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Non-positively curved complexes of groups and boundaries

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Given a complex of groups over a finite simplicial complex in the sense of Haefliger, we give conditions under which it is possible to build an $E\mathcal{Z}$–structure in the sense of Farrell and Lafont for its fundamental group out of such structures for its local groups. As an application, we prove a combination theorem that yields a procedure for getting hyperbolic groups as fundamental groups of simple complexes of hyperbolic groups. The construction provides a description of the Gromov boundary of such groups.

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In [2], Bestvina defined a fundamental notion of boundary that is relevant to geometric group theory. He showed how the topology of the boundary ∂G of a group G encodes the cohomology with group ring coefficients of G. This notion of boundary was further generalised to the notion of an equivariant compactification by the work of Farrell and Lafont [19], who proved the Novikov conjecture for groups admitting what they call an $E\mathcal{Z}$–structure, that is to say a classifying space for proper actions together with an equivariant $\mathcal{Z}$–compactification.

The existence of $E\mathcal{Z}$–structures, and their generalisation for groups with torsion, is known for groups that admit a classifying space for proper actions with a sufficiently nice geometry. For a group G acting properly and cocompactly on a CAT(0) space X, the compactification of X obtained by adding the visual boundary ∂X yields an $E\mathcal{Z}$–structure for G. In the case of a torsion-free hyperbolic group G, a classifying space is given by an appropriate Rips complex (see for instance Coornaert, Delzant and Papadopoulos [12]). Bestvina and Mess [4] proved that such a space can be compactified by adding the Gromov boundary of G to get an $E\mathcal{Z}$–structure for G. This result was further generalised in Meintrup and Schick [27] to the case of hyperbolic groups with torsion, where they show that such a compactification yields an $E\mathcal{Z}$–structure in the sense of Carlsson and Pedersen [11]. The existence of such an $E\mathcal{Z}$–structure is also known for systolic groups by work of Osajda and Przytycki [29].

In this article, we address the following combination problem: Given a group G acting cocompactly by simplicial isometries on a simplicial CAT(0) complex X, are there

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natural conditions under which it is possible to build an $E\mathcal{Z}$–structure for G, assuming that the stabilisers of simplices all admit such a structure?

There are already some special cases for which such a combination theorem is known to hold. For instance, Tirel [32] explained how to build a $\mathcal{Z}$–boundary for free and direct products of groups admitting $\mathcal{Z}$–boundaries. Furthermore, Dahmani [14] built an $E\mathcal{Z}$–structure for a torsion-free group that is hyperbolic relative to a group admitting an $E\mathcal{Z}$–structure.

This article deals with acylindrical actions on CAT(0) spaces. Recall that an action is called *acylindrical* if the diameter of sets with infinite pointwise stabiliser is uniformly bounded above.[1] This is a first step towards developing geometric tools to study groups through their cocompact actions on non-positively curved complexes of arbitrary dimension. This is particularly relevant for groups that do not split and lack the rich geometry of groups that are non-positively curved in a broad sense (hyperbolic, CAT(0), systolic). An example of such a phenomenon is the case of the mapping class group of a non-exceptional surface, acting on its curve complex. The action is acylindrical by a result of Bowditch [7] and the curve complex is hyperbolic by a celebrated result of Masur and Minsky [26]. However, it is known that the mapping class group is not relatively hyperbolic by work of Behrstock, Druţu and Mosher [1].

In this article, we consider a non-positively curved complex of groups $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ over a finite simplicial complex Y endowed with a M_κ–structure, $\kappa \leq 0$, in the sense of Bridson [8], such that the stabiliser of every simplex σ of Y admits an $E\mathcal{Z}$–structure $(\overline{EG}_\sigma, \partial G_\sigma)$. We further assume that these structures define an *$E\mathcal{Z}$–complex of space compatible with* $G(\mathcal{Y})$ (see Definition 2.2 for a precise definition), that is, there are embeddings $\phi_{\sigma,\sigma'}\colon \overline{EG}_{\sigma'} \hookrightarrow \overline{EG}_\sigma$, for all $\sigma \subset \sigma'$, that are equivariant with respect to the local maps of $G(\mathcal{Y})$, and such that the induced diagram of embeddings is commutative up to multiplication by twisting elements of $G(\mathcal{Y})$.

Combination theorem for boundaries of groups *Let $G(\mathcal{Y})$ be a non-positively curved complex of groups over a finite simplicial complex Y endowed with a M_κ–structure, $\kappa \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering[2] of $G(\mathcal{Y})$. Suppose that the following global condition holds:*

(i) *The action of G on X is acylindrical.*

[1] The original definition of acylindricity by Sela [31] considers nontrivial stabilisers instead of infinite ones. Here we use a more general notion of acylindricity introduced by Delzant [16] that is more suitable for proper actions.

[2] The simplicial complex X naturally inherits a M_κ–structure from that of Y, which makes it a complete geodesic metric space by work of Bridson [8]; the CAT(0) property follows from the Cartan–Hadamard theorem.

Further assume that there is an $E\mathcal{Z}$–complex of spaces compatible with $G(\mathcal{Y})$ that satisfies each of the following local conditions:

(ii) **The limit set property** *For every pair of simplices $\sigma \subset \sigma'$ of Y, the embedding $\overline{EG}_{\sigma'} \hookrightarrow \overline{EG}_{\sigma}$ realises an equivariant homeomorphism from $\partial G_{\sigma'}$ to the limit set $\Lambda G_{\sigma'} \subset \partial G_{\sigma}$. Furthermore, for every simplex σ of Y, and every pair of subgroups H_1, H_2 in the family*

$$\mathcal{F}_{\sigma} = \left\{ \bigcap_{i=1}^{n} g_i G_{\sigma_i} g_i^{-1} \,\middle|\, g_1, \dots, g_n \in G_{\sigma}, \sigma_1, \dots, \sigma_n \subset \mathrm{st}(\sigma), n \in \mathbb{N} \right\},$$

we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2) \subset \partial G_{\sigma}$.

(iii) **The convergence property** *For every pair of simplices $\sigma \subset \sigma'$ in Y and every sequence (g_n) of G_{σ} whose projection is injective in $G_{\sigma}/G_{\sigma'}$, there exists a subsequence such that $(g_{\varphi(n)}\overline{EG}_{\sigma'})$ uniformly converges to a point in $\overline{EG}_{\sigma}$.*

(iv) **The finite height property** *For every pair of simplices $\sigma \subset \sigma'$ of Y, $G_{\sigma'}$ has finite height in G_{σ}, that is, there exist an upper bound on the number of distinct cosets $\gamma_1 G_{\sigma'}, \dots, \gamma_n G_{\sigma'} \in G_{\sigma}/G_{\sigma'}$ such that the intersection $\gamma_1 G_{\sigma'} \gamma_1^{-1} \cap \dots \cap \gamma_n G_{\sigma'} \gamma_n^{-1}$ is infinite.*

Then G admits an $E\mathcal{Z}$–structure $(\overline{EG}, \partial G)$ in the sense of Farrell and Lafont.

Furthermore, the following properties hold:

(ii′) *For every simplex σ of Y, the map $\overline{EG}_{\sigma} \to \overline{EG}$ realises an equivariant embedding from ∂G_{σ} to $\Lambda G_{\sigma} \subset \partial G$. Moreover, for every pair H_1, H_2 of subgroups in the family*

$$\mathcal{F} = \left\{ \bigcap_{i=1}^{n} g_i G_{\sigma_i} g_i^{-1} \,\middle|\, g_1, \dots, g_n \in G,\ \sigma_1, \dots, \sigma_n \in S(Y), n \in \mathbb{N} \right\},$$

we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2) \subset \partial G$.

(iii′) *For every simplex σ of Y, the embedding $\overline{EG}_{\sigma} \hookrightarrow \overline{EG}$ satisfies the convergence property.*

(iv′) *For every simplex σ of Y, the local group G_{σ} has finite height in G.*

As an application of the previous construction, we prove a higher dimensional combination theorem for hyperbolic groups, in the case of acylindrical complexes of groups of arbitrary dimension.

Combination theorem for hyperbolic groups *Let $G(\mathcal{Y})$ be a strictly developable non-positively curved simple complex of groups over a finite simplicial complex Y endowed with a M_κ–structure, $\kappa \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering of $G(\mathcal{Y})$. Assume that:*

- *The universal covering X is hyperbolic.*[3]
- *The local groups are hyperbolic and all the local maps are quasiconvex embeddings.*
- *The action of G on X is acylindrical.*

Then G is hyperbolic. Furthermore, the local groups embed in G as quasiconvex subgroups.

Note that a complex of groups over a simply connected simplicial complex is developable if and only if it is strictly developable. Hence one might try to create new hyperbolic groups as fundamental groups of non-positively curved complexes of hyperbolic groups over a simply connected finite complex (see Bridson and Haefliger [9, Theorem II.12.28]).

Such a result is already known for acylindrical graphs of groups: the hyperbolicity is a direct consequence of the much more general combination theorem of Bestvina and Feighn [3], while the quasiconvexity of vertex stabilisers follows from a result of Kapovich [25]. Mj and Sadar [28] have, using a different approach, a combination theorem that deals with the case where all the local groups are the same.

Our construction follows the strategy of Dahmani [15], who applied this idea to amalgamate Bowditch boundaries of relatively hyperbolic groups in the case of acylindrical graphs of groups. The proofs in our case are significantly more involved as the topology of X can be much more complicated than that of a tree. Generalising an argument of Dahmani, we prove that G is a uniform convergence group on ∂G (see Section 6 for definitions), which implies the hyperbolicity of G by a celebrated result of Bowditch [5] and Tukia [33].

The article is organised as follows. In Section 1, we study complexes of spaces over a simplicial complex. These spaces are direct generalisations of graphs of spaces studied by Scott and Wall in the context of Bass–Serre theory [30]. In Section 2, we give conditions under which it is possible to build a classifying space for proper actions of the fundamental group of a complex of groups as a complex of spaces over its universal covering. We also define the boundary ∂G of G and the compactification

[3] For instance, when $\kappa < 0$.

$\overline{EG}$ of EG as sets. In Section 3, we investigate geometric properties of geodesic in CAT(0) M_κ–complexes. In Section 4, we study the geometry of some subcomplexes of X, called domains, which are used to define ∂G. Section 5 is devoted to the proof of some geometric results that are used throughout the paper. We define a topology on $\overline{EG}$ in Section 6 and we prove that it makes $\overline{EG}$ a compact metrisable space in Section 7. The proof of the combination theorem for boundaries of groups is completed in Section 8, where the properties of ∂G are investigated. Finally, Section 9 is devoted to the dynamics of G on its boundary and to the proof of the combination theorem for hyperbolic groups.

Notation Throughout this paper, X is a simplicial complex. For a point x of X, we denote by σ_x the unique simplex containing x in its interior. For a simplex σ of X, we denote by $\mathrm{st}(\sigma)$ the open star of σ, that is, the reunion of the open simplices whose closure contain σ. We also denote by $\overline{\mathrm{st}}(\sigma)$ its closed star, that is, the reunion of the closed simplices containing σ. We denote by $S(X)$ the set of simplices of X, and by $V(X)$ the set of its vertices. For a simplex σ of X and a constant $r > 0$, we denote by $B(\sigma, r)$ (resp. $\overline{B}(\sigma, r)$) the open (resp. closed) metric r–ball around σ.

All the types of classifying spaces we will consider in this paper are classifying spaces for proper actions (see Section 2 for definitions). Consequently, we will simply speak of *classifying spaces* rather than *classifying spaces for proper actions*. Moreover, although the notation $\underline{E}G$ is well spread in the literature to mean a classifying space for proper actions of a discrete group G, we will simply use the notation EG so as to avoid the somehow unaesthetic notation $\overline{\underline{E}G}$ when speaking of an $E\mathcal{Z}$–compactification of a classifying space for proper actions of G.

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1 Complexes of spaces and their topology

In this section, we study a class of spaces with a projection to a given simplicial complex X, called *complexes of spaces over* X, which are high-dimensional analogues of graphs of spaces studied in the context of actions on trees (see Scott and Wall [30]). This notion of complexes of spaces is close to the one studied by Corson [13] and Haefliger [24].

1.1 A few geometric facts about M_κ–simplicial complexes

Since the present article deals with nonproper actions of a group, the simplicial complex on which it acts is generally non locally finite. In [8], Bridson defined a class of spaces that is suitable for a geometric approach.

Definition 1.1 (M_κ–simplicial complexes with $\kappa \leq 0$ [8]) Let $\kappa \leq 0$. A simplicial complex X is called a M_κ–simplicial complex if it satisfies the following two conditions:

- Each simplex of X is modeled after a geodesic simplex in some M_κ^n, where M_κ^n is the simply connected n–Riemannian manifold of constant curvature κ.
- If σ and σ' are two simplices of X sharing a common face τ, the identity map from $\tau \subset \sigma$ to $\tau \subset \sigma'$ is an isometry.

To such a M_κ–complex X is associated a canonical simplicial metric.

Theorem 1.2 (Bridson [8]) *If X is a M_κ–simplicial complex, $\kappa \leq 0$, with finitely many isometry types of simplices, the simplicial metric is complete and geodesic.*

From now on, every simplicial complex will implicitly be given the structure of a M_κ–complex, $\kappa \leq 0$, with finitely many isometry types of simplices.

1.2 Complexes of spaces

Definition 1.3 A *complex of spaces $C(\mathcal{X})$ over X* consists of the following data:

- For every simplex σ of X, a pointed CW–complex C_σ, called a *fibre*.
- For every pair of simplices $\sigma \subset \sigma'$, an embedding $\phi_{\sigma',\sigma}\colon C_{\sigma'} \hookrightarrow C_\sigma$, called a *gluing map*, such that for every $\sigma \subset \sigma' \subset \sigma''$, we have $\phi_{\sigma,\sigma''} = \phi_{\sigma,\sigma'} \circ \phi_{\sigma',\sigma''}$.

Definition 1.4 (Realisation of a complex of spaces) Let $C(\mathcal{X})$ be a complex of spaces over X. The *realisation* of $C(\mathcal{X})$ is the quotient space

$$|C(\mathcal{X})| = \Big(\coprod_{\sigma \in S(X)} \sigma \times C_\sigma \Big) \Big/ \simeq,$$

where

$$(i_{\sigma,\sigma'}(x), s) \simeq (x, \phi_{\sigma,\sigma'}(s)) \quad \text{for } x \in \sigma \subset \sigma' \text{ and } s \in C_{\sigma'},$$

where $i_{\sigma,\sigma'}\colon \sigma \hookrightarrow \sigma'$ is the natural inclusion. The class in $|C(\mathcal{X})|$ of a point (x,s) will be denoted $[x,s]$.

Definition 1.5 A complex of spaces $C(\mathcal{X})$ will be called *locally finite* if for every simplex σ of X and every point $x \in C_\sigma$, there exists an open set U of C_σ containing x and such that there are only finitely many simplices σ' in the open star of σ satisfying $U \cap \mathrm{Im}(\phi_{\sigma,\sigma'}) \neq \varnothing$.

Proposition 1.6 *Let $C(\mathcal{X})$ be a locally finite complex of spaces. Then $|C(\mathcal{X})|$ admits a natural locally finite CW–complex structure, for which the $\sigma \times C_\sigma$ embed as subcomplexes.* □

1.3 Topology of complexes of spaces with contractible fibres

Definition 1.7 (Quotient complex of spaces) Let $C(\mathcal{X})$ be a complex of spaces over X and $Y \subset X$ a subcomplex. We denote $C_Y(\mathcal{X})$ the complex of spaces over X defined as follows:

- $(C_Y)_\sigma = C_\sigma$ if $\sigma \not\subseteq Y$, $(C_Y)_\sigma$ is the basepoint of C_σ otherwise.
- $\phi^Y_{\sigma,\sigma'}$ is the composition $(C_Y)_{\sigma'} \hookrightarrow C_{\sigma'} \xrightarrow{\phi_{\sigma,\sigma'}} C_\sigma \twoheadrightarrow (C_Y)_\sigma$.

We denote by $p_Y\colon |C(\mathcal{X})| \to |C_Y(\mathcal{X})|$ the canonical projection, and simply p for $p_X\colon |C(\mathcal{X})| \to X$. In the same way, if $Y \subset Y'$ are subcomplexes of X, we denote by $p_{Y,Y'}\colon |C_Y(\mathcal{X})| \to |C_{Y'}(\mathcal{X})|$ the canonical projection.

Lemma 1.8 *Let $C(\mathcal{X})$ be a locally finite complex of spaces over X with contractible fibres, and let Y be a finite subcomplex of X. Then $p_Y\colon |C(\mathcal{X})| \to |C_Y(\mathcal{X})|$ is a homotopy equivalence.*

Proof It suffices to prove the result for Y consisting of a single closed simplex σ. We have the following commutative diagram:

$$\begin{tikzcd}
|C(\mathcal{X})| \arrow[r, "p_Y"] \arrow[d, "\simeq"'] & |C_Y(\mathcal{X})| \arrow[d, "\simeq"] \\
|C(\mathcal{X})|/(\sigma \times C_\sigma) \arrow[r, "="'] & |C_Y(\mathcal{X})|/(\sigma \times \star)
\end{tikzcd}$$

The vertical arrows are homotopy equivalences, since we are quotienting by contractible subcomplexes, hence the result. □

Theorem 1.9 (Dowker [17]) *The (continuous) identity map $X \to X$ from X with its CW topology to X with its simplicial metric is a homotopy equivalence.* □

Proposition 1.10 *Let $C(\mathcal{X})$ be a locally finite complex of space over X with contractible fibres. If X has finitely many types of simplices and is contractible, then $|C(\mathcal{X})|$ is contractible.*

Proof By the previous theorem, it is enough to show that the projection $p\colon |C(\mathcal{X})| \to X$ induces isomorphisms on homotopy groups, when X is endowed with its CW topology. For that topology, a continuous map from a compact space to X has its image contained in a finite subcomplex, to which Lemma 1.8 applies. □

2 Construction of EG and ∂G for developable complexes of groups

In this section, given a developable simple complex of groups $G(\mathcal{Y})$ over a finite simplicial complex Y, we build a classifying space for its fundamental group.

Notation We choose once and for all a non-positively curved complex of groups $G(\mathcal{Y})$ over a finite simplicial complex endowed with a M_κ–structure, $\kappa \leq 0$ (where $\mathcal{Y}$ is the small category without loops whose vertices correspond to simplices of Y and whose oriented edges come from inclusion of simplices of Y). Recall that a complex of groups consists of the data $(G_\sigma, \psi_a, g_{a,b})$ of local groups (G_σ), local maps (ψ_a) and twisting elements $(g_{a,b})$. For the background on complexes of groups, we refer the reader to Bridson and Haefliger [9]. We fix a maximal tree T in the 1–skeleton of the first barycentric subdivision of Y, which allows us to define the fundamental group $G = \pi_1(G(\mathcal{Y}), T)$ and the canonical morphism $\iota_T\colon G(\mathcal{Y}) \to G$ given by the collection of injections $G_\sigma \to G$ [9, p. 553]. Finally, we define X as the universal covering of $G(\mathcal{Y})$ associated to ι_T. The simplicial complex X naturally inherits a M_κ–structure with finitely many isometry types of simplices from that of Y and the simplicial metric d on X makes it a complete geodesic metric space by work of Bridson [8]. This space is CAT(0) by the curvature assumption on $G(\mathcal{Y})$ [9, p. 562].

2.1 Construction of EG and $\partial_{\mathrm{Stab}} G$

Definition 2.1 ((Cofinite and finite-dimensional) classifying space for proper actions) Let Γ be a countable discrete group. A *cofinite and finite-dimensional classifying space for proper actions of* Γ (or briefly a *classifying space for* Γ) is a contractible CW–complex $E\Gamma$ with a proper cocompact and cellular action of Γ, and such that:

- For every finite subgroup H of Γ, the fixed point set $E\Gamma^H$ is nonempty and contractible.
- Every infinite subgroup H of Γ has an empty fixed point set.

Definition 2.2 A *complex of spaces $D(\mathcal{Y})$ compatible with the complex of groups* $G(\mathcal{Y})$ consists of the following:

- For every vertex σ of $\mathcal{Y}$, a space D_σ that is a model of classifying space for proper actions EG_σ of the local group G_σ.
- For every edge a of $\mathcal{Y}$ with initial vertex $i(a)$ and terminal vertex $t(a)$, an embedding $\phi_a\colon EG_{i(a)} \hookrightarrow EG_{t(a)}$ that is $G_{i(a)}$–equivariant, that is, for every $g \in G_{i(a)}$ and every $x \in EG_{i(a)}$, we have
$$\phi_a(g.x) = \psi_a(g).\phi_a(x),$$
and such that for every pair (a,b) of composable edges of $\mathcal{Y}$, we have:
$$g_{a,b} \circ \phi_{ab} = \phi_a\phi_b.$$

Note that a complex of spaces compatible with the complex of groups $G(\mathcal{Y})$ is *not* a complex of spaces over Y when the twisting elements $g_{a,b}$ are not trivial. Nonetheless, this data is used to build a complex of spaces over X, as explained in the following definition.

Definition 2.3 We define the space
$$Cl_{D(\mathcal{Y})} = \Big(G \times \coprod_{\sigma \in V(\mathcal{Y})} (\sigma \times EG_\sigma)\Big)\Big/ \simeq,$$
where $(g, i_{\sigma,\sigma'}(x), s) \simeq (g\iota_T([\sigma\sigma'])^{-1}, x, \phi_{(\sigma\sigma')}(s))$ if $[\sigma\sigma'] \in \mathrm{Edges}(\mathcal{Y})$, $s \in EG_\sigma$, $x \in \sigma'$, $g \in G$, $(gg', x, s) \simeq (g, x, g's)$ if $x \in \sigma$, $s \in EG_\sigma$, $g' \in G_\sigma$, $g \in G$.

The canonical projection $G \times \coprod_{\sigma \in V(\mathcal{Y})}(\sigma \times EG_\sigma) \to G \times \coprod_{\sigma\in V(\mathcal{Y})} \sigma$ yields a map
$$p\colon Cl_{D(\mathcal{Y})} \to X.$$

The action of G on $G \times \coprod_{\sigma \in V(\mathcal{Y})}(\sigma \times EG_\sigma)$ on the first factor by left multiplication yields an action of G on $Cl_{D(\mathcal{Y})}$, making the projection $p\colon Cl_{D(\mathcal{Y})} \to X$ a G–equivariant map.

Note that $Cl_{D(\mathcal{Y})}$ can be seen as a complex of spaces over X, the fibre of a simplex $[g,\sigma]$ being the classifying space EG_σ. Indeed, for en edge $[g,a]$ of the first barycentric subdivision of X, the gluing map $\phi_{[g\iota_T(a)^{-1}, i(a)],[g,t(a)]}\colon EG_{i(a)} \to EG_{t(a)}$ is defined as $\phi_{i(a),t(a)}$.

For every simplex σ of X, we denote by EG_σ the fibre over σ of that complex of space. For simplices σ, σ' of X such that $\sigma' \subset \sigma$, we denote by $\phi_{\sigma',\sigma}\colon EG_\sigma \to EG_{\sigma'}$ the associated gluing map.

Theorem 2.4 *The space $Cl_{D(\mathcal{Y})}$ is a classifying space for proper actions of G.*

From now on, we denote this classifying space by $\underline{E}G$.

Proof *Local finiteness* Let σ be a simplex of X and U be an open set of $\underline{E}G_\sigma$ that is relatively compact. It is enough to prove that for any injective sequence (σ_n) of simplices of X containing σ there are only finitely many n such that the image of ϕ_{σ,σ_n} meets U. By cocompactness of the action, we can assume that all the σ_n are in the same G–orbit, and let σ' be a simplex in that orbit. Since the action of G_σ on $\underline{E}G_\sigma$ is proper, it follows that for every compact subset K of $\underline{E}G_\sigma$, only finitely many distinct cosets $g\underline{E}G_{\sigma'}$ in $\underline{E}G_\sigma$ can meet K, hence the result.

Contractibility Since the complex of spaces associated to $Cl_{D(\mathcal{Y})}$ is locally finite and has contractible fibres, $Cl_{D(\mathcal{Y})}$ is contractible by Proposition 1.10.

Cocompact action For every simplex σ of Y, we choose a compact fundamental domain K_σ for the action of G_σ on $D_\sigma = \underline{E}G_\sigma$. Now the image in $Cl_{D(\mathcal{Y})}$ of $\bigcup_{\sigma \in S(Y)} \sigma \times K_\sigma$ clearly defines a compact subset of $Cl_{D(\mathcal{Y})}$ meeting every G–orbit.

Proper action As $Cl_{D(\mathcal{Y})}$ is a locally finite CW–complex, hence a locally compact space, it is enough to show that every finite subcomplex intersects only finitely many of its G–translates.

Let us first show that for every cell τ of $Cl_{D(\mathcal{Y})}$, there are only finitely many $g \in G$ such that $g\tau = \tau$. Indeed, let $g \in G$ such that $g\tau = \tau$. The canonical projection $Cl_{D(\mathcal{Y})} \to X$ is G–equivariant and sends a cell of $Cl_{D(\mathcal{Y})}$ on a simplex of X, thus g also stabilises the simplex $p(\tau) \subset X$. Since G acts without inversion on X, g pointwise stabilises the vertices of $p(\tau)$. Let s be such a vertex. Then $g \in G_s$ and, by construction of $Cl_{D(\mathcal{Y})}$, the restriction to G_s of the action of G on $Cl_{D(\mathcal{Y})}$ is just the action of G_s on $\underline{E}G_s$. Thus, by definition of a classifying space for proper actions, this implies that there are only finitely many possibilities for g.

Now, let F be a finite subcomplex of $Cl_{D(\mathcal{Y})}$ and $S(F)$ the (finite) set of pairs (τ, τ') of cells of F that are in the same G–orbit. The set $\{g \in G \mid gF \cap F \neq \varnothing\}$ is contained in $\bigcup_{(\tau,\tau') \in S(F)} \{g \in G \mid g\tau = \tau'\}$, and $\{g \in G \mid g\tau = \tau'\}$ has the same cardinality as the set $\{g \in G \mid g\tau = \tau\}$, which is finite by the previous argument.

Fixed sets Let H be a finite subgroup of G. As G acts without inversion on the CAT(0) complex X, the subset X^H is a nonempty convex subcomplex of X. Furthermore, for every simplex σ of X^H, the subcomplex $(\underline{E}G_\sigma)^H$ of $\underline{E}G_\sigma$ is nonempty and contractible. Thus $Cl^H_{D(\mathcal{Y})}$ is the realisation of a complex of spaces over the contractible complex X^H and with contractible fibres, hence it is nonempty and contractible by Proposition 1.10.

If H is an infinite subgroup of G, we prove by contradiction that $Cl^H_{D(\mathcal{Y})}$ is empty. If this was not the case, there would exist a simplex σ fixed pointwise under H and a point x of EG_σ that is fixed under $H \subset G_\sigma$. But this is absurd as $(EG_\sigma)^H = \varnothing$ by assumption. □

We now turn to the construction of a boundary of G. As introduced by Farrell and Lafont [19], the definition of an $E\mathcal{Z}$–structure only applies to torsion-free groups. Here we use a notion of $\mathcal{Z}$–structure suitable for groups with torsion, which was introduced by Dranishnikov [18].

Definition 2.5 ($\mathcal{Z}$–structures, $E\mathcal{Z}$–structures) Let Γ be a discrete group. A *$\mathcal{Z}$–structure* for Γ is a pair (Y, Z) of spaces such that:

- Y is a Euclidean retract, that is, a compact, contractible and locally contractible space with finite covering dimension.
- $Y \setminus Z$ is a classifying space for proper actions of Γ.
- Z is a $\mathcal{Z}$–set in Y, that is, Z is a closed subspace of Y such that for every open set U of Y, the inclusion $U \setminus Z \hookrightarrow U$ is a homotopy equivalence.
- Compact sets *fade at infinity*, that is, for every compact set K of $Y \setminus Z$, every point $z \in Z$ and every neighbourhood U of z in Z, there exists a subneighbourhood $V \subset U$ with the property that if a Γ–translate of K intersects V, then it is contained in U.

The pair (Y, Z) is called an *$E\mathcal{Z}$–structure* if in addition we have:

- The action of Γ on $Y \setminus Z$ continuously extends to Y.

Definition 2.6 We say that a complex of spaces $D(\mathcal{Y})$ compatible with a complex of groups $G(\mathcal{Y})$ extends to an *$E\mathcal{Z}$–complex of spaces* if it satisfies the following extra conditions:

- Each fibre $D_\sigma = EG_\sigma$ is endowed with an $E\mathcal{Z}$–structure $(\overline{EG}_\sigma, \partial G_\sigma)$.
- The equivariant embeddings (ϕ_a) extend to equivariant embeddings

  $$\phi_a\colon \overline{EG}_{i(a)} \to \overline{EG}_{t(a)},$$

  such that for every pair (a, b) of composable edges of $\mathcal{Y}$, we have:

  $$g_{a,b} \circ \phi_{ab} = \phi_a \phi_b.$$

Definition 2.7 We define the space

$$\Omega(\mathcal{Y}) = \Big(G \times \coprod_{\sigma \in V(\mathcal{Y})} (\{\sigma\} \times \partial G_\sigma)\Big) \Big/ \simeq,$$

where $(gg', (x,s)) \simeq (g, (x, g's))$ if $x \in \sigma, s \in EG_\sigma, g' \in G_\sigma, g \in G$. It should be noted here that $\{\sigma\}$ denotes a point labeled by σ and not the simplex itself. The set $\Omega(\mathcal{Y})$ comes with a natural projection to the set of simplices of X. If σ is a simplex of X, we denote by ∂G_σ the preimage of $\{\sigma\}$ under that projection. We now define

$$\partial_{\mathrm{Stab}} G = \Omega(\mathcal{Y}) / \sim,$$

where $\sim$ is the equivalence relation generated by the following identifications:

$$\big[g, \{\sigma\}, \xi\big] \sim \big[gF([\sigma\sigma'])^{-1}, \{\sigma'\}, \phi_{[\sigma\sigma']}(\xi)\big]$$

if $g \in G, [\sigma\sigma'] \in \mathrm{Edges}(\mathcal{Y})$ and $\xi \in \partial G_\sigma$.

The action of G on $G \times \coprod_{\sigma \in V(\mathcal{Y})} (\{\sigma\} \times \partial G_\sigma)$ by left multiplication on the first factor yields an action of G on $\Omega(\mathcal{Y})$ and on $\partial_{\mathrm{Stab}} G$.

Definition 2.8 We define the spaces $\partial G = \partial_{\mathrm{Stab}} G \cup \partial X$ and $\overline{EG} = EG \cup \partial G$.

Our aim is to endow $\overline{EG}$ with a topology that makes $(\overline{EG}, \partial G)$ an $E\mathcal{Z}$–structure for G.

Notation Since the $\phi_{\sigma,\sigma'}$ are embeddings, we will identify $\phi_{\sigma,\sigma'}(\overline{EG}_{\sigma'})$ with $\overline{EG}_{\sigma'}$. For instance, if U is an open subset of EG_σ we will simply write "we have $EG_{\sigma'} \subset U$ in EG_σ" instead of "we have $\phi_{\sigma,\sigma'}(EG_{\sigma'}) \subset U$ in EG_σ".

From now on, we assume that there is a complex of spaces $D(\mathcal{Y})$ that extends to an $E\mathcal{Z}$–complex of spaces compatible with the complex of groups $G(\mathcal{Y})$.

2.2 Further properties of $E\mathcal{Z}$–complexes of spaces

In this paragraph, we define additional properties of $E\mathcal{Z}$–complexes of spaces, which will enable us to study the properties of the equivalence relation $\sim$ previously defined.

2.2.1 The limit set property Recall that for a discrete group Γ together with an $E\mathcal{Z}$–structure $(\overline{E\Gamma}, \partial\Gamma)$ and a subgroup H, the *limit set* ΛH of H in $\partial\Gamma$ is the set $\overline{Hx} \cap \partial\Gamma$, where x is an arbitrary point of $E\Gamma$.

Definition 2.9 (Limit set property for an $E\mathcal{Z}$–complex of spaces) We say that the $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ compatible with the complex of groups $G(\mathcal{Y})$ satisfies the *limit set property* if the following conditions are satisfied:

- For every pair of simplices $\sigma \subset \sigma'$ of Y, the map $\phi_{\sigma,\sigma'}$ is an equivariant homeomorphism from $\partial G_{\sigma'}$ to the limit set $\Lambda G_{\sigma'} \subset \partial G_{\sigma}$.
- For every simplex σ of Y, and every pair of subgroups H_1, H_2 in the family $\mathcal{F}_\sigma = \{\bigcap_{i=1}^n g_i G_{\sigma_i} g_i^{-1} \mid g_1, \dots, g_n \in G_\sigma, \sigma_1, \dots, \sigma_n \subset \mathrm{st}(\sigma), n \in \mathbb{N}\}$, we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2)$.

Remark (i) *Let Γ be a hyperbolic group, and H a subgroup. Then H is quasiconvex if and only if its limit set in $\partial\Gamma$ is equivariantly homeomorphic to ∂H, by a result of Bowditch [6].*

(ii) *Let Γ be a hyperbolic group and $\partial\Gamma$ its Gromov boundary. Let H_1 and H_2 be two quasiconvex subgroups of Γ. Then $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2)$ by a result of Gromov [23].*

2.2.2 The finite height property Recall that, for Γ a discrete group and H a subgroup, the *height* of H is the supremum of the set of integers $n \in \mathbb{N}$ such that there exist distinct cosets $\gamma_1 H, \dots, \gamma_n H \in G/H$ such that the intersection $\gamma_1 H \gamma_1^{-1} \cap \cdots \cap \gamma_n H \gamma_n^{-1}$ is infinite. If such a supremum is infinite, we say that H is of *infinite height* in Γ. Otherwise, H is said to be of *finite height* in Γ. A quasiconvex subgroup of a hyperbolic group is of finite height, by a result of Gitik, Mitra, Rips and Sageev [21].

Definition 2.10 (Finite height property) We say that the $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ compatible with the complex of groups $G(\mathcal{Y})$ satisfies the *finite height property* if for every pair of simplices $\sigma \subset \sigma'$ of Y, $G_{\sigma'}$ is of finite height in G_σ.

3 Geodesics in M_κ–complexes

In this section, we study the geometry of the set of geodesics of an M_κ–complex. Recall that X is assumed to be a M_κ–complex, $\kappa \leq 0$, with finitely many isometry types of simplices.

3.1 The finiteness lemma

Definition 3.1 For subsets K, L of X, we define $\mathrm{Geod}(K, L)$ as the set of points lying on a geodesic segment from a point of K to a point of L.

Definition 3.2 (Simplicial neighbourhood) Let K be a subcomplex of X. The subcomplex spanned by the closed simplices that meet K is called the *closed simplicial neighbourhood* of K, and denoted $\overline{\mathrm{N}}(K)$. The union of the open simplices whose closure meets K is called the *open simplicial neighbourhood* of K, and denoted $\mathrm{N}(K)$.

We recall the following proposition of Bridson, which follows from the claim contained in the proof of Theorem 1.11 of [8].

Proposition 3.3 (Containment lemma, Bridson [8]) *For every n there exists a constant k such that for every finite subcomplex $K \subset X$ spanned by at most n simplices, any geodesic path contained in the open simplicial neighbourhood of K meets at most k simplices.* □

We also recall this useful related result, which follows from [8, Theorem 1.11].

Corollary 3.4 (Bridson [8]) *For every n there exists a constant k such that every geodesic segment of length at most n meets at most k simplices.* □

Lemma 3.5 (Finiteness lemma) *Let X be as before. For subcomplexes $K, K' \subset X$, $\mathrm{Geod}(K, K')$ meets only finitely many open simplices.*

Proof It is enough to prove the result when K and K' consist of two closed simplices σ and σ'. For every $x \in \sigma$ and every $x' \in \sigma'$, we consider the sequence of open simplices $\sigma_1, \dots, \sigma_n$ met by the geodesic segment $[x, x']$ and set $C_{x,x'} = \sigma \cup \sigma_1 \cup \dots \cup \sigma_n \cup \sigma'$. Note that by Corollary 3.4 there is a uniform k bound on the number of simplices contained in $C_{x,x'}$. Since there are only finitely many isometry types of simplices in X, there are, up to simplicial isometry fixing pointwise σ and σ', finitely many subcomplexes of the form $C_{x,x'}$. Following Bridson, we call such an equivalence class of subcomplexes a *model* (see Bridson and Haefliger [9, proof of I.7.57]).

We now claim that for every $x, y \in \sigma$ and every $x', y' \in \sigma'$ such that $C_{x,x'}$ and $C_{y,y'}$ are in the same model, we have $C_{x,x'} = C_{y,y'}$. Indeed, choose a simplicial isometry $\phi\colon C_{x,x'} \to C_{y,y'}$ that fixes pointwise σ and σ'. Then ϕ sends the geodesic segment $[x, x'] \subset C_{x,x'}$ to a simplicial path of the same length between $\phi(x) = x$ and $\phi(x') = x'$. As X is CAT(0), geodesic segments are unique, hence ϕ pointwise fixes $[x, x']$. We thus have $[x, x'] = \phi([x, x']) \subset C_{y,y'}$, hence $C_{x,x'} \subset C_{y,y'}$. The same reasoning applied to the geodesic segment $[y, y']$ yields $C_{y,y'} \subset C_{x,x'}$, hence $C_{x,x'} = C_{y,y'}$.

We have

$$\mathrm{Geod}(\sigma, \sigma') \subset \bigcup_{x \in \sigma, x' \in \sigma'} C_{x,x'}$$

and the previous discussion shows that this is a finite union, which concludes the proof. □

3.2 Paths of simplices of uniformly bounded length

Definition 3.6 A *path of simplices* is a sequence of open simplices $\sigma_1, \dots, \sigma_n$ such that $\overline{\sigma_i} \subset \overline{\sigma_{i+1}}$ or $\overline{\sigma_{i+1}} \subset \overline{\sigma_i}$ for every $i = 1, \dots, n-1$. Equivalently, it is a finite path in the first barycentric subdivision of X. The integer n is called the *length* of the path of simplices.

Up to rescaling the metric, we also make the following assumption:

From now on, we will assume that the distance from any simplex to the boundary of its (closed) simplicial neighbourhood is at least 1.

Here we prove the following lemma:

Lemma 3.7 (Short paths of simplices) *For every $n \geq 1$, there exists $m \geq 1$ such that the following holds: Let K be a convex subcomplex of X and K' a connected subcomplex of X, both containing at most n simplices. Let $x, y \in K$ and $x', y' \in K'$ and assume that there exists a path in K' between x' and y' that does not meet K. Let τ, τ' be two simplices of $\mathrm{N}(K) \setminus K$ such that the geodesic segment $[x, x']$ (resp. $[y, y']$) meets the interior of τ (resp. τ'). Then there exists a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m between τ and τ'.*

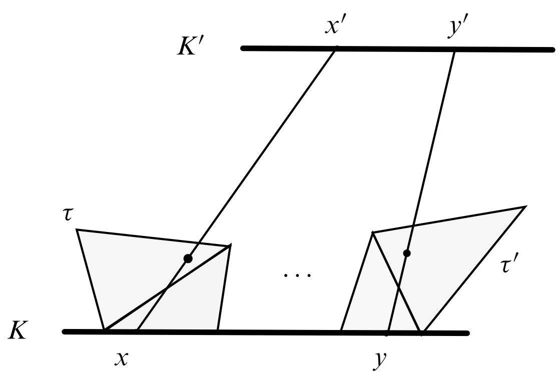

Figure 1

Definition 3.8 (Bridson and Haefliger [9, I.7.8]) For $x \in X$, let

$$\eta(x) = \inf\{d(x, \sigma) \mid \sigma \subset \overline{\mathrm{st}}(\sigma_x), x \notin \sigma\}.$$

The constant is such that for every $y \in B(x, \eta(x))$, we have $\sigma_x \subset \sigma_y$.

The following lemma is a controlled version of [9, Lemma I.7.54].

Lemma 3.9 *There exist constants $\eta_0 > \varepsilon_0 > 0$ such that:*

- *For every simplex σ of X, the $2\eta_0$–neighbourhood of σ is contained in the open simplicial neighbourhood of σ.*
- *For every point $x \in X$, there exists $y \in B(x, \eta_0)$ such that $B(x, \varepsilon_0) \subset B(y, \eta(y))$.*

Proof For a simplex σ of X, let

$$\eta(\sigma) = \inf\{d(\sigma, \tau) \mid \tau \subset \overline{\mathrm{N}}(\sigma),\ \sigma \cap \tau = \varnothing\}.$$

The above set of distances is finite since X has only finitely many isometry types of simplices, thus $\eta(\sigma) > 0$. For the same reason, we can define $\eta_0 = \frac{1}{2} \cdot \min \eta(\sigma) > 0$, where the minimum is taken over all the simplices of X.

Now that η_0 is defined, we construct constants $\eta_1, \dots, \eta_D$ by induction, where D is the maximal dimension of a simplex of X, as well as subsets $T_0, \dots, T_D$ of X, such that each T_k is an open neighbourhood of the k–skeleton $X^{(k)}$ of X.

Let

$$T_0 = \bigcup_{v \in V(X)} B(v, \eta_0),$$

where η_0 is as above. Suppose that $\eta_0, \dots, \eta_k$ and $T_0, \dots, T_k$ are defined. For each simplex $\sigma \subset X$ of dimension $k+1$, the function η (as defined in Definition 3.8) is continuous on the compact set $\sigma \setminus T_k$ and does not vanish, hence is bounded below by a constant $\eta_{k+1}(\sigma) > 0$. As X has finitely many isometry types of simplices, we define $\eta_{k+1} = \frac{1}{2} \cdot \min \eta_{k+1}(\sigma) > 0$, where the minimum is taken over all simplices of dimension $k+1$. We can further assume that $\eta_{k+1} \leq \eta_k$. Let

$$T_{k+1} = T_k \cup \Bigg(\bigcup_{\substack{\sigma \subset X, \\ \dim \sigma = k+1}} \ \bigcup_{x \in \sigma \setminus T_k} B(x, \eta_{k+1}) \Bigg).$$

Finally, let $\varepsilon_0 = \eta_D$. We have $T_0 \subset \cdots \subset T_D = X$. Let $x \in X$. There exists a unique k such that $x \in T_k \setminus T_{k-1}$. For such a k, there exists $y \in X^{(k)} \setminus T_{k-1}$ with $d(x, y) \leq \eta_k$ (in particular $d(x, y) \leq \eta_0$). As $\varepsilon_0 \leq \eta_k$, we get

$$B(x, \varepsilon_0) \subset B(x, \eta_k) \subset B(y, 2\eta_k) \subset B(y, \eta_k(\sigma_y)) \subset B(y, \eta(y)),$$

which concludes the proof. □

Proof of Lemma 3.7 First notice that since X has only finitely many isometry types of simplices, there exists a constant l, which depends only on n and X, points $x = x_0, \dots, x_l = y$ in K and $x' = x'_0, \dots, x'_l = y'$ in K' such that for every k, $d(x_k, x_{k+1}) < \varepsilon_0$, $d(x'_k, x'_{k+1}) < \varepsilon_0$, x_k, x_{k+1} belong to the same simplex of K, and x'_k, x'_{k+1} belong to the same simplex of K'. For every $k = 1 \dots, l-1$, let τ_k be a simplex of $\mathrm{N}(K) \setminus K$ whose interior meets $[x_k, x'_k]$. In order to prove Lemma 3.7, it is thus enough to consider the case where $d(x, y) < \varepsilon_0$, $d(x', y') < \varepsilon_0$, x, y belong to the same simplex σ of K, and x', y' belong to the same simplex σ' of K'. We treat two cases separately.

Case 1 Suppose that the geodesic segments $[x, x']$ and $[y, y']$ are both contained in the open η_0–neighbourhood of K. Recall that by definition of η_0, this implies that they are contained in the open simplicial neighbourhood of K. The geodesic segment $[x, x']$ yields a geodesic segment, contained in $\mathrm{N}(K) \setminus K$ by convexity of K, between a point in the interior of τ and x'. By Proposition 3.3, there exists a constant m_1 (which depends only on X and n) such that there exists a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m_1 between τ and σ'. Reasoning similarly for $[y, y']$, we get a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m_1 between τ' and σ'. We thus get a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most $2m_1$ between τ and τ'.

Case 2 Suppose that the geodesic segment $[x, x']$ is not contained in the η_0–neighbourhood of K. We then choose a point u on that geodesic segment that belongs to $B(K, 2\eta_0) \setminus B(K, \eta_0)$ (such a subset is contained in $\mathrm{N}(K)$ by definition of η_0). By Lemma 3.9, we can choose $z \in X \setminus K$ such that $B(u, \varepsilon_0) \subset B(z, \eta(z))$. Since $d(x, y) < \varepsilon_0$ and $d(x', y') < \varepsilon_0$, the CAT(0) geometry of X implies that $[y, y']$ meets the ball $B(u, \varepsilon_0) \subset B(z, \eta(z))$ at a point v. By definition of $\eta(z)$, we thus have $\sigma_z \subset \sigma_u$ and $\sigma_z \subset \sigma_v$, which yields the path of simplices $\sigma_u, \sigma_z, \sigma_v$ in $\mathrm{N}(K) \setminus K$ between σ_u and σ_v. Now the geodesic segment $[x, x']$ (resp. $[y, y']$) yields a path of simplices in $\mathrm{N}(K) \setminus K$ (by convexity of K) of length at most m_1 between τ and σ_u (resp. between τ' and σ_v). We thus get a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most $2m_1 + 1$ between τ and τ'. □

4 The geometry of the action

In this section, we gather a few geometric tools that will be used to construct a topology on $\overline{EG} = EG \cup \partial G$. From now on, we assume that the $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ compatible with $G(\mathcal{Y})$ satisfies the limit set property Definition 2.9 and the finite height property Definition 2.10. We further assume that the action of G on X is acylindrical and we fix an *acylindricity constant* $A > 0$, that is, a constant such that every subcomplex of X of diameter at least A has a finite pointwise stabiliser.

4.1 Domains and their geometry

In this section, we study the topological properties of the identifications made to build the boundary of G.

Definition 4.1 Let $\xi \in \partial_{\mathrm{Stab}} G$. We define $D(\xi)$, called the *domain* of ξ, as the subcomplex of X spanned by simplices σ such that $\xi \in \partial G_\sigma$. We denote by $V(\xi)$ the set of vertices of $D(\xi)$.

The aim of this paragraph is to prove the following:

Proposition 4.2 *Domains are finite convex subcomplexes of X whose diameters are uniformly bounded above.*

The containment lemma Proposition 3.3 and Proposition 4.2 imply the following:

Corollary 4.3 *For every $\xi \in \partial_{\mathrm{Stab}} G$, there exists an integer d_ξ such that $D(\xi)$ has at most d_ξ simplices, and such that a geodesic segment in the open simplicial neighbourhood of $D(\xi)$ meets at most d_ξ open simplices. Furthermore, there exists an upper bound $d_{\max}$ on the set of integers $d_\xi, \xi \in \partial_{\mathrm{Stab}} G$.*

Recall that $\Omega(\mathcal{Y})$ is defined in Definition 2.7 as the disjoint union of the ∂G_v $(v \in V(X))$ and that $\partial_{\mathrm{Stab}} G$ is a quotient of $\Omega(\mathcal{Y})$ defined by making identifications along edges of X. We start by proving the following proposition:

Proposition 4.4 *Let v be a vertex of X. Then the projection $\pi\colon \partial G_v \to \partial G$ is injective.*

Definition 4.5 Let $\xi \in \partial_{\mathrm{Stab}} G$. A *$\xi$–path* is the data $\{(v_i)_{0 \le i \le n}, (\xi_i)_{0 \le i \le n}, (x_i)_{1 \le i \le n}\}$ consisting of:

- a sequence $v_0, \dots, v_n$ of adjacent vertices of X,
- a sequence $\xi_0, \dots, \xi_n$ of elements of $\Omega(\mathcal{Y})$, such that $\xi_i \in \partial G_{v_i}$ for every i, and such that each ξ_i is in the equivalence class ξ,
- a sequence $x_1, \dots, x_n$ of elements of $\Omega(\mathcal{Y})$, such that $x_i \in \partial G_{[v_{i-1}, v_i]}$ for every i, and such that $\phi_{v_{i-1},[v_{i-1},v_i]}(x_i) = \xi_{i-1}$ (resp. $\phi_{v_i,[v_{i-1},v_i]}(x_i) = \xi_i$).

To lighten notation, a ξ–path will sometimes just be denoted $[v_0, \dots, v_n]_\xi$. The path in the 1–skeleton of X induced by a ξ–path is called the *support* of $[v_0, \dots, v_n]_\xi$, and denoted $[v_0, \dots, v_n]$. If $v_0 = v_n$, a ξ–path will rather be called a *ξ–loop*.

Lemma 4.6 *Let $v_0, \dots, v_n$ be vertices of X, $H = \bigcap_{0 \le i \le n} G_{v_i}$, and K be a connected subcomplex of X pointwise fixed by H. Suppose that H is infinite, and let $\xi \in \partial_{\mathrm{Stab}} G$ such that, in G_{v_0}, we have*

$$\xi \in \Lambda H \subset \partial G_{v_0}.$$

Then $\xi \in \Lambda H \subset \partial G_\sigma$ for every simplex σ of K, hence $K \subset D(\xi)$.

Proof As K is connected, it is enough to prove that for every path of simplices $\sigma_0 = v_0, \dots, \sigma_d$ contained in K, we have $\xi \in \partial H \subset \partial G_{\sigma_d}$. Now this follows from an easy induction on the number of simplices contained in such a path. □

Lemma 4.7 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $[v_0, \dots, v_n]_\xi$ a ξ–path and $H = \bigcap_{0 \le i \le n} G_{v_i}$. Then*

- *H is infinite,*
- *$\xi \in \Lambda H \subset \partial G_{v_i}$ for every $i = 0, \dots, n$.*

Proof We show the result by induction on $n \ge 1$. The result is immediate for $n = 1$ by definition of $\sim$. Suppose the result true up to rank n and let $\xi \in \partial_{\mathrm{Stab}} G$ together with a ξ–path $[v_0, \dots, v_{n+1}]_\xi$. By restriction, we get a ξ–path $[v_0, \dots, v_n]_\xi$ for which the result is true by the induction hypothesis. Thus $\xi \in \Lambda(\bigcap_{0 \le i \le n} G_{v_i}) \subset \partial G_{v_n}$. But since ξ is also in $\partial G_{[v_n, v_{n+1}]} = \Lambda G_{[v_n, v_{n+1}]}$ by assumption, we get

$$\xi \in \Lambda\Big(\bigcap_{0 \le i \le n} G_{v_i}\Big) \cap \Lambda G_{[v_n, v_{n+1}]} = \Lambda\Big(\bigcap_{0 \le i \le n+1} G_{v_i}\Big) \subset \partial G_{v_n},$$

the previous equality following from the limit set property Definition 2.9. Now, by Lemma 4.6, we get $\xi \in \Lambda\big(\bigcap_{0 \le i \le n+1} G_{v_i}\big) \subset \partial G_{v_i}$ for every $i = 0, \dots, n+1$, which concludes the induction. □

Proof of Proposition 4.4 Let ξ, ξ' be two elements of $\Omega(\mathcal{Y})$ in the image of ∂G_v that are equivalent for the equivalence relation $\sim$. Then there exists a ξ–loop

$$\{(v_i)_{0 \le i \le n}, (\xi_i)_{0 \le i \le n}, (x_i)_{1 \le i \le n}\}$$

with $\xi_0 = \xi$, $\xi_n = \xi'$. It is enough to prove the result when the support $[v_0, \dots, v_n]$ of that ξ–loop is injective. Let Y be the set of all points on a geodesic between two points of $[v_0, \dots, v_n]$. By the previous lemma, there is an infinite subgroup H of G stabilising pointwise $v_0, \dots, v_n$. As X is CAT(0), H also stabilises pointwise every point of Y. As $[v_0, \dots, v_n]$ is contractible inside Y, the finiteness lemma, Lemma 3.5, implies that we can choose a finite 2–complex F such that the loop $[v_0, \dots, v_n]$ is contractible inside F, and such that F is pointwise fixed by H. We call such a subcomplex a *hull* of the loop $[v_0, \dots, v_n]$. Hence the result will follow from the following fact, which we now prove by induction.

(H_d) For every $\xi \in \partial_{\mathrm{Stab}} G$ and every ξ–loop $\{(v_i)_{0\le i\le n}, (\xi_i)_{0\le i\le n}, (x_i)_{1\le i\le n}\}$ admitting a hull containing at most d triangles, we have $\xi_0 = \xi_n$.

If $d=1$, then $n=2$, and the hull considered is just a single triangle σ. Since $H \subset G_\sigma$ because H stabilises σ pointwise, we can choose $x \in \partial G_\sigma$ such that $\phi_{v_1,\sigma}(x) = \xi_1$. From the commutativity of the diagram of embeddings for a simplex, it follows that $\phi_{[v_0,v_1],\sigma}(x) = x_1$ and $\phi_{[v_1,v_2],\sigma}(x) = x_2$. Hence $\xi_0 = \phi_{v_0,[v_0,v_1]}(x_1) = \phi_{v_0,\sigma}(x) = \phi_{v_0,[v_2,v_0]}(x_2) = \xi_2$.

Suppose the result true up to rank d, and let $\xi \in \partial_{\mathrm{Stab}} G$, together with a ξ–loop $\{(v_i)_{0\le i\le n}, (\xi_i)_{0\le i\le n}, (x_i)_{1\le i\le n}\}$ admitting a hull F containing at most $d+1$ triangles. Choose any triangle σ of F containing the segment $[v_1, v_2]$. As σ is stabilised by H, we can find $x \in \partial G_\sigma$ such that $\phi_{v_1,\sigma}(x) = \xi_1$. There are now two possible cases, depending of the nature of σ:

- If another side of σ is contained in the support of the ξ–loop, for example $[v_2, v_3]$, we set $x' = \phi_{[v_1,v_3],\sigma}(x)$.
  Now the commutativity of the diagram of embeddings for σ yields the following new ξ–loop:
  $$\{(v_0, v_1, v_3, v_4, \dots, v_n), (\xi_0, \xi_1, \xi_3, \dots, \xi_n), (x_1, x', x_4, \dots, x_n)\}.$$
  A hull for that new loop is given by the closure of $F \setminus \sigma$, thus containing at most d triangles, and we are done by induction.
- If no other side of σ is contained in the support of the ξ–loop, we set a to be the remaining vertex of σ, $\alpha = \phi_{a,\sigma}(x)$, $x_2 = \phi_{[v_1,a],\sigma}(x)$ and $x_2' = \phi_{[a,v_2],\sigma}(x)$. The commutativity of the diagram of embeddings for σ yields the following new ξ–loop:
  $$\{(v_0, v_1, a, v_2, \dots, v_n), (\xi_0, \xi_1, \alpha, \xi_2, \dots, \xi_n), (x_1, x_2, x_2', x_3, \dots, x_n)\}.$$
  A hull for that new loop is given by the closure of $F \setminus \sigma$, thus containing at most d triangles, and we are done by induction. □

Proof of Proposition 4.2 *Convexity* Let x, x' be two points of $D(\xi)$. Let v (resp. v') be a vertex of σ_x (resp. $\sigma_{x'}$). We can thus find a ξ–path
$$\{(v_i)_{0\le i\le n}, (\xi_i)_{0\le i\le n}, (x_i)_{1\le i\le n}\}$$
with $v_0 = v$ and $v_n = v'$. As $\xi \in \partial G_{\sigma_x}$ and $\xi \in \partial G_{\sigma_{x'}}$, we can assume without loss of generality that its support $[v_0, \dots, v_n]$ contains all the vertices of σ_x and $\sigma_{x'}$. By Lemma 4.7, this implies that the subgroup $H = \bigcap_{0\le i\le p} G_{v_i}$ is infinite and that $\xi \in \Lambda H \subset \partial G_{v_0}$. Now since H fixes pointwise all the vertices of σ_x and $\sigma_{x'}$, and

since X is CAT(0), H also fixes pointwise the geodesic segment $[x,x']$. But by Lemma 4.6, the fixed-point set of H is contained in $D(\xi)$, hence so is $[x,x']$. Thus $D(\xi)$ is convex.

Finiteness Let σ be a simplex of $D(\xi)$ and $\sigma_1,\sigma_2,\dots$ be a (possibly empty) sequence of simplices strictly containing σ and contained in $D(\xi)$. It follows from the proof of Proposition 4.4 that $\xi\in\partial G_{\sigma_i}\subset\partial G_\sigma$ for every i. Thus, the limit set property Definition 2.9, the finite height property Definition 2.10, and the cocompactness of the action imply that there can be only finitely many such simplices. Thus $D(\xi)$ is locally finite. To prove that it is also bounded, consider x,x' two points of $D(\xi)$. By Lemma 4.7 the stabiliser of $\{x,x'\}$ is infinite. Thus the domain of ξ has a diameter bounded above by the acylindricity constant. The complex $D(\xi)$ is locally finite and bounded, hence finite. Moreover, it is clear from the above argument that the bound can be chosen uniform on ξ. □

4.2 Nestings and families

Definition 4.8 (Convergence property) We say that an $E\mathcal{Z}$–complex of spaces compatible with $G(\mathcal{Y})$ satisfies the *convergence property* if, for every pair of simplices $\sigma\subset\sigma'$ in Y and every injective sequence $(g_nG_{\sigma'})$ of cosets of $G_\sigma/G_{\sigma'}$, there exists a subsequence such that $(g_{\varphi(n)}\overline{EG}_{\sigma'})$ uniformly converges to a point in $\overline{EG}_\sigma$.

> *From now on, besides the limit set property Definition 2.9, the finite height property Definition 2.10 and the acylindricity assumption, we assume that the $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ satisfies the convergence property Definition 4.8.*

Definition 4.9 Let $\xi\in\partial_{\mathrm{Stab}}G$, v a vertex of $D(\xi)$, and U a neighbourhood of ξ in $\overline{EG}_v$. We say that a subneighbourhood $V\subset U$ containing ξ is *nested* in U if its closure is contained in U and for every simplex σ of $\mathrm{st}(v)$ not contained in $D(\xi)$, we have

$$\overline{EG}_\sigma\cap V\neq\varnothing\Rightarrow\overline{EG}_\sigma\subset U.$$

Lemma 4.10 (Nesting lemma) *Let $\xi\in\partial_{\mathrm{Stab}}G$, v a vertex of $D(\xi)$ and U a neighbourhood of ξ in $\overline{EG}_v$. Then there exists a subneighbourhood of ξ in $\overline{EG}_v$, $V\subset U$, which is nested in U.*

Proof We show this by contradiction. Consider a countable basis $(V_n)_n$ of neighbourhoods of ξ in $\overline{EG}_v$, and suppose that for every n, there exists a simplex $\sigma_n\in\mathrm{st}(v)\setminus D(\xi)$ such that $\overline{EG}_{\sigma_n}\cap V_n\neq\varnothing$ and $\overline{EG}_{\sigma_n}\subsetneq U$. Up to a subsequence, we can assume that $(\sigma_n)_n$ is injective. By cocompactness of the action, we can also assume that all the σ_n

cover a unique simplex $\overline{\sigma}$ of Y. Now the convergence property Definition 4.8 implies that there should exist a subsequence $\sigma_{\lambda(n)}$ such that $\overline{EG}_{\sigma_{\lambda(n)}}$ uniformly converges to a point in $\overline{EG}_v$, a contradiction. □

Since, in ∂G, boundaries of stabilisers of vertices are glued together along boundaries of stabilisers of edges, we will construct neighbourhoods in $\overline{EG}$ of a point $\xi \in \partial_{\mathrm{Stab}} G$ using neighbourhoods of the representatives of ξ in the various $\overline{EG}_v$, where v runs over the vertices of the domain of ξ.

Definition 4.11 (ξ–family) Let $\xi \in \partial_{\mathrm{Stab}} G$. A collection $\mathcal{U}$ of open sets $\{U_v, v \in V(\xi)\}$ is called a *ξ–family* if for every pair of vertices v, v' of X that are joined by an edge e and every $x \in \overline{EG}_e$,

$$\phi_{v,e}(x) \in U_v \iff \phi_{v',e}(x) \in U_{v'}.$$

Proposition 4.12 *Let $\xi \in \partial_{\mathrm{Stab}} G$. For every vertex v of $D(\xi)$, let U_v be a neighbourhood of ξ in $\overline{EG}_v$. Then there exists a ξ–family $\mathcal{U}'$ such that $U'_v \subset U_v$ for every vertex v of $D(\xi)$.*

Proof For every simplex σ of $D(\xi)$, we construct open sets U'_σ by induction on $\dim(\sigma)$, starting with simplices of maximal dimension, that we denote d.

If $\dim(\sigma) = d$, we set

$$U'_\sigma = \bigcap_{v \in \sigma} \phi_{v,\sigma}^{-1}(U_v).$$

Assume the U'_σ constructed for simplices of dimension at least $k \leq d$, and let σ_0 be of dimension $k-1$. If no simplex of dimension $\geq k$ contains σ_0, set

$$U'_{\sigma_0} = \bigcap_{v \in \sigma} \phi_{v,\sigma_0}^{-1}(U_v).$$

Otherwise, since the $\phi_{\sigma,\sigma'}$ are embeddings,

$$\bigcup_{\substack{\sigma_0 \subset \sigma \subset D(\xi) \\ \dim(\sigma) = k}} \phi_{\sigma_0,\sigma}(U'_\sigma)$$

is open in

$$\bigcup_{\substack{\dim(\sigma) = k \\ \sigma_0 \subset \sigma \subset D(\xi)}} \phi_{\sigma_0,\sigma}(\overline{EG}_\sigma).$$

We can thus write it as the trace of an open set U'_{σ_0} of $\overline{EG}_{\sigma_0}$. This yields for every vertex v of $D(\xi)$ a new open set U'_v. By intersecting it with U_v, we can further assume that $U'_v \subset U_v$. This new collection of neighbourhoods clearly satisfies the desired property. □

Definition 4.13 Let $\xi \in \partial_{\mathrm{Stab}} G$, together with two ξ–families $\mathcal{U}, \mathcal{U}'$. We say that $\mathcal{U}'$ is *nested* in $\mathcal{U}$ if for every vertex v of $D(\xi)$, U'_v is nested in U_v. Furthermore we say that $\mathcal{U}'$ is n–nested in $\mathcal{U}$ if there exist ξ–families

$$\mathcal{U}' = \mathcal{U}^{[0]} \subset \cdots \subset \mathcal{U}^{[n]} = \mathcal{U}$$

with $\mathcal{U}^{[i]}$ nested in $\mathcal{U}^{[i+1]}$ for every $i = 0, \dots, n-1$.

5 A geometric toolbox

We now prove some results that will be our main tools in studying $\overline{EG}$ and ∂G. Since the proofs in this section rely heavily on the geometry of X, we start with a few definitions.

Definition 5.1 Let $\xi \in \partial_{\mathrm{Stab}} G$, $x \in X$, $\eta \in \partial X$ and $\varepsilon \in (0,1)$.

Let d be the simplicial metric on X, and choose a basepoint $v_0 \in X$. We denote by $[v_0, x]$ the unique geodesic segment from v_0 to x, and by $\gamma_x \colon [0, d(v_0, x)] \to X$ its parametrisation. We denote by $[v_0, \eta)$ the unique geodesic ray from v_0 to η, and by $\gamma_\eta \colon [0, \infty) \to X$ its parametrisation.

We denote by $D^\varepsilon(\xi)$ the open ε–neighbourhood of $D(\xi)$.

We say that a geodesic in X parametrised by γ *goes through* (resp. *enters*) $D^\varepsilon(\xi)$ if there exist t_0 such that $\gamma(t_0) \in D^\varepsilon(\xi)$ and $t_1 > t_0$ such that $\gamma(t_1) \notin D^\varepsilon(\xi)$ (resp. if there exists t_0 such that $\gamma(t_0) \in D^\varepsilon(\xi)$).

If the geodesic $[v_0, x]$ goes through $D^\varepsilon(\xi)$, we define an *exit simplex* $\sigma_{\xi,\varepsilon}(x)$ as the first simplex touched by $[v_0, x]$ after leaving $D^\varepsilon(\xi)$. If $x \in D^\varepsilon(\xi)$, we set $\sigma_{\xi,\varepsilon}(x) = \sigma_x$.

Note that, by the assumption on the distance from a simplex to the boundary of its closed simplicial neighbourhood, we always have $D^\varepsilon(\xi) \subset \mathrm{N}(D(\xi))$.

Definition 5.2 Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. We define $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$ (resp. $\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$) as the set of points x of $X \setminus D(\xi)$ such that the geodesic $[v_0, x]$ goes through (resp. enters) $D^\varepsilon(\xi)$ and such that for some vertex v of $D(\xi)$ (hence for every by Definition 4.11) contained in the exit simplex $\sigma_{\xi,\varepsilon}(x)$, we have, in $\overline{EG_v}$:

$$\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset U_v$$

Definition 5.3 For $\xi \in \partial_{\mathrm{Stab}} G$ and $\mathcal{U}$ a ξ–family (Definition 4.11), we call $\mathrm{N}_{\mathcal{U}}(D(\xi))$ the subcomplex spanned by simplices $\sigma \subset \mathrm{N}(D(\xi))$ such that for some (hence for every) vertex v of $D(\xi) \cap \sigma$, we have, in $\overline{EG}_v$:

$$\overline{EG}_\sigma \cap U_v \neq \varnothing$$

5.1 The crossing lemma

Lemma 5.4 (Crossing lemma) *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}, \mathcal{U}'$ two ξ–families, and $\sigma_1, \dots, \sigma_n$ ($n \geq 1$) a path of open simplices contained in $\mathrm{N}(D(\xi)) \setminus D(\xi)$. Suppose that $\mathcal{U}'$ is n–nested in $\mathcal{U}$ (Definition 4.13), and that $\sigma_1 \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$. Then for every $k \in \{1, \dots, n\}$ and every vertex v of $D(\xi)$ contained in σ_k, we have $\overline{EG}_{\sigma_k} \subset U_v$ in $\overline{EG}_v$.*

Proof We prove the result by induction on n, by using the definition of nested families.

The result for $n = 1$ follows from the definition of a nested family. Suppose the result true for $1, \dots, n$, and let $\sigma_1, \dots, \sigma_{n+1}$ be a path of simplices in $\mathrm{N}(D(\xi)) \setminus D(\xi)$ and $\mathcal{U}^{[0]} \subset \cdots \subset \mathcal{U}^{[n+1]} = \mathcal{U}$. By induction, the result is true for the path $\sigma_1, \dots, \sigma_n$ and the filtration $\mathcal{U}^{[0]} \subset \cdots \subset \mathcal{U}^{[n]}$, so the only inclusion to be proved is the aforementioned one for σ_{n+1}.

If $\sigma_n \subset \sigma_{n+1}$, every vertex v of σ_n is also a vertex of σ_{n+1}, so the result is already true for vertices of $D(\xi)$ contained in σ_n. Now by the definition of ξ–families (see Definition 4.11), this implies the result for every vertex of $D(\xi) \cap \sigma_{n+1}$.

Suppose now that $\sigma_n \supset \sigma_{n+1}$, and let v be a vertex of $D(\xi)$ contained in σ_{n+1}. Since v is also in σ_n, $\overline{EG}_{\sigma_n} \subset U_{v_d}^{[n]}$ in $\overline{EG}_{\sigma_n}$, so we have

$$\overline{EG}_{\sigma_{n+1}} \cap U_{v_n}^{[n]} \neq \varnothing,$$

which in turn implies $\overline{EG}_{\sigma_{n+1}} \subset U_v^{[n+1]}$ since $\mathcal{U}^{[n]}$ is nested in $\mathcal{U}^{[n+1]}$. Now by the definition of ξ–families Definition 4.11, the same result holds for every vertex v of $D(\xi)$ contained in σ_{n+1}. □

5.2 The geodesic reattachment lemma

Recall that Corollary 4.3 yields for every $\xi \in \partial_{\mathrm{Stab}} G$ a constant $d_\xi \leq d_{\max}$ such that $D(\xi)$ contains at most d_ξ simplices and such that a geodesic contained in the open simplicial neighbourhood of $D(\xi)$ meets at most $d_{\max}$ open simplices.

Definition 5.5 (Refined families) Let $n \geq 1$. By Lemma 3.7, we can choose a constant m such that the following holds:

Let K be a convex subcomplex of X and K' a connected subcomplex of X, both containing at most $\max(n, d_{\max})$ simplices. Let $x, y \in K$ and $x', y' \in K'$ and assume that there exists a path in K' between x and y that does not meet K. Let τ, τ' be two simplices of $\mathrm{N}(K) \setminus K$ such that the geodesic segment $[x, x']$ (resp. $[y, y']$) meets the interior of τ (resp. τ'). Then there exists a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m between τ and τ'.

Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family. A ξ–family that is m–nested in $\mathcal{U}$ is said to be *n–refined* in $\mathcal{U}$. For n the number of simplices of $D(\xi)$, we denote by m_ξ such a choice of m.

Lemma 5.6 *Let $\xi \in \partial_{\mathrm{Stab}} G$. There exists a ξ–family $\mathcal{V}_\xi$ such that for every vertex v of $D(\xi)$ and every simplex σ of $(\mathrm{st}(v) \setminus D(\xi)) \cap \mathrm{Geod}(v_0, D(\xi))$, we have $(V_\xi)_v \cap \overline{EG}_\sigma = \varnothing$.*

Proof Let σ a simplex of $\mathrm{N}(D(\xi)) \setminus D(\xi)$ whose interior meets $\mathrm{Geod}(v_0, D(\xi))$. Let v be a vertex of $D(\xi) \cap \sigma$. Let U_v be a neighbourhood of ξ in $\overline{EG}_v$ that is disjoint from $\overline{EG}_\sigma$. For every other vertex w of $D(\xi)$, set $U_w = \overline{EG}_w$. By Proposition 4.12, we choose a ξ–family $\mathcal{V}_\xi$ that is $(d_\xi + 1)$–refined in the collection of open sets $\{U_w, w \in V(\xi)\}$. The result now follows from Definition 5.5. □

Lemma 5.7 *Let $\xi \in \partial_{\mathrm{Stab}} G$. Let $\mathcal{U}$ be a ξ–family that is m_ξ–nested in $\mathcal{V}_\xi$ (recall that $\mathcal{V}_\xi$ is assumed to satisfy Lemma 5.6). Let $x \in X \setminus D(\xi)$ be such that there exists a simplex $\sigma \subset (\mathrm{N}(D(\xi)) \setminus D(\xi))$ that meets $\mathrm{Geod}(x, D(\xi))$, and such that for some (hence any) vertex v of $\sigma \cap D(\xi)$ we have $\overline{EG}_\sigma \subset U_v$. Then $x \notin \mathrm{Geod}(v_0, D(\xi))$.*

Proof We prove the lemma by contradiction. Let x and σ be as in the statement of the lemma. Let $z \in D(\xi)$ be such that $x \in [v_0, z]$ and $z' \in D(\xi)$ be such that the geodesic segment $[x, z']$ meets σ. Let σ' be the last simplex touched by $[v_0, z]$ before meeting $D(\xi)$, and v' a vertex of σ'.

Since $\mathcal{U}$ is m_ξ–nested in $\mathcal{V}_\xi$, it follows from the inclusion $\overline{EG}_\sigma \subset U_v$ and Lemma 3.7 that $\overline{EG}_{\sigma'} \subset (V_\xi)_{v'}$, contradicting the definition of $\mathcal{V}_\xi$. □

The next lemma gives a useful criterion that ensures that a given path is a global geodesic.

Lemma 5.8 (Geodesic reattachment lemma) *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{V}$ a ξ–family satisfying Lemma 5.6, $\mathcal{U}$ a ξ–family which is $(m_\xi + d_\xi)$–nested in $\mathcal{V}$, and $x \in X \setminus D(\xi)$. Suppose that there exists a simplex $\sigma \subset \mathrm{N}(D(\xi)) \setminus D(\xi)$ that meets $\mathrm{Geod}(x, D(\xi))$ such that for some (hence any) vertex v of $\sigma \cap D(\xi)$ we have $\overline{EG}_\sigma \subset U_v$. Then $[v_0, x]$ meets $D(\xi)$ and $x \in \widetilde{\mathrm{Cone}}_{\mathcal{V},\varepsilon}(\xi)$ for every $\varepsilon \in (0, 1)$.*

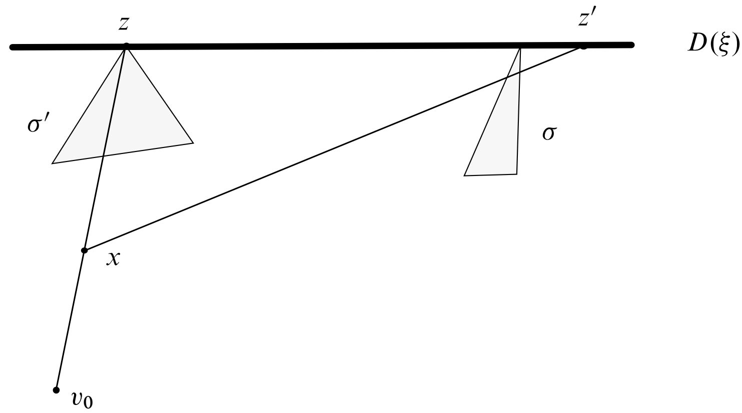

Figure 2

In such a case, the geodesic from v_0 to x meets $D(\xi)$, and is the concatenation of a geodesic segment in $\mathrm{Geod}(v_0, D(\xi))$ and a geodesic in $\mathrm{Geod}(D(\xi), x)$.

Proof Let $K = \mathrm{Geod}(v_0, D(\xi)) \cup \mathrm{Geod}(D(\xi), x)$ and let $[v_0, x]_K$ be the geodesic from v_0 to x in K (which meets finitely many simplices by Lemma 3.5). Our aim is to prove that $[v_0, x]_K = [v_0, x]$. By Lemma 5.7, $x \notin \mathrm{Geod}(v_0, D(\xi))$. As $D(\xi)$ is convex by Proposition 4.2, let $v_1, v_2 \in D(\xi)$ be such that $[v_0, x]_K = [v_0, v_1] \cup [v_1, v_2] \cup [v_2, x]$ and such that $[v_0, v_1)$ and $(v_2, x]$ do not meet $D(\xi)$. Let $\varepsilon \in (0, 1)$. Let $a \in [v_0, v_1]$ be such that $d(a, v_1) = \varepsilon$. If $x \notin D^{\varepsilon}(\xi)$ let $b \in [v_2, x]$ be such that $d(v_2, b) = \varepsilon$. Otherwise, let $b = x$. Since X is CAT(0), it is enough to prove that $[v_0, x]_K$ is a local geodesic at every point. We already have that $[v_0, v_1] \cup [v_1, v_2]$ and $[v_1, v_2] \cup [v_2, x]$ are geodesics, so it is sufficient to prove the result when $v_1 = v_2$. We thus have

$$[v_0, x]_K = [v_0, v_1] \cup [v_1, x],$$

with $[v_0, v_1] \subset \mathrm{Geod}(v_0, D(\xi))$ and $[v_1, x] \subset \mathrm{Geod}(D(\xi), x)$. Assume by contradiction that $[v_0, x]_K$ is not a local geodesic at v_1. Then the geodesic segment $[a, b]$ does not meet $D(\xi)$. This geodesic segment yields a path of simplices between σ_a and σ_b of length at most d_ξ in $\mathrm{N}(D(\xi)) \setminus D(\xi)$. Furthermore, there is a path of simplices between σ and σ_b of length at most m_ξ in $\mathrm{N}(D(\xi)) \setminus D(\xi)$ by Definition 5.5. Thus, there is a path of simplices between σ and σ_a of length at most $m_\xi + d_\xi$ in $\mathrm{N}(D(\xi)) \setminus D(\xi)$. But since $\overline{EG_b} \subset U_v$ and $\mathcal{U}$ is $(m_\xi + d_\xi)$–nested in $\mathcal{V}$, the crossing lemma, Lemma 5.4, implies $\overline{EG_a} \subset V_v$, which contradicts the fact that $\mathcal{V}$ satisfies Lemma 5.6.

Thus $[v_0, x]_K = [v_0, x]$ and $\sigma_b = \sigma_{\xi,\varepsilon}(x)$. It follows from the above discussion that for some (hence every) vertex v' of $\sigma_{\xi,\varepsilon}(x)$ we have $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset V_{v'}$, hence $x \in \widetilde{\mathrm{Cone}}_{\mathcal{V},\varepsilon}(\xi)$. □

From now on, every ξ–family will be assumed to be contained in a ξ–family $\mathcal{U}_\xi$ satisfying Lemma 5.8.

As a consequence, we get the following:

Corollary 5.9 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. Then for every $x \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic segment $[v_0, x]$ meets $D(\xi)$.*

Proof By Lemma 5.7 applied to x and $\sigma_{\xi,\varepsilon}(x)$, we get $x \notin \mathrm{Geod}(v_0, D(\xi))$. Let y be a point of $\sigma_{\xi,\varepsilon}(x) \cap [v_0, x] \cap D^\varepsilon(\xi)$. It follows from the geodesic reattachment Lemma 5.8 applied to y and $\sigma_{\xi,\varepsilon}(x)$ that $[v_0, y]$, hence $[v_0, x]$, meets $D(\xi)$. □

5.3 The refinement lemma

Lemma 5.10 (Refinement lemma) *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $n \geq 1$. Let $\mathcal{U}'$ be a ξ–family which is n–refined in $\mathcal{U}$. Then the following holds:*

For every $\varepsilon \in (0,1)$ and every path of simplices $\sigma_1, \dots, \sigma_n$ in $X \setminus D(\xi)$ such that there exists a point $x_1 \in \sigma_1$ such that $[v_0, x_1]$ enters $D^\varepsilon(\xi)$ and $\sigma_{\xi,\varepsilon}(x_1) \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$, we have

$$\sigma_1, \dots, \sigma_n \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi).$$

Proof Let us prove that for every $x \in \bigcup_{1 \leq i \leq n} \sigma_i$, the geodesic segment $[v_0, x]$ meets $D(\xi)$. Let $x_1 \in \sigma_1$ such that $\sigma_{\xi,\varepsilon}(x_1) \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$. Note that Corollary 5.9 implies that $[v_0, x_1]$ meets $D(\xi)$. Let v be a vertex of $D(\xi) \cap \sigma_{\xi,\varepsilon}(x_1)$.

Let $x \in \bigcup_{1 \leq i \leq n} \sigma_i$ and σ be a simplex of $\mathrm{N}(D(\xi)) \setminus D(\xi)$ touched by $[v, x]$ after leaving $D(\xi)$. Let also w be a vertex of $\sigma \cap D(\xi)$. We can apply Lemma 3.7 to the geodesic segments $[v, x]$ and (a portion of) $[v_0, x_1]$, and to simplices σ and $\sigma_{\xi,\varepsilon}(x_1)$. Since $\overline{EG}_{\sigma_{\xi,\varepsilon}(x_1)} \subset U'_v$ and $\mathcal{U}'$ is n–refined in $\mathcal{U}$, we get $\overline{EG}_\sigma \subset U_w$. Thus the geodesic reattachment Lemma 5.8 implies that $[v_0, x]$ meets $D(\xi)$.

Let $x \in \bigcup_{1 \leq i \leq n} \sigma_i$ and let w be a vertex of $\sigma_{\xi,\varepsilon}(x) \cap D(\xi)$. We apply once again Lemma 3.7, this time to portions of the geodesic segments $[v_0, x]$ and $[v_0, x_1]$, and to simplices $\sigma_{\xi,\varepsilon}(x)$ and $\sigma_{\xi,\varepsilon}(x_1)$. Now since $\mathcal{U}'$ is n–refined in $\mathcal{U}$ and $\overline{EG}_{\sigma_{\xi,\varepsilon}(x_1)} \subset U'_v$, we get $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset U_w$, hence $x \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$. □

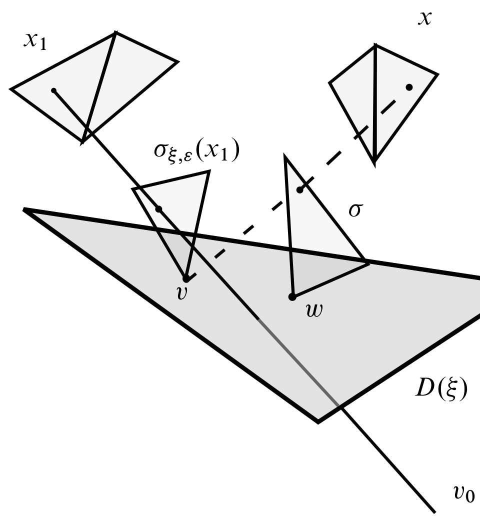

Figure 3

5.4 The star lemma

Lemma 5.11 (Star lemma) *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $x \in X \setminus D^{\varepsilon}(\xi)$ such that the geodesic segment $[v_0, x]$ goes through $D^{\varepsilon}(\xi)$. Then there exists $\delta > 0$ such that for every $y \in B(x,\delta) \setminus D^{\varepsilon}(\xi)$, the geodesic segment $[v_0, y]$ goes through $D^{\varepsilon}(\xi)$. Furthermore, for every $y \in B(x,\delta) \setminus D^{\varepsilon}(\xi)$, we have*

$$\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x)).$$

Proof Let $T = \mathrm{dist}(v_0, x)$, and let $\gamma_x\colon [0,T] \to X$ be the parametrisation of the geodesic segment $[v_0, x]$. Let $t_0 > 0$ such that $[v_0, x]$ leaves $D^{\varepsilon}(\xi)$ at time t_0. Since $D(\xi)$ is convex by Proposition 4.2, the map $z \mapsto \mathrm{dist}(z, D(\xi))$ is convex. Thus, there exists $r > 0$ such that

$$\gamma_x([t_0 - r, t_0)) \subset D^{\varepsilon}(\xi),$$
$$\gamma_x([t_0 - r, t_0]) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x)).$$

We also choose $\tau > 0$ such that for every y_-, y_+ in the τ–neighbourhood of $\gamma_x([t_0 - r, t_0])$, the geodesic segment $[y_-, y_+]$ is contained in $\mathrm{st}(\sigma_{\xi,\varepsilon}(x))$.

Let

$$k = \varepsilon - \mathrm{dist}(\gamma_x(t_0 - r), D(\xi)) > 0.$$

We set $\delta_1 = \frac{1}{10} \cdot \min(k, \tau, r)$. If $x \in \overline{D^{\varepsilon}(\xi)}$, set $\delta = \delta_1$. If $x \notin \overline{D^{\varepsilon}(\xi)}$, we can assume without loss of generality that $\delta_1 < \frac{1}{10} \cdot (T - t_0)$. By convexity of the distance, we have $d(\gamma_x(t_0 + \delta_1), D(\xi)) > \varepsilon$, and we set $\delta = \frac{1}{2} \cdot \min(\delta_1, d(\gamma_x(t_0 + \delta_1), D^{\varepsilon}(\xi))) > 0$.

Let $y \in B(x,\delta) \setminus D^\varepsilon(\xi)$, and let γ_y be its parametrisation.

Since $\delta \leq r$, we have $d(v_0, y) \geq t_0 - r$. Now, γ_x and γ_y parametrise geodesics starting at v_0 and such that $d(x,y) < \delta$, so since X is a CAT(0)-space, we get $d(\gamma_x(t_0-r), \gamma_y(t_0-r)) \leq 2\delta \leq \tau$. The inequality $10\delta \leq k$ now implies

$$\begin{aligned} d(\gamma_y(t_0-r), D(\xi)) &\leq d(\gamma_x(t_0-r), D(\xi)) + d(\gamma_x(t_0-r), \gamma_y(t_0-r)) \\ &\leq (\varepsilon - 10\delta) + 2\delta < \varepsilon, \end{aligned}$$

so $\gamma_y(t_0-r) \in D^\varepsilon(\xi)$. Since $y \notin D^\varepsilon(\xi)$, it follows that the geodesic segment $[v_0, y]$ goes through $D^\varepsilon(\xi)$ and leaves it for some $t_1 > t_0 - r$.

Moreover, after leaving $D^\varepsilon(\xi)$ the geodesic $[v_0, y]$ meets the τ–ball centred at $\gamma_x(t_0)$ for some $t_2 \geq t_1$. Indeed, this is clear if $x \in \overline{D^\varepsilon(\xi)}$ since $d(x,y) < \delta \leq \tau$. If $x \notin \overline{D^\varepsilon(\xi)}$, then $[v_0, y]$ meets the 2δ–ball centred at $\gamma_x(t_0+\delta_1)$, which is contained in $(X \setminus D^\varepsilon(\xi)) \cap B(\gamma_x(t_0), 2\delta_1)$ by definition of δ. Hence, $[v_0, y]$ meets $B(\gamma_x(t_0), \tau) \setminus D^\varepsilon(\xi)$ for some $t_2 \geq t_1$.

We thus have $d(\gamma_x(t_0-r), \gamma_y(t_0-r)) \leq \tau$ and $d(\gamma_x(t_0), \gamma_y(t_2)) \leq \tau$. By definition of τ, it follows that

$$\gamma_y([t_0 - r, t_2]) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x)),$$

which implies $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x))$. □

The star Lemma 5.11 immediately implies the following:

Corollary 5.12 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. Then the sets* $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$ *and* $\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ *are open in* X. □

6 The topology of $\overline{EG}$

In this section, we define a topology on $\overline{EG}$ and study its first properties.

6.1 Definition of the topology

In this paragraph, we define a topology on $\overline{EG}$, by defining a basis of open neighbourhoods at every point. Since points of $\overline{EG}$ are of three different kinds (EG, ∂X and $\partial_{\mathrm{Stab}} G$), we treat these cases separately.

Definition 6.1 Let $\widetilde{x} \in EG$. We define a basis of neighbourhoods of $\widetilde{x}$ in $\overline{EG}$, denoted $\mathcal{O}_{\overline{EG}}(\widetilde{x})$, as the set of open sets of EG containing $\widetilde{x}$.

We now turn to the case of points of the boundary of X. Recall that since X is a simplicial CAT(0) space with countably many simplices, the bordification $\overline{X} = X \cup \partial X$ has a natural metrisable topology, though not necessarily compact if X is not locally finite. For every $\eta \in \partial X$, a basis of neighbourhoods of η in that bordification is given by the family of

$$V_{r,\delta}(\eta) = \{x \in \overline{X} \mid d(v_0, x) > r \text{ and } \gamma_x(r) \in B(\gamma_\eta(r), \delta)\}, \quad r,\ \delta > 0.$$

Remark *For $r, \delta > 0$, $\eta \in \partial X$, and if γ is the parametrisation of a geodesic such that there exists $T \geq 0$ with $\gamma(T) \in V_{r,\delta}(\eta)$, then $\gamma(t) \in V_{r,\delta}(\eta)$ for every $t \geq T$.*

We denote this basis of neighbourhoods of η in $\overline{X}$ by $\mathcal{O}_{\overline{X}}(\eta)$. Endowed with that topology, $\overline{X}$ is a second countable metrisable space (see Bridson and Haefliger [9]).

Note that the topology of $\overline{X}$ satisfies the following properties:

Lemma 6.2 *Let $\eta \in \partial X$. Then there exists a basis of neighbourhoods (U_n) of η in $\overline{X}$ such that U_n and $U_n \setminus \partial X$ are contractible for every $n \geq 0$.*

Proof For $r, \delta > 0$, let $U_{r,\delta}(\eta) = V_{r,\delta}(\eta) \cup B(\gamma_\eta(r), \delta)$. This defines a basis of neighbourhoods of η in $\overline{X}$. As $U_{r,\delta}(\eta) \setminus \partial X$ can be retracted by strong deformation along geodesics starting at v_0 onto $B(\gamma_\eta(r), \delta)$, it is contractible. Furthermore, as $U_{r,\delta}(\eta)$ can be retracted by strong deformation onto $U_{r,\delta}(\eta) \setminus \partial X$, the same holds for $U_{r,\delta}(\eta)$. □

Lemma 6.3 *Let $\eta \in \partial X$, U a neighbourhood of η in $\overline{X}$ and $k \geq 0$. Then there exists a neighbourhood U' of η in $\overline{X}$ that is contained in U and such that $d(U' \cap X, X \setminus U) > k$.*

Proof The definition of the topology of $\overline{X}$ implies the following: if (x_n) and (y_n) are two sequences of X such that $d(x_n, y_n)$ is bounded, then (x_n) converges to a point of ∂X if and only if (y_n) converges to the same point. Reasoning by contradiction thus implies the lemma. □

Definition 6.4 Let $\eta \in \partial X$, and let U be a neighbourhood of η in $\overline{X}$. We set

$$V_U(\eta) = p^{-1}(U \cap X) \cup (U \cap \partial X) \cup \{\xi \in \partial_{\mathrm{Stab}} G \mid D(\xi) \subset U\}.$$

When U runs over the basis $\mathcal{O}_{\overline{X}}(\eta)$ of neighbourhoods of η in $\overline{X}$, the above formula defines a collection of neighbourhoods for η in $\overline{EG}$, denoted $\mathcal{O}_{\overline{EG}}(\eta)$.

We finally define open neighbourhoods for points in $\partial_{\mathrm{Stab}} G$.

Definition 6.5 Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U} \subset \mathcal{U}_\xi$ be a ξ–family, and $\varepsilon \in (0,1)$. A neighbourhood $V_{\mathcal{U},\varepsilon}(\xi)$ is defined in four steps as follows:

- Let $W_{\mathcal{U},\varepsilon}(\xi)$ be the set of points $\widetilde{x} \in EG$ whose projection $x \in X$ belongs to $D^\varepsilon(\xi)$ and is such that for some (hence every) vertex v of $D(\xi) \cap \sigma_x$, we have $\phi_{v,\sigma_x}(\widetilde{x}) \in U_v$.
- Let W_1 be the set of points of EG whose projection in X belongs to $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$.
- Let W_2 be the set of points of ∂X that belong to $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$.
- Let W_3 be the set of points $\xi' \in \partial_{\mathrm{Stab}} G$ such that $D(\xi') \setminus D(\xi) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ and for every vertex v of $D(\xi) \cap D(\xi')$ we have $\xi' \in U_v$.

Now set

$$V_{\mathcal{U},\varepsilon}(\xi) = W_{\mathcal{U},\varepsilon}(\xi) \cup W_1 \cup W_2 \cup W_3.$$

This collection of neighbourhoods of ξ in $\overline{EG}$ is denoted $\mathcal{O}_{\overline{EG}}(\xi)$. Note that these neighbourhoods depend on the chosen basepoint v_0. If we need to specify the basepoint used to define the various sets $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$, $V_{\mathcal{U},\varepsilon}(\xi)$, we will indicate it in superscript. In that case, we will speak of the topology (of $\overline{EG}$) *centred* at a given point.

Note that for ξ–families $\mathcal{U}' \subset \mathcal{U}$ and $\varepsilon' < \varepsilon$, we do not necessarily have the inclusion $V_{\mathcal{U}',\varepsilon'}(\xi) \subset V_{\mathcal{U},\varepsilon}(\xi)$ since these two neighbourhoods are defined by looking at the way geodesics leave two (a priori non related) different neighbourhoods of the domain $D(\xi)$. However, the crossing Lemma 5.4 immediately implies the following:

Lemma 6.6 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}, \mathcal{U}'$ two ξ–families, and $0 < \varepsilon' < \varepsilon$. If $\mathcal{U}'$ is d_ξ–nested in $\mathcal{U}$, then $V_{\mathcal{U}',\varepsilon'}(\xi) \subset V_{\mathcal{U},\varepsilon}(\xi)$.* □

Definition 6.7 We define a topology on $\overline{EG}$ by taking the topology generated by the elements of $\mathcal{O}_{\overline{EG}}(x)$, for every $x \in \overline{EG}$. We denote by $\mathcal{O}_{\overline{EG}}$ the set of elements of $\mathcal{O}_{\overline{EG}}(x)$ when x runs over $\overline{EG}$. Thus, any an open set in $\overline{EG}$ is a union of finite intersections of elements of $\mathcal{O}_{\overline{EG}}$.

We will show in the next subsection that $\mathcal{O}_{\overline{EG}}$ is actually a *basis* for the topology of $\overline{EG}$.

6.2 A basis of neighbourhoods

Here we prove that the set of neighbourhoods we just defined is a basis for the topology of $\overline{EG}$. In order to do that, we need the following:

Filtration Lemma Let $z, z' \in \overline{EG}$ and $U \in \mathcal{O}_{\overline{EG}}(z)$ an open neighbourhood of z. If $z' \in U$, then there exists an open neighbourhood of z', $U' \in \mathcal{O}_{\overline{EG}}(z')$, such that $U' \subset U$.

Since points of $\overline{EG}$ are of three different natures (EG, ∂X, and $\partial_{\mathrm{Stab}} G$), the proof breaks into six distinct cases. We first introduce a notation that will be useful for treating similar cases at once.

Definition 6.8 We extend the projection $p\colon EG \to X$ to a map $\overline{p}$ from $\overline{EG}$ to the set of subsets of $\overline{X}$ in the following way:

- For $\tilde{x} \in EG$, we define $\overline{p}(z)$ to be the singleton $\{p(\tilde{x})\}$.
- For $\eta \in \partial X$, we define $\overline{p}(\eta)$ to be the singleton $\{\eta\}$.
- For $\xi \in \partial_{\mathrm{Stab}} G$, we set $\overline{p}(\xi) = D(\xi)$.
- Finally, for $K \subset \overline{EG}$, we set $\overline{p}(K) = \bigcup_{z \in K} \overline{p}(z)$.

Lemma 6.9 *Let $\tilde{x}, \tilde{y} \in EG$ and $U \in \mathcal{O}_{EG}(\tilde{x})$ an open neighbourhood of $\tilde{x}$ in EG. If $\tilde{y} \in U$, then there exists an open neighbourhood of $\tilde{y}$ in EG, $U' \in \mathcal{O}_{\overline{EG}}(\tilde{y})$ such that $U' \subset U$.*

Proof By definition of the topology, we can take $U' = U$. □

Lemma 6.10 *Let $\eta, \eta' \in \partial X$ and $U \in \mathcal{O}_{\overline{X}}(\eta)$ an open neighbourhood of η in $\overline{X}$. If $\eta' \in V_U(\eta)$, then there exists an open neighbourhood U' of η' in $\overline{X}$, such that $V_{U'}(\eta') \subset V_U(\eta)$.*

Proof Since $\mathcal{O}_{\overline{X}}$ is a basis of neighbourhoods for $\overline{X}$, there exists a neighbourhood $U' \in \mathcal{O}_{\overline{X}}(\eta')$ such that $U' \subset U$. Now one clearly has $\eta \in V_{U'}(\eta') \subset V_U(\eta)$. □

Lemma 6.11 *Let $\tilde{x} \in EG, \eta \in \partial X$ and U an open neighbourhood of η in $\overline{X}$. If $\tilde{x} \in V_U(\eta)$, then there exists an open neighbourhood U' of $\tilde{x}$ in $\overline{EG}$, $U' \in \mathcal{O}_{EG}(\tilde{x})$, such that $U' \subset V_U(\eta)$.*

Proof It is enough to choose an arbitrary open neighbourhood U' of $\tilde{x}$ contained in $p^{-1}(U \cap X)$. □

Lemma 6.12 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\eta \in \partial X$ and $U \in \mathcal{O}_{\overline{X}}(\eta)$ an open neighbourhood of η in $\overline{X}$. If $\xi \in V_U(\eta)$, then there exist $\varepsilon \in (0,1)$ and a ξ–family $\mathcal{U}$ such that $V_{\mathcal{U},\varepsilon}(\xi) \subset V_U(\eta)$.*

Proof The subcomplex $D(\xi) \subset U$ is finite, hence compact, so choose $\varepsilon \in (0,1)$ such that $D^{\varepsilon}(\xi) \subset U$. Let $\mathcal{U}$ be any ξ–family. For every $x \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic segment $[v_0, x]$ meets $D(\xi)$ by Corollary 5.9. As $D(\xi)$ is contained in U, the same holds for x. It then follows that $V_{\mathcal{U}_{\xi},\varepsilon}(\xi) \subset V_U(\eta)$. □

Lemma 6.13 *Let $\eta \in \partial X$, $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. If $\eta \in V_{\mathcal{U},\varepsilon}(\xi)$, then there exists an open neighbourhood U of η in $\overline{X}$ such that $V_U(\eta) \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof Let $\gamma_\eta \colon [0,\infty) \to X$ be a parametrisation of the geodesic ray $[v_0, \eta)$. The subcomplex $D(\xi)$ being finite by Proposition 4.2, choose $R > 0$ such that $D(\xi) \subset B(v_0, R)$, and let $x = \gamma_\eta(R+1)$. Since $\eta \in V_{\mathcal{U},\varepsilon}(\xi)$, we have $x \in \mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$, which is open in X by Corollary 5.12. Let $\delta > 0$ such that $B(x,\delta) \subset \mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$. Now if we set $U' = V_{R+1,\delta}(\eta) \in \mathcal{O}_{\overline{X}}(\eta)$, it follows that $V_{U'}(\eta) \subset V_{\mathcal{U},\varepsilon}(\xi)$. □

Lemma 6.14 *Let $\tilde{x} \in EG$, $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. If $\tilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$, then there exists a $U \in \mathcal{O}_{\overline{EG}}(\tilde{x})$ such that $U \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof It is enough to prove that $V_{\mathcal{U},\varepsilon}(\xi) \cap EG$ is open in EG. First, since the maps $\phi_{\sigma,\sigma'}$ are embeddings, it is clear that $W_{\mathcal{U},\varepsilon}(\xi)$ is open in EG. Let $\tilde{y} \in V_{\mathcal{U},\varepsilon}(\xi) \cap EG$ with $y = p(\tilde{y}) \notin D^{\varepsilon}(\xi)$. The star Lemma 5.11 yields a $\delta > 0$ such that for every $z \in B(y,\delta) \setminus D^{\varepsilon}(\xi)$, the geodesic segment $[v_0, z]$ goes through $D^{\varepsilon}(\xi)$ and $\sigma_{\xi,\varepsilon}(z) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(y))$. We can further assume that $B(y,\delta) \subset \mathrm{st}(\sigma_y)$. It now follows immediately from the construction of $V_{\mathcal{U},\varepsilon}(\xi)$ that $p^{-1}(B(y,\delta))$ is an open neighbourhood of $\tilde{x}$ contained in $V_{\mathcal{U},\varepsilon}(\xi)$, which concludes the proof. □

Lemma 6.15 *Let $\xi, \xi' \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. If $\xi' \in V_{\mathcal{U},\varepsilon}(\xi)$, then there exists a ξ'–family $\mathcal{U}'$ and $\varepsilon' \in (0,1)$ such that $V_{\mathcal{U}',\varepsilon'}(\xi') \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

By Lemma 5.11, let $\delta \in (0,\varepsilon)$ be such that for all $y \in D^{\delta}(\xi') \setminus D^{\varepsilon}(\xi)$, the geodesic segment $[v_0, y]$ goes through $D^{\varepsilon}(\xi)$ and is such that $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x))$, for some $x \in D(\xi')$. We now define a ξ'–family using the following lemma.

Lemma 6.16 *There exist nested ξ'–families $\mathcal{U}^{[d_\xi]} \supset \cdots \supset \mathcal{U}^{[0]} = \mathcal{U}'$ such that the following holds: Let x be a point of $\mathrm{Cone}_{\mathcal{U}',\delta}(\xi')$ such that the geodesic from v_0 to x leaves $D^{\delta}(\xi')$ at a point that is still inside $D^{\varepsilon}(\xi)$. Let $\sigma_1 = \sigma_{\xi',\delta}(x), \ldots, \sigma_n = \sigma_{\xi,\varepsilon}(x)$ ($n \leq d_\xi$) be the path of simplices met by the geodesic segment $[v_0, x]$ inside $\overline{D^{\varepsilon}(\xi)}$ after leaving $D^{\delta}(\xi')$ (cf Figure 4).*

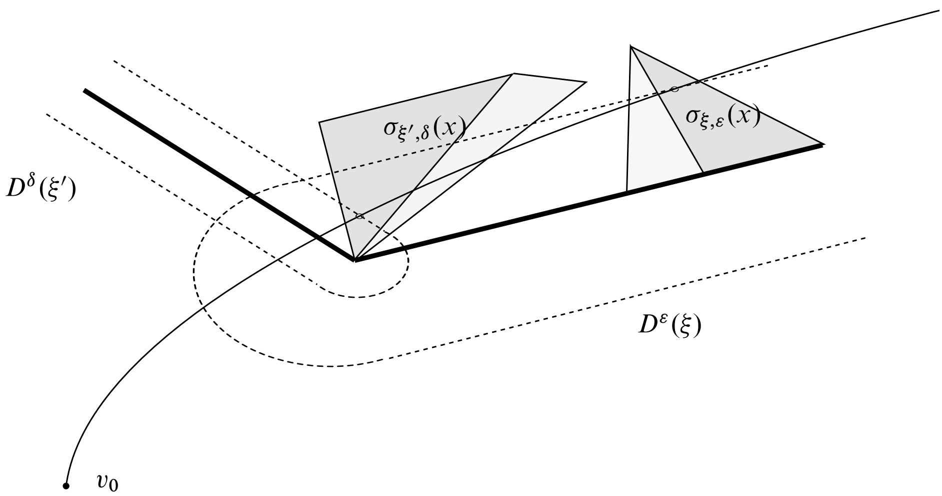

Figure 4

We then have the following, for every $1 \le k \le n$*:*

(i) *The simplex* σ_k *is contained in* $\bigcup_{v' \in V(\xi) \cap V(\xi')} \mathrm{st}(v')$*, but not contained in* $\bigcup_{v \in V(\xi) \setminus V(\xi')} \mathrm{st}(v)$*.*

(ii) *For every vertex* v' *of* σ_k *contained in* $D(\xi')$*, the inclusion* $\overline{EG_{\sigma_k}} \subset U_{v'}^{[k]}$ *holds in* $\overline{EG}_{v'}$*.*

Proof If v' is a vertex of $D(\xi) \cap D(\xi')$, then for every vertex v of

$$\overline{\mathrm{st}}(v') \cap (D(\xi) \setminus D(\xi')),$$

choose a neighbourhood $W_{v,v'}$ of ξ' in $\overline{EG}_{v'}$ missing $\overline{EG}_{[v,v']}$, and set

$$W_{v'} = \Bigg(\bigcap_{v \in \overline{\mathrm{st}}(v') \cap (V(\xi) \setminus V(\xi'))} W_{v,v'} \Bigg) \cap U_{v'}.$$

If v' is a vertex not in $D(\xi)$, set $W_v = \overline{EG}_{v'}$.

We now define $\mathcal{U}'$ to be a ξ'–family that is d_ξ–nested in the family of $W_{v'}$, $v' \in D(\xi')$, that is, $\mathcal{U}'$ is a ξ'–family such that there exists a sequence of nested ξ'–families $\mathcal{U}^{[d_\xi]} \supset \cdots \supset \mathcal{U}^{[0]} = \mathcal{U}'$ satisfying $W_{v'} \supset U_{v'}^{[d_\xi]} \supset \cdots \supset U_{v'}^{[0]} = U'_{v'}$ for every vertex v' of $D(\xi')$.

We now prove (i) and (ii) by induction on k. Since the geodesic segment $[v_0, x]$ leaves $D^\delta(\xi')$ while inside $D^\varepsilon(\xi)$, we have $\sigma_1 = \sigma_{\xi',\delta}(x) \subset \bigcup_{v' \in V(\xi) \cap V(\xi')} \mathrm{st}(v')$. To prove (i) for $k = 1$, we reason by contradiction. Suppose there exists a vertex v' of $D(\xi) \cap D(\xi')$ and a vertex v of $D(\xi) \setminus D(\xi')$ such that $\sigma_1 \subset \mathrm{st}([v, v'])$. Then we have $\overline{EG}_{\sigma_1} \subset \overline{EG}_{[v,v']}$ in $\overline{EG}_{v'}$. But the former set is contained in $U_{v'}$ since $\widetilde{x} \in V_{\mathcal{U}',\delta}(\xi')$, and the latter is disjoint from $U_{v'}$ by construction of $\mathcal{U}'$, which is absurd.

Suppose the result has been proved up to rank k. If $\sigma_{k+1} \subset \sigma_k$, the result is straightforward, so we suppose that $\sigma_k \subset \sigma_{k+1}$. We prove (i) by contradiction. Suppose there exists a vertex v' of $D(\xi) \cap D(\xi')$ and a vertex v of $D(\xi) \setminus D(\xi')$ such that $\sigma_{k+1} \subset \mathrm{st}([v, v'])$. Then by the induction hypothesis, we have $\overline{EG}_{[v,v']} \cap U_{v'}^{[k]} \neq \varnothing$ in $\overline{EG}_{v'}$, hence $\overline{EG}_{[v,v']} \subset U_{v'}^{[k+1]} \subset W_{v'}$ since $\mathcal{U}^{[k]}$ is nested in $\mathcal{U}^{[k+1]}$, and the last inclusion contradicts the definition of $\mathcal{U}'$.

We now prove (ii). Let v_k a vertex of $D(\xi) \cap D(\xi')$ contained in σ_k (hence in σ_{k+1}). Thus we have $\overline{EG}_{\sigma_{k+1}} \subset \overline{EG}_{\sigma_k} \subset U_{v_k}^{[k]} \subset U_{v_k}^{[k+1]}$ in $\overline{EG}_{v_k}$. Now let v' be another vertex of $D(\xi') \cap D(\xi)$ contained in σ_{k+1} (if any). We thus have $\overline{EG}_{[v_k,v']} \cap U_{v_k}^{[k]} \neq \varnothing$ in $\overline{EG}_{v_k}$, so $\overline{EG}_{[v_k,v']} \subset U_{v_k}^{[k+1]}$ in $\overline{EG}_{v_k}$. But by Proposition 4.12, this implies

$$\overline{EG}_{[v_k,v']} \subset U_{v'}^{[k+1]},$$

which proves (ii). □

Proof of Lemma 6.15 Let us show now that $V_{\mathcal{U}',\delta}(\xi') \subset V_{\mathcal{U},\varepsilon}(\xi)$. Let $z \in V_{\mathcal{U}',\delta}(\xi')$ and $x \in \overline{p}(z)$. The geodesic $[v_0, x]$ meets $D^\delta(\xi')$, hence $D^\varepsilon(\xi)$. To prove that $z \in V_{\mathcal{U},\varepsilon}(\xi)$, it is now enough to prove that $x \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$.

If $x \in W_{\mathcal{U}',\delta}(\xi') \cap D^\varepsilon(\xi)$, it follows from the definition of $\mathcal{U}'$ (defined in Lemma 6.16) that $z \in W_{\mathcal{U},\varepsilon}(\xi)$.

If the geodesic segment $[v_0, x]$ meets $D^\delta(\xi')$ outside $D^\varepsilon(\xi)$, it follows from the definition of δ that there exists $x' \in D(\xi') \setminus D(\xi)$ such that $\sigma_{\xi,\varepsilon}(x) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x'))$. But since $x' \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, the same holds for x.

Thus the only case left to consider is when the geodesic segment $[v_0, x]$ leaves $D^\delta(\xi')$ while still being inside $D^\varepsilon(\xi)$. Lemma 6.16 shows that for every vertex v' of $\sigma_{\xi,\varepsilon}(x)$ contained in $D(\xi)$, $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset U_v^{[n]} \subset U_v$ in $\overline{EG}_v$, which now implies $x \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$. This concludes the proof. □

Theorem 6.17 *$\mathcal{O}_{\overline{EG}}$ is a basis for the topology of $\overline{EG}$, which makes it a second countable space. For this topology, EG embeds as a dense open subset.*

Proof To prove that $\mathcal{O}_{\overline{EG}}$ is a basis for the topology of $\overline{EG}$, it is enough to show that for every pair of open sets U_1, U_2 of $\overline{EG}$ and every $z \in U_1 \cap U_2$, there exists an open neighbourhood $W \in \mathcal{O}_{\overline{EG}}$ such that $z \in W \subset U_1 \cap U_2$.

If $z \in EG$ By the results from the previous paragraph, there exists $V_1, V_2 \in \mathcal{O}_{EG}(z)$ such that $V_1 \subset U_1$ and $V_2 \subset U_2$. Then take W to be any element of $\mathcal{O}_{EG}(z) = \mathcal{O}_{\overline{EG}}(z)$ contained in $V_1 \cap V_2$.

If $z = \eta \in \partial X$ By the results from the previous paragraph, let $O_1, O_2 \in \mathcal{O}_{\overline{X}}(\eta)$ such that $V_{O_1}(\eta) \subset U_1$ and $V_{O_2}(\eta) \subset U_2$. Choosing a neighbourhood $W \in \mathcal{V}_{\overline{X}}(\eta)$ contained in $O_1 \cap O_2$, it follows that $V_W(\eta) \subset U_1 \cap U_2$.

If $z = \xi \in \partial_{\mathrm{Stab}} G$ By the results from the previous paragraph, let $V_{\mathcal{U}_1,\varepsilon_1}(\xi), V_{\mathcal{U}_2,\varepsilon_2}(\xi)$ such that $V_{\mathcal{U}_1,\varepsilon_1}(\xi) \subset U_1$ and $V_{\mathcal{U}_2,\varepsilon_2}(\xi) \subset U_2$. Let $\mathcal{U}$ be a ξ–family that is d_ξ–nested in $\{(U_1)_v \cap (U_2)_v, v \in V(\xi)\}$, and let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. It follows from Lemma 6.6 that $V_{\mathcal{U},\varepsilon}(\xi) \subset V_{\mathcal{U}_1,\varepsilon_1}(\xi) \cap V_{\mathcal{U}_2,\varepsilon_2}(\xi) \subset U_1 \cap U_2$.

To prove that this topology is second countable, we define countably many open sets $(U_n)_{n \geq 0}$ such that for every open set U in $\mathcal{O}_{\overline{EG}}$ and every x in U, there exists an integer m such that $x \in U_m \subset U$.

Since EG is the realisation of a complex of spaces over a simplicial complex with countably many simplices, and with fibres that have a CW–structure with countably many cells, it inherits a CW–complex structure with countably many cells. Thus its topology is second countable, and we can choose a countable basis of neighbourhoods $(U_n), n \geq 0$, of EG.

Since X is a simplicial complex with countably many cells, it is a separable space, hence so is the set Λ of points lying on a geodesic from v_0 to a point of ∂X (note that a given geodesic segment may not necessarily be extendable to a geodesic ray). Let Λ' be a dense countable subset of Λ. Now the family of open sets $V_{r,\varepsilon}(\eta)$ for $\eta \in \partial X$, $\gamma_\eta(r) \in \Lambda'$ and $\varepsilon \in \mathbb{Q}$ is a countable family, yielding a countable family of open neighbourhoods of $\overline{EG}$, denoted $(V_n)_{n \geq 0}$. Note that $(V_n)_{n \geq 0}$ contains a basis of neighbourhoods for every point of $\overline{EG}$ that belongs to ∂X.

A neighbourhood of a point ξ of $\partial_{\mathrm{Stab}} G$ is defined by choosing a constant $\varepsilon \in (0,1)$, a finite subcomplex of X (the domain of ξ), and for every vertex v of that subcomplex an open set of $\overline{EG_v}$. Since domains of points of $\partial_{\mathrm{Stab}} G$ are finite by Proposition 4.2, there are only countably many such subcomplexes. Furthermore, for every vertex v of X, $\overline{EG_v}$ has a countable basis of neighbourhoods. It is now clear that we can define a countable family $(W_n)_{n \geq 0}$ of open neighbourhoods, containing a basis of neighbourhoods of every element of $\partial_{\mathrm{Stab}} G$.

The family consisting of all the U_n, V_n, W_n is now a countable basis of neighbourhoods of $\overline{EG}$.

Finally, the subset EG, which is open by construction of the topology, is dense in $\overline{EG}$ since every open set in that basis of neighbourhoods meets EG by construction. □

Lemma 6.18 *The topology of $\overline{EG}$ does not depend on the choice of a basepoint. Moreover, the action of G on EG continuously extends to ∂G.*

Proof Choose x_0 and x_1 two points of X (note that we do not assume these points to be vertices). Throughout this proof, we will indicate the dependence on the basepoint by indicating it in superscript, as explained in Definition 6.5. It is a well known fact that the topology of $\overline{X}$ does not depend on the basepoint, so it is enough to consider neighbourhoods of points in $\partial_{\mathrm{Stab}} G$.

Recall that the number of simplices in a domain $D(\xi)$, $\xi \in \partial_{\mathrm{Stab}} G$ is uniformly bounded by the constant $d_{\max}$ defined in Corollary 4.3. Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}_0$ a ξ–family for the topology centred at x_0 and $\varepsilon > 0$. Now let $\mathcal{U}_1$ be a ξ–family for the topology centred at x_1, which is $2d_{\max}$–refined in $\mathcal{U}_0$. Let x be a point of $\widetilde{\mathrm{Cone}}^{x_1}_{\mathcal{U}_1,\varepsilon}(\xi)$. Then the geodesic reattachment Lemma 5.8 implies that $[x_0, x]$ meets $D(\xi)$. We can thus apply Lemma 3.7 to subsegments of $[x_0, x]$ and $[x_1, x]$, and to simplices $\sigma^{x_0}_{\xi,\varepsilon}(x)$ and $\sigma^{x_1}_{\xi,\varepsilon}(x)$. Since $\mathcal{U}_1$ is $2d_{\max}$–refined in $\mathcal{U}_0$, it follows that

$$x \in \widetilde{\mathrm{Cone}}^{x_0}_{\mathcal{U}_0,\varepsilon}(\xi), \quad \text{hence } \widetilde{\mathrm{Cone}}^{x_1}_{\mathcal{U}_1,\varepsilon}(\xi) \subset \widetilde{\mathrm{Cone}}^{x_0}_{\mathcal{U}_0,\varepsilon}(\xi).$$

Moreover, since $\mathcal{U}_1$ is contained in $\mathcal{U}_0$, we get $V^{x_1}_{\mathcal{U}_1,\varepsilon}(\xi) \subset V^{x_0}_{\mathcal{U}_0,\varepsilon}(\xi)$.

We extend the G–action on EG to ∂G as follows. First note that the action naturally extends to ∂X. Indeed, G acts on the CAT(0) space X by isometries, and those isometries naturally extend to homeomorphisms of the visual boundary ∂X. Furthermore, we defined in Section 2 a G–action on $\partial_{\mathrm{Stab}} G$. Thus we have an action of G on $\overline{EG}$, which we now prove to be continuous.

Let $g \in G$. Since EG is open in $\overline{EG}$ and the action of G on EG is continuous, it is enough to check the continuity at points of ∂G. For a point $z \in \partial G$, the element g sends a basis of neighbourhoods of z for the topology centred at v_0 to a basis of neighbourhoods of $g.z$ for the topology centred at $g.v_0$. Since the topology does not depend on the basepoint by the above discussion, the action of g is continuous at points of ∂G. □

6.3 Induced topologies

Proposition 6.19 *The topology of $\overline{EG}$ induces the natural topologies on EG, ∂X and $\overline{EG}_v$ for every vertex v of X.*

We first prove that for any open set U in the basis of neighbourhoods $\mathcal{O}_{\overline{EG}}$ previously defined, $U \cap EG$ is open in EG. For $x \in EG$, the result is obvious for points in $\mathcal{O}_{\overline{EG}}(x)$ since open sets in $\mathcal{O}_{\overline{EG}}(x)$ are open sets of EG by definition. For $\eta \in \partial X$ and U a neighbourhood of η in $\overline{X}$, we have $V_U(\eta) \cap EG = p^{-1}(U \cap X)$, which is open in EG. For $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ–family, it was proven in Lemma 6.14 that $V_{\mathcal{U},\varepsilon}(\xi) \cap EG$ is open in EG.

We now prove that for any open set U in the basis of neighbourhoods $\mathcal{O}_{\overline{EG}}$, $U \cap \partial X$ is open in ∂X. For a point $\eta \in \partial X$ and U a neighbourhood of η in $\overline{X}$, we have $V_U(\eta) \cap \partial X = U \cap \partial X$, which is open in ∂X. Now consider $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ–family. If $V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$ is empty there is nothing to prove. Otherwise, let $\eta \in V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$. By Lemma 6.13, let U' be a neighbourhood of η in $\overline{X}$ such that $\mathcal{V}_{U'}(\eta) \subset V_{\mathcal{U},\varepsilon}(\xi)$. Thus, $\eta \in U' \cap \partial X \subset V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$, and $V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$ is open in ∂X.

Before proving the analogous result for $\overline{EG}_v$, with v a vertex of X, we need the following lemma.

Lemma 6.20 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\mathcal{U}$ a ξ–family and $\varepsilon \in (0,1)$. Recall that $d_{\max}$ was defined in Corollary 4.3 as an integer such that domains of points of $\partial_{\mathrm{Stab}} G$ meet at most $d_{\max}$ simplices. Let $\mathcal{U}'$ be a ξ–family which is $d_{\max}$–refined in $\mathcal{U}$. Then we have $\bigcup_{v \in D(\xi)} U'_v \cap \partial G_v \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof Let $\xi' \in \bigcup_{v \in D(\xi)} U'_v \cap \partial G_v$ and $x \in D(\xi')$. If x is a vertex of $D(\xi) \cap D(\xi')$, the definition of a ξ–family implies that $\xi' \in U_x$. Otherwise, since $D(\xi')$ is convex by Proposition 4.2, let γ be a geodesic path in $D(\xi')$ from x to $D(\xi)$ and meeting $D(\xi)$ at a single point. This yields a path of open simplices from a simplex $\sigma \subset \mathrm{N}(D(\xi)) \setminus D(\xi)$ to σ_x of length at most $d_{\max}$ in $D(\xi') \setminus D(\xi)$. Since $\xi' \in \bigcup_{v \in D(\xi)} U'_v \cap \partial G_v$ also belongs to ∂G_σ, we have $\sigma \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$. Now since $\mathcal{U}'$ is $d_{\max}$–refined in $\mathcal{U}$, we get $\sigma_x \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ by Lemma 5.10. □

Proof of Proposition 6.19 Let v be a vertex of X. We now prove that for every open set U in the basis of neighbourhood $\mathcal{O}_{\overline{EG}}$, $U \cap \overline{EG}_v$ is open in $\overline{EG}_v$.

We proved already that the topology of $\overline{EG}$ induces the natural topology on EG. Now using the filtration lemmas Lemma 6.12 and Lemma 6.15, it is enough to show, for every $\xi \in \partial G_v$, every $\varepsilon \in (0,1)$ and every ξ–family $\mathcal{U}$, that $V_{\mathcal{U},\varepsilon}(\xi) \cap \overline{EG}_v$ contains a neighbourhood of ξ in $\overline{EG}_v$. By Lemma 6.20, let $\mathcal{U}'$ be a ξ–family contained in $\mathcal{U}$ and such that every point of $U'_v \cap \partial G_v$ belongs to $V_{\mathcal{U},\varepsilon}(\xi)$. Then we have $\xi \in U'_v \subset V_{\mathcal{U},\varepsilon}(\xi) \cap \overline{EG}_v$, and so $V_{\mathcal{U},\varepsilon}(\xi) \cap \overline{EG}_v$ is open in $\overline{EG}_v$. Thus the topology of $\overline{EG}$ induces the natural topology on $\overline{EG}_v$.

Finally, note that the map $\overline{EG}_v \to \overline{EG}$ is injective by Proposition 4.4. As $\overline{EG}_v$ is a compact space, that map is an embedding. □

In the exact same way, we can prove the following:

Lemma 6.21 *Let σ be a closed cell of X. Then the quotient map $\sigma \times \overline{EG_\sigma} \to \overline{EG}$ is continuous.* $\square$

7 Metrisability of $\overline{EG}$

In this section, we prove that $\overline{EG}$ is a compact metrisable space. Recall that by the classical metrisation theorem, it is enough to prove that $\overline{EG}$ is a second countable Hausdorff regular space (see below for definitions) that is sequentially compact.

7.1 Weak separation

In this paragraph, we prove the following:

Proposition 7.1 *The space $\overline{EG}$ satisfies the T_0 condition, that is, for every pair of distinct points, there is an open set of $\overline{EG}$ containing one but not the other.*

Note that this property does not imply that the space is Hausdorff. However, we will prove in the next subsection that $\overline{EG}$ is also *regular*, and it is a common result of point-set topology that a space that is T_0 and regular is also Hausdorff. As usual, the proof of Proposition 7.1 splits in many cases.

Lemma 7.2 *Let $\tilde{x}, \tilde{y}$ be two distinct points of $EG \subset \overline{EG}$. Then $\tilde{x}$ and $\tilde{y}$ admit disjoint neighbourhoods.*

Proof Open sets in EG are open in $\overline{EG}$ by definition. The result thus follows from the fact that EG is a Hausdorff space. $\square$

Lemma 7.3 *Let η, η' be two distinct points of $\partial X \subset \overline{EG}$. Then η and η' admit disjoint neighbourhoods.*

Proof The space $\overline{X}$ is metrisable, hence Hausdorff. Choosing disjoint neighbourhoods U of η in $\overline{X}$ (resp. U' of η' in $\overline{X}$) yields disjoint neighbourhoods $V_U(\eta), V_{U'}(\eta')$. $\square$

Lemma 7.4 *Let $\tilde{x} \in EG$ and $\eta \in \partial X$. Then $\tilde{x}$ and η admit disjoint neighbourhoods.*

Proof Let $x = p(\tilde{x}) \in X$. Since $\overline{X}$ is a Hausdorff space, we can choose a neighbourhood U of x in $\overline{X}$ and a neighbourhood U' of η' in $\overline{X}$ that are disjoint. Then $p^{-1}(U)$ is a neighbourhood of $\tilde{x}$ in $\overline{EG}$ and $V_{U'}(\eta)$ is a neighbourhood of η in $\overline{EG}$ that is disjoint from $p^{-1}(U)$. $\square$

Lemma 7.5 *Let $\xi \in \partial_{\mathrm{Stab}} G$ and $\eta \in \partial X$. Then there exists a neighbourhood of η in $\overline{EG}$ that does not contain ξ.*

Proof Since $D(\xi)$ is bounded, let $R > 0$ such that the $D(\xi)$ is contained in the R–ball centred at v_0. Now take a neighbourhood U of η in $\overline{X}$ that does not meet that R–ball. The subset $V_U(\eta)$ is a neighbourhood of η in $\overline{EG}$ to which ξ does not belong. □

Lemma 7.6 *Let $\widetilde{x} \in EG$ and $\xi \in \partial_{\mathrm{Stab}} G$. Then there exists a neighbourhood of $\widetilde{x}$ in $\overline{EG}$ that does not contain ξ.*

Proof Choose any neighbourhood of $\widetilde{x}$ in EG. This is by definition a neighbourhood of $\widetilde{x}$ in $\overline{EG}$, to which ξ does not belong. □

Lemma 7.7 *Let ξ, ξ' be two different points of $\partial_{\mathrm{Stab}} G$. Then there exists a neighbourhood of ξ in $\overline{EG}$ that does not contain ξ'.*

Proof If $D(\xi) \cap D(\xi') \neq \varnothing$, let v be a vertex in that intersection and let U_v be a neighbourhood of ξ in $\overline{EG}_v$ that does not contain ξ'. Now we can take a ξ–family $\mathcal{U}'$ small enough so that $U'_v \subset U_v$ and thus $\xi' \notin V_{\mathcal{U}', \frac{1}{2}}(\xi)$ by Proposition 6.19.

If $D(\xi) \cap D(\xi') = \varnothing$, let $x \in D(\xi')$. There are two cases to consider:

- If $[v_0, x]$ does not meet $D(\xi)$, then $V_{\mathcal{U}_\xi, \frac{1}{2}}(\xi)$ does not contain ξ' by Corollary 5.9.
- Otherwise, $[v_0, x]$ meets $D(\xi)$ and leaves it. Let σ be the first simplex touched by $[v_0, x]$ after leaving $D(\xi)$, v a vertex of $\sigma \cap D(\xi)$ and U_v a neighbourhood of ξ in $\overline{EG}_v$ that does not contain $\overline{EG}_\sigma$. Now let $\mathcal{U}'$ be ξ–family such that $U'_v \subset U_v$ and $\mathcal{U}''$ a ξ–family that is d_ξ–nested in $\mathcal{U}'$. It then follows from the crossing Lemma 5.4 that $\xi' \notin V_{\mathcal{U}'', \frac{1}{2}}(\xi)$. □

7.2 Regularity

In this paragraph, we prove the following:

Proposition 7.8 *The space $\overline{EG}$ is regular, that is, for every open set U in $\overline{EG}$ and every point $x \in U$, there exists another open set U' containing x and contained in U, and such that every point of $\overline{EG} \setminus U$ admits a neighbourhood that does not meet U'.*

Since we previously defined a basis of neighbourhoods for $\overline{EG}$, it is enough to prove such a proposition for open sets U in that basis. As usual, the proof of Proposition 7.8 splits in many cases, depending on the nature of the open sets U and points of U involved.

Lemma 7.9 *Let $\widetilde{x} \in EG$ and U an open neighbourhood of $\widetilde{x}$ in $\overline{EG}$. Then there exists a subneighbourhood U' of $\overline{EG}$ containing $\widetilde{x}$ and such that every point in $\overline{EG} \setminus U$ admits a neighbourhood that does not meet U'.*

Proof The space EG being a CW–complex, its topology is regular, so we can choose a neighbourhood U' of $\widetilde{x}$ in EG whose closure (in EG) is contained in U. Let us call V that closure, and let $x = p(\widetilde{x})$. Since EG is locally finite, we can further assume that $p(V)$ meets only finitely many simplices and that it is contained in $\mathrm{st}(\sigma_x)$. We now show that V is closed in $\overline{EG}$, which implies the proposition.

A point of $EG \setminus V$ clearly admits a neighbourhood in $\overline{EG}$ that does not meet V, since open subsets of EG are open in $\overline{EG}$. For a point $\eta \in \partial X$, choosing any neighbourhood of η in $\overline{X}$ that does not meet $p(V)$ yields a neighbourhood of η in $\overline{EG}$ not meeting V. Thus the only case left is that of a point $\xi \in \partial_{\mathrm{Stab}} G$. There are two cases to consider:

If $x \in D(\xi)$, then since $p(V)$ meets only finitely many simplices, it is easy to find a ξ–family $\mathcal{U}$ such that $W_{\mathcal{U},\frac{1}{2}}(\xi)$ misses V, which implies that the whole $V_{\mathcal{U},\frac{1}{2}}(\xi)$ misses V.

If $x \notin D(\xi)$, then Lemma 3.5 ensures the existence of a finite subcomplex $K \subset X$ containing $\mathrm{Geod}(v_0, p(V))$. We define a ξ–family $\mathcal{U}$ and a constant ε as follows. Let v be a vertex of $D(\xi)$. For every $\sigma \subset (\mathrm{st}(v) \cap K) \setminus D(\xi)$, let $U_{v,\sigma}$ be a neighbourhood of ξ in $\overline{EG_v}$ that is disjoint from $\overline{EG_\sigma}$. We now set

$$U_v = \bigcap_{\sigma \subset (\mathrm{st}(v) \cap K) \setminus D(\xi)} U_{v,\sigma}.$$

Let $\mathcal{U}$ be a ξ–family which is contained in $\{U_v, v \in V(\xi)\}$, and choose

$$\varepsilon = \min\left(\tfrac{1}{3} \operatorname{dist}(p(V), D(\xi)), 1\right),$$

which is positive since $p(V) \subset \mathrm{st}(\sigma_x)$.

We now show by contradiction that $V_{\mathcal{U},\varepsilon}(\xi) \cap V = \varnothing$. Suppose there exists a point $\widetilde{y}$ in that intersection and let $y = p(\widetilde{y})$. By Corollary 5.9, $[v_0, y]$ goes through $D(\xi)$. But since $\widetilde{y} \in V$, we have $\sigma_{\xi,\varepsilon}(y) \subset K$, which contradicts the construction of $\mathcal{U}$.

Thus every point of $\overline{EG} \setminus V$ admits a neighbourhood missing V, so V is closed in $\overline{EG}$. □

Lemma 7.10 *Let $\eta \in \partial X$ and U be an open neighbourhood of η in $\overline{X}$. Then there exists an open neighbourhood U' of η in $\overline{X}$ such that every point not in $V_U(\eta)$ admits a neighbourhood that does not meet $V_{U'}(\eta)$.*

Proof By Lemma 6.3, we first choose a neighbourhood W of η in $\overline{X}$ contained in U and such that $d(W \cap X, X \setminus U) > A+1$, where A is the acylindricity constant. Since $\overline{X}$ is metrisable, hence regular, we can further assume that $\overline{W} \subset U$. Finally, we can choose $R > 0$ and $\delta > 0$ such that $U' = V_{R,\delta}(\eta)$ is contained in W and $B(\gamma_\eta(R), \delta)$ is contained in the open star of the minimal simplex containing $\gamma_\eta(R)$ (recall that γ_η is a parametrisation of the geodesic ray $[v_0, \eta)$). We now show that every point not in $V_U(\eta)$ admits a neighbourhood that does not meet $V_{U'}(\eta)$.

Let $z \in EG \setminus V_U(\eta)$. Then $p(z)$ is not in U, hence not in $\overline{U'}$. Since $\overline{U'}$ is closed in $\overline{X}$, there exists an open set U'' of $\overline{X}$ containing $p(z)$ and such that $U'' \subset X \setminus \overline{U'}$. Then $p^{-1}(U'')$ is open in $\overline{EG}$ and $p^{-1}(U'')$ does not meet $V_U(\eta)$.

Let $\eta' \in \partial X \setminus V_U(\eta)$. Then $\eta' \notin U \cap \partial X$ hence $\eta' \notin \overline{U'}$. Since $\overline{U'}$ is closed in $\overline{X}$, we choose an open set U'' in $\mathcal{O}_{\overline{X}}(\eta)$ disjoint from U'. It is now clear that $V_{U''}(\eta')$ does not meet $V'_{\mathcal{U}}(\eta)$.

Let $\xi \in (\partial_{\mathrm{Stab}} G) \setminus V_U(\eta)$. To find a neighbourhood of ξ that does not meet $V_{U'}(\eta)$, is enough to find a ξ–family $\mathcal{U}'$ such that $U' \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}', \frac{1}{2}}(\xi) = \varnothing$. We define such a ξ–family as follows:

Let $x = \gamma_\eta(R)$. By Lemma 3.5, let K be the finite subcomplex of X spanned by open simplices meeting $\mathrm{Geod}(D(\xi), x)$. Let v be a vertex of $D(\xi)$. For every simplex σ contained in $(\mathrm{st}(v) \cap K) \setminus D(\xi)$, let $U_{v,\sigma}$ be an open neighbourhood of ξ in $\overline{EG_v}$ disjoint from $\overline{EG_\sigma}$. We then set

$$V_v = \bigcap_{\sigma \subset (\mathrm{st}(v) \cap K) \setminus D(\xi)} U_{v,\sigma}.$$

Now take $\mathcal{U}$ to be a ξ–family contained in $\{V_v, v \in V(\xi)\}$, and let $\mathcal{U}'$ be a ξ–family that is 2–refined in $\mathcal{U}$.

We now show by contradiction that $U' \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}', \frac{1}{2}}(\xi) = \varnothing$. Let y be an point of this intersection. Then $[v_0, y]$ meets $D(\xi)$ (by Corollary 5.9) and $B(x, \delta) \cap S(v_0, R)$ (by construction of U').

Since $d(U', X \setminus U) \geq A+1$ and $D(\xi)$ meets $X \setminus U$, it follows that $\mathrm{N}(D(\xi)) \cap U' = \varnothing$. Hence the geodesic segment $[v_0, y]$ enters $D(\xi)$ before meeting $B(x, \delta) \cap S(v_0, R)$. Let y' be the point of $[v_0, y]$ inside $B(x, \delta) \cap S(v_0, R)$. By construction of R and δ, it follows that $\sigma_{y'}$ is in the open star of σ_x. Now since $x \in \mathrm{Cone}_{\mathcal{U}', \frac{1}{2}}(\xi)$, the refinement Lemma 5.10 implies that $\sigma_{y'} \subset \mathrm{Cone}_{\mathcal{U}, \frac{1}{2}}(\xi)$, which contradicts the definition of $\mathcal{U}$. □

Lemma 7.11 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ–family. Then there exists a ξ–family $\mathcal{U}'$ such that every point not in $V_{\mathcal{U},\varepsilon}(\xi)$ admits a neighbourhood that misses $V_{\mathcal{U}',\varepsilon}(\xi)$.*

Proof Recall that domains of points of $\partial_{\mathrm{Stab}} G$ contain at most $d_{\max}$ simplices (see Corollary 4.3). Choose a ξ–family $\mathcal{U}'$ that is $d_{\max}$–refined and nested in $\mathcal{U}$. We now show that every point not in $V_{\mathcal{U},\varepsilon}(\xi)$ admits a neighbourhood that misses $V_{\mathcal{U}',\varepsilon}(\xi)$.

Let $\widetilde{x} \in EG \setminus V_{\mathcal{U},\varepsilon}(\xi)$, and $x = p(\widetilde{x})$.

- If $x \in \overline{D^\varepsilon(\xi)}$, let v be a vertex of $D(\xi) \cap \sigma_x$. We have $\phi_{v,\sigma_x}(\widetilde{x}) \notin U_v$, hence $\phi_{v,\sigma_x}(\widetilde{x}) \notin \overline{U_v'}$. Let W_x be a neighbourhood of $\phi_{v,\sigma_x}(\widetilde{x})$ in $\overline{EG_v}$ that does not meet U_v', and V be an open neighbourhood of x in X contained in $\mathrm{st}(\sigma_x)$. Let W be the neighbourhood of $\widetilde{x}$ consisting of those elements $\widetilde{y} \in EG$ whose projection $p(\widetilde{y})$ is in V and such that $\phi_{v,\sigma_x}(\widetilde{y})$ belongs to W_x. Since $\mathcal{U}'$ is refined in $\mathcal{U}$, it then follows that W is a neighbourhood of $\widetilde{x}$ that does not meet $V_{\mathcal{U}',\varepsilon}(\xi)$.
- If $x \notin \overline{D^\varepsilon(\xi)}$, let V be an open neighbourhood of x in $X \setminus D^\varepsilon(\xi)$ contained in $\mathrm{st}(\sigma_x)$. As $\mathcal{U}'$ is refined in $\mathcal{U}$ and $x \notin V_{\mathcal{U},\varepsilon}(\xi)$, Lemma 5.10 implies that $p^{-1}(V)$ is a neighbourhood of $\widetilde{x}$ that does not meet $V_{\mathcal{U}',\varepsilon}(\xi)$.

Let $\eta \in \partial X \setminus V_{\mathcal{U},\varepsilon}(\xi)$. We construct a neighbourhood V of η in $\overline{X}$ that does not meet $\widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi)$. First, since $D(\xi)$ is bounded, let $R > 0$ such that $D(\xi)$ is contained in the R–ball centred at v_0, and let $x = \gamma_\eta(R+1)$.

- If $[v_0, \eta)$ does not meet $D(\xi)$, let $\delta = \frac{1}{2}\,\mathrm{dist}(\gamma_\eta([0, R+1]), D(\xi)) > 0$, and let V be a neighbourhood of η in $\overline{X}$ that is contained in $V_{R+1,\delta}(\eta)$. For every y in V, $[v_0, y]$ does not meet $D(\xi)$, hence $V \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi) = \varnothing$.
- If $[v_0, \eta)$ goes through $D(\xi)$, then since x does not belong to $\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, let v be a vertex of $D(\xi)$ in $\sigma_{\xi,\varepsilon}(x)$ such that $\overline{EG}_{\sigma_{\xi,\varepsilon}}(x) \not\subseteq U_v$ in $\overline{EG_v}$. Lemma 5.11 yields a constant $\delta > 0$ such that for every $y \in B(x, \delta)$, $[v_0, y]$ goes through $D^\varepsilon(\xi)$ and $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x))$. Let $V := V_{R+1,\delta}(\eta)$ and $y \in V$. Then $[v_0, y]$ goes through $B(x, \delta)$, hence $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}(\sigma_{\xi,\varepsilon}(x))$. As $\mathcal{U}'$ is nested in $\mathcal{U}$ and $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \not\subseteq U_v$ in $\overline{EG_v}$, it follows that $\overline{EG}_{\sigma_{\xi,\varepsilon}(y)} \not\subseteq U_v'$, hence
$$y \notin \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon'}(\xi) \quad \text{and} \quad V \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon'}(\xi) = \varnothing.$$

Let $\xi' \in (\partial_{\mathrm{Stab}} G) \setminus V_{\mathcal{U},\varepsilon}(\xi)$. To find a neighbourhood of ξ' that misses $V_{\mathcal{U}',\varepsilon}(\xi)$, it is enough, since cones are open subsets of X by Corollary 5.12, to find a ξ'–family $\mathcal{U}''$ such that $\widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon}(\xi') \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi) = \varnothing$ and such that for every vertex v of $D(\xi) \cap D(\xi')$, we have $U_v' \cap U_v'' = \varnothing$. We define such a ξ'–family as follows. By Lemma 3.5, let K be a finite subcomplex containing $\mathrm{Geod}(v_0, D(\xi))$. Let v be a vertex of $D(\xi')$. For every $\sigma \subset (\mathrm{st}(v) \cap K) \setminus D(\xi')$, let $U''_{v,\sigma}$ be a neighbourhood of ξ' in $\overline{EG}_v$ that is disjoint from $\overline{EG}_\sigma$, and set

$$U''_v = \bigcap_{\sigma \subset (\mathrm{st}(v) \cap K \setminus D(\xi')} U''_{v,\sigma}.$$

If v is also in $D(\xi)$, note that since the closure of U'_v is contained in U_v, we can assume that $U'_v \cap U''_v = \varnothing$. Furthermore, we can assume by the convergence property Definition 4.8 that the only EG_σ inside EG_v meeting both U_v and U''_v contains ξ and ξ'. Now let $\mathcal{U}''$ be a ξ'–family which is $d_{\max}$–refined in $\{U''_v, v \in D(\xi')\}$.

Let us prove by contradiction that $\widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon'}(\xi') \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi) = \varnothing$. Let x be in such an intersection. Then, by Corollary 5.9, the geodesic $[v_0, x]$ goes through both $D(\xi)$ and $D(\xi')$. Note that, by construction of the various neighbourhoods U''_v, the geodesic segment $[v_0, x]$ cannot leave $D(\xi')$ before leaving $D(\xi)$, nor can it leave both $D(\xi)$ and $D(\xi')$ at the same time. If $D(\xi) \cap D(\xi') = \varnothing$, it follows from the fact that $\mathcal{U}'$ is $d_{\max}$–refined in $\mathcal{U}$ that $D(\xi') \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ by Lemma 5.10, hence $\xi' \in V_{\mathcal{U},\varepsilon}(\xi)$, which is absurd. Otherwise, let x' be the last point of $D(\xi')$ met by $[v_0, x]$ and let γ be a geodesic path in $D(\xi')$ from x' to a point of $D(\xi)$, such that γ meets $D(\xi)$ in exactly one point. Let σ be the last simplex touched by γ before touching $D(\xi)$. The fact that $\mathcal{U}'$ is $d_{\max}$–refined in $\mathcal{U}$ implies that $\overline{EG}_\sigma \subset U_v$ for some (hence every) vertex v of $\sigma \cap D(\xi)$ by Lemma 5.10, hence $\xi' \in U_v \subset V_{\mathcal{U},\varepsilon}(\xi)$, a contradiction.

Finally, for every vertex v of $D(\xi) \cap D(\xi')$, we have $U'_v \cap U''_v = \varnothing$ by construction of U''_v, hence the result. □

Theorem 7.12 *The space $\overline{EG}$ is separable and metrisable.*

Proof It is second countable by Theorem 6.17, regular by Proposition 7.8 and satisfies the T_0 condition by Proposition 7.1. Thus it is Hausdorff and the result follows from Urysohn's metrisation theorem. □

7.3 Sequential compactness

In this subsection, we prove the following:

Theorem 7.13 *The metrisable space $\overline{EG}$ is compact.*

First of all, note that since EG is dense in $\overline{EG}$ by Theorem 6.17, it is enough to prove that any sequence in EG admits a subsequence converging in $\overline{EG}$. Let $(\widetilde{x_n})_{n \geq 0} \in (EG)^{\mathbb{N}}$. For every $n \geq 0$, let $x_n = p(\widetilde{x_n})$. Furthermore, to every x_n we associate the finite sequence $\sigma_0^{(n)} = v_0, \sigma_1^{(n)}, \ldots$ of simplices met by $[v_0, x_n]$. Finally, let $l_n \geq 1$ be the number of simplices of such a sequence.

Lemma 7.14 *Suppose that for all $k \geq 0$, $\{\sigma_k^{(n)}, n \geq 0\}$ is finite.*

- *If (l_n) admits a bounded subsequence, then $(\widetilde{x_n})$ admits a subsequence that converges to a point of $EG \cup \partial_{\mathrm{Stab}} G$.*
- *Otherwise, $(\widetilde{x_n})$ admits a subsequence that converges to a point of ∂X.*

Proof Up to a subsequence, we can assume that there exist open simplices $\sigma_0, \sigma_1, \dots$ such that for all $k \geq 0$, $(\sigma_k^{(n)})_{n \geq 0}$ is eventually constant at σ_k. There are two cases to consider:

(i) Up to a subsequence, there exists a constant $m \geq 0$ such that each geodesic $[v_0, x_n]$ meets at most m simplices. This implies that the x_n live in a finite subcomplex. Up to a subsequence, we can now assume that there exists a (closed) simplex σ of X such that x_n is in the interior of σ for all $n \geq 0$. This in turn implies that $\widetilde{x_n}$ is in $\sigma \times \overline{EG_\sigma}$ (or more precisely in the image of $\sigma \times \overline{EG_\sigma}$ in $\overline{EG}$) for all $n \geq 0$. This space is compact since the canonical map $\sigma \times \overline{EG_\sigma} \hookrightarrow \overline{EG}$ is continuous by Lemma 6.21, hence we can take a convergent subsequence.

(ii) Up to a subsequence, we can assume that $l_n \to \infty$. For $r > 0$, let $\pi_r \colon X \to \overline{B}(v_0, r)$ be the retraction on $\overline{B}(v_0, r)$ along geodesics starting at v_0. By assumption, we have that for every $r > 0$, the sequence of projections $(\pi_r(x_n))_{n \geq 0}$ lies in a finite subcomplex of X. A diagonal argument then shows that, up to a subsequence, we can assume that all the sequences of projections $(\pi_m(x_n))_{n \geq 0}$ converge in X for every $m \geq 0$. As the topology of $\overline{X}$ is the topology of the projective limit
$$\overline{B}(v_0, 1) \xleftarrow{\pi_1} \overline{B}(v_0, 2) \xleftarrow{\pi_2} \cdots,$$
it then follows that (x_n) converges in $\overline{X}$. As $l_n \to \infty$, (x_n) converges to a point η of ∂X. The definition of the topology of $\overline{EG}$ now implies that $(\widetilde{x_n})$ converges to η in $\overline{EG}$. □

Lemma 7.15 *Suppose that there exists $k \geq 0$ such that $\{\sigma_k^{(n)}, n \geq 0\}$ is infinite. Then (x_n) admits a subsequence that converges to a point of $\partial_{\mathrm{Stab}} G$.*

Proof Without loss of generality, we can assume that such a k is minimal. Up to a subsequence, we can assume that there exist open simplices $\sigma_1, \dots, \sigma_{k-1}$ such that for all $n \geq 0$, $\sigma_0^{(n)} = \sigma_0, \dots, \sigma_{k-1}^{(n)} = \sigma_{k-1}$, and $(\sigma_k^{(n)})_{n \geq 0}$ is injective. By cocompactness of the action, we can furthermore assume (up to a subsequence) that the $\sigma_k^{(n)}$ are above a unique simplex of Y. This corresponds to embeddings $\overline{EG}_{\sigma_k^{(n)}} \hookrightarrow \overline{EG}_{\sigma_{k-1}}$. By the convergence property Definition 4.8, we can assume, up to a subsequence, that in

$\overline{EG}_{\sigma_{k-1}}$ the sequence of subspaces $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to a point $\xi \in \partial G_{\sigma_{k-1}}$. Let us prove that $(\widetilde{x_n})_{n\geq 0}$ converges to ξ in $\overline{EG}$.

Since $\overline{EG}$ has a countable basis of neighbourhoods, it is enough to prove that for every $\varepsilon \in (0,1)$ and every ξ–family $\mathcal{U}$ there exists a subsequence of $(\widetilde{x_n})$ lying in $V_{\mathcal{U},\varepsilon}(\xi)$. By construction of ξ, we have $\sigma_{k-1} \subset D(\xi)$, and there exists a vertex v_k of $D(\xi)$ such that $\sigma_k^{(n)} \subset \mathrm{st}(v_k)$ for all $n \geq 0$. Two cases may occur:

- Up to a subsequence, all the $[v_0, x_n]$ leave $D^\varepsilon(\xi)$ inside $\sigma_k^{(n)}$. Since $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG}_{\sigma_{k-1}}$ and thus in $\overline{EG}_{v_k}$, we can assume, up to a subsequence, that $\overline{EG}_{\sigma_k^{(n)}} \subset U_{v_k}$ inside $\overline{EG}_{\sigma_k}$. This implies that $\widetilde{x_n} \in V_{\mathcal{U},\varepsilon}(\xi)$, which is what we wanted.
- Up to a subsequence, all the $[v_0, x_n]$ remain inside $D^\varepsilon(\xi)$ when inside $\sigma_k^{(n)}$. Up to a subsequence, we can further assume that all the $\sigma_{k+1}^{(n)}, n \geq 0$ are above a unique simplex of Y. Thus there exists a vertex v_{k+1} of $D(\xi) \cap \overline{\mathrm{st}}(v_k)$ such that $\sigma_{k+1}^{(n)} \subset \mathrm{st}(v_{k+1})$ for all $n \geq 0$.
  In particular we have $\sigma_k^{(n)} \subset \mathrm{st}(v_k) \cap \mathrm{st}(v_{k+1})$ and thus $\xi \in \partial G_{v_{k+1}}$. Since $\mathcal{U}$ is a ξ–family, the fact that $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG}_{v_k}$ implies that $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG}_{v_{k+1}}$. Note that since the sequence $(\sigma_k^{(n)})_{n\geq 0}$ takes infinitely many values, the finiteness lemma Lemma 3.5 implies that $(\sigma_{k+1}^{(n)})_{n\geq 0}$ also takes infinitely many values. Up to a subsequence, we can thus assume by the convergence property Definition 4.8 that $\overline{EG}_{\sigma_{k+1}^{(n)}}$ uniformly converges in $\overline{EG}_{v_{k+1}}$. As $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG}_{v_{k+1}}$, the same holds for $\overline{EG}_{\sigma_{k+1}^{(n)}}$, and we are back to the previous situation.

By iterating this algorithm, two cases may occur:

- There is a value $k' \geq k$ such that, up to a subsequence, all the $[v_0, x_n]$ leave $D^\varepsilon(\xi)$ while being inside $\sigma_{k'}^{(n)}$ and the same argument as before shows that we can take a subsequence satisfying $\widetilde{x_n} \in V_{\mathcal{U},\varepsilon}(\xi)$.
- Up to a subsequence, at every stage $k' \geq k$ all the $[v_0, x_n]$ remain within $D^\epsilon(\xi)$.
  In the latter case, the containment lemma Proposition 3.3 implies that there exists an integer $m \geq 0$ such that each geodesic segment $[v_0, x_n]$ meets at most m simplices. Up to a subsequence, we can further assume that all the $[v_0, x_n]$ meet exactly m simplices. Thus we can iterate our algorithm up to rank m, which yields the existence of a vertex v_m of $D^\varepsilon(\xi)$ such that $\sigma_m^{(n)} \subset \mathrm{st}(v_m)$ for all $n \geq 0$ and such that $\overline{EG}_{\sigma_m^{(n)}}$ uniformly converges to ξ in $\overline{EG}_{v_m}$. Up to a subsequence, we can furthermore assume that $\overline{EG}_{\sigma_m^{(n)}} \subset U_m$ in $\overline{EG}_{v_{k+1}}$ for all $n \geq 0$. This in turn implies $\widetilde{x_n} \in W_{\mathcal{U},\varepsilon}(\xi)$, hence $\widetilde{x_n} \in V_{\mathcal{U},\varepsilon}(\xi)$ and we are done. $\square$

Proof of Theorem 7.13 This follows immediately from Theorem 7.12, Lemma 7.14 and Lemma 7.15. □

As a direct consequence, we get the following convergence criterion.

Corollary 7.16 *Let (K_n) be a sequence of subsets of $\overline{EG}$.*

- *K_n uniformly converges to a point $\eta \in \partial X$ if and only if the sequence of coarse projections $\overline{p}(K_n)$ uniformly converges to η in $\overline{X}$.*
- *Suppose that there exists $\xi \in \partial_{\mathrm{Stab}} G$ such that, for n large enough, every geodesic from v_0 to a point of $\overline{p}(K_n)$ goes through $D(\xi)$. For every such n and every $z \in K_n$, choose $x \in \overline{p}(z)$ and let $\sigma_{n,x}$ be the first simplex touched by the geodesic $[v_0, x]$ after leaving $D(\xi)$. If there exists a vertex $v \in D(\xi)$ contained in each $\sigma_{n,x}$ and such that for every neighbourhood U of ξ in $\overline{EG_v}$, there exists an integer $N \geq 0$ such that for every $(n, x) \in \bigcup_{n \geq N} \{n\} \times K_n$, we have $\overline{EG}_{\sigma_{n,x}} \subset U$, then (K_n) uniformly converges to ξ.* □

8 The properties of ∂G

In this section we prove the following:

Theorem 8.1 *$(\overline{EG}, \partial G)$ is an $E\mathcal{Z}$–structure in the sense of Farrell and Lafont.*

8.1 The $\mathcal{Z}$–set property

Here we prove the following:

Proposition 8.2 *∂G is a $\mathcal{Z}$–set in $\overline{EG}$.*

Proving this property is generally technical. However, Bestvina and Mess proved in [4] a useful lemma ensuring that a given set is a $\mathcal{Z}$–set in a bigger set, which we now recall.

Lemma 8.3 (Bestvina and Mess [4]) *Let $(\tilde{x}, Z)$ be a pair of finite-dimensional metrisable compact spaces with Z nowhere dense in $\tilde{x}$, and such that $X = \tilde{x} \setminus Z$ is contractible and locally contractible, with the following condition holding:*

(∗) *For every $z \in Z$ and every neighbourhood $\widetilde{U}$ of z in $\tilde{x}$, there exists a neighbourhood $\widetilde{V}$ contained in $\widetilde{U}$ and such that*

$$\widetilde{V} \setminus Z \hookrightarrow \widetilde{U} \setminus Z$$

is null-homotopic.

Then $\tilde{x}$ is an Euclidean retract and Z is a $\mathcal{Z}$–set in $\tilde{x}$.

We now use this lemma to prove that the boundary ∂G is a $\mathcal{Z}$–boundary in $\overline{EG}$.

Lemma 8.4 *$\overline{EG}$ and ∂G are finite-dimensional.*

Proof We have

$$\partial G = \Big(\bigcup_{v \in V(X)} \partial G_v \Big) \cup \partial X.$$

Each vertex stabiliser boundary is a $\mathcal{Z}$–boundary in the sense of Bestvina, hence finite-dimensional, and they are closed subspaces of ∂G by Proposition 6.19. As the action of G on X is cocompact, their dimension is uniformly bounded above, so the countable union theorem implies that $\bigcup_{v \in V(X)} \partial G_v$ is finite-dimensional. Furthermore, X is a CAT(0) space of finite geometric dimension, so its boundary has finite dimension by a result of Caprace [10]. Thus, the classical union theorem implies that ∂G is finite-dimensional. Now $\overline{EG} = EG \cup \partial G$. EG is a CW–complex that can be decomposed as the countable union of its closed cells, all of which have a dimension bounded above by $\dim(X) \cdot \sup_\sigma (\dim EG_\sigma)$. It follows from the countable union theorem in covering dimension theory that EG is finite-dimensional, and the same holds for $\overline{EG}$ by the classical union theorem. □

We now turn to the proof of the $\mathcal{Z}$–set property, using the lemma of Bestvina and Mess recalled above. As usual, the proof splits in two cases, depending on the nature of the point of ∂G that we consider.

Lemma 8.5 *Let $\eta \in \partial X$ and U be a neighbourhood of η in $\overline{X}$. Then there exists a subneighbourhood $U'' \subset U$ of η in $\overline{X}$ such that the inclusion*

$$V_{U''}(\eta) \setminus \partial G \hookrightarrow V_U(\eta) \setminus \partial G$$

is null-homotopic.

Proof By Lemma 6.3, there exists a neighbourhood U' of η in $\overline{X}$ such that $d(U' \cap X, X \setminus U) > 1$. In particular, $\mathrm{Span}(U' \setminus \partial X) \subset U$, and $p^{-1}(\mathrm{Span}(U' \setminus \partial X))$ can be seen as the realisation of a complex of spaces over $\mathrm{Span}(U' \setminus \partial X)$ the fibres of which are contractible. Thus Proposition 1.10 implies that the projection $p^{-1}(\mathrm{Span}(U' \setminus \partial X)) \to \mathrm{Span}(U' \setminus \partial X)$ is a homotopy equivalence. Now Lemma 6.2 yields another neighbourhood $U'' \subset U'$ of η in $\overline{X}$ such that $U'' \setminus \partial X$ is contractible. We thus have the following commutative diagram:

$$\begin{tikzcd}
V_U(\eta) \setminus \partial G & p^{-1}(\mathrm{Span}(U' \setminus \partial X)) \arrow[l, hook'] \arrow[d, "\simeq"] & V_{U''}(\eta) \setminus \partial G \arrow[l, hook'] \arrow[d] \\
& \mathrm{Span}(U' \setminus \partial X) & U'' \setminus \partial X \arrow[l, hook', "0"']
\end{tikzcd}$$

Now since $U'' \setminus \partial X$ is contractible, the inclusion $V_{U''}(\eta) \setminus \partial G \hookrightarrow V_U(\eta) \setminus \partial G$ is null-homotopic. $\square$

Lemma 8.6 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ–family. Then there exists a ξ–family $\mathcal{U}'$ such that $V_{\mathcal{U}',\varepsilon}(\xi)$ is a subneighbourhood of $V_{\mathcal{U},\varepsilon}(\xi)$ and such that the inclusion*

$$V_{\mathcal{U}',\varepsilon}(\xi) \setminus \partial G \hookrightarrow V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G$$

is null-homotopic.

Lemma 8.7 *There exists a ξ–family $\mathcal{U}''$, a subcomplex X' of X with $\widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon}(\xi) \subset X' \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, and a subset C' of EG with $V_{\mathcal{U}'',\varepsilon}(\xi) \setminus \partial G \subset C' \subset V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G$, such that $p(C') \subset X'$ and the projection map $C' \to X'$ is a homotopy equivalence.*

Proof Let $\mathcal{U}'$ be a ξ–family that is 2–refined in $\mathcal{U}$ and d_ξ–nested in $\mathcal{U}$. It follows from the refinement Lemma 5.10 that

$$\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi)) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi).$$

By Lemma 6.6, we have $V_{\mathcal{U}',\varepsilon}(\xi) \subset V_{\mathcal{U},\varepsilon}(\xi)$. Let

$$X' = \mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi)) \cup (\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)).$$

Note that it is possible to give $\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ a simplicial structure from that of X such that a vertex of $\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ for that structure either is a vertex of $D(\xi)$ or belongs to an edge in X between a vertex of $D(\xi)$ and a vertex of $X \setminus D(\xi)$. Furthermore, we can give $\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi))$ a simplicial structure that is finer than that of X, whose vertices are the vertices of $\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi))$ and vertices of $\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ (for its given simplicial structure), that is compatible with that of $\overline{D^\varepsilon(\xi)}$, and which turns X' into a simplicial complex such that an open simplex is completely contained either in $\overline{D^\varepsilon(\xi)}$ or in $X \setminus D^\varepsilon(\xi)$ (see Figure 5). Thus X' is endowed with a simplicial structure.

We now define a contractible open subset C'_σ of EG_σ for every open simplex σ of X'. This will allow us to define the following subset of EG,

$$C' = \bigcup_{\sigma \in S(X')} \sigma \times C'_\sigma.$$

Note that although C' is not naturally the realisation of a complex of spaces in the sense of the first section, it is nonetheless possible to endow it with one, so as to use Proposition 1.10.

We first define these spaces C'_σ for vertices of X'. Let v be such a vertex.

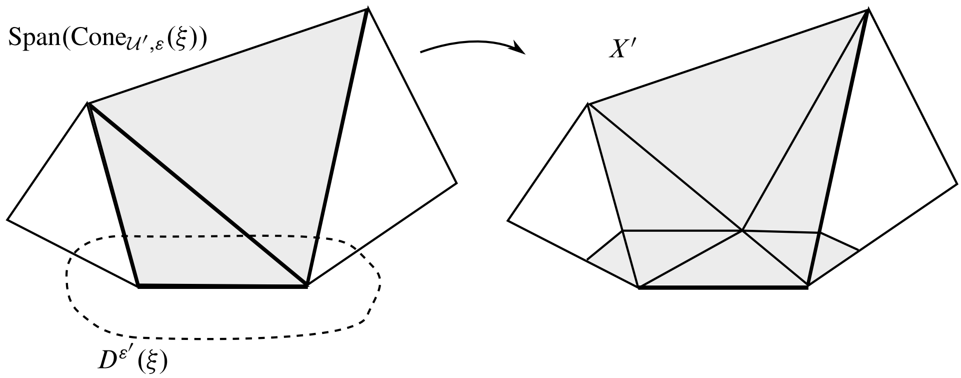

Figure 5

- If v is a vertex of $D(\xi)$, the compactification $\overline{EG_v}$ is locally contractible so we can choose a contractible open set U'_v of $\overline{EG_v}$ contained in $\overline{U}_v$ and containing ξ, and set $C'_v = U'_v \cap EG_v$. As ∂G_v is a $\mathcal{Z}$–boundary, C'_v is a contractible open subset.
- If v does not belong to $D^\varepsilon(\xi)$, set $C'_v = \overline{EG_v}$.
- If v is a vertex of $\overline{D^\varepsilon(\xi)} \setminus D(\xi)$ (for the chosen simplicial structure of $\overline{D^\varepsilon(\xi)} \subset X'$), then either v belongs to $\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi))$, in which case we set $C'_v = EG_v$, or it does not, in which case v belongs to a unique edge e (for the simplicial structure of X) between a vertex v' of $D(\xi)$ and a vertex of $X \setminus D(\xi)$. In that case, $\overline{EG_e}$ is contained in $U_{v'}$ since $\mathcal{U}'$ is nested in $\mathcal{U}$ and we set $C'_v = EG_e$.

We now define the subsets C'_σ for simplices $\sigma \subset X'$. Let σ be such a simplex, and let σ' be the unique open simplex of X such that $\sigma \subset \sigma'$ as subsets of X. We set $C'_\sigma = EG_{\sigma'}$.

We define the space $C' = \bigcup_{\sigma \in S(X')} \sigma \times C'_\sigma$. As explained above, the projection $C' \to X'$ is a homotopy equivalence. Furthermore, we can choose a ξ–family $\mathcal{U}''$ small enough so that the subset $V_{\mathcal{U}'',\varepsilon}(\xi) \setminus \partial G$ is contained in C'. □

Proof of Lemma 8.6 We apply the previous lemma twice to get the following commutative diagram:

$$\begin{tikzcd} V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G & C' \arrow[l, hook'] \arrow[d, "\simeq"] & V_{\mathcal{U}'',\varepsilon}(\xi) \setminus \partial G \arrow[l, hook'] \arrow[d] & C^{(3)} \arrow[l, hook'] \arrow[d, "\simeq"] \\ & X' & \widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon}(\xi) \arrow[l, hook'] & X^{(3)} \arrow[l, hook', "0"'] \end{tikzcd}$$

Since $X^{(3)}$ retracts by strong deformation (along geodesics starting at v_0) inside $\widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi)$ on the contractible subcomplex $D(\xi)$ (relatively to $D(\xi)$), the inclusion

$X^{(3)} \hookrightarrow \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi)$ is null-homotopic, hence $C^{(3)} \hookrightarrow V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G$ is null-homotopic. As there exists a ξ–family $\mathcal{U}^{(4)}$ such that $V_{\mathcal{U}^{(4)},\varepsilon}(\xi) \setminus \partial G \hookrightarrow C^{(3)}$, this concludes the proof. □

Proof of Proposition 8.2 Theorem 7.13 and Lemma 8.4 together with Lemma 8.5 and Lemma 8.6 yield the desired result. □

8.2 Compact sets fade at infinity

Here we prove the following:

Proposition 8.8 *Compacts subsets of EG fade at infinity in $\overline{EG}$, that is, for every $x \in \partial G$, every neighbourhood U of x in $\overline{EG}$ and every compact $K \subset EG$, there exists a subneighbourhood $V \subset U$ of x such that any G–translate of K meeting V is contained in U.*

As usual, we split the proof in two parts, depending on the nature of the points considered.

Proposition 8.9 *Let $\eta \in \partial X$. For every neighbourhood U of η in $\overline{X}$ and every compact subset $K \subset EG$, there exists a neighbourhood U' of η contained in U and such that any G–translate of K meeting $V_{U'}(\eta)$ is contained in $V_U(\eta)$.*

Proof By Lemma 6.3, let U' be a neighbourhood of η in $\overline{X}$ that is contained in U and such that

$$d(U', X \setminus U) > \text{diam}(p(K)).$$

Let $g \in G$ such that gK meets $V_{U'}(\eta)$. Since G acts on X by isometries, we have

$$\text{diam}(p(g.K)) = \text{diam}(g.p(K)) = \text{diam}(p(K)),$$

which implies that $gK \subset V_U(\eta)$. □

The proof for points of $\partial_{\text{Stab}} G$ is slightly more technical. We start by defining a class of compact sets of EG that are easy to handle.

Definition 8.10 Let F be a finite subcomplex of X, together with a collection $(K_\sigma)_{\sigma \in S(F)}$ of nonempty compact subsets of EG_σ for every simplex σ of F. Suppose that for every simplex σ of F and every face σ' of σ, we have $\phi_{\sigma',\sigma}(K_\sigma) \subset K_{\sigma'}$. Then the set

$$\bigcup_{\sigma \in \mathcal{S}(F)} \sigma \times K_\sigma.$$

is called a *standard compact subset of EG over F*. Every compact subset of EG obtained in such a way is called a standard compact of EG.

Note that the projection in X of any compact subset of EG meets finitely many simplices of X, so every compact subset of EG may be seen as a subset of a standard compact subset of EG.

Definition 8.11 Let $\xi \in \partial_{\mathrm{Stab}} G$ and $\mathcal{U}$ a ξ–family. We define $W_{\mathcal{U}}(\xi)$ as the set of points $\widetilde{x}$ of EG whose projection $x \in X$ belongs to the domain of ξ and is such that for some (hence any) vertex v of $\sigma_x \cap D(\xi)$ we have

$$\phi_{v,\sigma_x}(\widetilde{x}) \in U_v.$$

Before proving that compact sets fade near points of $\partial_{\mathrm{Stab}} G$, we prove the following lemma.

Lemma 8.12 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ–family. Let K be a compact subspace of EG. Then there exists a ξ–family $\mathcal{U}'$ contained in $\mathcal{U}$ such that for every point $g \in G$, the following holds:*

If gK meets $W_{\mathcal{U}'}(\xi)$, then $gK \cap p^{-1}(D(\xi))$ is contained in $W_{\mathcal{U}}(\xi)$.

Proof Let L be a standard compact subset of EG over the (finite) full subcomplex of X defined by Span $p(K)$. By choosing the L_σ big enough, we can assume that L contains K. Let $N \geq 0$ be such that any two vertices of L can be joined by a sequence of at most N adjacent vertices.

Since $D(\xi)$ and $p(L)$ meet finitely many vertices of X, there are only finitely many elements of G such that $g.p(L)$ meets $D(\xi)$ up to left multiplication by an element of $G_v, v \in V(\xi)$. Let $(g_\lambda.p(L))_{\lambda\in\Lambda}$ be such a finite family of cosets. For every vertex v of $V(\xi)$, $\{g_\lambda L \cap EG_v, \lambda \in \Lambda\}$ is a finite (possibly empty) collection of compact subsets of EG_v. Since ∂G_v is a Bestvina boundary for G_v, compact subsets fade at infinity in $\overline{EG}_v$, so there exists a subneighbourhood U'_v of U_v such that any G_v–translate of one of these $g_\lambda L$ meeting U'_v is contained in U_v. Repeating this procedure $N+1$ times, we get a sequence of ξ–families denoted

$$\{U_v, v \in V(\xi)\} \supset \mathcal{U}^{[N]} \supset \mathcal{U}^{[N-1]} \supset \cdots \supset \mathcal{U}^{[0]}.$$

Let $g \in G$ such that gK meets $W_{\mathcal{U}'}(\xi)$, and let w be a vertex of $D(\xi)$ such that gK, hence gL, meets $U_w^{[0]}$. In order to prove the lemma, it is enough to show by induction on $k = 0, \ldots, N$ the following:

(H_k) For every chain of adjacent vertices $w_0 = w, w_1, \dots, w_k$ of $D(\xi)$ such that gL meets $EG_{w_0}, \dots, EG_{w_k}$, we have $gL \cap EG_{w_k} \subset U_w^{[k+1]}$.

Since gL meets $D(\xi)$, let $\lambda \in \Lambda$ such that $gL = g_\lambda L$ pointwise. The result is true for $k = 0$ by definition of $\mathcal{U}^{[0]}$ and $\mathcal{U}^{[1]}$. Suppose we have proven it up to rank k, and let $w_0 = w, w_1, \dots, w_{k+1}$ a chain of vertices of $D(\xi)$ such that gL meets $EG_{w_0}, \dots, EG_{w_k}$. By induction hypothesis, we already have $gL \cap EG_{w_k} \subset U_{w_k}^{[k+1]}$. Since $p(L)$ is a full subcomplex of X, it follows from the fact that gL meets EG_{w_k} and $EG_{w_{k+1}}$ that gL also meets $EG_{[w_k, w_{k+1}]}$. In particular, since

$$gL \cap EG_{w_k} \subset U_{w_k}^{[k+1]},$$

it follows from the properties of ξ–families that $gL \cap EG_{w_{k+1}}$ meets $U_{w_{k+1}}^{[k+1]}$. This in turn implies that $gL \cap EG_{w_{k+1}} \subset U_{v_{k+1}}^{[k+2]}$, which concludes the induction. □

Proposition 8.13 *Let $\xi \in \partial_{\mathrm{Stab}} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ–family. Let K be a connected compact subset of EG. Then there exists a ξ–family $\mathcal{U}'$ contained in $\mathcal{U}$ and such that every G–translate of K meeting $V_{\mathcal{U}',\varepsilon}(\xi)$ is contained in $V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof Let k be the number of simplices met by $p(K)$, and let $\mathcal{U}'$ be a ξ–family that is k–refined in $\mathcal{U}$. Applying the previous proposition to $V_{\mathcal{U}',\varepsilon}(\xi)$ yields a ξ–family $\mathcal{U}''$. Finally, let $\mathcal{U}'''$ be a ξ–family that is k–refined in $\mathcal{U}''$.

Suppose that gK meets $V_{\mathcal{U}''',\varepsilon}(\xi)$, and let $\widetilde{x_0} \in gK \cap V_{\mathcal{U}''',\varepsilon}(\xi)$. Let $\widetilde{x} \in gK$, and let us prove that $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$. Since $p(K)$ is connected, let γ be a path from $x_0 = p(\widetilde{x_0})$ to $x = p(\widetilde{x})$ in $p(gK)$. This yields a path of open simplices $\sigma_1, \dots, \sigma_n$, with $n \le k$. If gK does not meet $D(\xi)$, the refinement Lemma 5.10 implies that $\sigma_n \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, and $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$.

Otherwise, let n_0 (resp. n_1) be such that σ_{n_0} (resp. σ_{n_1}) is the first (resp. the last) simplex contained in $D(\xi)$. If x_0 is not in $D(\xi)$, we can apply the refinement Lemma 5.10 to the path $\sigma_1, \dots, \sigma_{n_0-1}$, which implies $\sigma_{n_0-1} \subset \mathrm{N}_{\mathcal{U}''}(D(\xi))$. In particular, we see that gK meets $W_{\mathcal{U}''}(\xi)$, which is also true if x_0 is in $D(\xi)$. Now by definition of $\mathcal{U}''$, we have that $gK \cap p^{-1}(D(\xi)) \subset W_{\mathcal{U}'}(\xi)$. If γ goes out of $D(\xi)$ after σ_{n_1}, then $\sigma_{n_1+1} \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$, and we can apply the refinement Lemma 5.10 to the path of simplices $\sigma_{n_1+1}, \dots, \sigma_n$. In any case, we get in the end $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$, which concludes the proof. □

Proof of Proposition 8.8 This follows from Proposition 8.9 and Proposition 8.13. □

Proof of Theorem 8.1 This follows from Theorem 7.13, Lemma 6.18, Proposition 8.2, and Proposition 8.8. □

8.3 Proof of the main theorem

We are now ready to conclude the proof of the combination theorem for boundaries of groups.

Lemma 8.14 *Let X, Y and G as in the statement of the main theorem. Then for every simplex σ of Y, the embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ realises an equivariant homeomorphism from ∂G_σ to $\Lambda G_\sigma \subset \partial G$. Moreover, for every pair H_1, H_2 of subgroups in the family $\mathcal{F} = \{\bigcap_{i=1}^n g_i G_{\sigma_i} g_i^{-1} \mid g_1, \dots, g_n \in G,\ \sigma_1, \dots, \sigma_n \in S(Y),\ n \in \mathbb{N}\}$, we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2) \subset \partial G$.*

Proof The equivariant embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ induces an equivariant embedding $\partial G_\sigma \hookrightarrow \Lambda G_\sigma \subset \partial G$. But since $\overline{EG_\sigma}$ is a closed subspace of $\overline{EG}$ by Proposition 6.19, and is stable under the action of G_σ, the reverse inclusion $\Lambda G_\sigma \subset \partial G_\sigma$ follows.

Now let $\sigma_1, \dots, \sigma_n$ be simplices of X. The inclusion

$$\Lambda\Big(\bigcap_{1 \leq i \leq n} G_{\sigma_i} \Big) \subset \bigcap_{1 \leq i \leq n} \Lambda G_{\sigma_i}$$

is clear, and the reverse inclusion follows directly from Lemma 4.7. □

Lemma 8.15 *Let X and G be as in the statement of the main theorem. Then for every simplex σ of X, the embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ satisfies the convergence property Definition 4.8.*

Proof Let $(g_n G_\sigma)$ be a sequence of distinct G–cosets. This yields an injective sequence of simplices $(g_n \sigma)$ of X. Let $\widetilde{x}$ be any point of EG_σ. By compactness of $\overline{EG}$, we can assume up to a subsequence that $g_n \widetilde{x}$ converges to a point $l \in \overline{EG}$. But it follows immediately from Lemma 7.14 and Lemma 7.15 that $l \in \partial G$ and that $g_n \overline{EG_\sigma}$ uniformly converges to l. □

Lemma 8.16 *Let X and G be as in the statement of the main theorem. Then for every simplex σ of X, the group G_σ is of finite height in G.*

Proof Let $g_1 G_\sigma, \dots, g_n G_\sigma$ be distinct G–cosets such that $g_1 G_\sigma g_1^{-1} \cap \dots \cap g_n G_\sigma g_n^{-1}$ is infinite. Thus the simplices $g_1 \sigma, \dots, g_n \sigma$ of X are distinct and such that the boundary of their stabilisers have a nonempty intersection in $\partial_{\mathrm{Stab}} G$. But as there is a uniform bound on the number of simplices contained in the domain of a point of $\partial_{\mathrm{Stab}} G$ by Proposition 4.2, Lemma 4.6 implies that there is a uniform bound on the number of simplices whose stabilisers have an infinite intersection, hence the result. □

Proof of the combination theorem for boundaries of groups This follows from Theorem 8.1, Lemma 8.14, Lemma 8.15 and Lemma 8.16. □

8.4 Boundaries in the sense of Carlsson and Pedersen

So far we have been concerned with the notion of an $E\mathcal{Z}$–structure in the sense of Farrell and Lafont. We now turn to a slightly stronger notion of boundary, which also has stronger implications for the Novikov conjecture.

Definition 8.17 Let G be a group endowed with an $E\mathcal{Z}$–structure in the sense of Farrell and Lafont $(\overline{EG}, \partial G)$. We say that $(\overline{EG}, \partial G)$ is an *$E\mathcal{Z}$–structure in the sense of Carlsson and Pedersen* if in addition we have:

For every finite group H of G, the fixed point set $\overline{EG}^H$ is nonempty and admits EG^H as a dense subset.

The importance of such finer structures comes from the following implication.

Theorem 8.18 (Carlsson and Pedersen [11]) *If G admits an $E\mathcal{Z}$–structure in the sense of Carlsson and Pedersen, then G satisfies the integral Novikov conjecture.*

In our context, we will need an additional assumption on these $E\mathcal{Z}$–structures. As explained below, this is by no means a restrictive assumption.

Definition 8.19 We say that an $E\mathcal{Z}$–structure in the sense of Carlsson and Pedersen $(\overline{EG}, \partial G)$ is *strong* if in addition we have the following:

For every finite group H of G, $(\partial G)^H$ is either empty or a $\mathcal{Z}$–set in $\overline{EG}^H$.

Without any assumption of a strong $E\mathcal{Z}$–structure, it is still possible to prove the following partial result.

Lemma 8.20 *Let $H \subset G$ be a finite subgroup. Then the closure of EG^H in $\overline{EG}$ is exactly $\overline{EG}^H$.*

Proof As EG is a classifying space for proper actions of G, EG^H is nonempty. We now prove that it is dense in $\overline{EG}^H$.

Let $\xi \in \partial_{\mathrm{Stab}} G \cap \overline{EG}^H$. The domain $D(\xi)$ is thus stable under the action of H. As $D(\xi)$ is a finite convex subcomplex of X, the fixed point theorem for CAT(0) spaces implies that there is a point of $D(\xi)$ fixed by H. Since the action is without inversion, we can further assume that H fixes a vertex v of $D(\xi)$. Moreover, EG_v^H is dense in $\overline{EG_v^H}$. Thus, by definition of a basis of neighbourhoods at ξ, any neighbourhood of ξ in $\overline{EG}$ meets EG^H.

Now let $\eta \in \partial X \cap \overline{EG}^H$. Let γ be a geodesic from a point of X^H to η. Then γ is fixed pointwise by H. Let U be a neighbourhood of η in $\overline{X}$. Since the path γ eventually meets U, let σ be a simplex of X contained in U and met by γ. Thus σ is fixed pointwise by H. Now since EG_σ^H is nonempty by assumption, it follows that EG^H meets $V_U(\eta)$, and the result follows. □

However, the previous reasoning does not show the contractibility of $\overline{EG}^H$. We now reformulate our main theorem in the setting of $E\mathcal{Z}$–structures in the sense of Carlsson and Pedersen.

Definition 8.21 An $E\mathcal{Z}$–complex of spaces *in the sense of Carlsson and Pedersen* (compatible with the complex of groups $G(\mathcal{Y})$) is a complex of spaces over a fundamental domain for the action satisfying the axioms of a compatible $E\mathcal{Z}$–complex of spaces, with strong $E\mathcal{Z}$–structures in the sense of Carlsson and Pedersen instead of $E\mathcal{Z}$–structures in the sense of Farrell and Lafont.

Theorem 8.22 *The combination theorem for boundaries of groups remains true if one replaces "$E\mathcal{Z}$–complexes of spaces" with "$E\mathcal{Z}$–complexes of spaces in the sense of Carlsson and Pedersen".*

Proof The only thing to prove is that $(\overline{EG}, \partial G)$ is an $E\mathcal{Z}$–structure in the sense of Carlsson and Pedersen. We already know that it is an $E\mathcal{Z}$–structure in the sense of Farrell and Lafont by Theorem 0.1 in the case of $E\mathcal{Z}$–structures in the sense of Farrell and Lafont. Let H be a finite subgroup of G. To prove that $\overline{EG}^H$ is contractible, we want to apply the Lemma 8.3 of Bestvina and Mess to the pair $(\overline{EG}^H, \overline{EG}^H \setminus EG^H)$.

In order to do this, first notice that EG^H is nothing but the complex of spaces over X^H with fibres the subcomplexes EG_σ^H of EG_σ. Thus, it is possible to apply the exact same reasoning with X^H in place of X and the EG_σ^H in place of the EG_σ. As X^H is a convex, hence contractible subcomplex of X, this is enough to recover the fact that EG^H is contractible.

Now, notice that, because of Lemma 8.20, $\overline{EG}^H$ is obtained from EG^H by the same procedure as before, compactifying every EG_σ^H (for σ a simplex fixed under H) by $\overline{EG}_\sigma^H$ and adding the visual boundary of the CAT(0) subcomplex X^H, $\partial(X^H) = (\partial X)^H$. We now briefly indicate why this is enough to prove the $\mathcal{Z}$–set property for $(\overline{EG}^H, \overline{EG}^H \setminus EG^H)$. The only properties that were required are the fact that X is a CAT(0) space, the convergence properties of the embeddings between the various classifying spaces, and the fact that ∂G_σ is a $\mathcal{Z}$–set in $\overline{EG}_\sigma$. But since X^H is convex in a CAT(0) space, it is itself CAT(0). Moreover, the convergence properties of the embeddings are clearly still satisfied for simplices that are fixed under H. Finally, by assumption, $(\partial G_\sigma)^H$ is a $\mathcal{Z}$–set in $\overline{EG}_\sigma^H$. Thus, the same reasoning as in Lemma 8.5 and Lemma 8.6 shows that the Lemma 8.3 of Bestvina and Mess applies, thus implying that $(\overline{EG}^H, \overline{EG}^H \setminus EG^H)$ is a $\mathcal{Z}$–compactification, and we are done. □

9 A high-dimensional combination theorem for hyperbolic groups

In this section, we apply our construction of boundaries to get a generalisation of a combination theorem of Bestvina and Feighn to complexes of groups of arbitrary dimension.

This will be done by constructing an $E\mathcal{Z}$–structure for G and proving that G is a uniform convergence group on its boundary. Note that this proof has the advantage of yielding a construction of the Gromov boundary of G.

In the following, $G(\mathcal{Y})$ will be a complex of groups over a simplicial complex Y satisfying the conditions of the combination theorem for hyperbolic groups. We will denote by G the fundamental group of $G(\mathcal{Y})$ and by X a universal covering.

9.1 A few facts about hyperbolic groups and quasiconvex subgroups

We start by recalling here a few elementary facts about hyperbolic groups. There is an extensive literature about such groups, and we refer the reader to Coornaert, Delzant and Papadopoulos [12] and Gromov [22] for more details.

Lemma 9.1
- *Let $H_1 \leq H_2 \leq H$ be three hyperbolic groups. If H_1 is quasiconvex in H_2, and H_2 is quasiconvex in H, then H_1 is quasiconvex in H. If both H_1 and H_2 are quasiconvex in H, then H_1 is quasiconvex in H_2.*
- *(Gromov [23, page 164]) Let H be a hyperbolic group, and H_1, H_2 two quasiconvex subgroups. Then $H_1 \cap H_2$ is quasiconvex in H, and $\Lambda(H_1 \cap H_2) = \Lambda H_1 \cap \Lambda H_2$.* □

Corollary 9.2 *Let Γ be a finite connected graph contained in the 1–skeleton of X, and $\Gamma' \subset \Gamma$ a connected subgraph. Then $\bigcap_{v\in\Gamma} G_v$ is hyperbolic and quasiconvex in $\bigcap_{v\in\Gamma'} G_v$.*

Proof This follows from an easy induction on the number of vertices of Γ, together with Lemma 9.1. □

Recall that in the case of a hyperbolic group H, there is a very explicit example of a classifying space for proper actions, namely the Rips complex. Moreover, there is a natural notion of boundary, namely the Gromov boundary of H (see [12]).

Theorem 9.3 [4; 27] *Let H be a finitely generated hyperbolic group, H' a finitely generated subgroup, and S a finite generating set of H that contains a finite generating set of H'. For $n \gg 0$, the Rips complex $P_n(H)$ is contractible and there is a topology on $P_n(H) \cup \partial H$ such that $(P_n(H) \cup \partial H, \partial H)$ is an $E\mathcal{Z}$–structure for H. Furthermore, if H' is quasiconvex in H, the equivariant embedding $P_n(H') \hookrightarrow P_n(H)$ naturally extends to an equivariant embedding $P_n(H') \cup \partial H' \hookrightarrow P_n(H) \cup \partial H$.* □

9.2 Construction of an $E\mathcal{Z}$–complex of space compatible with $G(\mathcal{Y})$

We now define an $E\mathcal{Z}$–complex of spaces over Y as follows:

- We define inductively sets of generators for the local groups of the complex of groups $G(\mathcal{Y})$ induced over Y in the following way: Start with simplices σ of Y of maximal dimension, and choose for each of them a finite symmetric set of generators for G_σ. Suppose we have defined a set of generators for local groups over simplices of dimension at most k. If σ is a simplex of dimension $k-1$, choose a finite set of generators that contains all the generators of local groups of simplices strictly containing σ. This allows us to define for every simplex σ of Y a set of generator such that $\psi_{\sigma,\sigma'}(S_{\sigma'}) \subset S_\sigma$ whenever $\sigma \subset \sigma'$.
- Let $n \geq 1$ be an integer. Define D_σ as the Rips complex $P_n(G_\sigma)$ associated to the set of generators S_σ. Moreover, if $\sigma \subset \sigma'$, let $\phi_{\sigma,\sigma'}$ be the equivariant embedding $P_n(G_{\sigma'}) \hookrightarrow P_n(G_\sigma)$.
- Since there are only finitely many hyperbolic groups involved, choose $n \geq 0$ such that all the previously defined Rips complexes are contractible.

It follows from the above discussion that:

Proposition 9.4 *The complex of spaces $D(\mathcal{Y})$ is compatible with the complex of groups $G(\mathcal{Y})$.* □

Lemma 9.5 *The $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ satisfies the limit set property Definition 2.9.*

Proof For every pair of simplices $\sigma \subset \sigma'$ of Y, $G_{\sigma'}$ is a quasiconvex subgroup of G_σ, so the map $\phi_{\sigma,\sigma'}\colon \partial G_{\sigma'} \to \partial G_\sigma$ realises a $G_{\sigma'}$–equivariant homeomorphism $\partial G_{\sigma'} \to \Lambda G_{\sigma'} \subset \partial G_\sigma$ by a result of Bowditch [6].

For every simplex σ of Y, the family

$$\mathcal{F}_\sigma = \left\{ \bigcap_{i=1}^{n} g_i G_{\sigma_i} g_i^{-1} \;\middle|\; g_0, \dots, g_n \in G_\sigma, \sigma_1, \dots, \sigma_n \in \mathrm{st}(\sigma), n \in \mathbb{N} \right\}$$

is contained in the family of quasiconvex subgroups of G_σ. Indeed, let $g_0, \dots, g_n$ be elements of G. Then, as X is CAT(0), $\bigcap_{0 \le i \le n} g_i G_\sigma g_i^{-1} = \bigcap_{v \in \Gamma} g_i G_v g_i^{-1}$, where Γ is a graph containing all the vertices of the simplices $g_0\sigma, \dots, g_n\sigma$ and contained in the convex hull of the $g_0\sigma, \dots, g_n\sigma$. For such subgroups, the equality $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2)$ holds by Lemma 9.1. □

Lemma 9.6 *The $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ satisfies the convergence property Definition 4.8.*

Proof This is Dahmani's [15, Proposition 1.8]. □

Lemma 9.7 *The $E\mathcal{Z}$–complex of spaces $D(\mathcal{Y})$ satisfies the finite height property Definition 2.10.*

Proof A quasiconvex subgroup of a hyperbolic group has finite height by a result of Gitik, Mitra, Rips and Sageev [21]. □

The combination theorem for boundaries of groups now implies the following:

Corollary 9.8 *The fundamental group of $G(\mathcal{Y})$ admits a classifying space for proper actions and a strong boundary in the sense of Carlsson and Pedersen.* □

Note that this corollary does not use the hyperbolicity of X.

9.3 Background on convergence groups and hyperbolicity

Definition 9.9 (Convergence group) A group Γ acting on a compact metrisable space M with more than two points is called a *convergence group* if, for every sequence (γ_n) of elements of Γ, there exist two points ξ_+ and ξ_- in M and a subsequence $(\gamma_{\varphi(n)})$, such that for any compact subspace $K \subset M \setminus \{\xi_-\}$, the sequence $(\gamma_{\varphi(n)} K)$ of translates uniformly converges to ξ_+.

A hyperbolic group Γ is always a convergence group on $\Gamma \cup \partial\Gamma$ (see for instance Freden [20]). A direct consequence is the following:

Proposition 9.10 *Let Γ be a hyperbolic group and $\overline{E\Gamma}$ an $E\mathcal{Z}$–structure obtained as in Theorem 9.3. Then Γ is a convergence group on $\overline{E\Gamma}$.* □

Definition 9.11 (Conical limit point) Let Γ be a convergence group on a compact metrisable space M. A point ζ in M is called a *conical limit point* if there exists a sequence (γ_n) of elements of Γ and two points $\xi_- \neq \xi_+$ in M, such that $\gamma_n \zeta \to \xi_-$ and $\gamma_n \zeta' \to \xi_+$ for every $\zeta' \neq \zeta$ in M. The group Γ is called a *uniform convergence group* on M if M consists only of conical limit points.

Theorem 9.12 (Bowditch [5]) *Let Γ be a uniform convergence group on a compact metrisable space M with more than two points. Then Γ is hyperbolic and M is Γ–equivariantly homeomorphic to the Gromov boundary of Γ.* □

9.4 A combination theorem

We now prove that G is a hyperbolic group, by proving that it is a uniform convergence group on its boundary ∂G.

So far, the topology on $\overline{EG}$ and ∂G was defined by choosing a specific, although arbitrary, basepoint. In forthcoming proofs, we will choose neighbourhoods centred at points that are relevant to the geometry of the problem.

Definition 9.13 Let $\delta \geq 0$ be such that the space X is δ–hyperbolic. We denote by $\langle \cdot, \cdot \rangle$ the Gromov product on X and an extension to $\overline{X}$. For $z \in \overline{X}$, $k \geq 0$ and $x_0 \in X$ a basepoint, let

$$W_k(z) = \big\{x \in \overline{X} \text{ such that } \langle x, z\rangle_{x_0} \geq k\big\}.$$

For $\eta \in \partial X$ and $k \geq 0$, the family of subsets $(W_k(\eta))$ forms a basis of (not necessarily open) neighbourhoods of η in $\overline{X}$.

Recall that $d_{\max}$ was defined in Corollary 4.3 as a constant such that domains of points of $\partial_{\mathrm{Stab}} G$ have at most $d_{\max}$ simplices, and a geodesic segment contained in the open simplicial neighbourhood of the domain of a point of $\partial_{\mathrm{Stab}} G$ meets at most $d_{\max}$ simplices.

Lemma 9.14 *Let (g_n) be an injective sequence of elements of G, and suppose there exist vertices v_0 and v_1 of X such that $g_n v_0 = v_1$ for infinitely many n. Then there exist $\xi_+, \xi_- \in \partial G$ and a subsequence $(g_{\varphi(n)})$ such that for every compact subset K of $\partial G \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)} K$ uniformly converges to ξ_+.*

Proof It is enough to prove the result when $g_n v_0 = v_0$ for infinitely many n. Since G_{v_0} is hyperbolic, we can assume that there exists a subsequence of (g_n), that we still denote (g_n), and points $\xi_+, \xi_- \in \partial G_{v_0}$ such that for every compact subset K of $\overline{EG_{v_0}} \setminus \{\xi_-\}$, the sequence of translates $g_n K$ uniformly converges to ξ_+. Throughout this proof, we choose v_0 as the basepoint.

Let σ be a simplex of X containing v_0.

If σ is not contained in $D(\xi_-)$, then the convergence property Definition 4.8 implies that, up to a subsequence, we can assume that the sequence of $g_n \partial G_\sigma$ uniformly converges to ξ_+ in ∂G_{v_0}.

If σ is contained in $D(\xi_-)$, then the subset $\partial G_\sigma \subset \partial G_{v_0}$ consists of at least two points, among which there is ξ_-. Since for any other point α of ∂G_σ we have that $g_n \alpha$ tends to ξ_+, the convergence property Definition 4.8 implies that one of the following situations happens:

- $g_n G_\sigma$ only takes finitely many values of cosets, in which case we can find a subsequence (g_n) such that $g_n \partial G_\sigma$ is constant and contains ξ_+. This means that we can write $g_n = g'_n . g$ where g is in the stabiliser of v_0 and g'_n in a sequence in the stabiliser of σ. Up to replacing g_n by g'_n, we can assume that g_n fixes σ.
- $g_n G_\sigma$ takes infinitely many values of cosets, in which case we can find a subsequence (g_n) such that $g_n \partial G_\sigma$ uniformly converges to ξ_+.

As domains are finite subcomplexes of X by Proposition 4.2, we can iterate this procedure a finite number of times so as to obtain a subsequence (g_n) and a subcomplex $F \subset D(\xi_-) \cap D(\xi_+)$ such that:

- F is fixed pointwise under all the g_n.
- For every simplex σ in $(\mathrm{st}(F) \setminus F)$ and every vertex v of $\sigma \cap F$, we have that $g_n \partial G_\sigma$ uniformly converges to ξ_+ in ∂G_v.

For every vertex v of $D(\xi_-)$, choose U_v to be a neighbourhood of ξ_- in ∂G_{v_0}. Choose a ξ_-–family $\mathcal{U}'$ that is nested in $\{U_v, v \in D(\xi_-)\}$, and choose $\varepsilon \in (0,1)$. We can further assume that for every simplex σ of F and every vertex v of σ, the subset $\overline{EG_\sigma} \setminus U'_v$ is infinite. Let $K = \partial G \setminus V_{\mathcal{U}',\varepsilon}(\xi_-)$.

We now prove that, up to a subsequence, the sequence of translates $g_n K$ uniformly converges to ξ_+. Because of the definition of neighbourhoods of points of $\partial_{\mathrm{Stab}} G$, we need to treat different cases.

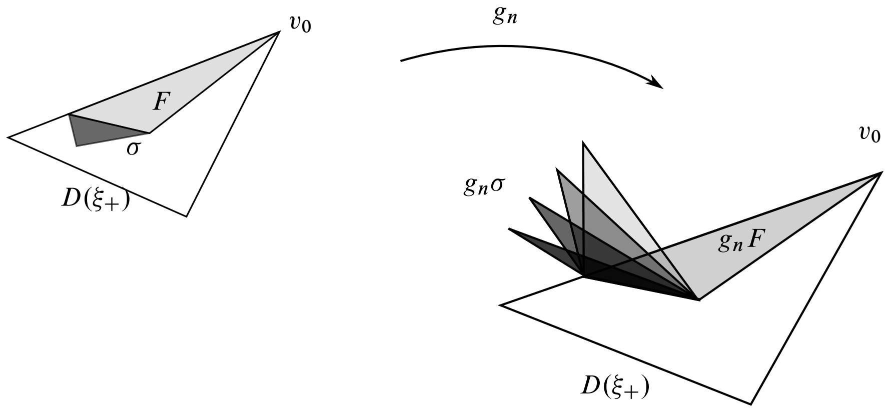

Figure 6

Let σ be a simplex of F containing v_0, so that $G_\sigma \subset G_{v_0}$, and v a vertex of σ distinct from v_0. Since G_v is hyperbolic, there exists a subsequence of (g_n), which we still denote (g_n), and points $\xi'_+, \xi'_- \in \partial G_v$ such that for every compact subset K' of $\overline{EG_v} \setminus \{\xi'_-\}$, the sequence of translates $g_n K'$ uniformly converges to ξ'_+. By definition of ξ_+ and ξ_-, we already have that the sequence $g_n(\overline{EG_{v_0}} \setminus U'_{v_0})$ uniformly converges to ξ_+ in ∂G_{v_0}. We thus have that $g_n(\overline{EG_\sigma} \setminus U'_{v_0})$ uniformly converges to ξ_+ in ∂G_v. Since $\overline{EG_\sigma} \setminus U'_{v_0}$ is infinite by construction, this implies that $\xi'_+ = \xi_+$. If we had $\xi'_- \neq \xi_-$, then $g_n\overline{EG_\sigma}$ would uniformly converge to ξ_+, contradicting the fact that $g_n\overline{EG_\sigma} = \overline{EG_\sigma}$ since g_n fixes σ. Therefore $\xi'_- = \xi_-$. This implies that $g_n(\partial G_v \setminus U'_v)$ uniformly converges to ξ_+ in ∂G_v. Since F is finite, an easy induction shows that there exists a subsequence, still denoted (g_n), such that $g_n(\partial G_v \setminus U'_v)$ uniformly converges to ξ_+ in ∂G_v for every vertex v of F.

Let $\tilde{x} \in K$, and $x \in \overline{p}(\tilde{x}) \setminus F$. Let σ be the first simplex touched by $[v_0, x]$ after leaving F. It follows from the definition of F that the sequence of simplices $(g_n\sigma)$ is such that for some (hence any) vertex v of $\sigma \cap F$, the sequence of $(\partial G_{g_n\sigma})$ uniformly converges to ξ_+ in ∂G_v. It follows from the convergence criterion Corollary 7.16 that the sequence $(g_n\tilde{x})$ converges to ξ_+. Since $\tilde{x} \notin V_{\mathcal{U},\varepsilon}(\xi_-)$, we have $\partial G_\sigma \not\subset U_v$ for some (hence any) vertex v of F. Since $\mathcal{U}'$ is nested in $\{U_w, w \in V(\xi_-)\}$, it follows that

$$\partial G_\sigma \cap U'_v = \varnothing,$$

this being true for every $\tilde{x} \in K$ and $x \in \overline{p}(\tilde{x}) \setminus F$. We already have that for every vertex v of F, the sequence of $g_n.(\partial G_v \setminus U_v)$ uniformly converges to ξ_+ by the above discussion. As F is a finite subcomplex of X, the convergence criterion Corollary 7.16 now shows that the sequence $(g_n.K)$ uniformly converges to ξ_+. $\square$

Lemma 9.15 *Let (g_n) be an injective sequence of elements of G. Suppose that for some (hence any) vertex v the sequence $(g_n v)$ is bounded, but there do not exist vertices v_0 and v_1 of X such that $g_n v_0 = v_1$ for infinitely many n. Then there exist $\xi_+, \xi_- \in \partial G$ and a subsequence $(g_{\varphi(n)})$ such that for every compact subset K of $\partial G \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)} K$ uniformly converges to ξ_+.*

Proof Choose an arbitrary vertex v and a point $\widetilde{x}$ of EG_v. As ∂G is compact by Theorem 7.13 and $(g_n v)$ is bounded, we can choose a subsequence, still denoted (g_n), and points $\xi_+, \xi_- \in \partial_{\mathrm{Stab}} G$ such that $g_n \widetilde{x} \to \xi_+$ and $g_n^{-1} \widetilde{x} \to \xi_-$. We choose a vertex v_0 of $D(\xi_+)$ to be the basepoint, and let $\widetilde{x_0} \in EG_{v_0}$. By Theorem 8.1, we still have $g_n \widetilde{x_0} \to \xi_+$ and $g_n^{-1} \widetilde{x_0} \to \xi_-$.

Claim 1

- *For every $\eta \in \partial X$, the geodesic ray $[g_n v_0, g_n \eta)$ does not meet $D(\xi_+)$ for n large enough.*
- *For every $\xi \in \partial_{\mathrm{Stab}} G$, the subset* $\mathrm{Geod}(g_n v_0, g_n D(\xi))$ *does not meet $D(\xi_+)$ for n large enough.*

Let $z \in \partial G$. If $z \in \partial X$, we denote by $D(z)$ the singleton $\{z\}$. By contradiction, suppose that there exists an infinite number of n for which there exist $y_n \in D(\xi_+)$ and $x_n \in \mathrm{Geod}(v_0, D(z))$ such that $g_n x_n = y_n$. As (y_n) is bounded by Proposition 4.2, the assumption on (g_n) implies that (x_n) is bounded too. Since x_n lies on $\mathrm{Geod}(v_0, D(z))$ for every n, the containment Proposition 3.3 and the finiteness Lemma 3.5 imply that, up to a subsequence, we can assume that x_n always lies in the same simplex σ of X. Furthermore, since $D(\xi_+)$ is finite by Proposition 4.2, we can assume, up to a subsequence, that y_n lies in a simplex σ' of X for every n. As the action of G on X is without inversion, this implies that $g_n \sigma = \sigma'$ for every n, which was excluded by assumption.

Claim 2 *For every ξ in ∂G, the sequence $g_n \xi$ converges to ξ_+.*

Let $\mathcal{U}$ be a ξ_+–family, $\mathcal{U}'$ a ξ_+–family that is $3d_{\max}$–nested in $\mathcal{U}$ and $\varepsilon > 0$. Recall that, by assumption on (g_n), the vertex $g_n v_0$ does not belong to $D(\xi_+)$ for n big enough. Furthermore, since $g_n \widetilde{x_0} \to \xi_+$, we have that $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n v_0)} \subset U'_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n v_0)$. We split the proof of the claim in two cases.

Let $\eta \in \partial X$. For n large enough, the path $[g_n v_0, g_n \eta)$ does not meet $D(\xi_+)$, by Claim 1. By Proposition 4.2 ,we can choose $y \in D(\xi_+)$ that minimises the distance to $\mathrm{Geod}(g_n v_0, g_n \eta)$. Let τ (resp. τ') be a simplex of $\mathrm{N}(D(\xi_+)) \setminus D(\xi_+)$ whose interior is crossed by $[y, g_n v_0]$ (resp. $[y, g_n \eta)$) at a point u (resp. u'). By convexity of the function $z \mapsto d(z, [g_n v_0, g_n \eta])$, it follows from the definition of y that the geodesic segment $[u, u']$ does not meet $D(\xi_+)$, thus yielding a path of simplices of length at most $d_{\max}$ between τ and τ' in $\mathrm{N}(D(\xi_+)) \setminus D(\xi_+)$. Lemma 3.7 implies that there exists a path of simplices of length at most $d_{\max}$ between τ and the exit simplex $\sigma_{\xi_+,\varepsilon}(g_n v_0)$ (resp. between τ' and the exit simplex $\sigma_{\xi_+,\varepsilon}(g_n \eta)$) in $\mathrm{N}(D(\xi_+)) \setminus D(\xi_+)$. Thus for n large enough, there is a path of simplices of length at most $3d_{\max}$ from $\sigma_{\xi_+,\varepsilon}(g_n v_0)$ to $\sigma_{\xi_+,\varepsilon}(g_n \eta)$ in $\mathrm{N}(D(\xi_+)) \setminus D(\xi_+)$. As $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n v_0)} \subset U'_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n v_0)$, it follows from the fact that $\mathcal{U}'$ is $3d_{\max}$–nested in $\mathcal{U}$ that $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n \eta)} \subset U_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n \eta)$. It thus follows that $(g_n \eta)$ converges to ξ_+.

Let $\xi \in \partial_{\mathrm{Stab}} G$. For n large enough, $\mathrm{Geod}(g_n v_0, g_n D(\xi))$ does not meet $D(\xi_+)$ by Claim 1. Let $x \in D(\xi)$ and, by Proposition 4.2, let y be a point of $D(\xi_+)$ which minimises the distance to $\mathrm{Geod}(g_n v_0, g_n x)$. Using the same reasoning as above, we get, for n large enough, a path of simplices of length at most $3d_{\max}$ from $\sigma_{\xi_+,\varepsilon}(g_n v_0)$ to $\sigma_{\xi_+,\varepsilon}(g_n x)$ in $\mathrm{N}(D(\xi_+)) \setminus D(\xi_+)$. As $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n v_0)} \subset U'_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n v_0)$, it follows from the fact that $\mathcal{U}'$ is $3d_{\max}$–nested in $\mathcal{U}$ that, for n large enough and for every $x \in D(\xi)$, $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n x)} \subset U_v$ for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n x)$. It thus follows that $(g_n \xi)$ converges to ξ_+.

In the same way, we prove that for every $\xi \in \partial G$, the sequence $g_n^{-1} \xi$ converges to ξ_-. To conclude the proof of the lemma, it remains to show that this convergence can be made uniform away from ξ_-:

Claim 3 *For every $\xi \neq \xi_-$ in ∂G, there is a subsequence (g_n) and a neighbourhood U of ξ in ∂G such that the sequence of $g_n U$ uniformly converges to ξ_+.*

Once again, we split the proof in two cases.

Let $\xi \in \partial_{\mathrm{Stab}} G$. We already have that $g_n \xi \to \xi_+$ by Claim 2. In order to find a ξ–family $\mathcal{U}$ and a constant ε such $g_n V_{\mathcal{U},\varepsilon}(\xi)$ uniformly converges to ξ_+, it is enough, using the same reasoning as in Claim 2, to find a ξ–family $\mathcal{U}$ and a constant ε such that for every x in $D(\xi) \cup \mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$.

By Claim 1, we already have that for n large enough, no geodesic from $g_n v_0$ to a point of $g_n D(\xi)$ meets $D(\xi_+)$. As $\xi \neq \xi_-$, we choose a ξ–family $\mathcal{U}$, a ξ_-–family $\mathcal{U}'$ and constants $\varepsilon, \varepsilon' \in (0,1)$ such that the neighbourhoods $V_{\mathcal{U},\varepsilon}(\xi)$ and $V_{\mathcal{U}',\varepsilon'}(\xi_-)$ are disjoint. Up to a subsequence, we have by the first claim that $g_n D(\xi)$ does not meet $D(\xi_+)$. It now follows from the definition of $\mathcal{U}$ and the fact that $g_n^{-1} \xi_+ \to \xi_-$ that $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$ does not meet the sets $g_n^{-1} D(\xi_+)$, hence the sets $g_n \mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$ do not meet $D(\xi_+)$. Now this implies that for every x in $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$: indeed, this geodesic must meet $g_n D(\xi)$ since the geodesic from v_0 to a point of $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$ must meet $D(\xi)$, and we already proved that a geodesic segment from $g_n v_0$ to a point of $g_n D(\xi)$ does not meet $D(\xi_+)$. Now the same proof as in Claim 2 shows that $g_n V_{\mathcal{U},\varepsilon}(\xi)$ uniformly converges to ξ_+.

Let $\eta \in \partial X$. We already know that $g_n \eta \to \xi_+$ by Claim 2. In order to find a neighbourhood U of η in $\overline{X}$ such that such $g_n V_U(\eta)$ uniformly converges to ξ_+, it is enough, using the same reasoning as in Claim 2, to find a neighbourhood U of η in $\overline{X}$ such that for every x in U, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$.

First, notice that the distance from the geodesic rays $[g_n v_0, g_n \eta)$ to $D(\xi_+)$ is uniformly bounded below: indeed, if this was not the case, the same reasoning as in Claim 1 would imply the existence of simplices σ, σ' of X such that $g_n \sigma \cap \sigma' \neq \varnothing$. This in turn would imply that, up to a subsequence, there exist subsimplices $\tau \subset \sigma$ and $\tau' \subset \sigma'$ such that $g_n \tau = \tau'$, which was excluded. Thus, let $\varepsilon > 0$ be such a uniform bound. Let also

$$M = \sup_{x \in D(\xi_+), n \geq 0} d(g_n v_0, x).$$

Now consider the neighbourhood $V_{M,\varepsilon}(\eta)$ of η in $\overline{X}$. Let $x \in X$ be in that neighbourhood, and let γ be a parametrisation of the geodesic from v_0 to x. Suppose by contradiction that the geodesic from $g_n v_0$ to $g_n x$ does meet $D(\xi_+)$. Then, by definition of M, the geodesic segment $g_n \gamma([0, M])$ meets $D(\xi_+)$. But as this geodesic segment is in the open ε–neighbourhood of $[g_n v_0, g_n \eta)$, we get our contradiction from the definition of ε.

Thus for every x in $V_{M,\varepsilon}(\eta)$, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$, and we are done. □

Lemma 9.16 *Let (g_n) be an injective sequence of elements of G, and suppose that for some (hence every) vertex v_0 of X, $d(v_0, g_n v_0) \to \infty$. Since $(\overline{EG}, \partial G)$ is an $E\mathcal{Z}$–structure for G by Theorem 8.1, we can assume up to a subsequence that there exist $\xi_+, \xi_- \in \partial G$ such that for every compact subset $K \subset EG$, we have $g_n K \to \xi_+$ and $g_n^{-1} K \to \xi_-$. Then there exists a subsequence $(g_{\varphi(n)})$ such that for every compact subset K of $\partial G \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)} K$ uniformly converges to ξ_+.*

Proof If $\xi_- \in \partial X$, let U be a neighbourhood of ξ_- in ∂X and $K = \partial G \setminus V_U(\xi_-)$. Since X has finitely many isometry types of simplices, it follows from Lemma 6.3 that we can choose a subneighbourhood U' of U containing ξ_- and such that any path from $U' \cap X$ to $X \setminus U$ meets at least $d_{\max}$ simplices.

If $\xi_- \in \partial_{\text{Stab}}G$, let $\mathcal{U}$ be a ξ_-–family, and $\varepsilon \in (0,1)$, and let $K = \partial G \setminus V_{\mathcal{U},\varepsilon}(\xi_-)$. We also choose another ξ_-–family $\mathcal{U}'$ which is $2d_{\max}$–refined in $\mathcal{U}$.

We want to prove that $(g_n K)$ uniformly converges to ξ_+. Recall that the sets $W_k(g_n v_0)$ were defined in Definition 9.13.

Claim 1 *For every k, the following holds:*

- *If $\xi_- \in \partial X$, we have $g_n(\overline{X} \setminus U') \subset W_k(g_n v_0)$ for n large enough.*
- *If $\xi_- \in \partial_{\text{Stab}}G$, we have $g_n(\overline{X} \setminus \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)) \subset W_k(g_n v_0)$ for n large enough.*

We split the proof in two cases.

Suppose that $\xi_- \in \partial X$. First notice that since $g_n^{-1} v_0 \to \xi_-$, there exists a constant C such that for every $n \geq 0$ and every $x \notin U$, we have $\langle g_n^{-1} v_0, x \rangle_{v_0} \leq C$. Since we also have $d(g_n^{-1} v_0, v_0) \to \infty$, the claim follows.

Suppose now that $\xi_- \in \partial_{\text{Stab}}G$. We start by proving by contradiction that there exists a constant C such that for every $n \geq 0$ and every $x \notin \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$, we have $\langle g_n^{-1} v_0, x \rangle_{v_0} \leq C$.

The containment lemma Proposition 3.3 yields a constant m such that a geodesic path of length at most δ meets at most m simplices, where δ is the hyperbolicity constant of X. Let $\mathcal{U}''$ be a ξ_-–family that is m–nested in $\mathcal{U}'$. Since we are reasoning by contradiction, then, up to a subsequence, there exist points $y_n \notin \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$ such that $\langle g_n^{-1} v_0, y_n \rangle_{v_0} \to \infty$. By hyperbolicity of X, the geodesic segments $[v_0, g_n^{-1} v_0]$ and $[v_0, y_n]$ stay δ–close until time $\langle g_n^{-1} v_0, y_n \rangle_{v_0} \to \infty$. Moreover, as $g_n^{-1} \widetilde{x_0} \to \xi_-$ for any point $\widetilde{x_0} \in EG_{v_0}$, we have $g_n^{-1} v_0 \in \text{Cone}_{\mathcal{U}'',\varepsilon}(\xi_-)$ for n large enough. Thus, for n large enough, there exist points $a_n \in [v_0, y_n]$ and $b_n \in [v_0, g_n^{-1} v_0] \cap \text{Cone}_{\mathcal{U}'',\varepsilon}(\xi_-)$ and a path of simplices of length at most m between a_n and b_n that is contained in $X \setminus D(\xi_-)$ (see Figure 7).

The refinement lemma, Lemma 5.10, now implies that a_n and y_n both are in $\text{Cone}_{\mathcal{U}',\varepsilon}(\xi_-)$ for n large enough, a contradiction.

Now the same reasoning as before shows that for every $k \geq 0$, there exists N such that for every $n \geq N$ and every $x \notin \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$, $\langle v_0, x \rangle_{g_n^{-1} v_0} \geq k$, hence $\langle g_n v_0, g_n x \rangle_{v_0} \geq k$.

Claim 2 *For every k, we have $g_n \overline{p}(K) \subset W_k(g_n v_0)$ for n large enough.*

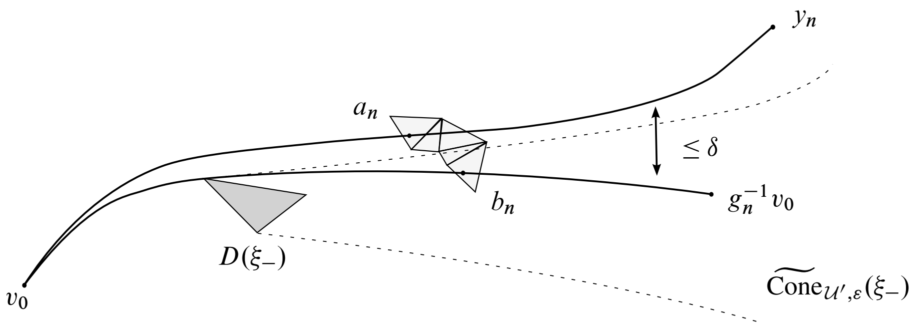

Figure 7

Suppose that $\xi_- \in \partial X$. By definition of U', we have that for every $z \in K$, $\overline{p}(z) \cap U' = \varnothing$. Thus $\overline{p}(K) \subset \overline{X} \setminus U'$ and the result follows from Claim 1.

Suppose now that $\xi_- \in \partial_{\mathrm{Stab}} G$, and let $z \in K$. Suppose by contradiction that $\overline{p}(z) \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-) \neq \varnothing$. If $\overline{p}(z)$ is contained in $X \setminus D(\xi_-)$, then the refinement Lemma 5.10 implies that $\overline{p}(z) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi_-)$; hence $z \in V_{\mathcal{U},\varepsilon}(\xi_-)$, which is absurd. If $\overline{p}(z)$ meets $D(\xi_-)$, then since $\mathcal{U}'$ is $2d_{\max}$–refined in $\mathcal{U}$ it follows from the refinement lemma Lemma 5.10 and Lemma 6.20 that $z \in V_{\mathcal{U},\varepsilon}(\xi_-)$, a contradiction. Thus $\overline{p}(K) \subset \overline{X} \setminus \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$ and the result follows from Claim 1.

Claim 3 *$g_n K$ uniformly converges to ξ_+.*

Once again, we split the proof in two cases.

Suppose that $\xi_+ \in \partial X$. Then, as $g_n v_0 \to \xi_+$, it follows from Claim 2 that for every k, $g_n \overline{p}(K) \subset W_k(\xi_+)$ for n large enough. It then follows from the convergence criterion Corollary 7.16 that $g_n K$ uniformly converges to ξ_+.

Suppose now that $\xi_+ \in \partial_{\mathrm{Stab}} G$. Let $\mathcal{U}_+$ be a ξ_+–family and $\varepsilon \in (0,1)$. Since X is δ–hyperbolic, let m be a constant such that a geodesic path of length at most δ meets at most m simplices, and let $\mathcal{U}_+'$ be a ξ_+–family that is m–nested in $\mathcal{U}_+$. As $g_n \widetilde{x_0} \to \xi_+$ for any $\widetilde{x_0} \in EG_{v_0}$, we have $g_n v_0 \in \mathrm{Cone}_{\mathcal{U}_+',\varepsilon}(\xi_+)$ for n large enough. For every $T \geq 0$, we can choose n large enough so that the geodesic segments $[v_0, g_n v_0]$ and $[v_0, g_n x]$, $x \in \overline{p}(K)$, remain δ–close up to time T (if we choose k large enough in Claim 2). In particular, we can choose k and N large enough so that, for every $n \geq N$ and every $x \in \overline{p}(K)$, there exists a path of simplices of length at most m in $X \setminus D(\xi_+)$ between a point of $[v_0, g_n v_0] \cap \mathrm{Cone}_{\mathcal{U}_+',\varepsilon}(\xi_+)$ and a point of $[v_0, g_n x]$. The refinement Lemma 5.10 now implies that

$$g_n \overline{p}(K) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U}_+,\varepsilon}(\xi_+)$$

for $n \geq N$, hence $g_n K \subset V_{\mathcal{U}_+,\varepsilon}(\xi_+)$ for $n \geq N$. Thus, $g_n K$ uniformly converges to ξ_+. □

Corollary 9.17 *The group G is a convergence group on ∂G.*

Proof This follows from Lemma 9.14, Lemma 9.15 and Lemma 9.16. □

To prove that G is hyperbolic, it remains to show that every point of ∂G is conical.

Lemma 9.18 *Every point of ∂G is a conical limit point for ∂G.*

Proof Consider first a point in ∂G_v for some vertex v of X. It is a conical limit point for G_v on ∂G_v, since G_v is hyperbolic. Therefore it is a conical point for G_v on ∂G, hence for G since G is a convergence group on ∂G by Corollary 9.17.

Now consider a point $\eta \in \partial X$. Since the action of G on X is cocompact, we can find a sequence (g_n) of elements of G and a simplex σ such that $(g_n\sigma)$ uniformly converges to η in $\overline{X}$ and such that for every n, the geodesic ray $[v_0, \eta)$ meets the interior of $g_n\sigma$. Let v be a vertex of σ and $\widetilde{x} \in EG_v$.

Claim Up to multiplying each g_n on the right by an element of G_v and taking a subsequence, we can further assume that $g_n^{-1}\widetilde{x}$ converges to a point $\xi_- \in \partial G \setminus \partial G_v$.

Consider the first simplex touched by the geodesic $[v, g_n^{-1}v]$ after leaving v. Since the action of G on X is cocompact, we can assume up to a subsequence that this sequence of simplices is in the same G–orbit. Now up to multiplying each g_n by an element of G_v, we can further assume that this sequence of simplices is constant at a unique simplex σ_1. Up to a subsequence, we can further assume that all the geodesic segments $[v, g_n^{-1}v]$ leave σ_1 along the same open simplex τ_1. Now consider the simplex $\sigma_2^{(n)}$ touched by $[v, g_n^{-1}v]$ after leaving τ_1 and choose a G_{σ_1}–orbit in EG_{τ_1}. Since G_{σ_1} is quasiconvex in G_{τ_1}, this orbit is a quasiconvex subset Q_1 of EG_{τ_1}; choose a basepoint y of Q_1. For every n, choose a point $x_n \in EG_{\sigma_2^{(n)}}$ and let y_n be a projection of x_n on the quasiconvex subset Q_1. By definition of Q_1, there exists an element $h_n \in G_{\sigma_1} \subset G_v$ such that $h_n y_n = y$. This implies that for every n, the subset $h_n EG_{\sigma_2^{(n)}}$ contains a point that projects to y. In particular, no subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ converges to a point of ∂G_{σ_1}. Suppose by contradiction that there exists a subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ that converges to a point $z \in \partial G_{\tau_1}$. Since G_{τ_1} is a convergence group on $\overline{EG}_{\tau_1}$ by Proposition 9.10, it follows that for every $x \in \overline{EG}_{\tau_1}$ except maybe one point, $h_n x$ converges to z. But as Q_1 is stable under all the h_n, such a z belongs to $\partial G_{\sigma_1} \subset \partial G_{\tau_1}$, contradicting the fact that no subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ converges to a point of ∂G_{σ_1}. Thus, no subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ converges to a point of ∂G_{τ_1} and the convergence property Definition 4.8 now implies that, up to a subsequence, we can assume that $h_n\overline{EG}_{\sigma_2^{(n)}}$ is constant. Up to a subsequence, we can further assume that $\sigma_2^{(n)}$ is constant at σ_2 and every geodesic segment $[v, g_n^{-1}v]$ leaves σ_2 along the same open simplex τ_2. In view of the above, we replace the sequence (g_n) by $(g_nh_n^{-1})$. Now one of the following happens:

(i) Suppose that $G_{\sigma_1} \cap G_{\sigma_2}$ is finite. By applying the same reasoning as in the proof of the compactness lemmas Lemma 7.14 and Lemma 7.15, either there exists a subsequence of (g_n) such that $g_n^{-1}\widetilde{x}$ converges to a point of ∂X and we are done, or the path of simplices σ_1, σ_2 extends to a path of simplices $\sigma_1, \dots, \sigma_m$ that are crossed by every geodesic segment $[v, g_n^{-1}v]$ and $g_n^{-1}\widetilde{x}$ converges to a point $\xi_- \in \partial G_{\sigma_m}$. As $D(\xi_-)$ is convex by Proposition 4.2 and $G_{\sigma_1} \cap G_{\sigma_2}$ is finite, it follows from Lemma 4.7 that $\xi_- \notin \partial G_v$ and we are done.

(ii) Suppose that $G_{\sigma_1} \cap G_{\sigma_2}$ is infinite. Let $\sigma_3^{(n)}$ be the simplex touched by $[v, g_n^{-1}v]$ after leaving τ_2, and let Q_2 be a $G_{\sigma_1} \cap G_{\sigma_2}$–orbit in EG_{τ_2}. Note that Q_2 is quasiconvex in EG_{τ_2} by Corollary 9.2. We are thus back to the previous situation with EG_{τ_2} instead of EG_{τ_1}, $\overline{EG}_{\sigma_3^{(n)}}$ instead of $\overline{EG}_{\sigma_2^{(n)}}$ and Q_2 instead of Q_1.

We claim that this procedure eventually stops. Indeed, the containment lemma Proposition 3.3 yields a constant m such that every geodesic meeting at least m simplices has length at least A, where A is the acylindricity constant. Thus, after at most m applications of this algorithm, we get to situation (i), which concludes the proof of the claim.

By the above discussion, we already have that $g_n^{-1}\widetilde{x} \to \xi_-$ for every $\widetilde{x} \in EG_v$. Thus, by Lemma 9.16, it is enough, in order to prove Lemma 9.18, to show that $g_n^{-1}\eta$ does not converge to ξ_-, which we now prove by contradiction.

Suppose $g_n^{-1}\eta$ were converging to ξ_-. For every n, let x_n be a point of $[g_n^{-1}v, g_n^{-1}\eta)$ that is contained in the interior of σ. Since the geodesic ray $[g_n^{-1}v, g_n^{-1}\eta)$ meets σ for every n, the Gromov product $\langle g_n^{-1}v, g_n^{-1}\eta\rangle_v$ is bounded. Thus, ξ_- cannot belong to ∂X, and $\xi_- \in \partial_{\mathrm{Stab}}G$.

Now since both $g_n^{-1}\eta$ and $g_n^{-1}\widetilde{x}$ converge to $\xi_- \in \partial_{\mathrm{Stab}}G$, both geodesics $[v, g_n^{-1}\eta)$ and $[v, g_n^{-1}v]$ must go through $D(\xi_-)$ for n large enough. But Lemma 3.7 and Lemma 5.8 imply that for n large enough and any $x \in \sigma$, both geodesic rays $[x, g_n^{-1}\eta)$ and $[x, g_n^{-1}v]$ also meet $D(\xi_-)$. In particular, $[x_n, g_n^{-1}\eta)$ and $[x_n, g_n^{-1}v]$ meet $D(\xi_-)$ for n large enough. As $D(\xi_-)$ is convex by Proposition 4.2, this implies that x_n belongs to $D(\xi_-)$, hence so does v, which is absurd by construction of (g_n). □

Corollary 9.19 *G is a hyperbolic group and ∂G is G–equivariantly homeomorphic to its Gromov boundary.*

Proof The group G is a convergence group on ∂G by Corollary 9.17, and every point of ∂G is conical by Lemma 9.18, thus the result follows from Theorem 9.12. □

To conclude the proof of the combination theorem for hyperbolic groups, it remains to show that stabilisers embed as quasiconvex subsets.

Proposition 9.20 *Stabilisers of simplices of X are quasiconvex subgroups of G.*

Proof It is enough to prove the result for the stabiliser of a vertex v of X. Notice that, by Proposition 6.19, the boundary of G_v embeds G_v–equivariantly in ∂G, the latter being G–equivariantly homeomorphic to the Gromov boundary of G by Corollary 9.19. Hence, the result follows from a result of Bowditch [6] recalled in the introduction. □

Proof of the combination theorem for hyperbolic groups This follows from Corollary 9.19 and Proposition 9.20. □

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