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Correction to the article: A cylindrical reformulation of Heegaard Floer homology

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This note corrects a serious mistake in the computation of the embedded index in "A cylindrical reformulation of Heegaard Floer homology." (The mistake is in a lemma about representing homology classes by surfaces.) It also corrects several smaller mistakes. The main results of the original paper are unchanged.

57R17; 57M27, 57R58

This note has two parts. The first part explains a serious gap in the proof of the index formula in our earlier work [1, Section 4], discovered by John Pardon. We explain the gap in Section 1.1 and how to correct the proof of the index formula in Section 1.2. Section 2 acknowledges and corrects four smaller errors, not affecting the main results of [1].

Acknowledgments I thank John Pardon, Clifford Taubes and Guangbo Xu for pointing out errors in [1], and for helpful conversations about how to correct these errors. I also thank John Pardon for helpful comments on a draft of this erratum. Finally, I thank the referee for helpful comments and corrections.

1 The index formula for embedded curves

1.1 The gap

1.1.1 What is correct In the cylindrical formulation, there are two steps to studying the expected dimensions of the moduli spaces. The first step is to consider the $\overline{\partial}$–operator for maps

(1) $$u\colon (S,\partial S)\longrightarrow (\Sigma\times[0,1]\times\mathbb{R}, (\boldsymbol{\alpha}\times\{1\}\times\mathbb{R})\cup(\boldsymbol{\beta}\times\{0\}\times\mathbb{R}))$$

for a fixed homeomorphism type of S. It is shown that the index of the $\overline{\partial}$–operator for such maps is given by

$$\mathrm{ind}(u) = g-\chi(S)+2e(A),$$

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where g is the genus of the Heegaard surface (or, more importantly, the number of negative (equivalently positive) ends of u), and $e(A)$ is the Euler measure of the domain A in Σ of the map u. (This formula holds whether or not u is holomorphic.)

The cylindrical formulation of Heegaard Floer homology corresponds to counting embedded holomorphic curves of the form (1). So, the second step in studying the index is to show that for embedded curves, $\chi(S)$ is determined by the homology class A. It is shown in [1, Proposition 4.2 and Corollary 4.3] that at an embedded holomorphic curve, $\chi(S)$ is given by

$$\chi(S) = g - n_{\vec{x}}(A) - n_{\vec{y}}(A) + e(A), \tag{2}$$

so

$$\mathrm{ind}(u) = e(A) + n_{\vec{x}}(A) + n_{\vec{y}}(A). \tag{3}$$

The proofs in [1] of (2) and (3) at an embedded holomorphic curve u, with respect to any almost complex structure satisfying the conditions [1, **(J1)**–**(J5)**, page 959]) (including nongeneric almost complex structures of this form), are correct.

Homology classes of curves in $\Sigma \times [0,1] \times \mathbb{R}$ correspond to homotopy classes of disks in the symmetric product. If A is represented by an embedded holomorphic curve with respect to the product complex structure on $\Sigma \times [0,1] \times \mathbb{R}$, it follows from the tautological correspondence that $\mathrm{ind}(u)$ agrees with the Maslov index in the symmetric product. So, in these cases, Formula (3) computes the Maslov index for disks in $\mathrm{Sym}^g(\Sigma)$.

1.1.2 What more one wants It is natural to be interested in the index at homology classes not represented by embedded holomorphic curves, for two reasons.

(1) One wants to know that the right hand side of Formula (3) is additive, so one can use it to (re)define and compute the relative grading on the Heegaard Floer complexes.

(2) It is tidier to know that Formula (3) always agrees with the Maslov index in $\mathrm{Sym}^g(\Sigma)$; the Maslov index is defined whether or not there is a holomorphic representative.

Note that S Sarkar has given a combinatorial proof that Formula (3) is additive, in the process of generalizing it to give a formula for the Maslov index of higher holomorphic polygons [5].

1.1.3 What is wrong To generalize Formula (2) to homology classes not admitting holomorphic representatives, we need some class of maps u which is broader than holomorphic maps but for which $\chi(S)$ is still determined. To show that the right hand side of Formula (3) agrees with the Maslov index in $\mathrm{Sym}^g(\Sigma)$ for homology classes without holomorphic representatives, we also want these maps u to correspond to disks in $\mathrm{Sym}^g(\Sigma)$. Such classes of maps u were proposed in [1, Lemmas 4.1 and 4.9], as follows.

Lemma 4.1 *Suppose $A \in \pi_2(\vec{x}, \vec{y})$ is a positive homology class. Then there is a Riemann surface with boundary and corners $\overline{S}$ and smooth map $u\colon S \to \Sigma \times [0,1] \times \mathbb{R}$ (where S denotes the complement in $\overline{S}$ of the corners of $\overline{S}$) in the homology class A such that*

1. $u^{-1}(C_\alpha \cup C_\beta) = \partial S$;
2. *for each i, $u^{-1}(\alpha_i \times \{1\} \times \mathbb{R})$ and $u^{-1}(\beta_i \times \{0\} \times \mathbb{R})$ each consists of one arc in ∂S;*
3. *the map u is J–holomorphic in a neighborhood of $(\pi_\Sigma \circ u)^{-1}(\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ for some J satisfying* **(J1)–(J5)** *(in fact, for $j_\Sigma \times j_{\mathbb{D}}$);*
4. *for each component of S, either*
   - *the component is a disk with two boundary punctures and the map is a diffeomorphism to $\{x_i\} \times [0,1] \times \mathbb{R}$ for some $x_i \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ (such a component is a* degenerate disk*) or*
   - *the map $\pi_\Sigma \circ u$ extends to a branched covering map $\overline{\pi_\Sigma \circ u}$, none of whose branch points map to points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$;*
5. *all the corners of S are acute;*
6. *the map u is an embedding.*

Lemma 4.9 *Suppose A is a positive homology class. Then we can represent $A + [\Sigma]$ by a map $u\colon S \to \Sigma \times [0,1] \times \mathbb{R}$ satisfying all the conditions of Lemma 4.1 and such that, additionally,*

- *the map $\pi_{\mathbb{D}} \circ u$ is a g–fold branched covering map with all its branch points of order 2;*
- *the map u is holomorphic near the preimages of the branch points of $\pi_{\mathbb{D}} \circ u$.*

The proof of Lemma 4.1 has two gaps.

(1) In the proof, one starts by gluing up the domain of u to produce a surface. One wants to ensure that the only corners of the resulting surface correspond to the points in $\vec{x} \cup \vec{y}$. The proof says to start with any maximal gluing and then make some local changes, but is imprecise or incorrect about how to do so.

(2) The argument for ensuring that the map is an embedding (Property (6)) is incorrect. First, some map, not necessarily an embedding, is constructed. Then, the proof says: "Modifying S_1 and $p_{\Sigma,1} \times p_{\mathbb{D},1}$ near the double points of $p_{\Sigma,1} \times p_{\mathbb{D},1}$ we can obtain a new map $u\colon S \to \Sigma \times [0,1] \times \mathbb{R}$ satisfying all of the stated properties". Typically, such a modification is not possible while keeping $\pi_\Sigma \circ u$ a branched map; see Example 1.

The proof of Lemma 4.9 builds on Lemma 4.1, and has the same gaps.

As we will see, the first point can be resolved by being more careful in the construction, following Ozsváth and Szabó [3, Lemma 2.17] (see also the author, Ozsváth and Thurston [2, Lemma 10.3]). The second point is more serious, as the following example (explained to me by J Pardon) shows.

Example 1 Consider the domain A shown in Figure 1. There is an obvious holomorphic representative $S \to \Sigma \times [0,1] \times \mathbb{R}$, where S is the disjoint union of two disks (bigons). This representative has a positive double point. Resolving the double point gives a map $S' \to \Sigma \times [0,1] \times \mathbb{R}$ where S' is an annulus. Indeed, Formula (2) predicts the embedded Euler characteristic $\chi = 2 - 6/4 - 6/4 + 1 = 0$.

But we can also find other, nonholomorphic representatives of this domain. For example, take S to be a surface of genus 1 with 2 boundary components. Then we can find branched maps $S \to \Sigma$ and $S \to \mathbb{D}$ representing the domain A. It is easy to arrange this map to satisfy the conditions in Lemma 4.1 except for being an embedding. Resolving double points decreases the Euler characteristic of S, which is already lower than the Euler characteristic predicted by Formula (2); so, if we could resolve them (without losing the other properties in Lemma 4.1), this would contradict Proposition 4.2. The difference with the previous case is that these double points are negative rather than positive.

1.2 Revised proofs of the main results

We can salvage the main result by weakening the conditions in Lemmas 4.1 and 4.9 to allow u to have double points, and extending Proposition 4.2, Corollary 4.3 and the proofs of Propositions 4.8 and Corollary 4.10 to curves with double points.

The revised Lemma 4.1 reads as follows.

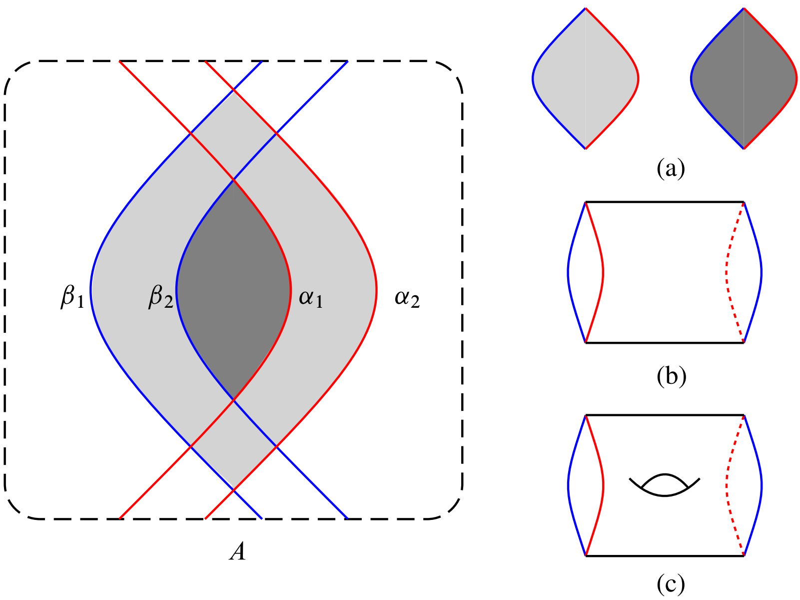

Figure 1: A domain A so that $\pi_\Sigma \circ u$ has branch points

Lemma 4.1′ *Suppose $A \in \pi_2(\vec{x}, \vec{y})$ is a positive homology class. Then there is a Riemann surface with boundary and corners $\overline{S}$ and smooth map $u\colon S \to \Sigma \times [0,1] \times \mathbb{R}$ (where S denotes the complement in $\overline{S}$ of the corners of $\overline{S}$) in the homology class A such that*

1. *$u^{-1}(C_\alpha \cup C_\beta) = \partial S$;*
2. *for each i, $u^{-1}(\alpha_i \times \{1\} \times \mathbb{R})$ and $u^{-1}(\beta_i \times \{0\} \times \mathbb{R})$ each consists of one arc in ∂S;*
3. *the map u is J–holomorphic in a neighborhood of ∂S for some J satisfying* **(J1)–(J5)** *(in fact, for $j_\Sigma \times j_{\mathbb{D}}$);*
4. *for each component of S, either*
   - *the component is a disk with two boundary punctures and the map is a diffeomorphism to $\{x_i\} \times [0,1] \times \mathbb{R}$ for some $x_i \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ (such a component is a degenerate* disk*) or*
   - *the map $\pi_\Sigma \circ u$ extends to a branched covering map $\overline{\pi_\Sigma \circ u}$, none of whose branch points map to points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$;*
5. *all the corners of S are acute;*
6. *the map u has at worst transverse double point singularities.*

(For convenience, we have also weakened (3); the resulting condition is sufficient for the other results to go through and requires one fewer step to achieve.)

The statement of Lemma 4.9 does not need any revisions, except that “all the conditions of Lemma 4.1” now refers to Lemma 4.1$'$, and we should have assumed that $g > 1$.

Lemma 4.9$'$ *Assume that $g > 1$. Suppose A is a positive homology class. Then we can represent $A + [\Sigma]$ by a map $u\colon S \to \Sigma \times [0,1] \times \mathbb{R}$ satisfying all the conditions of Lemma 4.1$'$ and such that, additionally,*

- *the map $\pi_{\mathbb{D}} \circ u$ is a g–fold branched covering map with all its branch points of order 2;*
- *the map u is holomorphic near the preimages of the branch points of $\pi_{\mathbb{D}} \circ u$.*

Proposition 4.2 now reads as follows.

Proposition 4.2$'$ *Let $u\colon S \to \Sigma \times [0,1] \times \mathbb{R}$ be a map satisfying the conditions enumerated in the Lemma 4.1$'$, representing a homology class A. Suppose that u has d_+ positive double points and d_- negative double points. Then the Euler characteristic $\chi(S)$ is given by*

$$\chi(S) = g - n_{\vec{x}}(A) - n_{\vec{y}}(A) + e(A) + 2(d_+ - d_-).$$

With these changes, it is clearer to state Proposition 4.8 as follows.

Proposition 4.8$'$ *We have that the Maslov index (in the symmetric product) of a domain $A \in \pi_2(\vec{x}, \vec{y})$ is given by*

$$\mu(A) = e(A) + n_{\vec{x}}(A) + n_{\vec{y}}(A). \tag{4}$$

This agrees with the index in the cylindrical setting at any embedded holomorphic curve (with respect to an almost complex structure satisfying conditions **(J1)**–**(J5)***).*

Proof of Lemma 4.1$'$ This construction is adapted from the proof of [3, Lemma 2.17]. Let $\{\mathcal{D}_i\}$ denote the components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Write $A = \sum n_i \mathcal{D}_i$; assume that we have ordered the $\mathcal{D}_i$ so that if $i < j$ then $n_i \le n_j$. Build a surface S_0 by taking, for each i, n_i copies of $\mathcal{D}_i$; denote these copies by $\mathcal{D}_i^{(j)}$. Glue these together as follows:

- if $\mathcal{D}_i$ and $\mathcal{D}_j$ ($i < j$) share a common α–arc a then for each $k = 1, \dots, n_i$ glue $\mathcal{D}_i^{(k)}$ to $\mathcal{D}_j^{(k+n_j-n_i)}$ along a;
- if $\mathcal{D}_i$ and $\mathcal{D}_j$ ($i < j$) share a common β–arc b then for each $k = 1, \dots, n_i$ glue $\mathcal{D}_i^{(k)}$ to $\mathcal{D}_j^{(k)}$ along b.

The resulting surface S_0 comes equipped with a map $u_{\Sigma,0}\colon S_0 \to \Sigma$. The surface S_0 is obviously a smooth surface with boundary away from $u_{\Sigma,0}^{-1}(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$. Next, consider the behavior of S_0 near a point $p \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$. Let $\mathcal{D}_{i_1}, \dots, \mathcal{D}_{i_4}$ be the four regions incident to p. (Some of the D_{i_j} might be the same.) If $p \notin \vec{x} \cup \vec{y}$ or $p \in \vec{x} \cap \vec{y}$ then the coefficients of the D_{i_j} in A have the form $a, a+k, a+\ell, a+k+\ell$ for some $a, k, \ell \geq 0$. Reordering the i_j if necessary, assume that D_{i_1} and D_{i_2} share a common α–arc (and $D_{i_1}^{(1)}$ and $D_{i_3}^{(3)}$ share a common β–arc). Then the following $D_{i_j}^{(m)}$ are glued together:

$$\{D_{i_1}^{(1)}, D_{i_2}^{(1+k)}, D_{i_3}^{(1)}, D_{i_4}^{(1+k)}\}, \dots, \{D_{i_1}^{(a)}, D_{i_2}^{(a+k)}, D_{i_3}^{(a)}, D_{i_4}^{(a+k)}\},$$
$$\{D_{i_2}^{(1)}, D_{i_4}^{(1)}\}, \dots, \{D_{i_2}^{(k)}, D_{i_4}^{(k)}\},$$
$$\{D_{i_3}^{(a+1)}, D_{i_4}^{(a+k+1)}\}, \dots, \{D_{i_3}^{(a+\ell)}, D_{i_4}^{(a+k+\ell)}\}.$$

In particular, near each point in $u_{\Sigma,0}^{-1}(p)$, S_0 is again a smooth surface with boundary and the map $u_{\Sigma,0}$ is a homeomorphism onto its image.

Now, suppose $p \in \vec{x} \setminus \vec{y}$ or $p \in \vec{y} \setminus \vec{x}$. Then the coefficients of A near p can be written as $\{a, a+k, a+\ell, a+k+\ell+1\}$ with $a, k, \ell \geq 0$ or $\{a, a+k, a+\ell, a+k+\ell-1\}$ with $a, k, \ell \geq 0$ and $k+\ell \geq 1$. In the first case, if $a \geq 1$, the glued regions are

$$\{D_{i_4}^{(a+k+1)}, D_{i_3}^{(a)}, D_{i_1}^{(a)}, D_{i_2}^{(a+k)}, D_{i_4}^{(a+k)}, D_{i_3}^{(a-1)}, D_{i_1}^{(a-1)}, \dots$$
$$, D_{i_4}^{(2+k)}, D_{i_3}^{(1)}, D_{i_1}^{(1)}, D_{i_2}^{(1+k)}, D_{i_4}^{(1+k)}\},$$
$$\{D_{i_2}^{(1)}, D_{i_4}^{(1)}\}, \dots, \{D_{i_2}^{(k)}, D_{i_4}^{(k)}\},$$
$$\{D_{i_3}^{(a+1)}, D_{i_4}^{(a+k+2)}\}, \dots, \{D_{i_3}^{(a+\ell)}, D_{i_4}^{(a+k+\ell+1)}\}.$$

(If $a = 0$, the first row is replaced with simply $\{D_{i_4}^{(k+1)}\}$.) In particular, the preimage of p consists of $k+\ell$ preimages which are smooth boundary points, and near which $u_{\Sigma,0}$ is a homeomorphism onto its image; and one preimage which looks like a boundary branch point. Call this last preimage a *bad point*. If we choose a smooth structure on S_0 making the bad point a $\pi/2$ corner then the map $u_{\Sigma,0}$ is of the form $z \mapsto z^{4a+1}$ near this point.

The other case — multiplicities $\{a, a+k, a+\ell, a+k+\ell-1\}$ — is similar. If $k, \ell \geq 1$ then the glued regions are

$$\{D_{i_3}^{(a+1)}, D_{i_4}^{(a+k)}, D_{i_2}^{(a+k)}, D_{i_1}^{(a)}, D_{i_3}^{(a)}, D_{i_4}^{(a+k-1)}, \dots, D_{i_3}^{(1)}, D_{i_4}^{(k)}, D_{i_2}^{(k)}\},$$
$$\{D_{i_2}^{(1)}, D_{i_4}^{(1)}\}, \dots, \{D_{i_2}^{(k-1)}, D_{i_4}^{(k-1)}\},$$
$$\{D_{i_3}^{(a+2)}, D_{i_4}^{(a+k+1)}\}, \dots, \{D_{i_3}^{(a+\ell)}, D_{i_4}^{(a+k+\ell-1)}\}.$$

If $k=0$ then the glued regions are

$$\{D_{i_3}^{(a+1)}, D_{i_4}^{(a)}, D_{i_2}^{(a)}, D_{i_1}^{(a)}, D_{i_3}^{(a)}, D_{i_4}^{(a-1)}, \dots, D_{i_3}^{(2)}, D_{i_4}^{(1)}, D_{i_2}^{(1)}, D_{i_1}^{(1)}, D_{i_3}^{(1)}\},$$
$$\{D_{i_3}^{(a+2)}, D_{i_4}^{(a+1)}\}, \dots, \{D_{i_3}^{(a+\ell)}, D_{i_4}^{(a+\ell-1)}\}.$$

If $\ell=0$ then the glued regions are

$$\{D_{i_2}^{(a+k)}, D_{i_1}^{(a)}, D_{i_3}^{(a)}, D_{i_4}^{(a+k-1)}, D_{i_2}^{(a+k-1)}, \dots, D_{i_2}^{(1+k)}, D_{i_1}^{(1)}, D_{i_3}^{(1)}, D_{i_4}^{(k)}, D_{i_2}^{(k)}\},$$
$$\{D_{i_2}^{(1)}, D_{i_4}^{(1)}\}, \dots, \{D_{i_2}^{(k-1)}, D_{i_4}^{(k-1)}\}.$$

In each case, all but one of the preimages of p lie on the smooth boundary of S, and near them the map $u_{\Sigma,0}$ is a local homeomorphism; and there is one remaining *bad point*. Near the bad point we can add a $\pi/2$ corner to S_0 and view $u_{\Sigma,0}$ as a branched map. In particular, for each α_i, $u_{\Sigma,0}^{-1}(\alpha_i)\cap\partial S_0$ consists of a union of some circles $C_{i,j}^{\alpha}$ and possibly a single arc A_i^{α}; and similarly for each β_i.

The surface S_0 has corners, which are in bijective correspondence with $(\vec{x}\cup\vec{y})\setminus(\vec{x}\cap\vec{y})$. The map $u_{\Sigma,0}$ may have branch points at some of these corners, say $p_1,\dots,p_k$. If p_i has total angle $n\pi/2$, make $(n-1)/2$ cuts in S_0 at p_i, as in Figure 2. Let S_1 be the resulting surface and $u_{\Sigma,1}$ the resulting map to Σ.

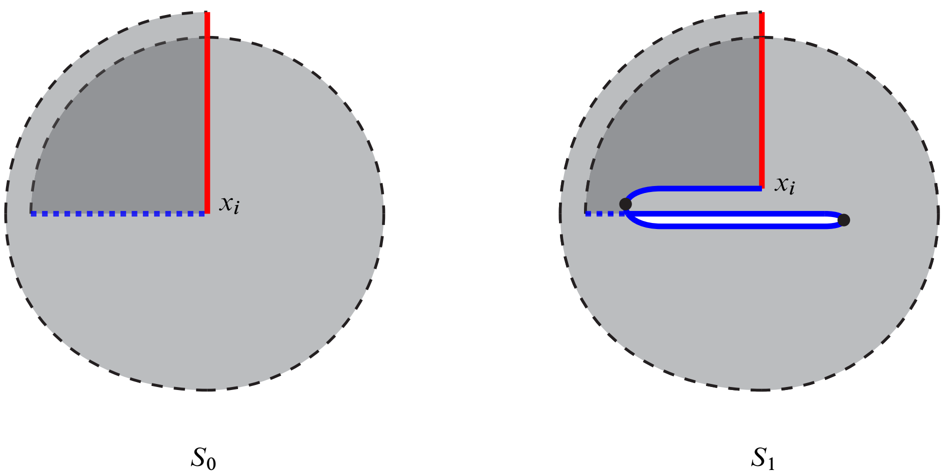

Figure 2: *Making cuts at the corners* The figure shows a region of S_0 (left) and S_1 (right); in S_0 there is a branch point at x_i. The darker region is covered with multiplicity 2. On the right, the two dots are boundary branch points. We made cuts along the β–arcs; we could equally well have made cuts along the α–arcs.

Next, we modify $(u_{\Sigma,1}, S_1)$ to a new surface whose corners correspond to $\vec{x} \amalg \vec{y}$; that is, we introduce corners corresponding to points in $\vec{x} \cap \vec{y}$. For each point $x_i \in \vec{x} \cap \vec{y}$, if x_i is disjoint from $u_{\Sigma,1}(\partial S_1)$ then take the disjoint union of S_1 with a twice punctured disk, and define u_{Σ_2} to map the twice punctured disk by a constant map to x_i. If x_i is not disjoint from $u_{\Sigma,1}(\partial S_1)$ then choose an arc in ∂S_1 covering x_i and make a small slit in the arc starting at x_i. (This introduces two new corners, both mapping to x_i, and a boundary branch point.) See Figure 3. After this modification, the surface has exactly $2g$ corners, corresponding to $\vec{x} \amalg \vec{y}$. Call the result $(S_2, u_{\Sigma,2})$.

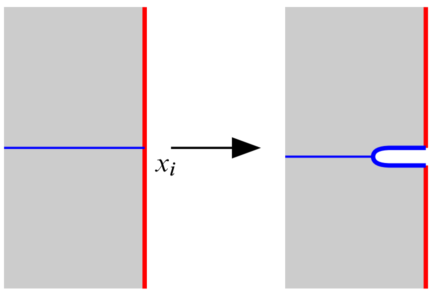

Figure 3: Adding slits at degenerate corners

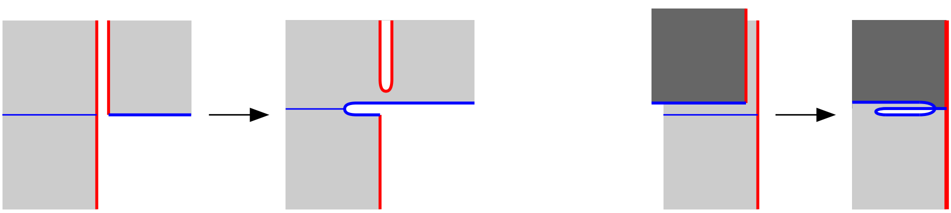

Figure 4: Splicing a corner and an edge

Next, we modify $(S_2, u_{\Sigma,2})$ to a new pair $(S_3, u_{\Sigma,3})$ so that for each i, $u_{\Sigma,3}^{-1}(\alpha_i) \cap \partial S_3$ (respectively $u_{\Sigma,3}^{-1}(\beta_i) \cap \partial S_3$) consists of a single arc (and no circles). In the process, we will introduce some more boundary branch points. Suppose that C is a boundary component of S_2 which is mapped entirely to α_i. Let $x_i \in \vec{x}$ be the corner on $\alpha_i \cap \beta_j$ (for some j) and p_i the corresponding corner of S_2. Make a small slit in S_2 along $u_{\Sigma,2}^{-1}(\beta_j)$ starting at $C \cap u_{\Sigma,2}^{-1}(x_i)$, and glue one edge of the resulting surface to the α– or β–arc near p_j in such a way that $u_{\Sigma,2}$ induces a branched map from the result. (There are two cases for the local geometry here; see Figure 4.) This reduces the number of boundary components of S mapped to α_i by 1; repeat for the other α–boundary circles of S_2. Modify boundary components mapped entirely to β_i

similarly. Call the result $(S_3, u_{\Sigma,3})$; this pair has the property that $u_{\Sigma,3}^{-1}(\alpha_i) \cap \partial S_3$ (respectively $u_{\Sigma,3}^{-1}(\beta_i) \cap \partial S_3$) consists of a single arc (and no circles).

The map $u_{\Sigma,3}$ and the complex structure on Σ induce a complex structure on S_3. Let U denote a tubular neighborhood of ∂S_3. Choose a holomorphic map $u_{\mathbb{D},3}\colon U \to [0,1]\times\mathbb{R}$ so that:

- $u_{\mathbb{D},3}$ sends each α–arc in ∂S_3 to $\{1\}\times\mathbb{R}$ and each β–arc to $\{0\}\times\mathbb{R}$;
- near each corner of S_3 corresponding to a point in $\vec{x}$, $u_{\mathbb{D},3}$ is asymptotic to $-\infty$;
- near each corner of S_3 corresponding to a point in $\vec{y}$, $u_{\mathbb{D},3}$ is asymptotic to $+\infty$;
- the map $u_{\mathbb{D},3}$ is a local diffeomorphism (ie, has nonvanishing derivative).

Extend $u_{\Sigma,3}$ arbitrarily to the rest of S_3. Then we have that $u_{\Sigma,3}\times u_{\mathbb{D},3}$ is a map to $\Sigma\times[0,1]\times\mathbb{R}$. By construction, this map satisfies Conditions (1), (2), (3) (4) and (5). Perturbing $u_{\Sigma,3}\times u_{\mathbb{D},3}$ slightly (without changing it near the boundary) gives a map $u\colon S\to\Sigma\times[0,1]\times\mathbb{R}$ satisfying (6), as well. □

Proof of Proposition 4.2′ The proof is essentially the same as the original proof of Proposition 4.2, noting that each double point leads to two intersections of u and u'. We spell this out.

First, note that each degenerate disk adds 1 to $\chi(S)$, 1 to g, 0 to $e(A)$ and 0 to $2(d_+ - d_-) - n_{\vec{x}}(A) - n_{\vec{y}}(A)$. Thus, each such disk changes the two sides of the formula in identical ways, and so we may assume there are no degenerate disks.

Next, by the Riemann–Hurwitz formula,

$$e(S) = e(A) - \mathrm{br}(\pi_{\mathbb{D}}\circ u),$$

where $\mathrm{br}(\pi_{\mathbb{D}}\circ u)$ denotes the ramification degree of $\pi_{\mathbb{D}}\circ u$. (For example, if all branch points of $\pi_{\mathbb{D}}\circ u$ have order 2 then $\pi_{\mathbb{D}}\circ u$ is just the number of branch points.) Moreover, since S has $2g$ $\pi/2$–corners,

$$\chi(S) = e(S) + g/2;$$

so, we want to compute $\mathrm{br}(\pi_{\mathbb{D}}\circ u)$.

Let

$$\tau_r\colon \Sigma\times[0,1]\times\mathbb{R}\to\Sigma\times[0,1]\times\mathbb{R}$$

be translation by r units in the $\mathbb{R}$–direction, ie, $\tau_r(p,s,t) = (p,s,t+r)$. Let $\partial/\partial t$ denote the tangent vector field to $\mathbb{R}$. Then, for ϵ sufficiently small, we have

$$\begin{aligned}\mathrm{br}(\pi_{\mathbb{D}}\circ u) &= \#\{\text{tangencies of } \partial/\partial t \text{ and } u\}\\ &= \#(u\cap(\tau_\epsilon\circ u)) - 2(d_+ - d_-),\end{aligned}$$

where all counts are with multiplicity. (Tangencies along the boundary, boundary double points, boundary intersection points and boundary branch points each count for $\frac{1}{2}$.) The term $2(d_+ - d_-)$ comes from the fact that each positive (respectively negative) double point of u contributes 2 (respectively -2) intersections between u and $\tau_\epsilon \circ u$.

The fact that u is holomorphic near its boundary implies that $\#(u \cap (\tau_\epsilon \circ u)) = \#(u \cap (\tau_R \circ u))$ for any $R \in \mathbb{R}$; see [1, page 978]. When R is sufficiently large,

$$\#(u \cap (\tau_R \circ u)) = n_{\vec{x}}(A) + n_{\vec{y}}(A) - g/2.$$

Collecting these equalities,

$$\begin{aligned}
\chi(S) &= e(S) + g/2 \\
&= e(A) - \mathrm{br}(\pi_{\mathbb{D}} \circ u) + g/2 \\
&= e(A) - \#(u \cap (\tau_R \circ u)) + 2(d_+ - d_-) + g/2 \\
&= e(A) - n_{\vec{x}}(A) - n_{\vec{y}}(A) + g + 2(d_+ - d_-),
\end{aligned}$$

as desired. □

Proof of Lemma 4.9′ Let $(S_3, u_{\Sigma,3})$ be as in the proof of Lemma 4.1′ applied to the homology class A. Build a pair $(S_4, u_{\Sigma,4})$ representing $A + [\Sigma]$, and with S_4 connected, as follows. Start with the disjoint union $S_3 \amalg \Sigma$. Forget each degenerate disk in S_3 and instead make three cuts starting from the corresponding $x_i \in \Sigma$, two along the α–circle and one along the β–circle, as in the left of Figure 5. At each remaining point $x_i \in \vec{x}$, cut open Σ in the same way and glue it to the corresponding corner of S_3, as shown in Figure 5. The result is a connected surface S_4 and map $u_{\Sigma,4}\colon S_4 \to \Sigma$ representing the homology class $A + [\Sigma]$.

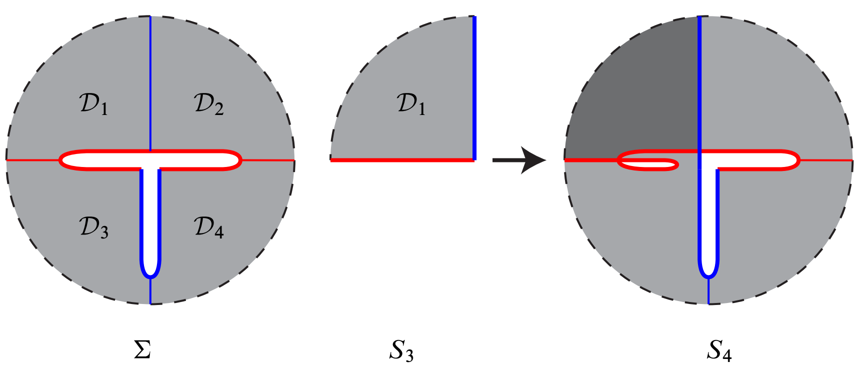

Figure 5: Cutting Σ and gluing to S_3

We claim that there is a g–fold branched covering $u_{\mathbb{D},4}\colon S_4 \to [0,1]\times\mathbb{R}$ sending the α–boundary of S_4 to $\{1\}\times\mathbb{R}$ and the β–boundary of S_4 to $\{0\}\times\mathbb{R}$. To see this, let $\partial S_4\times[0,\epsilon)$ be a collar neighborhood of ∂S_4 and let $C=\partial S_4\times\{\epsilon/2\}$. Collapsing the circles in C gives a surface S' consisting of a nonempty union of disks D — one for each boundary component of S — and a closed, connected surface E meeting each disk in a single point. If D_i has $2n_i$ corners (so $\sum n_i = g$) then we can choose an n_i–fold branched cover $v_i\colon D_i\to[0,1]\times\mathbb{R}$ with the specified boundary behavior. Choose a g–fold branched cover $w\colon E\to S^2$. (This is where we use the assumption that $g>1$.) Splicing together $\coprod_i v_i$ and w gives a g–fold branched cover $u_{\mathbb{D},4}\colon S_4\to[0,1]\times\mathbb{R}$ with the desired boundary behavior. Perturbing $u_{\mathbb{D},4}$ slightly, we can ensure that all branch points of $u_{\mathbb{D},4}$ are simple.

Consider the map $u=u_{\Sigma,4}\times u_{\mathbb{D},4}\colon S_4\to\Sigma\times[0,1]\times\mathbb{R}$. This satisfies (1), (2) (4) and (5) of Lemma 4.1′, and the projection to $[0,1]\times\mathbb{R}$ is a branched covering. Isotoping $u_{\mathbb{D},4}$ we can ensure (3) as well as making $u_{\mathbb{D},4}$ holomorphic near its branch points. Finally, deforming u slightly we can ensure that it has only double point singularities, ensuring (6). □

Proof of Proposition 4.8′ Using the fact that μ is unchanged by stabilization (see [3, Remark 10.5]), as is $e+n_{\vec{x}}+n_{\vec{y}}$ (obvious), we may assume that $g>1$. Since

$$\mu([\Sigma]) = 2 = e([\Sigma]) + n_{\vec{x}}([\Sigma]) + n_{\vec{y}}([\Sigma]),$$

adding or subtracting copies of $[\Sigma]$ changes both sides of Formula (4) in the same way. So, it suffices to prove Formula (4) after replacing A by $A+(n+1)[\Sigma]$ where $A+n[\Sigma]$ is positive. Let u be the map given by Lemma 4.9′ in the homology class $A+(n+1)[\Sigma]$. Via the tautological correspondence (see, for instance, [1, Section 13]), u corresponds to a map $\phi\colon\mathbb{D}^2\to\mathrm{Sym}^g(\Sigma)$ with the same domain as u. Rasmussen showed [4, Theorem 9.1] that

$$\mu(A) = \Delta\cdot\phi + 2e(A), \tag{5}$$

where Δ denotes the diagonal.

In terms of u, the intersections of ϕ with the Δ arise in two ways.

- Branch points of $\pi_{\mathbb{D}}\circ u$. Lemma 4.9′ guaranteed that these be order 2 branch points, and that u be holomorphic near each of them. It follows that each branch point corresponds to a positive, transverse intersection of ϕ and Δ.
- Double points of u. Each positive (respectively negative) double point corresponds to a positive (respectively negative), degree 2 tangency of ϕ and Δ.

So, we have

$$\Delta \cdot \phi = \operatorname{br}(\pi_{\mathbb{D}} \circ u) + 2(d_+ - d_-), \tag{6}$$

where br denotes the number of branch points. By the Riemann–Hurwitz formula,

$$\chi(S) = g\chi(\mathbb{D}^2) - \operatorname{br}(\pi_{\mathbb{D}} \circ u)$$

so

$$\begin{aligned} \operatorname{br}(\pi_{\mathbb{D}} \circ u) &= g - \chi(S) = g - (g - n_{\vec{x}}(A) - n_{\vec{y}}(A) + e(A) + 2(d_+ - d_-)) \\ &= n_{\vec{x}}(A) + n_{\vec{y}}(A) - e(A) - 2(d_+ - d_-). \end{aligned}$$

Combining this with Equations (5) and (6) gives

$$\begin{aligned} \mu(A) &= n_{\vec{x}}(A) + n_{\vec{y}}(A) - e(A) - 2(d_+ - d_-) + 2(d_+ - d_-) + 2e(A) \\ &= e(A) + n_{\vec{x}}(A) + n_{\vec{y}}(A), \end{aligned}$$

as desired. □

2 Smaller errata

(1) In Section 1, on page 959, Condition (J4) should read "$J(\partial/\partial s) = \partial/\partial t$", not "$J(\partial/\partial t) = \partial/\partial s$" as currently written. (Thanks to C Taubes for pointing out this mistake.)

(2) In Section 3, pages 966–972, the paper considers the space

$$W_k^{p,d}((S, \partial S), (W, C_\alpha \cup C_\beta)).$$

(See, for instance, Definition 3.5, page 968.) Here, $W = \Sigma \times [0,1] \times \mathbb{R}$. It is important that this space of $W_k^{p,d}$ maps be a Banach manifold; this is used, for instance, in the proof of Proposition 3.7. However, it is not clear that this space of maps is a Banach manifold, because W has boundary.
The easiest way to fix this is to replace W by $\Sigma \times \mathbb{R} \times \mathbb{R}$ (but leave the boundary conditions C_α and C_β unchanged). This larger space has the structure of a Banach manifold in an obvious way. Since the projection to $\mathbb{R} \times \mathbb{R}$ is holomorphic, the 0–set of the $\overline{\partial}$–operator on the larger space of maps is, in fact, contained in the smaller space of maps, so this has no effect on the space of holomorphic curves under consideration. (Thanks to J Pardon for pointing out this mistake.)

(3) Also in Section 3, in the definition of the universal moduli space $\mathcal{M}^\ell$, instead of considering the space of all almost complex structures on S (which is infinite-dimensional), one should consider the moduli space of complex structures on S (which is finite-dimensional). (Otherwise, in the proof of Proposition 3.8, the

fiber $\mathcal{M}$ of the projection $\mathcal{M} \to \mathcal{J}$ is the product of the desired moduli space with an infinite-dimensional space.) (Thanks to J Pardon for pointing out this mistake.)

(4) In Section 14.2, on page 1071, the form $df \wedge dg + \star df$ is only closed if the Morse function f is harmonic. To guarantee the existence of a harmonic Morse function, puncture the 3–manifold Y at two points, and consider functions which approach $+\infty$ at one of the punctures and $-\infty$ at the other puncture. (Thanks to G Xu for pointing out this mistake.)

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