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Pseudo-Anosov flows in toroidal manifolds

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We first prove rigidity results for pseudo-Anosov flows in prototypes of toroidal 3–manifolds: we show that a pseudo-Anosov flow in a Seifert fibered manifold is up to finite covers topologically equivalent to a geodesic flow and we show that a pseudo-Anosov flow in a solv manifold is topologically equivalent to a suspension Anosov flow. Then we study the interaction of a general pseudo-Anosov flow with possible Seifert fibered pieces in the torus decomposition: if the fiber is associated with a periodic orbit of the flow, we show that there is a standard and very simple form for the flow in the piece using Birkhoff annuli. This form is strongly connected with the topology of the Seifert piece. We also construct a large new class of examples in many graph manifolds, which is extremely general and flexible. We construct other new classes of examples, some of which are generalized pseudo-Anosov flows which have one-prong singularities and which show that the above results in Seifert fibered and solvable manifolds do not apply to one-prong pseudo-Anosov flows. Finally we also analyse immersed and embedded incompressible tori in optimal position with respect to a pseudo-Anosov flow.

37D20, 37D50; 57M60, 57R30

1 Introduction

The goal of this article is to start a systematic study of pseudo-Anosov flows in toroidal 3–manifolds. More specifically we analyse such flows in closed manifolds which are not hyperbolic or in pieces of the torus decomposition which are not hyperbolic and we obtain substantial results in Seifert fibered pieces. We also produce many new examples of pseudo-Anosov flows, including a large new class in graph manifolds and we study optimal position of tori with respect to arbitrary pseudo-Anosov flows.

The study of hyperbolic flows in toroidal manifolds was initiated by Ghys [29], who analysed Anosov flows in 3–dimensional circle bundles. Ghys showed that up to finite covers, the flow is topologically equivalent to the geodesic flow in the unit tangent bundle of a hyperbolic surface. This was later strengthened by the first author who showed that this holds if the manifold is Seifert fibered [2]. In the mid 70s

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a generalization of Anosov flows called pseudo-Anosov flows was introduced by Thurston [47]. He showed that these are extremely important for the study of surfaces and 3–manifolds [45; 46; 47]. The difference from Anosov flows is that one allows finitely many singularities which are each of p–prong type. In the applications to the topology of 3–manifolds there is a requirement that p is at least 3, which is the convention here as well. Pseudo-Anosov flows have been used very successfully to analyse the topology and geometry of 3–manifolds (see Mosher [37; 38; 39], Gabai and Kazez [27] and the second author [16; 20; 21]) and they are much more common than Anosov flows; see Fried [25], Roberts, Shareshian and Stein [43] and the second author [19]. They are much more flexible because for instance they survive most Dehn surgeries on closed orbits [25]; see also Section 8 for new examples. In addition as opposed to Anosov flows, pseudo-Anosov flows can be constructed transverse to Reebless foliations in vast generality if the manifold is atoroidal (see [37], the second author [17] and Calegari [10; 11; 12], yielding deep geometric information.

In this article we analyse several aspects of pseudo-Anosov flows in toroidal manifolds. In the presence of a general pseudo-Anosov flow the manifold is always irreducible; see the second author and Mosher [22]. By the geometrization theorem a three-manifold with a pseudo-Anosov flow is either hyperbolic, Seifert fibered, or a solv manifold, or the torus decomposition of the manifold is nontrivial; see Perelman [40; 42; 41].

Notice that there is an ongoing broad study of pseudo-Anosov flows in *closed*, hyperbolic manifolds by the second author [16; 20; 21], which is mostly orthogonal to this article. In our situation classical 3–dimensional topology will play a much bigger role.

A *topological equivalence* between two flows is a homeomorphism which sends orbits to orbits. We first analyse Seifert fibered manifolds. Despite the much bigger flexibility of pseudo-Anosov flows we prove a strong rigidity theorem, extending the result of [2] for Anosov flows.

Theorem 4.1 Let Φ be a pseudo-Anosov flow in a Seifert fibered 3–manifold. Then up to finite covers, Φ is topologically equivalent to a geodesic flow in the unit tangent bundle of a hyperbolic surface.

In particular the flow does not have singularities and is topologically Anosov. The proof of Theorem 4.1 splits into two cases depending on whether the fiber is homotopic to a closed orbit of the flow or not. In fact later on this dichotomy will be fundamental for the study of pseudo-Anosov flows restricted to an arbitrary Seifert fibered piece of the torus decomposition of the manifold. In the proof of Theorem 4.1 we start by showing that the first case cannot happen. In the other case we prove that there are no singularities and also that the stable/unstable foliations are slitherings as introduced by

Thurston [48; 49]. This produces two actions of the fundamental group on the circle, which are used to produce a π_1–invariant conjugacy of the orbit space with the orbit space of the geodesic flow. This is enough to prove Theorem 4.1. Here orbit space refers to the orbit space of the flow lifted to the universal cover. For a pseudo-Anosov flow, this orbit space is always homeomorphic to the plane [22] and hence the flow in the universal cover is topologically a product.

Next we analyse pseudo-Anosov flows in three-manifolds with virtually solvable fundamental group. Here again there is a very strong rigidity result.

Theorem 5.7 Suppose that Φ is a pseudo-Anosov flow in a three-manifold with virtually solvable fundamental group. Then Φ has no singularities and is topologically equivalent to a suspension Anosov flow.

The proof of Theorem 5.7 is roughly as follows. Suppose first that the fundamental group is solvable and consider a normal rank two abelian subgroup. The first case is that this subgroup acts nonfreely on the orbit space. In this case we show that the subgroup preserves a structure in the universal cover called a chain of lozenges (described below). By normality the whole fundamental group of the manifold will preserve this chain of lozenges. We also show that the stabilizer of a chain of lozenges is at most a finite extension of $\mathbb{Z}^2$, which leads to a contradiction. It follows that the rank two abelian subgroup acts freely on the orbit space and by previous results this implies that the flow is topologically equivalent to a suspension Anosov flow; see the second author [18]. If the manifold is virtually solvable then the flow is covered by a suspension Anosov flow and one can show that the original flow is also a suspension Anosov flow.

One difference between Theorems 4.1 and 5.7 is that in Theorem 5.7 the flow is topologically equivalent to a suspension Anosov flow, whereas in Theorem 4.1, we only prove it is equivalent to a geodesic flow up to finite covers. The condition on finite covers cannot be removed from Theorem 4.1 as can be seen by unwrapping the fiber direction. See detailed explanation after the proof of Theorem 4.1 in Section 4.

The proofs of both Theorems 4.1 and 5.7 use amongst other tools, the study of actions on the leaf spaces of the stable/unstable foliations in the universal cover. These topological spaces already have a key role in the context of Anosov flows; see [29; 2] and the second author [14; 15]. In the more general context of pseudo-Anosov flows, these leaf spaces are generalizations of both trees and non-Hausdorff simply connected one manifolds and are called non-Hausdorff trees; see the second author [18]. A key fact used, generalizing a previous result in the case of non-Hausdorff simply connected one manifolds (see the first author [5]), is that a group element acting freely on the non-Hausdorff tree has an axis; see [18] and Roberts and Stein [44]. Notice that for a pseudo-Anosov flow, the axis may not be properly embedded in the respective leaf space.

This theme of analysing the structure of the flow in the universal cover is prevalent in a lot of the study of pseudo-Anosov flows and is central to the results of this article. This is used to give topological and homotopic information about the manifold, and it also aids in answering questions of rigidity of the flows and large scale geometry of the flow and the manifold. This previous, extensive topological study of pseudo-Anosov flows substantially simplifies the proofs of Theorems 4.1 and 5.7.

Next we consider manifolds with nontrivial torus decomposition. The overarching goal is to understand the flow in each piece of the torus decomposition and then analyse how the pieces are glued. In this article we do a substantial analysis of one type of Seifert fibered piece (the periodic type; see below) and we study the tori in the boundary of the pieces of a torus decomposition. One of our main goals is to produce a large new class of examples. These examples are much more naturally understood after the structure of periodic Seifert pieces is analysed and the structure of tori is better understood.

In terms of the relation with pseudo-Anosov flows, Seifert fibered pieces in the torus decomposition fall in two categories: if the piece admits a Seifert fibration where the fiber is freely homotopic to a closed orbit of the flow we say that the piece is *periodic*, otherwise the piece is called a *free* piece. Equivalently the Seifert piece is free if and only if the action in the orbit space of a deck transformation corresponding to a fiber in any possible Seifert fibration is free. This dichotomy between free pieces and periodic pieces is fundamental. For example if the whole manifold is Seifert then one main step in the proof of Theorem 4.1 is to show that the piece is a free piece. For solvable manifolds, after cutting along a fiber, the piece is also free. For Anosov flows, the case of free Seifert pieces has been extensively analysed by the first author in [4], giving a nearly final conclusion in the following case: Anosov flows on graph manifolds where all Seifert fibered pieces are free. Recall that a *graph manifold* is an irreducible 3–manifold where the pieces of the torus decomposition are all Seifert.

To understand pseudo-Anosov flows in pieces of the torus decomposition one wants to cut the manifold along tori and analyse the flow in each piece. Therefore one wants the cutting torus to be in good position with respect to the flow. The best situation for a general given torus is that there is a torus isotopic to it which is transverse to the flow. But this is not always possible. A good representative of a much more common situation is the following: consider the geodesic flow in the unit tangent bundle of a closed hyperbolic surface (an Anosov flow). Let α be a simple closed geodesic and let T be the torus of unit vectors along α. Then T is embedded and incompressible but is not transverse to the flow: it contains two copies of α corresponding to the two directions along α and is otherwise transverse to the flow. This is the best position amongst all tori isotopic to T.

Hence it is essential to understand the interaction between π_1–injective tori and pseudo-Anosov flows. Consider a $\mathbb{Z}^2$ subgroup of the fundamental group: if it acts freely on the orbit space then the flow is topologically equivalent to a suspension Anosov flow [18]. Otherwise some element in $\mathbb{Z}^2$ does not act freely on the orbit space and is associated to a closed orbit of the flow. In the last case the $\mathbb{Z}^2$ describes a nontrivial free homotopy from a closed orbit to itself. Any free homotopy between closed orbits can be put in a canonical form as a union of immersed Birkhoff annuli; see eg the first author [3; 4]. A *Birkhoff annulus* is an immersed annulus so that each boundary component is a closed orbit of the flow and the interior of the annulus is transverse to the flow. A *Birkhoff torus* or *Birkhoff Klein bottle* is essentially a π_1–injective surface which is a union of Birkhoff annuli (see Section 6). Given an embedded incompressible torus T, one looks for an isotopic copy which is a Birkhoff torus.

A Birkhoff annulus lifts to a *lozenge* in the universal cover: the boundaries lift to periodic orbits and the interior lifts to a partial ideal quadrilateral region D in the orbit space: two opposite vertices of D are lifts of the boundary orbits, two vertices of D are ideal and the stable/unstable foliations in D form a product structure. The boundary orbits are the corners of the lozenge. Lozenges are the building blocks in the universal cover associated to free homotopies between closed orbits and they are fundamental for much of the theory of Anosov flows [3; 15] and more generally, of pseudo-Anosov flows [16; 18]. Unless the flow is suspension Anosov, then any $\mathbb{Z}^2$ in the fundamental group has associated to it an (essentially) unique *chain* of lozenges, where some elements of $\mathbb{Z}^2$ act fixing the corners and some elements act freely. In the next two results one goal is to look for the best position of embedded incompressible tori. In Proposition 6.4, we prove the following (see Definition 6.3 for the notion of a string of lozenges).

Theorem A *Let T be a π_1–injective torus and let $\mathcal{C}$ be a $\pi_1(T)$ invariant chain of lozenges. Suppose there is a corner α of $\mathcal{C}$ and a covering translation g with $g(\alpha)$ in the interior of a lozenge in $\mathcal{C}$. Then $\mathcal{C}$ is a string of lozenges. In addition T is homotopic into a free Seifert fibered piece.*

An important consequence of this result is that we also prove the following: if no corner of $\mathcal{C}$ is mapped into the interior of a lozenge in $\mathcal{C}$ then one can homotope T to a union of Birkhoff annuli so that the periodic orbits in the annuli do not intersect the union of the interiors of the Birkhoff annuli. This is half way to producing an embedded torus homotopic to T which is a union of Birkhoff annuli. The second conclusion of

Theorem A implies for instance that if T is the boundary torus between two hyperbolic pieces in the torus decomposition, then the situation of Theorem A cannot happen. The general result concerning best position of embedded tori is the following.

Theorem 6.10 Suppose that M is orientable and that the pseudo-Anosov flow is not topologically equivalent to a suspension Anosov flow. Let T be an embedded, incompressible torus in M. Then either

(1) T is isotopic to an embedded Birkhoff torus,

(2) T is homotopic to a weakly embedded Birkhoff torus T' and T (or T') is contained in a periodic Seifert fibered piece, or

(3) T is isotopic to the boundary of the tubular neighborhood of an embedded Birkhoff–Klein bottle contained in a free Seifert piece.

Weakly embedded means that T' is embedded except perhaps along the closed orbits contained in the Birkhoff annuli. All the possibilities in Theorem 6.10 indeed happen: (1) is the typical situation when the flow is a geodesic flow of an orientable surface (or more generally, a Handel–Thurston example [31]), (2) occurs in the Bonatti–Langevin examples [9] and (3) occurs in the geodesic flow on nonorientable closed surfaces (see the last remark of Section 4).

One consequence of this study of standard forms for tori is the following.

Proposition 6.9 Let α be a singular orbit of a pseudo-Anosov flow. Then α is homotopic into a piece of the torus decomposition of the manifold.

If the manifold is atoroidal or Seifert fibered the statement is trivial. Notice that the result is clearly not true for regular periodic orbits as there are many transitive Anosov flows in graph manifolds which are not Seifert fibered [31].

The results above help tremendously to understand canonical flow neighborhoods associated to periodic Seifert fibered pieces (Section 7).

Theorem B *Let Φ be a pseudo-Anosov flow in M orientable and let P be a periodic Seifert fibered piece of the torus decomposition of M. Then there is a finite union Z of Birkhoff annuli, which is embedded except perhaps at the boundaries of the Birkhoff annuli and which is a model for the core of P: a sufficiently small neighborhood of Z is a representative for the Seifert piece P. The finite union Z is well defined up to flow isotopy.*

This is a remarkably simple form for the flow in the piece P. It follows that the dynamics of the flow restricted to the piece is extremely simple: there are finitely many closed orbits; the union of the boundary of the Birkhoff annuli. All other orbits are either in the stable or unstable leaves of the closed orbits or enter and exit the manifold with boundary. In addition the theorem says that in periodic Seifert pieces the flow is intimately connected with the topology of the Seifert piece. This provides a strong relation between dynamics and topology. Notice for future reference that it is not true in general that one can make the boundary of a neighborhood of the Birkhoff annuli transverse to the flow.

The basic ideas of the proof of Theorem B are as follows: The fiber in P is represented by a closed orbit of the flow and any $\mathbb{Z}^2$ in $\pi_1(P)$ can be represented by a Birkhoff torus which has this closed orbit. The Seifert piece being periodic implies that the situation of Theorem A cannot happen, and we can adjust the Birkhoff annuli so that the interiors are embedded and disjoint. three-manifold topology and the study of chains of lozenges implies that we can choose finitely many of these Birkhoff annuli which carry all of $\pi_1(P)$. This produces Z and P can be represented by a small neighborhood of Z.

We are now ready to describe the main family of examples we produce; see Section 8. The construction uses the understanding of the structure given by Theorem B and it shows that the description given in Theorem B is actually realizable in a wide variety of cases, at least when one requires that the boundary of the periodic Seifert pieces are transverse to the flow.

In fact in the construction of Theorem C we *allow* one-prongs. If there are one-prongs, these generalized pseudo-Anosov flows are called *one-prong pseudo-Anosov flows*. Classically they originated in Thurston's work [47] since he constructed pseudo-Anosov homeomorphisms of the two sphere, having for example four one-prong singularities. A suspension of these homeomorphisms produces a one-prong pseudo-Anosov flow. In this case the universal cover is $S^2 \times S^1$ and hence M is not irreducible, but still the flow in the universal cover is topologically a product flow and the orbit space is S^2 which is a two manifold. Other examples with one-prongs are obtained doing Dehn surgery on periodic orbits of pseudo-Anosov flows [25], but here very little is known about the resulting one-prong pseudo-Anosov flows.

Theorem C *There is a large family of (possibly one-prong) pseudo-Anosov flows in graph manifolds and manifolds fibering over the circle with fiber a torus, where the flows are obtained by gluing simple building blocks. The building blocks are homeomorphic to solid tori and they are canonical flow neighborhoods of intrinsic (embedded) Birkhoff annuli. The building blocks have tangential boundary, transverse boundary and only* *two periodic orbits. A collection of blocks is first glued along annuli in their tangential boundary to obtain Seifert fibered manifolds with boundary, and which have a semiflow transverse to the boundary with finitely many periodic orbits. Under very general and specified conditions these can be glued along their boundaries (transverse to the flow) to produce (possibly one-prong) pseudo-Anosov flows in the resulting closed manifolds. In addition one can do any Dehn surgery (except for one) in the periodic orbits of the middle step to obtain new (possibly one-prong) pseudo-Anosov flows.*

This family is a vast generalization of the Bonatti–Langevin construction [9]. The constructions in Theorem C are very general producing for example one-prong pseudo-Anosov flows in all but one torus bundle over the circle. This shows that Theorem 5.7 also does not hold if one allows one-prongs. In the construction in Theorem C, if the middle step produces a flow without one-prong periodic orbits (this is immediate to check), then the resulting final flow in the closed manifold will be pseudo-Anosov in a graph manifold. All the Seifert fibered pieces are periodic pieces. This construction is extremely general producing a very large class of new examples.

An appealing way to describe the examples of Theorem C in the absence of the Dehn surgeries is the following: the manifolds with transverse boundary in the middle step are circle bundles, with fibers preserved by the local flow, and projecting to a local flow of Morse–Smale type on a surface S with boundary: there is a finite number of singular points (prong singularities) in S, stable and unstable manifolds joining the singular points to the boundary, and all other orbits go from one boundary component to another. This picture can be encoded in the combinatorial data of a fat graph satisfying some conditions.

In a subsequent article we will show that the examples of Theorem C without one-prongs (hence pseudo-Anosov) are rigid; that means that up to topological equivalence they are determined by topological data. By construction these flows have many tori transverse to the flow. We stress that the family of examples in Theorem C is entirely new and is constructed by assembling non pseudo-Anosov blocks of semiflows and gluing. Recall that the majority of constructions of pseudo-Anosov flows, besides those transverse to foliations, are obtained by modifying some original pseudo-Anosov flow:

(1) Dehn surgery on closed orbits of flows, introduced by Goodman for Anosov flows [30] and extended by Fried for pseudo-Anosov flows [25].

(2) The derived from Anosov construction of blow up of orbits and gluing by Franks and Williams [24].

(3) The shearing construction along tori, by Handel and Thurston [31].

When the flows of Theorem C do not have p–prong singularities or one-prongs, they are new examples of Anosov flows. In this case these Anosov flows are never contact. This is because all contact Anosov flows are $\mathbb{R}$–covered, ie the lift to the universal cover of the weak stable and unstable foliations have leaf space homeomorphic to the real line [14]; see the first author [7]. In addition if an Anosov flow is $\mathbb{R}$–covered and admits a transverse torus T, then it has to be topologically equivalent to a suspension and T must be a cross section [14; 2]. In our situation consider the transverse tori which are the boundary components of the middle gluing pieces: they do not intersect all orbits of the flow and cannot be cross sections. This proves that the flows are not contact.

We now describe additional families of new examples. At the end of Section 4, we produce some interesting new examples using branched cover constructions.

Theorem D (1) *There is an infinite family of one-prong pseudo-Anosov flows with two one-prong singular orbits and no other singular orbits where the manifold is Seifert fibered. They are doubly branched covered by the Handel–Thurston examples [31].*

(2) *There are also infinitely many examples of one-prong pseudo-Anosov flows which are doubly branched covered by a geodesic flow in a hyperbolic surface and where the original manifolds are not irreducible.*

As remarked above the Handel–Thurston examples are in graph manifolds which are not Seifert fibered. Part (1) of Theorem D shows that Theorem 4.1 does not hold in Seifert fibered manifolds if one allows one-prong orbits. The manifolds in part (2) are not irreducible and neither homeomorphic to $S^2 \times S^1$. At the beginning of Section 8, we improve these examples to show that a mixed behavior of Seifert fibered pieces is possible.

Theorem E *There are examples of pseudo-Anosov flows in graph manifolds with one periodic piece and an arbitrary number of free pieces.*

The flows in Theorem E are obtained as branched cover constructions of the example (2) in Theorem D.

At this point there is no good understanding of the general structure of one-prong pseudo-Anosov flows and they can be much less well behaved than pseudo-Anosov flows. In this article we do not analyse at all the structure of one-prong pseudo-Anosov flows, but only construct many examples of these, some of which highlight the differences with pseudo-Anosov flows in Seifert fibered manifolds, solvable manifolds and graph manifolds.

The first examples of an Anosov flow in a graph manifold where the pieces are periodic were constructed by Bonatti and Langevin [9]: they are extremely special cases of the examples provided by Theorem C. The general case requires different arguments to prove the pseudo-Anosov behavior. The systematic study of Anosov flows in graph manifolds was started by the first author in [4; 6].

In the final section of this article we discuss further questions/comments/conjectures concerning pseudo-Anosov flows in toroidal manifolds. In this article we do not really analyse free Seifert pieces, but in the final section we have some comments and questions about them.

2 Background

Pseudo-Anosov flows: Definitions

Definition 2.1 (Pseudo-Anosov flows) Let Φ be a flow on a closed 3–manifold M. We say that Φ is a *pseudo-Anosov flow* if the following conditions are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1 and not a single point, and the tangent vector bundle $D_t\Phi$ is C^0 in M.
- There are two (possibly) singular transverse foliations Λ^s, Λ^u which are two dimensional, with leaves saturated by the flow and so that Λ^s, Λ^u intersect exactly along the flow lines of Φ.
- There is a finite number (possibly zero) of periodic orbits γ_i, called singular orbits. A stable/unstable leaf containing a singularity is homeomorphic to $P \times I/f$ where P is a p–prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$; in addition p is at least 3.
- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic.

Basic references for pseudo-Anosov flows are [38; 39] and [1] for Anosov flows. A fundamental remark is that the ambient manifold supporting a pseudo-Anosov flow (without 1–prongs) is necessarily irreducible; the universal covering is homeomorphic to $\mathbb{R}^3$ [22].

Definition 2.2 (One-prong pseudo-Anosov flows) A flow Φ is a one-prong pseudo-Anosov flow in M^3 if it satisfies all the conditions of the definition of pseudo-Anosov flows except that the p–prong singularities can also be 1–prong ($p = 1$).

Torus decomposition Let M be an irreducible closed 3–manifold. If M is orientable, it has a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of M obtained by cutting along the tori is either atoroidal or Seifert-fibered (see Jaco [33] and Jaco and Shalen [34]) and the pieces are isotopically maximal with this property. If M is not orientable, a similar conclusion holds; the decomposition has to be performed along tori, but also along some incompressible embedded Klein bottles.

Hence the notion of maximal Seifert pieces in M is well-defined up to isotopy. If M admits a pseudo-Anosov flow, we say that a Seifert piece P is *periodic* if there is a Seifert fibration on P for which a regular fiber is freely homotopic to a periodic orbit of Φ. If not, the piece is called *free*.

Remark In a few circumstances, the Seifert fibration is not unique: it happens for example when P is homeomorphic to a twisted line bundle over the Klein bottle or P is $T^2 \times I$. We stress that our convention is to say that the Seifert piece is free if *no* Seifert fibration in P has fibers homotopic to a periodic orbit.

Orbit space and leaf spaces of a pseudo-Anosov flow

Notation/definition We denote by $\pi\colon \widetilde{M} \to M$ the universal covering of M, and by $\pi_1(M)$ the fundamental group of M, considered as the group of deck transformations on $\widetilde{M}$. The singular foliations lifted to $\widetilde{M}$ are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. If $x \in M$ let $W^s(x)$ denote the leaf of Λ^s containing x. Similarly one defines $W^u(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x)$. Similarly if α is an orbit of Φ define $W^s(\alpha)$, etc. Let also $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

We review the results about the topology of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ that we will need. We refer to [15; 16] for detailed definitions, explanations and proofs. The orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is the quotient space $\widetilde{M}/\widetilde{\Phi}$ and is denoted by $\mathcal{O}$. It is always homeomorphic to the plane $\mathbb{R}^2$ [22]. There is an induced action of $\pi_1(M)$ on $\mathcal{O}$. This action is not free, a fixed point in $\mathcal{O}$ corresponds to a closed orbit of Φ. Let

$$\Theta\colon \widetilde{M} \to \mathcal{O} \cong \mathbb{R}^2$$

be the projection map: it is naturally $\pi_1(M)$–equivariant. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbb{R}$ if L is regular, or is a union of p–rays all with the same starting point if L has a singular p–prong orbit. In addition L is a closed subset of $\widetilde{M}$ or equivalently L is properly embedded in $\widetilde{M}$. The foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ induce $\pi_1(M)$–invariant singular 1–dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$. Its leaves are the $\Theta(L)$ where L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then a *sector* (of L) is a component of $\widetilde{M} - L$. Similarly for $\mathcal{O}^s, \mathcal{O}^u$. If B is any subset of $\mathcal{O}$, we denote by $B \times \mathbb{R}$ the set $\Theta^{-1}(B)$. We stress that for pseudo-Anosov flows there are at least 3–prongs in any singular orbit ($p \geq 3$). For example, the fact that the orbit space in $\widetilde{M}$ is a 2–manifold is not true in general if one allows one-prongs.

Definition 2.3 Let L be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. A slice of L is $l \times \mathbb{R}$, where l is a properly embedded copy of the reals in $\Theta(L)$. For instance if L is regular then L is its only slice. If a slice is the boundary of a sector of L then it is called a line leaf of L. If a is a ray in $\Theta(L)$ then $A = a \times \mathbb{R}$ is called a half leaf of L. If ζ is an open segment in $\Theta(L)$ it defines a *flow band* L_1 of L by $L_1 = \zeta \times \mathbb{R}$. We use the same terminology of slices and line leaves for the foliations $\mathcal{O}^s, \mathcal{O}^u$ of $\mathcal{O}$.

If $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ then F and G intersect in at most one orbit.

We abuse convention and call a leaf L of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ *periodic* if there is a nontrivial covering translation g of $\widetilde{M}$ with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ. Observe that in general, the stabilizer of an element α of $\mathcal{O}$ is either trivial, or a cyclic subgroup of $\pi_1(M)$.

Product regions Suppose that a leaf $F \in \widetilde{\Lambda}^s$ intersects two leaves $G, H \in \widetilde{\Lambda}^u$ and so does $L \in \widetilde{\Lambda}^s$. Then F, L, G, H form a *rectangle* in $\widetilde{M}$, ie every stable leaf between F and L intersects every unstable leaf between G and H. In particular, there is no singularity in the interior of the rectangle [16].

There will be two generalizations of rectangles: perfect fit, which is a rectangle with one corner orbit removed (Definition 2.8) and lozenge, which is a rectangle with two opposite corners removed (Definition 2.9). We will also call rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{O} \cong \mathbb{R}^2$.

Definition 2.4 Suppose A is a flow band in a leaf of $\widetilde{\Lambda}^s$. Suppose that for each orbit α of $\widetilde{\Phi}$ in A there is a half leaf B_α of $\widetilde{W}^u(\alpha)$ defined by α so that for any two orbits γ, β in A then a stable leaf intersects B_β if and only if it intersects B_γ. This defines a stable product region which is the union of the B_γ. Similarly define unstable product regions.

The main property of product regions is the following: for any product region P, and for any $F \in \widetilde{\Lambda}^s$, $G \in \widetilde{\Lambda}^u$ so that $F \cap P \neq \varnothing$, $G \cap P \neq \varnothing$, then $F \cap G \neq \varnothing$. There are no singular orbits of $\widetilde{\Phi}$ in P.

Theorem 2.5 [16] *Let Φ be a pseudo-Anosov flow. Suppose that there is a stable or unstable product region. Then Φ is topologically equivalent to a suspension Anosov flow. In particular Φ is nonsingular.*

In particular, we have the following.

Definition 2.6 [14] A pseudo-Anosov flow is *product* (or *splitting* in the terminology of Franks [23]) if the entire $\widetilde{M}$ is a product region, ie if every leaf of its stable foliation $\widetilde{\Lambda}^s$ intersects every leaf of its unstable foliation $\widetilde{\Lambda}^u$.

Proposition 2.7 *A (topological) Anosov flow is product if and only if it is topologically equivalent to a suspension Anosov flow. In particular M fibers over the circle with fiber a torus and Anosov monodromy.*

Hence, in the sequel, we will use *product pseudo-Anosov flow* as an abbreviation for *pseudo-Anosov flow topologically equivalent to a suspension.*

Perfect fits, lozenges and scalloped chains Recall that a foliation $\mathcal{F}$ in M is $\mathbb{R}$–covered if the leaf space of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ is homeomorphic to the real line $\mathbb{R}$ [14].

Definition 2.8 (Perfect fits [15; 16]) Two leaves $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$, form a perfect fit if $F \cap G = \varnothing$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \widetilde{\Lambda}^s$ and $H_1 \subset H \in \widetilde{\Lambda}^u$, so that the set

$$\bar{F}_1 \cup \bar{H}_1 \cup \bar{L}_1 \cup \bar{G}_1$$

separates $\widetilde{M}$ and forms a rectangle R with a corner removed: The joint structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in R is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect of F and G which do not intersect.

We refer to Figure 1, a for perfect fits. There is a product structure in the interior of R: there are two stable boundary sides and two unstable boundary sides in R. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves F, G are *asymptotic*.

Definition 2.9 (Lozenges [15; 16]) A lozenge R is an open region of $\widetilde{M}$ containing no singularity, whose closure is a rectangle in $\widetilde{M}$ with two corners removed. More specifically two orbits p, q define the corners of a lozenge if there are half leaves A, B of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by p and C, D half leaves of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ defined by q, so that A and D form a perfect fit and so do B and C. The sides of R are A, B, C, D. The sides are not contained in the lozenge, but are in the boundary of the lozenge. There may be singularities in the boundary of the lozenge; see Figure 1(b).

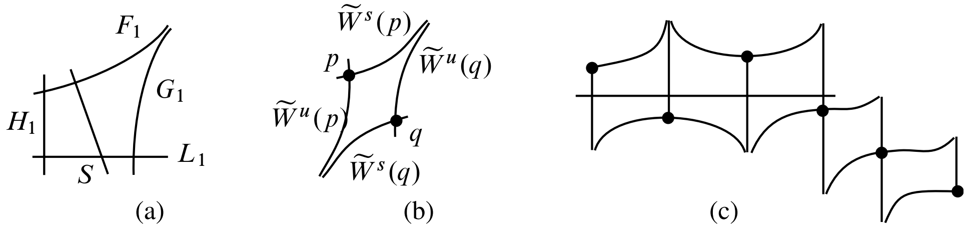

Figure 1: (a) perfect fits in $\widetilde{M}$, (b) a lozenge, (c) a chain of lozenges

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them; see Figure 1(c). Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{C}_i\}, i \in I$, where I is an interval (finite or not) in $\mathbb{Z}$; so that if $i, i+1 \in I$, then $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ share a corner; see Figure 1(c). Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 2.10 (Scalloped chain) Let $\mathcal{C}$ be a chain of lozenges. If any two successive lozenges in the chain are adjacent along one of their unstable sides (respectively stable sides), then the chain is called *s–scalloped* (respectively *u–scalloped*) (see Figure 2 for an example of a s–scalloped region). Observe that a chain is s–scalloped if and only if there is a stable leaf intersecting all the lozenges in the chain. Similarly, a chain is u–scalloped if and only if there is an unstable leaf intersecting all the lozenges in the chain. The chains may be infinite. A scalloped chain is a chain that is either s–scalloped or u–scalloped.

For simplicity when considering scalloped chains we also include any half leaf which is a boundary side of two of the lozenges in the chain. The union of these is called a *scalloped region* which is then a connected set.

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, if γ is a corner of $\mathcal{C}_1$ and α is a corner of $\mathcal{C}_n$.

Fat tree of lozenges

Definition 2.11 (Fat tree of lozenges $\mathcal{G}(\alpha)$) Let α be an orbit of $\widetilde{\Phi}$. We define $\mathcal{G}(\alpha)$ as the graph such that:

- The vertices $\mathcal{G}(\alpha)$ are orbits of $\widetilde{\Phi}$ connected to α by a chain of lozenges.
- There is an edge in $\mathcal{G}(\alpha)$ between β and γ if and only if there is a lozenge with corners γ, β.

One easily proves (see for example [15] for Anosov flows) the following.

Proposition 2.12 *For every α in $\mathcal{O}$, $\mathcal{G}(\alpha)$ is a tree.*

In particular for any two orbits δ, γ connected by a chain of lozenges, then there is a unique indivisible or minimal chain of lozenges, where no backtracking on lozenges is allowed.

The proposition implies that $\mathcal{G}(\alpha)$ is naturally embedded in the 2–plane $\mathcal{O}$. Hence, once an orientation is fixed on $\mathcal{O}$, there is, for every vertex α, a cyclic order on the set of edges of $\mathcal{G}(\alpha)$ adjacent to α. Moreover, $\mathcal{G}(\alpha)$ is naturally equipped with a structure of a *fat graph:* it is a retract of an orientable surface with boundary (the tubular neighborhood of its embedding in $\mathcal{O}$). This object will be extremely useful in this article.

If $\mathcal{C}$ is a lozenge with corner orbits β, γ and g is a nontrivial covering translation leaving β, γ invariant (and so also the lozenge), then $\pi(\beta), \pi(\gamma)$ are closed orbits of Φ which are freely homotopic to the *inverse* of each other [15]. Here we consider the closed orbits $\pi(\beta), \pi(\gamma)$ traversed in the positive flow direction and we allow $\pi(\beta), \pi(\gamma)$ to be nonindivisible closed orbits. In other words it is the closed orbit associated to the deck transformation g, which may not be indivisible.

Theorem 2.13 [15; 16] *Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\Lambda}^s$. Suppose there is a nontrivial covering translation g with $g(F_i) = F_i$, $i = 0, 1$. Let $\alpha_i, i = 0, 1$ be the periodic orbits of $\widetilde{\Phi}$ in F_i so that $g(\alpha_i) = \alpha_i$. Then α_0 and α_1 are connected by a finite chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, and g leaves invariant each lozenge $\mathcal{C}_i$ as well as their corners.*

In particular, we have the following.

Proposition 2.14 *Let g be a nontrivial element of $\pi_1(M)$ fixing two orbits α and γ. Then $\mathcal{G}(\alpha) = \mathcal{G}(\gamma)$.*

We think of a fat tree as a simplicial tree. Observe that g as above naturally acts simplicially on $\mathcal{G}(\alpha)$. It does not necessarily preserve the cyclic order on links of vertices in $\mathcal{G}(\alpha)$, since it does not necessarily preserve the orientation of $\mathcal{O}$.

Definition 2.15 (The tree $\mathcal{G}(g)$) Let g in $\pi_1(M)$ fixing an orbit α of $\widetilde{\Phi}$. The g–fixed points in $\mathcal{G}(\alpha)$ form a connected subtree because of simplicial action. This subtree is denoted by $\mathcal{G}(g)$.

From this observation we infer several interesting facts.

Proposition 2.16 *Let g be a nontrivial element of $\pi_1(M)$. All of the following statements are true:*

(1) *For any $n \neq 0$, g admits a fixed point in $\mathcal{O}$ if and only if g^n admits a fixed point in $\mathcal{O}$.*

(2) *Assuming that g fixes an orbit $\alpha \in \mathcal{O}$, then, some positive power g^p acts trivially on $\mathcal{G}(\alpha)$.*

(3) *Let p be an integer as in* (2)*, let $Z(g^p)$ be the pseudocentralizer of g^p in $\pi_1(M)$, ie the subgroup comprised of elements f such that $fg^pf^{-1} = g^{\pm p}$; then $Z(g^p)$ acts on the tree $\mathcal{G}(\alpha) = \mathcal{G}(g^p)$.*

(4) *Assuming that g preserves a lozenge $\mathcal{L}$, then, g preserves individually each corner of $\mathcal{L}$; moreover, g preserves the orientation of $\mathcal{O}$, and acts trivially on $\mathcal{G}(\alpha) = \mathcal{G}(\beta)$, where α and β are the corners of $\mathcal{L}$.*

Proof (1) Suppose $g^n(\alpha) = \alpha$ with α orbit of $\widetilde{\Phi}$. Then $g^n(g(\alpha)) = g(\alpha)$, so by Theorem 2.13, α and $g(\alpha)$ are connected by a chain of lozenges and therefore $\mathcal{G}(\alpha) = \mathcal{G}(g(\alpha)) = g(\mathcal{G}(\alpha))$. Hence g acts on $\mathcal{G}(\alpha)$. The result now follows easily from the fact that if g acts freely on a tree, then g^n acts freely on the tree.

(2) Let k be the number of prongs at α. Then g^2 preserves the orientation of $\mathcal{O}$, hence the cyclic ordering of the link of α. Hence g^{2k} fixes every vertex of $\mathcal{G}(\alpha)$ adjacent to α. But if g^{2k} fixes a point γ in $\mathcal{G}(\alpha)$ and an edge in $\mathcal{G}(\alpha)$ adjacent to γ, it fixes every vertex adjacent to γ (once more, due to the preservation of orientation of $\mathcal{O}$ by g^{2k}). Our claim follows by induction.

(3) Let f in $Z(g^p)$ and β a vertex in $\mathcal{G}(\alpha)$. Then $g^p f(\beta) = ff^{-1}g^p f(\beta) = f(g^{\pm p}(\beta)) = f(\beta)$ by (2). By Theorem 2.13 $f(\beta)$ is in $\mathcal{G}(\alpha)$ and so f acts on $\mathcal{G}(\alpha)$.

(4) Let α, β be the corners of $\mathcal{L}$. Assume by way of contradiction that $g(\alpha) = \beta$ and $g(\beta) = \alpha$. Let A, C be the stable half leaves of $\widetilde{W}^s(\alpha)$, $\widetilde{W}^s(\beta)$ contained in the closure of $\mathcal{L}$. Let $a = \Theta(A)$ and $c = \Theta(C)$. Considering g acting on $\mathcal{O}$, then $g(a) = c$, and composing g with the holonomy map from c to a along leaves of $\mathcal{O}^u$ defines an orientation reversing map from a onto itself. This map must admit a fixed point, hence there is a leaf U of $\widetilde{\Lambda}^u$ invariant under g and intersecting $\mathcal{L}$. Now g^2 fixes U and A and hence leaves invariant the orbit $U \cap A$. This produces two distinct periodic orbits in $\pi(A)$, contradiction.

Hence, g fixes α and β. Keeping the notation above, we have $g(A) = A$ and $g(B) = B$ (where B is the g invariant unstable half-leaf of $\widetilde{W}^u(\alpha)$ in the boundary of $\mathcal{L}$). It follows that g preserves the orientation of $\mathcal{O}$. It therefore preserves the cyclic ordering along vertices of $\mathcal{G}(\alpha)$. It follows as in item (2) that g acts trivially on $\mathcal{G}(\alpha)$. □

The main result concerning non-Hausdorff behavior in the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ is the following.

Theorem 2.17 [15; 16] *Let Φ be a pseudo-Anosov flow in M^3. Suppose that $F \neq L$ are not separated in the leaf space of $\widetilde{\Lambda}^s$. Then F is periodic and so is L. More precisely, there is a nontrivial element g of $\pi_1(M)$ such that $g(F) = F$ and $g(L) = L$. Moreover, let α, β be the unique g–invariant orbits of $\widetilde{\Phi}$ in F, L, respectively. Then, the chain of lozenges connecting α to β is s–scalloped (see Figure 2).*

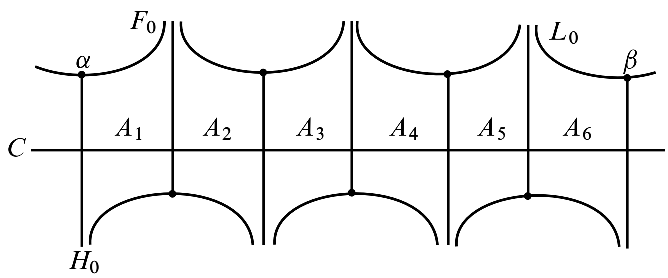

Figure 2: The correct picture between nonseparated leaves of $\widetilde{\Lambda}^s$

Non-Hausdorff trees A *segment* is a set with a linear order which is isomorphic to an interval in $\mathbb{R}$: $[0,1], [0,1), (0,1], (0,1)$ or $[0,0]$. Type $(0,1)$ is called an *open* segment and type $[0,0]$ is a *degenerate* segment. A *closed* segment is one of type either $[0,0]$ or $[0,1]$, ie admitting a minimal and a maximal element. A *half open* segment is one of type $[0,1)$ or $(0,1]$. If I is a segment then I with the reverse linear order is also considered a segment. A *subsegment* C is a subset of a segment I so that if x, y are in C and z in C satisfies $x < z < y$, then z is also in C. With the induced linear order, C is also a segment. If a set Z is a union of segments, then given x in Z, a *prong* at x is a segment I in Z of type $[0,1)$ or $[0,1]$ with $x \in I$ corresponding to 0. A subprong of a prong I at x is a subsegment of I of type $[0,1)$ with x corresponding to 0. We will say that two prongs I_1, I_2 at x are *distinct* if $I_1 \cap I_2 = \{x\}$, or equivalently they do not share a subprong at x.

Definition 2.18 (Non-Hausdorff tree [18]) A non-Hausdorff tree is a space $\mathcal{H}$ satisfying

(1) $\mathcal{H}$ is a union of open segments.

(2) $\mathcal{H}$ is arcwise connected: for each $x, y \in \mathcal{H}$, there is a finite chain of segments $I_1, \dots, I_n$ with $x \in I_1$, $y \in I_n$ and $I_i \cap I_{i+1} \neq \varnothing$ for any $1 \leq i < n$.

(3) Points separate $\mathcal{H}$ in the following way: for any $x \in \mathcal{H}$ and I_1, I_2 distinct prongs at x the following happens: given $y_1 \in I_1 - \{x\}$, $y_2 \in I_2 - \{x\}$, then any finite chain of segments from y_1 to y_2 (as in (2) above) must contain x in at least one of the segments.

If I_1, I_2 are two segments with $I_1 \cap I_2$ a single point which is an endpoint of both I_1 and I_2, then given compatible orders in I_1, I_2 we extend them to an order in $I_1 \cup I_2$, which is then a segment of $\mathcal{H}$.

A priori there may be infinitely or even uncountably many distinct prongs at x.

Definition 2.19 (Topology of $\mathcal{H}$ [18]) We say that a subset A of $\mathcal{H}$ is open in $\mathcal{H}$ if for any $x \in A$ the following happens: for any prong I at x, there is a subprong I' at x ($I' \subset I$) so that $I' \subset A$.

Equivalently A is open if for any open segment S and x in $A \cap S$, there is an open subsegment S' containing x and contained in A.

It follows from condition 3) of non-Hausdorff trees that if I_1 and I_2 are two segments, then $I_1 \cap I_2$ is either empty or is a subsegment of both I_1, I_2, which may be a point. A point $x \in \mathcal{H}$ is *regular* if given any two open segments I_1, I_2 with $x \in I_1 \cap I_2$, then $I_1 \cap I_2$ is an *open* segment in $\mathcal{H}$. Otherwise x is *singular* and $\mathcal{H}$ is "treelike" at x. Equivalently a point is regular if there are only two distinct prongs at x.

It is easy to check that if V is an interval in $\mathbb{R}$ with the standard topology and $f\colon V \to \mathcal{H}$ is an order preserving bijection to a segment in $\mathcal{H}$, then f is a continuous map.

Given $x \neq y$ then for any prong at y there is a subprong disjoint from x, hence contained in $\mathcal{H} - \{x\}$. It follows that $\mathcal{H} - \{x\}$ is an open set in $\mathcal{H}$ and therefore points are closed in $\mathcal{H}$, that is, $\mathcal{H}$ satisfies the T_1 property of topological spaces; see Kelley [36]. In general $\mathcal{H}$ does not satisfy the Hausdorff property T_2 [36]. Given $x \in \mathcal{H}$ and I a prong at x let

$$A_I = \{y \in \mathcal{H} - \{x\} \mid \text{there is a segment path } \gamma \subset \mathcal{H} - \{x\} \text{ from } y \text{ to some point in } I\}.$$

By the above remark, A_I is arcwise connected. If I, J are prongs at x which share a subprong then it is easy to see that $A_I = A_J$. If I, J are distinct prongs at x then $I \cup J$ is a segment of $\mathcal{H}$ with x in the interior of the segment. If there is a segment path $\gamma \subset \mathcal{H} - \{x\}$ from some $y \in A_I$ to some $z \in A_J$ then one constructs a segment path γ contained in $\mathcal{H} - \{x\}$ from some $y' \in I$ to some $z' \in J$. This contradicts condition (3) of the definition of non-Hausdorff tree. Hence $A_I \cap A_J = \varnothing$.

In addition given $y \in A_I$ and J a prong at y, there is a subprong $J' \subset \mathcal{H} - \{x\}$. Clearly $J' \subset A_I$. This implies that any A_I is open in $\mathcal{H}$ and hence A_I is also closed in $\mathcal{H} - \{x\}$. Each A_I is path connected hence connected, so the collection

$$\{A_I\}, \quad I \text{ distinct prongs at } x, \tag{1}$$

is the collection of connected components of $\mathcal{H} - \{x\}$.

In addition suppose that A_I, A_J are distinct, but there is a path α in $\mathcal{H} - \{x\}$ from a point in A_I to a point in A_J (notice here we consider a general path). Then since A_I, A_J are path connected, it follows that $A_I \cup A_J \cup \alpha$ is path connected and hence connected in $\mathcal{H} - \{x\}$ contradicting the fact that (1) is the family of connected components of $\mathcal{H} - \{x\}$. It follows that the collection (1) is also the collection of path components of $\mathcal{H} - \{x\}$.

Conclusion distinct prongs at x are in one to one correspondence with components (or path components) of $\mathcal{H} - \{x\}$. For instance, x has exactly p distinct prongs if and only if $\mathcal{H} - \{x\}$ has p components.

Given $x, y \in \mathcal{H}$ which are not separated from each other in $\mathcal{H}$ we write $x \approx y$. One says that z *separates* x from y if x, y are in distinct components of $\mathcal{H} - \{z\}$. Given any two $x, y \in \mathcal{H}$ there is a continuous path $\alpha(t), 0 \le t \le 1$, from x to y. Define

$$(x, y) = \{z \in \mathcal{H} \mid z \text{ separates } x \text{ from } y\}, \quad [x, y] = (x, y) \cup \{x\} \cup \{y\},$$

The first is the *open block* of $\mathcal{H}$ with endpoints x, y and the second is the *closed block* of $\mathcal{H}$ with endpoints x, y. In [18] it is proved that $[x, y]$ is the intersection of all continuous paths in $\mathcal{H}$ from x to y.

We remark that when x, y are the endpoints of a segment I of $\mathcal{H}$, the notation $[x, y]$ also suggests the segment I from x to y (there is a unique such segment). In fact I and $[x, y]$ are the same [18]. We will also use the notation $(x, y]$ for half open segments.

As $\mathcal{H}$ may not be Hausdorff it may be that $[x, y]$ is not connected. It turns out that $[x, y]$ is a union of finitely many closed segments of $\mathcal{H}$ homeomorphic to either $[0, 0]$ or $[0, 1]$.

Lemma 2.20 [18] *For any $x, y \in \mathcal{H}$ then there are $x_i, y_i \in \mathcal{H}$ with*

$$[x, y] = \bigcup_{i=1}^{n} [x_i, y_i], \quad x_1 = x, \quad y_n = y,$$

a disjoint union, where $[x_i, y_i]$ are closed segments in $\mathcal{H}$. In addition $y_i \approx x_{i+1}$ for any $1 \le i \le n-1$ and some or all segments $[x_i, y_i]$ may be degenerate, that is, points.

There is a natural pseudodistance in $\mathcal{H}$: $d(x,y) = \#(\text{components } [x,y]) - 1$ [5; 43]. So $d(x,y) = 0$ means there is a segment from x to y. Also $d(x,y)$ is the minimum number of nonimmersed points of any path from x to y.

We now consider group actions on non-Hausdorff trees. Let γ be a homeomorphism of $\mathcal{H}$. We say that γ *separates points* if $\gamma(x)$ is separated from x for any $x \in \mathcal{H}$, that is, they have disjoint neighborhoods in $\mathcal{H}$. In particular γ acts freely on $\mathcal{H}$. In [5], the first author constructed a fundamental axis $\mathcal{A}(\gamma)$ if γ separates points in $\mathcal{H}$ and $\mathcal{H}$ has no singularities. In that case $\mathcal{H}$ is a simply connected 1–dimensional manifold and hence is orientable.

Definition 2.21 (Fundamental axis [18]) Let γ be a homeomorphism of a non-Hausdorff tree $\mathcal{H}$ so that γ has no fixed points. The fundamental axis of γ, denoted by $\mathcal{A}(\gamma)$ is

$$\mathcal{A}(\gamma) = \{x \in \mathcal{H} \mid \gamma(x) \in [x, \gamma^2(x)]\}.$$

If $\gamma(x)$ is not separated from x in $\mathcal{H}$, we say that x is an *almost invariant* point under γ. In [18] the following easy fact is proved: Let γ be a homeomorphism of a non-Hausdorff tree $\mathcal{H}$ without fixed points. Then $x \in \mathcal{A}(\gamma)$ if and only if there is a component U to $\mathcal{H} - \{x\}$ so that $\gamma(U) \subset U$. The main result is the following.

Theorem 2.22 [18] *Let γ be a homeomorphism of a non-Hausdorff tree $\mathcal{H}$ without fixed points. Then $\mathcal{A}(\gamma)$ is nonempty.*

Clearly $\mathcal{A}(\gamma)$ is invariant under γ. Also applying γ^{-2} then $\gamma^{-1}(x)$ separates x from $\gamma^{-2}(x)$ and so $\mathcal{A}(\gamma) = \mathcal{A}(\gamma^{-1})$.

Proposition 2.23 *For any $x \in \mathcal{A}(\gamma)$, then $\mathcal{A}(\gamma) = \bigcup_{i \in \mathbb{Z}} [\gamma^i(x), \gamma^{i+1}(x)]$.*

Remark In general it is not true that if γ acts freely on $\mathcal{H}$, then powers of γ also do. For example let γ have an almost invariant point v with $\gamma(v) \neq v$, but $\gamma^2(v) = v$. In this case $\mathcal{A}(\gamma)$ is an open segment which is not properly embedded in $\mathcal{H}$.

Let $x \in \mathcal{A}(\gamma)$. If $d(x, \gamma(x)) = 0$, then $x, \gamma(x)$ are connected by a segment in $\mathcal{H}$. Since $\gamma(x)$ separates x from $\gamma^2(x)$, then $[x, \gamma(x)] \cup [\gamma(x), \gamma^2(x)] = [x, \gamma^2(x)]$ is a segment of $\mathcal{H}$. It follows that $\mathcal{A}(\gamma)$ is an open segment of $\mathcal{H}$, hence homeomorphic to $\mathbb{R}$. If $d(x, \gamma(x)) > 0$, then x and $\gamma(x)$ are connected by a chain of closed segments. It is easy to see that

$$\mathcal{A}(\gamma) = \bigcup_{n \in \mathbb{Z}} [z_i, w_i],$$

where w_i is not separated from z_{i+1}. Then γ acts as a translation on the set of segments, that is, there is $k \in \mathbb{Z}$, so that $\gamma([z_i, w_i]) = [z_{i+k}, w_{i+k}]$ for any $i \in \mathbb{Z}$. We abuse notation and say that γ acts on $\mathbb{Z}$.

Notice that if γ acts freely and γ leaves invariant an open segment I of $\mathcal{H}$, then $\mathcal{A}(\gamma) = I$. This is because for any $z \in I$, $\gamma(x)$ separates x from $\gamma^2(x)$ (free action on I), so $I \subset \mathcal{A}(\gamma)$. But $\mathcal{A}(\gamma) = \cup_{n \in \mathbb{Z}}[\gamma^n(x), \gamma^{n+1}(x)]$ so $I = \mathcal{A}(\gamma)$. Finally it is also not hard to prove the following: Let γ, α be two commuting homeomorphisms of $\mathcal{H}$ which act freely. Then $\mathcal{A}(\gamma) = \mathcal{A}(\alpha)$; see [18].

3 Actions and pseudo-Anosov flows

Let Φ be a pseudo-Anosov flow in M^3. The foliations Λ^s, Λ^u have the following local models: at a nonsingular point y there is a ball neighborhood U of y in M homeomorphic to $D^2 \times [0,1]$ where the leaves of (say) Λ^s are of the form $D^2 \times \{t\}$. Near a singular p prong orbit the picture is the same as a p–prong singularity of a pseudo-Anosov homeomorphism of a surface times an interval. For example consider the germ near zero of the foliation of the plane whose leaves are the fibers of the complex map $z \to \mathrm{Re}(z^{p-2})$. This foliation has a p–prong singularity at the origin. The 3–dimensional picture is obtained by multiplying this by an interval. Similarly for Λ^u. Let C be an interval in $\mathbb{R}$.

Definition 3.1 (Transverse curves) Let $\tau\colon C \to M$ be a continuous curve. Then τ is transverse to Λ^s if the following happens: given t in C there is a small neighborhood Z of $\tau(t)$ where τ is an injective map to the set of local sheets of Λ^s. The same definition works for $\Lambda^u, \widetilde{\Lambda}^s, \widetilde{\Lambda}^u$.

Equivalently the curve is always crossing local leaves. The foliations Λ^s, Λ^u blow up to essential laminations. Hence in $\widetilde{M}$ being transverse to $\widetilde{\Lambda}^s$ is equivalent to τ inducing an injective map in the leaf space of $\widetilde{\Lambda}^s$. For nonsingular points this is the usual notion of transversality.

We establish some notation. Let

$$\mathcal{H}^s = \text{the leaf space of } \widetilde{\Lambda}^s \text{ and } \nu_s\colon \widetilde{M} \to \mathcal{H}^s \text{ the projection map.}$$

Similarly define $\mathcal{H}^u$ and ν_u. The results below which will be proved for $\mathcal{H}^s$, obviously work also for $\mathcal{H}^u$. Notice the leaf space of $\mathcal{O}^s$ is canonically and naturally identified with $\mathcal{H}^s$. We will abuse notation and also denote the leaf space of $\mathcal{O}^s$ by $\mathcal{H}^s$; the context makes it clear which foliation is being considered. Similarly for $\widetilde{\Lambda}^u, \mathcal{O}^u$ and $\mathcal{H}^u$.

Lemma 3.2 *$\mathcal{H}^s$ has a natural structure as a non-Hausdorff tree, where the segments in $\mathcal{H}^s$ are projections of transversals to $\widetilde{\Lambda}^s$. Similarly for $\mathcal{H}^u$.*

Proof We prove properties (1)–(3) of the definition of non-Hausdorff tree. Given x in $\mathcal{H}^s$ let p in $\nu_s^{-1}(x)$ and τ an open transversal to $\widetilde{\Lambda}^s$ containing p. Then $\nu_s(\tau)$ is an open segment containing x. This proves (1). Let x, y in $\mathcal{H}^s$ and choose p in $\nu_s^{-1}(x)$, q in $\nu_s^{-1}(y)$. Connect p, q by a path in $\widetilde{M}$ and perturb it slightly to be a concatenation of transversals. This can be done because it can be done locally. Hence x, y are connected by a finite collection of segments in $\mathcal{H}^s$ and this proves (2).

Finally let I_1, I_2 be segments in $\mathcal{H}^s$ intersecting only in x. Let l_1, l_2 be transversals to $\widetilde{\Lambda}^s$ with $I_i = \nu_s(l_i)$, $i = 1, 2$. We can assume they share a point p in $\nu_s^{-1}(x)$. Any two transversals to $\widetilde{\Lambda}^s$ entering the same component of $\widetilde{M} - \widetilde{W}^s(p)$ will have subtransversals intersecting the same leaves of $\widetilde{\Lambda}^s$ because of the local picture. Therefore $l_1 - \{p\}$, $l_2 - \{p\}$ are contained in different components of $\widetilde{M} - \widetilde{W}^s(p)$. Let now $y_k \in I_k - \{x\}$, $k = 1, 2$. Let $J_i, 1 \leq i \leq n$ be a concatenation of segments from y_1 to y_2 in $\mathcal{H}^s$. There are transversals τ_i to $\widetilde{\Lambda}^s$ with $\nu_s(\tau_i) = J_i$. Let q_1 in $\tau_1 \cap \nu_s^{-1}(y_1)$ and q_2 in $\tau_n \cap \nu_s^{-1}(y_2)$. Since J_i and J_{i+1} intersect we can connect a point in τ_i to a point in τ_{i+1} by a path in a leaf of $\widetilde{\Lambda}^s$. The concatenation of parts of τ_i and paths in leaves of $\widetilde{\Lambda}^s$ produces a path from q_1 to q_2 in $\widetilde{M}$. Since $\widetilde{W}^s(p)$ separates $\widetilde{M}$ and q_1, q_2 are in different components of the complement, then this path has to intersect $\widetilde{W}^s(p)$. If it intersects $\widetilde{W}^s(p)$ in a path in $\widetilde{W}^s(p)$ then the endpoints of this path are in some τ_i and hence its projection, which is x is in J_i. This proves (3). □

We have two topologies in $\mathcal{H}^s$: the quotient topology from ν_s and the non-Hausdorff tree topology. These are the same as proved in the next lemma.

Lemma 3.3 *The quotient topology in $\mathcal{H}^s$ (from $\nu_s\colon \widetilde{M} \to \mathcal{H}^s$) is the same as the non-Hausdorff tree topology in $\mathcal{H}^s$.*

Proof Let $A \subset \mathcal{H}^s$ be an open set in the quotient topology and x in A. Let I be a prong at x. Then $I = \nu_s(\tau)$ for some transversal τ to $\widetilde{\Lambda}^s$ starting in some $p \in \nu_s^{-1}(x)$. Since $\nu_s^{-1}(A)$ is open in $\widetilde{M}$ and p is in $\nu_s^{-1}(A)$ there is a nondegenerate subtransversal τ' of τ starting at p and contained in $\nu_s^{-1}(A)$. Let $I' = \nu_s(\tau')$. Then I' is a prong at x which is a subprong of I. In addition I' is contained in A. Therefore A is open in the non-Hausdorff tree topology.

Conversely suppose that A is open in the non-Hausdorff tree topology. By way of contradiction suppose that there is p in $\nu_s^{-1}(A)$ which is not in the interior of $\nu_s^{-1}(A)$.

Then we can find a sequence $(p_n)_{n\in\mathbb{N}}$ in $\widetilde{M}$ converging to p and with p_n not in $\nu_s^{-1}(A)$ for any n. It follows that $p_n \notin \widetilde{W}^s(p)$ for any n as $\nu_s^{-1}(A)$ is $\widetilde{\Lambda}^s$ saturated. Up to a subsequence assume there is a component Z of $\widetilde{M} - \widetilde{W}^s(p)$ containing p_n for every n. Here the condition of finitely many prongs at singular points is used. Let τ be a transversal to $\widetilde{\Lambda}^s$ starting at p and entering the component Z. Let $x = \nu_s(p)$ and $I = \nu_s(\tau)$. Then I is a prong at p and since A is open in the non-Hausdorff tree topology, there is a subprong I' at x with I' contained in A. Let τ' be the subtransversal of τ corresponding to I'. For n sufficiently large $\widetilde{W}^s(p_n)$ intersects $\tau' \subset \nu_s^{-1}(A)$. Hence p_n is in $\nu_s^{-1}(A)$. This contradiction shows that $\nu_s^{-1}(A)$ is open in $\widetilde{M}$. Therefore A is open in the quotient topology. □

Remarks (1) A variation of the proof works for non-Hausdorff trees $\mathcal{H}$ which are "leaf spaces" of lifts of essential laminations. The difference is that it is very possible that there are singularities $\mathcal{H}$ which have infinitely many prongs.

(2) The study of group actions on non-Hausdorff trees has important precursors: first the fundamental work of Gabai and Oertel [28] who created essential laminations and introduced order trees. Previous study of group actions on order trees and related sets was done by Roberts and Stein [44] and Roberts, Shareshian and Stein [43] with some exceptional consequences for the existence of foliations in certain 3–manifolds.

We say that two leaves L, F of $\widetilde{\Lambda}^s$ are nonseparated from each other if there are p in L, q in F and a sequence of leaves (L_n) of $\widetilde{\Lambda}^s$ having points p_n, q_n in L_n with (p_n) converging to p and (q_n) converging to q. We call this condition (I) for L, F. Up to subsequence we may assume that (L_n) is a nested sequence of leaves of $\widetilde{\Lambda}^s$. By throwing out a few initial terms of the sequences $(p_n), (q_n)$, this is equivalent to the existence of transversals τ_L, τ_F to $\widetilde{\Lambda}^s$ with τ_L starting at p, τ_F starting at q with τ_L containing all p_n as above and τ_L containing all q_n. Project to $\mathcal{H}^s$: let

$$x = \nu_s(p), \quad y = \nu_s(q), \quad x_n = \nu_s(p_n), \quad y_n = \nu_s(q_n), \quad I = \nu_s(\tau_L), \quad J = \nu_s(\tau_F).$$

Here I, J are segments in $\mathcal{H}^s$, I is a prong at x and J is a prong at y. Also $x_n = y_n$. If I_n is the subsegment of I from x_1 to x_n and J_n the subsegment of J from y_1 to y_n then $I_n = J_n$ and therefore $I - \{x\} = J - \{y\}$. Conversely if x, y have prongs I, J so that $I - \{x\} = J - \{y\}$ it is easy to show that $L = \nu^{-1}(x)$ and $F = \nu^{-1}(y)$ are leaves of $\widetilde{\Lambda}^s$ nonseparated from each other.

We claim that condition (I) is also equivalent to condition (II): L, F do not have disjoint, open, $\widetilde{\Lambda}^s$ saturated neighborhoods in $\widetilde{M}$. In other words x, y do not have disjoint open neighborhoods in $\mathcal{H}^s$. Clearly condition (I) implies condition (II). Conversely suppose that condition (II) holds. If $x = y$ then clearly condition (I) holds. Suppose then x, y are distinct. We proved before that for any z in $\mathcal{H}^s$, then two points are in the same path component of $\mathcal{H}^s - \{z\}$ if and only if they are connected by a segment path in $\mathcal{H}^s$ which does not contain $\{z\}$ and these path components are open in $\mathcal{H}^s$. By condition (II) it follows that for any z in $\mathcal{H}^s - \{x, y\}$, the points x, y are in the same component of $\mathcal{H}^s - \{z\}$. Hence (x, y) is empty. There are prongs I at x and J at y so that $I - \{x\} = J - \{y\}$, by [18, Lemma 3.5, page 71]. This is condition (I).

If either of these conditions holds for x, y in $\mathcal{H}^s$ we write $x \sim y$.

For f in $\pi_1(M)$ let $\mathrm{Fix}(f)$ be those x in $\mathcal{H}^s$ with $f(x) = x$. Let $\mathrm{Fix}^{\sim}(f)$ be the set of x in $\mathcal{H}^s$ with $x \sim f(x)$. Considering the action of f on the orbit space $\mathcal{O}$, let $B(f)$ the set of u in $\mathcal{O}$, fixed by f.

Lemma 3.4 *Let Φ be a pseudo-Anosov flow and f in $\pi_1(M)$. Then $\mathrm{Fix}^{\sim}(f)$ is a closed subset of $\mathcal{H}^s$.*

Proof Let x not in $\mathrm{Fix}^{\sim}(f)$, so $x \not\sim f(x)$. Then x and $f(x)$ have disjoint open neighborhoods U, V in $\mathcal{H}^s$. By continuity of f, there is a smaller open neighborhood W of x so that $f(W)$ is contained in V. Hence any y in W satisfies $y \not\sim f(y)$ and $(\mathrm{Fix}^{\sim}(f))^c$ is open. □

Remark In general $\mathrm{Fix}(f)$ is not closed: a sequence (x_n) in $\mathrm{Fix}(f)$ may converge to x which is only in $\mathrm{Fix}^{\sim}(f)$.

The following will be useful later.

Lemma 3.5 *If f is in $\pi_1(M)$ and f is not the identity, then $\mathrm{Fix}^{\sim}(f)$ is countable.*

Proof First we show that $\mathrm{Fix}(f)$ is countable. Let L in $\widetilde{\Lambda}^s$ with $f(L) = L$. Then there is a periodic orbit in $\pi(L)$. If L_1, L_2 are in $\mathrm{Fix}(f)$ then their periodic orbits are connected by a finite chain of lozenges by Theorem 2.13. In addition the orbit space $\mathcal{O} \cong \mathbb{R}^2$ is countably compact. If $\mathrm{Fix}(f)$ were uncountable, then $B(f)$ would be uncountable and there would be accumulation points in $B(f)$. This is disallowed because any two points in $\mathrm{Fix}(f)$ are connected by a unique chain of lozenges.

Now let $N = \{x \in \mathcal{H}^s,\ \text{so that } x \text{ is nonseparated from some } y \in \mathcal{H}^s\}$. We will prove that N is countable, hence $\mathrm{Fix}^{\sim}(f)$ is countable. Assume by way of contradiction that N is uncountable. The space $\mathcal{H}^s$ is a union of countably many open segments and we fix one such countable collection. For each x in $\mathcal{H}^s$, let I_x be one such segment in the countable family containing x. Notice that $I_x = I_y$ with x, y distinct occurs a lot. If N is uncountable, then there is an open segment I in $\mathcal{H}^s$ containing uncountably many elements of N. Choose an order in I. For each z in $I \cap N$, there is y distinct from z with $y \sim z$. Suppose without loss of generality that for uncountably many such z the corresponding y is nonseparated from the z in their positive sides, with respect to the order in I. For any such z, z' in $I \cap N$, let y, y' be nonseparated from them respectively. We claim that $I_y, I_{y'}$ are different. Suppose for simplicity that $z < z'$ in I. Here $z' \sim y'$ and nonseparated on their positive sides, so $I_{y'}$ does not contain z' or any point in I smaller than z'. But by construction I_y contains y, so $I_y, I_{y'}$ are different. Hence all such I_y are different, contradicting the fact that there are only countably many of these. This finishes the proof of the lemma. □

4 Pseudo-Anosov flows in Seifert fibered spaces

This section is devoted to proving the following result.

Theorem 4.1 *If Φ is a pseudo-Anosov flow in M^3 which is a Seifert fibered space, then up to finite covers, Φ is topologically equivalent to a geodesic flow on a closed hyperbolic surface.*

Proof The new mathematical result is a reduction of the proof to the nonsingular case. The smooth Anosov case was originally proved in [2]. We also give an improved proof of the Anosov case, which may be useful in other contexts.

If necessary lift to a double cover so that the Seifert fibration is orientable, hence the center of $\pi_1(M)$ is nonempty (it contains for example the homotopy class of the regular fibers). Let h be in the center of $\pi_1(M)$. The cyclic subgroup $\langle h \rangle$ is a normal subgroup of $\pi_1(M)$. The proof splits in two cases, depending on whether $\mathrm{Fix}^{\sim}(h)$ is empty or not.

Case 1: $\mathbf{Fix}^{\sim}(h)$ is nonempty We show that this cannot happen. Notice that if $x \sim y$ in $\mathcal{H}^s$ and g is in $\pi_1(M)$ then $g(x) \sim g(y)$. Let g in $\pi_1(M)$ and x in $\mathrm{Fix}^{\sim}(h)$. Then $g^{-1}hg(x) = h(x) \sim x$, so $hg(x) \sim g(x)$ and $g(x)$ is in $\mathrm{Fix}^{\sim}(h)$. By Lemma 3.5, $\mathrm{Fix}^{\sim}(h)$ is countable. Therefore $\mathrm{Fix}^{\sim}(h)$ is a countable, closed, $\pi_1(M)$ invariant subset of $\mathcal{H}^s$. Consider the union Z of the leaves L in $\widetilde{\Lambda}^s$ with $\nu_s(L)$ in $\mathrm{Fix}^{\sim}(h)$. This set Z is closed, $\widetilde{\Lambda}^s$ saturated, $\pi_1(M)$ invariant and transversely countable. It projects to a sublamination of Λ^s which is transversely countable. Let $\mathcal{L}$ be a minimal sublamination of $\pi(Z)$. Any sufficiently small transversal to a minimal lamination intersects it in either a closed interval, a Cantor set or a point. The first two are disallowed by the transverse countability condition. The last option implies that there is an isolated leaf in Λ^s, which is not possible for pseudo-Anosov flows. This shows that Case 1 cannot happen.

Case 2: $\mathrm{Fix}^{\sim}(h)$ is empty By Theorem 2.22, we have that h has a nonempty axis $\mathcal{A}(h) = \{x \in \mathcal{H}^s \mid h(x) \text{ separates } x \text{ from } h^2(x)\}$. This axis has a linear order where h acts as a translation. Clearly, for every g in $\pi_1(M)$,

$$g\mathcal{A}(h) = \mathcal{A}(ghg^{-1}) = \mathcal{A}(h),$$

hence $\mathcal{A}(h)$ is $\pi_1(M)$–invariant.

Either $\mathcal{A}(h)$ is an infinite segment or a countable union of disjoint closed segments:

$$(*) \qquad \mathcal{A}(h) = \bigcup_{i \in \mathbb{Z}} [x_i, y_i] = \bigcup_{i \in \mathbb{Z}} B_i,$$

where $y_i \sim x_{i+1}$. We show that the second option cannot happen. Suppose by way of contradiction that $\mathcal{A}(h)$ is of form $(*)$. Every g in $\pi_1(M)$ permutes the components B_i, preserving or reversing the order on the set $\mathbb{Z}$ of labels. Hence there is a morphism $\pi_1(M) \to \mathrm{Aut}(\mathbb{Z})$, whose kernel is the subgroup made of elements g such that $gx_i = x_i$ for all i, ie a trivial or cyclic normal subgroup. Since $\mathrm{Aut}(\mathbb{Z})$ is the dihedral group, containing a cyclic subgroup of index 2, it follows that $\pi_1(M)$ contains a finite index subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$, which is not possible for an irreducible Seifert fibered space without boundary. We conclude that $\mathcal{A}(h)$ cannot be an infinite collection of disjoint closed segments.

Therefore $\mathcal{A}(h)$ is a real line parametrized as $\mathcal{A}(h) = \{l_t, t \in \mathbb{R}\}$. If $\mathcal{A}(h)$ is not properly embedded in $\mathcal{H}^s$, then (l_t) converges to a point x in $\mathcal{H}^s$ as t converges to infinity (and maybe other points as well). But then since $\mathcal{A}(h)$ is invariant under h, this implies that $h(x) \sim x$, which is not allowed in Case 2.

Next we show that $\mathcal{A}(h)$ is all of $\mathcal{H}^s$. Again suppose it is not and let l be a point of $\mathcal{H}^s$ not in $\mathcal{A}(h)$. Since $\mathcal{A}(h)$ is connected (as it is a line), then $\mathcal{A}(h)$ is contained in a single component of $\mathcal{H}^s - \{l\}$. Let B be another component of $\mathcal{H}^s - \{l\}$. Let $L = \nu_s^{-1}(l)$. It was proved in [20] that any complementary component of L covers M. This implies that given x in $\mathcal{A}(h)$, there is g in $\pi_1(M)$ with $g(x)$ in B, which is disjoint from $\mathcal{A}(h)$. This contradicts the $\pi_1(M)$ invariance of $\mathcal{A}(h)$.

We conclude that $\mathcal{H}^s$ is homeomorphic to $\mathbb{R}$ and similarly $\mathcal{H}^u$ is also homeomorphic to $\mathbb{R}$. Therefore there are no singularities of Φ and Λ^s, Λ^u are $\mathbb{R}$–covered.

Since there is no singularity, the flow is actually (topologically) Anosov. The result was then proved in [2]. We present a different proof here, which improves arguments in [2] and which follows arguments in the unpublished reference by the first author [8].

If there is a leaf of $\widetilde{\Lambda}^s$ intersecting all leaves of $\widetilde{\Lambda}^u$, then it is easy to see that this holds for all leaves and hence Φ is a product pseudo-Anosov flow. Proposition 2.7

shows that the manifold would have solv geometry and could not be Seifert fibered, contradiction.

It follows from [14; 2] that Φ has the skewed type: the orbit space $\mathcal{O}$ is homeomorphic to an infinite strip in $\mathbb{R}^2$ bounded by parallel lines, say with slope one. The stable foliation is the foliation by horizontal segments and the unstable foliation is the foliation by vertical segments (see Figure 3). Let (x, y) be the induced cartesian coordinates in $\mathcal{O} \subset \mathbb{R}^2$. The fact that Φ has skewed type implies in particular that M is orientable [15].

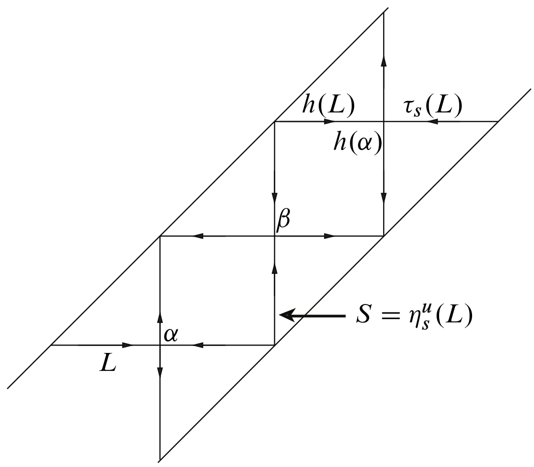

Figure 3: Orbit space of skewed type

Put a transverse orientation to $\widetilde{\Lambda}^s$ positive with increasing y and to $\widetilde{\Lambda}^u$ positive with increasing x. For each stable leaf L, there is in the positive side of L a unique unstable leaf S which makes a perfect fit with L: in this model it is equivalent to S sharing an endpoint with L. This produces a $\pi_1(M)$ equivariant map η_s^u from $\mathcal{H}^s$ to $\mathcal{H}^u$, which is a homeomorphism [2; 14]. Similarly for each S in $\widetilde{\Lambda}^u$ there is a unique E of $\widetilde{\Lambda}^s$ in the positive side of S and sharing an endpoint with S. The composition $L \to S \to E$ is a translation τ_s in $\mathcal{H}^s$ and $\mathcal{H}^s/\tau_s$ is a circle S^1_s. Similarly one has τ_u which is increasing from $\mathcal{H}^u$ to $\mathcal{H}^u$ and a circle $S^1_u = \mathcal{H}^u/\tau_u$. Both τ_s and τ_u are $\pi_1(M)$ equivariant homeomorphisms [2; 14], so $\pi_1(M)$ acts on S^1_s and S^1_u. We denote the first action by

$$\xi_s\colon \pi_1(M) \to \mathrm{Homeo}(S^1_s).$$

In addition the map $\eta_s^u\colon L \to S$ as above is also equivariant by the action of $\pi_1(M)$ and hence induces a canonical homeomorphism from S^1_s to S^1_u with inverse denoted by ζ. So we can identify $S^1_s \times S^1_u$ with $S^1_s \times S^1_s$ by $(z, w) \to (z, \zeta(w))$. This induces an action of $\pi_1(M)$ on $S^1_s \times S^1_s$.

For every orbit β of $\widetilde{\Phi}$, there are unique leaves L of $\widetilde{\Lambda}^s$ and G of $\widetilde{\Lambda}^u$ so that $\beta = L \cap G$. Using L and G, the orbit β generates a point in $S^1_s \times S^1_u$ and hence a point (p,q) in $S^1_s \times S^1_s$. We say that β projects to (p,q). This defines a map

$$\nu\colon \mathcal{O} \to S^1_s \times S^1_s.$$

The projection (p,q) is not in the diagonal Δ: points in the diagonal correspond to L in $\widetilde{\Lambda}^s$ and S in $\widetilde{\Lambda}^u$ so that $S = \eta^u_s(\tau_s)^n(L)$ for some integer n. In particular L and S do not intersect and neither does S intersect $(\tau_s)^m(L)$ for any integer m. Conversely if (p,q) is in $S^1_s \times S^1_s - \Delta$, then one can lift p to a leaf L of $\widetilde{\Lambda}^s$ and q lifts to a stable leaf, which after the identification S^1_s with S^1_u produces S in $\widetilde{\Lambda}^u$ with $S \cap L$ not empty.

Note that if g acts trivially on $\mathcal{H}^s$ then g is the identity in $\pi_1(M)$. This follows for instance because the set of fixed points of nontrivial elements of $\pi_1(M)$ is discrete in $\mathcal{H}^s$.

Claim 1 *h acts trivially on S^1_s.*

Let $\widetilde{\alpha}$ be a lift of a periodic orbit α associated to a covering translation g. Then $g^2h(\widetilde{\alpha}) = hg^2(\widetilde{\alpha}) = h(\widetilde{\alpha})$, so $\widetilde{\alpha}$ and $h(\widetilde{\alpha})$ are connected by a chain of n lozenges by Theorem 2.13. Replacing g^2 by g^{-2} if necessary, we can assume that $\widetilde{\alpha}$ is an attracting fixed point of the restriction of g^2 to the set of orbits in the stable leaf L through $\widetilde{\alpha}$. Then $h(\widetilde{\alpha})$ is also an attracting fixed point of the restriction of g^2 to $h(L)$. It follows (see Figure 3) that n is even. In the figure β is connected to α by one lozenge and $h(\alpha)$ is connected to α by a chain of two lozenges. Therefore $h(L) = (\tau_s)^i(L)$ for $i = n/2$.

This implies that the projections to S^1_s of periodic leaves are fixed points of $\xi_s(h)$. Since periodic leaves are dense, we conclude that $\xi_s(h)$ is the identity map on S^1_s. The claim is proved.

Recall that h was any element of the center of $\pi_1(M)$. First notice that $\pi_1(M)$ cannot be $\mathbb{Z}^3$ because M has a pseudo-Anosov flow: if $\mathbb{Z}^2 \times \{0\}$ subgroup of $\pi_1(M)$ acts freely on $\mathcal{O}$, then Φ is product and $\pi_1(M)$ is solvable and not $\mathbb{Z}^3$. Then $\mathbb{Z}^2 \times \{0\}$ leaves invariant a bi-infinite chain of lozenges $\mathcal{C}$ by Lemma 5.3. As $\pi_1(M))$ is abelian it follows that it preserves $\mathcal{C}$. But Lemma 5.2 states that the stabilizer has a finite index subgroup which is a subgroup of $\mathbb{Z}^2$. This is a contradiction.

Since M is Seifert fibered, orientable and $\pi_1(M)$ not $\mathbb{Z}^3$ it follows that the center of $\pi_1(M)$ is a cyclic subgroup; see Hempel [32] and [34]. The center is generated by an element associated with a regular fiber of the Seifert fibration, which is unique up to isotopy [32; 34]. From now on, we assume that h generates the center; and we denote by l the integer such that when acting on $\mathcal{H}^s$, then $\tau_s^l = h$. In order to simplify the presentation, we identify in the sequel $\mathcal{H}^s$ with $\mathbb{R}$ in a way that τ_s is the translation $x \mapsto x+1$.

Now let f be in the kernel of ξ_s. When acting on $\mathcal{H}^s$, $f(x) = x + j$ for some j in $\mathbb{Z}$. In addition given any g in $\pi_1(M)$ and considering the action on $\mathcal{H}^s$, it follows that for any x in $\mathcal{H}^s$, for any i in $\mathbb{Z}$, then $g(x+i) = g(x)+i$. Now, for any g in $\pi_1(M)$, again when considering the action on $\mathcal{H}^s$ we have

$$g^{-1}f^{-1}gf(x) = g^{-1}f^{-1}g(x+j) = g^{-1}f^{-1}(g(x)+j) = g^{-1}g(x) = x.$$

Therefore $g^{-1}f^{-1}gf$ acts trivially on $\mathcal{H}^s$ and is the identity in $\pi_1(M)$. Hence f is in the center of $\pi_1(M)$ which is $\langle h\rangle$.

Conclusion $\ker \xi_s = \langle h\rangle =$ center of $\pi_1(M)$.

Let $H = \langle h\rangle$ and $Q = \pi_1(M)/H$. Since H is the kernel of ξ_s, there is an induced action $\overline{\xi}_s$ of Q on S^1_s. Given g in $\pi_1(M)$ let $\overline{g}$ be its image in Q. By the conclusion above the action $\overline{\xi}_s$ is faithful.

Lemma 4.2 *The action $\overline{\xi}_s$ of Q on S^1_s is a convergence group action.*

Proof First we prove the following fact.

Claim 2 Two arbitrary orbits β_1, β_2 of $\widetilde{\Phi}$ are connected by a chain of lozenges if and only if β_1, β_2 project to either the same point of $S^1_s \times S^1_s - \Delta$ or one projects to some point (p,q) and the other projects to (q,p). In the first case they are connected by an even number of lozenges and in the second case they are connected by an odd number of lozenges.

Suppose first that β_1, β_2 are connected by a chain of lozenges. The first lozenge in the chain has a stable side L containing β_1. There is an unstable side S of the lozenge making a perfect fit with L. The other corner β of the lozenge is contained in S. Suppose without loss of generality that S is in the positive side of L. Then $S = \eta^u_s(L)$. In addition $\widetilde{W}^u(\beta_1), \widetilde{W}^s(\beta)$ also make a perfect fit and

$$\widetilde{W}^u(\beta_1) = \eta^u_s(\tau_s^{-1}(\widetilde{W}^s(\beta)).$$

So if β_1 projects to (p,q) then β projects to (q,p). Following the lozenges in the chain proves that β_2 projects to either (p,q) or (q,p). Using these arguments one sees that β_1 and $\alpha = \tau_s(L) \cap \tau_u(\widetilde{W}^u(\beta_1))$ are connected by a chain of two lozenges.

Conversely suppose β_1 and β_2 both project to (p,q). Let $F = \widetilde{W}^s(\beta_1)$, $G = \widetilde{W}^u(\beta_1)$ and let also $E = \widetilde{W}^s(\beta_2)$, $S = \widetilde{W}^u(\beta_2)$. Since the projections of both β_1 and β_2 have the same point p as first coordinate, there is n in $\mathbb{Z}$ so that $E = \tau_s^n(F)$. Similarly there is m in $\mathbb{Z}$ with $S = \tau_u^m(G)$. In the collection $\{\tau_u^i(G), i \in \mathbb{Z}\}$, there is only one element intersecting $\tau_s^n(F)$ and that is $\tau_u^n(G)$. It follows that $n = m$. In addition,

$$\beta_2 = \tau_s^n(F) \cap \tau_u^n(G).$$

As explained above β_1 and $\tau_s(F) \cap \tau_u(G)$ are connected by a chain of two lozenges and by induction β_1 and β_2 are connected by a chain with an even number of lozenges. The case that β_1 projects to (p,q) and β_2 projects to (q,p) is very similar and is left to the reader. This proves Claim 2.

Let α be an arbitrary closed orbit of Φ traversed once, let $\widetilde{\alpha}$ be a lift to $\widetilde{M}$, which is invariant under g in $\pi_1(M)$, with g associated to α in the positive direction. Let (p,q) in $S^1_s \times S^1_s - \Delta$ be $\nu(\widetilde{\alpha})$. We call such a (p,q) a *periodic* point. Recall that h in $\pi_1(M)$ represents the fiber of the Seifert fibration. Since h acts trivially on S^1_s, then claim 2 implies that $\widetilde{\alpha}$ and $h(\widetilde{\alpha})$ are connected by a chain of lozenges with an even number of lozenges [14]. Therefore the set of orbits in the complete chain of lozenges from $\widetilde{\alpha}$ is finite modulo the action by $\langle h \rangle$ and this set projects to a finite set V of orbits of Φ in M. But α is closed, so V is a finite set of closed orbits and hence discrete in M. Hence $\pi^{-1}(V)$ is a discrete, $\pi_1(M)$ invariant set of orbits of $\widetilde{\Phi}$. We conclude that $\nu(\Theta(\pi^{-1}(V)))$ is a discrete set in $S^1_s \times S^1_s - \Delta$. It is also $\pi_1(M)$ invariant. This is the "orbit" of (p,q) under the action of $\pi_1(M)$.

Now given $\alpha, \widetilde{\alpha}, g$ as above, let $L = \widetilde{W}^s(\widetilde{\alpha})$. Then $g(L) = L$ and since g is associated to the positive direction of α then L is a contracting fixed point of g acting on $\mathcal{H}^s$. In the same way $S = \widetilde{W}^u(\alpha)$ is also fixed by g and it is a repelling fixed point of g acting on $\mathcal{H}^u$ and hence p is the attracting fixed point of g acting on S^1_s and q is the repelling fixed point. There are no other fixed points.

In order to prove the convergence group property for the action $\overline{\xi}_s$ of Q on S^1_s, we now consider a sequence d_n of distinct elements of Q and let g_n in $\pi_1(M)$ with $d_n = \overline{g}_n$. In the arguments below we abuse notation and also denote by ξ_s the action of $\pi_1(M)$ on $S^1_s \times S^1_s$; the context makes clear which one is being used.

Consider a closed orbit α as above, with a given lift $\widetilde{\alpha}$. Let $(p,q) = \nu(\widetilde{\alpha})$ and $L = \widetilde{W}^s(\widetilde{\alpha})$. Suppose first that up to subsequence

$$\xi_s(g_n)((p,q)) = (p,q) \quad \text{or} \quad \xi_s(g_n)((p,q)) = (q,p) \quad \text{for all } n.$$

Notice that it does not matter if we consider $\xi_s(g_n)$ or $\overline{\xi}_s(\overline{g}_n)$. First a reduction: if $\xi_s(g_n)((p,q)) = (q,p)$ for all n, then replace $\widetilde{\alpha}$ by $g_1(\widetilde{\alpha})$ and g_n by $g_n g_1^{-1}$. The new collection satisfies $\xi_s(g_n)((p,q)) = (p,q)$ for all n. Claim 2 implies that for every n, $g_n(\widetilde{\alpha})$ is connected to $\widetilde{\alpha}$ by a chain of lozenges, with an even number of lozenges. For each n there is a_n so that $g_n(L) = \tau_s^{a_n}(L)$. Recall the integer l above so that $h = \tau_s^l$ when acting on $\mathcal{H}^s$. There are unique b_n and c_n in $\mathbb{Z}$ with $0 \leq c_n < l$ and $a_n = b_n l + c_n$. Up to another subsequence we assume that c_n is constant. Again up to taking $g_1(\widetilde{\alpha})$ instead of $\widetilde{\alpha}$ and $g_n g_1^{-1}$ instead of g_n we may assume that $c_n = 0$ for all n. The above facts imply that for each n there is i_n in $\mathbb{Z}$ so that $h^{i_n} g_n(\widetilde{\alpha}) = \widetilde{\alpha}$ (in fact $i_n = -b_n$). Therefore $h^{i_n} g_n = f^{j_n}$, for some j_n in $\mathbb{Z}$ where f is a generator of the isotropy group of $\widetilde{\alpha}$ in the forward direction. Notice that $\xi_s(h^{i_n})$ acts as the identity on S^1_s (and also on $S^1_s \times S^1_s - \Delta$). If there is a subsequence (j_{n_k}) which is constant, then the formula

$$g_{n_k} = h^{-i_{n_k}} f^{j_{n_k}}$$

shows that all $\xi_s(g_{n_k})$ act in exactly the same way on S^1_s. Then $\overline{\xi}_s(\overline{g}_{n_k})$ is constant and since $\overline{\xi}_s$ is faithful, then the sequence $(\overline{g}_{n_k})$ is also constant, a contradiction to the hypothesis. So up to subsequence we may assume (say) that j_n converges to infinity (as opposed to converging to minus infinity) when $n \to \infty$. Then

$$\xi_s(g_n) = \xi_s(h^{-i_n} f^{j_n}) = \xi_s(f^{j_n})$$

and p is the sink for the sequence $\xi_s(g_n)$ acting on S^1_s and q is the source. This proves the convergence group property in this case.

From now on we assume up to subsequence that $\xi_s(g_n)((p,q)) \neq (p,q), (q,p)$ for all n. In fact by the same arguments we can assume that all $\xi_s(g_n)((p,q))$ are distinct. Since the orbit of (p,q) under $\pi_1(M)$ is discrete in $S^1_s \times S^1_s - \Delta$, then up to subsequence $\xi_s(g_n)((p,q))$ converges to a point (z,z) in $S^1_s \times S^1_s$. These arguments work for *any* closed orbit α.

We now show that $\xi_s(g_n)$ has a subsequence with the source/sink behavior. Fix an identification of S^1_s with the unit circle S^1. Since Φ is $\mathbb{R}$–covered, then the set of closed orbits is dense [2]. Find a pair p_1, q_1 so that (p_1, q_1) is a periodic point, very close to $(-1,1)$ and p_1, q_1 not disconnecting $-1, 1$ in S^1. By the above arguments, up to subsequence $\xi_s(g_n)((p_1,q_1))$ converges to a single point (z,z) in $S^1 \times S^1$. Therefore one interval I_1 of S^1 defined by the pair p_1, q_1 converges to z under $\xi_s(g_n)$. The interval I_1 has length close to half the length of the circle S^1. We work by induction assuming that an interval I_i has been produced. Let J_i be the closed complementary interval to I_i. Find a periodic point (p_i, q_i) so that: q_i is in J_i and almost cuts it in half and p_i is in the interior of I_i (switch p_i and q_i if necessary). We already know that $\xi_s(g_n)(p_i))$ converges to z. As before up to another subsequence one of the intervals defined by (p_i, q_i) converges to a point under $\xi_s(g_n)$, which then must be z as p_i is in I_i. Adjoin this interval to I_i to produce I_{i+1} which converges to z under $\xi_s(g_n)$. Let J_{i+1} be the closed complementary interval. Since each step roughly reduces the size of the remaining interval by a factor of $1/2$, then the intervals J_i converge to a single point w. Use a diagonal process and obtain a sequence $\xi_s(g_{n_k})$ with source w and sink z. This finishes the proof of the convergence group property.

Notice that as we mentioned before, we denoted by ξ_s the action on both S^1_s and $S^1_s \times S^1_s - \Delta$. □

Convention We lift to a double cover if necessary so that $\widetilde{\Lambda}^s$ is transversely orientable. Every orientation preserving convergence group acting on the circle is equivalent in $\mathrm{Homeo}^+(S^1)$ to a Fuchsian group; see Gabai [26] and Casson and Jungreis [13]. Let Γ be $\overline{\xi}_s(Q)$. Hence Γ is equivalent to a Fuchsian group T. Here $O = \mathbb{H}^2/T$ is a hyperbolic 2–dimensional orbifold.

We have a conjugation $\psi\colon S^1_s \to S^1$ between the action of Γ on S^1_s and a Fuchsian action T on S^1. Lift ψ to a homeomorphism $\widetilde{\psi}\colon \mathcal{H}^s \to \mathbb{R}$. Let g in $\pi_1(M)$ and we also think of g as acting on $\mathcal{H}^s$. Then

$$\psi \circ \overline{\xi}_s(\overline{g}) \circ \psi^{-1} = \psi \circ \xi_s(g) \circ \psi^{-1}$$

is the ideal map of a Möbius transformation and hence $\widetilde{\psi} g (\widetilde{\psi})^{-1}$ is a projective transformation of $\mathbb{R}$. This shows that the foliation Λ^s is transversely projective. As shown by the first author in [2], this implies that the flow Φ is up to a finite cover, topologically equivalent to a geodesic flow in the unit tangent bundle of a hyperbolic surface. This finishes the proof of Theorem 4.1. □

Remark One may ask whether Theorem 4.1 can be improved to remove the finite covers condition, perhaps by considering geodesic flows in unit tangent bundles of hyperbolic orbifolds. But this is not possible in general, because one can unwrap the fiber direction. We explain this: suppose Φ' is the geodesic flow in $T_1 S$, where S is a closed hyperbolic orbifold. Let α be a closed orbit of Φ', that is, it comes from a closed geodesic γ in S, where for simplicity we assume that γ does not pass through a singularity of S. Then the vectors in $T_1 S$ projecting to γ in S form a torus T' in $T_1 S$ and there are *exactly* two closed orbits in T' corresponding to the two directions along γ. To get a counterexample start with $M = T_1 R$ where again for simplicity R is a nonsingular hyperbolic surface. Let M_1 be a finite cover of M obtained by unwrapping the fiber direction some number n of times. Then M_1 is Seifert fibered and the geodesic flow in $T_1 R$ lifts to an Anosov flow in M_1. Any torus in T in M_1 projects to a torus in $T_1 R$ and this is homotopic to a torus over a closed geodesic of R, but traversed n times in the fiber direction. This implies that the original torus is homotopic to one which has $2n$ closed orbits, and hence cannot be a torus of the geodesic flow of a hyperbolic orbifold. Hence the Anosov flow in M_1 cannot be topologically equivalent to a geodesic flow, but is equivalent to a finite cover of a geodesic flow.

Examples and counterexamples

Recall that in a one-prong pseudo-Anosov flow we allow the existence of one-prongs. One–prong pseudo-Anosov flows can behave completely differently from pseudo-Anosov flows. In particular it is well known that there are one-prong pseudo-Anosov flows in $S^2 \times S^1$, so the manifold M need not be irreducible and the universal cover need not be $\mathbb{R}^3$.

Here we introduce two new classes of examples of one-prong pseudo-Anosov flows.

(1) Let R be a closed hyperbolic surface with an order two symmetry σ which is an isometric reflection along a separating simple closed geodesic α of R. Let M_1 be the unit tangent bundle of R and Φ_1 be the geodesic flow in M_1. The isometry σ sends geodesics of R to geodesics and preserves length along geodesics. It induces a map σ_* in M_1 which has order 2. Let M be the quotient of M_1 by the map σ_*. The map σ_* does not act freely: the fixed points correspond exactly to the tangent vectors to α; there are two closed orbits α_1, α_2 of Φ_1 which are fixed pointwise by σ_*. These correspond to the two directions in α. Hence M is an orbifold, but admitting a natural manifold structure so that the projection map $M_1 \to M$ is an order 2 branched covering map. The flow Φ_1 induces a flow Φ in M because σ sends geodesics to geodesics and preserves length. The stable/unstable foliations of Φ_1 are invariant under σ_* so induce stable/unstable foliations of Φ. The stable leaf of Φ_1 through α_1 folds in two, producing a one-prong singularity of Φ and similarly for α_2. The flow Φ is an example of a one-prong pseudo-Anosov flow. Alternatively the manifold M is obtained as follows: let R_1, R_2 be the closures of the two components of $R - \alpha$. The unit tangent bundle of R_1 is homeomorphic to $R_1 \times S^1$, with boundary a torus Z with two closed curves corresponding to α_1 and α_2. The map σ_* identifies one complementary annulus of α_1, α_2 in Z to the other one with no shearing. This is obtained by a Dehn filling of Z where $\{t\} \times S^1$ (a Seifert fiber) is the meridian. Therefore M is homeomorphic to the union of $N_1 = R_1 \times S^1$ and a solid torus. This is almost a graph manifold: it is the union of Seifert fibered spaces, but M is not irreducible: Take a nonperipheral arc l in R_1. Then $l \times S^1$ is an annulus in $R_1 \times S^1$ which is capped off with two discs in the solid torus to produce a sphere which is nonseparating in M and hence clearly does not bound a ball in M.

Remark This example and the next work whenever the hyperbolic surface R admits an isometric reflection along a collection of simple closed geodesics $\{\alpha_i\}$. For simplicity of exposition we describe the examples in (1) and (2) with a single geodesic α.

(2) The second class of examples is obtained by a modification of example (1) in order to be in a Seifert fibered manifold. The modification is that the gluing of the annuli in ∂N_1 is done with a shearing. The notation is the same as in example 1): R is the hyperbolic surface with a geodesic α of symmetry and R_1, R_2 the closures of the components of $R - \alpha$. The unit tangent bundle of R is M_1 and N_1, N_2 are the restrictions to unit vectors in R_1 and R_2 respectively. We use two tori: $\partial N_1 = T_1$ and $\partial N_2 = T_2$. These are glued to form M_1. Put coordinates (θ_1, θ_2) in T_1, (a_1, a_2) in T_2 as follows: T_1 consists of the unit vectors along α. Parametrize α by arc length parameter t where $0 \leq t \leq l_0$ and l_0 is the length of α. Let $\theta_1 = 2\pi t / l_0$. Let θ_2 be the angle between the unit tangent vector to α and the vector v, where $\theta_2 = 0$ corresponds to the forward direction of α_1. Also $\theta_2 = \pi$ corresponds to α_2 and $0 < \theta_2 < \pi$ are the vectors exiting N_1 and entering N_2. Put coordinates (a_1, a_2) in T_2 so that the gluing map to create M_1 is η: $T_1 \to T_2$ given by $a_1 = \theta_1, a_2 = \theta_2$ (essentially the same coordinates). Notice that vectors with $0 < a_2 < \pi$ are entering N_2 and vectors with $\pi < a_2 < 2\pi$ are entering N_1.

In N_1 we consider the restriction of the geodesic flow of R. We collapse $\partial N_1 = T_1$ to an annulus as follows. Let A_1 be the strip $0 \leq \theta_2 \leq \pi$ in T_1 and let A_2 be the strip $\pi \leq \theta_2 \leq 2\pi$ in T_1. Let n be an integer. We glue A_1 to A_2 by

$$(*) \qquad f(\theta_1, \theta_2) = (\theta_1 + 2n\theta_2, 2\pi - \theta_2).$$

Let M be the quotient of N_1 by this gluing and let Φ be the induced flow from the geodesic flow in N_1. Notice that the flow in N_1 is outgoing in the interior of A_1 and incoming in the interior of A_2. In addition, the angle between flow lines and T_1 depends only on θ_2 and not on θ_1 (by definition) and so by formula $(*)$ this produces a flow Φ in M which is smooth outside of the closed orbits α_1, α_2. Here we abuse notation and continue to call α_1, α_2 their projections to M.

Let A be the annulus which is the quotient of A_1, A_2 by the gluing. Let M_2 be the double branched cover of M obtained by double branched cover (opening up) along A. This M_2 can be cut along the torus T which is the preimage of A. The closure of the two complementary components of T are homeomorphic to N_1 and N_2 and still denoted by N_1, N_2. We think of N_1 as the unit tangent bundle of R_1. We can also think of N_2 as the unit tangent bundle of R_2: this is because N_2 under the branched cover is another copy of N_1, which is isometric to N_2 by the map σ_* induced by the symmetry σ of the surface R. Let T_1, T_2 be the corresponding boundaries of N_1, N_2, with the corresponding coordinates (θ_1, θ_2) and (a_1, a_2) as above. Therefore M_2 is obtained by a certain gluing of map g from T_1 to T_2.

We first extend the map f to an involution on the entire torus T_1: in A_2 (which is the region $\pi \le \theta_2 \le 2\pi$), the map f has the same formula $f(\theta_1, \theta_2) = (\theta_1 + 2n\theta_2, 2\pi - \theta_2)$. Clearly f is an involution in T_1.

Claim In order to obtain the flow Φ in M, the gluing from T_1 to T_2 in the (θ_1, θ_2), (a_1, a_2) coordinates is given by

$$g\colon T_1 \to T_2, \quad g(\theta_1, \theta_2) = (\theta_1 + 2n\theta_2, \theta_2).$$

In order to prove the claim we need to show that when restricted to the annulus A_1 then $f = \sigma_* g$. Recall that σ_* restricted to T_2 (which is identified with T) has the form $\sigma_*\colon T_2 \to T_1$, $\sigma_*(a_1, a_2) = (a_1, 2\pi - a_2)$. It is now clear that $f = \sigma_* g$ in A_1. By the extension of f to A_2, this also holds in A_2. This proves the claim.

Let Φ_2 be the lift of the flow Φ to M_2. This flow Φ_2 is the geodesic flow in R_1 when restricted to N_1 and the geodesic flow of R_2 when restricted to N_2. The gluing is given by the map g described above. The map g is a shearing. In a very nice result, Handel and Thurston [31] studied exactly this example and they proved that the flow Φ_2 in M_2 is an Anosov flow which is volume preserving. Therefore this flow has stable and unstable foliations which project to stable/unstable foliations of Φ: this is because if two orbits in M_2 are asymptotic then their projections to M are asymptotic and vice versa. The projection from M_2 to M is locally injective and smooth except along α_1 and α_2, where it is branched 2 to 1. Hence the stable/unstable foliations in M are nonsingular except possibly at α_1, α_2. Since the projection is 2 to 1 and stable leaves go to stable leaves, then along the stable leaf of α_1 the stable leaf folds in two and similarly for the unstable leaf and likewise for α_2. Therefore Φ is smooth everywhere except at α_1, α_2 which are one-prong singularities. We conclude that Φ is a one-prong pseudo-Anosov flow.

Finally M can be thought as a Dehn filling of N_1 along T_1. We determine the new meridian. Under the map f from A_1 to A_2, the segment $\theta_1 = 0$, $0 \le \theta_2 \le \pi$ in A_1 is glued to the segment $(2n\theta_2, 2\pi - \theta_2)$, $0 \le \theta_2 \le \pi$ in A_2. This last segment goes from $(0, 2\pi)$ to $(2n\pi, \pi)$ linearly. It follows that this is the new meridian which is then the $(-n, 1)$ curve.

When $n = 0$, this is exactly the same construction as in the first example which makes the fiber in N_1 nullhomotopic. When $n \ne 0$, the curve which becomes nullhomotopic is not $\{p\} \times S^1$. It follows that the resulting manifold M is Seifert fibered.

Conclusion If one allows 1–prongs, then Seifert fibered manifolds can admit one-prong pseudo-Anosov flows with singularities as opposed to what happens with pseudo-Anosov flows. Theorem 4.1 does not hold for one-prong pseudo-Anosov flows.

This poses the following questions: Suppose that Φ is a one-prong pseudo-Anosov flow in M Seifert fibered (closed). Can one show that there are no p–prongs with $p \geq 3$? Can one show that Φ has a branched cover to an Anosov flow in a Seifert manifold?

Remark With this description of geodesic flows we now mention the following, which will be extremely useful later on in the article. Here is an explicit example of an embedded, incompressible Klein bottle in a manifold with an Anosov flow. Let Φ be the geodesic flow of a nonorientable hyperbolic surface S and α an orientation reversing simple geodesic. Let A be the unit tangent bundle of S restricted to α and α_1, α_2, the two orbits of Φ associated to the two directions of α. Consider tubular neighborhoods of α_1, α_2. These are solid tori, and A in these neighborhoods wraps around each of these periodic orbits twice producing a Möbius band, which contains the periodic orbit, and with boundary a closed curve homotopic to the double of the periodic orbit. It follows that the closure of A is the union of an annulus (outside the solid tori) and two Möbius strips and therefore A is a Klein bottle. This Klein bottle is embedded and incompressible. This is a typical example of *Birkhoff–Klein bottle*; see the formal definition in Section 6. A tubular neighborhood of this Klein bottle is homeomorphic to the twisted line bundle over the Klein bottle.

5 Pseudo-Anosov flows in manifolds with virtually solvable fundamental group

In this section we first do a detailed analysis of maximal subgroups of $\pi_1(M)$ stabilizing a given chain of lozenges. Conversely given a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$ we analyse the uniqueness of chains of lozenges invariant under this subgroup. These results are foundational for understanding any $\mathbb{Z}^2$ subgroup of $\pi_1(M)$ and they are fundamental for the analysis of pseudo-Anosov flows in manifolds with virtually solvable fundamental groups. The results are later used for other results in this article. We also expect that these results will be useful for further study of pseudo-Anosov flows in toroidal manifolds.

In this section let K denote the Klein bottle. We first need a result from 3–dimensional topology. Let F be a compact surface with a free involution τ. Then we have that $M = (F \times I)/(x,t) \sim (\tau(x), 1-t)$ is a *twisted I–bundle* over the surface $F' = F/x \sim \tau(x)$ and F is the associated 0–sphere bundle; see [32, page 97].

Lemma 5.1 *Let N be an irreducible, compact 3–manifold with finitely generated fundamental group which is torsion free and has a finite index subgroup isomorphic to $\mathbb{Z}^2$. Then N is either an I–bundle or a twisted I–bundle over a surface of zero Euler characteristic. In particular $\pi_1(N)$ is isomorphic to either $\mathbb{Z}^2$ or $\pi_1(K)$. In addition if N is orientable, then either $N = T^2 \times I$ or $N = (T^2 \times I)/(x,t) \sim (\tau(x), 1-t)$ is a twisted I–bundle over the Klein bottle $T^2/x \sim \tau(x)$ which is one sided in N.*

Proof Suppose first that N is closed. Then take a finite cover N' with $\pi_1(N') = \mathbb{Z}^2$. Since the finite cover is irreducible, this is not possible [32]. Hence ∂N is not empty. Suppose that boundary of N is compressible. By the loop theorem [32] there is a curve in ∂N, not nullhomotopic in ∂N, but bounding an embedded disc D in N. Cutting along D, shows that $\pi_1(N)$ is either a free product or an amalgamated free product along a trivial group, hence a free product with $\mathbb{Z}$. In either case the free product would either not contain a $\mathbb{Z}^2$ (it would be infinite cyclic) or would contain a free group of rank greater than or equal to 2, in which case it could not contain $\mathbb{Z}^2$ with finite index. Hence ∂N is incompressible. If it has a component of genus greater than or equal to 2 then as above it would have a rank 2 free subgroup, again contradiction. If it has a component which is a projective plane, then $\pi_1(N)$ has elements of order 2, contrary to hypothesis. Since N is irreducible, no component of ∂N is a sphere, as $\pi_1(N)$ is not trivial. We conclude that every boundary component of N is either a torus or a Klein bottle.

Let F be one such component. Because F is incompressible and $\pi_1(N)$ has a finite index subgroup isomorphic to $\mathbb{Z}^2$, then we have that $\pi_1(F)$ has finite index in $\pi_1(N)$. By [32, Theorem 10.5], either

(i) $\pi_1(N) = \mathbb{Z}$,

(ii) $\pi_1(N) = \pi_1(F)$ with $N \cong F \times I$ or

(iii) $\pi_1(F)$ has index 2 in $\pi_1(N)$ and N is a twisted I–bundle over a compact manifold F', with F the associated 0–sphere bundle.

In our situation case (i) cannot happen. In case (ii) $\pi_1(N)$ is either $\mathbb{Z}^2$ or $\pi_1(K)$ and we are done. In case (iii) $\pi_1(N)$ is isomorphic to $\pi_1(F')$ as there is a deformation retract from N to F'. Here F' is a closed surface which has a double cover either the torus or the Klein bottle. Hence again F' is the torus or the Klein bottle and we also conclude that $\pi_1(N)$ is either $\mathbb{Z}^2$ or $\pi_1(K)$. The last statement is easy given the above. This finishes the proof of the lemma. □

Note that both the torus and the Klein bottle have double covers homeomorphic to themselves. The manifolds in Lemma 5.1 can be either orientable or not. It is easy to construct a compact manifold N which is a twisted I–bundle over the Klein bottle (with quotient surface a Klein bottle). This manifold has boundary a Klein bottle and an orientation double cover N_2 which is a twisted I–bundle over the torus (with quotient surface a Klein bottle, which is one sided in N_2). Finally N has an order 4 cover homeomorphic to $T^2 \times I$.

Lemma 5.2 *Suppose that M has a pseudo-Anosov flow Φ. Let $\mathcal{C}$ be a bi-infinite chain of lozenges of Φ. Let G be the stabilizer of $\mathcal{C}$ in $\pi_1(M)$. Then G is isomorphic to a subgroup of $\pi_1(K)$. In particular, G is torsion free and it contains a unique maximal abelian subgroup, which has index at most two, and which is either trivial, (infinite) cyclic or isomorphic to $\mathbb{Z}^2$.*

Proof The proof will reveal the structure of the stabilizer of $\mathcal{C}$ and not just show that it is isomorphic to a subgroup of $\pi_1(K)$. In this proof cyclic means infinite cyclic. Let α be a corner in $\mathcal{C}$. The chain $\mathcal{C}$ corresponds to a linear subtree T_0 of the tree $\mathcal{G}(\alpha)$. It defines a homomorphism ρ: $G \to \mathrm{Aut}(T_0)$. The kernel $\mathcal{K}$ of ρ stabilizes every corner of $\mathcal{C}$, and thus, is either cyclic or trivial.

Assume first that G preserves the orientation on T_0. Then $\rho(G)$ is a group of translations along T_0, ie trivial or cyclic. In the former case, $G = \mathcal{K}$ is either trivial or cyclic. In the latter case, if $\mathcal{K}$ is trivial then G is isomorphic to $\rho(G)$ and hence trivial or cyclic. If $\mathcal{K}$ is cyclic then G is an extension of $\mathbb{Z}$ by $\mathbb{Z}$. It is an elementary fact that any such extension splits and hence G is either $\mathbb{Z}^2$ or $\pi_1(K)$. We are done.

Hence from now on assume that some element g of G reverses the orientation of T_0. Hence g leaves either a vertex or an edge of T_0 invariant. According to Proposition 2.16(4), g preserves a corner α. That is g does not act as an inversion on an edge. Let s be a generator of the G–stabilizer of α; in particular this stabilizer is not the identity and is isomorphic to $\mathbb{Z}$. Then s reverses the orientation of T_0 (otherwise all elements in G leaving α invariant would preserve orientation) and s^2 is in $\mathcal{K}$. On the other hand, every element of $\mathcal{K}$ fixes α and preserves the orientation: it must be a power of s^2, which therefore generates $\mathcal{K}$. As before there are two options for $\rho(G)$. One option is that $\rho(G) = \rho(\langle s \rangle)$ and therefore G is generated by s and is cyclic. Otherwise $\rho(G)$ has at least one translation. Select h in G such that $\rho(h)$ is a translation along T_0 of minimal length. In this case it is easy to see that s, h generate G.

By considering the action on the set of vertices of T_0 one sees that hsh preserves α. It is also in G so $hsh = s^i$ where i is odd. Similarly $h^{-1}sh^{-1} = s^j$, j odd. Now we use 3–manifold topology.

Let G' be the subgroup of G preserving the orientation on T_0. The first case of the proof shows that G' has a subgroup of index less than or equal to 2 isomorphic to $\mathbb{Z}^2$, so G has a subgroup of index less than or equal to 4 isomorphic to $\mathbb{Z}^2$ (we stress that we want a subgroup of index 2 isomorphic to $\mathbb{Z}^2$, so more work is needed). Let U be the cover of M associated to G. Then U is irreducible and $\pi_1(U)$ is torsion free, because $\pi_1(M)$ is torsion free, as its universal cover is homeomorphic to $\mathbb{R}^3$. By Scott's Core Theorem [32] there is a compact core N for U. We can assume that no boundary component of N is a sphere, by attaching 3 balls to such components, without affecting the fundamental group. Now apply the previous lemma to show that $G = \pi_1(N)$ is isomorphic to either $\mathbb{Z}^2$ or $\pi_1(K)$.

Finally if G is not abelian then G is isomorphic to $\pi_1(K)$ and it is an elementary algebra fact that G has a unique maximal abelian subgroup, and that this maximal abelian subgroup has index 2 and is isomorphic to $\mathbb{Z}^2$. □

Remark Consider the most complicated case (G contains a $\mathbb{Z}^2$ subgroup). Using graphs of groups one can quickly produce a short exact sequence $1 \to \mathbb{Z} \to G \to A \to 1$, where A is either $\mathbb{Z}$ or D_∞, the infinite dihedral group. The difficulty occurs in the dihedral group case: in particular in our situation the exact sequence does not split, for otherwise G would have torsion. Hence the graph of groups analysis gets more involved and has many possibilities.

Conversely, we have the following.

Lemma 5.3 *Let G be a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$. Assume that Φ is not product. Then G preserves a bi-infinite chain of lozenges.*

Proof If $G \sim \mathbb{Z} \oplus \mathbb{Z}$ acts freely on the orbit space $\mathcal{O}$, then it was proved in [18] that Φ is product, contrary to hypothesis. Hence there is g in G with a fixed point in $\mathcal{O}$. If $g = (g')^n$ where g' is in G and $|n| > 1$, then g' also does not act freely on $\mathcal{O}$ (Proposition 2.16(1)). Hence we may assume that g is indivisible in G. Choose h in G so that h, g form a basis of G. Consider the tree $\mathcal{T} = \mathcal{G}(g)$: since G is abelian, then G acts on $\mathcal{T}$. If f is an element of G admitting a fixed point in $\mathcal{T}$, then some power of f leaves invariant all vertices of $\mathcal{T}$ and likewise for g. It follows that g and f admit a common power: $g^p = f^q$. Since f, g are in $G \cong \mathbb{Z}^2$ then f, g generate a cyclic group. But g is indivisible in G, implying that f is a power of g. Hence, $G/\langle g\rangle \sim \mathbb{Z}$ is a cyclic group acting freely on the vertices of the tree $\mathcal{T}$. According to Proposition 2.16(4), an element in $G/\langle g\rangle$ cannot reverse an edge of $\mathcal{T}$. It follows that $G/\langle g\rangle$ acts freely on $\mathcal{T}$, and that there is an invariant axis for this cyclic group therein. It provides a bi-infinite G–invariant chain of lozenges $\mathcal{C}$. In particular the arguments show that g fixes all the vertices in $\mathcal{C}$. □

Definition 5.4 [18] Let $\mathcal{C}$ be an s–scalloped bi-infinite chain of lozenges. The s–scalloped region defined by $\mathcal{C}$ is the union of all lozenges in $\mathcal{C}$ with the half-leaves of $\mathcal{O}^u$ common to two adjacent lozenges in $\mathcal{C}$. One defines similarly u–scalloped regions. A scalloped region is a s–scalloped or u–scalloped region; it is an open subset of $\mathcal{O}$.

It may happen in the situation of Lemma 5.3 that the G–invariant chain is not unique, but only in a very special situation as shown by the next lemma.

Lemma 5.5 *Let G be a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$. Assume that G preserves two different chains of lozenges. Then, one of these chains is s–scalloped, and the other is u–scalloped. Moreover the associated u–scalloped and s–scalloped regions are the same, that is, they are the same subset of the orbit space $\mathcal{O}$.*

Proof In this proof we consider all objects in $\mathcal{O}$. Let $\mathcal{C}$, $\mathcal{C}'$ be two different G–invariant chains of lozenges. First of all notice that the set of vertices of a chain is a linear set isomorphic to a subset of $\mathbb{Z}$ and the stabilizer of any vertex is at most $\mathbb{Z}$. Since $G \cong \mathbb{Z}^2$ there is an element of G acting freely on the set of vertices of $\mathcal{C}$ (or $\mathcal{C}'$). It follows that $\mathcal{C}, \mathcal{C}'$ are bi-infinite chains of lozenges.

Let g be an element of G fixing every corner in $\mathcal{C}$, and let f be an element of G fixing every corner of $\mathcal{C}'$; see proof of the previous lemma. Suppose first that g and f share a common nontrivial power: $g^p = f^q$, $p, q \neq 0$. Since G is abelian it acts on $\mathcal{G}(g^p)$ and also $\mathcal{G}(g) \subset \mathcal{G}(g^p)$, so $\mathcal{C}$ is an invariant axis for G acting on $\mathcal{G}(g^p)$. Similarly $\mathcal{C}'$ is a G–invariant axis in $\mathcal{G}(f^q)$. Since these trees are the same, it now follows that $\mathcal{C} = \mathcal{C}'$, contradiction.

Hence, replacing G by a finite index subgroup if necessary, one can assume that f, g form a basis of $G \sim \mathbb{Z}^2$.

Let β be a corner of $\mathcal{C}'$. We claim that β cannot be in $\mathcal{C}$ or in one of its boundary sides. Suppose not. There is h nontrivial in G fixing β and therefore fixing every corner of $\mathcal{C}'$. As h leaves $\mathcal{C}$ invariant, then β has to be a corner of $\mathcal{C}$. This would produce an element in G fixing every corner of $\mathcal{C}$ and every corner of $\mathcal{C}'$ and hence some powers of f and g coincide. The previous paragraph shows this is impossible. Let now c be a path in $\mathcal{O}$ joining β to an element δ in the union of the lozenges in $\mathcal{C}$, and disjoint from the corners of $\mathcal{C}$. We assume that c avoids the singular orbits in $\mathcal{O}$. Notice that the union of corners of $\mathcal{C}$ forms a discrete set in $\mathcal{O}$. Consider the intersection V between c and the union of stable and unstable half-leaves contained in the boundary of the lozenges of $\mathcal{C}$. By the above this intersection is nonempty. Assume first that V is finite. Let γ be the first element of V met while traveling along c

from β to δ. Then γ lies on the boundary of a lozenge C of $\mathcal{C}$, for instance the boundary component is a stable half leaf L containing a corner α of $\mathcal{C}$. Let C' be the other lozenge in $\mathcal{C}$ admitting also α as a corner: there is a half leaf K, contained in the boundary of C' and such that the union $L \cup K \cup \alpha$ is an embedded line in $\mathcal{O}$, which moreover disconnects $\mathcal{C}$ from β. In addition this properly embedded line is unique with these properties. Since $\mathcal{C}$ and β are f–invariant, it now follows that $L \cup K \cup \alpha$ is f–invariant, and hence $f(\alpha) = \alpha$, where α is a corner of $\mathcal{C}$. Contradiction.

Therefore V is not finite: it admits an accumulation point γ. Since $\mathcal{O}^s$ and $\mathcal{O}^u$ are transverse to each other outside the singular points, γ is an accumulation point of a sequence $F_n \cap c$, where the F_n are leaves in the boundary of lozenges in $\mathcal{C}$. In addition we may assume that all F_n have all the same type, for example all F_n are leaves of $\mathcal{O}^s$. Let L be the leaf of $\mathcal{O}^u$ through γ: it intersects all the F_n for n sufficiently big. It follows that $\mathcal{C}$ contains an infinite u–scalloped subchain. Since $\mathcal{C}$ is G–invariant, the entire chain $\mathcal{C}$ has to be a bi-infinite u–scalloped chain. Hence it defines a u–scalloped region U.

Similarly, $\mathcal{C}'$ has to be scalloped, and defines a scalloped region U'.

Now the key point is the following: in [18] the following facts are shown.

(i) We can choose h in G acting freely on $\mathcal{O}$.

(ii) The leaves of $\mathcal{O}^s$ (respectively $\mathcal{O}^u$) intersecting U define a G–invariant subline I^s in $\mathcal{H}^s$ (respectively a G–invariant subline I^u in $\mathcal{H}^u$); here we are thinking of $\mathcal{H}^s$ as the leaf space of $\mathcal{O}^s$.

(iii) Every leaf in I^s intersects every leaf in I^u, and these intersections occur in U.

(iv) Every point in U is the intersection of a leaf in I^s and a leaf in I^u.

Similarly, the open scalloped region U' provides G–invariant sublines J^s and J^u in $\mathcal{H}^s$ and $\mathcal{H}^u$, such that every leaf in J^s intersects every leaf in J^u at a point in U'. But since h acts freely, h–invariant lines in $\mathcal{H}^s$, $\mathcal{H}^u$ are unique [18]. Thus, $I^s = J^s$ and $I^u = J^u$. The equality $U = U'$ follows.

If the chain $\mathcal{C}'$ was u–scalloped, as $\mathcal{C}$, then it would be equal to $\mathcal{C}$ since it defines the same scalloped region. Hence, $\mathcal{C}'$ is s–scalloped. The lemma follows. □

Corollary 5.6 *Let G be a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$ and h an element of $\pi_1(M)$ such that $hG'h^{-1} = G'$, where G' is a finite index subgroup of G. Then h preserves any G–invariant chain of lozenges.*

Proof Let $\mathcal{C}$ be a G–invariant chain of lozenges. Then, $\mathcal{C}$ is G'–invariant, and $h(\mathcal{C})$ is $hG'h^{-1} = G'$–invariant. According to Lemma 5.5, if $\mathcal{C}$ is not scalloped, then $\mathcal{C}$ is the unique G'–invariant chain: hence we have $h(\mathcal{C}) = \mathcal{C}$. If not, $\mathcal{C}$ is scalloped, for example suppose that $\mathcal{C}$ is s–scalloped. Again by Lemma 5.5, $\mathcal{C}$ is the unique s–scalloped G'–invariant chain, and since $h(\mathcal{C})$ is also s–scalloped, the equality $h(\mathcal{C}) = \mathcal{C}$ follows. □

As a corollary of these results, we get the description of pseudo-Anosov flows in manifolds with virtually solvable fundamental group (Theorem 5.7).

Theorem 5.7 *Let Φ be a pseudo-Anosov flow in M^3 with $\pi_1(M)$ virtually solvable. Then Φ has no singularities and is product. In particular Φ is topologically equivalent to a suspension Anosov flow.*

Proof First notice that the fact that each leaf of $\widetilde{\Lambda}^s$ intersects every leaf of $\widetilde{\Lambda}^u$ is invariant up to taking finite covers and so is the existence of singularities. Hence we can take finite covers at will. Up to a finite cover, one can assume that $\pi_1(M)$ is solvable. Notice that as M has a pseudo-Anosov flow then M is irreducible. Since $\pi_1(M)$ is solvable, classical 3–manifold topology results [32] imply that M fibers over the circle with fiber a surface S which has solvable fundamental group. The surface S can only be the torus or the Klein bottle K. Up to another finite cover one can assume that S is actually the torus.

Assume that Φ is not product. Then, according to Lemma 5.3, $\pi_1(S)$ preserves a chain of lozenges. Since $\pi_1(S)$ is normal in $\pi_1(M)$, it follows from Corollary 5.6 that this chain of lozenges is $\pi_1(M)$–invariant. According to Lemma 5.2, $\pi_1(M)$ is a finite index extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. This contradicts the fact that M fibers over the circle with fiber T^2. This finishes the proof. □

6 π_1–injective tori in optimal position

Given a π_1–injective torus in a 3–manifold M with a pseudo-Anosov flow Φ, we look for a representative in its homotopy class which is in optimal position with respect to the flow Φ. First we define Birkhoff annuli.

Definition 6.1 A *Birkhoff annulus* is an immersed annulus in M so that each boundary component is a periodic orbit of the flow Φ, and such that the interior of the annulus is transverse to the flow. If the interior is embedded, then the annulus is called *weakly embedded.*

The interior of a Birkhoff annulus is transverse to the flow, and hence is also transverse to the foliations Λ^s, Λ^u. They therefore induce foliations on the annulus denoted by l^s, l^u. These foliations can both be extended to the boundary of the annulus as foliations tangent to the boundary. A singular orbit with p prongs (here again we use that for pseudo-Anosov flows $p \geq 3$) induces a singularity of l^s (or l^u) in the interior of the annulus having negative index $1 - p/2$. Since the Euler characteristic of the annulus is zero, Poincaré–Hopf index formula implies that the interior of the annulus intersects no singular orbits.

Definition 6.2 A Birkhoff annulus is *elementary* if l^s, l^u do not have closed leaves in the interior.

Observe that in the definition of weakly embedded Birkhoff annuli, we did not require the whole annulus to be embedded: it may wrap around each periodic orbit in its boundary, an arbitrary (finite) number of times. It may also be that the two boundary orbits are the same orbit. Notice however that the boundary orbits cannot intersect the interior, as otherwise points near the boundary would produce self intersections in the interior.

Good position of a torus with respect to a pseudo-Anosov flow means that the torus is a union of Birkhoff annuli. If the initial torus is embedded we want to study when the optimal position torus is also embedded. This is tremendously important if one wants to cut the manifold along the tori which separate pieces in the torus decomposition.

We first study under which conditions a chain of lozenges $\mathcal{C}$ may admit a corner α such that for some element g of $\pi_1(M)$ the image $g(\alpha)$ is contained in a lozenge of $\mathcal{C}$. Later on we explain how this concerns the intersections of corner orbits in the Birkhoff annuli with the interior of the annuli.

Definition 6.3 Let $\mathcal{C}$ be a chain of lozenges. If for any element g of $\pi_1(M)$ and for every corner α of $\mathcal{C}$ then the orbit $g(\alpha)$ is not in the interior of a lozenge in $\mathcal{C}$, then $\mathcal{C}$ is called *simple*. The chain $\mathcal{C}$ is called a *string of lozenges* if no corner orbit is singular and consecutive lozenges are never adjacent.

Proposition 6.4 *Let G be a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$ and let $\mathcal{C}$ be a G–invariant chain of lozenges. Suppose that $\mathcal{C}$ is not simple. Then $\mathcal{C}$ is a string of lozenges. In addition G is contained in the fundamental group of a free Seifert fibered piece and no corner of $\mathcal{C}$ is the lift of a singular orbit of Φ.*

Proof The chain $\mathcal{C}$ is not simple. Therefore there is a corner orbit α of $\mathcal{C}$ and an element g in $\pi_1(M)$ so that $g(\alpha)$ is in the interior of a lozenge in $\mathcal{C}$. Once and for all in this proof the transformation g is fixed. The α may change to another corner orbit of $\mathcal{C}$. We stress that the element g is not used all the time in this proof, but whenever it is, it refers to this fixed element. Notice that g is NOT in G as G preserves $\mathcal{C}$ and its corners.

Proof that $\mathcal{C}$ is a string of lozenges We denote by $\{\alpha_i, i \in \mathbb{Z}\}$ the corners of $\mathcal{C}$ and by $\{C_i, i \in \mathbb{Z}\}$ the lozenges of $\mathcal{C}$, so that α_i, α_{i+1} are the corners of C_i for each integer i. Moreover, by shifting the indices ($i \in \mathbb{Z}$) we may assume that $\alpha = \alpha_0$. By assumption there is an integer k so that $\beta = g(\alpha)$ is contained in C_k. We will prove that both corners α_k, α_{k+1} of C_k are in the interior of lozenges in $g(\mathcal{C})$. Since the orbit β is in the interior of a lozenge, then β is nonsingular and $\widetilde{W}^s(\beta), \widetilde{W}^u(\beta)$ define exactly 4 quadrants in $\widetilde{M}$. Two of the quadrants contain the corners of C_k. Let W be one of the remaining quadrants. It contains a perfect fit between two sides of the lozenge C_k. Without loss of generality we may assume that these sides are $S = \widetilde{W}^s(\alpha_k)$ and $U = \widetilde{W}^u(\alpha_{k+1})$.

We claim that W does not contain a lozenge with corner in β. Suppose not and call this lozenge D_1. Then D_1 has two sides in $\widetilde{W}^u(\beta)$ and $\widetilde{W}^s(\beta)$. There is a unstable side of D_1, call it E which is contained in an unstable leaf and makes a perfect fit with $\widetilde{W}^s(\beta)$. Since $\widetilde{W}^s(\beta)$ intersects $U = \widetilde{W}^u(\alpha_{k+1})$ transversely, it follows that U separates E from the lozenge C_k. Therefore E cannot intersect any leaf which makes a perfect fit with $\widetilde{W}^u(\beta)$. This is a contradiction and proves the claim.

It follows that the two quadrants defined by β which contain respectively α_k and α_{k+1}, also contain lozenges in $g(\mathcal{C})$. Let D_2, D_3 be these lozenges which are in $g(\mathcal{C})$ and have a corner in β (where g is the fixed element in this proof). Since $\widetilde{W}^s(\alpha_{k+1})$ intersects $\widetilde{W}^u(\beta)$ and $\widetilde{W}^u(\alpha_{k+1})$ intersects $\widetilde{W}^s(\beta)$, the definition of lozenges implies that α_{k+1} is in the interior of one of these lozenges, say D_3. As in the argument above it now follows that the other corners of D_2, D_3 are in the interior of C_{k-1}, C_{k+1} respectively. This procedure can be iterated indefinitely and in both directions. It now follows that *all* $g(\alpha_i)$ are in the interior of lozenges in $\mathcal{C}$. In particular this implies that each $g(\alpha_i)$ (and consequently the same for the orbits α_i) is nonsingular and C_i, C_{i+1} are not adjacent. This shows that $\mathcal{C}$ is a string of lozenges.

In order to conclude, we have to show that up to conjugation G is contained in the fundamental group of a free Seifert piece. Let H be the stabilizer of $\mathcal{C}$ in $\pi_1(M)$, and let H_0 be the maximal abelian subgroup of H (see Lemma 5.2 which shows that H_0 exists and has index less than or equal to 2 in H. Apply that Lemma to the stabilizer H of $\mathcal{C}$ in $\pi_1(M)$). Then $G \subset H_0$; hence we can assume $G = H_0$, ie that G has index at most two in H.

We stress the following very important fact: the above arguments show that for any corner γ of $\mathcal{C}$ there are exactly two lozenges which have corner γ. The remaining quadrants of γ do NOT have lozenges with corner γ. As a corollary, we obtain that the tree $\mathcal{G}(\alpha)$ coincides with $\mathcal{C}$. Similarly, $\mathcal{G}(\beta) = g(\mathcal{C})$. In particular $\mathcal{C} = \mathcal{G}(\alpha)$ is a simplicial linear tree. This implies the following fact:

> If $z \in \pi_1(M)$ and for some corner γ of $\mathcal{C}$, the orbit $z(\gamma)$ is also a corner of $\mathcal{C}$, then $z(\mathcal{C}) = \mathcal{C}$.

Claim 1 One can assume that the manifold M is orientable.

Suppose that M is not orientable and let $p^{or}\colon M_2 \to M$ be the orientation double cover of M, with lifted flow Φ_2. Let l^s be the set of stable leaves either intersecting a lozenge in $\mathcal{C}$ or containing a corner orbit in $\mathcal{C}$. This set is order isomorphic to the reals $\mathbb{R}$. Similarly define l^u. One can use the arguments above to show that l^s, l^u are invariant under g. This is because every $g(\alpha_i)$ is in the interior of a lozenge in $\mathcal{C}$, so the arguments above show that if q is any corner of $\mathcal{C}$, then $g(q)$ is also in the interior of a lozenge in $\mathcal{C}$. This implies the g invariance of l^s, l^u. If g preserves the order in l^s then the arguments above imply that g also preserves the order in l^u: this is because one can order l^s, l^u so that "high elements" in l^s intersect high elements in l^u. Since intersection is preserved by the action of g the statement follows. This implies that g preserves orientation in $\mathcal{O} \cong \mathbb{R}^2$. If on the other hand g reverses order in l^s, the same argument shows that g also reverses order in l^u and hence g again preserves orientation in $\mathcal{O}$. Since clearly g preserves the flow direction it follows that in any case g preserves orientation in M. Therefore g is an element of $\pi_1(M_2)$.

Similarly, one proves for every element a of G that if a reverses the orientation of l^s, it also reverses the orientation of l^u: G is contained in $\pi_1(M_2)$. Now if P_2 is a free Seifert piece of M_2 whose fundamental group contains G, then $P = p^{or}(P_2)$ is a free Seifert piece in M whose fundamental group contains G. Hence we may assume that $M = M_2$ in the statement of the proposition. Claim 1 is proved.

Notice that it is not true that any $\mathbb{Z}^2$ subgroup of any 3–manifold group consists entirely of orientation preserving elements. For instance consider the twisted I–bundle over the torus T^2. The one sided torus in the middle has orientation reversing elements. Glue two copies of this to produce examples in closed 3–manifolds.

Assumption From now on we can assume that M is orientable.

Since g preserves l^s, there are two options.

Case I: g preserves orientation in l^s Then there is k in $\mathbb{Z}$ so that $g(\alpha_i)$ is always in the interior of C_{k+i}.

Case II: g reverses orientation in l^s In this case we will reindex the α_i. This forces a reindexing of the C_i as C_i is always the lozenge with corners in α_i and α_{i+1}. Here there are two possibilities: First if k is even, then shift the $i's$ (by $i \to i - k/2$) so that $g(\alpha_i)$ is in the interior of C_{-i} for all i. If on the other hand k is odd, then first shift the $i's$ by $i \to i - (\frac{k+1}{2})$ which results in $g(\alpha_i)$ is in C_{-i-1}, then do a reflection $i \to -i$, which results in $g(\alpha_i)$ is in C_{-i}. So regardless of k even or odd in Case II, we can adjust the indices so that $g(\alpha_i)$ is always in C_{-i}.

Then up to choosing a new α_0 and perhaps changing i to $-i$, it follows that $g(\alpha_i)$ is in the interior of C_{-i} for all i.

Claim 2 There is an element h_0 of G such that the centralizer $Z(h_0)$ (in $\pi_1(M)$) is not abelian.

Let f denote a generator of the stabilizer in G of every α_i, and let h be an element of G acting freely on $\mathcal{C}$: there is an integer p so that $h(\alpha_j) = \alpha_{j-p}$, $h(C_j) = C_{j-p}$.

Assume first that we are in Case I. For every integer i, $g(\alpha_{pi})$ is contained in C_{k+pi}, hence all the $h^i g(\alpha_{pi})$ lie in C_k. On the other hand, one can produce as in [3] a f–invariant proper embedding of $[0,1] \times \mathbb{R}$ into $\widetilde{M}$, so that $\{0,1\} \times \mathbb{R}$ maps into the corner orbits of C_k, $(0,1) \times \mathbb{R}$ maps into the interior of the lozenge and transversely to $\widetilde{\Phi}$. The image of this embedding projects to an embedded annulus $\widehat{A}$ in $\widetilde{M}/\langle f \rangle$, which itself projects to an *immersed* annulus A in M, transverse in its interior to the flow Φ. The key point is that A is compact, hence the periodic orbit $\pi(\beta)$ intersects A only a finite number of times. It follows that $\pi(\beta) = \pi(\alpha_0) = \pi(\alpha)$ admits only finitely many lifts in $\widetilde{M}/\langle f \rangle$ intersecting $\widehat{A}$. In other words, there must be distinct positive integers i, j and an integer q such that

$$h^i g(\alpha_{pi}) = f^q(h^j g(\alpha_{pj})),$$

because $\pi(\alpha_{pi}) = \pi(\alpha)$ as $\alpha_{pi} = h^{-i}(\alpha_0)$. Let

$$\alpha' = \alpha_{pj} = h^{-j}(\alpha) \quad \text{so} \quad \alpha_{pi} = h^{-i}(\alpha) = h^{j-i}(\alpha').$$

Hence

$$h^i g h^{j-i}(\alpha') = f^q h^j g(\alpha').$$

So there is n for which $h^i g h^{j-i} = f^q h^j g s^n$, where s is the stabilizer *in* $\pi_1(M)$ of $\alpha_{pj} = \alpha'$. Let $m = i - j$. Here n may be zero, but m is never zero. Since f and h are both in G they commute, so the last equation implies

$$g^{-1} f^{-q} h^m g = s^n h^m. \tag{2}$$

Notice that s preserves $\mathcal{G}(\alpha) = \mathcal{C}$, by the fact stated just before Claim 1 in this proof, because $s(\alpha') = \alpha'$ and α' is a corner of $\mathcal{C}$. Hence s belongs to H. Let

$$h_0 = (s^n h^m)^2, \quad v = (f^{-q} h^m)^2.$$

Since m is not zero then h_0 is not the identity. Equation (2) implies that

$$g^{-1} v g = h_0. \tag{3}$$

Since H_0 has index less than or equal to 2 in H then h_0 is in H_0. The elements f, h are in G so v is in G. Equation (3) means that $h_0 \in g^{-1} G g$. As this group is abelian, then $g^{-1} G g \subset Z(h_0)$.

We conclude that h_0 is a nontrivial element of G whose centralizer $Z(h_0)$ in $\pi_1(M)$ contains G, and it also contains $g^{-1} G g$.

Now suppose we are in Case II and we want to achieve the same conclusion. This is similar to Case I and some details are left to the reader. Here $g(\alpha_{pi})$ is in C_{-pi} and $h^{-i}(C_{-pi}) = C_0$. As in Case I there are i, j positive and distinct and q an integer so that

$$h^{-i} g(\alpha_{pi}) = f^q h^{-j} g(\alpha_{pj}).$$

So if $\alpha' = \alpha_{pj}$ then $h^{-i} g h^{j-i} = f^q h^{-j} g s^n$, with s as in Case I, leading finally to

$$g^{-1}(f^{-q} h^{-m}) g = s^n h^m, \quad \text{where } m = i - j \neq 0.$$

Here take $h_0 = (s^n h^m)^2$ nontrivial in H_0 because $m \neq 0$. Let $v = (f^{-q} h^{-m})^2$. So as before $g^{-1} v g = h_0$, so again h_0 is a nontrivial element of H_0 whose centralizer contains G and also $g^{-1} G g$.

Now assume by way of contradiction that $Z(h_0)$ is abelian. According to Lemma 5.5, since the chain $\mathcal{C}$ is not scalloped, it is the unique G–invariant chain of lozenges. Since $g^{-1} G g$ is a subgroup of $Z(h_0)$, it commutes with G as $Z(h_0)$ is abelian. It follows that $\mathcal{C}$ is $g^{-1} G g$–invariant.

As $\mathcal{C}$ is the unique G invariant chain of lozenges, then $g^{-1}(\mathcal{C})$ is the unique $g^{-1} G g$–invariant chain of lozenges. Hence $g^{-1}(\mathcal{C}) = \mathcal{C}$. This is a contradiction since $\beta = g(\alpha)$ is not a corner of $\mathcal{C}$. This finishes the proof of Claim 2.

Since $Z(h_0)$ is not abelian, [34, Lemma VI.1.5] shows that there is a Seifert fibered piece P of the torus decomposition of M [34; 33] and Johannson [35] so that up to conjugation $Z(h_0) \subset \pi_1(P)$. We may conjugate and assume that this in fact holds. The hypotheses of [34, Lemma VI.1.5] require

(i) M is irreducible,

(ii) M is orientable,

(iii) M has an incompressible surface.

Condition (i) holds because M has a pseudo-Anosov flow [22]. Condition (ii) holds because of Claim 1. As for condition (iii) we know that $\pi_1(M)$ has a $\mathbb{Z}^2$ subgroup. Work of Gabai [26] or Casson and Jungreis [13] implies that either M has an embedded incompressible torus or M is a small Seifert fibered space. But it M is Seifert fibered, then Theorem 4.1 shows that the fiber in M acts freely on $\mathcal{O}$ and we are done. So we can assume that condition (iii) also holds. An example of a nonsimple chain of lozenges in Seifert fibered spaces is the following: let Φ be a geodesic flow, γ a nonsimple geodesic and T the torus associated to γ with corresponding chain $\mathcal{C}$. Then $\mathcal{C}$ is not simple.

In order to conclude, we just have to show that P is a free piece. Assume this is not the case: let t be the fiber of a Seifert fibration in P admitting fixed points in $\mathcal{O}$.

Claim 3 For any ν in $\pi_1(P)$, $\nu(\mathcal{C}) = \mathcal{C}$.

Since $G \subset \pi_1(P)$, for every a in G we have $ata^{-1} = t^{\pm 1}$. Let G' be the subgroup of G made of elements a^2 where a is an arbitrary element of G. Then G' is isomorphic to $\mathbb{Z}^2$ (it has index 4 in G) and G' is contained in the centralizer $Z(t)$. The chain $\mathcal{C}$ is the unique G'–invariant chain of lozenges (Lemma 5.5). But since $G' \subset Z(t)$, the chain $t(\mathcal{C})$ is G'–invariant, hence equal to $\mathcal{C}$. Then t has a fixed point which is a corner of $\mathcal{C}$ and so $\mathcal{G}(t) \subset \mathcal{G}(\alpha)$.

Consider now the action of G' on the tree $\mathcal{G}(t)$. Since $\mathcal{G}(t)$ is contained in a linear tree and G' is isomorphic to $\mathbb{Z}^2$, there is an element b of G' acting freely on $\mathcal{G}(t)$. Since $\mathcal{G}(t) \subset \mathcal{G}(\alpha) = \mathcal{C}$ and the last one is a simplicial linear tree, it now follows that $\mathcal{G}(t) = \mathcal{C}$. Claim 3 follows since $\mathcal{G}(t)$ is obviously $\pi_1(P)$–invariant.

The fundamental group $\pi_1(P)$ contains $Z(h_0)$ which itself contains $g^{-1}Gg$: it follows that $g^{-1}Gg$ preserves $\mathcal{C}$. We have already observed, while proving that $Z(h_0)$ is not abelian (Claim 2), that this is impossible. This contradiction proves that t acts freely on $\mathcal{O}$. This finishes the proof of Proposition 6.4. □

Remark The same arguments as in the section "Proof that $\mathcal{C}$ is a string of lozenges" of the above proposition prove the following: suppose that $\mathcal{C}$ is a infinite chain of lozenges. Suppose that there is a corner p in $\mathcal{C}$ and an element g of $\pi_1(M)$ so that $g(p)$ is in (the interior of) a lozenge in $\mathcal{C}$. Then C is a string of lozenges with their corners.

Let $\Upsilon\colon A \hookrightarrow M$ be a Birkhoff annulus (embedded or not). It lifts as an immersion $\widetilde{\Upsilon}\colon \widetilde{A} \sim \mathbb{R}\times[0,1] \hookrightarrow \widetilde{M}$ such that $\mathbb{R}\times\{0\}$, and $\mathbb{R}\times\{1\}$ are orbits of $\widetilde{\Phi}$, and such that the image by $\widetilde{\Upsilon}$ of $\mathbb{R}\times(0,1)$ is transverse to $\widetilde{\Phi}$: we call $\widetilde{\Upsilon}\colon \widetilde{A} \hookrightarrow \widetilde{M}$ a *Birkhoff band.* Moreover, this image is invariant under the action of the cyclic subgroup $\Upsilon_*(\pi_1(A)) \sim \mathbb{Z}$. Finally, every orbit of $\widetilde{M}$ intersects the image of the interior in at most one point, and if $\Upsilon\colon A \hookrightarrow M$ is elementary, then the projection in $\mathcal{O}$ is a $\Upsilon_*(\pi_1(A))$–invariant lozenge union its two corners but without the sides [3, Proposition 5.1]. This set is neither closed nor open in $\mathcal{O}$.

Conversely, and as we already mentioned in the proof of Proposition 6.4, Claim 2, every lozenge C in $\mathcal{O}$ invariant by a cyclic subgroup H of $\pi_1(M)$ is the projection in $\mathcal{O}$ of an embedded Birkhoff band in $\widetilde{M}$ that is H invariant and it projects in M to an elementary Birkhoff annulus. To get more embeddedness assume moreover that the lozenge is *simple*, ie its interior contains no iterate of its corner, and also that H is the group of all elements of $\pi_1(M)$ preserving C. Observe that according to Proposition 2.16(4), elements of H preserve every corner of C, hence each boundary component of the H–invariant Birkhoff band is preserved. Then, the Birkhoff annulus in M which is the projection of the Birkhoff band invariant under the maximal such H can be selected weakly embedded [3, Theorem D].

More generally, let $\mathcal{C}$ be a string of lozenges invariant under a subgroup G of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$. Then, there is a cyclic subgroup H of G fixing every lozenge in $\mathcal{C}$. We lift all the lozenges to $\widetilde{M}$, so that the lift of every two successive lozenges share a common H–invariant orbit. This can be done in a G–equivariant way. We also lift the entire corner orbits of the lozenges. The union is a set which is G invariant in $\widetilde{M}$ and projects in the quotient of $\widetilde{M}$ by G to an embedded torus. This torus projects to an immersed torus in M which is a union of elementary Birkhoff annuli.

Definition 6.5 A *Birkhoff torus* is an immersion $\Upsilon\colon T \to M$ of a torus T, such that T is a union of distinct annuli A_i for which every restriction $\Upsilon\colon A_i \to M$ is an elementary Birkhoff annulus. In addition we require the following: if A_i and A_{i+1} are two consecutive annuli abutting the common closed orbit γ, then locally near γ the annuli A_i and A_{i+1} are in distinct quadrants defined by γ. Similarly, a *Birkhoff–Klein bottle* is an immersion of the Klein bottle whose image is a union of elementary Birkhoff annuli. We have the same restriction on abutting annuli as in the tori case.

Notice the restriction to elementary Birkhoff annuli.

In the sequel, a *closed Birkhoff surface* means a Birkhoff torus or a Birkhoff–Klein bottle. As is commonly done in topology, unless it is necessary, we do not distinguish between a Birkhoff surface and its image. A Birkhoff surface is a union of Birkhoff annuli. It contains a finite number of periodic orbits of Φ, called the *tangent orbits,* and is transverse to Φ outside these periodic orbits.

The reason for the added condition about quadrants is the following: Without it we could have started with say an embedded Birkhoff annulus A and flow A forward slightly to an annulus A' which is disjoint from A in the interior. The union $T = A \cup A'$ is a torus which is NOT incompressible as it bounds an obvious solid torus. We now explain the added condition on quadrants. Let γ be a closed orbit in the boundary of an elementary Birkhoff annulus A. Denote this boundary component of A by $\partial_1 A$. The stable and unstable leaves of γ define quadrants: in a neighborhood of γ they are the components of the complement of the union of the local sheets of $W^s(\gamma)$ and $W^u(\gamma)$ near γ. The interior of the annulus A cannot intersect these local sheets of $W^s(\gamma)$ or $W^u(\gamma)$. This is because the interior is transverse to Φ and A is compact; an intersection would create a closed curve intersection. This is disallowed because all Birkhoff annuli are elementary. Therefore near $\partial_1 A$, the annulus A enters a unique quadrant defined by γ. In the universal cover this means that the lift $\widetilde{A}$ enters a well defined, unique lozenge with corner $\widetilde{\gamma}$.

Definition 6.6 A closed Birkhoff surface $\Upsilon\colon S \to M$ is called *weakly embedded* if the Birkhoff annuli $\Upsilon\colon A_i \to M$ are all weakly embedded, with interiors two-by-two disjoint. If moreover $\Upsilon\colon S \to M$ is an embedding, then the closed Birkhoff surface is said to be *embedded.*

As explained above, the condition that interiors are embedded and two by two disjoint implies that none of the tangent periodic orbits of $\Upsilon(S)$ intersects the interior of any of the annuli.

Proposition 6.7 *Let $\mathcal{C}$ be a string of lozenges in $\mathcal{O}$ invariant under a subgroup G of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$ or $\pi_1(K)$. Then the union of $\mathcal{C}$ and its corner orbits is the projection in $\mathcal{O}$ of the lift to $\widetilde{M}$ of a closed Birkhoff surface $\Upsilon\colon S \to M$. More precisely, $\Upsilon\colon S \to M$ is the composition $\widehat{p} \circ \widehat{\Upsilon}$ of an embedding $\widehat{\Upsilon}\colon S \to \widehat{M}$ and the covering map $\widehat{p}\colon \widehat{M} \to M$, where $\widehat{M}$ is the quotient of $\widetilde{M}$ by G.*

Moreover, assume that we have the following additional properties:

- *$\mathcal{C}$ is simple, ie no element of $\pi_1(M)$ maps a corner of $\mathcal{C}$ in the interior of a lozenge of $\mathcal{C}$.*
- *The only elements of $\pi_1(M)$ mapping a lozenge of $\mathcal{C}$ to itself are the elements of G.*

Then the closed Birkhoff surface can be selected weakly embedded.

Proof The first part has been explained before in the case where G is abelian, and is easily generalized to the case $G \sim \pi_1(K)$: the matter is to find a fundamental domain of the action of G on the set of lozenges in $\mathcal{C}$, to lift each lozenge in this fundamental domain to a Birkhoff band, and then to lift all other lozenges in $\mathcal{C}$ as Birkhoff bands in a G–equivariant way.

Assume now that the additional properties hold: every lozenge in it is simple, and elements of $\pi_1(M)$ preserving a lozenge in $\mathcal{C}$ are in G. Then the closed Birkhoff surface is a union of weakly embedded Birkhoff annuli, whose interiors are all disjoint from the tangent periodic orbits. Since the chain is simple, we can prove, using the techniques in [3, Section 7] that through some isotopy along the flow, the interiors of the elementary annuli can be made disjoint from each other, that is, the Birkhoff surface is weakly embedded. □

All of these results in [3] were stated and proved for smooth Anosov flows. However, exactly the same techniques work for general pseudo-Anosov flows.

More generally, using the results above, then according to Lemma 5.3, we have the following.

Lemma 6.8 *Let G be a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}^2$. Suppose that the pseudo-Anosov flow Φ is not product. Then G is the image $\Upsilon_*(\pi_1(T))$ of the fundamental group of a Birkhoff torus $\Upsilon\colon T \to M$.*

From the conclusion of Theorem B (to be proved in the next section) it is easy to construct many weakly embedded Birkhoff surfaces that are not homotopic to an embedded Birkhoff surface. Observe that weakly embedded closed Birkhoff surfaces may fail to be embedded for various reasons, as follows.

(I) A Birkhoff subannulus may be non-embedded, wrapping around one or both of the tangent periodic orbits in its boundary. It means that some element g of $\pi_1(M)$ (corresponding to the periodic orbit) is not in G, but g preserves a corner in $\mathcal{C}$ (where $\mathcal{C}$ is the $G \cong \mathbb{Z}^2$ invariant chain of lozenges).

(II) An element of $\pi_1(M)$ may map a corner α of $\mathcal{C}$ to another corner β of $\mathcal{C}$ which is not in the G–orbit of α, ie a tangent periodic orbit can be the boundary of more than two Birkhoff subannuli. This is the case in the Bonatti–Langevin example [9].

(III) Even an element g of $\pi_1(M)$ not in G could map a lozenge in $\mathcal{C}$ to another lozenge in $\mathcal{C}$. At the Birkhoff surface level this implies the existence of two different elementary Birkhoff annuli (in the torus) sharing the same boundary components and homotopic one to the other along the orbits of Φ. This situation typically arises in Proposition 6.7 if G is a finite index subgroup of a bigger group preserving the chain $\mathcal{C}$.

Remark Let us first stress that possibility (I) can certainly happen. For example let Φ be the geodesic flow in the unit tangent bundle of an *orientable* hyperbolic surface and let T be the set of unit vectors along a simple closed geodesic. Let γ be one closed orbit in T. Put coordinates in the torus $\partial N(\gamma)$ so that $(0,1)$ is the meridian and $(1,0)$ is the trace of say the stable foliation. The construction here is more general, the key fact used is that the trace of the stable foliation intersects the meridian once. Do Dehn surgery on γ so that the new meridian is $(n,1)$, where n is an integer greater than 1. Isotoping the old torus slightly to a torus T' avoiding γ we see that it survives the Dehn surgery. After Dehn surgery T' is homotopic to a Birkhoff torus, with Birkhoff annuli which wrap n times around the orbit γ. Since it is a Birkhoff torus, it is π_1–injective and so is T'. This gives the desired examples. In fact the surgery procedure can be done by blowing up the orbit γ into a boundary torus and then blowing back using the new meridian information [25]. Therefore the new Birkhoff torus can be taken as the result of the original Birkhoff torus under this procedure.

A Birkhoff torus is π_1–injective because of the following: a closed curve is homotopic to either a closed orbit in the Birkhoff torus or to a curve transverse to say the stable foliation in the torus. In the first case the curve represents a power of a closed orbit, which is not nullhomotopic [22]. In the second case, the curve is transverse to the stable foliation *in the torus*. The condition that consecutive annuli abut the closed orbit from distinct quadrants of that closed orbit implies that this curve is also transverse to the stable foliation *in the manifold*. It now follows from the theory of essential laminations that this curve is not nullhomotopic in the manifold [16; 28].

The notion of weakly embedded tori is sufficient to analyse the relationship between (possible) singular orbits of the flow and the torus decomposition of M.

Proposition 6.9 *Let α be a singular orbit of a pseudo-Anosov flow Φ in M. Then α is homotopic into a piece of the torus decomposition of M.*

Remarks **(1)** Clearly this is not true for regular periodic orbits: for example there are (non-Seifert) graph manifolds with Anosov flows which are transitive, for example the flows constructed by Handel and Thurston [31], which are actually volume preserving. Then there are dense orbits and hence periodic orbits which are not homotopic into any Seifert fibered piece.

(2) If M is atoroidal, Lemma 6.8 is trivial.

(3) Notice that α may be homotopic into several pieces. If that happens then α is homotopic into a torus T which is boundary of two pieces. Finally, if α is homotopic into a third piece, then α has to be homotopic through a piece P. The piece P cannot be atoroidal, because an atoroidal piece is a cusped hyperbolic manifold and consequently acylindrical. Hence P is Seifert and since α is homotopic to two distinct boundary components of P, it follows that α and the fiber of P have common powers. In that case the piece P has to be a periodic piece. Then α is homotopic into any other piece P_1 which intersects P. Notice that α cannot be homotopic into any additional piece: otherwise α would be homotopic through a second Seifert piece P'. That would force the fibers in P and P' to be the same, which is impossible.

Proof of Proposition 6.9 Let $T_1, \dots, T_a$ be the cutting tori in a torus decomposition of M with complementary components $P_1, \dots, P_b$, which are either Seifert fibered or atoroidal. By a small isotopy assume that α is transverse to the collection $\{T_i\}$. Fix a lift $\widetilde{\alpha}$ to $\widetilde{M}$ and let g in $\pi_1(M)$ be associated to α so that $g(\widetilde{\alpha}) = \widetilde{\alpha}$. Consider the collection of all lifts of the $\{T_i\}$.

Case 1 Suppose that $\widetilde{\alpha}$ eventually stops intersecting lifts of the $\{T_i\}$.

Since α is closed, this shows α is contained in a component of the complement of $\{T_i\}$.

Case 2 Suppose that $\widetilde{\alpha}$ keeps intersecting a fixed lift $\widetilde{T}$ in points $p_k = \widetilde{\Phi}_{t_k}(p_0)$ where t_k converges to infinity.

Let V be the tree, whose vertices are the components $\widetilde{M}-$ (lifts of $\{T_i\}$) and edges are the lifts of $\{T_i\}$. Then $\pi_1(M)$ acts on V.

By transversality, the intersection of α and $\{T_i\}$ is finite. Up to subsequence we may assume that $\pi(p_k)$ is a single point. The projection to M of $\widetilde{\Phi}_{[t_k, t_{k'}]}(p_0)$ is the orbit α being traversed a number n of times. This shows that α^n is freely homotopic into some torus T_i. It follows that g^n preserves an edge in V and so does not act freely on V. Therefore g also does not act freely on V. There are two options: If g acts as an inversion in the tree V, then it fixes an edge associated to a lifted torus $\widetilde{T}_*$ and then α is homotopic into the torus $T_* = \pi(\widetilde{T}_*)$. Then we are done. Otherwise g fixes a vertex in V and hence α is homotopic into a piece of the torus decomposition.

Case 3 $\widetilde{\alpha}$ intersects distinct lifts $\widetilde{T}^j$, $j \in \mathbb{N}$ of elements in $\{T_i\}$.

By the proof of Case 2, it follows that the assumption of Case 2 does not hold. Therefore $\widetilde{\alpha}$ eventually stops intersecting any single lift $\widetilde{T}$ of one of the $\{T_i\}$. In addition if distance between $\widetilde{\alpha}$ and any single lift $\widetilde{T}$ does not converge to infinity as time goes to infinity then: up to subsequence we may assume there are p_k in $\widetilde{\alpha}$ with $d(p_k, \widetilde{T})$ bounded. We may then assume that $\pi(p_k)$ converges in M and up to a small adjustment and subsequence we may assume that $\pi(p_k)$ is constant. In addition p_k is a bounded distance from z_k in $\widetilde{T}$. Up to another subsequence assume that $\pi(z_k)$ converges in M and since $\pi(T)$ is compact, we may assume that $\pi(z_k)$ is constant. The projection of $\widetilde{\Phi}_{[t_k,t_{k'}]}$ is α being traversed n times. The projection of an arc in $\widetilde{T}$ from z_k to $z_{k'}$ is a closed curve in T. Up to another subsequence assume that the geodesic arcs from z_k to p_k have images in M which are very close. This produces a free homotopy from α^n and a closed curve in $\pi(T)$. Now the proof is exactly as in Case 2.

Hence assume that $d(p_k, \widetilde{T})$ converges to infinity for any fixed lift $\widetilde{T}$. If $\widetilde{\alpha}$ keeps returning to the same component of $\widetilde{M}-$ (lifts of $\{T_i\}$), then some power of α preserves this component and an argument as in Case 2 finishes the proof.

Finally we can assume that $\widetilde{\alpha}$ crosses $\widetilde{T}^j$ for each j and eventually switches from one component of $\widetilde{M} - \widetilde{T}^j$ to the other. There is a smallest separation distance $a_0 > 0$ between any two lifts of $\{T_i\}$. Homotope each T_i to a Birkhoff torus, union of Birkhoff annuli $\{B_m\}$ and lift these homotopies to $\widetilde{M}$. Each point is moved at most a constant a_1. Fix j and let j' vary. The fact that $d(\widetilde{T}^j, \widetilde{T}^{j'})$ goes to infinity means that $\widetilde{\alpha}$ has to cross some lift $\widetilde{B}_m$ of some Birkhoff annulus B_m and cannot be contained in $\widetilde{B}_m$. But this is a contradiction because the orbits intersecting the interior of a Birkhoff annulus are never singular. This finishes the proof of Proposition 6.9. □

Remark One can also prove this by using group actions on trees more extensively: the element g either fixes a point, or g has an inverted edge or acts freely on V. To use this, further work is needed, for instance in the first case, one needs to find a vertex p in V, fixed by g so that $\widetilde{\alpha}$ has a point in the region associated to p and similarly for the other cases.

Theorem 6.10 *Suppose that M is orientable and that Φ is not product. Let T be an embedded, incompressible torus in M. Then either*

(1) *T is isotopic to an embedded Birkhoff torus,*

(2) *T is homotopic to a weakly embedded Birkhoff torus and contained in a periodic Seifert fibered piece, or*

(3) *T is isotopic to the boundary of the tubular neighborhood of an embedded Birkhoff–Klein bottle contained in a free Seifert piece.*

Proof Let T^2 denote the 2–dimensional torus. Let $\Upsilon_0\colon T^2 \to M$ be an immersed Birkhoff torus homotopic to T and let $T_* = \Upsilon_0(T^2)$, using Proposition 6.7. Let $\mathcal{C}$ be the chain of lozenges invariant under $\pi_1(T)$ and associated with the torus T^* (a priori there could be two $\pi_1(T)$–invariant chains, if they are scalloped). The proof of this theorem has similarities with that of Proposition 6.4, but notice that some of the conclusions are *opposite*.

Step 1: $\mathcal{C}$ is simple The proof of Step 1 does not assume that M is orientable.

Suppose this is not true. Then there is a corner orbit α in $\mathcal{C}$ and f in $\pi_1(M)$ with $f(\alpha) = \beta$ intersecting the interior of a lozenge in $\mathcal{C}$. Let g be a generator of the isotropy group of β. Let $\widetilde{T}_*$ be the lift of T_* to $\widetilde{M}$ which is invariant under $\pi_1(T)$. Similarly let $\widetilde{T}$ be the lift of T invariant under $\pi_1(T)$. Then β intersects $\widetilde{T}_*$ in a single point p. Let β^+, β^- be the two rays of β defined by p. Notice that $\widetilde{T}_*$ is embedded and separates $\widetilde{M}$. Hence β^+ and β^- are in distinct components of $\widetilde{M} - \widetilde{T}_*$. In addition $\widetilde{T}$ also separates $\widetilde{M}$.

Assume that β^+ and β^- are not at bounded distance from $\widetilde{T}_*$: for any $R > 0$, there are points q^-_R, q^+_R in β^-, β^+, each at distance greater than R from $\widetilde{T}_*$. But T and T_* are freely homotopic, hence there is some R_0 such that $\widetilde{T}$ is contained in the R_0–neighborhood of $\widetilde{T}_*$. It follows that for any $R > 2R_0$, any path joining a point q^- to a point q^+ such that $d(q^{\pm}, q^{\pm}_R) < R/2$ must intersect $\widetilde{T}$.

On the other hand, the closed orbit $\pi(\beta)$ is freely homotopic in M to a curve in T. But T is embedded. If T is two sided then $\pi(\beta)$ is freely homotopic to a curve disjoint from T (it is crucial that T is embedded here). If T is one sided then $(\pi(\beta))^2$ is freely homotopic to a curve disjoint from T. In either case lift this to a homotopy from β to a curve β_1 disjoint from $\widetilde{T}$. The homotopies from β to β_1 move points a bounded distance. Hence, there is a positive number r such that for every $R > 0$, there are points $m^{\pm}_R$ on β_1 such that $d(m^{\pm}_R, q^{\pm}_R) < r$. Take $R > 2R_0$, $R > 2r$: according to the above, the segment in β_1 between m^-_R and m^+_R must intersect $\widetilde{T}$. This is a contradiction.

Therefore, one of the two rays (say β^+) is at bounded distance less than or equal to a_1 from $\widetilde{T}_*$. Consider a sequence of points $p_i = g^{n_i}(p)$ in β^+ which all project to the same point $\pi(p_1)$ in M.

Let q_i in $\widetilde{T}_*$ a distance less than or equal to a_1 from p_i. Up to subsequence assume that $\pi(q_i)$ converges in M. Since T_* is compact, we can assume that $\pi(q_i)$ is constant. Now up to another subsequence assume that there are geodesic segments u_i in $\widetilde{M}$ from p_i to q_i so that $\pi(u_i)$ converges in M. Again by small adjustments we can assume that $\pi(u_i)$ is constant for i big. Consider the following closed curve in $\widetilde{M}$: a segment in β from p_i to p_k, $k > i$, then the segment u_k, then a segment in $\widetilde{T}_*$ from q_k to q_i and finally a segment from q_i to p_i along u_i. Since $\pi(u_i) = \pi(u_k)$ this projects to a free homotopy from a power of the loop $\pi(\alpha)$ to a closed curve in T_*.

In other words, $g^n(q_i) = q_k$ for some n in $\mathbb{Z}$. Hence for some n different from 0, g^n leaves $\widetilde{T}_*$ invariant.

But this implies that g^n leaves $\mathcal{C}$ invariant. Since $g^n(\beta) = \beta$, then g^n leaves invariant the lozenge C of $\mathcal{C}$ containing β in its interior and g^n is not the identity. But then g^n does not leave invariant any orbit in the interior of C, a contradiction to it leaving β invariant. This finishes the proof of Step 1.

Let $G = \pi_1(T)$. As we already observed, if this Birkhoff torus is not embedded, some element g of the set $(\pi_1(M) - \pi_1(T))$ maps a corner of $\mathcal{C}$ to a corner of $\mathcal{C}$. Our strategy is to enlarge G to a bigger subgroup of $\pi_1(M)$, containing all these elements.

Let $\mathcal{G}$ denote the tree $\mathcal{G}(\alpha)$ where α is a corner in $\mathcal{C}$. The chain $\mathcal{C}$ corresponds to a G–invariant line in $\mathcal{G}$. Let H be the subset of $\pi_1(M)$ of those h such that there is a vertex β of $\mathcal{G}$ such that $h(\beta)$ is also a vertex of $\mathcal{G}$. In particular $\mathcal{G}(\beta) = \mathcal{G}(\alpha) = \mathcal{G}(h(\beta))$. Then, for every h in H,

$$h(\mathcal{G}(\alpha)) = h(\mathcal{G}(\beta)) = \mathcal{G}(h(\beta)) = \mathcal{G}(\alpha),$$

hence H is the stabilizer of $\mathcal{G}$ in $\pi_1(M)$. In particular H is a subgroup of $\pi_1(M)$.

Let H_0 be the subgroup of H acting trivially on $\mathcal{G}$: H_0 is a cyclic normal subgroup of H, generated by an element h_0. Let H' be the centralizer of H_0 (or h_0) in H: it is a normal subgroup of H of index at most 2. Since G leaves invariant the collection of corners of $\mathcal{C}$, it contains an element h_0^n of H_0 with $n \neq 0$. Since G is contained in H, for every g in G we have $gh_0g^{-1} = h_0^{\pm 1}$. On the other hand, G is abelian, hence we have $gh_0^ng^{-1} = h_0^n$, and $G \subset H'$.

Step 2: The case where H' is abelian Then H' has no torsion, contains $G \cong \mathbb{Z}^2$, and is contained in $\pi_1(M)$, the only possibility is $H' \cong \mathbb{Z}^2$. Then $G \subset H'$ has finite index in H': it follows that $\mathcal{C}$ is H'–invariant (Corollary 5.6). Now since H' is normal in H and $H' \cong \mathbb{Z}^2$, the same result shows that $\mathcal{C}$ is H–invariant. By Lemma 5.2, H is isomorphic to $\mathbb{Z}^2$ or $\pi_1(K)$, since it contains a $\mathbb{Z}^2$. This is the crucial conclusion in this case.

Apply Proposition 6.7 to H using that $\mathcal{C}$ is simple and that elements of $\pi_1(M)$ preserving a lozenge of $\mathcal{C}$ are in H: it implies that there is a weakly embedded closed Birkhoff surface $\Upsilon_1\colon S \to M$ with $(\Upsilon_1)_*(\pi_1(S)) = H$. It follows from the discussion following Lemma 6.8 that $\Upsilon_1\colon S \to M$ is an embedding, since any element of $\pi_1(M)$ mapping a corner of $\mathcal{C}$ to a corner of $\mathcal{C}$ lies in H.

Suppose first that S is a torus, that is, H is isomorphic to $\mathbb{Z}^2$. If S is one sided, then M is nonorientable, contrary to hypothesis. Therefore there is a neighborhood N of S homeomorphic to $S \times [0,1]$. As the initial embedded torus $T \subset M$ is homotopic into N, it now follows from classical 3–dimensional topology [32] that T is homotopic and in fact isotopic to the embedded Birkhoff torus $\Upsilon_1(S)$. In other words, T is isotopic to an embedded Birkhoff torus: we are done here. This is case (1) of the statement of the theorem.

Consider now the case where S is the Klein bottle. Since M is oriented, $\Upsilon_1(S)$ admits a tubular neighborhood U in M diffeomorphic to the nontrivial line bundle over K. The boundary of U is an embedded torus T'. As above T is homotopic into U and has to be homotopic and in fact isotopic to T'.

Now observe that U is a Seifert submanifold which is not a product of surface cross interval. It follows that U is contained in a Seifert piece P of the torus decomposition of M (that is S is not in the boundary of two intersecting atoroidal/hyperbolic pieces). If P is periodic then Proposition 6.7 implies that T is homotopic to a weakly embedded Birkhoff torus; this is case (2) of the statement of the theorem. If P is not periodic then we are in case (3). We are done in this case.

Step 3: The case where H' is not abelian Since H', contained in the centralizer of H_0, is not abelian, [34, Lemma VI.1.5] shows that there is a Seifert fibered piece P of the torus decomposition of M so that $H' \subset \pi_1(P)$ (after conjugation if necessary). Let $t \in \pi_1(P)$ be a representative of the regular fibers of a fibration of P. Then the centralizer $Z(t)$ of t in $\pi_1(P)$ (the characteristic subgroup) has index at most two in $\pi_1(P)$. Observe that we have $G \subset H' \subset \pi_1(P)$. Therefore T is homotopic, hence isotopic, to an embedded torus in P. But incompressible tori in P are vertical: it implies that t lies in G, hence $G \subset Z(t)$.

Assume first that the action of t on $\mathcal{G}$ is free. Then $\mathcal{C}$ is the unique axis of t in $\mathcal{G}$. Since H' is contained in $\pi_1(P)$, some finite index normal subgroup H'' of H is contained in $Z(t)$. For any g in H'', $tg(\mathcal{C}) = g(\mathcal{C})$ so by the uniqueness above, $g(\mathcal{C}) = \mathcal{C}$, hence $\mathcal{C}$ is the unique chain in $\mathcal{G}$ preserved by H''. Since H'' is normal in H then again it follows that H preserves $\mathcal{C}$. Now we conclude almost as in Step 2: if H is isomorphic to $\mathbb{Z}^2$ then H' is abelian, contradiction to assumption in case (3). If H is isomorphic to $\pi_1(K)$ then T is isotopic to the boundary of a tubular neighborhood of an embedded Birkhoff–Klein bottle contained in P, which can be periodic (case (2)) or free (case (3)). Is the periodic case possible? That is, can t act nonfreely on $\mathcal{O}$ if it acts freely on $\mathcal{C}$? This is possible if $\mathcal{C}$ is a scalloped region, preserved by t and t has fixed points in the boundary of the scalloped region.

Assume now that t has a fixed point in $\mathcal{G}$. Then P is periodic, and some nontrivial power t^k lies in H_0. Since $t \in G$, the line in $\mathcal{G}$ corresponding to $\mathcal{C}$ is preserved by t. If t acts freely on $\mathcal{C}$, it acts on it by translation, and it contradicts our hypothesis that t acts non-freely on $\mathcal{G}$. Since there is f in G acting freely on $\mathcal{C}$ and $ft = tf$ it follows that every corner of $\mathcal{C}$ is a fixed point of t. According to Proposition 2.16(4), t lies in H_0.

The transformation t fixes every corner of $\mathcal{G}$ and only those points in $\mathcal{O}$. It follows that $\pi_1(P)$ preserves $\mathcal{G}$. Therefore, H_0 is a cyclic normal subgroup of $\pi_1(P)$. By Hempel [32, Theorem 12.8] H_0 is contained in the group generated by a regular fiber of a Seifert fibration of P. A little bit of care is needed if P admits two nonisotopic Seifert fibrations. Here P is compact, orientable and with boundary. [33, Lemma VI.20] states that the free homotopy class of the fiber is unique unless P is homeomorphic to (1) $D^2 \times [0,1]$, (2) $T^2 \times [0,1]$ or (3) a twisted I–bundle over the Klein bottle. But P is a piece of the torus decomposition of an irreducible manifold, so P cannot be $D^2 \times [0,1]$. If P were $T^2 \times [0,1]$ it would be included in a peripheral part of an adjoining Seifert piece, that is, this cannot happen either. The only possibility is that P is a twisted I–bundle over the Klein bottle. There are two Seifert fibrations of P in this case. But since t lies in H_0 it now follows that H_0 is the group generated by t and therefore H_0 is contained in G.

On the other hand if P is not as in (3), then the Seifert fibration in P is unique up to isotopy. It now follows that $H_0 \subset \langle t \rangle$ and so H_0 is contained in G as well.

Let now g be an element of $\pi_1(M)$ preserving a lozenge in $\mathcal{C}$. According to Proposition 2.16(4), g lies in H_0, hence in G. Since $\mathcal{C}$ is simple, the additional properties in Proposition 6.7 are satisfied: it follows that T is homotopic to a weakly embedded Birkhoff torus, contained (up to homotopy) in P. We are in case (2) of Theorem 6.10. Notice that in general there may be identifications in the boundary orbits as already described. A priori any of problems (I), (II) or (III) described after Lemma 6.8 may occur. This finishes the proof of Theorem 6.10. □

Step 1 of this theorem proves the following.

Corollary 6.11 *Suppose that T is an incompressible, embedded torus or Klein bottle in M which admits a pseudo-Anosov flow Φ. If $\mathcal{C}$ is a chain lozenges invariant under $\pi_1(T)$ then $\mathcal{C}$ is simple.*

Step 1 proves this for tori. The same proof of Step 1 can be applied to Klein bottles.

Remark We remark that tori homotopic to a double cover of a Birkhoff–Klein bottle appearing in Step 2 and 3 actually occur in the free case and in the periodic case too. The periodic case occurs for example in the Bonatti–Langevin flow [9]. An example of the free case was described in the remark at the end of Section 4.

Remark The hypothesis of orientability for M in Theorem 6.10 occurs because several results for torus decompositions and maps of Seifert spaces into manifolds are only clearly stated in the literature for orientable manifolds, for example [34].

7 Periodic Seifert fibered pieces

This section is devoted to the proof of Theorem B; in particular we assume that M is orientable. Let P be a (nontrivial) Seifert fibered piece of a 3–manifold M with a pseudo-Anosov flow Φ. We will analyse here only the case that the regular fiber h_0 in $\pi_1(P)$ does not act freely on $\mathcal{O}$, that is P is a periodic piece. By Theorem 4.1 this implies that P is not all of M. We start by constructing a canonical tree of lozenges associated to P. First consider the action on $\mathcal{O}$: there is α in $\mathcal{O}$ with $h_0(\alpha) = \alpha$. Let $\mathcal{T}$ be the fat tree $\mathcal{G}(\alpha)$. Given g in $\pi_1(P)$, then $gh_0g^{-1} = h_0^{\pm 1}$ so $h_0 g(\alpha) = g(\alpha)$ and $g(\alpha)$ is in $\mathcal{G}(\alpha)$. It follows that $\mathcal{T} = \mathcal{G}(\alpha)$ is a $\pi_1(P)$–invariant tree. The kernel of the $\pi_1(P)$–action on $\mathcal{T}$ is a normal cyclic subgroup H_0 of $\pi_1(M)$, which contains a nontrivial power h_0^n of h_0 (cf Proposition 2.16).

Notice that there is at least a $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(P)$ so there are elements in $\pi_1(P)$ acting freely on $\mathcal{T}$. We now go through several steps to produce a normal form of the flow in P.

Pruning the tree $\mathcal{T}$ We first construct a subtree of $\mathcal{T}$ which is still $\pi_1(P)$–invariant and has no vertices of valence one. Given g in $\pi_1(P)$ acting freely on $\mathcal{T}$ let $A(g)$ be the axis of g in $\mathcal{T}$. Let now $\mathcal{T}'$ be the union of all axes $A(g)$, for all g in $\pi_1(P)$ acting freely on $\mathcal{O}$. Clearly $\mathcal{T}'$ is $\pi_1(P)$–invariant and has no vertices of valence one, since they are all in axes. All that is left to prove is that $\mathcal{T}'$ is connected and hence a subtree.

Let c_0, c_1 in $\mathcal{T}'$ so that there are f, g in $\pi_1(P)$ with c_0 in $A(f)$, c_1 in $A(g)$. If $A(f), A(g)$ intersect, then there is a path in $\mathcal{T}'$ from c_0 to c_1. Suppose then that they do not intersect. There is a well-defined bridge in $\mathcal{T}$ from $A(f)$ to $A(g)$ denoted by $[x, y]$; it is a closed segment intersecting $A(f)$ only in the extremity x and intersecting $A(g)$ only in y. Let $z = f^{-1}(x)$. Consider the element gf which is in $\pi_1(P)$. Then x separates z from y and so separates z from $gf(x)$ which is in $gA(f)$. Also $gf(z) = g(x)$ separates x from $gf(x)$ which is in $gA(f)$. It follows that $z, x, gf(z)$ and $gf(x)$ are all distinct and linearly ordered in a segment contained in $\mathcal{T}$. Hence gf acts freely on $\mathcal{T}$ and $x, gf(x)$ are in $A(gf)$. In particular x and y are in $A(gf)$ contained in $\mathcal{T}'$ so there is a path in $\mathcal{T}'$ from c_0 to c_1. This shows that $\mathcal{T}'$ is connected.

Weakly embedded Birkhoff annuli Suppose there is a vertex q of $\mathcal{T}'$ and an element g of $\pi_1(M)$ (necessarily not in $\pi_1(P)$) and a lozenge C in $\mathcal{T}'$ with $g(q)$ intersecting the interior of C. This is called condition (I). The lozenge C is part of an axis $A(f)$ for some f in $\pi_1(P)$. By the remark after Proposition 6.4, it follows that the subtree $\mathcal{T}'$ is a string of lozenges, which is denoted by $\mathcal{C}$. Then f, h_0^{2n} generate a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup G of $\pi_1(M)$ preserving this string of lozenges. Moreover, q is a vertex of $\mathcal{T}'$ and $g(q)$ is in the interior of C. Proposition 6.4 again implies that G is also a subgroup of the fundamental group of a free Seifert piece P'. Therefore the pieces P and P' are adjacent and there is a boundary torus T of both of them so that G is a subgroup of $\pi_1(T)$. In particular $\pi_1(T)$ also preserves the string of lozenges $\mathcal{C}$. But now T is embedded, so Corollary 6.11 implies that $\mathcal{C}$ is simple, contradiction. This shows that condition (I) cannot happen.

We first prove that each lozenge in $\mathcal{T}'$ corresponds to a weakly embedded elementary Birkhoff annulus in M. As in the proof of Theorem 6.10, we consider the stabilizer H in $\pi_1(M)$ of $\mathcal{T}'$. The action of H on $\mathcal{T}'$ is not faithful, since the kernel contains a nontrivial group. Therefore, H contains an infinite cyclic normal subgroup but is not cyclic. Again by [34, Lemma VI.1.5] H is contained (up to conjugation) in $\pi_1(P')$ for some Seifert fibered piece P' of M. But H also contains $\pi_1(P)$. It follows that $P = P'$ and $H = \pi_1(P)$. In addition the same arguments show that the stabilizer in $\pi_1(M)$ of $\mathcal{T}$ is also $\pi_1(P)$.

Suppose that g in $\pi_1(M)$ maps a vertex α of $\mathcal{T}'$ to a vertex of $\mathcal{T}'$. Hence it also sends a vertex of $\mathcal{T}$ to a vertex of $\mathcal{T}$. In that case we already observed during the proof of Theorem 6.10 that g stabilizes $\mathcal{T}$ and hence belongs to $\pi_1(P)$. We can now apply Proposition 6.7 to conclude that each lozenge in $\mathcal{T}'$ corresponds to a weakly embedded elementary Birkhoff annulus in M.

Weakly embedded union of Birkhoff annuli We want to show that the union of the Birkhoff annuli can be adjusted to be embedded in the interiors.

Consider the quotient of the tree $\mathcal{T}'$ by $\pi_1(P)$. It is a graph, that we denote by A. Since it is a graph, the fundamental group of A is a free group, and since $\pi_1(P)$ is finitely generated, then the fundamental group of A has finite rank. Moreover, by construction, A does not contain an infinite ray (since every element of $\mathcal{T}'$ lies on the axis of some element of $\pi_1(P)$ acting freely on $\mathcal{T}'$). In particular this is one reason why we do the pruning $\mathcal{T}$ to $\mathcal{T}'$. It follows that A is a finite graph.

Consider a fundamental domain of the action of $\pi_1(P)$ on $\mathcal{T}'$. We lift every lozenge of this fundamental domain to a Birkhoff band in $\widetilde{M}$, and then lift all other lozenges in $\mathcal{T}'$ in a $\pi_1(P)$–equivariant way. By the previous subsection we can choose the

Birkhoff bands so that the union projects in M to a union of weakly embedded Birkhoff annuli in M. Once more, we can then use cut and paste techniques of [3] to have the union of the Birkhoff annuli to be embedded in the interior of the annuli, with possible identifications in the boundary orbits.

Flow adapted neighborhoods of periodic pieces Let B be the union of the weakly embedded elementary Birkhoff annuli as in the previous item. Let U be the neighborhood of B obtained by taking a tubular neighborhood of every periodic orbit contained in B (the "tangent periodic orbits"), attaching to them tubular neighborhoods of the elementary Birkhoff annuli. Then U is a submanifold of M with boundary. Topologically, this corresponds to the following: start with a finite collection of solid tori and attach several handles diffeomorphic to $[-1,1]\times[-1,1]\times S^1$, where in each handle, $\{0\}\times[-1,1]\times S^1$ is contained in the corresponding weakly embedded Birkhoff annulus. Handles attached to a given solid torus (corresponding to one of the tangent periodic orbits) are pairwise disjoint. One can perform a Dehn surgery on U along tangent periodic orbits so that now the handles are attached along longitudes of the solid tori: we get a 3–manifold U' which is clearly a circle bundle over a surface with boundary Σ. We can think of A as naturally embedded in the interior of Σ. Moreover, Σ retracts to the graph A. Notice that U' is not contained in any way in M.

It follows that U is diffeomorphic to a Seifert manifold, obtained by Dehn surgeries around fibers in U' above vertices of A. More precisely, there is a Seifert fibration $\xi\colon U\to\Sigma_*$ where Σ_* is an orbifold, whose singularities correspond to some vertices of A. The singular fibers of ξ are the tangent periodic orbits where the attached Birkhoff annuli wrap more than once around this orbit.

Now observe that U is the projection of a "tubular neighborhood" in $\widetilde{M}$ of a countable union Birkhoff bands. This union is homeomorphic to the product of the tree $\mathcal{T}'$ by $\mathbb{R}$. This neighborhood is therefore simply connected. This implies that U has incompressible boundary in M. Hence U is a Seifert submanifold of M with fundamental group isomorphic to P (in the appropriate conjugacy class). Therefore, P is isotopic to U. This completes the proof of Theorem B.

Remark The only periodic orbits contained in U correspond to the projections of the vertices of $\mathcal{T}'$.

Here is why: The interiors of the finitely many Birkhoff annuli in question are transverse to Φ and so orbits intersecting these interiors exit U if U is sufficiently small. The other orbits are in the solid tori neighborhoods. If these neighborhoods are small enough then the only orbits entirely contained in them are the core orbits.

In particular a singular orbit γ cannot intersect the interior of the Birkhoff annuli, hence either γ is one of the periodic orbits in U or can be chosen disjoint from U if U is small. Previously we had proved that a singular orbit is homotopic into a piece of the torus decomposition. In a graph manifold, if a singular orbit is homotopic into a free piece Z, we conjecture that it must be homotopic into the boundary of the piece Z.

8 New classes of examples of pseudo-Anosov flows in graph manifolds

In Section 4 we described some new examples of (one-prong) pseudo-Anosov flows. In this section we will describe two new classes of examples, which are extremely interesting: The first class consists of actual pseudo-Anosov flows. The examples in the second class, which is a much larger class, may have one-prongs.

(1) Consider the class of examples (1) of Section 4. Each example had a 2–fold branched cover which is the geodesic flow in $T_1 S$, where S is closed, hyperbolic and has a reflection along finitely many geodesics. For simplicity we assume here that S has a single closed geodesic α of symmetry. Let N be the quotient manifold. In N, there is a quotient annulus C which is the branched quotient of the unit tangent bundle of α. Now for any integer $n > 0$ we can do the n–fold branched cover of N along C. If $n = 2$ this recovers the original geodesic flow. Otherwise the boundary of C lifts to two closed orbits which are n–prongs. Let M_n be this n–fold cover and C' be the lift of the annulus C. The set C' cuts M_n into Seifert fibered pieces, each a copy of $T_1 S'$, where S' is one component of S cut along α (notice both components of $S - \alpha$ are isometric by the reflection along α). Each of these components is a component (up to isotopy) of the torus decomposition of M_n. In each of these components the fiber acts freely on the orbit space, so these are free pieces. There is one additional Seifert component which is a small neighborhood of C'. There is a planar graph X which has two vertices (corresponding to the two directions on the geodesic α) and n edges from one vertex to the other. The set C' is homeomorphic to $X \times S^1$. This is a Seifert fibered piece of M_n, where the fiber corresponds to a periodic orbit; this is a periodic piece.

This highlights an important fact: there are examples of graph manifolds M supporting pseudo-Anosov flow Φ, so that in the torus decomposition of M there are periodic pieces glued to free pieces.

(2) The next class of examples will be on graph manifolds where all pieces are periodic. It is much more involved and much more interesting.

In the previous section we proved that the periodic Seifert pieces of orientable manifolds with pseudo-Anosov flows can be obtained as neighborhoods of unions of Birkhoff annuli. Here we will introduce standard models for certain neighborhoods of Birkhoff annuli and then use them to produce many examples.

Model of neighborhood of an embedded Birkhoff annulus Let $I = [-\pi/2, \pi/2]$. Let $N = I \times S^1 \times I$ with coordinates (x, y, z). Think of S^1 as $[0, 1]/0 \sim 1$. Convention: the increasing or positive direction in S^1 corresponds to increasing in $[0, 1]$.

For every positive real number λ, we consider the C^∞ vector field X_λ defined by

$$\begin{aligned}\dot{x} &= 0,\\ \dot{y} &= \lambda \sin(x)\cos^2(z),\\ \dot{z} &= \cos^2(x) + \sin^2(z)\sin^2(x).\end{aligned}$$

Let ψ_λ be the local flow in N generated by X_λ. It has the following properties.

- It preserves the fibration by circles $(x, y, z) \mapsto (x, z)$.
- There are only two closed orbits:

  $$\alpha_1 = \{-\pi/2\} \times S^1 \times \{0\}, \quad \alpha_2 = \{\pi/2\} \times S^1 \times \{0\}.$$

  In α_1 the flow is decreasing the y coordinate (in the flow forward direction) and in α_2 the flow is increasing the y coordinate. Hence as oriented orbits, α_1 is freely homotopic in N to $(\alpha_2)^{-1}$.
- The flow is incoming and perpendicular to the boundary $I \times S^1 \times \{-\pi/2\}$ and outgoing and perpendicular to $I \times S^1 \times \{\pi/2\}$. The flow is tangent to $\partial I \times S^1 \times I$.
- The annuli $x =$ constant are flow saturated.
- The orbits in $\{-\pi/2\} \times S^1 \times \{-\pi/2\}$ enter N and spiral towards α_1 in the negative y direction. Hence in N, $W^s(\alpha_1) = \{-\pi/2\} \times S^1 \times [-\pi/2, 0]$. In $\{-\pi/2\} \times S^1 \times (0, \pi/2]$ the orbits spiral (flow backwards) to α_1 in the positive y direction, so $W^u(\alpha_1) = \{-\pi/2\} \times S^1 \times [0, \pi/2]$. We have a similar behavior (with the y coordinate increasing when moving flow forward) in $\{\pi/2\} \times S^1 \times I$.
- The flow is invariant under any rotation in the y coordinate: $(x, y, z) \to (x, y + a, z)$ where the y coordinate is mod 1. The flow is invariant under $(x, y, z) \to (-x, -y \pmod 1, z)$. This is symmetry (I).
- Let $F_0 = (-\pi/2, \pi/2) \times S^1 \times \{-\pi/2\}$, $F_1 = (-\pi/2, \pi/2) \times S^1 \times \{\pi/2\}$, both parametrized by the x, y coordinates. In $(-\pi/2, \pi/2) \times S^1 \times I$ all orbits enter N

in F_0 and exit N in F_1. An easy computation shows that the variation of time spent between the entrance and the exit is

$$\Delta t = \frac{\pi}{|\cos(x)|}.$$

There is an induced homeomorphism $f\colon F_0 \to F_1$ given by the exit point in the x, y coordinates. It has the form

$$f(x, y) = (x, y + a(x)),$$

where the function $a(x)$ is C^∞ and depends only on x. It can also be computed:

$$a(x) = \lambda\pi[\tan(x) - \tan(x/2)].$$

Observe that $a(0) = 0$. In fact, the orbits in the center annulus have y coordinate constant. Also, $a(x)$ converges to minus infinity when x converges to $-\pi/2$ and $a(x)$ converges to infinity when x converges to $\pi/2$. In addition, $a(-x) = -a(x)$. Finally,

$$a'(x) = \lambda\pi\left[\tfrac{1}{2} + (\tan^2(x) - \tfrac{1}{2}\tan^2(x/2))\right] \geq \frac{\lambda\pi}{2}.$$

By the formula above, the map f is a nonlinear shearing in the y direction. The bigger the λ the stronger the shearing.

One canonical Birkhoff annulus associated to the block N is $B = [-\pi/2, \pi/2] \times S^1 \times 0$. If α_1, α_2 are traversed in the positive flow direction then B is a free homotopy from α_1 to $(\alpha_2)^{-1}$. The flow is transverse to B outside of α_1, α_2. The formulas above are convenient and give explicit models, but they are not essential: Up to topological equivalence any embedded Birkhoff annulus has a neighborhood with this description.

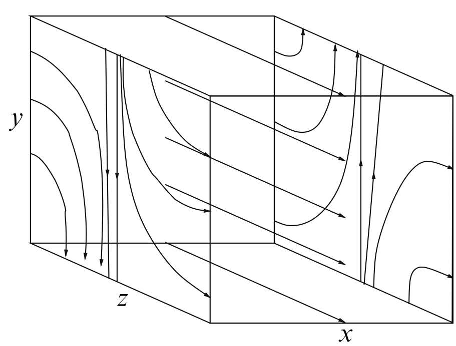

Figure 4: The local flow (N, ψ_λ). The top and bottom are identified, that is, the vertical coordinate y is defined modulo 1.

Gluing the tangential boundaries of the blocks Observe that the same formula defines a vector field $\widetilde{X}_\lambda$ on $N_* := \mathbb{R} \times S^1 \times I$, which is 2π–periodic on the coordinate x. Actually, due to the invariance of X_λ under the symmetry (I), we see that the transformation $\tau(x, y, z) = (x + \pi, -y, z)$ preserves $\widetilde{X}_\lambda$. We can take the quotient of N_* by the group generated by τ is a Seifert manifold N_1, homeomorphic to the product $K \times I$, where K is the Klein bottle. The induced local flow has a single 1-prong singular orbit: N_1 has two boundary components, one which is a incoming Klein bottle, the other an outgoing Klein bottle.

More generally, we can take the quotient of N_* by the group generated by τ^k, where k is a positive integer. We get a Seifert 3–manifold P_k, diffeomorphic to $K \times I$ or $T^2 \times I$ (according to the parity of k), with one incoming boundary component and one outgoing component, containing exactly k singular 1-prong periodic orbits.

Now, more generally, we can glue several copies of (N, X_λ) in a much more involved way. The blueprint encoding such a gluing will be a finite fat graph (X, Σ) that is, a graph embedded in the interior of a surface Σ with boundary, such that X is a retract of Σ; here, we do not require that Σ be oriented.

We moreover require two additional conditions. Basic topology of surfaces implies that each component A of $\Sigma - X$ is a half open annulus homeomorphic to $[0, 1) \times S^1$. Then $\partial A = \gamma$ is a component of $\partial\Sigma$. If e is an edge of X we define "sides" of e as follows: if p is an interior point of e and D is a small disk around p intersecting e in a properly embedded arc, then $D - e$ has two components. Each such component B defines a "local side" of e in Σ. Then B is contained in an annulus component A as above and we say that the *side* of e associated to B is the boundary curve γ of A. The two additional on conditional conditions on fat graphs are the following.

(I) The valence of every vertex of X is an even number.

(II) The set of boundary components of Σ can be partitioned in two subsets so that for every edge e of X, the two sides of e in Σ lie in different subset of this partition.

Use as labels "incoming" and "outgoing" for this partition of the set of boundary components of Σ. Now every edge has an incoming side, and an outgoing side.

Given such information we construct a flow in a 3–manifold. Associate to every edge e of X a copy N_e of N as above. Then, every *incoming* boundary component c of Σ corresponds to a cyclic sequence of edges $(e_1, e_2, \ldots, e_k)$. We glue all the associated N_{e_i} along the stable manifolds $\{\pm\pi/2\} \times S^1 \times [-\pi/2, 0)$ in the same cyclic order; more precisely, we map every point of coordinate $(\pi/2, y, z)$, $(z < 0)$ in N_{e_i}

to the point of coordinate $(-\pi/2, -y, z)$ in the following copy $N(e_{i+1})$. The result, for each boundary component c, is a Seifert 3–manifold (with boundary and corner). The Seifert 3–manifold has interior diffeomorphic to P_k with the unstable manifolds $\{\pm\pi/2\} \times S^1 \times [-\pi/2, 0]$ and the incoming and outgoing boundaries removed. It has an incoming boundary component, obtained by gluing copies of closures of the incoming annulus F_0 for each N_{e_i}. This boundary component is diffeomorphic to the torus if k is even and to the Klein bottle if k is odd. This manifold also has "outgoing" annular components. Observe that up to diffeomorphism, the result depends only on the cyclic order $(e_1, e_2, \dots, e_k)$.

Next we do the similar gluing along outgoing boundary components, but now gluing the copies of N along the unstable annuli. The result is a Seifert manifold $N(X)$, with incoming and outgoing components, but no tangential boundary components. Moreover, to every vertex v of X corresponds a tubular neighborhood of the periodic orbit which is homeomorphic to a solid torus. The flow is obviously homeomorphic to a p–branched cover of a tubular neighborhood of the singular orbit in P_1; here $2p$ is the valence of v. This is a compact Seifert manifold.

At first it may seem that $N(X)$ is orientable if and only if all the integers k in the description above are even. This is because if k is even then going around each transverse boundary component can always be achieved by orientation preserving gluing maps along tangential annuli. In addition all gluing to produce $N(X)$ is done tangential boundary annuli. We will have more to say about that at the end of this section.

By construction, $N(X)$ is equipped with a vector field X_λ for every $\lambda > 0$. The boundary of $N(X)$ is a union of incoming components and outgoing components, which are tori or Klein bottles. Due to the final process in the construction, this vector field is not smooth along the vertical orbits corresponding to the vertices of X, except if the valence of the vertex is 2 or 4, a special situation where we can perform the gluing so that the vector field is smooth in the neighborhood of the associated singular orbit. In particular, if all vertices have valence 4, then there is no singular orbit.

This is exactly the case in the Bonatti–Langevin [9] example, where the fat tree X is a figure eight (with one vertex) embedded in a once-punctured Möbius strip.

Remark Notice that $N(X)$ is a circle bundle over the surface Σ, with fibers the vertical circles with constant x, z components. Moreover, the local flow generated by X_λ preserves this fibration, hence there is an induced vector field $\overline{X}_\lambda$ on Σ. The vector field $\overline{X}_\lambda$ is Morse–Smale. Its singularities are the vertices of X; it is transverse to $\partial\Sigma$. There are three types of nonsingular trajectories of $\overline{X}_\lambda$:

- Trajectories in the stable line of a singularity, entering Σ.
- Trajectories in the unstable line of a singularity, exiting Σ.
- Trajectories joining two boundary components.

Observe that the data (Σ, X) is equivalent to the data $(\Sigma, \overline{X}_\lambda)$ up to isotopy.

Gluing the transverse boundary components

The next step is to glue outgoing boundary components to incoming boundary components. Observe that these components are naturally isomorphic to boundary components in the manifolds N_k, and thus admit natural coordinates (x, y).

Let T' be the union of the incoming boundary components and let T be the union of the outgoing boundary components. Let also

μ denote the line field in T or T' associated to x being constant.

In order to perform the gluing, we have one obvious condition: there must be the same number of outgoing and incoming tori, and the same number of outgoing and incoming Klein bottles.

Under this condition, we can select a map $A\colon T \to T'$. The only assumption we will have is that A does not preserve any of the line fields μ. Equivalently A does not send any unstable manifold of the periodic orbits to a curve isotopic into the stable manifold of a periodic orbit.

Given this condition we first show that there are no components of T which are Klein bottles. Suppose there is one such component denoted by K_1 to be glued to a component K_2 of T'. Notice that up to isotopy there are only two foliations by circles of the Klein bottle K. One foliation has two circles which are orientation reversing and the nearby leaves cover such a leaf two to one. The leaf space is a 1–dimensional orbifold, with two "boundary" orbifold points of order 2. This is type I. The other foliation comes from a product foliation by circles of the annulus and gluing the boundaries by an orientation reversing homeomorphism. This is type II. Since there are only two such foliations up to isotopy and they are intrinsically different (one has orientation reversing leaves and the other does not), then: any homeomorphism between a Klein bottle K and another K' has to preserve each type up to isotopy.

The construction of the flow shows that the line field μ induces foliations of type II in K_1 and K_2. By the above explanation A has to preserve the line field μ up to isotopy, which we do not want. Hence we have the following necessary condition.

Conclusion In order for the last step to produce a pseudo-Anosov flow, then all the components of T have to be tori.

Since all components of T, T' are tori we can use the natural linear coordinates (x, y) to choose the gluing map $A\colon T \to T'$ which is linear in the x, y–coordinates on each component. By gluing T' onto T by A we obtain a closed 3–manifold $M = M(X, A)$ equipped with a family of vector fields Y_λ. Hence it provides a flow Ψ_λ on M for each $\lambda > 0$. The periodic orbits of X_λ provide a finite number of periodic orbits of Ψ_λ that we call *vertical orbits*. Observe that since X_λ is orthogonal to the boundary, Y_λ is smooth outside of the vertical orbits.

Our goal is to prove that, if λ is big enough, then Ψ_λ is pseudo-Anosov, perhaps with one-prongs.

Let l_0^u be the union of the circles in T contained in the local unstable manifolds of the vertical orbits (they are associated to the circles $x = \pm\pi/2$, $z = \pi/2$ in each block), and similarly let l_0^s be the union of the circles in T' contained in the local stable manifolds of the vertical orbits. Let φ be the first return map of Ψ_λ from a maximal subset of T to itself. Its domain is the complement in T of $\mu_0^s = A^{-1}(l_0^s)$. For every $n > 0$, let μ_n^s be the preimage of μ_0^s by φ^n: $T \setminus \mu_n^s$ is the domain of φ^{n+1}. Each component of $\mu_n^s - \mu_0^s$ is a curve in T, intersecting every circle in l_0^u, and spiraling around two circles in μ_0^s. The complement Ω^+ of the union μ_∞^s of all μ_n^s is the domain of points where all the positive iterates φ^n, $(n \geq 0)$ are defined. Observe that μ_∞^s is a union of countably many 1–manifolds: the intersection with T of the stable manifolds of the vertical orbits.

Let C_0 be a smooth small cone field on T, centered around μ, and constant in the coordinates x, y. If C_0 is small enough, then $A(C_0)$ is a cone field in T' whose closure avoids the line field μ in T'. If in addition $\lambda > \lambda_0 \gg 1$, that is, $a'(x) \geq \lambda\pi/2 > a_0 \gg 1$, then the image of $A(C_0)$ across the fundamental blocks will be very close to the constant x direction, that is μ. This is because A is a linear map, so $A(C_0)$ is a definite positive distance away from the line field μ. In addition if the shearing is strong enough as above then the first return of $A(C_0)$ will be very close to the line field μ. This implies that whenever φ is defined, then $\varphi_*(C_0)$ is strictly contained in C_0. Moreover, this contraction from C_0 inside itself is uniform, since the bound from below of $a'(x)$ is uniform. Furthermore: $\varphi_*(C_0) \subset C_0$ is close to μ, hence every tangent vector in C_0 has a nontrivial y–component, which is uniformly expanded by the differential of φ. It follows that, again increasing λ_0 if necessary, all vectors in $\varphi_*(C_0)$ have a norm uniformly expanded under the differential of φ, let us say have norm at least multiplied by two.

Similarly, let C_0' be a small cone field defined on the entire T, centered around $A^{-1}(\mu)$ and constant in the x, y coordinates. If the shearing along the blocks is strong enough then we can choose C_0' so that $C_0' \supset (\varphi^{-1})_*(C_0')$. In addition as above we can choose the shearing strong enough so that any vector in C_0' has norm multiplied by two under $(\varphi^{-1})_*$. It implies that any vector in $(\varphi^{-1}) * (C_0')$ has its norm divided by two under φ_*. We can select the cone fields C_0 and C_0' so that there are disjoint.

Given these properties, standard arguments (see for example [31]) show that at every point p of Ω^+, the intersection of all iterates $\varphi_*^{-n}(C_0'(\varphi^n(p)))$ defines an invariant direction $E^s(p)$. Vectors in this direction are uniformly exponentially contracted under the action of φ_*.

Similarly, at a point p where φ^{-n} is defined for every positive integer n, the intersection of all the iterated cones $\varphi_*^n(C_0(\varphi^{-n}(p)))$ defines an invariant direction $E^u(p)$ whose elements are uniformly exponentially expanded under the action of φ_*.

Consider now more closely the set μ^s_∞. Let F be a component of the complement in T of l^u_0: it is a copy of the annulus F_1 (from the definition of model neighborhoods of Birkhoff annuli). The intersection between F and μ^s_0 (after the gluing by A) is a union of straight segments, with tangent vectors lying in the cone field C_0', and joining the two boundary components of F. The second generation curves, that is, the components of $\mu^s_1 = \varphi^{-1}(\mu^s_0))$ are obtained by pushing backward the first generation lines through all blocks. These become curves in T' with direction very close to μ if the curves are close to l^s_0. Then apply A^{-1}: in every annular component F they are still a union of curves joining the boundary of F, and these curves are nearly horizontal, that is, with tangent directions inside C_0'. Iterating the argument, we get that every connected component of μ^s_∞ has these properties: in every annular component F, it is a disjoint union of graphs $y = g(x)$ of smooth functions, with uniformly bounded derivative g'. They are of course all included in the stable manifold of vertical orbits.

Claim Ω^+ has empty interior.

This is the key property. Suppose this is not true and let q be a point in the interior of Ω^+. Its positive orbit intersects T infinitely many times; hence there is an annular component F of $T - l^u_0$ visited infinitely many times.

Consider now all paths c in $\mathrm{Int}(\Omega^+)$, with tangent directions contained in C_0. Due to the description above, the length of these paths is uniformly bounded from above.

On the other hand, let c be such a path containing q. There are infinitely many iterates $\varphi^{n_k}(q)$ contained in F. Since c is connected, and since the image of φ avoids l^u_0, the paths $\varphi^{n_k}(c)$ are all contained in $F \cap \mathrm{Int}(\Omega^+)$. But they all have tangent vectors contained in C_0 as $\varphi_*(C_0) \subset C_0$, and their length is exponentially increasing as proved above: contradiction. The claim is proved.

It follows that μ^s_∞ is dense. In every annular component F, every point p is a limit of points p_i lying in μ^s_∞, which is in the graph of a function g_i. We may assume that p_i are in different components of μ^s_∞ and that these components are nested. Recall that every component of $\mu^s_\infty \cap F$ is the graph of a function g with bounded derivative and therefore uniformly Lipschitz. The Arzelà–Ascoli Theorem implies that g_i converge to a Lipschitz map g^p_∞ whose graph contains p. First the Arzelà–Ascoli Theorem implies the convergence of a subsequence, but the nested, disjoint condition implies the convergence of the whole sequence. It follows that F is foliated by graphs of these Lipschitz functions.

We claim that every g^p_∞ is C^1. In order to prove that, first observe that since the set μ^s_n of iterated stable curve of generation n is locally finite inside F, we can assume that for every n, the graph of g_n lies in an element of $\mu^s_\infty \setminus \mu^s_n$, meaning that every iterate $\varphi^k(q)$, for $1 \leq k \leq n$ and every q in the graph of g_n, is well-defined. Hence, the direction tangent to the graph of g_n at q, which is obviously contained in $C'_0(q)$, is actually contained in $\varphi^{-n}_*(C'_0(\varphi^n(q)))$. It means that the uniform control on the derivative g'_n of g_n increases with n, so that the sequence of tangent vectors $(1, g'_n(x))$ converges to a vector $(1, \alpha)$ contained in the direction $E^s(p)$. Our claim follows, and moreover, the curve g^p_∞ is tangent to the stable direction E^s that we have defined in Ω^+.

Pushing along the flow, we obtain a foliation Λ^s in M of codimension one which is C^1 outside the vertical orbits. Observe that this foliation induces a C^1 one-dimensional foliation on T. This foliation admits closed leaves (the circles μ^s_0) and all other leaves in T spiral towards these closed leaves. There is no Reeb component.

Reversing the flow direction, we construct a codimension-one foliation Λ^u. These two foliations are transverse to T and T'. Moreover, there are transverse one to the other: indeed, in T, near l^u_0 the foliation Λ^s is very close to $A^{-1}(\mu)$, whereas Λ^u is very close to μ. Iterating by powers of φ this works in all of T. Moreover, the stable (respectively unstable) manifolds of the vertical orbits are leaves of Λ^s (respectively Λ^u), and their union is dense in M. The foliations Λ^s and Λ^u are the natural candidates for being the stable and unstable foliations of Ψ_λ.

Let q be a point in T. If q is in μ^s_∞, ie the stable manifold of a vertical orbit, then the leaf of Λ^s containing q is obviously in the stable manifold of q: for t big enough, the vectors tangent to $\Lambda^s(q)$ at q are divided at least by two by the differential of Ψ^t_λ.

Now assume that q lies in Ω^+, ie that all iterates $\varphi^n(q)$ are defined. Vectors tangent to the leaf of q lies in $C'_0(\varphi^n(q)))$, hence are exponentially contracted. It follows that Λ^s is the stable foliation for Ψ_λ, and similarly, Λ^u is the unstable foliation.

Conclusion There are stable and unstable foliations of Ψ_λ, which is a (possibly one-prong) pseudo-Anosov flow.

Observe that the flow is a 1–prong pseudo-Anosov flow if and only if X admits vertices of degree 2. If there are only 2–prong orbits before the last gluing, ie if all vertices of X have valence 4, then Ψ_λ is an Anosov flow. If there are no 1–prong orbits, then Ψ_λ is a pseudo-Anosov flow.

This proves Theorem C.

Remark Notice that this produces infinitely many examples of pseudo-Anosov flows in nonorientable graph manifolds. These are obtained by appropriate arrangements of orientation reversing gluing maps from tori T to T'.

An interesting subclass of the class of flows constructed here is the class where the graph X is a circle: all the vertices have degree two, that is all the vertical orbits are 1–prong. Observe that condition (II) implies that the surface Σ must be an annulus. The intermediate gluing $N(X)$ is then one of the manifolds N_k. The resulting manifold $M(X,A)$ is then a torus bundle over the circle (k must be even by the discussion above).

Since the only requirement on A is that it does not preserve the vertical direction, we obtain in particular the following.

Corollary 8.1 *In any torus bundle over S^1 which is not T^3 there are 1–prong pseudo-Anosov flows with any even number of 1–prong orbits.*

In particular notice that there are infinitely many one-prong such examples in nil manifolds. The fundamental groups of these manifolds have polynomial growth as opposed to exponential growth, which is obtained by taking a hyperbolic linear map A.

Remark In the construction of periodic Seifert fibered pieces in this section the following happens: For every vertical orbit δ in the piece and for every quadrant W associated to δ, then W contains a lozenge Z with a corner in δ. This is not true for every periodic Seifert fibered piece with respect to a pseudo-Anosov flow. It follows that the construction in this section does not attain all possible periodic Seifert fibered pieces. In particular in the construction in this section the neighborhoods of the periodic pieces always have boundaries which are transverse to the flow. This does not occur in general periodic pieces. For instance it does not occur for the class of examples (1) in the beginning of this section.

Dehn surgery Once the examples in family (2) are constructed, then one can perform any Dehn surgery on the vertical orbits. As long as the new meridian is not the original longitude, the resulting flow will be a (possibly one-prong) pseudo-Anosov flow. In addition each middle step manifold is still Seifert fibered, so the resulting manifolds are still graph manifolds. This tremendously expands the class of examples in graph manifolds.

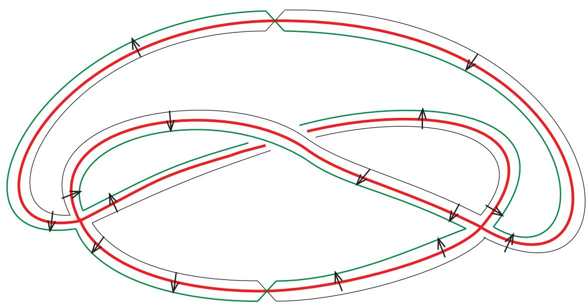

Figure 5: A fat graph surface generating a Seifert piece $N(X)$ with one sided incompressible tori, hence $N(X)$ is nonorientable.

Orientability of the manifolds $N(X)$ in the middle gluing step Here we produce some examples so that $N(X)$ has all boundary components tori (hence orientable) and still $N(X)$ is not orientable. Before we describe the examples, recall that a cycle in X corresponds to a chain of Birkhoff annuli in $N(X)$. If the cycle has even length then the orientations agree in the end and we obtain a Birkhoff torus in the corresponding manifold $N(X)$. If the cycle has odd length we obtain a Birkhoff Klein bottle.

For the first example consider the fat graph in Figure 5. The graph X has 2 vertices and 4 edges. Each embedded cycle in X has length two, hence each such cycle generates an embedded, incompressible Birkhoff torus in $N(X)$. The drawing is a figure of an immersion of Σ in the plane, except for the two crosses which mean that the band does a half twist. It is easy to check that this surface has Euler characteristic -2, the surface is nonorientable and it has two boundary components. Hence Σ is a Klein bottle minus two disks. Each boundary cycle has edge length 4, generating an embedded torus in the boundary of $N(X)$. There is one incoming boundary component and one outgoing boundary component. The direction incoming to outgoing is given by the arrows across any "fat edge" in the fat graph; see Figure 5. These constructions were introduced by Russ Waller in [50]. The cycles of length two in X which traverse exactly one band with a half twist generate a torus in $N(X)$ which is one sided. A half twist exchanges the boundary sides. There are 4 such tori. Hence $N(X)$ is not orientable. In this example $N(X)$ is a product bundle over Σ, that is $N(X)$ is the product (Klein bottle minus two disks) times the circle.

This is an example where every gluing which follows along a boundary component preserves orientation, but they also induce other gluings, some of which are orientation reversing.

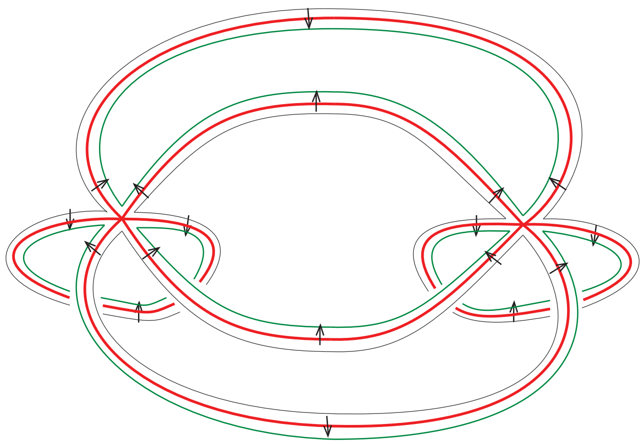

Figure 6: A fat graph surface generating a Seifert piece $N(X)$ with two sided, incompressible Klein bottles, hence $N(X)$ is nonorientable.

The second example has fat graph described in Figure 6. The graph X has 2 vertices and 6 edges. Four edges go from one vertex to the other and each of the other two edges are self connecting from a vertex to itself. The surface Σ is orientable and the figure is an immersion of Σ in the plane. The surface is a torus minus four disks. There are two outgoing boundary components, of edge length six and two respectively. It follows that each of them generates a torus in the boundary of $N(X)$. Similarly for the incoming boundary components. Each self connecting edge generates an incompressible, two sided Klein bottle, hence again $N(X)$ is nonorientable. The Seifert fibration in $N(X)$ does not have singular fibers, but is not orientable.

It is also very easy to construct examples with cycles of odd length 3 or more and which are two sided. This generates more complicated two sided Klein bottles. For the sake of space we do not describe those in detail here. Clearly all these examples can be glued to other pieces along the transverse boundary to generate examples of pseudo-Anosov flows.

Corollary 8.2 *There are examples of pseudo-Anosov flows in graph manifolds having one sided incompressible tori. There are also examples having two sided, incompressible Klein bottles.*

We already alluded to examples of embedded, incompressible Klein bottles before: the geodesic flow of a nonorientable hyperbolic surface has those. For example consider the collection of unit vectors along an orientation reversing geodesic. But the manifold is orientable and the Klein bottle is one sided. Likewise this is also what happens in the Bonatti–Langevin example [9].

9 Questions and comments

Some of the important questions not directly addressed in this paper are the following.

(1) Free Seifert fibered pieces Let P be one such piece. One fundamental question is the following: Is there is a representative for P with boundary a union of Birkhoff tori so that the flow restricted to P is up to finite covers topologically equivalent to the geodesic flow on a hyperbolic surface with boundary? A geodesic flow on a surface with boundary is the *restriction* of the geodesic flow to the unit tangent bundle of a compact surface with boundary a union of closed geodesics. Nothing is known in the case of general pseudo-Anosov flows. In the case of smooth Anosov flows, this has been analysed by the first author and proved to be true in almost all circumstances when the Anosov flow Φ is $\mathbb{R}$–covered [4].

Along these lines one very important, but vaguely phrased question is: suppose that P is a free Seifert piece. Is there no singular orbit in the middle of P? The geodesic flow on the unit tangent bundle of a surface with boundary has no singular orbits, so any singular orbit would have to be in the "boundary" of this piece. Perhaps the formulation should be that any singular orbit in the piece has to be homotopic into the boundary of the piece.

Recall the structure and examples of periodic Seifert pieces: certainly they can have singular orbits which are in some way not removable from the piece. Notice also that periodic pieces and free pieces can occur in the same flow: we described examples in the beginning of Section 8.

In any case the dynamics in free Seifert fibered pieces should be much more complex than in periodic Seifert pieces. For example the Handel–Thurston example flows [31] are obtained from geodesic flows, by cutting along a Birkhoff torus, the set of unit vectors along a separating geodesic of the surface and gluing with a shearing. Each piece is a free Seifert piece. Notice that there are infinitely many closed orbits entirely contained in each piece: they correspond to all closed geodesics contained in that piece of the surface. In fact there are uncountable many orbits of the flow entirely contained in this piece.

(2) Periodic Seifert pieces In Section 8 we produced many examples with periodic Seifert pieces where the boundary of each piece is transverse to the flow. What happens in general? We obtained partial answers in Section 6, but the general picture is not known yet.

(3) The atoroidal case The first author has done extensive work [16; 20] in the *closed* atoroidal case (where in fact by Perelman's work [40; 41; 42], M is hyperbolic). The questions addressed in that analysis were more of a geometric nature: Exactly when is the flow is quasigeodesic? Can the flow yield geometric information about the asymptotic or large scale geometric structure of the universal cover? There is not a general structure theorem in such manifolds, even in particular manifolds.

As mentioned in the introduction the atoroidal, nonclosed case is effectively unknown. Still there are examples in manifolds with two atoroidal pieces; the Franks–Williams examples are obtained as follows: start with a suspension Anosov flow in a manifold N and do a derived from Anosov construction [24]. This transforms a periodic orbit (which is hyperbolic type) into (say) a repelling orbit. Remove a neighborhood of this orbit to produce a manifold N_1 with a semiflow which is incoming along the boundary. Let N_2 be a copy of N_1 with a reversed flow. Franks and Williams show examples of gluings of N_1 to N_2 which yield Anosov flows in the resulting manifold M. N_1 and N_2 are both atoroidal and the torus decomposition of M is $N_1 \cup N_2$. Notice that these flows are not transitive.

Surely the Franks and Williams examples can be generalized to a certain extent. One fundamental remaining question is whether there are examples of pseudo-Anosov or Anosov flows in toroidal manifolds, so that the flow is transitive and there are nontrivial atoroidal pieces. Are there also examples with mixed behavior? That is, examples with atoroidal and Seifert pieces? Finally: what is the general structure of pseudo-Anosov flows restricted to atoroidal pieces? For example can one always show that the boundary tori are isotopic to transverse tori? This is not the case for Seifert pieces.

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