Erratum for “An elementary construction of Anick’s fibration”

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A misstatement in the key proof in our paper “An elementary construction of Anick’s fibration” led to an erroneous proof. This is repaired by a slightly longer argument.

55P35, 55P40, 55P45; 55Q51, 55Q52

In [3] we gave an elementary construction of Anick’s space $T_{2n-1}$. This is a space that lies in a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1},$$

where the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is the $(p^r)^{\text{th}}$ power map. This construction was carried out for any $p \geq 3$ and $r \geq 1$.

We also proved in [3, Theorem 4.3] that there is an $H$–space structure on $T_{2n-1}$ such that (1) is an $H$–fibration. The proof of 4.3 involved induction over the skeleton of $T_{2n-1}$ and cycled through 14 steps ((a), . . . , (n)). The argument given for the proof of [3, Theorem 4.3(l)] contained an incorrect statement and is not valid. The purpose of this note is to supply a correct proof for 4.3(l).

We will abbreviate $T_{2n-1}$ as $T$ and write $T^m$ for the $m$ skeleton of $T$. Recall that $\mathcal{W}^b_a$ is the collection of all spaces that are of the homotopy type of a simply connected locally finite wedge of mod $p^s$ Moore spaces for $a \leq s \leq b$.

At the point in the induction that we need to prove 4.3(l), we have established the following facts:

(A) $\Sigma T^{2np^k} \simeq G_k \vee W_k$ with $W_k \in \mathcal{W}^{r+k-1}_{r}$ (4.3(j))

(B) $G_k = G_{k-1} \cup_{\alpha_k} \mathbb{C}P^{2np^k}(p^r+k)$ (4.3(c))

(C) $\alpha_k: P^{2np^k}(p^r+k) \rightarrow G_{k-1}$ is divisible by $p^r+k-1$ (4.3(c), (e))

(D) $\Sigma^2 T^{2np^k} \in \mathcal{W}^{r+k}_r$ (4.3(b))
We will also use various steps in the induction at level \(k - 1\). Our task here is to prove:

**Theorem 4.3(l)** \(G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}\)

In the discussion of 4.3(l), steps 1 and 2 correctly conclude that \(G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}\). To complete the proof of 4.3(l), we need to analyze the cofibration sequence from (B) above:

\[
P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^k} \rightarrow G_k \wedge T^{2np^k}.
\]

\(P^{2np^k}(p^{r+k})\) is a double suspension and consequently \(P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}\) by (D) above. Since \(G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}\) it suffices to show that \(\alpha_k \wedge 1\) is null homotopic. A key ingredient in establishing this is the fact that \(\alpha_k \wedge 1\) is divisible by \(p^{r+k-1}\).

To clarify the situation, we recall from [3, 4.2] the following:

**Lemma 1** [3] Suppose \(W \in \mathcal{W}_d^{b}\) and \(f: P^k(p^s) \rightarrow W\) is divisible by \(p^b\). Write \(W \cong W_1 \vee W_2\) with \(W_1 \in \mathcal{W}_a^{b-1}\) and \(W_2 \in \mathcal{W}_b^{b}\). Then:

(a) \(f\) factors through \(W_2\) up to homotopy.

(b) Suppose that \(W_2\) is \((d - 1)\) connected and \(k < pd\). Then \(f \sim \ast\).

**Proof** This follows from the results of Cohen, Moore and Neisendorfer [1] and uses the Hilton–Milnor Theorem. Details are in [3].

In order to apply this, we need to know the exponents of the torsion in the integral cohomology of \(T\).

**Lemma 2** Let \(v_p(m)\) be the number of powers of \(p\) in \(m\). Then

\[
H^k(T) = \begin{cases} \mathbb{Z} / p^{r + v_p(m)} & \text{if } k = 2mn, \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof** This is implicit in [3] and follows from the integral cohomology Serre spectral sequence for the fibration

\[
S^{2n-1} \rightarrow T \rightarrow \Omega S^{2n+1}
\]

using the divided power relations in \(H^*(\Omega S^{2n+1})\).
We now apply the Lemma 1 with \( a = r, b = r + k - 1 \) and \( W = G_{k-1} \wedge T^{2np^k} \). We write \( W = W_1 \vee W_2 \) and, by Lemma 2, we have

\[
W_2 = p^{2np^{k-1}+1}(p^{r+k-1}) \wedge \left( \bigvee_{i=1}^{p-1} p^{2np^{k-1}i}(p^{r+k-1}) \vee p^{2np^k}(p^{r+k}) \right)
\]

and by Lemma 1(a), \( \alpha_k \wedge 1 \) factors through \( W_2 \).

Define \( A = p^{2np^{k-1}+1}(p^{r+k-1}) \wedge p^{2np^{k-1}}(p^{r+k-1}) \) and write \( W_2 \simeq A \vee B \), where \( B \) is \( 6np^{k-1} - 2 \) connected. We now apply the splitting:

\[
\Omega(A \vee B) \simeq \Omega A \times \Omega(B \times \Omega A).
\]

Since \( B \times \Omega A \) is \( 6np^{k-1} - 2 \) connected, the component of \( \alpha_k \wedge 1 \) in \( B \times \Omega A \) is null homotopic by part (b) of Lemma 1; this implies that \( \alpha_k \wedge 1 \) factors through \( A \), ie

\[
p^{2np^{k-1}+1}(p^{r+k-1}) \wedge p^{2np^{k-1}}(p^{r+k-1}) \simeq p^{4np^{k-1}}(p^{r+k-1}) \vee p^{4np^{k-1}+1}(p^{r+k-1}).
\]

**Lemma 3** Suppose \( f : P^m(p^{s+1}) \to P^n(p^s) \) has order \( p^{s+1} \). Then \( m \geq (4n - 2)p \).

**Proof** By [1], there is a decomposition

\[
\Omega P^{2n}(p^s) \simeq S^{2n-1}(p^s) \times \Omega \left( \bigvee_{k \geq 2} P^{(2n-2)k+3}(p^s) \right).
\]

Since the identity of \( S^{2n-1}(p^s) \) has order \( p^s \) (see [4]), it follows that some component of the second factor has order \( p^{s+1} \). By Lemma 1(b) we have \( m \geq ((2n - 2)k + 2)p \) for some \( k \geq 2 \). \( \square \)

It now follows that \( \alpha_k \wedge 1 \) has no essential component in \( p^{4np^{k-1}}(p^{r+k-1}) \) and we conclude that \( \alpha_k \wedge 1 \) factors as

\[
p^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha} p^{4np^{k-1}+1}(p^{r+k-1}) \xrightarrow{\beta} G_{k-1} \wedge T^{2np^k},
\]

where \( \beta \) induces a monomorphism in mod \( p \) homology.

We will describe a map \( \gamma : G_{k-1} \wedge T^{2np^k} \to P^{4np^{k-1}+1}(p^{r+k-1}) \) with the property that \( \gamma \beta \) is a homotopy equivalence and \( \gamma(\alpha_k \wedge 1) \sim * \). This will complete the proof that \( \alpha_k \wedge 1 \) is null homotopic and \( G_k \wedge T^{2np^k} \in W_{r+k-1} \).

The map \( \gamma \) is the composition

\[
G_{k-1} \wedge T^{2np^k} \longrightarrow G_{k-1} \wedge \Omega S^{2n-1} \xrightarrow{1^\wedge H_p^{k-1}} G_{k-1} \wedge \Omega S^{2np^{k-1}+1} \xrightarrow{\xi} G_{k-1} \wedge S^{2np^{k-1}} \longrightarrow p^{4np^{k-1}+1}(p^{r+k-1}).
\]
The first map in the composition comes from the projection in the fibration (1), and the second is the James Hopf invariant. We will describe \( \xi \) in the next proposition. The fourth map comes from the splitting of \( \Sigma G_{k-1} \) (from (B) and (D)).

**Proposition 4** Suppose \( G \) is a co–\( H \) space. Then the inclusion

\[
G \wedge X \longrightarrow G \wedge \Omega \Sigma X
\]

has a left homotopy inverse \( \xi: G \wedge \Omega \Sigma X \to G \wedge X \) and \( \xi \) commutes with co–\( H \) maps.

**Proof** Let \( \xi \) be the composition

\[
G \wedge \Omega \Sigma X \xrightarrow{\nu^1} \Sigma \Omega G \wedge \Omega \Sigma X \simeq \Omega G \wedge \Sigma \Omega \Sigma X \xrightarrow{1^1 \epsilon} \Omega G \wedge \Sigma X \xrightarrow{\epsilon^1} G \wedge X,
\]

where \( \nu \) is the co–\( H \) structure map and \( \epsilon \) is an evaluation. Clearly \( \xi \) is a left homotopy inverse to the inclusion. If \( \phi: G \to H \) is a co–\( H \) map, there is a homotopy commutative square:

\[
\begin{array}{ccc}
G & \xrightarrow{\nu} & \Sigma \Omega G \\
\phi \downarrow & & \Sigma \Omega \phi \\
H & \xrightarrow{\nu'} & \Sigma \Omega H \\
\end{array}
\]

(see [2]), so the map \( \xi \) is natural for co–\( H \) maps. \( \Box \)

We now show that \( \gamma \) induces an isomorphism in mod \( p \) homology in dimension \( 4np^{k-1} + 1 \). We first note that

\[
H_{4np^{k-1}+1}(G_{k-1} \wedge T^{2np^k}) \cong H_{2np^{k-1}+1}(G_{k-1}) \otimes H_{2np^k}(T^{2np^k}) \cong \mathbb{Z}/p.
\]

From the description of \( \gamma \) and the fact that \( \xi_* \) is an epimorphism, it follows that \( \gamma \) induces an isomorphism in this dimension. Since \( \beta \) induces a monomorphism, \( \gamma \beta \) induces an isomorphism in dimension \( 4np^{k-1} + 1 \). This implies that \( \gamma \beta \) is a homotopy equivalence.

It suffices, then, to show that \( \gamma(\alpha_k \wedge 1) \) is null homotopic. We appeal to the construction of \( \gamma \). We will show that the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
p^{2np^k}(p^{r+k}) \wedge T^{2np^k} & \xrightarrow{\alpha_k \wedge 1} & p^{2np^k} \wedge \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \\
G_{k-1} \wedge T^{2np^k} & \xrightarrow{\alpha_k \wedge 1} & G_{k-1} \wedge \Omega S^{2n+1} \\
\end{array}
\]

\[
\begin{array}{ccc}
p^{2np^k} \wedge S^{2np^{k-1}} & \xrightarrow{\alpha_k \wedge 1} & p^{2np^k} \wedge S^{2np^{k-1}+1} \\
\downarrow & & \downarrow & & \downarrow \\
G_{k-1} \wedge S^{2np^{k-1}} \wedge S^{2np^{k-1}+1} & \xrightarrow{\alpha_k \wedge 1} & G_{k-1} \wedge S^{2np^{k-1}+1} \\
\end{array}
\]

The composition of the bottom row with the splitting

\[
G_{k-1} \wedge S^{2np^{k-1}} \longrightarrow P^{4np^{k-1}+1}(p^{r+k})
\]
is the map $\gamma$, and the right hand vertical arrow is a suspension of $\alpha_k$. By (B) and (D) in case $k-1$, $\Sigma G_{k-1}$ is a retract of $\Sigma G_k$ so $\Sigma \alpha_k$ is null homotopic. Consequently it suffices to show that the diagram is homotopy commutative. The only issue is resolved by:

**Proposition 5** \( \alpha_k \) is a co–H map.

The proof of this result relies on:

**Lemma 6** \( \Omega G_{k-1} \ast \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1} \)

**Proof** We use (A) in case $k-1$ to see that the space $\Omega G_{k-1} \ast \Omega G_{k-1}$ is a retract of $\Omega \Sigma T^{2np^{k-1}} \ast \Omega \Sigma T^{2np^{k-1}}$. Using the James splitting of $\Sigma \Omega \Sigma X$, we have for any $X$

\[
\Omega \Sigma X \ast \Omega \Sigma X \simeq \bigvee_{i \geq 1} \bigvee_{j \geq 1} \Sigma X^{(i)} \wedge X^{(j)},
\]

so it suffices to show that $\Sigma T^{2np^{k-1}} \wedge T^{2np^{k-1}} \in \mathcal{W}_r^{r+k-1}$. However by (A) in case $k-1$,

\[
\Sigma T^{2np^{k-1}} \wedge T^{2np^{k-1}} \simeq (G_{k-1} \vee W_{k-1}) \wedge T^{2np^{k-1}} \simeq G_{k-1} \wedge T^{2np^{k-1}} \vee W_{k-1} \wedge T^{2np^{k-1}},
\]

which is in $\mathcal{W}_r^{r+k-1}$ by 4.3(l) and (D) in case $k-1$. \qed

**Proof of Proposition 5** It is required to show that there is a homotopy commutative diagram:

\[
\begin{array}{ccc}
P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} \\
\downarrow \gamma_k & & \downarrow \gamma_k \\
P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k \vee \alpha_k} & G_{k-1} \vee G_{k-1}
\end{array}
\]

Let $\Delta: P^{2np^k}(p^{r+k}) \to G_{k-1} \vee G_{k-1}$ be the difference between the two sides. Since $\gamma'$ is a suspension, $\Delta$ is divisible by $p^{r+k-1}$. The composition

\[
P^{2np^k}(p^{r+k}) \xrightarrow{\Delta} G_{k-1} \vee G_{k-1} \xrightarrow{\alpha_k - \alpha_k} G_{k-1} \times G_{k-1}
\]

is null homotopic, since each component is $\alpha_k - \alpha_k$. However there is a splitting [2]

\[
\Omega(G_{k-1} \vee G_{k-1}) \simeq \Omega(G_{k-1} \times G_{k-1}) \times \Omega(\Omega G_{k-1} \ast \Omega G_{k-1})
\]

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so $\Delta$ factors through $\Omega G_{k-1} \ast \Omega G_{k-1}$ and is divisible by $p^{r+k-1}$ in this space:

$$p^{2np^k}(p^{r+k}) \xrightarrow{p^{r+k-1}} p^{2np^k}(p^{r+k-1}) \xrightarrow{} \Omega G_{k-1} \ast \Omega G_{k-1} \rightarrow G_{k-1} \vee G_{k-1}$$

Since $\Omega G_{k-1} \ast \Omega G_{k-1} \in W_r^{r+k-1}$ by Lemma 6, we can apply Lemma 1 with $a = r$ and $b = r + k - 1$. In this case

$$W_2 = \Omega P^{2np_k+1}(p^{r+k-1}) \ast \Omega P^{2np_k+1}(p^{r+k-1}),$$

which is $4np^{k-1} - 2$ connected. Since $2np_k < p(4np^{k-1} - 1)$, $\Delta$ is null homotopic. $\square$

References


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