$$

for $(a, g) \in D^n(\exp(L), G_L)$, and $Dg \in L^1$.

Remark 4.25 If L is a nilpotent DGLA in nonnegative degrees and $G_L = \exp(L^0)$ as in Examples 3.5, then observe that $D(\exp(L), G_L) \cong \exp(DL)$, with the isomorphism given in level n by

$$(a, g) \mapsto a \cdot (\partial^1)^n g.$$

The only difficult part of the comparison is checking that the isomorphism preserves ∂^0 . This follows because for $v \in L^0$, we have $\partial^0 v = \partial^1 v + dv$. Since $\llbracket dv, \partial^1 v \rrbracket = 0$, this gives

$$\begin{aligned} \exp(\partial^0 v) &= \exp(\partial^1 v) + d \exp(v) \\ &= (1 + (d \exp(v)) \exp(-v)) \cdot \exp(\partial^1 v) \\ &= (1 + D \exp(v)) \cdot \exp(\partial^1 v) \\ &= \exp(D \exp(v)) \cdot \exp(\partial^1 v). \end{aligned}$$

Lemma 4.26 Given a simplicial DGLA L with gauge G_L for L_0 , the isomorphism $\text{MC}(D(\exp(L), G_L)) \cong \text{MC}(\text{Tot} \Pi N^s L)$ of Lemma 4.23 is G_L -equivariant for the respective gauge actions.

Proof This is a consequence of the proof of [21, Theorem 4.44], which deals with the case when $G_L = \exp(L^0)$ is defined as in Examples 3.5. □

Corollary 4.27 Given a DGLA L over R , with gauges $G_L(A)$ for $L \otimes_R A$, functorial in $A \in \text{Alg}_R$, there are canonical isomorphisms

$$\begin{aligned} \text{MC}(L \otimes_R -) &\cong \text{MC}(D(\exp(L \otimes_R -), G_L(-))), \\ [\text{MC}(L \otimes_R -)/G_L(-)] &\cong \mathcal{D}\text{el}(D(\exp(L \otimes_R -), G_L(-))) \end{aligned}$$

of functors on $dg_+ \text{Alg}_R$.

Thus cosimplicial groups generalise DGLAs, with the added advantage that they can also give functors on $s\text{Alg}_R$, and hence work in all characteristics.

4.4 Sheafification

Another advantage of cosimplicial groups over DGLAs is that they have a good notion of sheafification.

Definition 4.28 Given some class \mathbf{P} of covering morphisms in Alg_R and a levelwise formally smooth, homogeneous functor $G: \text{Alg}_R \rightarrow c\text{Gp}$ preserving finite products, define the sheafification G^\sharp of G with respect to \mathbf{P} by first defining a cosimplicial commutative A -algebra $(B/A)^\bullet$ for every \mathbf{P} -covering $A \rightarrow B$, as $(B/A)^n := B \otimes_A \cdots \otimes_A B$

($n + 1$ times), then setting

$$G^\sharp(A) = \text{diag} \lim_{\substack{\longrightarrow \\ B}} G((B/A)^\bullet),$$

where diag is the diagonal functor $(\text{diag} X)^n = X^{nn}$ from bicosimplicial groups to cosimplicial groups.

Definition 4.29 Given a groupoid valued functor $\Gamma: d\mathcal{N}_R^\flat \rightarrow \text{Gpd}$, and a class \mathbf{P} of covering morphisms in Alg_R , define $\Gamma^\sharp: d\mathcal{N}_R^\flat \rightarrow \text{Gpd}$ to be the stackification of the groupoid presheaf Γ in the strict \mathbf{P} -topology of Definition 2.17.

Lemma 4.30 For G and \mathbf{P} as above, there is a canonical morphism

$$\mathfrak{D}\text{el}(\underline{G})^\sharp \rightarrow \mathfrak{D}\text{el}(\underline{G}^\sharp)$$

of groupoid valued functors on $d\mathcal{N}_R^\flat$, inducing an equivalence

$$\pi^0 \mathfrak{D}\text{el}(\underline{G})^\sharp \rightarrow \pi^0 \mathfrak{D}\text{el}(\underline{G}^\sharp).$$

Proof An object of $\mathfrak{D}\text{el}(\underline{G})^\sharp$ is a pair $(\omega, g) \in \text{MC}(\underline{G}(B \otimes_{A_0} A)) \times G^0(B \otimes_{A_0} B)$, for $A_0 \rightarrow B$ a \mathbf{P} -covering, satisfying the conditions

- (1) $g \star (\text{pr}_1^* \omega) = (\text{pr}_0^* \omega) \in \text{MC}(\underline{G}(A \otimes_{A_0} B \otimes_{A_0} B))$
- (2) $\text{pr}_{02}^* g = (\text{pr}_{01}^* g) \cdot (\text{pr}_{12}^* g) \in G^0(B \otimes_{A_0} B \otimes_{A_0} B)$

and we now describe the image of (ω, g) .

First, form the cosimplicial A_0 -algebra $(B/A_0)^\bullet$ as in Definition 4.28. We now map (B, ω, g) to the object ω' of $\text{MC}(\text{diag} \underline{G}(A \otimes_{A_0} (B/A_0)^\bullet))$ given by

$$\omega'_n := (\text{pr}_{01}^* (\partial^1_G)^{n+1} g) \cdot \text{pr}_1^* \omega_n \in G^{n+1}((A^{\Delta^n})_0 \otimes_{A_0} (B/A_0)^{n+1}).$$

An isomorphism from (B, ω, g) to (C, ν, h) is a \mathbf{P} -covering $B \otimes_{A_0} C \rightarrow D$ with $A \rightarrow D$ a \mathbf{P} -covering, together with an element $\alpha \in G^0(D)$ such that we have $\alpha \star \omega = \nu \in \text{MC}(\underline{G}(A \otimes_{A_0} D))$, with $(\text{pr}_0^* \alpha) \cdot g = h \cdot (\text{pr}_1^* \alpha) \in G^0(D \otimes_{A_0} D)$. We just map this to $\alpha \in G^0(D)$.

To see that this induces an equivalence $\mathfrak{D}\text{el}(\underline{G})^\sharp(A) \rightarrow \mathfrak{D}\text{el}(\underline{G}^\sharp(A))$ for $A \in \text{Alg}_R$, we appeal to [23, Lemma 1.21], which shows that objects of

$$\text{MC}(\text{diag} G((B/A)^\bullet))$$

correspond to pairs $(\omega, g) \in \text{MC}(G(B)) \times \text{MC}(G^0((B/A)^\bullet))$ which satisfy the condition $\partial_B^1 \omega \cdot \partial_G^0 g = \partial^1 g \cdot \partial_B^0 \omega$. Since $\partial_B^1 \omega = \text{pr}_0^* \omega$ and $\partial_B^0 \omega = \text{pr}_1^* \omega$, this amounts to

saying that g and ω satisfy condition (1) above, while condition (2) is equivalent to saying that $g \in \text{MC}(G^0((B/A)^\bullet))$. Morphisms in $\text{MC}(\text{diag}G((B/A)^\bullet))$ are given by the gauge actions of $G^0(B^0)$, so the equivalence of groupoids follows. \square

Now recall that \mathbf{P} -covering morphisms are all assumed faithfully flat.

Lemma 4.31 *Take $G: \text{Alg}_R \rightarrow c\text{Gp}$ satisfying the conditions of Definition 4.28, a ring $A \in \text{Alg}_R$, an A -module M , and an object of $\mathcal{D}\text{el}(G)^\sharp(A)$ represented by (ω, g) for $\omega \in \text{MC}(G(B))$. If the maps*

$$H_\omega^*(G, M \otimes_A B) \otimes_B B' \rightarrow H_\omega^*(G, M \otimes_A B')$$

are isomorphisms for all \mathbf{P} -coverings $B \rightarrow B'$, then the morphism α of Lemma 4.30 induces an isomorphism

$$D_{(\omega, g)}^*(B\mathcal{D}\text{el}(G)^\sharp, M) \rightarrow D_{\alpha(\omega, g)}^*(B\mathcal{D}\text{el}(G)^\sharp, M).$$

Proof We begin by calculating the cohomology groups $D_{(\omega, g)}^*(B\mathcal{D}\text{el}(G)^\sharp, M)$. It follows from Lemma 4.20 that these are given by first taking a cosimplicial A -module $K(B')$ given by the equaliser of

$$C_\omega^\bullet(G, M \otimes_A B') \begin{matrix} \xrightarrow{\text{pr}_1^*} \\ \xrightarrow[\text{ad}_g \text{pr}_0^*]{} \end{matrix} C_{\text{pr}_1^* \omega}^\bullet(G, M \otimes_A B' \otimes_A B'),$$

then getting

$$D_{(\omega, g)}^i(B\mathcal{D}\text{el}(G)^\sharp, M) \cong \varinjlim_{B'} H^{i+1} K(B'),$$

where B' ranges over all \mathbf{P} -hypercoverings $B \rightarrow B'$.

Now, the requirement that H_ω^* commute with base change ensures that ad_g gives an effective descent datum on cohomology, giving an isomorphism

$$H^* K(B') \otimes_A B' \cong H_\omega^*(G, M \otimes_A B').$$

Thus taking the colimit over B' does not affect the calculation, so

$$D_{(\omega, g)}^i(B\mathcal{D}\text{el}(G)^\sharp, M) \cong H^{i+1} K(B).$$

Meanwhile, g allows us to extend the fork above to form a bicosimplicial complex $\check{C}^\bullet(B'/A, C_\omega^\bullet(G, M))$, with

$$\check{C}^i(B'/A, C_\omega^\bullet(G, M)) = C_{\text{pr}_i^* \omega}^\bullet(G, M \otimes_A (B'/A)^i)$$

and horizontal cohomology $\check{H}^0(B'/A, C_\omega^\bullet(G, M)) = K(B')$. Lemma 4.20 then shows that

$$D_{\alpha(\omega, g)}^i(B\mathcal{D}el(\underline{G}^\sharp), M) \cong \varinjlim_{B'} H^{i+1}(\text{diag}\check{C}^\bullet(B'/A, C_\omega^\bullet(G, M))),$$

and the Eilenberg–Zilber Theorem allows us to replace diag with the total complex functor Tot .

Now, there are canonical maps $K(B) \otimes_A (B'/A)^n \rightarrow \check{C}^n(B'/A, C_\omega^\bullet(G, M))$, and we know that these give isomorphisms on cohomology, so

$$D_{\alpha(\omega, g)}^i(B\mathcal{D}el(\underline{G}^\sharp), M) \cong \varinjlim_{B'} H^{i+1}(\text{Tot}K(B) \otimes_A (B'/A)^\bullet).$$

Since $H^j((B'/A)^\bullet) = 0$ for all $j > 0$, this becomes

$$D_{\alpha(\omega, g)}^i(B\mathcal{D}el(\underline{G}^\sharp), M) \cong \varinjlim_{B'} H^{i+1}K(B),$$

as required. □

Remark 4.32 In particular, this means the groupoid valued functor \mathcal{G} of Definition 3.29 can be replaced by a functor coming straight from a cosimplicial group valued functor. Explicitly, set $G_{[p, q]} := D(\exp(L_{[p, q]}), G_{L_{[p, q]}})$; then $G := \varinjlim_p \varprojlim_q G_{[p, q]}$ satisfies the conditions of Definition 4.28, so Lemma 4.30 gives a map

$$\mathcal{G} \rightarrow \mathcal{D}el(G^\sharp),$$

inducing an equivalence on π^0 , and isomorphisms on D^i for all points of $\mathcal{M} \subset \pi^0\mathcal{G}$.

Remark 4.33 In Definition 4.28, instead of just taking hypercovers $A_0 \rightarrow B^\bullet$ coming from \mathbf{P} -covering morphisms $A_0 \rightarrow B$, we could have taken the filtered colimit over the category of all simplicial \mathbf{P} -hypercovers $\text{Spec } \tilde{A}^\bullet \rightarrow \text{Spec } A$. This corresponds to a kind of hypersheafification, rather than just sheafification, and all the results above still carry over, by faithfully flat descent.

4.5 Derived moduli of G -torsors

We now show how cosimplicial groups can govern derived moduli of torsors. Fix a smooth algebraic group space G over R , and a Deligne–Mumford stack X over R .

Definition 4.34 Define $C_{\text{ét}}^\bullet(X, G): \text{Alg}_R \rightarrow c\text{Gp}$ as follows. Given $A \in \text{Alg}_R$, let $\text{hyp}_{\text{ét}}(X, A)$ be the homotopy inverse category whose objects are simplicial étale hypercovers Y_\bullet of $X \times \text{Spec } A$. Then set

$$C_{\text{ét}}^n(X, G)(A) := \varinjlim_{Y_\bullet \in \text{hyp}_{\text{ét}}(X, A)} \text{Hom}(Y_n, G).$$

Remark 4.35 Since G is finitely presented, we can work just as well with pro-étale hypercovers. This has the advantage that Lemma 2.13 can be applied to provide a weakly initial object among simplicial hypercovers, giving a smaller, nonfunctorial, model for $C_{\text{ét}}^\bullet(X, G)(A)$.

Lemma 4.36 For $A \in \text{Alg}_R$, the groupoid $\mathcal{D}\text{el}(C_{\text{ét}}^n(X, G)(A))$ is equivalent to the groupoid of G -torsors on $X \times \text{Spec } A$.

Proof An object of $\text{MC}(C_{\text{ét}}^\bullet(X, G)(A))$ is a descent datum $\omega \in \text{Hom}(Y_1, G)$, with the Maurer–Cartan relations giving the gluing conditions. Thus ω gives rise to an étale G -torsor \mathbb{B}_ω on $X \times \text{Spec } A$, and it is straightforward to check that the gauge action of $\text{Hom}(Y_0, G)$ corresponds to isomorphisms of torsors. Every torsor is trivialised by some étale cover, so this functor is an equivalence. \square

Lemma 4.37 Given $A \in \text{Alg}_R$, an A -module M , and $\omega \in \text{MC}(C_{\text{ét}}^\bullet(X, G)(A))$ corresponding to a G -torsor \mathbb{B}_ω on $X \times \text{Spec } A$, there are canonical isomorphisms

$$H_\omega^i(C_{\text{ét}}^\bullet(X, G), M) \cong H_{\text{ét}}^i(X \times \text{Spec } A, M \otimes_A \text{ad}\mathbb{B}_\omega),$$

where $\text{ad}\mathbb{B}_\omega$ is the adjoint bundle

$$(\mathfrak{g} \otimes_R \mathbb{O}_X \otimes_R A) \times_{G(\mathbb{O}_X \otimes_R A)} \mathbb{B}_\omega,$$

for \mathfrak{g} the Lie algebra of G , equipped with its adjoint G -action.

Proof First, observe that

$$G(\mathbb{O}_X \otimes_R (A \oplus M)) \cong G(\mathbb{O}_X \otimes_R A) \ltimes (\mathfrak{g} \otimes_R \mathbb{O}_X \otimes_R M),$$

which gives functorial isomorphisms

$$C_\omega^i(C_{\text{ét}}^\bullet(X, G), M) \cong \varinjlim_{Y_\bullet} \Gamma(Y_i, M \otimes_A \text{ad}\mathbb{B}_\omega).$$

Since étale hypercovers compute cohomology, this gives the required isomorphism. \square

Proposition 4.38 The functor $\underline{\text{Del}}(C_{\text{ét}}^\bullet(X, G))$ is canonically weakly equivalent to the derived stack of étale derived G -torsors on X from [25, Example 3.38].

Proof Combining Lemma 4.20 and Proposition 4.15 with Remark 1.30, it suffices to construct a functorial natural transformation from $\mathcal{D}\text{el}(C_{\text{ét}}^\bullet(X, G))(A)$ to the ∞ -groupoid $\text{Tors}(X, G)(A)$ of G -torsors on $X \times \text{Spec } A$, and to show that this is an equivalence for $A \in \text{Alg}_R$, inducing isomorphisms on cohomology groups D^i .

Our first key observation is that for $A \in d\mathcal{N}_R^{\flat}$, the ring $(A^{\Delta^n})_0$ is a nilpotent extension of H_0A , so its étale site is isomorphic to that of A_0 . In particular, every simplicial étale hypercover of $X \times \text{Spec}(A^{\Delta^n})_0$ is of the form $Y_{\bullet} \otimes_{A_0} (A^{\Delta^n})_0$, for Y_{\bullet} a simplicial étale hypercover of $X \times \text{Spec} A_0$.

Therefore an element $\omega \in \text{MC}(\mathbf{C}_{\text{ét}}^{\bullet}(X, G)(A))$ lies in MC applied to the simplicial cosimplicial group $\Gamma(Y_{\bullet} \otimes_{A_0} (A^{\Delta^{\bullet}}), G)$ given by $(i, j) \mapsto \text{Hom}(Y_i \otimes_{A_0} (A^{\Delta^j})_0, G)$, for some simplicial étale hypercover Y_{\bullet} of $X \times \text{Spec} A_0$. Now,

$$\begin{aligned} \text{MC}(\Gamma(Y_{\bullet} \otimes_{A_0} (A^{\Delta^{\bullet}}), G)) &= \text{Hom}_{cS}(\Delta, \overline{W}\Gamma(Y_{\bullet} \otimes_{A_0} (A^{\Delta^{\bullet}}), G)) \\ &\cong \text{Hom}_{s\text{Pr}(d\mathcal{N}_R^{\flat})}(Y_{\bullet} \times_{\text{Spec} A_0} \text{Spec} A, \overline{W}\underline{G}), \end{aligned}$$

where $s\text{Pr}(d\mathcal{N}_R^{\flat})$ denotes the category of functors $d\mathcal{N}_R^{\flat} \rightarrow \mathbb{S}$.

For a simplicial group Γ , and \overline{W} as in Definition 3.15, there is a universal principal Γ -space $W\Gamma$ over $\overline{W}\Gamma$, as in [9, Section V.4], given by $W\Gamma = \overline{W}[\Gamma/\Gamma]$, (whereas $\overline{W}\Gamma = \overline{W}[\bullet/\Gamma]$, regarding a group as a groupoid on one object). Thus $W\Gamma$ has a group structure inherited from Γ , and $\Gamma = \overline{W}[\Gamma/\{1\}]$ is a subgroup; the Γ -action on $W\Gamma$ is then given by left multiplication, with $\overline{W}\Gamma = \Gamma \backslash W\Gamma$.

Thus we may associate the \underline{G} -space $P_{\omega} := (Y_{\bullet} \times_{\text{Spec} A_0} \text{Spec} A) \times_{\overline{W}\underline{G}} \underline{W}\underline{G}$ to ω . Since \underline{G} is the derived stack $\mathbf{R}G$ associated to G , the derived stack $\mathbf{R}P_{\omega}$ associated to P_{ω} is a derived $\mathbf{R}G$ -torsor on $\mathbf{R}(Y_{\bullet} \times_{\text{Spec} A_0} \text{Spec} A) \simeq \mathbf{R}(X \times_{\text{Spec} A_0} \text{Spec} A)$, so we have defined our functor on objects.

Now, the constant group Γ_0 is a simplicial subgroup of Γ , giving a simplicial group homomorphism $W\Gamma_0 \rightarrow W\Gamma$. Moreover, $W\Gamma_0$ is the 0-coskeleton $\text{cosk}_0\Gamma_0$ of Γ_0 , so $\text{Hom}_{\mathbb{S}}(Y, W\Gamma_0) \cong \text{Hom}_{\text{Set}}(Y_0, \Gamma_0)$. From the proof of [21, Proposition 3.9], the gauge action (Definition 4.11) of $\text{Hom}(Y_0, \Gamma_0)$ on $\text{MC}(\mathbf{C}^{\bullet}(Y, \Gamma)) = \text{Hom}(Y, \overline{W}\Gamma)$ corresponds to the right multiplication by $W\Gamma_0$ on $\overline{W}\Gamma = \Gamma \backslash W\Gamma$.

Hence, given $\omega, \omega' \in \text{Hom}_{s\text{Pr}(d\mathcal{N}_R^{\flat})}(Y_{\bullet} \times_{\text{Spec} A_0} \text{Spec} A, \overline{W}\underline{G})$ and $g \in \text{Hom}_{\text{Pr}(d\mathcal{N}_R^{\flat})}(Y_0, G)$ with $g \star \omega = \omega'$, this means that $\omega'(y) = \omega(y) \cdot g(y)^{-1}$. We therefore construct an isomorphism $P_{\omega} \rightarrow P_{\omega'}$ by $(y, w) \mapsto (y, w \cdot g(y)^{-1})$, for $y \in Y_{\bullet} \times_{\text{Spec} A_0} \text{Spec} A$ and $w \in \underline{W}\underline{G}$.

We have thus constructed a morphism $\mathfrak{Del}(\mathbf{C}_{\text{ét}}^{\bullet}(X, G)) \rightarrow \text{Tors}(X, G)$. The nerve of $\text{Tors}(X, G)$ is weakly equivalent to the derived stack $\mathcal{H}om(X, BG)$ from [25, Example 3.38]. Since $\mathfrak{g} = \text{Hom}_R(e^* \Omega_{G/R}, R)$, the calculation of [25, Example 3.38] gives

$$D_{\mathbb{B}}^i(\mathcal{H}om(X, BG), M) \cong H_{\text{ét}}^{i+1}(X \times \text{Spec} A, M \otimes_A \text{ad}\mathbb{B}_{\omega}).$$

On taking nerves, we thus get maps

$$\underline{\text{Del}}(\underline{\text{C}}_{\text{ét}}^\bullet(X, G)) \leftarrow \text{Del}(\underline{\text{C}}_{\text{ét}}^\bullet(X, G)) \rightarrow \overline{W}\text{Tors}(X, G) \simeq \text{Hom}(X, BG),$$

all of which give isomorphisms on D^i and equivalences on π^0 . Remark 1.30 thus shows that

$$\underline{\text{Del}}(\underline{\text{C}}_{\text{ét}}^\bullet(X, G)) \simeq \underline{\overline{W}\text{Del}}(\underline{\text{C}}_{\text{ét}}^\bullet(X, G)) \simeq \text{Hom}(X, BG). \quad \square$$

Remark 4.39 If $G = \text{GL}_r$, this gives us a construction for derived moduli of rank r vector bundles on X . If instead $G = \text{SL}_r$, we get derived moduli of determinant 1, rank r vector bundles.

5 Moduli from quasicomonoids

Although cosimplicial groups can be used to construct derived moduli in all characteristics for many problems, they are insufficiently flexible to arise in the generality we need. Instead, we use the quasicomonoids introduced in [23]

5.1 Quasicomonoids

The following is a special case of [23, Lemma 1.5].

Definition 5.1 Define a quasicomonoid E to consist of sets E^n for $n \in \mathbb{N}_0$, together with maps

$$\begin{aligned} \partial^i: E^n &\rightarrow E^{n+1}, & 1 \leq i \leq n, \\ \sigma^i: E^n &\rightarrow E^{n-1}, & 0 \leq i < n, \end{aligned}$$

an associative product $*$: $E^m \times E^n \rightarrow E^{m+n}$, with identity $1 \in E^0$, such that

- (1) $\partial^j \partial^i = \partial^i \partial^{j-1} \quad i < j$;
- (2) $\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \quad i \leq j$;
- (3) $\sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1} & i < j, \\ \text{id} & i = j, i = j + 1, \\ \partial^{i-1} \sigma^j & i > j + 1; \end{cases}$
- (4) $\partial^i(e) * f = \partial^i(e * f)$;
- (5) $e * \partial^i(f) = \partial^{i+m}(e * f)$, for $e \in E^m$;
- (6) $\sigma^i(e) * f = \sigma^i(e * f)$;
- (7) $e * \sigma^i(f) = \sigma^{i+m}(e * f)$, for $e \in E^m$.

Denote the category of quasicomonoids by QM^* .

Example 5.2 Given a cosimplicial group G (or even a cosimplicial monoid G), there is an associated quasicomonoid $\mathcal{E}(G)$ given by $\mathcal{E}(G)^n = G^n$, with identity $1 \in G^0$, operations $\partial_{\mathcal{E}(G)}^i = \partial_G^i, \sigma_{\mathcal{E}(G)}^i = \sigma_G^i$, and Alexander–Whitney product

$$g * h = ((\partial_G^{m+1})^n g) \cdot ((\partial_G^0)^m h),$$

for $g \in G^m, h \in G^n$.

5.1.1 Maurer–Cartan We now construct a Maurer–Cartan functor analogous to the one for cosimplicial groups.

Definition 5.3 Define $MC: QM^*(\text{Set}) \rightarrow \text{Set}$ by

$$MC(E) = \{\omega \in E^1 \mid \sigma^0 \omega = 1, \partial^1 \omega = \omega * \omega\}.$$

Now let $QM^*(\mathbb{S})$ be the category of simplicial objects in QM^* . Then the following is [23, Definition 3.5].

Definition 5.4 Define $\underline{MC}: QM^*(\mathbb{S}) \rightarrow \mathbb{S}$ by

$$\underline{MC}(E) \subset \prod_{n \geq 0} (E^{n+1})^{I^n}$$

(where $I = \Delta^1 \in \mathbb{S}$), consisting of those ω satisfying

$$\begin{aligned} \omega_m(s_1, \dots, s_m) * \omega_n(t_1, \dots, t_n) &= \omega_{m+n+1}(s_1, \dots, s_m, 0, t_1, \dots, t_n), \\ \partial^i \omega_n(t_1, \dots, t_n) &= \omega_{n+1}(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n), \\ \sigma^i \omega_n(t_1, \dots, t_n) &= \omega_{n-1}(t_1, \dots, t_{i-1}, \min\{t_i, t_{i+1}\}, t_{i+2}, \dots, t_n), \\ \sigma^0 \omega_n(t_1, \dots, t_n) &= \omega_{n-1}(t_2, \dots, t_n), \\ \sigma^n \omega_n(t_1, \dots, t_n) &= \omega_{n-1}(t_1, \dots, t_{n-1}), \\ \sigma^0 \omega_0 &= 1. \end{aligned}$$

Define $MC: QM^*(\mathbb{S}) \rightarrow \text{Set}$ by $MC(E) = \underline{MC}(E)_0$, noting that this agrees with Definition 5.3 when $E \in QM^*(\text{Set})$.

Definition 5.5 Given a quasicomonoid E , define the n^{th} matching object $M^n E$ to be the set

$$M^n E = \{(e_0, e_1, \dots, e_{n-1}) \in (E^{n-1})^n \mid \sigma^i e_j = \sigma^{j-1} e_i \forall i < j\}.$$

The Reedy matching map $E^n \rightarrow M^n E$ sends e to $(\sigma^0 e, \sigma^1 e, \dots, \sigma^{n-1} e)$.

Then [23, Definition 3.2] gives a model structure on $QM^*(\mathbb{S})$ in which a morphism $E \rightarrow F$ is a (trivial) fibration whenever the canonical maps

$$E^n \rightarrow F^n \times_{M^n F} M^n E$$

are (trivial) fibrations in \mathbb{S} for all $n \geq 0$.

Lemma 5.6 *If $f: E \rightarrow F$ is a (trivial) fibration in $QM^*(\mathbb{S})$, then the map*

$$\underline{MC}(f): \underline{MC}(E) \rightarrow \underline{MC}(F)$$

is a (trivial) fibration in \mathbb{S} . In particular, if $f: E \rightarrow F$ is a trivial fibration, then $\underline{MC}(f)$ is surjective.

Proof This is a direct consequence of [23, Corollary 3.12], which shows that \underline{MC} is a right Quillen functor. □

Lemma 5.7 *There is an equivalence between the category of abelian group objects in QM^* , and the category cAb of cosimplicial complexes of abelian groups.*

Proof This is [23, Lemma 4.1]. The equivalence is given by the functor \mathcal{E} of Example 5.2. □

Lemma 5.8 *For a cosimplicial simplicial abelian group A , there are canonical isomorphisms*

$$\pi_n \underline{MC}(\mathcal{E}(A)) \cong H_{n-1}(\text{Tot}^{\Pi} \sigma^{\geq 1} N^s N_c A),$$

where $\sigma^{\geq 1}$ denotes brutal truncation in cochain degrees greater than or equal to 1, Tot^{Π} is the product total functor of Definition 4.9, and N^s and N_c are the normalisation functors of Definitions 1.1 and 4.6.

Proof This is [23, Proposition 4.12]. □

5.1.2 The gauge action

Definition 5.9 For $E \in QM^*(\mathbb{S})$, let $(E^0)_n^\times \subset E_n^0$ be the submonoid of invertible elements. There is then an action of the simplicial group $(E^0)^\times$ on the simplicial set $\underline{MC}(E)$, called the *gauge action*, and given by setting

$$(g \star \omega)_n = (\sigma_0)^n g * \omega_n * (\sigma_0)^n g^{-1},$$

as in [21, Definition 3.8], with $(\sigma_0)^n$ denoting the canonical map $(E^0) \rightarrow (E^0)^{\Delta^n}$.

Definition 5.10 Given an element $E \in QM^*(\mathbb{S})$, define the Deligne groupoid by $\mathcal{D}\text{el}(E) := [\underline{\text{MC}}(E)/(E_0^0)^\times]$ In other words, $\mathcal{D}\text{el}(E)$ has objects $\text{MC}(E)$, and morphisms from ω to ω' consist of $\{g \in (E_0^0)^\times \mid g * \omega = \omega' * g\}$.

Define the derived Deligne groupoid to be the simplicial object in groupoids given by $\underline{\mathcal{D}\text{el}}(E) := [\underline{\text{MC}}(E)/(E^0)^\times]$, so $\mathcal{D}\text{el}(E) = \underline{\mathcal{D}\text{el}}(E)_0$.

Define the simplicial sets $\text{Del}(E), \underline{\text{Del}}(E) \in \mathbb{S}$ to be the nerves $B\mathcal{D}\text{el}(E), \overline{W}\underline{\mathcal{D}\text{el}}(E)$, respectively.

Lemma 5.11 *If A is a simplicial cosimplicial abelian group, then*

$$\pi_n \underline{\text{Del}}(\mathcal{E}(A)) \cong H_{n-1}(\text{Tot}^\Pi N^s N_c A),$$

whereas $\pi_1 \text{Del}(\mathcal{E}(A)) \cong H^0(A_0)$, with

$$\pi_0 \text{Del}(\mathcal{E}(A)) \cong Z_{-1}(\text{Tot}^\Pi N^s N_c A)/d_c(A_0^0).$$

Proof This is [23, Proposition 4.13]. □

5.1.3 Comparison with cosimplicial groups

Lemma 5.12 *Given a simplicial cosimplicial group G , and associated simplicial quasicomonoid $\mathcal{E}(G)$ as in Example 5.2, there are G^0 -equivariant weak equivalences*

$$\underline{\text{MC}}(\mathcal{E}(G)) \simeq \underline{\text{MC}}(G)$$

and hence weak equivalences

$$\text{Del}(\mathcal{E}(G)) \simeq \text{Del}(G)$$

in \mathbb{S} , functorial in objects $G \in \text{csGp}$. Here, the functors $\underline{\text{MC}}$ on the left and right are those from Definitions 5.4 and 4.3 respectively, while the functors Del are those from Definitions 5.10 and 4.12.

Proof This is [23, Propositions 6.8 and 6.11]. □

5.2 Constructing quasicomonoids

5.2.1 Monads

Definition 5.13 A monad (or triple) on a category \mathcal{B} is a monoid in the category of endofunctors of \mathcal{B} (with the monoidal structure given by composition of functors).

Example 5.14 Given an adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \\ \top \end{array} \mathcal{B}$$

with unit $\eta: \text{id} \rightarrow UF$ and counit $\varepsilon: FU \rightarrow \text{id}$, the associated monad on \mathcal{B} is given by $\top = UF$, with unit $\eta: \text{id} \rightarrow \top$ and multiplication $\mu := U\varepsilon F: \top^2 \rightarrow \top$.

Definition 5.15 Given a monad (\top, η, μ) on a category \mathcal{B} , define the category of \top -algebras, \mathcal{B}^\top , to have objects

$$\top B \xrightarrow{\theta} B,$$

such that $\theta \circ \eta_B = \text{id}$ and $\theta \circ \top\theta = \theta \circ \mu_B: \top^2 B \rightarrow B$.

A morphism

$$g: (\top B_1 \xrightarrow{\theta} B_1) \rightarrow (\top B_2 \xrightarrow{\phi} B_2)$$

of \top -algebras is a morphism $g: B_1 \rightarrow B_2$ in \mathcal{B} such that $\phi \circ \top g = g \circ \theta$.

Example 5.16 Let $\top := \text{Sym}_R$ be the symmetric functor on Mod_R , with unit $\eta_M: M \rightarrow \top M$ given by the inclusion of degree 1 monomials, and $\mu_M: \top^2 M \rightarrow \top M$ given by expanding out polynomials of polynomials. Then $(\text{Mod}_R)^\top$ is equivalent to the category of unital commutative R -algebras.

Given a monad (\top, μ, η) on a category \mathcal{B} , and an object $B \in \mathcal{B}$, there is a quasi-comonoid $E(B)$ given by

$$E^n(B) = \text{Hom}_{\mathcal{B}}(\top^n B, B)$$

in (Set, \times) , with product $g * h = g \circ \top^n h$, and for $g \in E^n(B)$,

$$\partial^i(g) = g \circ \top^{i-1} \mu_{\top^{n-i} B},$$

$$\sigma^i(g) = g \circ \top^i \eta_{\top^{n-i-1} B}.$$

Note that these constructions also all work for a comonad $(\perp, \Delta, \varepsilon)$, by contravariance. There is even a generalisation to bialgebras for a distributive monad-comonad pair; see [23, Proposition 2.12].

Lemma 5.17 Given an object $B \in \mathcal{B}$, the set of \top -algebra structures on B is $\text{MC}(E(B))$, while $\mathcal{D}\text{el}(E(B))$ is equivalent to the groupoid of \top -algebras overlying B .

Proof This follows immediately from the explicit description in Definition 5.3. \square

5.2.2 Diagrams

Definition 5.18 Given a category \mathcal{B} equipped with a monad \top , together with $K \in \mathbb{S}$ and a map $B: K_0 \rightarrow \text{Ob } \mathcal{B}$, define the quasicomonoid $E_K(B)$ by

$$E^n(B/K) = \prod_{x \in K_n} \text{Hom}_{\mathcal{B}}(\top^n B((\partial_0)^n x), B((\partial_1)^n x)),$$

with operations

$$\begin{aligned} \partial^i(e)(x) &:= e(\partial_i x) \circ \top^{i-1} \mu_{\top^{n-i} B((\partial_0)^{n+1} x)}, \\ \sigma^j(e)(y) &:= e(\sigma_j y) \circ \top^i \eta_{\top^{n-i-1} B((\partial_0)^{n-1} x)}, \\ (f * e)(z) &:= f((\partial_{m+1})^n z) \circ \top^m e((\partial_0)^m z), \end{aligned}$$

for $f \in E^m(B/K)$, $e \in E^n(B/K)$.

Definition 5.19 Given a category \mathcal{B} equipped with a monad \top , together with a small category \mathbb{I} and a map $B: \text{Ob } \mathbb{I} \rightarrow \text{Ob } \mathcal{B}$, define the quasicomonoid $E(B/\mathbb{I})$ by

$$E(B/\mathbb{I}) := E(B/B\mathbb{I}),$$

where $B\mathbb{I}$ is the nerve of \mathbb{I} .

Lemma 5.20 Given $\mathcal{B}, \top, \mathbb{I}$ and $B: \text{Ob } \mathbb{I} \rightarrow \text{Ob } \mathcal{B}$ as above,

$$\text{MC}(E(B/\mathbb{I}))$$

is isomorphic to the set of functors $\mathbb{D}: \mathbb{I} \rightarrow \mathcal{B}^\top$ with $U\mathbb{D}(i) = B(i)$ for all $i \in \mathbb{I}$, where $U: \mathcal{B}^\top \rightarrow \mathcal{B}$ is the forgetful functor.

Meanwhile $\mathcal{D}\text{el}(E(B/\mathbb{I}))$ is equivalent to the groupoid of diagrams $\mathbb{D}: \mathbb{I} \rightarrow \mathcal{B}^\top$ with $\mathbb{D}(i)$ overlying $B(i)$ for all $i \in \mathbb{I}$.

Proof This is [23, Lemma 1.36]. Given $\omega \in \text{MC}(E(B/\mathbb{I}))$, the algebra structure $\top B(i) \rightarrow B(i)$ is given by $\omega(\text{id}: i \rightarrow i) \in \text{Hom}_{\mathcal{B}}(\top^n B(i), B(i))$, while the morphism $\mathbb{D}(f): \mathbb{D}(i) \rightarrow \mathbb{D}(j)$ is given by $\omega(f: i \rightarrow j) \circ \eta_{B(i)} \in \text{Hom}_{\mathcal{B}}(B(i), B(j))$. \square

Corollary 5.21 Take a category \mathcal{B} equipped with a monad \top , together with a small category \mathbb{I} and a subcategory \mathbb{J} . Assume that we have a functor $\mathbb{F}: \mathbb{J} \rightarrow \mathcal{B}^\top$, and a map $B: \text{Ob } \mathbb{I} \rightarrow \text{Ob } \mathcal{B}$ extending $U\text{Ob } \mathbb{F}: \text{Ob } \mathbb{J} \rightarrow \text{Ob } \mathcal{B}$.

For $\omega_{\mathbb{F}} \in \text{MC}(E(B|_{\mathbb{J}}/\mathbb{J}))$ corresponding to \mathbb{F} in Lemma 5.20, we can form a quasicomonoid E by

$$E^n := E^n(B/\mathbb{I}) \times_{E^n(B|_{\mathbb{J}}/\mathbb{J})} \overbrace{\{\omega_{\mathbb{F}} * \omega_{\mathbb{F}} * \dots * \omega_{\mathbb{F}}\}}^n.$$

Then $\text{MC}(E)$ is isomorphic to the set of functors $\mathbb{D}: \mathbb{I} \rightarrow \mathcal{B}^\top$ with $U\mathbb{D}(i) = B(i)$ for all $i \in \mathbb{I}$ and $\mathbb{D}|_{\mathbb{J}} = \mathbb{F}$, while $\mathfrak{D}\text{el}(E)$ is the groupoid of such functors.

Proof This follows immediately from the observation that MC preserves limits. \square

Example 5.22 The main applications of these results are to moduli of morphisms. In that case, \mathbb{I} is the category $0 \rightarrow 1$, $B\mathbb{I} \cong \Delta^1$, and we define $E(B_0, B_1) := E(B/\mathbb{I})$, where $B: \text{Ob } \mathbb{I} \rightarrow \mathcal{B}$ is given by $B(i) := B_i$.

The category \mathbb{J} will be \emptyset , $\{0\}$, $\{1\}$ or $\{0, 1\}$, depending on which endpoints we wish to fix (if any), and so the quasicomonoid E of the corollary is the fibre of $E(B_0, B_1) \rightarrow \prod_{j \in \mathbb{J}} E(B_j)$ over products of ω_j .

Remark 5.23 In [23, Definition 3.25] the construction $E(B/\mathbb{I})$ (for a simplicial category \mathcal{B} equipped with a monad \top) is used to extend the simplicial set $\underline{\text{MC}}(E)$ to a bisimplicial set $\mathcal{MC}(E)$. Explicitly, $\mathcal{MC}(E)_n$ is given by taking \mathbb{I} to be the category associated to the poset $[0, n]$, and setting

$$\mathcal{MC}(E)_n := \coprod_{B: [0, n] \rightarrow \text{Ob } \mathcal{C}} \underline{\text{MC}}(E(B/\mathbf{n})).$$

By [23, Proposition 5.7], $\mathcal{MC}(E)$ is a Segal space whenever \mathcal{C} satisfies suitable fibrancy conditions. Segal spaces are a model for ∞ -categories (whereas simplicial sets are a model for ∞ -groupoids), and [23, Propositions 5.15 and 5.24] show that $\mathfrak{D}\text{el}(E)$ is effectively the core of $\mathcal{MC}(E)$. This means that for all of our moduli constructions based on quasicomonoids in Section 6, we could construct derived moduli as ∞ -categories, rather than just ∞ -groupoids.

Definition 5.24 Say that an ordered pair B, B' of objects in \mathcal{B} induces fibrant quasisdescent data if $E(B)$ and $E(B')$ are fibrant simplicial quasicomonoids, and the matching maps $\underline{\text{Hom}}_{\mathcal{B}}(\top^n B, B') \rightarrow M^n \underline{\text{Hom}}_{\mathcal{B}}(\top^\bullet B, B')$ are also Kan fibrations for all $n \geq 0$.

Note that this is the same as regarding \mathcal{B} as a simplicial quasidescent datum (in the sense of [23, Proposition 2.9], then restricting to objects B, B' , discarding morphisms $B' \rightarrow B$, and requiring that the resulting simplicial quasidescent datum \mathcal{D} be fibrant.

Lemma 5.25 *Take objects $D_0, D_1 \in \mathcal{B}^\top$, with $UD_0, UD_1 \in \mathcal{B}$ inducing fibrant quasidescent data. Then there is a natural weak equivalence*

$\underline{\text{MC}}(UD_0, UD_1) \times_{\underline{\text{MC}}(E(UD_0)) \times \underline{\text{MC}}(E(UD_1))} \{(D_0, D_1)\} \simeq \text{Tot} \underline{\text{Hom}}_{\mathcal{B}}(\top^\bullet UD_0, UD_1)$,
 where $\text{Tot}: c\mathbb{S} \rightarrow \mathbb{S}$ the total space functor of [9, Chapter VIII], and the cosimplicial structure on $\underline{\text{Hom}}_{\mathcal{B}}(\top^\bullet UD_0, UD_1)$ is the usual cotriple resolution [33, Section 8.7] defined via the isomorphisms

$$\underline{\text{Hom}}_{\mathcal{B}}(\top^n UD_0, UD_1) \cong \underline{\text{Hom}}_{\mathcal{B}^\top}((FU)^{n+1} D_0, D_1).$$

Proof This is [23, Proposition 5.10]. □

5.2.3 A bar construction Now fix a simplicial category \mathcal{B} equipped with a monad (\top, μ, η) respecting the simplicial structure. Assume that \mathcal{B}^\top is a cocomplete simplicial category, equipped with a functor $\mathbb{S} \times \mathcal{B}^\top \xrightarrow{\otimes} \mathcal{B}^\top$ for which

$$\text{Hom}_{\mathcal{B}^\top}(K \otimes D, D') \cong \text{Hom}_{\mathbb{S}}(K, \underline{\text{Hom}}_{\mathcal{B}^\top}(D, D')).$$

Write $F: \mathcal{B} \rightarrow \mathcal{B}^\top$ for the free algebra functor sending M to $(\mu_M: \top^2 M \rightarrow \top M)$, and $U: \mathcal{B}^\top \rightarrow \mathcal{B}$ for the forgetful functor sending $(\theta: \top M \rightarrow M)$ to M . In particular, $UF = \top$.

Definition 5.26 Given an object $M \in \mathcal{B}$ and an element $\omega \in \text{MC}(E(B))$ (for MC as in Definition 5.4), define $\beta_F^*(M) \in \mathcal{B}^\top$ by the property that

$$\text{Hom}(\beta_F^*(M), D) \subset \prod_{q \geq 0} \text{Hom}_{\mathcal{B}^\top}((\Delta^1)^q \otimes F\top^q M, D)$$

consists of ϕ satisfying

$$\begin{aligned} \phi_q(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{q-1}) &= \partial^i \phi_{q-1}(t_1, \dots, t_{q-1}), \\ \phi_q(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1}) &= \phi_{i-1}(t_1, \dots, t_{i-1}) \circ F\top^{i-1} \omega_{q-i}(t_i, \dots, t_{q-1}), \\ \sigma^i \phi_q(t_1, \dots, t_q) &= \phi_{q-1}(t_1, \dots, \min(t_i, t_{i+1}), \dots, t_q), \quad 1 \leq i \leq q, \\ \sigma^q \phi_q(t_1, \dots, t_q) &= \phi_{q-1}(t_1, \dots, t_{q-1}), \end{aligned}$$

where $\partial^0 = \lambda_{\top^{q-1} M}^*$, $\partial^i = F\top^{i-1} \mu_{\top^{q-i} M}^*$, and $\sigma^i = F\top^{i-1} \eta_{\top^{q-i} M}^*$.

Beware that, unlike Definition 5.4, there is no relation for σ^0 .

Remarks 5.27 Note that $\beta_F^*(M, \omega)$ is a \top -algebra R generated by $\phi_q(\top^q M \otimes I^q)$, subject to various conditions, all of which are linear on generators except for

$$\phi_q(1, t_1, \dots, t_{q-1}) = U\varepsilon_R \circ \top\phi_{q-1}(t_1, \dots, t_{q-1}).$$

Also observe that β_F^* defines a functor $\mathcal{D}\text{el}(E(M)) \rightarrow \mathcal{B}^\top$.

This construction is inspired by Lada’s bar construction in [7], which uses similar data to define a bar construction as an object of \mathcal{B} , then shows that it carries a canonical \top -algebra structure. However, Lada’s construction only applies when \top is an operad and \mathcal{B} is the category of topological spaces, since it uses special properties of both. Beware that although similar expressions arise in both β_F^* and in Lada’s bar construction, they are not directly comparable.

Proposition 5.28 Fix $\mathcal{B}, \top, M, \omega$ as above, and take any $D \in \mathcal{B}^\top$ for which the pair M, UD induces fibrant quasidescent data (Definition 5.24). Then there is a functorial weak equivalence between $\underline{\text{Hom}}_{\mathcal{B}^\top}(\beta_F^*(M, \omega), D)$ and the fibre of

$$\underline{\text{MC}}(E(M, UD)) \rightarrow \underline{\text{MC}}(E(M)) \times \underline{\text{MC}}(E(UD))$$

over (ω, μ_D) , where $D = (\mu_D: \top UD \rightarrow UD)$, and $E(M, UD)$ is the quasicomonoid defined in Example 5.22.

Proof We adapt the proof of [23, Proposition 5.10]. In the notation of [23], we have $\mathcal{D} \in sQ\text{Dat}_1$ given by $\mathcal{D}(0, 0) = E(M)$, $\mathcal{D}(1, 1) = E(UD)$, $\mathcal{D}(1, 0) = \emptyset$ and $\mathcal{D}(0, 1)^n = \underline{\text{Hom}}_{\mathcal{B}}(\top^n M, UD)$. Using further notation from [23], we can define $\mathcal{A} \in sQ\text{Dat}_1$ by

$$\mathcal{A} := (\mathbb{E} \times \text{alg}^* \mathbf{1}) \cup_{\mathbb{E} \times \{1\}} (\text{alg}^* \mathbf{0} \times \{1\}),$$

and we then have

$$\underline{\text{MC}}(E(M, UD)) \times_{\underline{\text{MC}}(E(UD))} \underline{\text{MC}}(E(D)_0) \cong \underline{\text{Hom}}_{sQ\text{Dat}_1}(\mathcal{A}, \mathcal{D}).$$

Meanwhile, we can construct another object $\mathcal{C} \in sQ\text{Dat}_1$ with $\mathcal{C}(0, 0) = \mathbb{E}$ (so we have $\mathcal{C}(0, 0)^n = I^{n-1}$), $\mathcal{C}(1, 1) = \bullet$, $\mathcal{C}(1, 0) = \emptyset$ and $\mathcal{C}(0, 1)^n = I^n$, where $I = \Delta^1$. The multiplication operation $\mathcal{C}(0, 0)^m \times \mathcal{C}(0, 1)^n \rightarrow \mathcal{C}(0, 1)^{mn}$ is given by $I^{m-1} \times I^n \rightarrow I^{m-1} \times \{0\} \times I^n$. The operations on $\mathcal{C}(0, 1)$ are

$$\begin{aligned} \partial^i(t_1, \dots, t_n) &= (t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n), \quad 1 \leq i \leq n, \\ \sigma^i(t_1, \dots, t_n) &= (t_1, \dots, \min(t_i, t_{i+1}), \dots, t_n), \quad 0 \leq i < n. \end{aligned}$$

The multiplication operation $\mathcal{C}(0, 1)^m \times \mathcal{C}(1, 1)^n \rightarrow \mathcal{C}(0, 1)^{m+n}$ is given by

$$(t_1, \dots, t_m, \bullet) \mapsto (t_1, \dots, t_m, 1, 1, \dots, 1).$$

Now, the inclusion $\Xi = \mathcal{C}(0, 0) \rightarrow \mathcal{C}$ gives a Kan fibration

$$\begin{aligned} \underline{\text{Hom}}_{sQ\text{Dat}_1}(\mathcal{C}, \mathcal{D}) &\rightarrow \underline{\text{Hom}}_{sQM^*(\mathbb{S})}(\Xi, E(M)) \times \underline{\text{Hom}}(\text{alg}^*\mathbf{0}, E(UD)) \\ &= \underline{\text{MC}}(E(M)) \times \text{MC}(E(D)_0), \end{aligned}$$

whose fibre over (ω, D) is $\underline{\text{Hom}}_{\mathcal{B}^\top}(\beta_F^*(M, \omega), D)$. It therefore suffices to show that

$$\underline{\text{Hom}}_{sQ\text{Dat}_1}(\mathcal{C}, \mathcal{D}) \times_{\text{MC}(E(UD)_0)} \{D\} \simeq \underline{\text{MC}}(E(M, UD)) \times_{\text{MC}(E(UD)_0)} \{D\}$$

as fibrant objects over $\underline{\text{MC}}(E(M))$, since taking the fibre over ω yields the required result.

Now, $\text{alg}^*\mathbf{1} \in sQ\text{Dat}_1$ is given by $(\text{alg}^*\mathbf{1})(i, j)^n = \bullet$ for all n and for all $0 \leq i \leq j \leq 1$, while $(\text{alg}^*\mathbf{1})(1, 0) = \emptyset$. Thus the unique maps $\mathcal{C} \rightarrow \text{alg}^*\mathbf{1}$ and $\mathcal{A} \rightarrow \text{alg}^*\mathbf{1}$ are both weak equivalences, and hence both are cofibrant replacements for $\text{alg}^*\mathbf{1}$ in the comma category $(\text{alg}^*\mathbf{0} \times \{1\}) \downarrow sQ\text{Dat}_1$.

If we were to regard \mathcal{D} as an object of $(\text{alg}^*\mathbf{0} \times \{1\}) \downarrow sQ\text{Dat}_1$ via the morphism $D: (\text{alg}^*\mathbf{0} \times \{1\}) \rightarrow \mathcal{D}$, this means that

$$\begin{aligned} \underline{\text{Hom}}_{sQ\text{Dat}_1}(\mathcal{C}, \mathcal{D}) \times_{\text{MC}(E(UD)_0)} \{D\} &\simeq \underline{\text{Hom}}_{(\text{alg}^*\mathbf{0} \times \{1\}) \downarrow sQ\text{Dat}_1}(\mathcal{C}, \mathcal{D}) \\ &\simeq \underline{\text{RHom}}_{(\text{alg}^*\mathbf{0} \times \{1\}) \downarrow sQ\text{Dat}_1}(\text{alg}^*\mathbf{1}, \mathcal{D}) \\ &\simeq \underline{\text{Hom}}_{(\text{alg}^*\mathbf{0} \times \{1\}) \downarrow sQ\text{Dat}_1}(\mathcal{A}, \mathcal{D}) \\ &\simeq \underline{\text{MC}}(E(M, UD)) \times_{\text{MC}(E(UD))} \{D\}, \end{aligned}$$

which completes the proof. □

Definition 5.29 Define $\perp_\bullet: \mathcal{B}^\top \rightarrow \mathcal{B}^\top$ by the property that

$$\underline{\text{Hom}}_{\mathcal{B}^\top}(\perp_\bullet A, D) \cong \text{Tot} \underline{\text{Hom}}_{\mathcal{B}^\top}((FU)^{\bullet+1} A, D).$$

Explicitly, we form the simplicial diagram $n \mapsto (FU)^{n+1} A$ in \mathcal{B}^\top (the cotriple resolution), then let $\perp_\bullet A$ be the coend

$$\int^{n \in \Delta} \Delta^n \otimes (FU)^{n+1}.$$

Corollary 5.30 If $A, D \in \mathcal{B}^\top$, with UA, UD inducing fibrant quasidescent data, then there are functorial weak equivalences

$$\underline{\text{Hom}}_{\mathcal{B}^\top}(\beta_F^*(UA, \mu_A), D) \simeq \underline{\text{Hom}}_{\mathcal{B}^\top}(\perp_\bullet A, D).$$

Proof This just combines Lemma 5.25 with Proposition 5.28. □

5.3 Linear quasicomonoids

Definition 5.31 Say that a quasicomonoid A is linear if each A^n is an abelian group, with the operations ∂^i, σ^i being linear, and $*$: $A^m \times A^n \rightarrow A^{m+n}$ bilinear. As explained in [23, Section 4.4], this corresponds to working with the monoidal structure \otimes rather than \times .

Denote the category of linear quasicomonoids by $QM^*(\text{Ab}, \otimes)$, and the category of simplicial objects in $QM^*(\text{Ab}, \otimes)$ by $QM^*(s\text{Ab}, \otimes)$.

Example 5.32 The quasicomonoid $E(B)$ constructed in Section 5.2.1 is a linear quasicomonoid whenever \mathcal{B} is a preadditive category and \top is an additive functor.

Lemma 5.33 Given $A \in QM^*(\text{Ab}, \otimes)$, the normalisation NA has the natural structure of a (not necessarily commutative) DG ring.

Proof The normalisation is given by $(NA)^n = A^n \cap \bigcap_{i=0}^{n-1} \ker \sigma^i$. If we write 0_m for the additive identity in A^m , then define $\partial^0, \partial^{n+1}: A^n \rightarrow A^{n+1}$ by $\partial^0 a := 0_1 * a$, $\partial^{n+1} a := a * 0_1$. This makes A into a cosimplicial complex, so NA is a chain complex, with $d: N^n A \rightarrow N^{n+1} A$ given by

$$da := \sum_{i=0}^{n+1} (-1)^i \partial^i.$$

For $a \in N^m A$ and $b \in N^n A$, the product $a * b$ lies in $N^{m+n} B$, and we have that $d(a * b) = (da) * b + (-1)^m a * (db)$, so NA is indeed a DG ring, with unit $1 \in N^0 A$. \square

Definition 5.34 Given a DG ring B in nonnegative cochain degrees with multiplication denoted by \wedge , we now define the cosimplicial ring DB . As a cosimplicial complex, DB is given by the formula of Definition 4.21, with multiplication

$$(\partial^I a) \cdot (\partial^J b) := \begin{cases} \partial^{I \cap J} (-1)^{(J \setminus I, I \setminus J)} a \wedge b & a \in L^{|J \setminus I|}, b \in L^{|I \setminus J|}, \\ 0 & \text{otherwise,} \end{cases}$$

for $(-1)^{(S,T)}$ defined as in Definition 4.21.

Lemma 5.35 Given $A \in QM^*(\text{Ab}, \otimes)$, then the quasicomonoid $\mathcal{E}(DNA)$ is isomorphic to A (for \mathcal{E} as in Example 5.2, regarding DNA as a cosimplicial multiplicative monoid).

Proof Since $DNA \cong A$ as a cosimplicial complex, it is automatic that we have isomorphisms $\mathcal{E}(DNA)^n \cong A^n$, compatible with the structural operations σ^i, ∂^i , as well as with the operations $a \mapsto 0_1 * a$ and $a \mapsto a * 0_1$ (corresponding to the additional operations $\partial^0, \partial^{n+1}$). Now, since Eilenberg–Zilber shuffles are left inverse to Alexander–Whitney, it follows that for $a, b \in NA$, $a *_{\mathcal{E}} b = a * b$. Since A is spanned by elements of the form $\partial^I a$ for $a \in NA$, the defining equations of a quasicomonoid ensure that $a *_{\mathcal{E}} b = a * b$ for all $a, b \in A$. \square

Now, we have that any DG R -algebra B has an underlying DGLA over R , with bracket $[a, b] = ab - (-1)^{\deg a \deg b} ba$. If a nonunital DG \mathbb{Q} -algebra B is pro-nilpotent (ie $B \cong \varprojlim_n B/(B)^n$), we can thus define a group $\exp(DN^{>0} B)$ as in Lemma 4.23.

Lemma 5.36 *For any nonunital pro-nilpotent DG \mathbb{Q} -algebra B , there is a canonical isomorphism*

$$\exp(DB) \cong 1 + DB$$

of groups.

Proof Since B is pro-nilpotent, DB is as well. The isomorphism is given by evaluating the exponential in the ring DB , with inverse given by \log . \square

Definition 5.37 Given $A \in QM^*(\text{Ab}, \otimes)$, make $(A^0)^\times$ into a gauge on the DGLA underlying NA by giving it the obvious adjoint action, and with $D: (A^0)^\times \rightarrow N^1 A$ given by $Da = da \cdot a^{-1}$.

Lemma 5.38 *For $A \in QM^*(\text{Ab}, \otimes)$ with A^0 a \mathbb{Q} -algebra, there is a canonical isomorphism*

$$D(\exp(NA), (A^0)^\times) \cong (DNA) \times_{A^0} (A^0)^\times$$

of cosimplicial groups.

Proof By applying Lemma 5.36 to $N^{>0} A$,

$$\begin{aligned} D(\exp(NA), (A^0)^\times)^n &\cong (A^0)^\times \times (1 + \ker(A^n \rightarrow A^0)) \\ &= (A^0)^\times \times (A^n \times_{A^0} 1) \cong A^n \times_{A^0} (A^0)^\times, \end{aligned}$$

and this is automatically compatible with the cosimplicial operations in higher degrees. A short calculation show that it is also compatible with $\partial^0, \partial^1: A^0 \rightarrow A^1$. \square

Definition 5.39 Given a simplicial DGLA L , define the simplicial set $\underline{\text{MC}}(L)$ as follows. For any simplicial set K , define the simplicial DGLA L^K by $(L^K)^n := (L^n)^K$, defined with the formula of Definition 1.22. Then $\underline{\text{MC}}(L)$ is given by

$$\underline{\text{MC}}(L)_n := \text{MC}(\text{Tot}^\Pi N^s(L^{\Delta^n})),$$

where the normalisation $N^s(L)$ has a bracket $N^s(L)_m^i \times N^s(L)_n^j \rightarrow N^s(L)_{m+n}^{i+j}$ given by the Eilenberg–Zilber shuffle product [33, 8.5.4].

Proposition 5.40 For $A \in \text{QM}^*(s\text{Ab}, \otimes)$, there is a canonical $(A^0)^\times$ -equivariant weak equivalence

$$\underline{\text{MC}}(A) \simeq \underline{\text{MC}}(NA),$$

of simplicial sets, and hence a canonical weak equivalence

$$\text{Del}(A) \simeq \overline{W}[\underline{\text{MC}}(NA)/(A^0)^\times].$$

Proof By combining Lemma 4.23 with Lemma 5.38, we have an $(A^0)^\times$ -equivariant isomorphism

$$\underline{\text{MC}}(NA) \cong \underline{\text{MC}}((DNA) \times_{A^0} (A^0)^\times).$$

Now, Lemma 5.12 combines with Lemma 5.35 to give an $(A^0)^\times$ -equivariant weak equivalence

$$\underline{\text{MC}}((DNA) \times_{A^0} (A^0)^\times) \simeq \underline{\text{MC}}(A \times_{A^0} (A^0)^\times).$$

Since $\underline{\text{MC}}(A \times_{A^0} (A^0)^\times) = \underline{\text{MC}}(A)$ and $\text{Del}(A) = \overline{W}[\underline{\text{MC}}(A)/(A^0)^\times]$, this completes the proof. \square

5.4 Moduli functors from quasicomonoids

Definition 5.41 Given a ring $A \in \text{Alg}_R$, an A -module M , a homogeneous, level-wise formally smooth functor $E: \text{Alg}_R \rightarrow \text{QM}^*$ and $\omega \in \text{MC}(E(A))$, define the cosimplicial A -module $C_\omega^\bullet(E, M)$ by

$$C_\omega^n(E, M) := T_{\omega^n}(E^n, M)$$

with operations on $a \in C_\omega^n(E, M)$ given by

$$\begin{aligned} \sigma^i a &= \sigma_E^i a, \\ \partial^i a &= \begin{cases} \omega * a & i = 0, \\ \partial_E^i a & 1 \leq i \leq n, \\ a * \omega & i = n + 1. \end{cases} \end{aligned}$$

Proposition 5.42 *If $E: \text{Alg}_R \rightarrow \text{QM}^*$ is a homogeneous functor, with each E^n formally smooth, then the functor*

$$\underline{\text{MC}}(\underline{E}): d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$$

is homogeneous and formally quasismooth, so

$$\text{MC}(\underline{E}): d\mathcal{N}_R^\flat \rightarrow \text{Set}$$

is homogeneous and prehomotopic.

Proof This proceeds along the same lines as Proposition 4.14. Homogeneity is automatic, as $\underline{\text{MC}}$ preserves arbitrary limits.

We can extend E^n to a functor $E^n: d\mathcal{N}_R^\flat \rightarrow \text{Set}$, given by $E^n(A) := E^n(A_0)$. Then formal smoothness of E^n implies that the extended E^n is prehomotopic. It is automatically formally quasipresmooth, as all discrete morphisms are fibrations. Thus Proposition 1.27 implies that \underline{E}^n is formally smooth, and hence formally quasismooth.

For our next step, there is no analogue of Lemma 4.7 for quasicomonoids, so we have to work a little harder. We wish to show that $\underline{\text{MC}}(\underline{E})(A') \rightarrow \underline{\text{MC}}(\underline{E})(A)$ is a (trivial) fibration for all (acyclic) square zero extensions $A' \rightarrow A$. Write $C := H_0 A$, and note that $\underline{E}(C) = E(C)$, so $\underline{\text{MC}}(\underline{E}(C)) = \text{MC}(E(C))$ is a set. For any $\omega \in \text{MC}(E(C))$, it thus suffices to show that the morphism $\underline{\text{MC}}(\underline{E})(A')_\omega \rightarrow \underline{\text{MC}}(\underline{E})(A)_\omega$ of fibres over ω is a (trivial) fibration.

Now, on the subcategory of $d\mathcal{N}^\flat \downarrow C$ consisting of nilpotent extensions $B \rightarrow C$, define a functor E_ω by $E_\omega^n(B) = E^n(B) \times_{E^n(C)} \omega^n$. Thus $E_\omega(B) \in \text{QM}^*$; since $C^{\Delta^n} = C$, we also have $\underline{E}_\omega(B) \in \text{QM}^*(\mathbb{S})$. Now, the crucial observation is that

$$\underline{\text{MC}}(\underline{E})(B)_\omega = \underline{\text{MC}}(\underline{E}_\omega(B)),$$

so by Lemma 5.6, it suffices to show that $\underline{E}_\omega(A') \rightarrow \underline{E}_\omega(A)$ is a (trivial) fibration in $\text{QM}^*(\mathbb{S})$.

This amounts to saying that $\mu_n: \underline{E}_\omega^n \rightarrow M^n \underline{E}_\omega$ is formally quasismooth for all $n \geq 0$. In fact, it is formally smooth; we prove this inductively on n . For $n = 0$, this follows immediately from the observation above that \underline{E}^0 is formally smooth. If this holds up to level $n - 1$, then $M^n \underline{E}_\omega$ is also formally quasismooth, being a pullback of formally quasismooth maps. Since \underline{E}_ω^n is formally smooth, it follows from Corollary 1.16 that μ_n will be formally smooth provided

$$D_\omega^i(\underline{E}^n, M) \rightarrow D_\omega^i(M^n \underline{E}, M)$$

is surjective for $i = 0$, and an isomorphism for $i > 0$, for all C -modules M . Now, Lemma 1.28 shows that $D_\omega^i(\underline{E}^n, M) = D_\omega^i(E^n, M)$, and similarly for $M^n E$. These are 0 for $i \neq 0$, and $D_\omega^0(E^n, M) = C_\omega^n(E, M)$. Hence we need only show that

$$C_\omega^n(E, M) \rightarrow M^n C_\omega^\bullet(E, M)$$

is surjective, which follows from Lemma 4.7. □

Corollary 5.43 *If $E: \text{Alg}_R \rightarrow QM^*$ is a homogeneous functor, with each E^n formally smooth, then the functor*

$$\underline{\text{Del}}(E): d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$$

is homogeneous and formally quasismooth, while

$$\text{Del}(E): s\mathcal{N}_R^\flat \rightarrow \mathbb{S}$$

is homogeneous, prehomotopic and formally quasipresmooth.

Proof The proof of Proposition 4.15 adapts, if we substitute Proposition 5.42 for Proposition 4.14. □

5.4.1 Cohomology

Definition 5.44 Given a ring $A \in \text{Alg}_R$, an A -module M , a homogeneous, levelwise formally smooth functor $E: \text{Alg}_R \rightarrow QM^*$ and $\omega \in \text{MC}(E(A))$, define

$$H_\omega^i(E, M) := H^i C_\omega^\bullet(E, M),$$

for $C_\omega^\bullet(E, M)$ as in Definition 5.41.

The following is immediate.

Lemma 5.45 *Given a ring $A \in \text{Alg}_R$, an A -module M , a homogeneous, levelwise formally smooth functor $E: \text{Alg}_R \rightarrow QM^*(\mathbb{S})$ and $\omega \in \text{MC}(E_0(A))$, the fibre of $\text{MC}(E(A \oplus M)) \rightarrow \text{MC}(E(A))$ over ω is canonically isomorphic to $\text{MC}(C_\omega^n(G, M))$.*

Lemma 5.46 *If $E: \text{Alg}_R \rightarrow QM^*$ is a homogeneous, levelwise formally smooth functor, with $A \in \text{Alg}_R$ and M an A -module, then*

$$D_\omega^i(\underline{\text{MC}}(E), M) \cong D_\omega^i(\text{MC}(E), M) \cong \begin{cases} H_\omega^{i+1}(E, M) & i > 0, \\ Z^1 C_\omega^\bullet(E, M) & i = 1. \end{cases}$$

Proof First, observe that there is a constant quasicomonoid ι_\bullet , given by the one point set in every level. The element ω thus defines a morphism $\omega^\bullet: \iota_\bullet \rightarrow E(A)$ of quasicomonoids, given by ω^n in level n , and the fibre product

$$T_{\omega^\bullet}(E, L) := E(A \oplus L) \times_{E(A), \omega^\bullet} \iota_\bullet$$

is thus an abelian quasicomonoid, for all $L \in d\text{Mod}_A$. By Lemma 5.7, this corresponds to a simplicial cosimplicial abelian group, which is just given in simplicial level n by

$$T_{\omega^\bullet}(E, L)_n = C_\omega^\bullet(E, (L^{\Delta^n})_0).$$

Therefore

$$T_\omega(\underline{\text{MC}}(E), L) \cong \underline{\text{MC}}(T_{\omega^\bullet}(E, L)),$$

and the result is an immediate consequence of Lemma 5.8. □

Lemma 5.47 *If $E: \text{Alg}_R \rightarrow \mathcal{QM}^*$ is a homogeneous, levelwise formally smooth functor, with $A \in \text{Alg}_R$ and M an A -module, then*

$$D_\omega^i(\underline{\text{Del}}(E), M) \cong D_\omega^i(\text{Del}(E), M) \cong H_\omega^{i+1}(E, M).$$

Proof The proof of Lemma 4.20 carries over. □

5.5 Sheafification

Definition 5.48 Given a cosimplicial diagram $C^\bullet(E)$ in \mathcal{QM}^* , define the quasicomonoid $\text{diag}C^\bullet(E)$ by

$$\text{diag}C^\bullet(E)^n := C^n(E^n),$$

with operations $\partial^i = \partial_C^i \partial_E^i$, $\sigma^i = \sigma_C^i \sigma_E^i$, identity $1 \in C^0(E)$, and multiplication

$$a * b := (\partial_C^{m+1})^n a *_E (\partial_C^0)^m b,$$

for $a \in C^m(E^m)$, $b \in C^n(E^n)$.

Definition 5.49 Given some class \mathbf{P} of covering morphisms in Alg_R and a levelwise formally smooth, homogeneous functor $E: \text{Alg}_R \rightarrow \mathcal{QM}^*$ preserving finite products, define the sheafification E^\sharp of E with respect to \mathbf{P} by first defining a cosimplicial commutative A -algebra $(B/A)^\bullet$ for every \mathbf{P} -covering $A \rightarrow B$, as $(B/A)^n := B \otimes_A \cdots \otimes_A B$ ($n + 1$ times), then setting

$$E^\sharp(A) = \text{diag} \lim_{\substack{\longrightarrow \\ B}} E((B/A)^\bullet).$$

Lemma 5.50 For E and \mathbf{P} as above, there is a canonical morphism

$$\mathcal{D}\text{el}(\underline{E})^\# \rightarrow \mathcal{D}\text{el}(\underline{E}^\#)$$

of groupoid valued functors on $d\mathcal{N}_R^\flat$, for sheafification $\mathcal{D}\text{el}(\underline{E})^\#$ which is defined as in Definition 4.29. This induces an equivalence

$$\pi^0 \mathcal{D}\text{el}(\underline{E})^\# \rightarrow \pi^0 \mathcal{D}\text{el}(\underline{E}^\#).$$

Proof An object of $\mathcal{D}\text{el}(\underline{E})^\#$ is a pair $(\omega, g) \in \text{MC}(\underline{E}(A \otimes_{A_0} B)) \times E^0(B \otimes_{A_0} B)^\times$, for $A_0 \rightarrow B$ a \mathbf{P} -covering, satisfying the conditions

- (1) $g * (\text{pr}_1^* \omega) = (\text{pr}_0^* \omega) * g \in \text{MC}(\underline{E}(A \otimes_{A_0} B \otimes_{A_0} B))$,
- (2) $\text{pr}_{02}^* g = (\text{pr}_{01}^* g) \cdot (\text{pr}_{12}^* g) \in E^0(B \otimes_{A_0} B \otimes_{A_0} B)^\times$,

and we now describe the image of (ω, g) .

First, form the cosimplicial A_0 -algebra $(B/A_0)^\bullet$ as in Definition 5.49. We now map (B, ω, g) to the object ω' of $\text{MC}(\text{diag} \underline{E}(A \otimes_{A_0} (B/A_0)^\bullet))$ given by

$$\omega'_n := (\text{pr}_{0,n+1}^* \omega) * \text{pr}_1^* \omega_n \in E^{n+1}((A^{\Delta^n})_0 \otimes_{A_0} (B/A_0)^{n+1}).$$

The remainder of the proof now follows exactly as for that of Lemma 4.30. □

Now recall that \mathbf{P} -covering morphisms are all assumed faithfully flat.

Lemma 5.51 Take $E: \text{Alg}_R \rightarrow \text{QM}^*$ satisfying the conditions of Definition 5.49, a ring $A \in \text{Alg}_R$, an A -module M , and an object of $\mathcal{D}\text{el}(E)^\#(A)$ represented by (ω, g) for $\omega \in \text{MC}(E(B))$. If the maps

$$\text{H}_\omega^*(E, M \otimes_A B) \otimes_B B' \rightarrow \text{H}_\omega^*(E, M \otimes_A B')$$

are isomorphisms for all \mathbf{P} -coverings $B \rightarrow B'$, then the morphism α of Lemma 5.50 induces an isomorphism

$$\text{D}_{(\omega, g)}^*(B \mathcal{D}\text{el}(\underline{E})^\#, M) \rightarrow \text{D}_{\alpha(\omega, g)}^*(B \mathcal{D}\text{el}(\underline{E}^\#), M).$$

Proof The proof of Lemma 4.31 carries over verbatim. □

6 Examples of derived moduli via quasicomonoids

We now show how to apply the machinery of the previous section to construct derived moduli stacks for several specific examples. The approach is the same as that used to construct derived deformations by the author in [20; 24], by finding suitable monads to construct a quasicomonoid, and then taking the Deligne groupoid.

Throughout this section, R will be a G -ring admitting a dualising complex in the sense of [13, Chapter V]. Examples are \mathbb{Z} , any field, or any Gorenstein local ring.

6.1 Finite schemes

For a fixed $r \in \mathbb{N}$, we now study a quasicomonoid governing finite schemes of rank r . For any commutative R -algebra A , our moduli groupoid consists of algebra homomorphisms $A \rightarrow B$, with B a locally free A -module of rank r .

In order to construct a quasicomonoid, we take the approach of Section 5.2.1. Let $\mathcal{B}(A)$ be the category of A -modules, and define the monad \mathbb{T}_A on $\mathcal{B}(A)$ to be Sym_A , so $\mathcal{D}(A) := \mathcal{B}(A)^{\mathbb{T}_A}$ is the category of commutative A -algebras.

Definition 6.1 Working in $\mathcal{B}(A)$, define $E_r: \text{Alg}_R \rightarrow \mathcal{Q}M^*$ by setting $E_r(A)$ to be the quasicomonoid $E_r(A) := E(A^r)$, given by

$$E_r^n(A) = \text{Hom}_A(\mathbb{T}_A^n(A^r), A^r).$$

The following is then just Lemma 5.17 in this context.

Lemma 6.2 For any commutative R -algebra A , $\mathcal{D}\text{el}(E_r(A))$ is canonically isomorphic to the groupoid of commutative A -algebra structures on the A -module A^r .

For our next step, note that $\mathcal{D}\text{el}(E_r)$ is not a stack, although the core of \mathcal{D} is. However, the stackification $\mathcal{D}\text{el}(E_r)^\sharp$ in the Zariski topology is equivalent to the subgroupoid of $\mathcal{D}(A)$ consisting of commutative A -algebras B which are locally free A -modules of rank r .

Definition 6.3 Define $\mathcal{D}\text{el}(E_r)^\sharp: d\mathcal{N}_R^\flat \rightarrow \text{Gpd}$ to be the stackification of $\mathcal{D}\text{el}(E_r)$ in the strict Zariski topology of Definition 2.17. Likewise, define the simplicial groupoid valued functor $\mathcal{D}\text{el}(E_r)^\sharp$ on $d\mathcal{N}_R^\flat$ by stackifying levelwise, so we have that $(\mathcal{D}\text{el}(E_r)^\sharp)_n = (\mathcal{D}\text{el}(E_r)_n)^\sharp$.

Proposition 6.4 *The functor $\overline{W}\mathcal{D}\text{el}(\underline{E}_r)^\sharp \rightarrow \mathbb{S}$ is representable by an almost finitely presented derived geometric 1–stack. Moreover,*

$$\overline{W}\mathcal{D}\text{el}(\underline{E}_r)^\sharp \simeq \overline{W}\mathcal{D}\text{el}(E_r)^\sharp \simeq \underline{\text{Del}}(E_r)^\sharp,$$

where the last is defined using the quasicomonoid sheafification of Definition 5.49.

Proof For the first statement, we just show that $B\mathcal{D}\text{el}(E_r)^\sharp$ satisfies the conditions of Theorem 1.31. Corollary 5.43 ensures $B\mathcal{D}\text{el}(E_r)^\sharp$ is homogeneous and prehomotopic. It follows from Lemma 5.47 and the cotriple characterisation of André–Quillen cohomology [33, Section 8.8] that

$$D_C^i(B\mathcal{D}\text{el}(E_r)^\sharp, M) \cong \text{Ext}^{i+1}(\mathbb{L}_{\bullet}^{C/A}, C \otimes_A M),$$

so the finiteness conditions of Theorem 1.31 all hold, making $\overline{W}\mathcal{D}\text{el}(\underline{E}_r)^\sharp$ representable.

Similar arguments show that $\overline{W}\mathcal{D}\text{el}(E_r)^\sharp$ satisfies the conditions of Theorem 1.19, so is representable by a derived geometric 1–stack. For the first equivalence, we thus apply Corollary 1.29 to the morphism

$$B\mathcal{D}\text{el}(E_r)^\sharp \rightarrow \overline{W}\mathcal{D}\text{el}(E_r)^\sharp$$

coming from the map $\mathcal{D}\text{el}(E_r)^\sharp \rightarrow \underline{\mathcal{D}\text{el}}(E_r)^\sharp$ of simplicial diagrams of groupoids.

For the second equivalence, we just use Lemmas 5.50 and 5.51 to show that the composite map

$$B\mathcal{D}\text{el}(E_r)^\sharp \rightarrow B\underline{\mathcal{D}\text{el}}(E_r)^\sharp \rightarrow \overline{W}\mathcal{D}\text{el}(E_r)^\sharp$$

satisfies the conditions of Corollary 1.29. □

Remark 6.5 Alternatively, we can describe the associated derived geometric 1–stack explicitly. The functor $A \mapsto \text{MC}(E_r(A))$ is an affine dg scheme, $(E_r^0)^\times = \text{GL}_r$, and $\overline{W}\mathcal{D}\text{el}(\underline{E}_r)^\sharp$ is just the hypersheafification of the quotient $B[\text{MC}(E_r)/(E_r^0)^\times]$ in the homotopy Zariski (and indeed homotopy étale) topologies. In the terminology of [27], the simplicial affine dg scheme $B[\text{MC}(E_r)/(E_r^0)^\times]$ is a homotopy derived Artin 1–hypergroupoid representing $\overline{W}\mathcal{D}\text{el}(\underline{E}_r)^\sharp$.

Proposition 6.6 *For $A \in s\mathcal{N}_R^b$, the space $\underline{\text{Del}}(E_r)^\sharp(A)$ is functorially weakly equivalent to the nerve $\overline{W}\mathfrak{G}(A)$ of the ∞ –groupoid $\mathfrak{G}(A)$ of simplicial commutative A –algebras B for which $B \otimes_A^L \pi_0 A$ is weakly equivalent to a locally free module of rank r over $\pi_0 A$.*

Proof For $F: s\text{Mod}(A) \rightarrow s\text{Alg}(A)$ the free commutative algebra functor on simplicial A -modules, the functor β_F^* of Definition 5.26 maps from $\mathcal{D}\text{el}(E_r^\sharp)(A)$ to $s\text{Alg}(A)$, functorially in $A \in s\mathcal{N}_R^b$, and all objects in its image are cofibrant.

When $A \in \text{Alg}_R$, Corollary 5.30 implies that $\beta_F^*(C)$ is homotopy equivalent to the cotriple resolution $\perp_\bullet C$ of C . Thus $\pi_0(\beta_F^*(C)) \cong C$, and $\pi_i(\beta_F^*(C)) = 0$ for all $i > 0$, so we indeed have a functor

$$\beta_F^*: \mathcal{D}\text{el}(E_r^\sharp)(A) \rightarrow \mathfrak{G}(A)$$

for all $A \in s\mathcal{N}_R^b$ (using compatibility with base change). Moreover, this functor is an equivalence of ∞ -groupoids when $A \in \text{Alg}_R$.

Now, [25, Corollary 3.10 Example 3.11] and give

$$D_{[B]}^i(\overline{W}\mathfrak{G}, N) \cong \text{Ext}_B^{i+1}(\mathbb{L}^{B/A}, N \otimes_A B),$$

and so β_F^* satisfies the conditions of Remark 1.30, which gives an equivalence $\overline{W}\mathcal{D}\text{el}(E_r^\sharp) \rightarrow \overline{W}\mathfrak{G}$. If we combine this with Proposition 6.4, then we get the required equivalence $\overline{\text{Del}}(E_r^\sharp) \simeq \overline{W}\mathfrak{G}$. □

6.2 Coherent sheaves

Take a projective scheme X over R , and fix a numerical polynomial h . We now consider moduli of coherent sheaves on X with Hilbert polynomial h . In other words, our underived moduli functor \mathcal{M}_h will set $\mathcal{M}_h(A)$ to be the groupoid of coherent sheaves \mathcal{F} on $X \times \text{Spec } A$ with $\Gamma(X \times \text{Spec } A, \mathcal{F}(n))$ locally free of rank n for $n \gg 0$.

Definition 6.7 For $A \in \text{Alg}_R$, we now define our base category $\mathcal{B}(A)$ as follows. First form the category $\mathcal{C}(A)$ of graded A -modules $M = \bigoplus_{n \geq 0} M\{n\}$ in nonnegative degrees, then let $\mathcal{B}(A) := \text{pro}(\mathcal{C}(A))$. Explicitly, objects of $\mathcal{B}(A)$ are inverse systems $\{M^\alpha\}_\alpha$ in $\mathcal{C}(A)$, with

$$\text{Hom}_{\mathcal{B}(A)}(\{M^\alpha\}_\alpha, \{N^\beta\}_\beta) = \lim_{\leftarrow \beta} \lim_{\rightarrow \alpha} \text{Hom}_{\mathcal{C}(A)}(M^\alpha, N^\beta).$$

If we let $S := \bigoplus_{n \geq 0} \Gamma(X, \mathbb{O}_X(n))$, then there is a monad \mathbb{T} on $\mathcal{C}(A)$ given by $\mathbb{T}M = S \otimes_R M$, with multiplication $\mu: \mathbb{T}^2 \rightarrow \mathbb{T}$ coming from the multiplication $S \otimes_R A \rightarrow S$, and unit $\text{id} \rightarrow \mathbb{T}$ coming from $R \rightarrow S$.

This extends naturally to a monad on $\mathcal{B}(A)$, and we are now in the setting of Section 5.2.1, since the category $\mathcal{D}(A) := \mathcal{B}(A)^\mathbb{T}$ of \mathbb{T} -algebras is just the pro-category of graded

$S \otimes_R A$ -modules. We are interested in pro-modules of the form $M = \{\bigoplus_{n \geq p} M\{n\}\}_p$; for M, M' any two such, we get

$$\begin{aligned} \text{Hom}_{\mathcal{D}(A)}(M, M') &= \lim_{\leftarrow q} \lim_{\rightarrow p} \text{Hom}_{S \otimes_R A}^{\mathbb{G}^m}(M\{\geq p\}, M'\{\geq q\}) \\ &\cong \lim_{\leftarrow q} \lim_{\rightarrow p \geq q} \text{Hom}_{S \otimes_R A}^{\mathbb{G}^m}(M\{\geq p\}, M'\{\geq q\}) \\ &\cong \lim_{\leftarrow q} \lim_{\rightarrow p \geq q} \text{Hom}_{S \otimes_R A}^{\mathbb{G}^m}(M\{\geq p\}, M') \\ &\cong \lim_{\rightarrow p} \text{Hom}_{S \otimes_R A}^{\mathbb{G}^m}(M\{\geq p\}, M'), \end{aligned}$$

which ties in with [30].

Definition 6.8 Let $R^h \in \mathcal{B}(R)$ be the inverse system $\{\bigoplus_{n \geq p} R^{h(n)}(n)\}_p$ of graded modules, and form the quasicomonoid $E_h(A) := E(R^h \otimes_R A)$ given by

$$E_h^n(A) = \text{Hom}_{\mathcal{B}(A)}(\top^n(R^h \otimes_R A), R^h \otimes_R A).$$

Lemma 5.17 then implies that $\mathcal{D}\text{el}(E_h(A))$ is the subgroupoid of $\mathcal{D}(A)$ consisting of S -module structures on $R^h \otimes_R A$, and all isomorphisms between them.

For our next step, note that $\mathcal{D}\text{el}(E_h)$ is not a stack, although the core of \mathcal{D} is. However, Lemma 3.25 adapts to show that the stackification $\mathcal{D}\text{el}(E_h)^\sharp$ in the pro-Zariski topology is equivalent to the subgroupoid of $\mathcal{D}(A)$ consisting of pro- $(S \otimes_R A)$ -modules $M = \{\bigoplus_{n \geq p} M\{n\}\}_p$, with $M\{n\}$ locally free of rank $h(n)$ over A for $n \gg 0$.

Now, there is a functor $\mathcal{F} \mapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$ from coherent sheaves to $\mathcal{D}(A)$, and the essential image just consists of finitely generated $(S \otimes_R A)$ -modules.

Definition 6.9 Define $\text{MC}_f(E_h): \text{Alg}_R \rightarrow \text{Set}$ by letting $\text{MC}_f(E_h, A) \subset \text{MC}(E_h, A)$ consist of $(S \otimes_R A)$ -modules isomorphic to finitely generated modules. Next, define $\underline{\text{MC}}_f(\underline{E}_h): d\mathcal{N}_R^\flat \rightarrow \mathbb{S}$ by

$$\underline{\text{MC}}_f(\underline{E}_h, A) := \underline{\text{MC}}(\underline{E}_h, A) \times_{\text{MC}(E_h, H_0 A)} \text{MC}_f(E_h, H_0 A),$$

with $\text{MC}_f(\underline{E}_h, A) = \underline{\text{MC}}_f(\underline{E}_h, A)_0$.

Likewise, define $\mathcal{D}\text{el}_f(E_h) := [\text{MC}_f(E_h)/(E^0)^\times]$, $\mathcal{D}\text{el}_f(\underline{E}_h) := [\underline{\text{MC}}_f(\underline{E}_h)/(E^0)^\times]$ and $\underline{\mathcal{D}\text{el}}_f(\underline{E}_h) := [\underline{\text{MC}}_f(\underline{E}_h)/(\underline{E}^0)^\times]$.

Definition 6.10 Let $E_h^\#$ be the quasicomonoid sheafification (see Definition 5.49) of E_h in the pro-Zariski topology, and recall $\mathcal{D}el(E_h^\#) \simeq \mathcal{D}el(E_h)^\#$ from Lemma 5.50. Define $MC_f(E_h^\#) \subset MC(E_h^\#)$ to be the essential image of $\mathcal{D}el_f(E_h)^\#$ (which consists of modules isomorphic to finitely generated modules).

Then define $\underline{MC}_f(E_h^\#), \underline{MC}(E_h^\#), \underline{\mathcal{D}el}_f(E_h^\#), \underline{\mathcal{D}el}(E_h^\#)$ and $\underline{\mathcal{D}el}_f(\underline{E}_h^\#)$ by adapting the formulae of Definition 6.9.

Lemma 3.25 adapts to show the substack $\mathcal{D}el_f(E_h)^\#$ of $\mathcal{D}el_f(E_h)^\#$ is equivalent to the stack $\mathcal{M}_h(A)$ defined at the beginning of the section. Note that $MC_f(E_h) \rightarrow MC(E_h)$ is formally étale, and so $\underline{\mathcal{D}el}_f(E_h) \rightarrow \underline{\mathcal{D}el}(E_h)$ is also formally étale, as well as $\underline{\mathcal{D}el}_f(\underline{E}_h^\#) \rightarrow \underline{\mathcal{D}el}(\underline{E}_h^\#)$.

Proposition 6.11 *The functor $\overline{W}\mathcal{D}el_f(E_h)^\# \rightarrow \mathcal{S}$ is representable by an almost finitely presented derived geometric 1–stack. Moreover,*

$$\overline{W}\underline{\mathcal{D}el}_f(E_h)^\# \simeq \overline{W}\underline{\mathcal{D}el}_f(\underline{E}_h)^\# \simeq \underline{\mathcal{D}el}_f(\underline{E}_h)^\#.$$

Proof By exploiting the fact that $MC_f(E_h) \rightarrow MC(E_h)$ is formally étale, the proof of Proposition 6.4 carries over. The only substantial differences lie in the calculation of cohomology,

$$D_L^i(B\mathcal{D}el(\underline{E}_h)^\#, M) \cong \text{Ext}_{S \otimes_R A}^{i+1}(L, L \otimes_A M),$$

and in establishing local finite presentation, which we get from the adaptation of Lemma 3.28. □

Proposition 6.12 *For $A \in s\mathcal{N}_R^b$, the space $\underline{\mathcal{D}el}_f(\underline{E}_h)^\#(A)$ is functorially weakly equivalent to the nerve of the ∞ –groupoid \mathfrak{M}_h of derived quasicoherent sheaves \mathcal{F} on $X \times \text{Spec } A$ for which*

- (1) $\mathcal{F} \otimes_A^L \pi_0 A$ is weakly equivalent to a quasicoherent sheaf $\overline{\mathcal{F}}$ on $X \times \text{Spec } \pi_0 A$,
- (2) for all $n \gg 0$, $\Gamma(X \times \text{Spec } \pi_0 A, \overline{\mathcal{F}}(n))$ is a locally free A –module of rank $h(n)$.

Proof Taking $F: s\mathcal{B}(A) \rightarrow s\mathcal{D}(A)$ to be the functor $M \mapsto S \otimes_R M$ from simplicial graded A –modules to simplicial graded $S \otimes_R A$ –modules, Definition 5.26 gives us a functor

$$\beta_F^*: \mathcal{D}el(E_h^\#)(A) \rightarrow s\mathcal{D}(A),$$

preserving cofibrant objects. In particular, if $A \in \text{Alg}_R$, and $L \in \mathcal{D}(A)$ has $L\{n\}$ locally free for all n , then $\beta_F^*(L)$ is cofibrant, and Corollary 5.30 implies that it is homotopy equivalent to the cotriple resolution $S^{\otimes_R \bullet+1} \otimes_R L$ of L (as in [33, Section 8.7.1]).

This ensures that for any $A \in s\mathcal{N}_R^b$, β_F^* maps objects of $\mathcal{D}\text{el}_f(E_h^\#)(A)$ to modules associated to objects of $\mathfrak{M}_h(A)$. Explicitly, let $\tilde{X} := (\text{Spec } S) - \{0\}$ be the canonical \mathbb{G}_m -bundle over $X = \text{Proj } S$, with $\pi: \tilde{X} \rightarrow X$ the projection and $j: \tilde{X} \rightarrow \text{Spec } S$ the open immersion. Then our functor $\mathcal{D}\text{el}_f(E_h^\#)(A) \rightarrow \mathfrak{M}_h(A)$ is

$$(M, \omega) \mapsto \beta_F^*(M, \omega)^\# = (\pi_* j^{-1} \beta_F^*(M, \omega))^{\mathbb{G}_m}.$$

Now, for any $A \in \text{Alg}_R$, we have

$$\mathcal{D}_L^i(\mathcal{B}\mathcal{D}\text{el}(E_h^\#), M) \cong \text{Ext}_{\mathcal{D}(A)}^{i+1}(L, L \otimes_A M);$$

adapting Serre’s Theorem [30, Section 59] as in the proof of Proposition 3.32, this is isomorphic to $\text{Ext}_{X \times \text{Spec } A}^{i+1}(L^\#, L^\# \otimes_A M)$.

The remainder of the proof follows as for the proof of Proposition 6.6, but replacing [25, Example 3.11] with [25, Theorem 4.12]. □

Remark 6.13 (Associated DGLAs) Since the functor $\top = S \otimes_R$ is linear, our quasicomonoid $E_h(A)$ is linear in the sense of Section 5.3, so its normalisation NE_h has the natural structure of a DGLA. We could thus use Proposition 5.40 to rewrite all these results in terms of the groupoid $[\text{MC}(\text{NE}_h)/(E_h^0)^\times]$, thereby making them consistent with the approach of [5].

Remark 6.14 (Quot schemes) If we wished to work with quotients of a fixed coherent sheaf \mathcal{M} on X , there are two equivalent approaches we could take. One is to choose $M \in \mathcal{D}(R)$ with $M^\# = \mathcal{M}$, and then to replace $\mathcal{D}(A)$ with the comma category $(M \otimes_R A) \downarrow \mathcal{D}(A)$. The monad \top would then be given by $\top(L) = (M \otimes_R A) \oplus (S \otimes_A L)$, with unit $L \rightarrow S \otimes_A L$, and multiplication $\top^2(L) \rightarrow \top(L)$ being the map

$$\begin{aligned} (M \otimes_R A) \oplus (S \otimes_R M \otimes_R A) \oplus (S \otimes_R S \otimes_R L) &\rightarrow (M \otimes_R A) \oplus (S \otimes_A L), \\ (m_1, s_1 \otimes m_2, s_2 \otimes s_3 \otimes l) &\mapsto (m_1 + s_1 m_2, (s_2 s_3) \otimes l). \end{aligned}$$

We would also have to replace $\text{MC}_f(E_h)$ with the subset of $\text{MC}(E_h)$ consisting of finitely generated $S \otimes_R A$ -modules under $M \otimes_R A$ for which the map $M \otimes_R A \rightarrow L$ is surjective.

The alternative approach would be to work with the construction of Example 5.22, first forming the quasicomonoid

$$E := E(A^{h'}, A^h) \times_{E(A^{h'})} \{M\},$$

where h' is the Hilbert polynomial of M , then taking the subset of $\text{MC}(E)$ cut out by the finiteness and surjectivity conditions above.

Beware, however, that in both of these approaches, the resulting quasicomonoid is no longer linear, so cannot naturally be replaced with a DGLA.

6.3 Polarised projective schemes

For a fixed numerical polynomial $h \in \mathbb{Q}[t]$, we will now study the moduli of polarised projective schemes $(X, \mathcal{O}_X(1))$, with $\mathcal{O}_X(1)$ ample, for which $\Gamma(X, \mathcal{O}_X(n))$ is locally free of rank $h(n)$ for $n \gg 0$.

We take $\mathcal{B}(A)$ to be the pro-category of graded A -modules from Section 6.2, with monad

$$\mathbb{T} = \text{Symm}_A: \mathcal{B}(A) \rightarrow \mathcal{B}(A).$$

Setting $\mathcal{D}(A) := \mathcal{B}(A)^\mathbb{T}$ gives us the pro-category of \mathbb{G}_m -equivariant commutative A -algebras in nonnegative degrees.

Definition 6.15 Let $R^h \in \mathcal{B}(R)$ be the inverse system $\{\bigoplus_{n \geq p} R^{h(n)}(n)\}_p$ of graded modules, and form the quasicomonoid $E_h(A) := E(R^h \otimes_R A)$ given by

$$E_h^n(A) = \text{Hom}_{\mathcal{B}(A)}(\mathbb{T}^n(R^h \otimes_R A), R^h \otimes_R A).$$

Lemma 5.17 then implies that $\mathcal{D}\text{el}(E_h(A))$ is the subgroupoid of $\mathcal{D}(A)$ consisting of commutative ring structures on $R^h \otimes_R A$, and all isomorphisms between them. By Lemma 3.25, the stackification $\mathcal{D}\text{el}(E_h)^\sharp$ in the pro-Zariski topology is equivalent to the subgroupoid of $\mathcal{D}(A)$ consisting of commutative pro- A -algebras $B = \{\bigoplus_{n \geq p} B\{n\}\}_p$, with $B\{n\}$ locally free of rank $h(n)$ over A for $n \gg 0$.

We now proceed as in Definition 6.9, letting $\text{MC}_f(E_h) \subset \text{MC}(E_h)$ consist of finitely generated A -algebras, and so on for $\underline{\text{MC}}_f(\underline{E}_h)$ etc. Note that $\text{MC}_f(E_h) \rightarrow \text{MC}(E_h)$ is formally étale.

Proposition 6.16 For $A \in \text{Alg}_{\mathbb{Q}}$, $\mathcal{D}\text{el}_f(E_h(A))$ is equivalent to the groupoid of flat polarised schemes $(X, \mathcal{O}_X(1))$ of finite type over A , with $\mathcal{O}_X(1)$ ample and the A -modules $\Gamma(X, \mathcal{O}_X(n))$ locally free of rank $h(n)$ for all $n \gg 0$.

Proof This is essentially just Proposition 3.30. □

Proposition 6.17 The functor $\overline{W}\underline{\mathcal{D}\text{el}}_f(\underline{E}_h)^\sharp \rightarrow \mathbb{S}$ is representable by an almost finitely presented derived geometric 1-stack. Moreover,

$$\overline{\underline{\mathcal{D}\text{el}}_f(\underline{E}_h)^\sharp} \simeq \overline{W}\underline{\mathcal{D}\text{el}}_f(\underline{E}_h)^\sharp \simeq \underline{\text{Del}}_f(\underline{E}_h)^\sharp.$$

Proof The proofs of Propositions 6.4 and 6.11 carry over, substituting the relevant finiteness properties from Proposition 3.33. In particular, Proposition 3.32 adapts to show that

$$D_{[C]}^i(\underline{\text{Del}}_f(E_h^\sharp), M) \cong \mathbb{E}xt_X^{i+1}(\mathbb{L}^{X/B\mathbb{G}_m \otimes A}, \mathbb{O}_X \otimes_A M),$$

where $X = \text{Proj}(A \oplus C)$. □

Proposition 6.18 For $A \in s\mathcal{N}_{\mathbb{Q}}^b$, the space $\underline{\text{Del}}_f(E_h^\sharp)$ is functorially weakly equivalent to the nerve $\overline{W}\mathfrak{M}(A)$ of the ∞ -groupoid $\mathfrak{M}(A)$ of derived geometric 0-stacks \mathfrak{X} over $B\mathbb{G}_m \times \text{Spec } A$ for which $X := \mathfrak{X} \otimes_A^{\mathbb{L}} H_0 A$ is weakly equivalent to a flat projective scheme over $H_0 A$, with the polarisation $X \rightarrow B\mathbb{G}_m \otimes H_0 A$ ample with Hilbert polynomial h .

Proof This is essentially the same as Proposition 3.34, replacing β^* with the functor $\beta_F^*: \mathfrak{D}el(E_h(A)) \rightarrow s\mathcal{D}(A)$ on simplicial objects from Definition 5.26 (constructed similarly to those of Propositions 6.6 and 6.12). □

Remark 6.19 Note that the constructions of Section 5.2.2 immediately allow us to adapt E_h to work with moduli of diagrams of polarised projective schemes, and in particular with morphisms of such schemes. For moduli over a fixed base $\text{Proj } S$, an alternative approach is to replace \top with the monad $M \mapsto S \otimes_R \text{Sym}_A M$. Either of these approaches can be used to construct derived Hilbert schemes (by taking $\text{MC}_f(E)$ to be the subset of $\text{MC}(E)$ consisting of finitely generated A -algebras B for which $S \otimes_R A \rightarrow B$ is surjective). Propositions 6.18 and 3.34 ensure that these approaches give equivalent derived stacks, as does [6].

6.4 Finite group schemes

To study moduli of finite group schemes, we follow the approach of [25, Example 3.41], by noting that the nerve functor gives a full and faithful inclusion of the category of group schemes into the category of pointed simplicial schemes.

Given a finite group scheme G over $\text{Spec } A$, with $O(G) := \Gamma(G, \mathbb{O}_G)$ locally free of rank r , we thus look at the simplicial group scheme BG (for an explicit description, note this is the same as $\overline{W}G$ in Definition 3.15). If we write $O(BG)^n := \Gamma(BG_n, \mathbb{O}_{BG_n})$, then $O(BG)$ is a commutative cosimplicial augmented A -algebra, with $O(BG)^n$ locally free of rank r^n .

Lemma 6.20 The functor $G \mapsto BG$ from group schemes to pointed simplicial schemes is formally étale.

Proof A simplicial scheme X_\bullet over A is of the form BG if and only if

- (1) $X_0 = \text{Spec } A$;
- (2) for all $n > 1$ and all $0 \leq k \leq n$, the maps $X_n \rightarrow \text{Hom}_{\mathbb{S}}(\Lambda^{n,k}, X)$ are isomorphisms, where $\Lambda^{n,k} \subset \Delta^n$ is the k^{th} horn, obtained by removing the k^{th} face from $\partial\Delta^n$.

Since any deformation of an isomorphism is an isomorphism, the result follows. \square

We could now combine Section 5.2.2 with Section 6.1 to obtain a quasicomonoid functor governing moduli of such diagrams, but there is a far more efficient choice. If A is a local ring, then not only are the modules $O(BG)^n$ independent of G : we can also describe all operations except ∂^0 .

The following is adapted from [21, Definition 3.6].

Definition 6.21 Define $\bar{V}: s\text{Gp} \rightarrow \mathbb{S}$ by setting

$$\bar{V}G_n := G_{n-1} \times G_{n-2} \times \cdots \times G_0$$

with operations

$$\begin{aligned} \partial_0(g_{n-1}, \dots, g_0) &= ((\partial_0 g_{n-1})^{-1} g_{n-2}, \dots, (\partial_0^{n-1} g_{n-1})^{-1} g_0), \\ \partial_i(g_{n-1}, \dots, g_0) &= (\partial_{i-1} g_{n-1}, \dots, \partial_1 g_{n-i+1}, g_{n-i-1}, g_{n-i-2}, \dots, g_0), \\ \sigma_i(g_{n-1}, \dots, g_0) &= (\sigma_{i-1} g_{n-1}, \dots, \sigma_0 g_{n-i}, g_{n-i}, g_{n-i-1}, \dots, x_0). \end{aligned}$$

Lemma 6.22 *There is a natural isomorphism $\bar{\phi}: \bar{V} \rightarrow \bar{W}$.*

Proof As in [21, Proposition 3.9], the map $\bar{\phi}_G: \bar{V}G \rightarrow \bar{W}G$ is given by

$$\bar{\phi}(g_{n-1}, \dots, g_0) = (g_{n-1}, (\partial_0 g_{n-1})^{-1} g_{n-2} \dots, (\partial_0 g_1)^{-1} g_0). \quad \square$$

We can therefore replace B with the functor \bar{V} , and consider the simplicial scheme $\bar{V}G$, which has the property that ∂_0 is the only simplicial operation to depend on the group structure of G . We now proceed along the same lines as the author [22, Section 5.1].

Definition 6.23 Define Δ_* to be the subcategory of the ordinal number category Δ containing only those morphisms fixing 0. Given a category \mathcal{C} , define the category $c_+\mathcal{C}$ of almost cosimplicial diagrams in \mathcal{C} to consist of functors $\Delta_* \rightarrow \mathcal{C}$. Thus an almost cosimplicial object X^* consists of objects $X^n \in \mathcal{C}$, with all of the operations ∂^i, σ^i of a cosimplicial complex except ∂^0 , satisfying the usual relations.

Definition 6.24 Define the functor $F_\partial: c_+\text{Mod}(A) \rightarrow c\text{Mod}(A)$ from almost cosimplicial A -modules to A -modules by

$$(F_\partial M^*)^n = M^n \oplus M^{n-1} \oplus \dots \oplus M^0,$$

with operations

$$\begin{aligned} \partial^i(v_n, \dots, v_0) &= (\partial^i v_n, \partial^{i-1} v_{n-1}, \dots, \partial^1 v_{n-i+1}, 0, v_{n-i}, \dots, v_1, v_0), \\ \sigma^i(v_n, \dots, v_0) &= (\sigma^i v_n, \dots, \sigma^1 v_{n-i+1}, \sigma^0 v_{n-i} + v_{n-i-1}, v_{n-i-2}, \dots, v_0). \end{aligned}$$

By the argument of [22, Lemma 4.12], we have that F_∂ is left adjoint to the functor $U_\partial: c\text{Mod}(A) \rightarrow c_+\text{Mod}(A)$ given by forgetting ∂^0 . Moreover, this adjunction is monadic, so for the monad $\top_\partial := F_\partial U_\partial$, there is a natural equivalence

$$c\text{Mod}(A) \simeq c_+\text{Mod}(A)^{\top_\partial}.$$

In fact, we can go further than this. By [22, Section 5.1], the monad Symm distributes over \top_∂ , so the composite monad $\text{Symm} \circ \top_\partial$ is another monad. Moreover,

$$c\text{Alg}(A) \simeq c_+\text{Mod}(A)^{(\text{Symm} \circ \top_\partial)}.$$

We wish to modify this slightly, since we are only interested in augmented cosimplicial A -algebras, or equivalently nonunital cosimplicial A -algebras (taking augmentation ideals). We thus replace Symm with $\text{Symm}^+ := \bigoplus_{n>0} S^n$, and set $\top := \text{Symm}^+ \circ \top_\partial$. Then $c_+\text{Mod}(A)^\top$ is equivalent to the category $c\text{NAlg}(A)$ of nonunital commutative cosimplicial A -algebras.

Definition 6.25 Define a functor $E_r: \text{Alg}_R \rightarrow QM^*$ by first forming the almost cosimplicial module $\bar{V}(A^r) \in c_+\text{Mod}(A)$ as

$$\bar{V}(A^r)^n := A^{r^n} = \overbrace{(A^r) \otimes_A (A^r) \otimes_A \dots \otimes_A (A^r)}^n,$$

with operations dual to the operations of Definition 6.21, but without ∂^0 . We then let $\bar{V}_+(A^r) := \ker(\bar{V}(A^r) \rightarrow A)$, where the A is the constant diagram $\bar{V}(A^0)$, so

$$\bar{V}_+(A^r)^n = \ker((\partial^1)^n: \bar{V}(A^r)^n \rightarrow \bar{V}(A^r)^0).$$

We now set $E_r(A)$ to be the quasicomonoid $E_r(A) := E(\bar{V}_+(A^r))$ of Section 5.2.1, given by

$$E_r^n(A) = \text{Hom}_{c_+\text{Mod}(A)}(\top_A^n \bar{V}_+(A^r), \bar{V}_+(A^r)).$$

Definition 6.26 Define $\mathcal{D}\text{el}(\underline{E}_r)^\sharp: d\mathcal{N}_R^\flat \rightarrow \text{Gpd}$ to be the stackification of $\mathcal{D}\text{el}(\underline{E}_r)$ in the strict Zariski topology of Definition 2.17. Likewise, we define the simplicial groupoid valued functor $\underline{\mathcal{D}\text{el}}(\underline{E}_r)^\sharp$ on $d\mathcal{N}_R^\flat$ by stackifying levelwise, so we have that $(\underline{\mathcal{D}\text{el}}(\underline{E}_r)^\sharp)_n = (\underline{\mathcal{D}\text{el}}(\underline{E}_r)_n)^\sharp$.

Definition 6.27 By Lemma 5.17, $\text{MC}(E_r(A))$ is equivalent to the groupoid of pointed simplicial affine schemes X over A for which $U_\partial O(X) \cong \bar{V}(A^r) \in c_+\text{Mod}(A)$. We then define

$$\text{MC}_g(E_r(A)) \subset \text{MC}(E_r(A))$$

to consist of simplicial schemes of the form BG , for G a group scheme. By Lemma 6.20, this inclusion of functors is formally étale.

We define $\mathcal{D}\text{el}_g(E_r)$, $\underline{\text{MC}}_g(\underline{E}_r)$, $\text{MC}_g(E_r^\sharp)$ and so on similarly.

Proposition 6.28 *The functor $\bar{W}\underline{\mathcal{D}\text{el}}_g(\underline{E}_r)^\sharp \rightarrow \mathbb{S}$ is representable by an almost finitely presented derived geometric 1–stack. Moreover,*

$$\bar{W}\underline{\mathcal{D}\text{el}}_g(\underline{E}_r)^\sharp \simeq \bar{W}\underline{\mathcal{D}\text{el}}_g(\underline{E}_r)^\sharp \simeq \underline{\text{Del}}_g(E_r^\sharp),$$

where the last is defined using the quasicomonoid sheafification of Definition 5.49.

Proof The proofs of Propositions 6.4 and 6.17 carry over. The only differences lie in a straightforward check that $\mathcal{D}\text{el}_g(E_r)^\sharp$ is locally of finite presentation, and in the calculation of cohomology groups.

For the comonad $\perp := \text{Symm}^+ \circ F_\partial U_\partial$ on $c\text{NAlg}(A)$, we get a canonical simplicial resolution $\perp_\bullet S$, given by $\perp_n S := \perp^{n+1} S$. For $A \in \text{Alg}_R$, the proof of [22, Lemma 5.7] then shows that $A \oplus (\perp_\bullet S)^m$ is a cofibrant resolution of $A \oplus S^m$ for all m , whenever S^m is projective as an A –module. If we set $\mathbb{L}_\bullet^\perp(S) := \Omega((A \oplus \perp_\bullet S)/A)$, this means that the simplicial complex $\mathbb{L}_\bullet^\perp(S)^m$ is a model for the cotangent complex of $A \oplus S^m$.

If S is levelwise projective as an A –module, then by [22, Lemma 5.8], $\mathbb{L}_n^\perp(S)$ is a projective object of $c\text{Mod}(S)$. Thus, for $S \in E_r^\sharp(A)$, Lemma 5.47 gives that

$$D_{[S]}^i(\underline{\text{Del}}_g(\underline{E}_r)^\sharp, M) \cong \mathbb{E}\text{xt}_S^{i+1}(\mathbb{L}_\bullet^\perp(S), S \otimes_A M),$$

which satisfies the finiteness conditions of Theorem 1.19. □

Definition 6.29 Given a flat group scheme over a ring A , follow Illusie [16, Section 2.5.1] in defining the A –complex $\chi^{G/A}$ by

$$\chi^{G/A} := \mathbf{L}e^* \mathbb{L}^{G/A},$$

where $e: \text{Spec } A \rightarrow G$ is the unit of the group structure. This has a canonical G -action, and we write $\chi^{G/A}$ for the associated complex on BG .

As in [16, Section 4.1], $\chi^{G/A}$ is perfect and concentrated in chain degrees 0, 1. We set $\omega^{G/A} := H_0(\chi^{G/A})$, with $\omega^{G/A}$ the associated sheaf on BG .

Definition 6.30 For a cosimplicial ring S , we make $c\text{Mod}(S)$ into a simplicial category by setting (for $K \in \mathbb{S}$)

$$(M^K)^n := (M^n)^{K_n},$$

as an S^n -module. This has a left adjoint, which we denote by $M \mapsto M \otimes K$. Given a cofibration $K \hookrightarrow L$ in \mathbb{S} , we write $M \otimes (L/K) := (M \otimes L)/(M \otimes K)$.

Given $M \in c\text{Mod}(S)$, define \underline{M} to be the bicosimplicial complex given in horizontal level i by $\underline{M}^{i\bullet} = M \otimes \Delta^i$. Let $N_c \underline{M}$ be the cochain complex in $c\text{Mod}(S)$ given by taking the horizontal cosimplicial normalisation of Definition 4.6.

Lemma 6.31 Given an affine group scheme G over A , with $\Gamma(G, \mathbb{O}_G)$ locally free of rank r over A , let $S \in \mathcal{D}\text{el}(E_r)^\sharp(A)$ be the associated cosimplicial ring

$$S^n := \ker(\Gamma((BG)_n, \mathbb{O}_{(BG)_n}) \rightarrow A).$$

Then for $M \in \text{Mod}_A$,

$$D_{[S]}^i(\underline{\text{Del}}_g(\underline{E}_r^\sharp), M) \cong \mathbb{E}\text{xt}_{BG}^i(\mathbb{L}^{BG/A}, \ker(\mathbb{O}_{BG} \otimes_A M \rightarrow M)).$$

In particular,

$$D_{[S]}^i(\underline{\text{Del}}_g(\underline{E}_r^\sharp), M) \cong \mathbb{E}\text{xt}_{BG}^{i+2}(\chi^{G/A}, \mathbb{O}_{BG} \otimes_A M)$$

for $i \geq 1$. For low degrees, there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{BG}(\omega^{G/A}, \mathbb{O}_{BG} \otimes_A M) & \longrightarrow & \text{Hom}_A(\omega^{G/A}, M) & & \\ D_{[S]}^{-1}(\underline{\text{Del}})_g(\underline{E}_r^\sharp, M) & \longrightarrow & \text{Ext}_{BG}^1(\chi^{G/A}, \mathbb{O}_{BG} \otimes_A M) & \longrightarrow & \text{Ext}_A^1(\chi^{G/A}, M) & & \\ D_{[S]}^0(\underline{\text{Del}})_g(\underline{E}_r^\sharp, M) & \longrightarrow & \text{Ext}_{BG}^2(\chi^{G/A}, \mathbb{O}_{BG} \otimes_A M) & \longrightarrow & 0. & & \end{array}$$

Proof Write $L := \mathbb{L}_\bullet^\perp(S)$, defined as in the proof of Proposition 6.28, and recall that this is a projective object of $c\text{Mod}(S)$. Observe that in the terminology of [27], $\text{Spec}(A \oplus S)$ is a derived fppf 1-hypermultipoid, and a derived Artin 1-hypermultipoid whenever G is smooth.

If G is smooth, then [27, Proposition 7.21] shows that

$$\mathbb{E}\text{xt}_S^i(\text{Tot}N_c \underline{L}, S \otimes_A M) \cong \mathbb{E}\text{xt}_{BG}^i(\mathbb{L}^{BG/A}, \ker(\mathbb{O}_{BG} \otimes_A M \rightarrow M)),$$

where Tot is the total complex functor. For general G , the same formula holds, since [27, Proposition 7.11] only uses the Artin hypothesis to prove that $\text{Tot}N_c N^s \underline{L}$ is projective, while the descent argument from the proof of [27, Proposition 7.13] works for all faithfully flat morphisms.

Now, [27, Lemmas 2.18 and 7.4] combine to show that $N_c^i \underline{L}$ is acyclic for $i > 1$, while $N_c^1 \underline{L}$ is the pullback to BG of the cotangent complex of a trivial relative derived 1–hypergroupoid. It then follows from [27, Lemma 2.9] that there are canonical isomorphisms

$$\text{Ext}_S^*(N_c^1 \underline{L}, P) \cong \text{Ext}_{S^0}^*((N_c^1 \underline{L})^0, P^0)$$

for all $P \in c\text{Mod}(S)$. Since $S^0 = 0$, this means that

$$\mathbb{E}xt_S^i(\text{Tot}N_c N^s \underline{L}, S \otimes_A M) \cong \mathbb{E}xt^i(L, S \otimes_A M).$$

Thus, combined with the proof of Proposition 6.28, we get

$$D_{[S]}^i(\underline{\text{Del}}_g(E_r^\sharp), M) \cong \mathbb{E}xt_{BG}^i(\mathbb{L}^{BG/A}, \ker(\mathbb{O}_{BG} \otimes_A M \rightarrow M)).$$

Finally, $\mathbb{L}^{BG/A} \simeq \chi^{G/A}[1]$, so the exact sequence $0 \rightarrow S \rightarrow A \oplus S \rightarrow A \rightarrow 0$ of S –modules gives the required long exact sequence. □

Proposition 6.32 *For $A \in s\mathcal{N}_R^\flat$, the space $\underline{\text{Del}}_g(E_r^\sharp)(A)$ is functorially weakly equivalent to the nerve $\overline{W}\mathfrak{M}(A)$ of the ∞ –groupoid $\mathfrak{M}(A)$ of pointed derived geometric 1–stacks X over A for which $X \otimes_A^{\mathbb{L}} \pi_0 A$ is weakly equivalent to the nerve of a flat rank r group scheme over $\pi_0 A$.*

Proof We work along the same lines as Proposition 6.6. As a consequence of Proposition 6.28 and Remark 1.30, it suffices to construct a natural transformation

$$\Phi: \mathfrak{D}\epsilon\mathfrak{l}_g(E_r)^\sharp \rightarrow \mathfrak{M}$$

of ∞ –groupoids, inducing equivalences on π^0 and isomorphisms on D^i of the nerves.

Given an object of $\mathfrak{D}\epsilon\mathfrak{l}_g(E_r)^\sharp(A)$, we get $M \in c_+\text{Mod}(A_0)$, locally isomorphic (over A_0) to $\overline{V}_+(A_0^r)$, together with elements

$$\omega_n \in \text{Hom}_{c_+\text{Mod}(A_0)}(\mathbb{T}^{n+1}M, M \otimes_{A_0} (A^{I^n})_0),$$

which satisfy the Maurer–Cartan relations of Definition 5.4. For the free functor $F: sc_+\text{Mod}(A) \rightarrow sc\text{NAlg}(A)$ from simplicial almost cosimplicial A –modules to nonunital simplicial cosimplicial commutative A –algebras, Definition 5.26 thus gives us a functor

$$\beta_F^*: \mathfrak{D}\epsilon\mathfrak{l}(E_r)^\sharp(A) \rightarrow sc\text{NAlg}(A).$$

We therefore get a functor $\phi: \mathcal{D}\epsilon\ell(E_h(A)) \rightarrow sc\text{Alg}(A) \downarrow A$ to the category of augmented simplicial cosimplicial commutative A -algebras, given by $\omega \mapsto A \oplus \beta_F^*(\omega)$. Moreover, it follows from Definition 5.26 that all objects in the image of ϕ are Reedy cofibrant. If $A \in \text{Alg}_R$ and ω corresponds to a group scheme G , then arguing as in the proof of Proposition 6.6, $\phi(\omega)$ is a cofibrant resolution of $O(BG)$ as a simplicial augmented cosimplicial commutative A -algebra. Therefore for arbitrary $A \in s\mathcal{N}_R^{\text{fb}}$,

$$\text{Spec}(\phi(\omega) \otimes_A^L \pi_0 A)$$

is a pointed fppf 1-hypergroupoid, and so $\text{Spec} \phi(\omega)$ is a pointed derived fppf 1-hypergroupoid.

We therefore define $\Phi(\omega)$ to be the homotopy fppf hypersheafification of $\text{Spec} \phi(\omega)$. By [27, Theorem 4.15], $\Phi(\omega)$ is a pointed derived geometric fppf 1-stack whenever $\omega \in \mathcal{D}\epsilon\ell_g(\underline{E}_r)^\sharp(A)$. By Toën [31, Theorem 0.1], this is the same as a derived geometric Artin 1-stack.

When $A \in \text{Alg}_R$, with ω corresponding to G , we have seen that $\Phi(\omega)$ is just the classifying stack BG . For arbitrary $A \in s\mathcal{N}_R^{\text{fb}}$, this means that $\Phi(\omega) \otimes_A^L \pi_0 A$ is of the form BG for some flat rank r group scheme G over $\pi_0 A$. Thus Φ indeed gives a functor $\Phi: \mathcal{D}\epsilon\ell_g(\underline{E}_r)^\sharp \rightarrow \mathfrak{M}$.

The arguments above have shown that $\pi^0 \Phi$ is an equivalence, since the space of group homomorphisms $G \rightarrow G'$ corresponds to the space of pointed morphisms $BG \rightarrow BG'$. To see that Φ gives isomorphisms

$$D_\omega^i(B\mathcal{D}\epsilon\ell(\underline{E}_r)^\sharp, M) \rightarrow D_{\Phi(\omega)}^i(\overline{W}\mathfrak{M}, M),$$

we combine Lemma 6.31 with [25, Theorem 3.35]. □

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