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The Gromoll filtration, *KO*–characteristic classes and metrics of positive scalar curvature

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Let X be a closed m–dimensional spin manifold which admits a metric of positive scalar curvature and let $\mathcal{R}^+(X)$ be the space of all such metrics. For any $g \in \mathcal{R}^+(X)$, Hitchin used the KO–valued α–invariant to define a homomorphism $A_{n-1}\colon \pi_{n-1}(\mathcal{R}^+(X), g) \to KO_{m+n}$. He then showed that $A_0 \neq 0$ if $m = 8k$ or $8k+1$ and that $A_1 \neq 0$ if $m = 8k-1$ or $8k$.

In this paper we use Hitchin's methods and extend these results by proving that

$$A_{8j+1-m} \neq 0 \quad \text{and} \quad \pi_{8j+1-m}(\mathcal{R}^+(X)) \neq 0$$

whenever $m \geq 7$ and $8j - m \geq 0$. The new input are elements with nontrivial α–invariant deep down in the Gromoll filtration of the group $\Gamma^{n+1} = \pi_0(\mathrm{Diff}(D^n, \partial))$. We show that $\alpha(\Gamma^{8j+2}_{8j-5}) \neq \{0\}$ for $j \geq 1$. This information about elements existing deep in the Gromoll filtration is the second main new result of this note.

57R60; 53C21, 53C27, 58B20

1 Introduction

Let n be greater than 4, let Θ_{n+1} denote the group of homotopy $(n+1)$–spheres and let $\Gamma^{n+1} = \pi_0(\mathrm{Diff}(D^n, \partial))$ denote the group of isotopy classes of orientation-preserving diffeomorphisms of the n–disc which are the identity near the boundary. There is the standard isomorphism $\Sigma\colon \Gamma^{n+1} \cong \Theta_{n+1}$, due to Smale [26] and Cerf [6]. Moreover, for all $0 < i \leq j$ there are homomorphisms

$$\lambda^n_{i,j}\colon \pi_j(\mathrm{Diff}(D^{n-j}, \partial)) \longrightarrow \pi_{j-i}(\mathrm{Diff}(D^{n-j+i}, \partial)).$$

The definitions of Σ and $\lambda^n_{i,j}$ are recalled in Section 2.1.

We denote $\lambda := \lambda^n_{i,i}$. In [8, Abschnitt 1], Gromoll defined the group

$$\Gamma^{n+1}_{i+1} := \lambda(\pi_i(\mathrm{Diff}(D^{n-i}, \partial)) \subset \Gamma^{n+1}$$

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and the corresponding filtration

$$0 = \Gamma^{n+1}_{n-2} \subset \Gamma^{n+1}_{n-3} \subset \cdots \subset \Gamma^{n+1}_{3} \subset \Gamma^{n+1}_{2} = \Gamma^{n+1}.$$

We say that $f \in \Gamma^{n+1}$ has Gromoll filtration i if $f \in \Gamma^{n+1}_i \setminus \Gamma^{n+1}_{i+1}$. The identity $\Gamma^{n+1} = \Gamma^{n+1}_2$ is due to Cerf [6], as pointed out by Antonelli, Burghelea and Kahn [2]. The equality $\Gamma^{n+1}_{n-2} = 0$ follows from Hatcher's proof [10] of the Smale Conjecture.

Starting with Novikov [22], authors have used the homomorphisms $\lambda^n_{i,j}$ to explore the homotopy type of $\mathrm{Diff}(D^n, \partial)$. For example, Burghelea and Lashof [5, Theorem 7.4] show that there is an infinite sequence $\{(p_i, k_i, m_i)\}$ of integer triples with p_i odd primes, $\lim_{i\to\infty} m_i/k_i = 0$ and

$$\pi_{k_i}(\mathrm{Diff}(D^{m_i}, \partial)) \otimes \mathbb{Z}/p_i \neq 0.$$

Later, Hitchin [12, Section 4.4] used the homomorphisms $\lambda^n_{i,j}$ to investigate the homotopy type of the space of positive scalar curvature metrics on a closed manifold. In this paper we extend the results of [5] and [12, Section 4.4].

Hitchin's main tool is the α–invariant, the KO–valued index of the real Dirac operator of a closed spin manifold. Since an exotic sphere carries a unique spin structure, we get an induced homomorphism

$$\alpha\colon \Gamma^{m+1} \xrightarrow{\cong} \Theta_{m+1} \longrightarrow KO_{m+1}.$$

Our first main result shows that the Gromoll filtration of some $(8k+2)$–dimensional exotic spheres with nontrivial α–invariant is quite deep.

1.1 Theorem *For all $j \geq 1$ there is an element $f_j \in \pi_{8j-6}(\mathrm{Diff}(D^7, \partial))$ such that $\alpha(\lambda(f_j)) \neq 0$ and $2f_j = 0$. Hence $\alpha(\Gamma^{8j+2}_{8j-5}) \neq \{0\}$ and for all $0 \leq i \leq 8j-6$, $\lambda^{8j+1}_{i,8j-6}(f_j) \in \pi_{8j-6-i}(\mathrm{Diff}(D^{7+i}, \partial))$ is a nontrivial element of order 2.*

1.1 Positive scalar curvature

Let X be a closed spin manifold of dimension m and let $\mathcal{R}^+(X)$ denote the space of positive scalar curvature metrics on X. The Lichnerowicz formula entails that the first obstruction to the existence of a positive scalar curvature metric on X is the index of the Dirac operator defined by its spin structure. This is an element $\mathrm{ind}(X) \in KO_m$ which gives rise to a ring homomorphism

$$\alpha\colon \Omega^{\mathrm{spin}}_* \longrightarrow KO_*, \quad [X] \longmapsto \mathrm{ind}(X).$$

When X is simply connected of dimension ≥ 5, Stolz [27] proved that $\mathcal{R}^+(X) \neq \phi$ if and only if $\alpha(X) = 0$. In general, the question of whether $\mathcal{R}^+(X) \neq \phi$ is a deep problem which remains open; see for example Rosenberg [24] and the second author [25].

If $\mathcal{R}^+(X) \neq \varnothing$ we equip it with the C^∞–topology and go on to investigate this topological space. Note that $\mathrm{Diff}(X)$ acts on $\mathcal{R}^+(X)$ via pull-back of metrics and so fixing g defines a map $T\colon \mathrm{Diff}(X) \to \mathcal{R}^+(X)$, $h \mapsto h^*g$. Moreover, fixing $D^m \subset X$ defines an inclusion $i\colon \mathrm{Diff}(D^m, \partial) \to \mathrm{Diff}(X)$ via extension by the identity.

Hitchin observed in his thesis [12, Theorem 4.7] that sometimes nonzero elements in $\pi_*(\mathrm{Diff}(D^m, \partial))$ yield, via the induced action of $\mathrm{Diff}(D^m, \partial)$ on $\mathcal{R}^+(X)$, nonzero elements in $\pi_*(\mathcal{R}^+(X)) := \pi_*(\mathcal{R}^+(X), g)$. More precisely, Hitchin [12, Proposition 4.6] (see Section 2.5), defines a homomorphism

$$A_{n-1}\colon \pi_{n-1}(\mathcal{R}^+(X)) \longrightarrow KO_{m+n}$$

and shows that the composition

$$C_{n-1}\colon \pi_{n-1}(\mathrm{Diff}(D^m, \partial)) \xrightarrow{i_*} \pi_{n-1}(\mathrm{Diff}(X)) \xrightarrow{T_*} \pi_{n-1}(\mathcal{R}^+(X)) \xrightarrow{A_{n-1}} KO_{m+n}$$

is nontrivial for $n = 1$ and $m = 8k, 8k+1$ and for $n = 2$ and $n = 8k-1, 8k$.

Hitchin's method exploited the at the time known facts that $\alpha(\Gamma_1^{8j+1}) \neq \{0\}$ and $\alpha(\Gamma_2^{8j+2}) \neq \{0\}$. With our refined knowledge about the nonzero images $\alpha(\Gamma_{8j-5}^{8j+2})$, we obtain the following corollary using the same method as Hitchin.

1.2 Corollary *Let X be a spin manifold of dimension $m \geq 7$ with $g \in \mathcal{R}^+(X)$ and let f_j be as in Theorem 1.1. Then for all $j \in \mathbb{Z}$ such that $8j + 1 - m \geq 0$, $C_{8j+1-m}(\lambda^{8j+1}_{m-7,8j-6}(f_j)) \neq 0 \in KO_{8j+2}$. In particular, the homomorphism*

$$A_{8j+1-m}\colon \pi_{8j+1-m}(\mathcal{R}^+(X)) \longrightarrow KO_{8j+2}$$

is a split surjection and for all such (X, g) the graded group $\pi_*(\mathcal{R}^+(X))$ contains nontrivial two-torsion in infinitely many degrees.

To our knowledge, these examples and those of Hanke, Steimle and the second author [9] are the first examples where $\pi_k(\mathcal{R}^+(X))$ is shown to be nontrivial when $k > 1$. In contrast to [9], Corollary 1.2 also shows that $\pi_*(\mathcal{R}^+(X))$ is nontrivial in infinitely many degrees. However, note that by construction the elements of $\pi_*(\mathcal{R}^+(X))$ found in Corollary 1.2 vanish under the action of $\mathrm{Diff}(X)$, ie in $\pi_*(\mathcal{R}^+(X)/\,\mathrm{Diff}(X))$. In contrast to this in [9] the first examples of elements $x \in \pi_k(\mathcal{R}^+(X))$ which remain nontrivial by pullback with arbitrary families in $\mathrm{Diff}(X)$ are constructed for arbitrarily large k. That $\mathcal{R}^+(X)/\mathrm{Diff}(X)$ often has infinitely many components is already proved in Botvinnik and Gilkey [3], Lawson Jr and Michelsohn [17] and Piazza and the second author [23].

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2 The Gromoll filtration of Hitchin spheres

In this Section we prove Theorem 1.1 and Corollary 1.2. Section 2.1 recalls methods from smoothing theory which give a second definition of the Gromoll filtration. Section 2.2 reviews the Kervaire–Milnor analysis of the group of homotopy spheres. Section 2.3 recalls results of Adams from stable homotopy theory and their relation to the KO–index theory due to Milnor. Section 2.4 shows how nontrivial compositions in the stable homotopy groups of spheres lead to nonzero elements deeper in the Gromoll filtration and so proves Theorem 1.1.

2.1 The groups Θ_{n+1}, Γ^{n+1} and $\pi_{n+1}(PL/O)$

Let $n \geq 5$. Recall that Θ_{n+1} is the group of oriented diffeomorphism classes of homotopy $(n+1)$–spheres, that by definition $\Gamma^{n+1} = \pi_0(\mathrm{Diff}(D^n, \partial))$ and recall also the space PL/O which will be defined below. In this subsection we review the three fundamental isomorphisms Σ, Ψ and M_* appearing in the following diagram:

$$\begin{tikzcd}
\Gamma^{n+1} \arrow[rr, "\Sigma"] \arrow[dr, "M_*"'] & & \Theta_{n+1} \arrow[dl, "\Psi"] \\
& \pi_{n+1}(PL/O) &
\end{tikzcd}$$

We then prove that the diagram commutes: a point which seems to have been implicit in the literature.

Given a mapping class $f \in \Gamma^{n+1}$ we may build a homotopy $(n+1)$–sphere Σ_f by first extending f by the identity map to a diffeomorphism $\bar{f}\colon S^n \to S^n$ and then setting $\Sigma_f := D^{n+1} \cup_{\bar{f}} D^{n+1}$. In this way we obtain the map, which is well known to be a homomorphism,

$$\Sigma\colon \Gamma^{n+1} \longrightarrow \Theta_{n+1}, \quad f \longmapsto \Sigma_f. \tag{2.1}$$

By [26] Σ is onto and by [6] Σ is injective.

Next let O_k and PL_k denote the k–dimensional orthogonal group and the group of piecewise linear homeomorphisms of k–dimensional Euclidean space fixing the origin and let $O := \lim_{k\to\infty} O_k$ and $PL := \lim_{k\to\infty} PL_k$ denote the corresponding stable groups. There are inclusions $O_k \to PL_k$ with quotients PL_k/O_k and we obtain the space $PL/O = \lim_{k\to\infty}(PL_k/O_k)$ along with stabilization maps $S\colon PL_k/O_k \to PL/O$. The fundamental theorem of smoothing theory applied to the $(n+1)$–sphere (see [11], [16] and also [15, Theorem 7.3]) states that there is an isomorphism

$$\Psi_{n+1}\colon \Theta_{n+1} \cong \pi_{n+1}(PL/O). \tag{2.2}$$

A third fundamental result is due to Morlet (unpublished) and Burghelea and Lashof [5, Theorems 4.4, 4.6].

2.3 Theorem [5, Theorem 4.4] *There is a homotopy equivalence of commutative H–spaces*

$$M_n\colon \mathrm{Diff}(D^n, \partial) \simeq \Omega^{n+1}(PL_n/O_n)$$

such that the composition

$$\pi_0 \,\mathrm{Diff}(D^n, \partial) \xrightarrow{M_{n*}} \pi_0\Omega^{n+1}(PL_n/O_n) \xrightarrow{S_*} \pi_0\Omega^{n+1}(PL/O) = \pi_{n+1}(PL/O)$$

yields an isomorphism

$$M_*\colon \Gamma^{n+1} \cong \pi_{n+1}(PL/O).$$

Here S_* is induced by the stabilization map $\Omega^{n+1}(PL_n/O_n) \to \Omega^{n+1}(PL/O)$.

To give the alternative description of the Gromoll filtration, we use the homomorphisms

$$\lambda^n_{i,j}\colon \pi_j(\mathrm{Diff}(D^{n-j}, \partial)) \longrightarrow \pi_{j-i}(\mathrm{Diff}(D^{n-j+i}, \partial))$$

from Section 1. Here we represent $a \in \pi_j(\mathrm{Diff}(D^{n-j}, \partial))$ by a map

$$a\colon [0,1]^j \longrightarrow \mathrm{Diff}([0,1]^{n-j}, [0,1]^{n-j})$$

such that the value of a is the identity map near the boundary of $[0,1]^j$ and such that each $a(x)$ is a diffeomorphism which restricts to the identity near the boundary of $[0,1]^{n-j}$. The class $\lambda^n_{i,j}(a)$ is then represented by the map

$$\lambda^n_{i,j}(a)\colon [0,1]^{j-i} \longrightarrow \mathrm{Diff}([0,1]^{n-j}\times[0,1]^i, [0,1]^{n-j}\times[0,1]^i) \tag{2.4}$$

with $\lambda^n_{i,j}(a)(x)(t,y) = (a(x,y)(t), y)$. Indeed the formula (2.4) implies that if we use Ω to denote the space of differentiable loops, then there are maps

$$\Lambda^n_{i,j}\colon \Omega^j \,\mathrm{Diff}(D^{n-j}, \partial) \longrightarrow \Omega^{j-1}\,\mathrm{Diff}(D^{n-j+i}, \partial)$$

which induce the homomorphisms $\lambda^n_{i,j}$.

2.5 Lemma (cf [4, Theorem 1.3]) *Let $i_n\colon PL_n/O_n \to PL_{n+1}/O_{n+1}$ be the canonical inclusion and let ΩM_n be the map of smooth loop spaces induced by M_n and assume $n \geq 4$. Then the following diagram is homotopy commutative:*

$$\begin{tikzcd}
\Omega\,\mathrm{Diff}(D^n,\partial) \arrow[r, "\Omega M_n"] \arrow[d, "\lambda^n_{1,1}"] & \Omega^{n+2}(PL_n/O_n) \arrow[d, "\Omega^{n+2}(i_n)"] \\
\mathrm{Diff}(D^{n+1},\partial) \arrow[r, "M_{n+1}"] & \Omega^{n+2}(PL_{n+1}/O_{n+1})
\end{tikzcd}$$

Proof The corresponding statement for $n \neq 4$ with PL_n replaced by Top_n is given in [4, Theorem 1.3] where Burghelea considers the map $h_n\colon \mathrm{Diff}(D^n,\partial) \to \Omega^{n+1}(Top_n/O_n)$. And indeed Burghelea remarks [4, page 9] that the analogous versions of his results hold when Top_n is replaced by PL_n.

We give a somewhat indirect argument based on the work of Kirby and Siebenmann which deduces the commutativity of the diagram above from [4, Theorem 1.3]. By definition the map h_n factors through M_n and the canonical map $\pi_n\colon PL_n/O_n \to Top_n/O_n$:

$$h_n = \pi_n \circ M_n\colon \mathrm{Diff}(D^n,\partial) \longrightarrow \Omega^{n+1}(PL_n/O_n) \longrightarrow \Omega^{n+1}(Top_n/O_n).$$

Now there is a fibration sequence

$$\Omega^{n+1}(PL_n/O_n) \longrightarrow \Omega^{n+1}(Top_n/O_n) \longrightarrow \Omega^{n+1}(Top_n/PL_n)$$

and for $n \geq 5$ there is, by [14, Essay V, 5.0 (1)], a homotopy equivalence

$$Top_n/PL_n \simeq K(\mathbb{Z}/2,3).$$

Hence the space $\Omega^{n+1}(Top_n/PL_n)$ is contractible and the map π_n above is a homotopy equivalence. It follows that the commutativity of Burghelea's diagram [4, Theorem 1.3] entails the commutativity of the diagram above. □

An immediate consequence of Theorem 2.3 and [4, Theorem 1.3] is the following alternative definition of the Gromoll filtration.

2.6 Corollary $\Gamma^{n+1}_{k+1} = M_*^{-1} S_*\big(\pi_{n+1}(PL_{n-k}/O_{n-k})\big)$.

The following lemma is presumably well known and in particular is implicit in [5]. Since we could not find a reference, we give a proof.

2.7 Lemma $M_* = (\Psi \circ \Sigma)\colon \Gamma^{n+1} \xrightarrow{\cong} \pi_{n+1}(PL/O)$.

Proof We use the description of $\Psi\colon \Theta_{n+1} \cong \pi_{n+1}(PL/O)$ given in the proof of [19, Theorem 6.48]. Given an exotic sphere Σ_f obtained from a diffeomorphism $f \in \mathrm{Diff}(D^n, \partial)$, take the PL–homeomorphism $u\colon \Sigma_f \cong S^{n+1}$ to the standard sphere coming from the Alexander trick. There is an associated "derivative" map between the PL–microbundles of Σ_f and S^{n+1}. Using the smooth structures, these PL–bundles are induced from the smooth tangent bundles which are of course vector bundles. Pulling back with u to S^{n+1}, we then have two O_{n+1}–structures on the same PL_{n+1}–bundle over S^{n+1}, and the difference of the lifts of structure group gives a pointed map $S^{n+1} \to PL_{n+1}/O_{n+1}$. By stabilization we get an element of $\pi_{n+1}(PL/O)$, which is by definition $\Psi(\Sigma_f)$.

On the other hand, the map $M_*\colon \pi_0(\mathrm{Diff}(D^n, \partial)) \to \pi_{n+1}(PL/O)$ from [5] is defined (after we strip off the technicalities associated to the use of simplicial methods) by first looking at the path $\gamma\colon [0,1] \to PL(D^n, \partial)$ obtained by applying the Alexander trick to f, with induced loop $\bar{\gamma}\colon [0,1] \to PL(D^n, \partial)/\mathrm{Diff}(D^n, \partial)$. The latter corresponds to the inverse of f under the boundary map of the fibration with fiber $\mathrm{Diff}(D^n, \partial)$:

$$PL(D^n, \partial) \longrightarrow B\mathrm{Diff}(D^n, \partial) = PL(D^n, \partial)/\,\mathrm{Diff}(D^n, \partial);$$

compare the proof of [5, Theorem 4.2]. The path of PL–derivatives $t \mapsto D(\gamma_t)$ gives, as above by comparing the pullbacks of the vector bundle structure on the PL–microbundle of D^n to the standard vector bundle structure, a loop of maps from (D^n, ∂) to PL_n/O_n, ie a map $S^{n+1} \to PL_n/O_n$. By [5, proof of 4.2 and Section 1], its stabilization represents $M_*(\psi) \in \pi_{n+1}(PL/O)$.

Observe that the family of PL–homeomorphisms $D^n \to D^n$ just constructed, extended by the identity over a "second hemisphere", patch together to the PL–homeomorphism between the homotopy sphere Σ_f and S^{n+1} used in the definition of $\Psi \circ \Sigma$. Moreover, if we stabilize the family of differentials by the identity of the vertical direction, we obtain the differential of that PL–homeomorphism. Finally, the underlying vector bundle structures on the PL–microbundles patch together and stabilize to the vector bundle structures on the PL–microbundles of Σ_f and S^{n+1} encountered above. It follows that the stable comparison maps $S^{n+1} \to PL/O$ coincide, ie $M_* = \Psi \circ \Sigma$. □

2.8 Remark It is interesting to observe that $\Psi \circ \Sigma$ factors by construction through $\pi_{n+1}(PL_{n+1}/O_{n+1})$, whereas M_* even factors through $\pi_{n+1}(PL_n/O_n)$.

2.2 Homotopy spheres

In this subsection we review a number of important isomorphisms used to study the group of homotopy spheres Θ_{n+1}. More information and proofs can be found in

[19, 6.6] and [18, Appendix]. Let $G := \lim_{k\to\infty} G(k)$ denote the stable group of homotopy self-equivalences of spheres, let π_i^S denote the i^{th} stable stem and let Ω_i^{fr} denote the i–dimensional framed bordism group. We have isomorphisms

$$\pi_i(G) \cong \pi_i^S \cong \Omega_i^{\text{fr}},$$

where the first isomorphism may be found in [20, Corollary 3.8] and the second is the Pontrjagin–Thom isomorphism.

The canonical map $O \to G$ induces the stable J–homomorphism on homotopy groups $J_i\colon \pi_i(O) \to \pi_i(G)$. The group $\mathrm{im}(J_i) \subset \pi_i(G)$ is a cyclic summand and the group $\mathrm{coker}(J_i)$ maps isomorphically onto the torsion subgroup of $\pi_i(G/O)$ under the canonical map $q\colon G \to G/O$. Moreover there is an isomorphism $\pi_i(G/O) \cong \Omega_i^{\text{alm}}$, where Ω_*^{alm} denotes almost framed bordism (cycles are manifolds with a chosen base point and a framing of the stable normal bundle on the complement of this base point).

2.9 Theorem [13, Section 4] *For $n \geq 4$ the abelian group Θ_{n+1} is finite and lies in an exact sequence*

$$0 \longrightarrow bP_{n+2} \longrightarrow \Theta_{n+1} \xrightarrow{\Phi} \mathrm{coker}(J_{n+1}),$$

where bP_{n+2} is the finite cyclic subgroup of homotopy spheres bounding parallelizable manifolds. By [13, Theorem 6.6], Φ is surjective if n is odd.

2.10 Proposition *The canonical map $p\colon PL/O \to G/O$ satisfies*

$$q_* \circ \Phi = p_* \circ \Psi\colon \Theta_{n+1} \longrightarrow \pi_{n+1}(G/O).$$

Proof The statement follows from the commutativity of the squares

$$\begin{tikzcd}
 & \pi_{n+1}(PL/O) \arrow[d, "p_*"] & \Theta_{n+1} \arrow[l, "\Psi"', "\cong"] \arrow[d] \\
 & \pi_{n+1}(G/O) & \Omega_{n+1}^{\text{alm}} \arrow[l, "\cong"'] \\
\pi_{n+1}^S \arrow[r, "\cong"] & \pi_{n+1}(G) \arrow[u, "q_*"] & \Omega_{n+1}^{\text{fr}}, \arrow[l, "\cong"'] \arrow[u]
\end{tikzcd}$$

which is explained in [19, Theorem 6.48]. The homomorphism Φ is geometrically defined as the composition of the upper right homomorphism, the isomorphism $\Omega_{n+1}^{\text{alm}} \cong \pi_{n+1}(G/O)$ and the inverse of the isomorphism induced by q_* from $\mathrm{coker}(J_{n+1})$ to the torsion subgroup of $\pi_{n+1}(G/O)$. □

2.3 The α–invariant

Recall from [12, Section 4.2] that the α–invariant is the ring homomorphism $\alpha\colon \Omega_*^{\mathrm{Spin}} \to KO_*$ which associates to a spin bordism class the KO–valued index of the Dirac operator of a representative spin manifold. We also write α for the corresponding invariant on framed bordism:

$$\alpha\colon \Omega_*^{\mathrm{fr}} \longrightarrow \Omega_*^{\mathrm{Spin}} \longrightarrow KO_*. \tag{2.11}$$

Under the Pontrjagin–Thom isomorphism $\Omega_*^{\mathrm{fr}} \cong \pi_*^S$ the α–invariant has the following interpretation as Adams' d–invariant [1, Section 7], $d_{\mathbb{R}}\colon \pi_*^S \to KO_*$, which was used already in [12, page 44]; compare [21, Section 3].

2.12 Lemma *Under the Pontryagin–Thom isomorphism $\Omega_*^{\mathrm{fr}} \cong \pi_*^S$ the α–invariant $\alpha\colon \Omega_{8j+1}^{\mathrm{fr}} \to KO_{8j+1}$ may be identified with $d_{\mathbb{R}}\colon \pi_{8j+1}^S \to KO_{8j+1}$.*

Recall that KO_* satisfies Bott periodicity of period 8 with Bott generator $\beta \in KO_8 \cong \mathbb{Z}$. By [1, Theorems 7.18 and 12.13], for all $k \geq 1$ there are (not uniquely defined) Adams' elements $\mu_{8k+1} \in \pi_{8k+1}^S = \Omega_{8k+1}^{\mathrm{fr}}$ satisfying

$$\alpha(\mu_{8k+1}) = \alpha(\eta)\beta^k \neq 0 \in KO_{8k+1},$$

where $\eta \in \pi_1^S$ generates the 1–stem and $\alpha(\eta)$ generates KO_1. Since α is a ring homomorphism we see that $\alpha(\eta\mu_{8k+1}) = \alpha(\eta^2)\beta^k \neq 0 \in KO_{8k+2}$, and combining Lemma 2.12 with [1, Proposition 12.14] we have

$$\alpha(\mu_{8j+1} \cdot \mu_{8k+1}) = \alpha(\eta^2)\beta^{j+k} \neq 0 \in KO_{8(j+k)+2}. \tag{2.13}$$

Recall that an element $x \in \pi_j^S = \lim_k \pi_{j+k}(S^k)$ is said to *live on* S^k if there is $x_k \in \pi_{j+k}(S^k)$ which maps to x under the canonical homomorphism.

The next crucial property of the elements μ_{8k+1} is that (at least if we make suitable choices here) they all live on S^5.

2.14 Lemma *For suitable choices, the (not uniquely defined) homotopy class $\mu_{8j+1} \in \pi_{8j+1}^S$ lives on the 5–sphere and moreover there is $\mu_{8j+1,5} \in \pi_{8j+5}(S^5)$ with $2\mu_{8j+1,5} = 0$. It follows that there is a corresponding homotopy class $\mu_{8j+1,9} \in \pi_{8j+10}(S^9)$ of order 2.*

Proof The statement follows by carefully inspecting Adams' construction of the homotopy class $\mu_{8j+1} \in \pi_{8j+1}^S$, involving Toda brackets.

Let us recall that, given homotopy classes of maps $u\colon S^a \to S^b$, $v\colon S^b \to S^c$ and $w\colon S^c \to S^d$ such that $[v \circ u] = 0$ and $[w \circ v] = 0$, there is a set $\{w, v, u\}$ of homotopy classes of maps $S^{a+1} \to S^c$, the Toda brackets of w, v, u, a kind of secondary composition. The elements of the set depend on choices of null-homotopies for $v \circ u$ and $w \circ v$, and indeed (for $a \geq 1$) $\{w, v, u\}$ is a coset of $[Eu] \circ \pi_{b+1}(S^c) + \pi_{a+1}(S^b) \circ [w] \in \pi_{a+1}(S^c)$, where E denotes suspension.

Now, for the construction of the $\mu_{8j+1,5}$ on starts with a homotopy class $\alpha_1\colon S^{k+7} \to S^k$ of order 2 such that $\{2, \alpha_1, 2\}$ contains 0. Here 2 stands for the self map of the sphere of degree 2.

One then chooses inductively for $s > 1$ $\alpha_s\colon S^{k+8s-1} \to S^k$ to be any element in the Toda bracket $\{\alpha_{s-1}, 2, \alpha_1\}$. For notational simplicity we write α also instead of the appropriate suspension of it. Note that in this proof we follow Adams and use α_s to refer to a certain homotopy class. This should not be confused with the α–invariant of (2.11).

For the induction to work we have to show that $[2\alpha_s] = 0 \in \pi_{k+8s-1}(S^k)$. For this we use [28, Proposition 1.2 IV]: $\{\alpha_{s-1}, 2, \alpha_1\}2 = \alpha_{s-1} \circ \{2, \alpha_1, 2\} = 0$. The latter follows because by our induction hypothesis $[\alpha_{s-1} \circ 2] = 0$ and $\{2, \alpha_1, 2\}$ contains by assumption only multiples of 2.

Finally, we define $\mu_{8j+1,k-1}$ as any element in the Toda bracket $\{\eta_{,k-1}, 2, \alpha_j\}$. Here, we let $\eta_{,n}\colon S^{n+1} \to S^n$ represent (for $n \geq 3$) the generator of $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$.

To see that $\mu_{8j+1,k-1}$ is of order 2 we need some preparation:

If for $a \in \pi_{k+s}(S^k)$ we have that $\{2, a, 2\} = 2\pi_{k+s+1}(S^k) \subset \pi_{k+s+1}(S^k)$, then for arbitrary $x\colon S^r \to S^{k+s}$ and $y\colon S^k \to S^b$ also $\{2, a, 2x\} \subset 2\pi_{r+1}(S^k)$ and $\{2y, a, 2\} \subset 2\pi_{k+s+1}(S^b)$. Note that $\{2, a, 2x\}$ is a coset of

$$2\pi_{r+1}(S^k) + \pi_{k+s+1}(S^r) \circ 2Ex \subset 2\pi_{r+1}(S^k)$$

so it suffices to show that $0 \in \{2, a, 2x\}$, and similarly for $\{2y, a, 2\}$. Now the module property [28, Proposition 1.2 IV] implies $0 = 0 \circ x \in \{2, a, 2\} \circ x \subset \{2, a, 2x\}$, and in the same way $0 \in \{2, a, 2y\}$.

Now we show by induction that $\{2, \alpha_s, 2\}$ consists of the multiples of 2. By assumption this is true for $s = 1$. For the induction, we apply the Leibniz rule [28, Proposition 1.5], which says that $\{2, \alpha_s, 2\} = \{2, \{\alpha_{s-1}, 2, \alpha_1\}, 2\}$ (which is a coset of the multiples of 2) is congruent to the set

$$\{\{2, \alpha_{s-1}, 2\}, \alpha_1, 2\} + \{2, \alpha_{s-1}, \{2, \alpha_1, 2\}\}.$$

By the induction hypothesis and the above consideration, both these iterated Toda brackets only contain multiples of 2, and so $\{2, \alpha_s, 2\}$ must be the coset of 0 of the multiples of 2.

Next, using again [28, Proposition 1.2]

$$2\mu_{8j+1,k-1} \in \{\eta_{,k-1}, 2, \alpha_j\}2 = \eta_{,k-1} \circ \{2, \alpha_j, 2\} \subset \eta_{,k-1} \circ 2\pi_{k+8j}(S^k) = 0,$$

because $2\eta_{,k-1} = 0$ as long as $k \geq 4$.

Finally, we follow literally one of the proofs Adams gives to show that $\alpha(\mu_{8j+1})$ is nontrivial. This uses the fact, established in [1, page 68] that for the relevant dimension α coincides with Adams' homomorphism $e_{\mathbb{C}}$ (both considered to be maps to $\mathbb{R}/\mathbb{Z}$). To compute $e_{\mathbb{C}}(\mu_{8j+1})$ one can inductively apply [1, Theorem 11.1]. This theorem states that $e_{\mathbb{C}}\{x, 2, y\} = 2e_{\mathbb{C}}(x)e_{\mathbb{C}}(y)$ modulo $\mathbb{Z}$. Finally, one only has to use that $e_{\mathbb{C}}(\eta) = 1/2$ and $e_{\mathbb{C}}(\alpha) = 1/2$, which is established in the proof of [1, Theorem 12.13].

For the choice of α_1 we follow again the proof of [1, Theorem 12.13] which uses corresponding results of Toda. Indeed, in [28, Lemma 5.13] Toda checks that the element $\sigma''' \in \pi_{5+7}(S^5)$ of order 2 stabilizes to the element of order 2 in π_7^S. Moreover, with E still denoting the suspension, Toda shows in [28, Corollary 3.7] that $\{2, E\sigma''', 2\} \ni E\sigma'''\eta_{,13} = 2\sigma''\eta_{,13} = 0$ since $\eta_{,13}$ has order 2. Therefore, an appropriate choice is $\alpha_1 := E(\sigma''') = 2\sigma'' \in \pi_{6+7}(S^6)$. Here σ'' is Toda's notation for an element of order 4 in $\pi_{6+7}(S^6) \cong \mathbb{Z}/60$. □

2.15 Remark On the face of it, our construction of α_s and therefore μ_{8j+1} is slightly more general than Adams' construction which does seem not to allow for arbitrary elements in the Toda brackets involved in the inductive construction. Note, however that we have to use unstable Toda brackets, which means that the same construction, starting with larger k, might give rise to more elements in π_{8j+1}^S which do not live on the 5–sphere.

2.16 Remark Another proof of the existence of $\mu_{8j+1,5}$ comes from [7], where Curtis calculated the sphere of origin for many examples using the Adams spectral sequence and the restricted lower central series spectral sequence. In fact Curtis shows that elements of nontrivial d–invariant live on S^3. We gave an independent proof to avoid the task of checking how the notations from [7] match with those of [1] and to show that there is a $\mu_{8j+1,5}$ of order two.

2.4 Proof of Theorem 1.1

In this subsection we prove our main theorem. Since every homotopy sphere has a unique spin-structure we obtain the α–invariant on $\Gamma^{n+1} \cong \Theta_{n+1}$:

$$\alpha\colon \Gamma^{n+1} \longrightarrow \Omega^{\mathrm{Spin}}_{n+1} \longrightarrow KO_{n+1}.$$

Combining [21, Theorem 2 and proof], [1, Theorems 7.18 and 12.13] and Theorem 2.9 we see that for each $j > 1$ there is a homotopy $(8j-7)$–sphere $\Sigma_{\mu_{8j-7}} \in \Theta_{8j-7}$ representing $[\mu_{8j-7}] \in \operatorname{coker}(J_{8j-7})$. In particular we have the equation $\alpha(\Sigma_{\mu_{8j-7}}) = \alpha(\eta)\beta^{j-1} \neq 0 \in KO_{8j-7}$. By Cerf's Theorem [6], $\Gamma^9_2 = \Gamma^9_1$ and so we can find $g \in \pi_1(\mathrm{Diff}(D^7, \partial))$ such that $\Sigma(\lambda(g)) = \Sigma_{\mu_9}$. By (2.13) above,

$$\alpha(\Sigma_{\mu_9} \times \Sigma_{\mu_{8j-7}}) = \alpha(\eta^2)\beta^j \neq 0 \in KO_{8j+2}. \tag{2.17}$$

Recall the homotopy equivalence $M\colon \mathrm{Diff}(D^7, \partial) \simeq \Omega^8(PL_7/O_7)$ of Theorem 2.3 and consider the induced isomorphism

$$M_{7*}\colon \pi_1(\mathrm{Diff}(D^7, \partial)) \cong \pi_9(PL_7/O_7).$$

With $g \in \pi_1(\mathrm{Diff}(D^7, \partial))$ as above we have $M_{7*}(g) \in \pi_9(PL_7/O_7)$. Now let $\mu_{8j-7,9} \in \pi_{8j+2}(S^9)$ be an element of order 2 with $S(\mu_{8j-7,9}) = \mu_{8j-7} \in \pi^S_{8j-7}$ whose existence is proven in Lemma 2.14. The composition

$$M_{7*}(g) \circ \mu_{8j-1,9} \in \pi_{8j+2}(PL_7/O_7)$$

has order 2 and we define

$$f_j := M_{7*}^{-1}(M_{7*}(g) \circ \mu_{8j-7,9}) \in \pi_{8j-6}(\mathrm{Diff}(D^7, \partial))$$

so that $\lambda(f_j) \in \Gamma^{8j+2}_{8j-5}$. For $\Sigma_{f_j} := \Sigma(\lambda(f_j))$ we show below that

$$\alpha(\Sigma_{f_j}) = \alpha(\Sigma_{\mu_9} \times \Sigma_{\mu_{8j-7}}) \tag{2.18}$$

and so by (2.17) we have that $\alpha(\lambda(f_j)) = \alpha(\Sigma_{f_j}) = \alpha(\eta^2)\beta^j \neq 0 \in KO_{8j+2}$, which proves Theorem 1.1.

We prove Equation (2.18) using the following diagram, where $k = 8j+2$. We obtain the diagram by combining [5, page 14] and [19, Theorems 6.47, 6.48] and we claim that it commutes:

(2.19)

$$\begin{tikzcd}
\pi_1(\mathrm{Diff}(D^7,\partial))\times\pi_k(S^9) \arrow[d, "M_*\times\mathrm{id}"] & \pi_{k-8}(\mathrm{Diff}(D^7,\partial)) \arrow[r, "\Sigma\circ\lambda"] \arrow[d, "M_*", "\cong"'] & \Theta_k \arrow[d, "="] & \\
\pi_9(PL_7/O_7)\times\pi_k(S^9) \arrow[r, "\circ"] \arrow[d, "S_*\times\mathrm{id}"] & \pi_k(PL_7/O_7) \arrow[r, "\Psi^{-1}\circ S_*"] \arrow[d, "S_*"] & \Theta_k \arrow[d, "="] & \\
\pi_9(PL/O)\times\pi_k(S^9) \arrow[r, "\circ"] \arrow[d, "p_*\times\mathrm{id}"] & \pi_k(PL/O) \arrow[d, "p_*"] & \Theta_k \arrow[l, "\Psi"', "\cong"] \arrow[d] & \\
\pi_9(G/O)\times\pi^S_{k-9} \arrow[r, "\circ"] & \pi_k(G/O) & \Omega^{\mathrm{alm}}_k \arrow[l, "\cong"'] \arrow[r, "\alpha"] & KO_k \\
\pi_9(G)\times\pi^S_{k-9} \arrow[r, "\circ"] \arrow[u, "q_*\times\mathrm{id}"] & \pi_k(G) \arrow[u, "q_*"] & \Omega^{\mathrm{fr}}_k \arrow[l, "\cong"'] \arrow[r, "\alpha"] \arrow[u] & KO_k \arrow[u, "="] \\
\pi^S_9\times\pi^S_{k-9} \arrow[r, "\circ"] \arrow[u, "\cong"] & \pi^S_k \arrow[u, "\cong"] & \Omega^{\mathrm{fr}}_k \arrow[l, "\cong"'] \arrow[r, "\alpha"] \arrow[u, "="] & KO_k \arrow[u, "="]
\end{tikzcd}$$

Using the claimed commutativity of diagram (2.19) let us start in the second row with the pair

$$(M_{7*}(g),\mu_{8j-7,9})\in\pi_9(PL_7/O_7)\times\pi_{8j+2}(S^9).$$

Since $\Sigma(\lambda(g))=\Sigma_{\mu_9}$, the pair $(\mu_9,\mu_{8j-7})\in\pi^S_9\times\pi^S_{k-9}$ maps to the same element in $\pi_9(G/O)\times\pi^S_{k-9}$ as $(M_{7*}(g),\mu_{8j-7,9})$. We already checked in Equation (2.17) that (μ_9,μ_{8j-7}) is mapped in the bottom row to $\alpha(\eta^2)\beta^j\in KO_{8j+2}$. Finally, Σ_{f_j} is obtained from the element $\Sigma\circ\lambda\circ M_{7*}^{-1}(M_{7*}(g)\circ\mu_{8j-7,9})\in\Theta_{8k+2}$ in the top right corner of the diagram. By commutativity, its α–invariant is as desired.

Now we prove the commutativity of (2.19). The left part is taken from [5], the identification of the homotopy groups of PL/O, G/O, G with the bordism groups or Θ_k and the corresponding commutativity from [19, Section 6]. The only assertions which are not contained in those two references are the compatibility with α, which is clear, and, although implicitly stated in [5], the commutativity of the diagram

$$\begin{tikzcd}
\pi_{k-8}(\mathrm{Diff}(D^7,\partial)) \arrow[r, "\Sigma\circ\lambda"] \arrow[d, "M_*"', "\cong"] & \Theta_k \arrow[d, "="] \\
\pi_k(PL_7/O_7) \arrow[r, "\Psi^{-1}\circ S_*"] & \Theta_k.
\end{tikzcd}$$

This commutativity we have essentially proved in Lemma 2.7, one has additionally only to apply compatibility of the constructions with suspension.

2.20 Remark The argument above started from the statement $\Sigma^{-1}(\Sigma_{\mu_9}) \in \Gamma^9_2$. If one knew that a 9–dimensional Hitchin sphere Σ_{μ_9} had Gromoll filtration Γ^9_k for $2 < k \le 5$ then we could repeat the argument to conclude that $\alpha(\Gamma^{8j+2}_{8j-7+k}) \neq 0$. As of writing, it seems that nothing is known about the Gromoll filtration of 9–dimension Hitchin spheres beyond the Cerf–Hatcher bounds $\Sigma^{-1}(\Sigma_{\mu_9}) \in \Gamma^9_2$ and $\Gamma^9_6 = \{0\}$.

2.21 Remark In our construction, we crucially use the ring structure of KO_* and the nontriviality of the product of generators in KO_{8k+1}. This means that the interesting elements (with nontrivial α–invariant) we obtain are in $\pi_k(\mathrm{Diff}(D^n, \partial))$ with $k+n \equiv 1 \pmod 8$.

We expect that one can use Toda brackets (of an element in $\pi_*(PL_k/O_k)$ with elements of $\pi_*(S^n)$) to construct such elements in $\pi_k(\mathrm{Diff}(D^n, \partial))$ with $k+n \not\equiv 1 \pmod 8$. This we leave for future work.

2.5 Positive scalar curvature metrics: Corollary 1.2

To prove Corollary 1.2 one need only recall the arguments following [12, Proposition 4.6]: Let X be a closed m–dimensional spin-manifold ($m \ge 7$) and let $\mathcal{R}^+(X)$ be the space of positive scalar curvature metrics on X which we assume to be nonempty. Observe that the group of diffeomorphisms of X, $\mathrm{Diff}(X)$, acts on $\mathcal{R}^+(X)$ by pullback. In particular, fixing a metric $g \in \mathcal{R}^+(X)$, define the map

$$T\colon \mathrm{Diff}(X) \longrightarrow \mathcal{R}^+(X), \quad h \longmapsto h^* g.$$

Moreover, by fixing a k–disc $D^m \subset X$ and extending diffeomorphisms by the identity we obtain a map $i\colon \mathrm{Diff}(D^m, \partial) \to \mathrm{Diff}(X)$.

In [12, Proposition 4.6] Hitchin defines a homomorphism

$$A_{n-1}\colon \pi_{n-1}(\mathcal{R}^+(X)) \longrightarrow KO_{m+n}.$$

He shows then that the composite homomorphism

$$B_{n-1} := A_{n-1} \circ T_*\colon \pi_{n-1}(\mathrm{Diff}(X)) \longrightarrow \pi_{n-1}(\mathcal{R}^+(X), g_0) \longrightarrow KO_{m+n}$$

assigns to $\phi\colon S^{n-1} \to \mathrm{Diff}(X)$ the family index of the bundle of spin manifolds $X \to Z_\phi \to S^n$ obtained by the usual clutching construction. Moreover, in [12, Section 4.3, Proposition 4.4] Hitchin shows that if we start with $\phi\colon S^{n-1} \to \mathrm{Diff}(D^m, \partial)$ then $B(i_*(\phi)) = \alpha(\Sigma_\phi)$, where Σ_ϕ is the exotic $(n+m)$–sphere defined by $\lambda(\phi) \in \Gamma^{n+m}_n$.

Fix j with $8j+1 > m \geq 7$. We apply the argument above starting from f_j as in Theorem 1.1 and with

$$\phi := \lambda^{8j+1}_{m-7,8j-6}(f_j) \in \pi_{8j+1-m}(\mathrm{Diff}(D^m, \partial)).$$

By Theorem 1.1 we have that $2\phi = 0$ and that $\lambda(\phi) \in \Gamma^{8j+2}_{8j-5}$ satisfies $\alpha(\lambda(\phi)) \neq 0$. Pulling back the metric g by ϕ we obtain a continuous family of metrics in $\mathcal{R}^+(X)$ parameterized by S^{8j+1-m} and hence the homotopy class $T_* i_*(\phi) \in \pi_{8j+1-m}(\mathcal{R}^+(X))$ of order 2. By [12, Proposition 4.4], $A_{8j+1-m}(T_* i_*(\phi)) = \alpha(\lambda)$ and so generates $KO_{8j+2} \cong \mathbb{Z}/2$. This proves Corollary 1.2.

Appendix A: The Gromoll filtration: table of values

We think that our results about the Gromoll filtration and the existence of elements rather deep down with nontrivial α–invariant are interesting in their own right. In this appendix we place them in context by assembling some results from the literature about the Gromoll filtration.

$\Gamma^7_2 \cong \mathbb{Z}/28$	$\Gamma^7_2 \neq \Gamma^7_3 \supset 0 = \Gamma^7_4$. The inequality for $\Gamma^7_3 \neq \Gamma^7_2$ is due to Weiss [30] who proved that Γ^7_3 has at most 14 elements.
$\Gamma^8_2 \cong \mathbb{Z}/2$	Nothing known
$\Gamma^9_2 \cong (\mathbb{Z}/2)^3$	
$\Gamma^{10}_2 \cong \mathbb{Z}/6$	$\Gamma^{10}_3 \supset \mathbb{Z}/2$ by Theorem 1.1
$\Gamma^{11}_2 \cong \mathbb{Z}/992$	$\Gamma^{11}_3 \subset \mathbb{Z}/496$ by [29]
$\Gamma^{12}_2 = 0$	
$\Gamma^{13}_2 \cong \mathbb{Z}/3$	$\Gamma^{13}_2 = \Gamma^{13}_3 = \Gamma^{13}_4$ by [2]
$\Gamma^{14}_2 \cong \mathbb{Z}/2$	Nothing known
$\Gamma^{15}_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8{,}128$	$\Gamma^{15}_3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4{,}064$ by [2] and [29]
$\Gamma^{16}_2 \cong \mathbb{Z}/2$	Nothing known, conjecturally $\Gamma^{16}_3 = 0$
$\Gamma^{17}_2 \cong (\mathbb{Z}/2)^2$	If Remark 2.21 could be implemented we would be able to conclude that $\alpha(\Gamma^{17}_9) \neq 0$ or perhaps even $\alpha(\Gamma^{17}_{10}) \neq 0$, in particular Γ^{17}_9 or even Γ^{17}_{10} would contain $\mathbb{Z}/2$.
$\Gamma^{18}_2 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$	By Theorem 1.1, $\alpha(\Gamma^{18}_{11}) \neq 0$. Because $\mathbb{Z}/8 = \ker(\alpha)$, $\Gamma^{18}_{11} \supset \mathbb{Z}/2$.

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