

A variation of McShane's identity for 2–bridge links

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We give a variation of McShane's identity, which describes the cusp shape of a hyperbolic 2–bridge link in terms of the complex translation lengths of simple loops on the bridge sphere. We also explicitly determine the set of end invariants of $SL(2, \mathbb{C})$ –characters of the once-punctured torus corresponding to the holonomy representations of the complete hyperbolic structures of 2–bridge link complements.

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Dedicated to Professor Caroline Series on the occasion of her 60th birthday

1 Introduction

In his Ph D thesis [18], McShane proved the following remarkable identity concerning the lengths of simple closed geodesics on a once-punctured torus with a complete hyperbolic structure of finite area (see also Bowditch [5] and Nakanishi [21]):

$$(1-1) \quad \sum_{\beta \in \mathcal{C}} \frac{1}{1 + e^{l(\beta)}} = \frac{1}{2}.$$

Here \mathcal{C} denotes the set of all simple closed geodesics on a hyperbolic once-punctured torus, and $l(\beta)$ denotes the length of a closed geodesic β . This identity has been generalized to cusped hyperbolic surfaces by McShane himself [19], to hyperbolic surfaces with cusps and geodesic boundary by Mirzakhani [20], and to hyperbolic surfaces with cusps, geodesic boundary and conical singularities by Tan, Wong and Zhang [29]. A wonderful application to the Weil–Petersson volume of the moduli space of bordered hyperbolic surfaces was found by Mirzakhani [20]. Bowditch [7] showed that the identity (1-1) is also valid for all quasifuchsian punctured torus groups, where $l(\beta)$ is regarded as the complex translation length of the hyperbolic isometry corresponding to the closed geodesic β . Moreover, he gave a nice variation of the identity for hyperbolic once-punctured torus bundles, which describes the cusp shape in terms of the complex translation lengths of simple loops on the fiber torus [6]. Other

3–dimensional variations have been obtained by Akiyoshi, Miyachi and the second author [2; 3] and Tan, Wong and Zhang [28; 29; 30; 32; 31; 33].

The main purpose of this paper is to prove yet another 3–dimensional variation of McShane’s identity, which describes the cusp shape of a hyperbolic 2–bridge link in terms of the complex translation lengths of essential simple loops on the bridge sphere (Theorems 2.2 and 2.3). This proves a conjecture proposed by the second author in [23]. The proof of the main results is applied to the study of the end invariants of $\mathrm{SL}(2, \mathbb{C})$ –characters of the once-punctured torus introduced by Bowditch [7] and Tan, Wong and Zhang [28]. In fact, we explicitly determine the sets of end invariants of $\mathrm{SL}(2, \mathbb{C})$ –characters of the once-punctured torus corresponding to the holonomy representations of the complete hyperbolic structures of 2–bridge link complements (Theorems 8.4 and 8.5).

This paper is organized as follows. In Section 2, we give an explicit statement of the main result. In Section 3, we recall basic facts concerning the orbifold fundamental group of the $(2, 2, 2, \infty)$ –orbifold, which connects the once-punctured torus and the 4–punctured sphere. In Section 4, we recall basic facts concerning the type-preserving $\mathrm{PSL}(2, \mathbb{C})$ –representations of the orbifold fundamental group. In Section 5, we describe a certain natural triangulation of the cusps of hyperbolic 2–bridge complements, following Guéritaud [11] and Weeks and the second author [25]. In Section 6, we give a proof of the main result. In Section 7, we give an explicit homological description of the longitudes of 2–bridge links in the definition of the cusp shapes in Theorems 2.2 and 2.3. In Section 8, we apply the proof of the main results to the study of the set of end invariants. In Section 9, we explain another approach to the main results pursued by Hirotaka Akiyoshi and the second author. In the appendix, we give numerical calculations concerning McShane’s identity and its variation.

2 Statement of the main result

Let \mathbf{T} , \mathbf{S} and \mathbf{O} , respectively, be the once-punctured torus, the 4–times punctured sphere and the $(2, 2, 2, \infty)$ –orbifold (ie the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle π). They have $\mathbb{R}^2 - \mathbb{Z}^2$ as a common covering space. To be precise, let H and \tilde{H} , respectively, be the groups of transformations on $\mathbb{R}^2 - \mathbb{Z}^2$ generated by π –rotations about points in \mathbb{Z}^2 and $(\frac{1}{2}\mathbb{Z})^2$. Then $\mathbf{T} = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$, $\mathbf{S} = (\mathbb{R}^2 - \mathbb{Z}^2)/H$ and $\mathbf{O} = (\mathbb{R}^2 - \mathbb{Z}^2)/\tilde{H}$. In particular, there are a \mathbb{Z}_2 –covering $\mathbf{T} \rightarrow \mathbf{O}$ and a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ –covering $\mathbf{S} \rightarrow \mathbf{O}$: the pair of these coverings is called the *Fricke diagram* and each of \mathbf{T} , \mathbf{S} and \mathbf{O} is called a *Fricke surface* (see Sheingorn [27]).

A simple loop in a Fricke surface is said to be *essential* if it does not bound a disk, a disk with one puncture, or a disk with one cone point. Similarly, a simple arc in a Fricke surface joining punctures is said to be *essential* if it does not cut off a “monogon”, ie a disk minus a point on the boundary. Then the isotopy classes of essential simple loops (essential simple arcs with one end in a given puncture, respectively) in a Fricke surface are in one-to-one correspondence with $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\}$: a representative of the isotopy class corresponding to $r \in \widehat{\mathbb{Q}}$ is the projection of a line in \mathbb{R}^2 (the line being disjoint from \mathbb{Z}^2 for the loop case, and intersecting \mathbb{Z}^2 for the arc case). The element $r \in \widehat{\mathbb{Q}}$ associated to a loop or an arc is called its *slope*. An essential simple loop of slope r in \mathbf{T} or \mathbf{O} is denoted by β_r , while that in \mathbf{S} is denoted by α_r . The notation reflects the following fact: after an isotopy, the restriction of the projection $\mathbf{T} \rightarrow \mathbf{O}$ to $\beta_r (\subset \mathbf{T})$ gives a homeomorphism from $\beta_r (\subset \mathbf{T})$ to $\beta_r (\subset \mathbf{O})$, while the restriction of the projection $\mathbf{S} \rightarrow \mathbf{O}$ to $\alpha_r (\subset \mathbf{S})$ gives a two-fold covering from $\alpha_r (\subset \mathbf{S})$ to $\beta_r (\subset \mathbf{O})$.

Now we recall the definition of a 2-bridge link. To this end, set $(\mathbf{S}^2, \mathbf{P}) = (\mathbb{R}^2, \mathbb{Z}^2)/H$ and call it the *Conway sphere*. Then \mathbf{S}^2 is homeomorphic to the 2-sphere, \mathbf{P} consists of four points in \mathbf{S}^2 , and $\mathbf{S}^2 - \mathbf{P}$ is the 4-punctured sphere \mathbf{S} . We also call \mathbf{S} the Conway sphere. A *trivial tangle* is a pair (B^3, t) , where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3 . By a *rational tangle*, we mean a trivial tangle (B^3, t) which is endowed with a homeomorphism from $\partial(B^3, t)$ to $(\mathbf{S}^2, \mathbf{P})$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined. We define the *slope* of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t . (Such a loop is unique up to isotopy on $\partial B^3 - t$ and so the slope of a rational tangle is well-defined.)

For each $r \in \widehat{\mathbb{Q}}$, the *2-bridge link* $K(r)$ of slope r is defined to be the sum of the rational tangles of slopes ∞ and r , namely, $(\mathbf{S}^3, K(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map on the Conway sphere $(\mathbf{S}^2, \mathbf{P})$. (Recall that the boundaries of rational tangles are identified with the Conway sphere.) $K(r)$ has one or two components according to whether the denominator of r is odd or even. We call $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, the *upper tangle* and *lower tangle* of the 2-bridge link.

The topology of 2-bridge links is nicely described by using the *Farey tessellation* \mathcal{D} , the tessellation of the hyperbolic plane \mathbb{H}^2 obtained from the ideal triangle $\langle 0, 1, \infty \rangle$ by successive reflection along its edges. The vertex set of the Farey tessellation is equal to $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\} \subset \partial \mathbb{H}^2$ and so identified with the set of isotopy classes of essential simple loops on a Fricke surface, via the correspondence $s \mapsto \alpha_s$ or β_s . Schubert's Classification Theorem of 2-bridge links [26] shows that two 2-bridge links $K(r)$

and $K(r')$ are equivalent (ie there is a self-homeomorphism of S^3 carrying one to the other) if and only if there is an automorphism of \mathcal{D} which maps the set $\{\infty, r\}$ to $\{\infty, r'\}$.

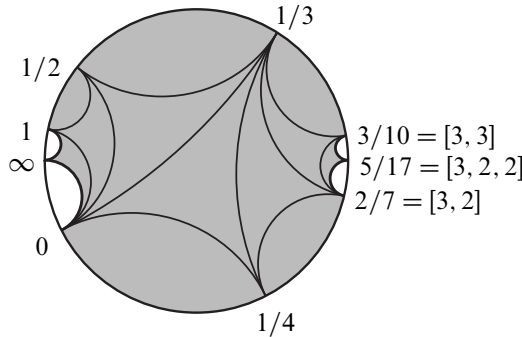


Figure 1: A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$

For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of \mathcal{D} generated by reflections along the edges of \mathcal{D} with an endpoint r , and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞ . Then the region, R , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on \mathbb{H}^2 (see Figure 1). Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R . Suppose that r is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2–component link.) Write

$$r = \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}}$$

$$=: [a_1, a_2, \dots, a_n],$$

where $n \geq 1$, $(a_1, \dots, a_n) \in (\mathbb{Z}_+)^n$ and $a_n \geq 2$. Then the above intervals are given by $I_1(r) = [0, r_1]$ and $I_2(r) = [r_2, 1]$, where

$$r_1 = \begin{cases} [a_1, a_2, \dots, a_{n-1}] & \text{if } n \text{ is odd,} \\ [a_1, a_2, \dots, a_{n-1}, a_n - 1] & \text{if } n \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [a_1, a_2, \dots, a_{n-1}, a_n - 1] & \text{if } n \text{ is odd,} \\ [a_1, a_2, \dots, a_{n-1}] & \text{if } n \text{ is even.} \end{cases}$$

The following theorem was established in [22] and [13].

Theorem 2.1 (1) [22, Proposition 4.6] *If two elements s and s' of $\widehat{\mathbb{Q}}$ belong to the same orbit $\widehat{\Gamma}_r$ -orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

(2) [13, Lemma 7.1] *For any given $s \in \widehat{\mathbb{Q}}$, there is a unique rational number s_0 in $I_1(r) \cup I_2(r) \cup \{\infty, r\}$ such that s is contained in the $\widehat{\Gamma}_r$ -orbit of s_0 . In particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$, then α_s is null-homotopic in $S^3 - K(r)$.*

(3) [13, Main Theorem 2.3] *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\widehat{\Gamma}_r$ -orbit of ∞ or r . In particular, if $s \in I_1(r) \cup I_2(r)$, then α_s is not null-homotopic in $S^3 - K(r)$.*

Moreover, it is proved by the authors in [14; 15; 16] that generically two simple loops α_s and $\alpha_{s'}$ with $s, s' \in I_1(r) \cup I_2(r)$ are homotopic in the link complement only when $s = s'$ (see Theorem 6.3).

Now assume $r = q/p$, where p and q are relatively prime positive integers such that $q \not\equiv \pm 1 \pmod{p}$. This is equivalent to the condition that $K(r)$ is hyperbolic, namely the link complement $S^3 - K(r)$ admits a complete hyperbolic structure of finite volume. Let ρ_r be the $\text{PSL}(2, \mathbb{C})$ -representation of $\pi_1(\mathcal{S})$ obtained as the composition

$$\pi_1(\mathcal{S}) \longrightarrow \pi_1(\mathcal{S}) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(S^3 - K(r)) \longrightarrow \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}),$$

where the last homomorphism is the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. Since $\pi(S^3 - K(r))$ is generated by two meridians, $\rho_r(\pi_1(\mathcal{S}))$ is generated by two parabolic transformations. Hence the hyperbolic manifold $S^3 - K(r)$ admits an isometric $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action (see [34, Section 5.4] and Figure 2), and so the $\text{PSL}(2, \mathbb{C})$ -representation ρ_r of $\pi_1(\mathcal{S})$ extends to that of $\pi_1(\mathcal{O})$. Moreover, this extension is unique (see [4, Proposition 2.2]). So we obtain, in a unique way, a $\text{PSL}(2, \mathbb{C})$ -representation of $\pi_1(\mathcal{T})$ by restriction. We continue to denote it by ρ_r .

Note that each cusp of the hyperbolic manifold $S^3 - K(r)$ carries a Euclidean structure, well-defined up to similarity, and hence it is identified with the quotient of \mathbb{C} (with the natural Euclidean metric) by the lattice $\mathbb{Z} \oplus \mathbb{Z}\lambda$, generated by the translations $[\zeta \mapsto \zeta + 1]$ and $[\zeta \mapsto \zeta + \lambda]$ corresponding to the meridian and a (suitably chosen) longitude, respectively. This λ does not depend on the choice of the cusp, because when $K(r)$ is a two-component 2-bridge link there is an orientation-preserving isometry of $S^3 - K(r)$ interchanging the two cusps (see Figure 2). We call λ the *modulus* of the cusp and denote it by $\lambda(K(r))$. In this paper, we prove the following variation of McShane's identity which describes the modulus $\lambda(K(r))$ in terms of the complex translation lengths of the images by ρ_r of essential simple loops on \mathcal{T} .

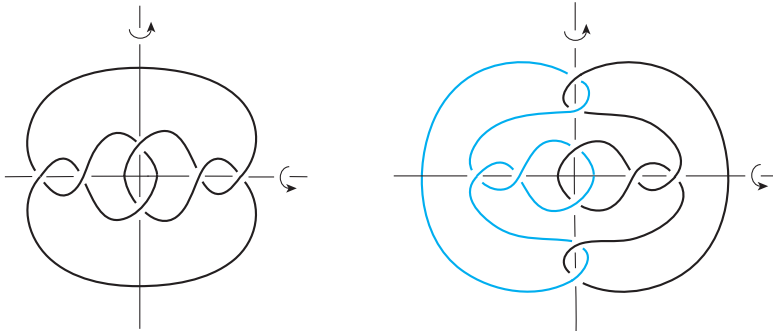


Figure 2: The symmetries of $S^3 - K(r)$

Theorem 2.2 For a hyperbolic 2-bridge link $K(r)$, the following identity holds:

$$2 \sum_{s \in \hat{\mathbb{Q}} \cap \text{int}I_1(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} + 2 \sum_{s \in \hat{\mathbb{Q}} \cap \text{int}I_2(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} + \sum_{s \in \partial I_1(r) \cup \partial I_2(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} = -1,$$

where $l_{\rho_r}(\beta_s)$ denotes the complex translation length of the hyperbolic isometry $\rho_r(\beta_s)$. Furthermore, the modulus $\lambda(K(r))$ of the cusp torus of the cusped hyperbolic manifold $S^3 - K(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$\begin{aligned} \lambda(K(r)) &= \frac{4}{|K(r)|} \left\{ 2 \sum_{s \in \hat{\mathbb{Q}} \cap \text{int}I_1(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} + \sum_{s \in \partial I_1(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} \right\} \\ &= \frac{-4}{|K(r)|} \left\{ 2 \sum_{s \in \hat{\mathbb{Q}} \cap \text{int}I_2(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} + \sum_{s \in \partial I_2(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} + 1 \right\}, \end{aligned}$$

where $|K(r)|$ denotes the number of components of $K(r)$.

We note that the additional term $s \in \partial I_1(r) \cup \partial I_2(r)$ in the first identity in the above theorem consists of four summands (because $\partial I_1(r) = \{0, r_1\}$ and $\partial I_2(r) = \{r_2, 1\}$).

In the above theorem and in the remainder of this paper, we abuse notation to use the symbol β_s to denote an element of $\pi_1(\mathbf{T})$ represented by the simple loop β_s with an arbitrary orientation. Though such an element of $\pi_1(\mathbf{T})$ is determined only up to conjugacy and inverse, the complex translation length of $\rho_r(\beta_s)$ is uniquely determined by the slope s . Here the complex translation length of an orientation-preserving isometry of \mathbb{H}^3 is an element of $\mathbb{C}/2\pi i\mathbb{Z}$ whose real part is the translation length along the axis of the isometry and whose imaginal part is the rotation angle

around the axis. If $\rho_r(\beta_s)$ is represented by a matrix $A \in \text{SL}(2, \mathbb{C})$, then the complex translation length $l_{\rho_r}(\beta_s)$ is determined by the formula $\text{tr}A = 2 \cosh(l_{\rho_r}(\beta_s)/2)$ and the condition that the real translation length $L(\rho_r(\alpha_s)) := \Re(l_{\rho_r}(\beta_s))$ is nonnegative.

At the end of this section, we restate the above theorem in terms of the hyperbolic 3-orbifold $\mathbf{O}(r) := S^3 - K(r)/(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$, the quotient of the hyperbolic manifold $S^3 - K(r)$ by the isometric $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action. To this end, note that the action extends to an action on $(S^3, K(r))$ satisfying the following conditions (see Figure 2):

- (1) If we regard S^3 as the one-point compactification of \mathbb{R}^3 , then the action on S^3 consists of the π -rotations around the three coordinate axes. In particular, the singular set, F , is the union of the three axes and the point at infinity.
- (2) If $K(r)$ is a knot, then both of the $\mathbb{Z}/2\mathbb{Z}$ -factors act on $K(r)$ as reflections, and so $K(r)$ intersects F in 4-points. Each of the four subintervals of $K(r)$ divided by $K(r) \cap F$ forms a fundamental domain of the restriction of the action to $K(r)$.
- (3) If $K(r)$ has two components, then one of the $\mathbb{Z}/2\mathbb{Z}$ -factors interchanges the components of $K(r)$ and the other factor preserves both components of $K(r)$ and acts on each component as a reflection. In particular, each component of $K(r)$ intersects F in two points, and each of the four subintervals of $K(r)$ divided by $K(r) \cap F$ forms a fundamental domain of the restriction of the action to $K(r)$.

Hence the orbifold $\mathbf{O}(r)$ has a single cusp, which forms a Euclidean orbifold of type $(2, 2, 2, 2)$, ie the orbifold with underlying space S^2 and with four cone points of cone angle π . Recall that the cusp torus of $S^3 - K(r)$ is identified with the quotient of \mathbb{C} by the lattice $\mathbb{Z} \oplus \mathbb{Z}\lambda$, generated by the translations $[\zeta \mapsto \zeta + 1]$ and $[\zeta \mapsto \zeta + \lambda]$ corresponding to the meridian and a (suitably chosen) longitude, respectively. By the above description of the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action, we see that the cusp of $\mathbf{O}(r)$ is the quotient of \mathbb{C} by the group generated by π -rotations around the origin 0, the point $\frac{1}{2}$ and the point $\frac{1}{4}|K(r)|\lambda$. It should be noted that the line segment in \mathbb{C} joining 0 and $\frac{1}{4}|K(r)|\lambda$ projects homeomorphically onto a simple arc joining two cone points in $\partial\mathbf{O}(r)$, whose inverse image in $S^3 - K(r)$ forms a longitude if $K(r)$ is a knot; it forms a union of longitudes of the two components if $K(r)$ is a 2-component link. We call the simple arc in $\partial\mathbf{O}(r)$ a *longitude*. We set

$$\lambda(\mathbf{O}(r)) := \frac{\frac{|K(r)|\lambda}{4}}{\frac{1}{2}} = \frac{|K(r)|\lambda}{2},$$

and call it the *modulus* of the cusp of $\mathbf{O}(r)$ with respect to the longitude. Then Theorem 2.2 is paraphrased as follows.

Theorem 2.3 For a hyperbolic 2-bridge link $K(r)$, the modulus of the cusp of the quotient orbifold $\mathcal{O}(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$\begin{aligned} \frac{1}{2}\lambda(\mathcal{O}(r)) &= 2 \sum_{s \in \widehat{\mathbb{Q}} \cap \text{int}I_1(r)} \frac{1}{1 + e^{l_{\rho r}(\beta_s)}} + \sum_{s \in \partial I_1(r)} \frac{1}{1 + e^{l_{\rho r}(\beta_s)}} \\ &= -2 \sum_{s \in \widehat{\mathbb{Q}} \cap \text{int}I_2(r)} \frac{1}{1 + e^{l_{\rho r}(\beta_s)}} - \sum_{s \in \partial I_2(r)} \frac{1}{1 + e^{l_{\rho r}(\beta_s)}} - 1. \end{aligned}$$

An explicit description of the longitude in Theorems 2.2 and 2.3 is in Proposition 7.1.

3 The orbifold fundamental group $\pi_1(\mathcal{O})$

The orbifold fundamental group of \mathcal{O} has the presentation

$$(3-1) \quad \pi_1(\mathcal{O}) = \langle A, B, C \mid A^2 = B^2 = C^2 = 1 \rangle,$$

where $D := (ABC)^{-1}$ is represented by the puncture of \mathcal{O} . We call D the *distinguished element*. Since \mathcal{T} and \mathcal{S} are finite regular coverings of the orbifold \mathcal{O} , the fundamental groups of \mathcal{T} and \mathcal{S} are regarded as normal subgroups of $\pi_1(\mathcal{O})$ of indices 2 and 4, respectively.

By picking a complete hyperbolic structure of \mathcal{O} (and hence of \mathcal{T}), we identify the universal covering space $\widetilde{\mathcal{O}} = \widetilde{\mathcal{T}}$ with (the upper half-space model of) the hyperbolic plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, and identify $\pi_1(\mathcal{O})$ with a Fuchsian group (see Figure 3). We assume that D is identified with the following parabolic transformation having the ideal point ∞ of \mathbb{H}^2 as the parabolic fixed point:

$$D(z) = z + 1.$$

Then the points $A(\infty)$, $B(\infty)$ and $C(\infty)$ lie on \mathbb{R} from left to right in this order. After a coordinate change, we may assume that the images of the three geodesics joining ∞ with these three points, in the universal abelian cover $\mathbb{R}^2 - \mathbb{Z}^2$ of \mathcal{T} , are open arcs of slopes 0, 1 and ∞ , joining the puncture $(0, 0)$ with $(1, 0)$, $(1, 1)$ and $(0, 1)$, respectively. Thus the images of these three geodesics in \mathcal{T} are mutually disjoint arcs properly embedded in \mathcal{T} , which divide \mathcal{T} into two ideal triangles, and thus they determine an ideal triangulation of \mathcal{T} .

We now recall the well-known correspondence between the ideal triangulations of \mathcal{T} and the *Farey triangles*, ie a triangle in the Farey tessellation \mathcal{D} . The vertex set of \mathcal{D} is equal to $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\} \subset \partial\mathbb{H}^2$ and each vertex s determines a properly embedded

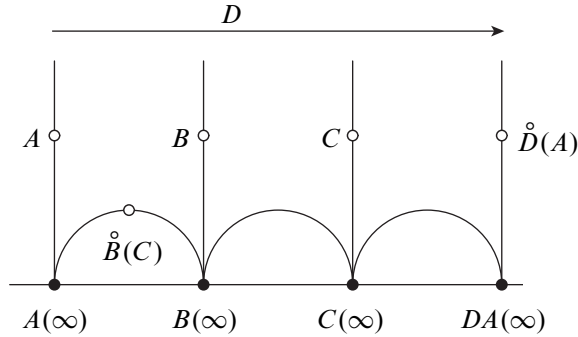


Figure 3: The Fuchsian group $\langle A, B, C \rangle$. The symbols $A, B, C, \mathring{D}(A)$ and $\mathring{B}(C)$ are situated near the fixed points in \mathbb{H}^2 of the involutions they denote.

arc δ_s in T of slope s , ie the arc in T obtained as the image of the straight arc of slope r in $\mathbb{R}^2 - \mathbb{Z}^2$ joining punctures. If $\sigma = \langle s_0, s_1, s_2 \rangle$ is a Farey triangle, then the arcs $\delta_{s_0}, \delta_{s_1}$ and δ_{s_2} are mutually disjoint and they determine an ideal triangulation of T . In the following we assume that the orientation of $\sigma = \langle s_0, s_1, s_2 \rangle$ is coherent with the orientation of the Farey triangle $\langle 0, 1, \infty \rangle$, where the orientation is determined by the order of the vertices. Then the oriented simple loop in T around the puncture representing D^2 meets the edges of the ideal triangulation $\text{trg}(\sigma)$ of slopes s_0, s_1, s_2 in this cyclic order, for every Farey triangle $\sigma = \langle s_0, s_1, s_2 \rangle$.

By using the above notation, the generators A, B and C in (3-1) are described as follows. Consider the ideal triangulation $\text{trg}(\sigma)$ of T determined by the Farey triangle $\sigma = \langle 0, 1, \infty \rangle$. It lifts to a $\pi_1(\mathbf{O})$ -invariant tessellation of the universal cover $\tilde{T} = \mathbb{H}^2$. Let $\{e_j\}_{j \in \mathbb{Z}}$ be the edges of the tessellation emanating from the ideal vertex ∞ , lying in \mathbb{H}^2 from left to right in this order. For each e_j , there is a unique order 2 element, P_j , in $\pi_1(\mathbf{O})$ which inverts e_j . We may assume after a shift of indices that e_{3j}, e_{3j+1} and e_{3j+2} project to the arcs in T of slopes 0, 1 and ∞ , respectively, for every $j \in \mathbb{Z}$. Then any triple of consecutive elements $(P_{3j}, P_{3j+1}, P_{3j+2})$ serves as (A, B, C) . Throughout this paper, (A, B, C) represents the triple of specific elements of $\pi_1(\mathbf{O})$ obtained in this way. We call $\{P_j\}_{j \in \mathbb{Z}}$ the *sequence of elliptic generators* associated with the Farey triangle σ .

The above construction works for every Farey triangle $\sigma = \langle s_0, s_1, s_2 \rangle$, and the sequence of elliptic generators associated with it is defined. (Here we use the assumption that the orientation of $\langle s_0, s_1, s_2 \rangle$ is coherent with the orientation of $\langle 0, 1, \infty \rangle$.) Any triple of three consecutive elements in a sequence of elliptic generators is called an *elliptic generator triple*. A member, P , of an elliptic generator triple is called an *elliptic*

generator, and its slope $s(P) \in \widehat{\mathbb{Q}}$ is defined to be the slope of the arc in T obtained as the image of the geodesic $\langle \infty, P(\infty) \rangle$. (Here it should be noted that ∞ is the parabolic fixed point of the distinguished element D .) For example, we have

$$(s(A), s(B), s(C)) = (0, 1, \infty).$$

When we say that $\{P_j\}_{j \in \mathbb{Z}}$ is the sequence of elliptic generators associated with a Farey triangle $\sigma = \langle s_0, s_1, s_2 \rangle$, we always assume that

$$(s(P_{3m}), s(P_{3m+1}), s(P_{3m+2})) = (s_0, s_1, s_2).$$

Thus the index j is well defined modulo a shift by a multiple of 3. We summarize the properties of elliptic generators (cf [4, Section 2.1]), by using the following nonstandard notation which was introduced in [8]:

(3-2) For elements X, Y of a group G , $\overset{\circ}{X}(Y)$ denotes XYX^{-1} .

We view $\overset{\circ}{X}$ as an element of the automorphism group of G .

Proposition 3.1 (1) Let $\{P_j\}_{j \in \mathbb{Z}}$ be the sequence of elliptic generators associated with a Farey triangle σ . Then the following hold for every $j \in \mathbb{Z}$:

- (i) $\pi_1(\mathbf{O}) \cong \langle P_j, P_{j+1}, P_{j+2} \mid P_j^2 = P_{j+1}^2 = P_{j+2}^2 = 1 \rangle$.
- (ii) $P_{j+2}P_{j+1}P_j$ is equal to the distinguished element D of $\pi_1(\mathbf{O})$.
- (iii) With the notation of (3-2), $P_{j+3m} = \overset{\circ}{D}^m(P_j)$ for every $m \in \mathbb{Z}$.
- (iv) $\langle s(P_j), s(P_{j+1}), s(P_{j+2}) \rangle$ is a Farey triangle and its orientation is coherent with $\langle 0, 1, \infty \rangle$.

(2) Let P and P' be elliptic generators of the same slope. Then $P' = \overset{\circ}{D}^m(P)$ for some $m \in \mathbb{Z}$. Let $\sigma = \langle s_0, s_1, s_2 \rangle$ and $\sigma' = \langle s'_0, s'_1, s'_2 \rangle$ be Farey triangles sharing the edge $\langle s_0, s_1 \rangle = \langle s'_0, s'_2 \rangle$, and let $\{P_j\}$ and $\{P'_j\}$, respectively, be the sequences of elliptic generators associated with σ and σ' . Then the following identity holds after a shift of indices by a multiple of 3 (see Figure 4):

$$(P'_{3j}, P'_{3j+1}, P'_{3j+2}) = (P_{3j}, \overset{\circ}{P}_{3j+1}(P_{3j+2}), P_{3j+1}).$$

The above proposition motivates us to introduce the following definition, which is used in the description of a cusp triangulation of the 2-bridge link complement.

Definition 3.2 (Chain) By a chain of Farey triangles, we mean a (nonempty) finite sequence $\Sigma = (\sigma_1, \dots, \sigma_m)$ of mutually distinct Farey triangles such that σ_{i+1} is adjacent to σ_i for each i ($1 \leq i \leq m-1$).

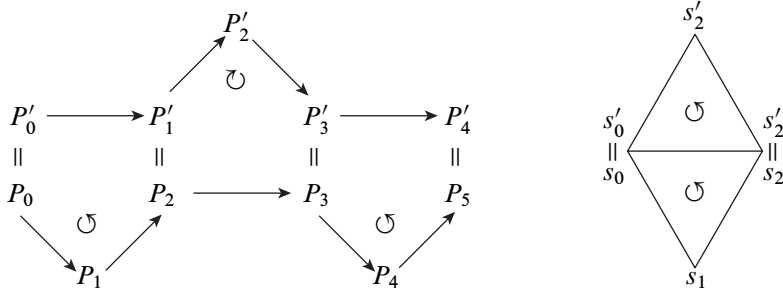


Figure 4: Adjacent sequences of elliptic generators. The symbol \curvearrowright (\curvearrowleft , respectively) indicates a triangle in which coherent reading of the vertices is counterclockwise (clockwise, respectively).

Definition 3.3 (Elliptic generator complex) Let $\Sigma = (\sigma_1, \dots, \sigma_m)$ be a chain of Farey triangles.

(1) $\mathcal{L}(\Sigma)$ (or $\mathcal{L}(\sigma_1, \dots, \sigma_m)$) denotes the simplicial complex constructed as follows, and we call it the *elliptic generator complex* associated with the chain Σ :

- (i) The vertex set $\mathcal{L}(\Sigma)^{(0)}$ is identified with the set of elliptic generators whose slope is contained in $\Sigma^{(0)}$.
- (ii) The edge set $\mathcal{L}(\Sigma)^{(1)}$ is identified with the set of the ordered pairs (P, Q) of elliptic generators, which appears (successively in this order) in a sequence of elliptic generators associated with a triangle of Σ .
- (iii) The set $\mathcal{L}(\Sigma)^{(2)}$ of the 2-simplices is identified with the set of the elliptic generator triples (P, Q, R) such that (P, Q) , (Q, R) and (P, R) are edges of $\mathcal{L}(\Sigma)$.

(2) The self-map $P \mapsto \mathring{D}(P)$ on $\mathcal{L}(\Sigma)^{(0)}$ induces a simplicial automorphism on $\mathcal{L}(\Sigma)$, and we denote it by the symbol D .

(3) $\mathcal{L}(\Sigma)/\langle D \rangle$ and $\mathcal{L}(\Sigma)/\langle D^2 \rangle$, respectively, denote the abstract cell complex obtained as the quotient of $\mathcal{L}(\Sigma)$ by the group $\langle D \rangle$ and $\langle D^2 \rangle$.

The 1-skeleton of $\mathcal{L}(\Sigma)$ is obtained as the union of the 1-dimensional simplicial complex $\mathcal{L}(\sigma_i)$ ($\sigma_i \in \Sigma^{(2)}$), which is obtained by joining the vertices $\{P_j\}$, the sequence of elliptic generators associated with σ_i , successively by edges.

4 Markoff maps and type-preserving representations

In this section, we recall basic facts concerning type-preserving $\text{PSL}(2, \mathbb{C})$ -representations of $\pi_1(\mathcal{O})$, and explain a key proposition, Proposition 4.4, which was established

by Bowditch [7] and generalized by Akiyoshi, Hirotaka and the second author [2] and Tan, Wong and Zhang [30].

A $\mathrm{PSL}(2, \mathbb{C})$ -representation of $\pi_1(\mathbf{T})$ or $\pi_1(\mathbf{O})$ is *type-preserving* if it is irreducible (equivalently, it does not have a common fixed point in $\partial\mathbb{H}^3$) and sends peripheral elements to parabolic transformations. When we mention a type-preserving representation $\rho: \pi_1(\mathbf{O}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, we always assume that the image of the distinguished element D is given by

$$(4-1) \quad \rho(D) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $\pi_1(\mathbf{T})$ is a free group, any type-preserving representation $\rho: \pi_1(\mathbf{T}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ lifts to a presentation $\tilde{\rho}: \pi_1(\mathbf{T}) \rightarrow \mathrm{SL}(2, \mathbb{C})$, which is *type-preserving*, in the sense that it is irreducible and sends peripheral elements to parabolic transformations. For a type-preserving $\mathrm{SL}(2, \mathbb{C})$ -representation $\tilde{\rho}: \pi_1(\mathbf{T}) \rightarrow \mathrm{SL}(2, \mathbb{C})$, let $\phi = \phi_{\tilde{\rho}}$ be the map from $\mathcal{D}^{(0)} = \hat{\mathbb{Q}}$ to \mathbb{C} defined by $\phi(s) = \mathrm{tr}(\tilde{\rho}(\beta_s))$. Then it is a nontrivial *Markoff map* in the sense of [7], that is:

- (i) For any Farey triangle $\langle s_0, s_1, s_2 \rangle$, the triple $(\phi(s_0), \phi(s_1), \phi(s_2))$ is a nontrivial *Markoff triple*, that is, it is a nontrivial solution of the Markoff equation

$$x^2 + y^2 + z^2 = xyz.$$

Here, being nontrivial means $(x, y, z) \neq (0, 0, 0)$.

- (ii) For any pair of triangles $\langle s_0, s_1, s_2 \rangle$ and $\langle s_1, s_2, s_3 \rangle$ of \mathcal{D} sharing a common edge $\langle s_1, s_2 \rangle$, we have

$$\phi(s_0) + \phi(s_3) = \phi(s_1)\phi(s_2).$$

Lemma 4.1 *Let $\rho: \pi_1(\mathbf{O}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a type-preserving representation satisfying the normalization condition (4-1), and let ϕ be a Markoff map induced by a type-preserving representation $\tilde{\rho}: \pi_1(\mathbf{T}) \rightarrow \mathrm{SL}(2, \mathbb{C})$ which is a lift of the restriction of ρ to $\pi_1(\mathbf{T})$.*

- (1) *Let P be an elliptic generator of slope s .*

- (i) *If $\phi(s) \neq 0$, then $\rho(P)$ is the π -rotation about the geodesic with endpoints $c(\rho(P)) \pm i/\phi(s)$, where $c(\rho(P)) = \rho(P)(\infty) \in \mathbb{C}$.*
- (ii) *If $\phi(s) = 0$, then $\rho(P)$ is the π -rotation about a vertical geodesic, ie a geodesic in the upper half-space model of the hyperbolic space which has ∞ as an endpoint.*

(2) Let $\{P_j\}$ be the sequence of elliptic generators associated with a Farey triangle $\sigma = \langle s_0, s_1, s_2 \rangle$.

(i) If $\phi(s_1)\phi(s_2) \neq 0$, then

$$c(\rho(P_2)) - c(\rho(P_1)) = \frac{\phi(s_0)}{\phi(s_1)\phi(s_2)}.$$

(ii) If $\phi(s_0)\phi(s_1)\phi(s_2) \neq 0$, then $\{c(\rho(P_j))\}$ is a sequence of points in \mathbb{C} such that $c(\rho(P_j)) + 1 = c(\rho(P_{j+3}))$.

(iii) If $\phi(s_0) = 0$, then $\phi(s_1)\phi(s_2) \neq 0$ and $c(\rho(P_{1+3k})) = c(\rho(P_{2+3k}))$ and $\rho(P_{3k})$ is a π -rotation about a vertical geodesic, where the end point of its axis in \mathbb{C} is the midpoint between $c(\rho(P_{-2+3k})) = c(\rho(P_{-1+3k}))$ and $c(\rho(P_{1+3k})) = c(\rho(P_{2+3k}))$.

(iv) Suppose $\phi(s'_0) = 0$, where s'_0 is the vertex of \mathcal{D} opposite to s_0 with respect to the edge $\langle s_1, s_2 \rangle$. Then $\phi(s_0)\phi(s_1)\phi(s_2) \neq 0$ and $c(\rho(P_{-1+3k})) = c(\rho(P_{1+3k}))$.

Proof The assertions of the lemma, except for (2-iv), are contained in [4, Proposition 2.2.4]. To prove (2-iv), note that the assumption $\phi(s'_0) = 0$ implies that $\phi(s_2) = \pm i\phi(s_1)$ and $\phi(s_0) = \pm i\phi(s_1)^2$. Since $(\phi(s'_0), \phi(s_1), \phi(s_2))$ is nontrivial, this implies that $\phi(s_1) \neq 0$ and so none of $\phi(s_j)$ ($j \in \{0, 1, 2\}$) is 0. \square

The following definition is used in a description of cusp triangulations of hyperbolic 2-bridge link complements.

Definition 4.2 Let $\rho: \pi_1(\mathbf{O}) \rightarrow \text{PSL}(2, \mathbb{C})$ be a type-preserving representation satisfying the normalization condition (4-1).

(1) Let $\sigma = \langle s_0, s_1, s_2 \rangle$ be a Farey triangle such that $\phi(s_0)\phi(s_1)\phi(s_2) \neq 0$. Then $\mathcal{L}(\rho, \sigma)$ denotes the (possibly singular) biinfinite zigzag line in \mathbb{C} which is obtained by successively joining the points $\{c(\rho(P_j))\}$, where $\{P_j\}$ is the sequence of elliptic generators associated with σ . We note that $\mathcal{L}(\rho, \sigma)$ is invariant by the transformation $z \mapsto z + 1$.

(2) We say that $\mathcal{L}(\rho, \sigma)$ is *simple*, if the underlying space $|\mathcal{L}(\rho, \sigma)|$ is homeomorphic to the real line \mathbb{R} and $\{c(\rho(P_j))\}$ sits on it in the order of the suffix $j \in \mathbb{Z}$.

(3) We say that $\mathcal{L}(\rho, \sigma)$ is *simply folded* at the vertex $c(\rho(P_j))$, if $c(\rho(P_{j-1})) = c(\rho(P_{j+1}))$ and the horizontal line, L , passing through this point does not contain $c(\rho(P_j))$. In this case, the underlying space $|\mathcal{L}(\rho, \sigma)|$ is the union of L and the "spikes" joining $c(\rho(P_{j+3k}))$ and $c(\rho(P_{j-1+3k})) = c(\rho(P_{j+1+3k})) \in L$, where k runs over all integers. We call L the *horizontal line* determined by $\mathcal{L}(\rho, \sigma)$. If s is the slope of

the elliptic generator P_j , we also say that $\mathcal{L}(\rho, \sigma)$ is *simply folded* at the slope s . We say that $\mathcal{L}(\rho, \sigma)$ is *simply folded* if it is simply folded at some slope (which is a vertex of σ).

(4) Let $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$ be a chain of Farey triangles such that $\phi^{-1}(0) \cap \Sigma^{(0)} = \emptyset$. Then $\mathcal{L}(\rho, \Sigma)$ denotes the union of the biinfinite zigzag lines $\{\mathcal{L}(\rho, \sigma_i)\}$ in \mathbb{C} .

Remark 4.3 By Lemma 4.1(2), $\mathcal{L}(\rho, \sigma)$ is simply folded at the slope s only when $\phi(s') = 0$, where s' is the vertex of \mathcal{D} opposite to s with respect to the edge of σ which does not contain s .

Let \mathcal{T} be a binary tree (a countably infinite simplicial tree all of whose vertices have degree 3) properly embedded in \mathbb{H}^2 dual to \mathcal{D} . A *directed edge*, \vec{e} , of \mathcal{T} can be thought of an ordered pair of adjacent vertices of \mathcal{T} , referred to as the *head* and *tail* of \vec{e} . Following [7], we use the notation $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ to mean that s_0, s_1, s_2 and s_3 are the ideal vertices of \mathcal{D} such that

- (i) the Farey edge $\langle s_1, s_2 \rangle$ is the dual to \vec{e} , and
- (ii) the Farey triangle $\langle s_0, s_1, s_2 \rangle$ ($\langle s_1, s_2, s_3 \rangle$, respectively) is dual to the head (tail, respectively) of \vec{e} , if $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$.

If $\phi(s_1)\phi(s_2) \neq 0$, then we set

$$\psi(\vec{e}) := \frac{\phi(s_0)}{\phi(s_1)\phi(s_2)}.$$

Then Lemma 4.1(2-i) is rephrased as follows:

$$(4-2) \quad c(\rho(P_2)) - c(\rho(P_1)) = \psi(\vec{e}).$$

We regard $\psi = \psi_\phi$ as a map from the set of oriented edges $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ of \mathcal{T} such that $\phi(s_1)\phi(s_2) \neq 0$, and we call it the *complex probability map* associated with the Markoff map ϕ . We note that this map is determined by the type-preserving representation $\rho: \pi_1(\mathbf{T}) \rightarrow \text{PSL}(2, \mathbb{C})$ obtained from the type-preserving $\text{SL}(2, \mathbb{C})$ -representation of $\pi_1(\mathbf{T})$ inducing the Markoff map ϕ . So we also call ψ the complex probability map associated with ρ .

By a *complementary region* of \mathcal{T} , we mean the closure of a connected component of $\mathbb{H}^2 - \mathcal{T}$. Let Ω be the set of complementary regions of \mathcal{T} . Then there is a natural bijection from Ω to $\hat{\mathbb{Q}}$. In the following we identify Ω with $\hat{\mathbb{Q}}$. Let $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ be a directed edge of \mathcal{T} . If we remove the interior of e from \mathcal{T} , we are left with two disjoint subset, which we denote by $\mathcal{T}^\pm(\vec{e})$, so that $e \cap \mathcal{T}^+(\vec{e})$ is the head of \vec{e}

and $e \cap \mathcal{T}^-(\vec{e})$ is its tail. Let $\Omega^\pm(\vec{e}) \subset \Omega$ be the set of regions whose boundaries lie in $\Sigma^\pm(\vec{e})$, and set $\Omega^0(e) = \{s_1, s_2\}$. We see that Ω can be written as the disjoint union: $\Omega = \Omega^0(e) \cup \Omega^+(\vec{e}) \cup \Omega^-(\vec{e})$. Set $\Omega^{0-}(\vec{e}) = \Omega^0(e) \cup \Omega^-(\vec{e})$ and $\Omega^{0+}(\vec{e}) = \Omega^0(e) \cup \Omega^+(\vec{e})$. We quote the following key result from [2, Proposition 5.2] which is a slight extension of [7, Proposition 3.13] (see [30] for further extension).

Proposition 4.4 *Let ϕ be a Markoff map and \vec{e} a directed edge of \mathcal{T} satisfying the following conditions:*

- (i) *The set $\{s \in \Omega^-(\vec{e}) \mid |\phi(s)| \leq 2\}$ is finite.*
- (ii) *$\Omega^{0-}(\vec{e}) \cap \phi^{-1}(-2, 2) = \emptyset$.*

Then

$$\psi(\vec{e}) = \sum_{s \in \Omega^0(e)} h(\phi(s)) + 2 \sum_{s \in \Omega^-(\vec{e})} h(\phi(s)).$$

Moreover, the above sum converges absolutely.

Here, $h: \mathbb{C} - [-2, 2] \rightarrow \mathbb{C}$ is defined by $h(x) = \frac{1}{2}(1 - \sqrt{1 - 4/x^2})$, where we adopt the convention that the real part of a square root is always nonnegative. For each $s \in \Omega = \hat{\mathbb{Q}}$, let $l(\rho(\beta_s))$ be the complex translation length of the isometry $\rho(\beta_s)$ of \mathbb{H}^3 , where we abuse notation to denote by β_s an element of $\pi_1(\mathbf{T})$ represented by the simple loop β_s of slope s . Then we have the following (see [7, page 721]):

$$h(\phi(s)) = \frac{1}{1 + e^{l(\rho(\beta_s))}}.$$

At the end of this section, we give a necessary and sufficient condition for a type-preserving $\text{PSL}(2, \mathbb{C})$ -representation to descend to a representation of the 2-bridge link group $\pi_1(S^3 - K(r))$.

Lemma 4.5 *Let ϕ be a nontrivial Markoff map, and let $\rho: \pi_1(\mathbf{O}) \rightarrow \text{PSL}(2, \mathbb{C})$ be a type-preserving representation induced by ϕ . Then the restriction of ρ to $\pi_1(\mathbf{S})$ descends to a representation of the 2-bridge link group $\pi_1(S^3 - K(r))$ if and only if $\phi(\infty) = \phi(r) = 0$.*

Proof Though this is proved in [24, Proof of Proposition 4.1], we give a proof for completeness. Since $\pi_1(S^3 - K(r)) = \pi_1(\mathcal{S}) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$, a representation ρ descends to a representation of the link group if and only if $\rho(\alpha_\infty) = \rho(\alpha_r) = 1$. Since $\alpha_s = \beta_s^2$, this condition is equivalent to the condition that either $\rho(\beta_s)$ is the identity or an elliptic transformation of order 2 for each $s = \infty$ and r . However, since ρ is irreducible by the assumption, we have $\rho(\beta_s) \neq 1$ for any $s \in \hat{\mathbb{Q}}$. Thus the above condition is equivalent to the condition that both $\rho(\beta_\infty)$ and $\rho(\beta_r)$ are elliptic of order 2, which in turn is equivalent to the condition $\phi(\infty) = \phi(r) = 0$. \square

5 The canonical decomposition of $S^3 - K(r)$ and the induced cusp triangulation

In this section, we describe the canonical decompositions of hyperbolic 2-bridge link complements and the induced cusp triangulations, following [11] and [25]. Let $K(r)$ be a hyperbolic 2-bridge link. Then we may assume $r = q/p$, where p and q are relatively prime integers such that $2 \leq q < p/2$, and so r has the continued fraction expansion $[a_1, a_2, \dots, a_n]$, where $(a_1, \dots, a_n) \in (\mathbb{Z}_+)^n$, $a_1 \geq 2$, $a_n \geq 2$ and $n \geq 2$. Set $c = \sum_{i=1}^n a_i$, and let $\Sigma(r) = (\sigma_1, \sigma_2, \dots, \sigma_c)$ be the chain of Farey triangles which intersect the hyperbolic geodesic joining ∞ with r in this order (see Figure 5).

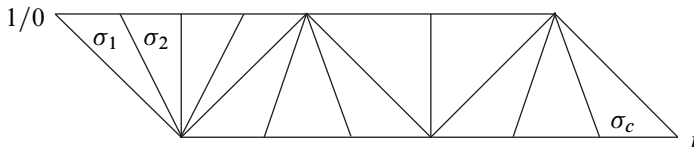


Figure 5: The chain $\Sigma(r)$ of Farey triangles

Just as with the once-punctured torus T , each Farey triangle $\sigma = \langle s_0, s_1, s_2 \rangle$ determines a (topological) ideal triangulation of the 4-times punctured sphere $\mathcal{S} = (\mathbb{R}^2 - \mathbb{Z}^2)/H$. To be precise, the union of the lines in $\mathbb{R}^2 - \mathbb{Z}^2$ passing through the punctures of slopes $\{s_0, s_1, s_2\}$ determines an H -invariant ideal triangulation of $\mathbb{R}^2 - \mathbb{Z}^2$, and this descends to an ideal triangulation of \mathcal{S} . The 1-skeleton of this ideal triangulation consists of three pairs of edges corresponding to the three vertices $\{s_0, s_1, s_2\}$ of σ . In the remainder of this paper, we abuse notation and use the symbol $\text{trg}(\sigma)$ to denote this ideal triangulation of \mathcal{S} .

If σ and σ' are adjacent Farey triangles, then $\text{trg}(\sigma')$ is obtained from $\text{trg}(\sigma)$ by a “diagonal exchange”, ie deleting a pair of edges of slope s and adding a pair of edges of slope s' , where s (s' , respectively) is the vertex of σ (σ' , respectively) which is

not contained in σ' (σ , respectively). As illustrated in Figure 6, $\text{trg}(\sigma)$ and $\text{trg}(\sigma')$ can be regarded as the bottom and top faces of an immersed pair of topological ideal tetrahedra in $\mathcal{S} \times \mathbb{R}$, which we denote by $\text{trg}(\sigma, \sigma')$.

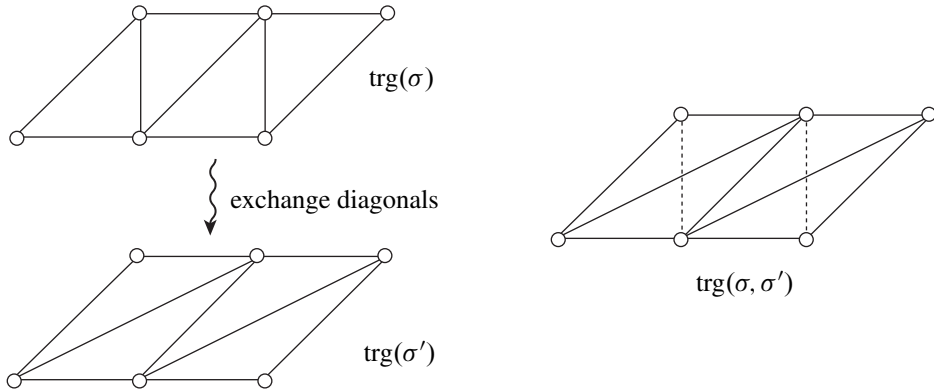


Figure 6: A diagonal exchange of an ideal triangulation of \mathcal{S} determines an immersed pair of ideal tetrahedra in $\mathcal{S} \times \mathbb{R}$

The immersed topological pairs of ideal tetrahedra $\{\text{trg}(\sigma_i, \sigma_{i+1})\}_{1 \leq i \leq c-1}$ can be stacked up to form a topological ideal triangulation, $\widehat{\mathcal{D}}(r)$, of $\mathcal{S} \times [-1, 1]$. The restriction of $\widehat{\mathcal{D}}(r)$ to $\mathcal{S} \times \{-1\}$ ($\mathcal{S} \times \{+1\}$, respectively) is $\text{trg}(\sigma_1)$ ($\text{trg}(\sigma_c)$, respectively), and each $\text{trg}(\sigma_i)$ can be regarded as (a triangulation of) a pleated surface in $\mathcal{S} \times [-1, 1]$. Let $\mathcal{D}(r)$ be the topological ideal simplicial complex obtained from $\widehat{\mathcal{D}}(r)$ by collapsing each edge of slope ∞ and r into an ideal vertex. To be precise, $\mathcal{D}(r)$ is constructed as follows. Since each edge of slope ∞ is collapsed into an ideal vertex, the subcomplex $\text{trg}(\sigma_1)$ of $\widehat{\mathcal{D}}(r)$ is collapsed into a single ideal edge, and $\text{trg}(\sigma_2)$ is folded along the pair of edges of slope $1/2$ to a pair ideal triangles as illustrated in Figure 7. (Note that the slope $1/2$ is the vertex of σ_2 which is not contained in σ_1 .) Similarly, since each edge of slope r is collapsed into an ideal vertex, the subcomplex $\text{trg}(\sigma_c)$ of $\widehat{\mathcal{D}}(r)$ is collapsed into a single ideal edge, and $\text{trg}(\sigma_{c-1})$ is folded along the pair of edges of slope $[a_1, \dots, a_n - 2]$ into a pair of ideal triangles. (Note that the slope $[a_1, \dots, a_n - 2]$ is the vertex of σ_{c-1} which is not contained in σ_c .) In other words, $\mathcal{D}(r)$ is obtained from the subcomplex $\widehat{\mathcal{D}}_0(r) := \{\text{trg}(\sigma_i, \sigma_{i+1})\}_{2 \leq i \leq c-2}$ of $\widehat{\mathcal{D}}(r)$ by folding the bottom surface $\text{trg}(\sigma_2)$ in the pair of edges of slope $1/2$ and by folding the top surface $\text{trg}(\sigma_{c-1})$ in the pair of edges of slope $[a_1, \dots, a_n - 2]$, as described in [25, page 408]. Hence $\mathcal{D}(r)$ gives a topological ideal triangulation of $S^3 - K(r)$ by [25, Theorem II.2.4].

It should be noted that the edges of $\text{trg}(\sigma_2)$ of slopes $0/1$ and $1/1$ are identified into a single edge in $\mathcal{D}(r)$, which forms the “core tunnel” of the rational tangle $(B^3, t(\infty))$

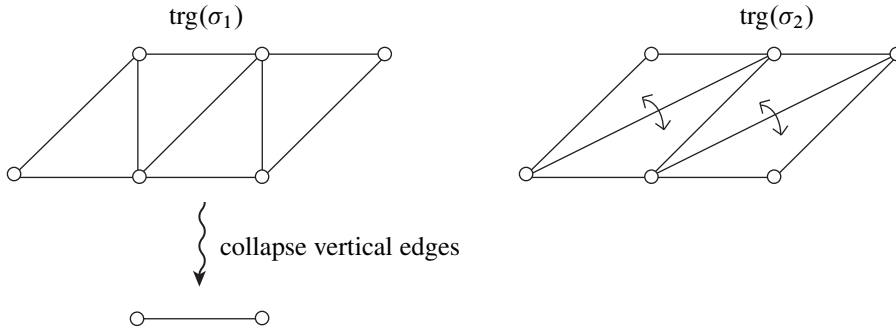


Figure 7: The effect of the collapsing of the edges of slope ∞ in the ideal triangulation $\widehat{\mathcal{D}}(r)$ of $\mathcal{S} \times [-1, 1]$: The subcomplex $\text{trg}(\sigma_1)$ is collapsed into a single ideal edge, and the subcomplex $\text{trg}(\sigma_2)$ is folded along the edges of slope $1/2$ into a pair of ideal triangles.

in $(S^3, K(r))$ (see [25, Figure II.2.7] and [11, Figure 17]). Similarly, the edges of $\text{trg}(\sigma_{c-1})$ of slopes $[a_1, \dots, a_{n-1}]$ and $[a_1, \dots, a_n - 1]$ are identified into a single edge in $\mathcal{D}(r)$, which forms the core tunnel of the rational tangle $(B^3, t(r))$ in $(S^3, K(r))$.

Now we describe the triangulation of the peripheral torus of $S^3 - K(r)$ induced by $\mathcal{D}(r)$. To this end, we identify the underlying space of the subcomplex $\widehat{\mathcal{D}}_0(r)$ of $\widehat{\mathcal{D}}(r)$ with $\mathcal{S} \times [-1, 1]$, and we first describe the triangulation of the peripheral annuli of $\mathcal{S} \times [-1, 1]$ induced by $\widehat{\mathcal{D}}_0(r)$. Since the combinatorics of the four peripheral annuli are identical, let us focus on a single peripheral annulus, A . Since $\text{trg}(\sigma_i)$ is an ideal triangulation of a level 4-punctured sphere, it induces a triangulation, $C(\sigma_i)$, of a core circle in A . The triangulation $C(\sigma_i)$ consists of 3 vertices and 3 edges. By recalling the definition of the slopes of elliptic generators, we may identify $C(\sigma_i)$ with the quotient complex $\mathcal{L}(\sigma_i)/\langle D \rangle$ (see Definition 3.3(3)). The region in A bounded by $C(\sigma_i)$ and $C(\sigma_{i+1})$ consists of 2 triangles as illustrated in Figure 8, and the triangulation of the region can be identified with $\mathcal{L}(\sigma_i, \sigma_{i+1})/\langle D \rangle$ (compare Figure 8 with Figure 4). The family $\{C(\sigma_i)\}_{2 \leq i \leq c-1}$ forms the 1-skeleton of the triangulation of A , which is identified with $\mathcal{L}(\Sigma_0(r))/\langle D \rangle$, where $\Sigma_0(r) := (\sigma_2, \dots, \sigma_{c-1})$.

Next, we explain the effect, to the triangulation $\mathcal{L}(\Sigma_0(r))/\langle D \rangle$ of the peripheral annulus A , of the folding of the pleated surfaces $\text{trg}(\sigma_2)$ and $\text{trg}(\sigma_{c-1})$. To this end, let $\{P_j\}$ be the sequence of elliptic generators associated with $\sigma_2 = (0/1, 1/2, 1/1)$ such that $(s(P_0), s(P_1), s(P_2)) = (0/1, 1/2, 1/1)$. Since the edges of slope $0/1$ and $1/1$ in $\widehat{\mathcal{D}}(r)$ are identified into a single edges by the folding of $\text{trg}(\sigma_2)$ along the edges of slope $1/2$, the vertices $[P_0]$ and $[P_2]$ of $\mathcal{L}(\Sigma_0(r))/\langle D \rangle$ are identified. In the infinite cyclic cover $\mathcal{L}(\Sigma_0(r))$, the boundary line $\mathcal{L}(\sigma_1)$ is deformed into a zigzag line which has a “hairpin curve” at the vertices P_{1+3k} , where the vertices P_{3k} and P_{2+3k} are

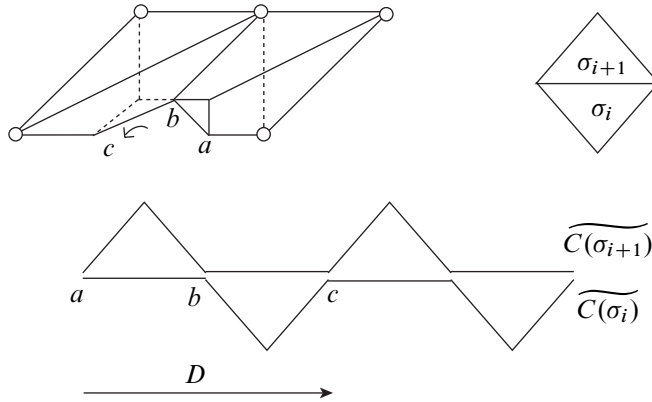


Figure 8: Local picture of the triangulation of the peripheral annuli of $S \times [-1, 1]$. In the infinite cyclic cover of the peripheral annulus A , the inverse images of $C(\sigma_i)$ and $C(\sigma_{i+1})$ form periodic zigzag lines, and the triangles bounded by them project to the two triangles in A .

identified into a single vertex for each $k \in \mathbb{Z}$ (see Figure 9). Furthermore, since the folding joins the punctures of $S \times [-1, 1]$, the resulting triangulation of the peripheral annulus A is joined to the corresponding triangulation of another peripheral annulus as illustrated in Figure 9 (see [11, Figure 19]). Similarly, the folding of the pleated surface $\text{trg}(\sigma_{c-1})$ causes a similar effect on the other side of A .

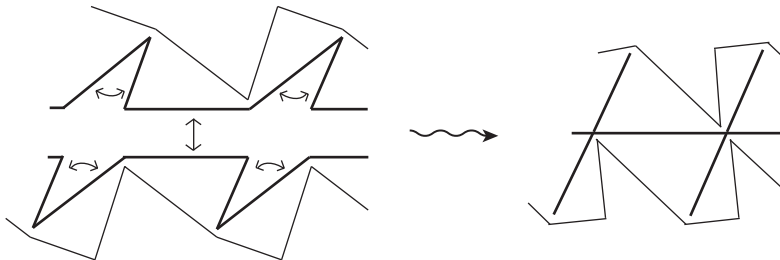


Figure 9: The effect of the folding in the cusp triangulation, viewed in the infinite cyclic cover

In [11], Futer applied Guéritaud's technique based on angled structures to prove that the topological ideal triangulation $\mathcal{D}(r)$ is geometric, namely, $\mathcal{D}(r)$ is homeomorphic to a geometric ideal triangulation of the complete hyperbolic manifold $S^3 - K(r)$. (Moreover, Guéritaud [10] proved that $\mathcal{D}(r)$ is homeomorphic to the canonical decomposition of $S^3 - K(r)$ in the sense of [9] and [37], proving the conjecture in [25]. In the second author's joint work with Akiyoshi, Wada and Yamashita [4], an approach using cone manifold deformation toward the same conclusion was announced.) The

geometric ideal triangulation $\mathcal{D}(r)$ induces a geometric triangulation of each cusp torus of $S^3 - K(r)$. Since $\mathcal{D}(r)$ is preserved by the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action described in Section 2 (see Figure 2), this triangulation does not depend on a choice of a cusp, and we call it *the triangulation of the cusp of $S^3 - K(r)$ induced by the geometric ideal triangulation $\mathcal{D}(r)$* , or simply *the cusp triangulation induced by $\mathcal{D}(r)$* . The cusp triangulation is preserved by the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action on $S^3 - K(r)$, and so it induces a “triangulation” of the cusp of the quotient orbifold $\mathbf{O}(r)$. We also call it *the triangulation of the cusp of $\mathbf{O}(r)$ induced by the geometric ideal triangulation $\mathcal{D}(r)$* , or simply *the cusp triangulation induced by $\mathcal{D}(r)$* . The following proposition describes the geometric structure of the cusp triangulation. (See Figure 10 for the actual geometric picture of the cusp triangulation, which is produced by SnapPea [36]. See also Figure 11 which illustrates the periodic zigzag lines $\mathcal{L}(\rho_r, \sigma_i)$ in the proposition.)

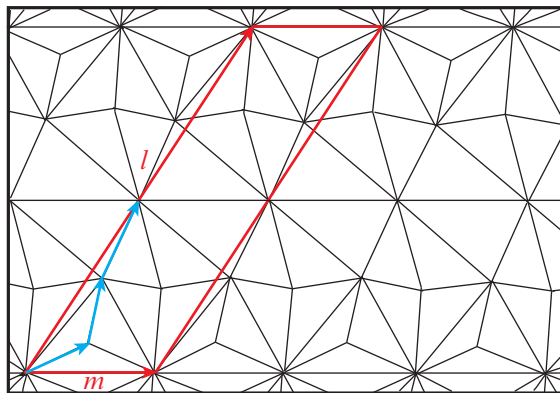


Figure 10: The actual cusp triangulation of $S^3 - K([3, 2, 2])$. The oriented zigzag line segment represent that obtained by joining the points $c(\rho_r(P_0)), c(\rho_r(P_1)), \dots, c(\rho_r(P_d))$ in the proof of Proposition 6.1.

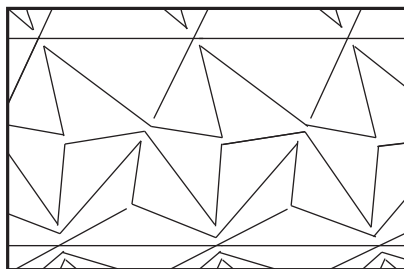


Figure 11: The periodic zigzag lines $\{\mathcal{L}(\rho_r, \sigma_i)\}$ in the cusp triangulation

Proposition 5.1 *Let $K(r)$ be a hyperbolic 2-bridge link and $\rho_r: \pi_1(\mathbf{O}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a type-preserving representation induced by the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$ satisfying the normalization condition (4-1). Then the following hold:*

- (1) We have $\phi_r^{-1}(0) \cap \Sigma(r)^{(0)} = \{\infty, r\}$, where $\Sigma(r)^{(0)}$ is the vertex set of $\Sigma(r)$ and ϕ_r is the Markoff map induced by a lift of the restriction of ρ_r to $\pi_1(\mathbf{T})$ to an $\mathrm{SL}(2, \mathbb{C})$ -representation.
- (2) The zigzag line $\mathcal{L}(\rho_r, \sigma_2)$ is simply folded at the slope $1/2$; $\mathcal{L}(\rho_r, \sigma_{c-2})$ is simply folded at the slope $[a_1, a_2, \dots, a_n - 2]$.
- (3) Let L_- and L_+ be the horizontal lines determined by the simply folded zigzag lines $\mathcal{L}(\rho_r, \sigma_2)$ and $\mathcal{L}(\rho_r, \sigma_{c-2})$, respectively. Then $\mathcal{L}(\rho_r, \Sigma_0(r))$ forms a 1-skeleton of a triangulation of the strip, \tilde{A} , in \mathbb{C} bounded by L_- and L_+ . This triangulation descends to the triangulation of the cusp of $\mathbf{O}(r)$ induced by $\mathcal{D}(r)$. To be precise, the following hold:
 - (i) Let P_- and P_+ be elliptic generators of slope ∞ and r , respectively. Then $\rho_r(P_-)$ ($\rho_r(P_+)$, respectively) acts on \mathbb{C} as the π -rotation about the center of an edge of $\mathcal{L}(\rho_r, \sigma_2)$ ($\mathcal{L}(\rho_r, \sigma_{c-2})$, respectively) contained in L_- (L_+ , respectively). In particular, \tilde{A} forms a fundamental domain of the action on \mathbb{C} of the infinite dihedral group generated by $\rho_r(P_-)$ and $\rho_r(P_+)$.
 - (ii) The orbifold fundamental group, $\pi_1(\partial\mathbf{O}(r))$, of the cusp of $\mathbf{O}(r)$ is identified with the group $\langle \rho_r(D), \rho_r(P_-), \rho_r(P_+) \rangle$. Moreover, $\rho_r(D)$ corresponds to a meridian of $K(r)$, whereas $\rho_r((P_+P_-)^2)$ or $\rho_r(P_+P_-)$ corresponds to a longitude of $K(r)$ according to whether $K(r)$ has one or two components.
 - (iii) The images of the triangulation $\mathcal{L}(\rho_r, \Sigma_0(r))$ of \tilde{A} by the infinite dihedral group $\langle \rho_r(P_-), \rho_r(P_+) \rangle$ form a $\pi_1(\partial\mathbf{O}(r))$ -invariant triangulation of \mathbb{C} , which projects to the triangulation of the cusp of $\mathbf{O}(r)$ induced by $\mathcal{D}(r)$.

Proof (1) By Lemma 4.5, we have $\phi_r(\infty) = \phi_r(r) = 0$. Let s be a vertex of $\Sigma_0(r)$. Since a simple arc of slope s in \mathcal{S} is realized as a geodesic edge in the geometric triangulation $\mathcal{D}(r)$ of the hyperbolic manifold $S^3 - K(r)$, it follows that if P is an elliptic generator of slope s then $\rho_r(P)(\infty) \neq \infty$. By Lemma 4.1(2), this implies that $\phi(s) \neq 0$. Hence we have $\phi_r^{-1}(0) \cap \Sigma(r)^{(0)} = \{\infty, r\}$.

(2) This follows from the fact that $\phi_r(\infty) = \phi_r(r) = 0$ and Lemma 4.1(2-iv) (cf Remark 4.3).

(3) By the preceding description of the combinatorial structure of the cusp triangulation and the fact that $\mathcal{D}(r)$ is geometric, we see that $\mathcal{L}(\rho_r, \Sigma_0(r))$ forms a 1-skeleton of a

triangulation of the strip, \tilde{A} , in \mathbb{C} bounded by L_- and L_+ . Since $\phi_r(\infty) = \phi_r(r) = 0$, we see by Lemma 4.1(2-iii) that $\rho_r(P_-)$ ($\rho_r(P_+)$, respectively) acts on \mathbb{C} as the π -rotation about the center of an edge of $\mathcal{L}(\rho_r, \sigma_2)$ ($\mathcal{L}(\rho_r, \sigma_{c-1})$, respectively) contained in L_- (L_+ , respectively). It is obvious that \tilde{A} forms a fundamental domain of the infinite dihedral group $\langle \rho_r(P_-), \rho_r(P_+) \rangle$, and so we have (i). Since \tilde{A} projects to one of the four peripheral annuli of $S \times [-1, 1]$ and since the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action on $S^3 - K(r)$ lifts to a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action on $S \times [-1, 1]$ which acts transitively on the set of the four peripheral annuli, we see that \tilde{A} is a fundamental domain of the action of $\pi_1(\partial \mathbf{O}(r))$ modulo the action of the meridian $\rho_r(D)$. Since \tilde{A} is a fundamental domain of the infinite dihedral group $\langle \rho_r(P_-), \rho_r(P_+) \rangle$, we have $\pi_1(\partial \mathbf{O}(r)) = \langle \rho_r(D), \rho_r(P_-), \rho_r(P_+) \rangle$. Thus we obtain the first assertion of (ii). The remaining assertion of (ii) follows from the description of the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ -action on $K(r)$ given at the end of Section 2. Assertion (iii) follows from (i), (ii) and the description of the combinatorial structure of the cusp triangulation, together with the fact that $\mathcal{D}(r)$ is geometric. □

6 Proof of Theorems 2.2 and 2.3

Throughout this section and in the remainder of this paper, $K(r)$ denotes a hyperbolic link, $\rho_r: \pi_1(\mathbf{O}) \rightarrow \text{PSL}(2, \mathbb{C})$ denotes the type-preserving representation induced by the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$, ϕ_r denotes a Markoff map determined by a lift $\tilde{\rho}_r: \pi_1(\mathbf{T}) \rightarrow \text{SL}(2, \mathbb{C})$ of the restriction of ρ_r to $\pi_1(\mathbf{T})$, and ψ_r denotes the complex probability map determined by ρ_r .

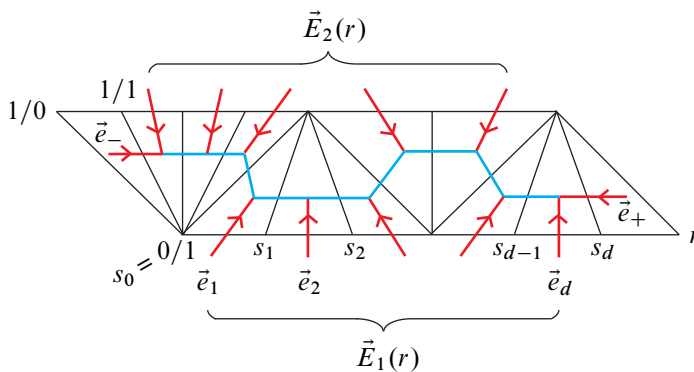


Figure 12: Dual oriented edges

Let $\mathcal{T}_0(r)$ be the subtree of \mathcal{T} dual to the chain $\Sigma_0(r)$, and let $\vec{E}(r)$ be the set of the oriented edges of $\mathcal{T} - \mathcal{T}_0(r)$ whose head is contained in $\mathcal{T}_0(r)$. For each interval $I_j(r)$

($j = 1, 2$), we consider the following set of oriented edges:

$$\vec{E}_j(r) = \{\vec{e} \in \vec{E}(r) \mid \Omega^{0^-}(\vec{e}) \subset I_j(r)\}.$$

It should be noted that $\vec{E}(r) = \vec{E}_1(r) \sqcup \vec{E}_2(r) \sqcup \{\vec{e}_-, \vec{e}_+\}$, where \vec{e}_- and \vec{e}_+ are the elements of $\vec{E}(r)$ with tails dual to σ_1 and σ_c , respectively (see Figure 12).

Proposition 6.1 (1) *The following identity holds:*

$$\sum_{\vec{e} \in \vec{E}_1(r)} \psi_r(\vec{e}) + \sum_{\vec{e} \in \vec{E}_2(r)} \psi_r(\vec{e}) = -1.$$

(2) *The cusp shape $\lambda(\mathbf{O}(r))$ with respect to a suitable choice of a longitude is given by the following formula:*

$$\frac{1}{2}\lambda(\mathbf{O}(r)) = \sum_{\vec{e} \in \vec{E}_1(r)} \psi_r(\vec{e}) = -1 - \sum_{\vec{e} \in \vec{E}_2(r)} \psi_r(\vec{e}).$$

Proof (1) By Proposition 5.1(1), $\psi_r(\vec{e})$ is defined for all $\vec{e} \in \vec{E}(r)$. Hence, we have the following identity by [5, Lemma 1]:

$$\sum_{\vec{e} \in \vec{E}(r)} \psi_r(\vec{e}) = 1.$$

On the other hand, since $\phi_r(\infty) = \phi_r(r) = 0$, we have $\psi_r(\vec{e}_-) = \psi_r(\vec{e}_+) = 1$. Hence, by using the fact that $\vec{E}(r) = \vec{E}_1(r) \sqcup \vec{E}_2(r) \sqcup \{\vec{e}_-, \vec{e}_+\}$, we obtain the desired identity.

(2) Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d$ be the members of $\vec{E}_1(r)$ whose heads lie in $\mathcal{T}_0(r)$ in this order, and let s_0, s_1, \dots, s_d be the vertices of $\Sigma_0(r)$ such that \vec{e}_i is dual to the Farey edge $\langle s_{i-1}, s_i \rangle$. Let P_0, P_1, \dots, P_d be elliptic generators satisfying the following conditions:

- (i) The slope of P_i is s_i for each $i \in \{0, 1, \dots, d\}$.
- (ii) For each $i \in \{1, 2, \dots, d\}$, the two elliptic generators P_{i-1} and P_i appear successively in the sequence of elliptic generators associated with the Farey triangle in $\Sigma_0(r)$ which contains $\langle s_{i-1}, s_i \rangle$.

Then the zigzag line segment in \mathbb{C} obtained by joining the points

$$c(\rho_r(P_0)), c(\rho_r(P_1)), \dots, c(\rho_r(P_d))$$

is an edge path in the triangulation of the strip \tilde{A} in \mathbb{C} bounded by the horizontal lines L_- and L_+ in Proposition 5.1(3) (see Figure 10). Moreover, since the

slopes $s_0 = s(P_0)$ and $s_d = s(P_d)$ are vertices of the Farey triangles σ_2 and σ_{c-2} , respectively, we see by Proposition 5.1(2) that the initial point $c(\rho_r(P_0))$ and the terminal point $c(\rho_r(P_d))$ lie in the horizontal lines L_- and L_+ , respectively. Hence the zigzag line segment projects to a longitude of the orbifold $\mathbf{O}(r)$. In particular, $c(\rho_r(P_d)) - c(\rho_r(P_0))$ is equal to the complex number $\frac{1}{4}|K(r)|\lambda$ introduced at the end of Section 2. Hence the modulus of the cusp of $\mathbf{O}(r)$ with respect to the longitude is given by the following formula:

$$\begin{aligned} \frac{1}{2}\lambda(\mathbf{O}(r)) &= \frac{|K(r)|}{4}\lambda = c(\rho_r(P_d)) - c(\rho_r(P_0)) \\ &= \sum_{i=1}^d (c(\rho_r(P_i))) - c(\rho_r(P_{i-1})) = \sum_{i=1}^d \psi(e_i). \end{aligned}$$

Here, the last equality follows from formula (4-2) in Section 4. Thus we have proved the first identity in (2). The second identity follows from (1). □

By the above proposition, the proof of Theorems 2.2 and 2.3 is reduced to the following key lemma.

Key Lemma 6.2 *Every member \vec{e} of $\vec{E}_1(r) \cup \vec{E}_2(r)$ satisfies the conditions of Proposition 4.4, namely, it satisfies the following conditions:*

- (1) *The set $\{s \in \Omega^-(\vec{e}) \mid |\phi_r(s)| \leq 2\}$ is finite.*
- (2) *$\Omega^{0-}(\vec{e}) \cap \phi_r^{-1}(-2, 2) = \emptyset$.*

Proof of Theorems 2.2 and 2.3 assuming Key Lemma 6.2 By Proposition 4.4 and Key Lemma 6.2, we have the following identity for each $j = 1, 2$:

$$\begin{aligned} \sum_{\vec{e} \in \vec{E}_j(r)} \psi_r(\vec{e}) &= \sum_{\vec{e} \in \vec{E}_j(r)} \left\{ \sum_{s \in \Omega^0(e)} h(\phi_r(s)) + 2 \sum_{s \in \Omega^-(\vec{e})} h(\phi(s)) \right\} \\ &= 2 \sum_{s \in \widehat{\mathbb{Q}} \cap \text{int}I_j(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}} + \sum_{s \in \partial I_j(r)} \frac{1}{1 + e^{l_{\rho_r}(\beta_s)}}. \end{aligned}$$

By applying this identity to the identities in Proposition 6.1, we obtain the desired results. □

Key Lemma 6.2 is proved by using the results obtained in the series of our papers [13; 14; 15; 16] (see also the announcement [12]), which gives a complete answer to the following questions concerning the simple loops in 2-bridge sphere \mathbf{S} of a 2-bridge link $K(r)$:

- (1) Which simple loop on S is null-homotopic or peripheral on $S^3 - K(r)$?
- (2) For given two simple loops on S , when are they homotopic?

In particular, we have the following theorem.

Theorem 6.3 For a hyperbolic 2-bridge link $K(r)$, the following hold:

- (1) For any rational number s in $I_1(r) \cup I_2(r)$, α_s is not null-homotopic in $S^3 - K(r)$.
- (2) There are at most two rational numbers s in $I_1(r) \cup I_2(r)$ such that α_s is peripheral.
- (3) Except for at most two pairs of rational numbers in $I_1(r) \cup I_2(r)$, the simple loops $\{\alpha_s \mid s \in I_1(r) \cup I_2(r)\}$ are not mutually homotopic in $S^3 - K(r)$.

Proof of Key Lemma 6.2 Since $\Omega^{0-}(\vec{e}) \subset I_1(r) \cup I_2(r)$ for any $\vec{e} \in \vec{E}_1(r) \cup \vec{E}_2(r)$, the lemma is reduced to the following assertions:

- (i) The set $\{s \in I_1(r) \cup I_2(r) \mid |\phi_r(s)| \leq 2\}$ is finite.
- (ii) $(I_1(r) \cup I_2(r)) \cap \phi_r^{-1}(-2, 2) = \emptyset$.

We first prove (ii). Let s be a rational number contained in $I_1(r) \cup I_2(r)$. Then, by Theorem 6.3(1), α_s determines a nontrivial element of $\pi_1(S^3 - K(r))$. Since ρ_r is induced by the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$, we see that $\rho_r(\alpha_s) = \rho_r(\beta_s^2)$ is neither trivial nor elliptic. Thus $\rho_r(\beta_s)$ is not elliptic, and so $\phi_r(s) = \text{tr}(\tilde{\rho}_r(\beta_s))$ is not contained in $(-2, 2)$. Hence we obtain (ii).

Next we prove (i). Suppose on the contrary that the set $\{s \in I_1(r) \cup I_2(r) \mid |\phi_r(s)| \leq 2\}$ contains infinitely many elements $\{s_j\}_{j \in \mathbb{Z}}$. By Theorem 6.3(1) and (2), we may assume that $\rho(\alpha_{s_j})$ is neither trivial nor parabolic, and hence, the simple loop α_{s_j} is homotopic to a closed geodesic in the hyperbolic manifold $S^3 - K(r)$. By Theorem 6.3(3), we may also assume that $\{\alpha_{s_j}\}$ are not mutually homotopic in $S^3 - K(r)$ and so the corresponding closed geodesics are mutually distinct. On the other hand, the condition $|\phi(s_j)| \leq 2$ implies that the real length $L(\rho(\alpha_{s_j})) = 2L(\rho(\beta_{s_j}))$ is bounded from above. This contradicts the discreteness of marked length spectrum of geometrically finite hyperbolic 3-manifolds (see Lemma 6.4 below). Hence we obtain (i). This completes the proof of Key Lemma 6.2. □

Since we could not find a proof of Lemma 6.4 below in a literature, we give a proof, for completeness, imitating the argument in [1, Proof of Theorem 1 on page 73], and we refer to [17] for terminologies for Kleinian groups.

Lemma 6.4 *Let M be a geometrically finite complete hyperbolic 3–manifold. Then the marked length spectrum of M is discrete, namely, for any positive real number L , there are only finitely many closed geodesics in M with length at most L .*

Proof Suppose on the contrary that there is an infinite sequence $\{\gamma_j\}$ of mutually distinct closed geodesics in a geometrically finite complete hyperbolic 3–manifold $M = \mathbb{H}^3/\pi_1(M)$ such that the lengths $L(\gamma_j)$ are bounded above by a positive constant L . Pick a base point b in the convex core, $C(M)$, of M and pick a lift \tilde{b} of b in the universal cover \mathbb{H}^3 . Let b_j a point in γ_j such that $d(b, b_j) = d(b, \gamma_j)$, where d denotes the hyperbolic distance. Let $\tilde{b}_j \in \mathbb{H}^3$ be the lift of b_j such that $d(\tilde{b}, \tilde{b}_j) = d(b, b_j)$, and let $\tilde{\gamma}_j$ be the geodesic in \mathbb{H}^3 passing through \tilde{b}_j which projects to γ_j . We abuse notation to continue to denote by γ_j the element of $\pi_1(M)$ ($\subset \text{Isom}(\mathbb{H}^3)$) represented by the closed geodesic γ_j whose axis is $\tilde{\gamma}_j$. Then

$$d(\tilde{b}, \gamma_j(\tilde{b})) \leq 2d(b, b_j) + L(\gamma_j) \leq 2d(b, b_j) + L.$$

So we have

$$d(b, b_j) \geq \frac{1}{2}(d(\tilde{b}, \gamma_j(\tilde{b})) - L).$$

Since $\pi_1(M)$ acts discontinuously on \mathbb{H}^3 , we see $d(\tilde{b}, \gamma_j(\tilde{b})) \rightarrow \infty$, and hence the sequence $\{b_j\}$ in $C(M)$ diverges. Since M is geometrically finite, the convex core $C(M)$ is a union of a compact submanifold and a finite union of cusp neighborhoods. So, we may assume, after taking a subsequence, that $\{b_j\}$ converges to a cusp of C . Let U be a neighborhood of the cusp in M , obtained as the image of a horoball H . Then the stabilizer of H is a parabolic subgroup, P_U , of $\pi_1(M)$ and H is precisely invariant by $(\pi_1(M), P_U)$. Since $\{b_j\}$ converges to the cusp, we can find, for a sufficiently large j , a lift $\tilde{b}'_j \in H$ of b_j such that $d(\tilde{b}'_j, \partial H) > L$. Let γ'_j be the element of $\pi_1(M)$ represented by the closed geodesic γ_j whose axis passes through \tilde{b}'_j . Then $d(\tilde{b}'_j, \gamma'_j(\tilde{b}'_j)) = l(\gamma_j) \leq L$, and therefore the point $\gamma'_j(\tilde{b}'_j)$ is also contained in H . Since $\gamma_j \in \pi_1(M)$ is not parabolic (and therefore it does not belong to P_U), this contradicts the assumption that H is precisely invariant by $(\pi_1(M), P_U)$. \square

7 Homological description of the longitude in Theorem 2.2

In this section, we give an explicit homological description of the longitude of $K(r)$ in Theorem 2.2. To this end, we fix an arbitrary orientation of $K(r)$, and employ the following notation.

- (1) If $K(r)$ is a knot, then ℓ_0 and m denote the preferred longitude and the meridian of $K(r)$, namely, ℓ_0 and m are oriented essential simple loops on a peripheral

torus, satisfying the following conditions: ℓ_0 is homologous to $K(r)$ in a regular neighborhood, $N(K(r))$, of $K(r)$ and $\text{lk}(\ell_0, K(r)) = 0$; m bounds a disk in $N(K(r))$ and $\text{lk}(m, K(r)) = 1$. The symbol ℓ denotes the longitude of $K(r)$ in Theorem 2.2, which is oriented so that it is homologous to $K(r)$ in $N(K(r))$.

- (2) If $K(r)$ consists of two components, K_1 and K_2 , then $\ell_{i,0}$ and m_i denote the preferred longitude and the meridian of K_i for $i = 1, 2$. The symbol ℓ_i denotes the longitude of K_i in Theorem 2.2, which is oriented so that it is homologous to K_i in $N(K_i)$ for $i = 1, 2$. We denote by ℓ the union $\ell_1 \cup \ell_2$, and call it the longitude of $K(r)$ in Theorem 2.2.

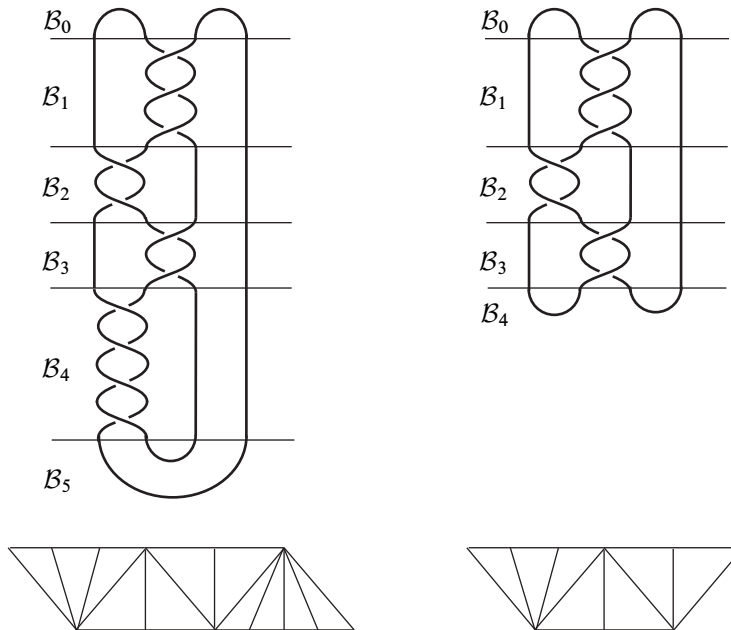


Figure 13: The decomposition $\{\mathcal{B}_k\}_{k=0}^{n+1}$ of $(S^3, K(r))$

We express ℓ in terms of ℓ_0 and m when $K(r)$ is a knot, and express ℓ_i in terms of $\ell_{i,0}$ and m_i when $K(r)$ is a two-component link. To this end, we use the alternating diagram of $K(r)$ associated with the continued fraction expansion described at the beginning of Section 5, and consider the decomposition of $(S^3, K(r))$ into $n + 2$ blocks, $\{\mathcal{B}_k\}_{k=0}^{n+1}$, as illustrated in Figure 13. The *first block* \mathcal{B}_0 and the *last block* \mathcal{B}_{n+1} are upper and lower bridges, respectively, and a *middle block* \mathcal{B}_k ($1 \leq k \leq n$) is a 4-braid, where the second and the third strings form a_k right-hand half-twists when k is odd; the first and the second strings form a_k left-hand half-twists when k

is even. For each middle block \mathcal{B}_k , we define δ_k to be 0 or 1 according to whether the orientations of the strings forming the half-twists in \mathcal{B}_k are parallel or not (see Figure 18). Then the longitude ℓ in Theorem 2.2 is given by the following proposition.

Proposition 7.1 *Under the above notation, the linking number $\text{lk}(\ell, K(r))$, for the longitude ℓ in Theorem 2.2, is given by the following formula:*

$$\text{lk}(\ell, K(r)) = \begin{cases} 2 \left(\sum_{k=1}^n \delta_k (-1)^{k-1} a_k \right) & \text{if } n \text{ is odd,} \\ 2 \left(1 + \sum_{k=1}^n \delta_k (-1)^{k-1} a_k \right) & \text{if } n \text{ is even.} \end{cases}$$

From this linking number, ℓ is determined by the following formula:

(1) If $K(r)$ is a knot, then

$$[\ell] = [\ell_0] + \text{lk}(\ell, K(r))[m] \in H_1(\partial N(K(r))).$$

(2) If $K(r)$ is a link, then for $i = 1, 2$,

$$[\ell_i] = [\ell_{i,0}] + \frac{1}{2}(\text{lk}(\ell, K(r)) - 2 \text{lk}(K_1, K_2))[m_i] \in H_1(\partial N(K_i)).$$

Remark 7.2 If every a_i is even and n is even, then $K(r)$ is a knot and the longitude ℓ is given by the following simple formula:

$$[\ell] = [\ell_0] + 2[m].$$

Our task is to draw the longitude ℓ on the boundary of the link exterior $E(K(r)) := S^3 - \text{int } N(K(r))$. To this end, let $\{\check{\mathcal{B}}_k\}_{k=0}^{n+1}$ be the decomposition of the knot exterior $E(K(r))$ induced by the decomposition $\{\mathcal{B}_k\}_{k=0}^{n+1}$ of $(S^3, K(r))$. We denote by $\partial_p \check{\mathcal{B}}_k$ the intersection of $\partial E(K(r))$ with $\partial \check{\mathcal{B}}_k$. Then $\partial_p \check{\mathcal{B}}_k$ is a disjoint union of four or two annuli according to whether $k \in \{1, 2, \dots, n\}$ or $k \in \{0, n+1\}$.

Set $c_0 = 0$, $c_k = \sum_{i=1}^k a_i$ and $\sigma_i^{(k)} = \sigma_{c_{k-1}+i}$ ($1 \leq i \leq a_k$). Then the k^{th} middle block \mathcal{B}_k ($1 \leq k \leq n$) corresponds to the subchain

$$\Sigma^{(k)}(r) := (\sigma_1^{(k)}, \sigma_2^{(k)}, \dots, \sigma_{a_k}^{(k)})$$

of $\Sigma(r)$. The subchain $\Sigma^{(k)}(r)$ contains a unique *pivot*, $s_*^{(k)}$, ie a vertex of $\Sigma(r)$ which is shared by at least three triangles in $\Sigma(r)$. We denote the remaining vertices of $\Sigma^{(k)}(r)$ by $s_0^{(k)}, s_1^{(k)}, \dots, s_{a_k}^{(k)}$, where they sit in $\Sigma^{(k)}(r)$ in this order. The Farey triangle $\sigma_i^{(k)}$ is spanned by the vertices $\{s_*^{(k)}, s_{i-1}^{(k)}, s_i^{(k)}\}$. By the description of $\mathcal{D}(r)$ as a quotient of $\hat{\mathcal{D}}(r)$ in Section 5, we may regard $\hat{\mathcal{D}}(r)$ as a decomposition of the middle part $\bigcup_{i=1}^k \check{\mathcal{B}}_k$ into truncated ideal tetrahedra: the collapsing of the edges

of slopes ∞ and r has the effect of adding the first and the last blocks $\check{\mathcal{B}}_0$ and $\check{\mathcal{B}}_{n+1}$. To be precise, the following hold:

- (1) For each $k \in \{1, 2, \dots, n\}$, the pair of edges of the pivot slope $s_*^{(k)}$ are represented by the pair of straight horizontal arcs in the upper boundary of the k^{th} middle block $\check{\mathcal{B}}_k$ as illustrated in Figure 14. They are isotopic to the straight horizontal arcs in the lower boundary of $\check{\mathcal{B}}_k$ as illustrated in Figure 14. An isotopy between these two representatives determines four mutually disjoint arcs, ξ_k , properly embedded in $\partial_p \check{\mathcal{B}}_k$. (To be precise, ξ_k is the intersection with $\partial_p \check{\mathcal{B}}_k$ of two properly embedded disks in $\check{\mathcal{B}}_k$ realizing the isotopy.)
- (2) For each $k \in \{1, 2, \dots, n\}$ and $i \in \{0, 1, \dots, a_k\}$, the pair of edges of slope $s_i^{(k)}$ are represented by the pair of straight horizontal arcs which sit in the level sphere between the i^{th} and $(i + 1)^{\text{st}}$ crossings in \mathcal{B}_k , as illustrated in Figure 14. Each of the four corners of the ideal triangles in $\text{trg}(\sigma_i^{(k)})$ bounded by an edge of slope $s_{i-1}^{(k)}$ and an edge of slope $s_i^{(k)}$ determines an arc in $\partial_p \check{\mathcal{B}}_k$ joining an end point of an edge of slope $s_{i-1}^{(k)}$ and an end point of an edge of slope $s_i^{(k)}$. The union of these arcs, where $\sigma_i^{(k)}$ runs over the Farey triangles of $\Sigma^{(k)}(r)$, forms four properly embedded arcs in $\partial_p \check{\mathcal{B}}_k$, which is disjoint from ξ_k . We denote by η_k the union of these four arcs in $\partial_p \check{\mathcal{B}}_k$.
- (3) In the top block $\check{\mathcal{B}}_0$, each of the two edges of slope ∞ in $\check{\mathcal{B}}_0 \cap \check{\mathcal{B}}_1$ shrinks to a point, and the two edges of slope 0 in $\check{\mathcal{B}}_0 \cap \check{\mathcal{B}}_1$ are isotopic to the upper tunnel. Let ξ_0 be the pair of arcs properly embedded in $\partial_p \check{\mathcal{B}}_0$ determined by the isotopy (see Figure 15).
- (4) In the bottom block $\check{\mathcal{B}}_{n+1}$, each of the two edges of slope r in $\check{\mathcal{B}}_n \cap \check{\mathcal{B}}_{n+1}$ shrinks to a point, and the two edges of slope $[a_1, a_2, \dots, a_{n-1}]$ in $\check{\mathcal{B}}_n \cap \check{\mathcal{B}}_{n+1}$ are isotopic to the lower tunnel. Let ξ_{n+1} be the pair of arcs properly embedded in $\partial_p \check{\mathcal{B}}_k$ determined by the isotopy.

To be more precise, the (actual) cusp torus is identified with a quotient of the “model” torus $\bigcup_{k=0}^{n+1} \partial_p \check{\mathcal{B}}_k$, where each arc component of ξ_k ($k = 0, \dots, n + 1$) shrinks to a point.

Lemma 7.3 *Suppose that n is an odd integer $2n_0 + 1$. Then under the above identification of the cusp torus with a quotient of the model torus $\bigcup_{k=0}^{n+1} \partial_p \check{\mathcal{B}}_k$, the longitude ℓ is equal to (the image of) the union of the following arcs (see Figure 16):*

$$\xi_0 \cup \xi_1, \eta_2, \xi_3, \eta_4, \dots, \xi_{2n_0-1}, \eta_{2n_0}, \xi_{2n_0+1} \cup \xi_{2n_0+2}.$$

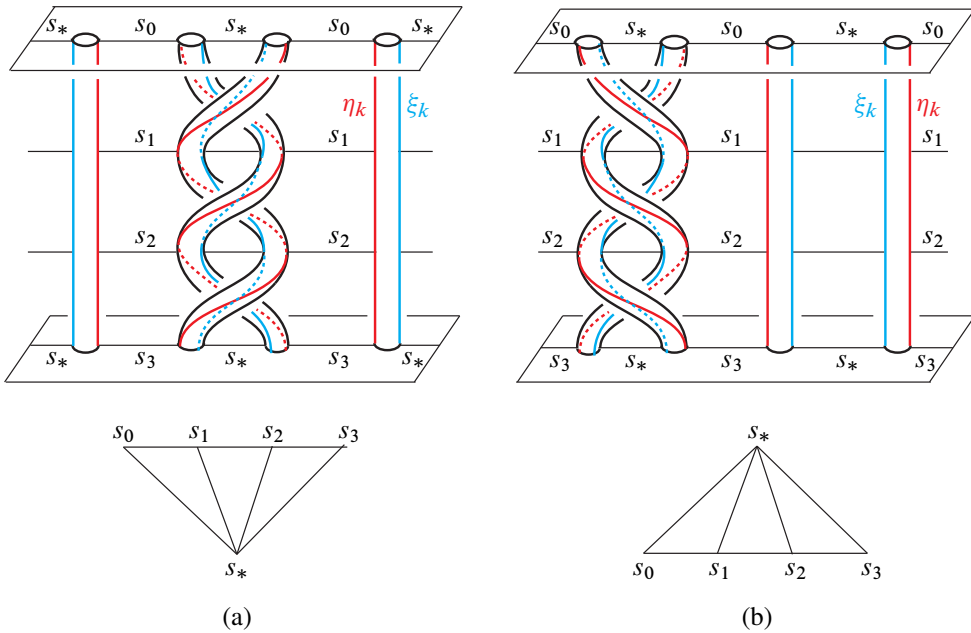


Figure 14: The middle block \check{B}_k (a) when k is odd, and (b) when k is even. The labels s_* and s_j for the straight horizontal arcs are the slopes of the edges, where s_* and s_j are abbreviations for $s_*^{(k)}$ and $s_j^{(k)}$, respectively. The set ξ_k consists of four vertical arcs joining the endpoints of the edges of slope $s_*^{(k)}$ in the upper boundary of \check{B}_k and the endpoints of the edges of slope $s_*^{(k)}$ in the lower boundary of \check{B}_k . The set η_k consists of four vertical arcs joining the endpoints of the edges of slope $s_0^{(k)}$ in the upper boundary of \check{B}_k and the endpoints of the edges of slope $s_{a_k}^{(k)}$ in the lower boundary of \check{B}_k , passing through the endpoints of the edges of slope $s_i^{(k)}$ ($0 < i < a_k$).

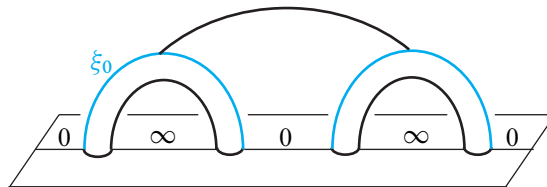


Figure 15: The top block \check{B}_0 . The set ξ_0 is the union of the two arcs on $\partial_p \check{B}_0$ joining the endpoints of the edges of slope 0 in the lower boundary $\check{B}_0 \cap \check{B}_1$ of \check{B}_0 .

Proof Let s_0, s_1, \dots, s_d be the slopes in the proof of Proposition 6.1. Recall that the longitude of $\partial O(r)$ is obtained as the image of the zigzag line in \mathbb{C} spanned by

$c(\rho_r(P_0)), c(\rho_r(P_1)), \dots, c(\rho_r(P_d))$, where P_i is an elliptic generator with $s(P_i) = s_i$ as in the proof of Proposition 6.1. We note that the following hold:

- (1) The set $\{s_0, s_1, \dots, s_d\}$ consists of the vertices of a subchain $\Sigma^{(2k)}(r)$ different from the pivot $s_*^{(2k)}$, where $2k$ runs over the set $\{2, 4, \dots, 2n_0\}$.
- (2) The first slope s_0 is equal the pivot $s_*^{(1)}$ of the subchain $\Sigma^{(1)}(r)$.
- (3) The last vertex s_d is equal the pivot $s_*^{(n)}$ of the subchain $\Sigma^{(n)}(r)$.

In particular, the first $(a_2 + 1)$ vertices s_0, s_1, \dots, s_{a_2} are equal to the vertices $s_0^{(2)}, s_1^{(2)}, \dots, s_{a_2}^{(2)}$ of the subchain $\Sigma^{(2)}(r)$ different from the pivot $s_*^{(2)}$. Thus the subzigzag line in \mathbb{C} spanned by

$$c(\rho_r(P_0)), c(\rho_r(P_1)), \dots, c(\rho_r(P_{a_1}))$$

corresponds to (a component of) $\eta_2 \subset \partial_p \check{\mathcal{B}}_2$. Similarly, for each even integer $2k \in \{2, 4, \dots, 2n_0\}$, the subzigzag line in \mathbb{C} spanned by the $c(\rho_r(P_i))$, where $s(P_i)$ runs over the vertices of the subchain $\Sigma^{(2k)}(r)$ different from the pivot $s_*^{(2k)}$, corresponds to (a component of) $\eta_{2k} \subset \partial_p \check{\mathcal{B}}_{2k}$.

For each $2k \in \{2, 4, \dots, 2n_0 - 2\}$, the final slope $s_{a_{2k}}^{(2k)}$ of the subchain $\Sigma^{(2k)}(r)$ is equal to the first slope $s_0^{(2k+2)}$ of the subchain $\Sigma^{(2k+2)}(r)$, and they are equal to the pivot $s_*^{(2k+1)}$ of the subchain $\Sigma^{(2k+1)}(r)$. Thus each edge of slope $s_{a_{2k}}^{(2k)}$, which lies in the lower boundary of $\partial_p \check{\mathcal{B}}_{2k}$, is isotopic in $\check{\mathcal{B}}_{2k+1}$ to an edge of slope $s_0^{(2k+2)}$, which lies in the upper boundary of the block $\check{\mathcal{B}}_{2k+2}$. So, in order to describe the longitude in the model torus $\bigcup_{k=0}^{n+1} \partial_p \check{\mathcal{B}}_k$, the end points of η_{2k} in the lower boundary of $\partial_p \check{\mathcal{B}}_{2k}$ should be joined with the end points of η_{2k+2} in the upper boundary of $\partial_p \check{\mathcal{B}}_{2k+2}$ by the trace, $\xi_{2k+1} \subset \partial_p \check{\mathcal{B}}_{2k+1}$, of the isotopy (see Figure 16).

The first slope $s_0^{(2)}$ of the subchain $\Sigma^{(2)}(r)$ is equal to the pivot $s_*^{(1)}$ of the subchain $\Sigma^{(1)}(r)$. Thus the edges of slope $s_0^{(2)}$, which lie in the upper boundary of the block $\check{\mathcal{B}}_2$, are equal to the edges of slope $s_*^{(1)}$ which lie in the lower boundary of the block $\check{\mathcal{B}}_1$. The latter edges are isotopic in $\check{\mathcal{B}}_1$ to the edges of slope $s_*^{(1)}$ which lie in the upper boundary of $\check{\mathcal{B}}_1$. These in turn are isotopic in $\check{\mathcal{B}}_0$ to the upper bridges. So, the endpoints of η_2 in the upper boundary are joined each other by the trace, $\xi_0 \cup \xi_1 \subset \partial_p \check{\mathcal{B}}_0 \cup \partial_p \check{\mathcal{B}}_1$, of the isotopy (see Figure 16). Similarly, the end points of η_{2n_0} in the lower boundary of $\partial_p \check{\mathcal{B}}_{2n_0}$ should be joined by the trace,

$$\xi_{2n_0+1} \cup \xi_{2n_0+2} \subset \partial_p \check{\mathcal{B}}_{2n_0+1} \cup \partial_p \check{\mathcal{B}}_{2n_0+2},$$

of the isotopy. Hence, the longitude ℓ is as described in the lemma. □

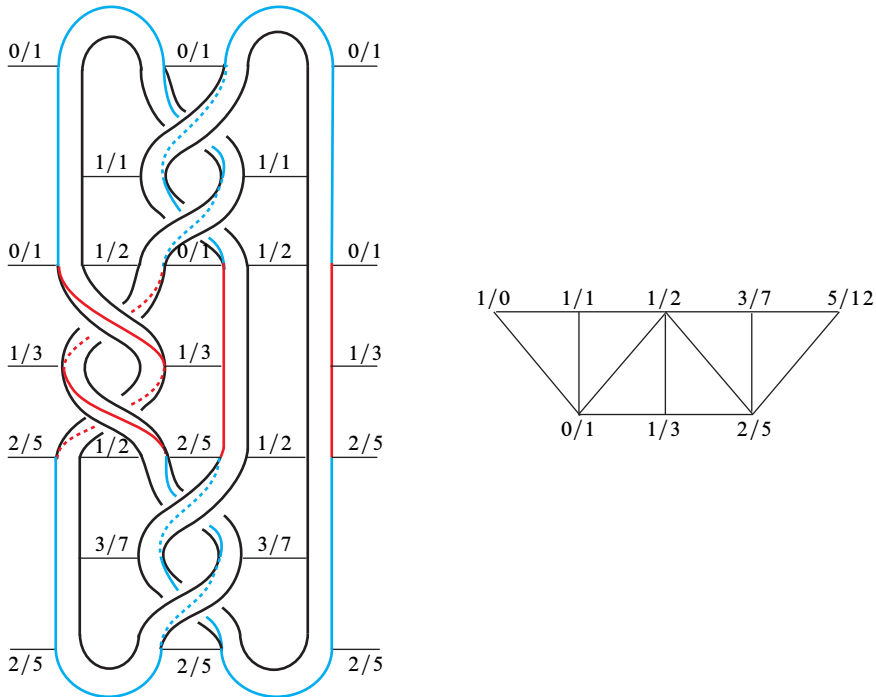


Figure 16: The whole picture of the longitude ℓ when $n = 3$. In this case, ℓ is the union of $\xi_0 \cup \xi_1$, η_2 and $\xi_4 \cup \xi_4$.

In order to treat the case when n is an even integer $2n_0$, we need to introduce the following notation (see Figure 17):

- (1) By the symbol η'_n , we denote the subarcs of η_n bounded by the end points of η_n in the upper boundary of $\partial_p \check{\mathcal{B}}_n$ and the endpoints of the edges of slope $r' := [a_1, a_2, \dots, a_{n-1}, a_n - 1]$.
- (2) The edges of slope r' in $\check{\mathcal{B}}_n$ is isotopic in $\check{\mathcal{B}}_n \cup \check{\mathcal{B}}_{n+1}$ to the lower tunnel. The symbol ξ'_{n+1} denotes the arcs in $\partial_p \check{\mathcal{B}}_n \cup \partial_p \check{\mathcal{B}}_{n+1}$ obtained from the isotopy.

By the proof of Lemma 7.3, we obtain the following lemma.

Lemma 7.4 *Suppose that n is an even integer $2n_0$. Then under the identification of the cusp torus with a quotient of the model torus $\bigcup_{k=0}^{n+1} \partial_p \check{\mathcal{B}}_k$, the longitude ℓ is equal to (the image of) the union of the arcs*

$$\xi_0 \cup \xi_1, \eta_2, \xi_3, \eta_4, \dots, \xi_{2n_0-1}, \eta'_{2n_0}, \xi'_{2n_0+1}.$$

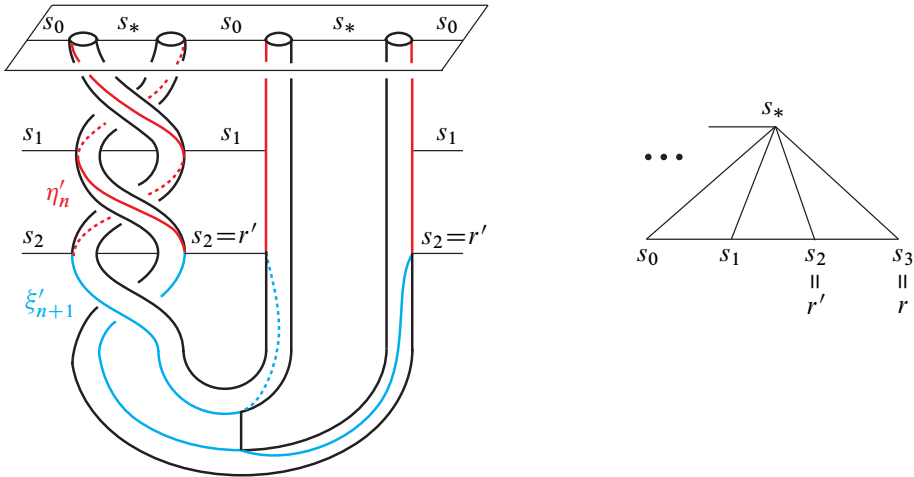


Figure 17: $\check{B}_n \cup \check{B}_{n+1}$ when n is even

Proof of Proposition 7.1 We evaluate the linking number $\text{lk}(\ell, K(r))$ by counting the number of crossings with sign where ℓ runs below $K(r)$. Then the formula for the linking number follows from the following observations:

- (i) If $2 \leq k \leq n - 1$, then we can easily check that the contribution of the crossings in \mathcal{B}_k is equal to $2\delta_k(-1)^{k-1}a_k$ (see Figure 18).
- (ii) The contribution of the crossings in $\mathcal{B}_0 \cup \mathcal{B}_1$ is equal to $2\delta_k(-1)^{k-1}a_k$ with $k = 1$.
- (iii) The contribution of the crossings in $\mathcal{B}_n \cup \mathcal{B}_{n+1}$ is equal to $2\delta_n(-1)^{n-1}a_n$ or $2(\delta_n(-1)^{n-1}a_n + 1)$ according to whether n is odd or even.

The formula for the longitude is obvious if $K(r)$ is a knot. Suppose $K(r) = K_1 \cup K_2$. Then by using the symmetry, we have the following identity for $\{i, j\} = \{1, 2\}$:

$$\begin{aligned} \text{lk}(\ell, K(r)) &= \text{lk}(\ell_1 \cup \ell_2, K_1 \cup K_2) = 2 \text{lk}(\ell_i, K_i) + 2 \text{lk}(\ell_i, K_j) \\ &= 2 \text{lk}(\ell_i, K_i) + 2 \text{lk}(K_1, K_2). \end{aligned}$$

Hence we obtain the desired formula for ℓ_j . □

8 Application to end invariants of $\text{SL}(2, \mathbb{C})$ -characters

Bowditch [7] introduced the notion of the end invariants of a type-preserving $\text{SL}(2, \mathbb{C})$ -representation of $\pi_1(\mathcal{T})$ by extending the concept of a geometrically infinite end of

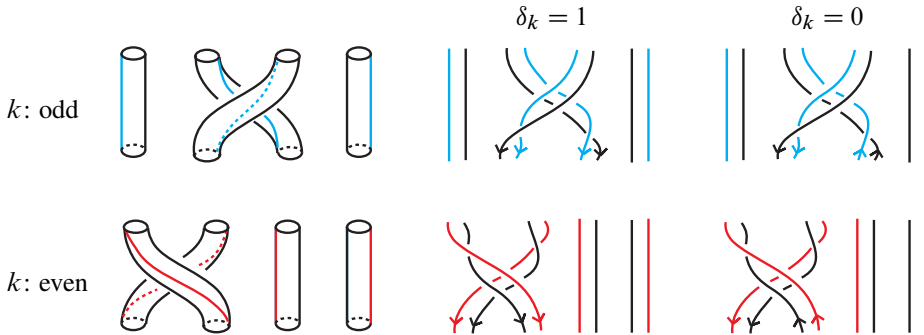


Figure 18: If $\delta_k = 1$, then the contribution of each of the crossings in \mathcal{B}_k to the linking number is equal to 2 or -2 according as k is odd or even. If $\delta_k = 0$, then there is no contribution.

a Kleinian group. Tan, Wong and Zhang [28; 31] extended this notion (with slight modification) to an arbitrary $SL(2, \mathbb{C})$ -representation of $\pi_1(\mathbf{T})$.

To recall the definition of end invariants, let \mathcal{C} be the set of free homotopy classes of essential simple loops on \mathbf{T} . Then as described in Section 2, \mathcal{C} is identified with $\hat{\mathbb{Q}}$, the vertex set of the Farey tessellation \mathcal{D} by the rule $s \mapsto \beta_s$. The projective lamination space \mathcal{PL} of \mathbf{T} is then identified with $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and contains \mathcal{C} as the dense subset of rational points.

Definition 8.1 Let ρ be an $SL(2, \mathbb{C})$ -representation of $\pi_1(\mathbf{T})$.

(1) An element $X \in \mathcal{PL}$ is an *end invariant* of ρ if there exists a sequence of distinct elements $X_n \in \mathcal{C}$ such that

- (i) $X_n \rightarrow X$, and
- (ii) $\{|\text{tr}\rho(X_n)|\}_n$ is bounded from above.

(2) $\mathcal{E}(\rho)$ denotes the set of end invariants of ρ .

Note that $\mathcal{E}(\rho)$ is actually determined by the $PSL(2, \mathbb{C})$ -representation induced by ρ , because $|\text{tr}\rho(X_n)|$ is determined by the $PSL(2, \mathbb{C})$ -representation. Note also that the condition that $\{|\text{tr}\rho(X_n)|\}_n$ is bounded from above is equivalent to the condition that the (real) hyperbolic translation lengths of the isometries $\rho(X_n)$ of \mathbb{H}^3 are bounded from above.

Tan, Wong and Zhang [28; 31] showed that $\mathcal{E}(\rho)$ is a closed subset of \mathcal{PL} and proved various interesting properties of $\mathcal{E}(\rho)$, including a characterization of those representations ρ with $\mathcal{E}(\rho) = \emptyset$ or \mathcal{PL} , generalizing corresponding results of Bowditch [7]

for type-preserving representations. They also proposed an interesting conjecture [31, Conjecture 1.8] concerning possible homeomorphism types of $\mathcal{E}(\rho)$. The following is a modified version of the conjecture which Tan informed the authors of.

Conjecture 8.2 *Suppose $\mathcal{E}(\rho)$ has at least two accumulation points. Then $\mathcal{E}(\rho)$ is either a Cantor set of $\mathcal{P}\mathcal{L}$ or all of $\mathcal{P}\mathcal{L}$.*

They constructed a family of representations ρ which have Cantor sets as $\mathcal{E}(\rho)$, and proved the following supporting evidence of the conjecture (see [31, Theorem 1.7]).

Theorem 8.3 *Let the representation $\rho: \pi_1(\mathbf{T}) \rightarrow \mathrm{SL}(2, \mathbb{C})$ be discrete in the sense that the set $\{\mathrm{tr}(\rho(X)) \mid X \in \mathcal{C}\}$ is discrete in \mathbb{C} . Then if $\mathcal{E}(\rho)$ has at least three elements, then $\mathcal{E}(\rho)$ is either a Cantor set of $\mathcal{P}\mathcal{L}$ or all of $\mathcal{P}\mathcal{L}$.*

The above theorem together with Lemma 6.4 implies that $\mathcal{E}(\rho_r)$ of the representation ρ_r induced by the holonomy representation of a hyperbolic 2–bridge link $K(r)$ is a Cantor set. But it does not give us the exact description of $\mathcal{E}(\rho_r)$. By using the proof of the main results in Section 6, we can explicitly determine the end invariants $\mathcal{E}(\rho_r)$. To state the theorem, recall that the *limit set* $\Lambda(\widehat{\Gamma}_r)$ of the group $\widehat{\Gamma}_r$ is the set of accumulation points in the closure of \mathbb{H}^2 of the $\widehat{\Gamma}_r$ –orbit of a point in \mathbb{H}^2 .

Theorem 8.4 *For a hyperbolic 2–bridge link $K(r)$, the set of end invariants $\mathcal{E}(\rho_r)$ of the holonomy representation ρ_r is equal to the limit set $\Lambda(\widehat{\Gamma}_r)$ of the group $\widehat{\Gamma}_r$.*

Proof Since $\phi_r(\infty) = \phi_r(r) = 0$, we see that both ∞ and r belong to $\mathcal{E}(\rho_r)$ (cf [31, Lemma 3.5(b)]). By Theorem 2.1(1), $\mathcal{E}(\rho_r)$ is invariant by $\widehat{\Gamma}_r$, and so it is a $\widehat{\Gamma}_r$ –invariant closed set. Thus $\mathcal{E}(\rho_r)$ contains the closure of the $\mathcal{E}(\rho_r)$ –orbit of ∞ and r . Since $\Lambda(\widehat{\Gamma}_r)$ is the smallest nonempty $\widehat{\Gamma}_r$ –invariant closed set, $\mathcal{E}(\rho_r)$ must contain $\Lambda(\widehat{\Gamma}_r)$. On the other hand, Key Lemma 6.2 says that ϕ_r satisfies the *extended BQ–condition* on $I_1(r) \cup I_2(r)$ in the sense of [31, Section 4]. Hence $\mathcal{E}(\rho_r)$ is disjoint from $I_1(r) \cup I_2(r)$ by [31, Propositions 4.1 and 4.3]. Since $I_1(r) \cup I_2(r)$ is a fundamental domain of the action of $\Lambda(\widehat{\Gamma}_r)$ on the domain of discontinuity, $\widehat{\mathbb{R}} - \Lambda(\widehat{\Gamma}_r)$, and since $\mathcal{E}(\rho_r)$ is a $\widehat{\Gamma}_r$ –invariant, we see that $\mathcal{E}(\rho_r)$ is disjoint from the $\widehat{\mathbb{R}} - \Lambda(\widehat{\Gamma}_r)$, ie $\mathcal{E}(\rho_r) \subset \Lambda(\widehat{\Gamma}_r)$. Hence we have $\mathcal{E}(\rho_r) = \Lambda(\widehat{\Gamma}_r)$. □

In Bowditch’s original definition of the “set of end invariants” [7, page 729], the accidental parabolics are also regarded as an end invariant. (He denotes the set by the symbol $L(\phi)$, where ϕ is a Markoff map.) By using the classification of the essential simple loops in the 2–bridge sphere which are peripheral in hyperbolic 2–bridge links complements (see [15; 16] and [12, Theorem 2.6(1)]), we have the following theorem for Bowditch’s set of end invariants $L(\rho_r) := L(\phi_r)$.

Theorem 8.5 For a hyperbolic 2-bridge link $K(r)$ with $0 < r < 1/2$, Bowditch's set of end invariants $L(\rho_r)$ of the holonomy representation ρ_r is equal to the limit set $\Lambda(\widehat{\Gamma}_r)$ of the group $\widehat{\Gamma}_r$, except in the following cases:

- (1) If $r = 2/5$, then $L(\rho_r) = \Lambda(\widehat{\Gamma}_r) \cup \widehat{\Gamma}_r\{1/5, 3/5\}$.
- (2) If $r = n/(2n+1)$ for some $n \geq 3$, then $L(\rho_r) = \Lambda(\widehat{\Gamma}_r) \cup \widehat{\Gamma}_r\{(n+1)/(2n+1)\}$.
- (3) If $r = 2/(2n+1)$ for some $n \geq 3$, then $L(\rho_r) = \Lambda(\widehat{\Gamma}_r) \cup \widehat{\Gamma}_r\{1/(2n+1)\}$.

In the exceptional cases, $L(\rho_r)$ is the union of the Cantor set $\Lambda(\widehat{\Gamma}_r)$ and infinitely many isolated points.

At the end of this section, we would like to propose the following conjecture, which is a variation of a special case of [7, Question D] and [31, Conjecture 1.9].

Conjecture 8.6 Let $\rho: \pi_1(\mathbf{T}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a type-preserving representation such that $\mathcal{E}(\rho) = \Lambda(\widehat{\Gamma}_r)$. Then ρ is conjugate to the representation ρ_r induced by the holonomy representation of a hyperbolic 2-bridge link $K(r)$.

9 Further discussion

As noted in Section 5, in the second author's joint work with Akiyoshi, Wada and Yamashita [4], it is announced that there is a continuous family of hyperbolic cone manifolds $\{M(r; \theta^-, \theta^+)\}_{0 \leq \theta^\pm \leq \pi}$ satisfying the following conditions (see [4, Preface, in particular Figures 0.22–0.26] and the demonstration in Wada's software OPTi [35]):

- (1) The underlying space of $M(r; \theta^-, \theta^+)$ is $S^3 - K(r)$.
- (2) The cone axis of $M(r; \theta^-, \theta^+)$ consists of the core tunnel of $(B^3, t(\infty))$ and that of $(B^3, t(r))$, where the cone angles are $2\theta^-$ and $2\theta^+$, respectively. In particular, $M(r; \pi, \pi)$ is the complete hyperbolic manifold $S^3 - K(r)$.
- (3) If $0 < \theta^\pm < \pi$, then the combinatorial dual of the "Ford domain" of $M(r; \theta^-, \theta^+)$ is homeomorphic to $\widehat{\mathcal{D}}(r)$. If $\theta^- = \theta^+ = \pi$, the combinatorial dual of the Ford domain of $M(r; \pi, \pi) = S^3 - K(r)$ is homeomorphic to $\mathcal{D}(r)$, ie the canonical decomposition of $S^3 - K(r)$ is homeomorphic to $\mathcal{D}(r)$.

Thus the announcement in [4] says that the collapsing of the edges of slopes ∞ and r in $\widehat{\mathcal{D}}(r)$ is realized geometrically by a continuous family of hyperbolic cone manifolds.

Akiyoshi and the second author tried to prove the main results in this paper, by establishing the following natural generalization of Theorem 8.4.

Conjecture 9.1 Let $\rho = \rho_{(r; \theta^-, \theta^+)}: \pi_1(\mathbf{O}) \rightarrow \text{PSL}(2, \mathbb{C})$ be the type-preserving $\text{PSL}(2, \mathbb{C})$ -representation induced by the holonomy representation of the hyperbolic cone manifold $M(r; \theta^-, \theta^+)$ with $0 \leq \theta_{\pm} \leq \pi$. Then the set $\mathcal{E}(\rho)$ is disjoint from the fundamental intervals $I_1(r) \cup I_2(r)$.

Consider the subset, J , of $[0, \pi] \times [0, \pi]$ consisting of those points (θ^-, θ^+) for which the conjecture is valid. It is obvious that $(0, 0)$ belongs to J and so J is nonempty. Tan pointed out that [28] implies that the set J is open. So what we need to show is that J is closed. Though computer experiments seem to support the conjecture, it is still open.

Appendix A: Explicit numerical calculations

Here we include explicit numerical calculations concerning the Key Proposition 4.4, which is essentially due to Bowditch [7].

Example 1 Let ϕ be the integral Markoff map determined by

$$(\phi(s_0), \phi(s_1), \phi(s_2)) = (3, 3, 3),$$

and let $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ be as in Section 4. Then ϕ satisfies the condition in Proposition 4.4, and so we have

$$\sum_{s \in \Omega^0(e)} h(\phi(s)) + 2 \sum_{s \in \Omega^-(\vec{e})} h(\phi(s)) = \psi(\vec{e}) = \frac{1}{3}.$$

We numerically calculate the partial finite sums of the series according to the “depths” of the regions. Here the depth of a region (corresponding to a rational number) $s \in \Omega^0(\vec{e})$ is defined as follows. If $s \in \{s_1, s_2, s_3\}$, then the depth of s is 0. Otherwise, the depth of s is the length d of the shortest oriented edge path $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d$ in the tree \mathcal{T} such that the tail of \vec{e}_1 corresponds to s and the head of s_d is the tail of \vec{e} . For each integer $d \geq 0$, let $\Omega_d^{0-}(\vec{e})$ be the subset of $\Omega^0(\vec{e})$ consisting of those regions with depth $\leq d$, and let $H_d(\phi, \vec{e})$ be the finite subseries of the above infinite series corresponding to the finite subset $\Omega_d^{0-}(\vec{e})$, which has cardinality $3 + 2 + 2^2 + \dots + 2^d = 1 + 2^{d+1}$. Then, by numerical calculation using Mathematica, we obtain the following, which shows that the sequence converges very rapidly:

$$H_0(\phi, \vec{e}) \doteq 0.311835,$$

$$H_3(\phi, \vec{e}) \doteq 0.333267,$$

$$H_6(\phi, \vec{e}) \doteq 0.333333.$$

Example 2 Let ϕ be the Markoff map determined by

$$(\phi(1/0), \phi(0/1), \phi(1/1)) = (0, \omega, i\omega), \quad \omega = \frac{\sqrt{3} + i}{2}.$$

Then ϕ corresponds to the holonomy representation of the complete hyperbolic structure of the complement of the figure-eight knot, $K(2/5)$. The oriented edge

$$\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3) := (1/1, 1/2; 0/1, 2/3)$$

is the unique element of the edge set $\vec{E}_2(2/5)$ in Figure 12. Hence, Key Lemma 6.2 implies

$$\sum_{s \in \Omega^0(e)} h(\phi(s)) + 2 \sum_{s \in \Omega^-(\vec{e})} h(\phi(s)) = \psi(\vec{e}) = \frac{-1 + \sqrt{3}i}{2} \doteq -0.5 + 0.866025i.$$

By numerical calculation using Mathematica, we obtain the following:

$$\begin{aligned} H_0(\phi, \vec{e}) &\doteq -0.903348 + 0.77871i, \\ H_3(\phi, \vec{e}) &\doteq -0.232475 + 0.806591i, \\ H_6(\phi, \vec{e}) &\doteq -0.33507 + 0.852655i, \\ H_9(\phi, \vec{e}) &\doteq -0.388262 + 0.863634i, \\ H_{12}(\phi, \vec{e}) &\doteq -0.416476 + 0.865674i. \end{aligned}$$

Thus the convergence is very slow in this case. This reflects the fact that the values of ϕ diverge relatively slowly, as shown in Figure 19. In particular, since $\phi(3/5) = 2$ (cf Theorem 8.5), the values of ϕ at the regions around the region $3/5$ diverge only linearly (see [31, Lemma 3.5]).

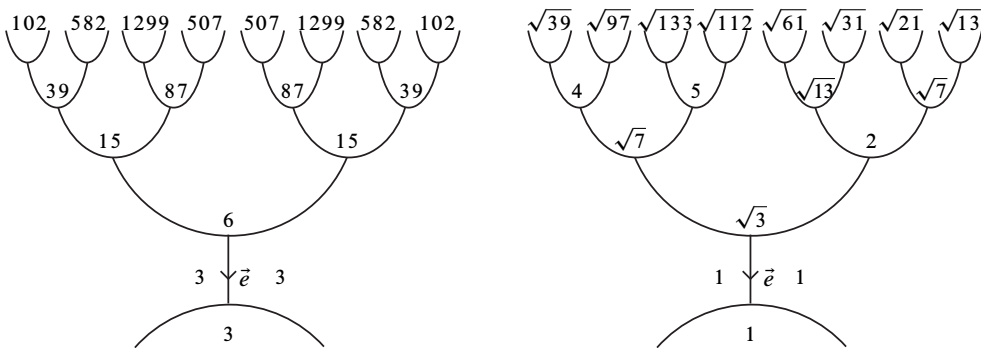


Figure 19: The absolute values of the Markoff maps in Example 1 (left) and Example 2 (right)

Late addition Makoto Tamura and Yasushi Yamashita made further numerical calculation for Example 2. The following result was obtained by using a computer program written by Yamashita [38].

$$H_{100}(\phi, \vec{e}) \doteq -0.490000 + 0.866025i,$$

$$H_{10000}(\phi, \vec{e}) \doteq -0.499900 + 0.866025i.$$

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