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Commutative ring objects in pro-categories and generalized Moore spectra

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We develop a rigidity criterion to show that in simplicial model categories with a compatible symmetric monoidal structure, operad structures can be automatically lifted along certain maps. This is applied to obtain an unpublished result of M J Hopkins that certain towers of generalized Moore spectra, closely related to the $K(n)$–local sphere, are E_∞–algebras in the category of pro-spectra. In addition, we show that Adams resolutions automatically satisfy the above rigidity criterion. In order to carry this out we develop the concept of an operadic model category, whose objects have homotopically tractable endomorphism operads.

55P43, 55U35; 18D20, 18D50, 18G55

1 Introduction

One of the canonical facts that distinguishes stable homotopy theory from algebra is the fact that the mod 2 Moore spectrum does not admit a multiplication. There are numerous consequences and generalizations of this fact: there is no Smith–Toda complex $V(1)$ at the prime 2; the Smith–Toda complex $V(1)$ does not admit a multiplication at the prime 3; the mod 4 Moore spectrum admits no multiplication which is either associative or commutative; the mod p Moore spectrum admits the structure of an $A(p-1)$–algebra but not an $A(p)$–algebra; and so on. (A discussion of the literature on multiplicative properties of Moore spectra can be found in Thomason [42, A.6], while multiplicative properties of $V(1)$ can be found in Oka [30]. The higher structure on Moore spectra plays an important role in Schwede [37].)

These facts and others form a perpetual sequence of obstructions to the existence of strict multiplications on generalized Moore spectra, and it appears to be the case that essentially no generalized Moore spectrum admits the structure of an E_∞–algebra.

Despite this, the goal of the current paper is to show the following:

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For any prime p and any $n \geq 1$, let $\{M_I\}_I$ be a tower of generalized Moore spectra of type n, with homotopy limit the p–complete sphere (as in Hovey and Strickland [27, 4.22]). Then $\{M_I\}_I$ admits the structure of an E_∞–algebra in the category of pro-spectra.

Roughly, the multiplicative obstructions vanish when taking the inverse system as a whole (by analogy with the inverse system of neighborhoods of the identity in a topological group).

This statement is due to Mike Hopkins, and it is referenced in Rognes [35, 5.4.2]. Mark Behrens gave a proof that the tower admits an H_∞ structure, based on Hopkins' unpublished argument, in [6]. As discussed in [3, 2.7], Ausoni, Richter and Rognes worked out a version of Hopkins' statement for the pro-spectrum $\{ku/p^\nu\}_{\nu \geq 1}$ for any prime p as an object in the category of pro-ku–modules. (Here, ku is the connective complex K–theory spectrum.)

It has been understood for some time that the $K(n)$–local category should, in some sense, be a category with some pro-structure. For example, as in Hovey [22, Section 2], if X is any spectrum, then

$$L_{K(n)}(X) \simeq \operatorname*{holim}_I (L_n X \wedge M_I);$$

ie the $K(n)$–localization of X is the homotopy limit of the levelwise smash product in pro-spectra of $L_n X$ with the tower $\{M_I\}_I$. In applications, the Morava E–theory homology theory $E(k,\Gamma)_*(-)$ defined below is often replaced by the more tractable completed theory which again involves smashing with the pro-spectrum $\{M_I\}_I$:

$$E(k,\Gamma)^\vee_*(X) = \pi_*\big(L_{K(n)}(E(k,\Gamma) \wedge X)\big) \cong \pi_*\big(\operatorname*{holim}_I (E(k,\Gamma) \wedge X \wedge M_I)\big).$$

(For example, see the work of Goerss, Henn, Mahowald and Rezk [20, Section 2], Hovey [24] and Rezk [34].) Thus, in some sense our goal is to establish appropriate multiplicative properties of this procedure.

We give several applications of our results. Let $n \geq 1$ and let p be a fixed prime. As in Rezk [32], let $\mathcal{FG}$ be the category that consists of pairs (k,Γ), where k is any perfect field of characteristic p and Γ is a height n formal group law over k. The morphisms are pairs $(r,f)\colon (k,\Gamma) \to (k',\Gamma')$, where $r\colon k' \to k$ is a ring homomorphism and $f\colon \Gamma \to r^*(\Gamma')$ is an isomorphism of formal group laws.

By Goerss and Hopkins [21] (see also Goerss [19, 2.7]), the Goerss–Hopkins–Miller Theorem says that there is a presheaf

$$E\colon \mathcal{FG}^{\mathrm{op}} \longrightarrow \mathcal{S}p_{E_\infty}, \quad (k,\Gamma) \mapsto E(k,\Gamma),$$

where $\mathcal{S}p_{E_\infty}$ is the category of commutative symmetric ring spectra and

$$E(k,\Gamma)_* \cong W(k)[\![u_1,\dots,u_{n-1}]\!][u^{\pm 1}].$$

Here $W(k)$ is the ring of Witt vectors of the field k, each u_i has degree zero, and the degree of u is -2. The E_∞–algebra $E(k,\Gamma)$ is a *Morava E–theory*, whose formal group law is a universal deformation of Γ. In Section 6 we show that each $E(k,\Gamma)$ lifts to the E_∞–algebra $\{E(k,\Gamma)\wedge M_I\}_I$ in the category of pro-spectra.

Also, in Section 7 we show that various completions that are commonly employed in homotopy theory also have highly multiplicative structures. In particular, these include classical Adams resolutions. This has the amusing consequence that, in homotopy theory, the completion of a commutative ring object with respect to a very weak notion of an ideal (whose quotient is only assumed to have a left-unital binary operation) automatically inherits a commutative ring structure.

It should be noted that some care is required in the definition of an E_∞–algebra structure when working with pro-objects. In this paper, we use a definition in terms of endomorphism operads in simplicial sets: an E_∞–algebra structure is a map from an E_∞–operad to the endomorphism operad of the pro-object. If $X=\{x_\alpha\}_\alpha$ is a pro-object, note that this does *not* define maps of pro-objects

$$\{(E\Sigma_n)_+\wedge_{\Sigma_n}(x_\alpha)^{\wedge n}\}_\alpha \longrightarrow X.$$

Roughly, the issue is that the levelwise smash product only commutes with finite colimits in the pro-category. In particular, it does not represent the tensor of pro-objects with spaces; see Isaksen [28, Section 4.1].

The starting point for the proof that Moore towers admit E_∞ structures is the following algebraic observation.

Proposition 1.1 *Suppose that R is a commutative ring, S is an R–module, and $e\in S$ is an element such that the evaluation map $\mathrm{Hom}_R(S,S)\to S$ is an isomorphism. Then S admits a unique binary multiplication such that $e^2=e$, and under this multiplication S becomes a commutative R–algebra with unit e.*

The proof consists of iteratively applying the adjunction

$$\mathrm{Hom}_R(S^{\otimes_R n},S)\cong \mathrm{Hom}_R(S^{\otimes_R (n-1)},\mathrm{Hom}_R(S,S))$$

to show that a map $S^{\otimes_R n}\to S$ is equivalent to a choice of image of $e^{\otimes n}$; existence shows that $e\otimes e\mapsto e$ determines a binary multiplication, and uniqueness forces the commutativity, associativity and unitality properties. In particular, this applies whenever

S is the localization of a quotient of R. One notes that, while this proof only requires studying maps $S^{\otimes_R n} \to S$ for $n \leq 3$, it is implicitly an operadic proof.

This proof almost carries through when S is the completion of R with respect to an ideal $\mathfrak{m}$. However, in this case the topology on $R^\wedge_{\mathfrak{m}}$ needs to be taken into account. The tensor product over R needs to be replaced by a completed tensor product of inverse limits of modules, which does not have a right adjoint in general. However, when restricted to objects which are inverse limits of finitely presented modules, smallness gives the completed tensor product a right adjoint (cf Bauer [5, B.3]).

The paper follows roughly this line of argument, mixed with the homotopy theory of pro-objects developed by Isaksen and Fausk [28; 18].

Unfortunately, the "levelwise" tensor product for pro-objects does not usually have a right adjoint. This means that the constructions of model categories of rings and modules, from Schwede and Shipley [39] and Hovey [23, Section 4], do not apply in this circumstance. Understanding these homotopical categories should be a topic worth further investigation.

Outline

We summarize the portions of this paper not previously described. Our work begins in Section 2 by collecting definitions and results on the homotopy theory of operads and spaces of operad structures on objects. A more detailed outline is at the beginning of that section.

In Section 3 we flesh out the proof outlined in the introduction. In model categories with amenable symmetric monoidal structure, as well as a weak variant of internal function objects, certain "rigid" maps automatically allow one to lift algebra structures uniquely from the domain to the target.

Section 4 assembles together enough of the homotopy theory of pro-objects to show that pro-dualizable objects behave well with respect to a weak function object, allowing the results of Section 3 to be applied. To obtain the main results of this section, we place several strong assumptions on the behavior of filtered colimits with respect to the homotopy theory. In particular, we require that filtered colimits represent homotopy colimits and preserve both fibrations and finite limits. The main reason for restricting to this circumstance is that we need to gain homotopical control over function spaces of the form $\lim_\beta \operatorname{colim}_\alpha \operatorname{Map}(x_\alpha, y_\beta)$, as well as other function objects. (Functors such as $\operatorname{Map}(-, y)$ are rarely assumed to have good behavior on towers of fibrations.)

Section 5 verifies all these necessary assumptions in the case of modules over a commutative symmetric ring spectrum. (The category of modules over a commutative differential graded algebra is Quillen equivalent to such a category.)

The main result of the paper appears in Section 6, which shows (Theorem 6.3) that a tower of generalized Moore spectra (constructed by Hovey and Strickland based on previous work of Devinatz, Hopkins and Smith) automatically obtains an E_∞–algebra structure from the sphere. This is then applied to show that certain chromatic localizations of the sphere, as well as all the Morava E–theories $E(k,\Gamma)$, are naturally inverse limits of highly multiplicative pro-objects.

Section 7 carries out the aforementioned study of multiplicative structure on completions.

Notation and assumptions

As various model categories of pro-objects are very large and do not come equipped with functorial factorization, there are set-theoretic technicalities. These include being able to define either derived functors or a homotopy category with the same underlying object set. We refer the reader to Dwyer, Hirschhorn, Kan and Smith [13] (eg Section 8) for one solution, which involves employing a larger universe in which one constructs equivalence relations and produces canonical definitions which can be made naturally equivalent to constructions in the smaller universe.

For a functor F with source a model category, the symbol $\mathbb{L}F$ (resp. $\mathbb{R}F$) will be used to denote the derived functor, with domain the homotopy category of cofibrant-fibrant objects, when F takes acyclic cofibrations (resp. acyclic fibrations) to weak equivalences. For inline operators such as $\otimes$, this will be replaced by a superscript. We use $[-,-]$ to denote the set of maps in the homotopy category.

The generic symbol $\oslash$ denotes a monoidal product, while $\otimes$ is reserved for actual tensor products and categories tensored over simplicial sets.

For a pro-object X, we will often write an isomorphic diagram using lowercase symbols $\{x_\alpha\}_\alpha$ without comment.

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2 Operads

In this section, we will discuss some background relating to operads and their actions. In essence, we would like to establish situations where we have a model category $\mathcal{D}$ supporting enough structure so that objects of $\mathcal{D}$ have endomorphism operads, and we would like to ensure that these endomorphism operads are invariant under both weak equivalences and appropriate Quillen equivalences.

This requires us to dig our way through several layers of terminology.

Endomorphism objects are functorial under isomorphisms. Our goal is to produce "derived" endomorphism objects which are functorial under weak equivalences. While our attention is turned towards endomorphism operads, the methods apply when we have a very general enriching category $\mathcal{V}$. We give a functorial construction of derived endomorphism objects in $\mathcal{V}$–monoids, which mostly relies on an SM7 axiom, in Section 2.1. As a side benefit, we obtain a definition of endomorphism objects for diagrams which will prove necessary later.

We then turn our attention to the construction of endomorphism operads. By its very nature, this requires our category to carry a symmetric monoidal structure, a model structure, and an enrichment in spaces, and all of these must obey compatibility rules. This presents us with a significant number of adjectives to juggle. We study this compatibility in Section 2.2, finally encoding it in the notion of an *operadic model category*.

The motivation for operadic model categories is the ability to extend our enrichment, from simplicial sets under cartesian product to symmetric sequences under the composition product. The work of Section 2.1 then produces derived endomorphism operads. To ensure that these constructions make sense in homotopy theory, we show that they are invariant under an appropriate notion of operadic Quillen equivalence.

Once this is in place, in Section 2.4 we are able to study a space parametrizing $\mathcal{O}$–algebra structures on a fixed object, and be assured that if $\mathcal{O}$ is cofibrant it is an invariant under equivalences of the homotopy type and equivalences of the model category.

In this paper, operads are assumed to have symmetric group actions, and no assumptions are placed on degrees 0 or 1. We will write Com for the commutative operad, which is terminal among simplicial operads and consists of a single point in each degree.

Both the definitions and the philosophy here draw heavily from Rezk [31].

2.1 Enriched endomorphisms

In this section we assume that $\mathcal{V}$ is a monoidal category with a model structure, and that $\mathcal{D}$ is a model category with a $\mathcal{V}$–enriched structure. For $a, b \in \mathcal{D}$ we write $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(a,b)$ for the enriched mapping object.

We assume that the following standard axiom holds.

Axiom 2.1 (SM7) *Given a cofibration $i\colon a \rightarrowtail b$ and a fibration $p\colon x \twoheadrightarrow y$ in $\mathcal{D}$, the map*

$$\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(b,x) \longrightarrow \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(a,x) \times_{\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(a,y)} \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(b,y)$$

is a fibration in $\mathcal{V}$, which is acyclic if either i or p is.

Write $\mathcal{V}\text{–}\mathrm{Mon}$ for the category of monoids in $\mathcal{V}$. For an object $c \in \mathcal{D}$, we have a $\mathcal{V}$–endomorphism object $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(c) \in \mathcal{V}\text{–}\mathrm{Mon}$.

Definition 2.2 A map in $\mathcal{V}\text{–}\mathrm{Mon}$ is a *fibration* or a *weak equivalence* if the underlying map is a fibration or weak equivalence in the category $\mathcal{V}$.

This definition may or may not come from a model structure on the category of $\mathcal{V}$–monoids. However, under amenable circumstances it makes sense to form the localization of $\mathcal{V}\text{–}\mathrm{Mon}$ with respect to the weak equivalences.

Our goal is to prove that endomorphism objects are functorial in weak equivalences, at least on the level of homotopy categories (Theorem 2.11). To construct this functor, it is useful to first note that the subcategory of isomorphisms in the homotopy category of $\mathcal{D}$ is naturally equivalent to a category formed by a restricted localization.

Lemma 2.3 *Let $\mathcal{M}$ be a model category and $\mathcal{A} \subset \mathcal{M}$ be the subcategory of acyclic fibrations between cofibrant-fibrant objects. Then the natural functor*

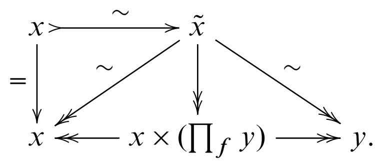
$$\mathcal{A}^{-1}\mathcal{A} \longrightarrow \mathrm{ho}(\mathcal{M})^{w},$$

from the groupoid completion to the subcategory of isomorphisms in the homotopy category of $\mathcal{M}$, is fully faithful and essentially surjective.

Remark 2.4 The dual result clearly holds for inverting acyclic cofibrations.

Proof Any object in $\mathcal{M}$ is equivalent to a cofibrant-fibrant one, so the functor is obviously essentially surjective.

Let x, y be cofibrant-fibrant objects in $\mathcal{M}$, and consider the map $x \to x \times (\prod_f y)$, where the product is indexed by weak equivalences $f\colon x \to y$. We can factor this map into an acyclic cofibration $x \to \tilde{x}$ followed by a fibration, and for any such weak equivalence f this yields a diagram in $\mathcal{M}$ of the form



This shows that $\mathcal{A}^{-1}\mathcal{A} \to \mathrm{ho}(\mathcal{M})^w$ is full. Moreover, all maps in the homotopy category are realized in $\mathcal{A}^{-1}\mathcal{A}$ by the inverse of the map $\tilde{x} \twoheadrightarrow x$ followed by a map $\tilde{x} \twoheadrightarrow y$. Therefore, to complete the proof it suffices to show that right homotopic acyclic fibrations $\tilde{x} \to y$ become equal in $\mathcal{A}^{-1}\mathcal{A}$.

Let $y \rightarrowtail z \twoheadrightarrow y \times y$ be a path object for y, with $p_0, p_1\colon z \twoheadrightarrow y$ the component projections (which are acyclic fibrations). Let $h = z \times_y z$, with the product taken over p_0 on both factors, and $j_0, j_1\colon h \twoheadrightarrow z$ the component projections. The maps $p_1 j_0$ and $p_1 j_1$ make the object h into another path object for y. However, we have an identity of acyclic fibrations $p_0 j_0 = p_0 j_1$, and so in the category $\mathcal{A}^{-1}\mathcal{A}$ we have $p_1 j_0 = p_1 j_1$. □

Definition 2.5 For a small category I, the functor category $\mathcal{D}^I$ is a $\mathcal{V}$–enriched category, with $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}^I}(F,G)$ described by the equalizer diagram

$$\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}^I}(F,G) \to \prod_i \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(F(i),G(i)) \rightrightarrows \prod_{i\to j} \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(F(i),G(j)).$$

In the particular case where I is the poset $\{0 < 1\}$ and $\mathcal{D}^I$ is the category of arrows $Ar(\mathcal{D})$, we will abuse notation by writing $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(f,g)$ as an enriched mapping object between two morphisms f and g of $\mathcal{D}$. Similarly, in the case where J is the poset $\{0 < 1 < 2\}$ and $\mathcal{D}^J$ is the category of composable pairs of arrows of $\mathcal{D}$, for J–diagrams

$$(\cdot \xrightarrow{f} \cdot \xrightarrow{f'} \cdot) \quad \text{and} \quad (\cdot \xrightarrow{g} \cdot \xrightarrow{g'} \cdot)$$

we will similarly write $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}((f',f),(g',g))$ as an enriched mapping object.

Remark 2.6 When $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{V}$ preserves limits, we can say more. For Reedy categories I, the category $\mathcal{D}^I$ then inherits a $\mathcal{V}$–enriched Reedy model structure that satisfies an SM7 axiom. (Compare Angeltveit [1], which assumes a symmetric monoidal closed structure on $\mathcal{V}$.)

For maps $f\colon a \to b$ and $p\colon x \to y$ in $\mathcal{D}$, the categorical equalizer $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(f,p)$ can be alternatively described in $\mathcal{V}$ as a fiber product

$$\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(a,x) \times_{\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(a,y)} \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(b,y).$$

This makes the following proposition a straightforward consequence of the SM7 axiom.

Proposition 2.7 (cf Dwyer and Hess [12, 6.6]) *Suppose that in $\mathcal{D}$, $f\colon a \to b$ is a map with cofibrant domain and $p\colon x \twoheadrightarrow y$ is a fibration between fibrant objects. Then:*

- *The map $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(f,p) \to \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(b,y)$ is a fibration.*
- *If p is an acyclic fibration, then the map $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(f,p) \to \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(b,y)$ is an acyclic fibration.*
- *If p is an acyclic fibration and f is a weak equivalence between cofibrant objects, then the map $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(f,p) \to \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(a,x)$ is a weak equivalence.*

Restricting to the case where f and p coincide, we deduce the following consequences for endomorphisms.

Corollary 2.8 *Suppose that in $\mathcal{D}$, $f\colon x \twoheadrightarrow y$ is an acyclic fibration between fibrant objects. If x is cofibrant, then the map $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f) \to \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(y)$ is an acyclic fibration in $\mathcal{V}\text{–}\mathrm{Mon}$, and if both x and y are cofibrant, then the map $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f) \to \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(x)$ is a weak equivalence in $\mathcal{V}\text{–}\mathrm{Mon}$.*

Similar analysis yields the following.

Proposition 2.9 *Suppose that in $\mathcal{D}$, $a \xrightarrow{f} b \xrightarrow{g} c$ are maps between cofibrant objects and $x \overset{p}{\twoheadrightarrow} y \overset{q}{\twoheadrightarrow} z$ are fibrations with z fibrant. Then:*

- *The map $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}((g,f),(q,p)) \to \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(g,q)$ is a fibration.*
- *If p is an acyclic fibration, the map $\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}((g,f),(q,p)) \to \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(g,q)$ is an acyclic fibration.*
- *If p and q are acyclic fibrations and g is a weak equivalence, then the following maps are weak equivalences:*

$$\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}((g,f),(q,p)) \longrightarrow \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(f,p),$$
$$\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}((g,f),(q,p)) \longrightarrow \mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(gf,qp).$$

Corollary 2.10 *Suppose that in $\mathcal{D}$, $f\colon x \twoheadrightarrow y$ and $g\colon y \twoheadrightarrow z$ are acyclic fibrations between cofibrant-fibrant objects. Then the map $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}((g,f)) \to \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(g)$ is an acyclic fibration in $\mathcal{V}\text{–}\mathrm{Mon}$, and the following maps are weak equivalences in $\mathcal{V}\text{–}\mathrm{Mon}$:*

$$\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}((g,f)) \longrightarrow \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f),$$
$$\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}((g,f)) \longrightarrow \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(gf).$$

Theorem 2.11 *Suppose that the category of $\mathcal{V}$–monoids has a homotopy category $\mathrm{ho}(\mathcal{V}\text{–}\mathrm{Mon})$. Then there is a derived functor*

$$\mathbb{R}\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}\colon \mathrm{ho}(\mathcal{D})^{w} \longrightarrow \mathrm{ho}(\mathcal{V}\text{–}\mathrm{Mon})^{w}$$

from isomorphisms in the homotopy category of $\mathcal{D}$ to isomorphisms in the homotopy category of $\mathcal{V}$–monoids.

This lifts the composite of the antidiagonal

$$\mathrm{ho}(\mathcal{D})^{w} \longrightarrow \mathrm{ho}(\mathcal{D})^{\mathrm{op}} \times \mathrm{ho}(\mathcal{D})$$

with the functor

$$\mathbb{R}\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}\colon \mathrm{ho}(\mathcal{D})^{\mathrm{op}} \times \mathrm{ho}(\mathcal{D}) \longrightarrow \mathrm{ho}(\mathcal{V}).$$

The monoid $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(c)$ represents the derived homotopy type in $\mathrm{ho}(\mathcal{V}\text{–}\mathrm{Mon})$ on cofibrant-fibrant objects.

Remark 2.12 In the case of the mapping space between two objects in a model category, this is most easily accomplished using the Dwyer–Kan [14] simplicial localization (generalized by Dundas [11]). This constructs a simplicially enriched category, with the correct mapping spaces, where the weak equivalences have become isomorphisms.

However, as natural transformations can only be recovered in the simplicial localization using simplicial homotopies, study of the interaction between the symmetric monoidal structure and simplicial localization would require extra work. The shortest path is likely through ∞–category theory, which would take us too far afield.

Proof of 2.11 Let $\mathcal{A} \subset \mathcal{D}$ be the category of acyclic fibrations between cofibrant-fibrant objects of $\mathcal{D}$. By Lemma 2.3, it suffices to define the functor $\mathcal{A} \to \mathrm{ho}(\mathcal{V}\text{–}\mathrm{Mon})^{w}$.

For an acyclic fibration $f\colon x \to y$ in $\mathcal{A}$, Corollary 2.8 gives a diagram of weak equivalences

$$\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(x) \xleftarrow{\sim} \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f) \xrightarrow{\sim} \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(y)$$

in $\mathcal{V}\text{–Mon}$, representing a composite map in $\mathrm{ho}(\mathcal{V}\text{–Mon})^w$. For a composition $g \circ f$ we apply Corollary 2.10 to obtain a commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}((g,f)) & & \\
 & \swarrow & \downarrow & \searrow & \\
\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f) & & \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(gf) & & \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(g) \\
\downarrow & \searrow\ \swarrow & & \searrow\ \swarrow & \downarrow \\
\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(x) & & \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(y) & & \mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(z)
\end{array}$$

of weak equivalences in $\mathcal{V}\text{–Mon}$, which shows that the resulting assignment respects composition. □

By replacing $\mathcal{D}$ with the category $Ar(\mathcal{D})$ of arrows in $\mathcal{D}$, equipped with the projective model structure, we obtain the following consequence of Theorem 2.11.

Proposition 2.13 *Suppose that the category of $\mathcal{V}$–monoids has a homotopy category* $\mathrm{ho}(\mathcal{V}\text{–Mon})$. *Then there is a derived functor*

$$\mathbb{R}\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}\colon \mathrm{ho}(Ar(\mathcal{D}))^w \longrightarrow \mathrm{ho}(\mathcal{V}\text{–Mon})^w,$$

from isomorphisms in the homotopy category of $Ar(\mathcal{D})$ to isomorphisms in the homotopy category of $\mathcal{V}$–monoids, together with natural transformations

$$\mathbb{R}\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(x) \longleftarrow \mathbb{R}\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f) \longrightarrow \mathbb{R}\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(y)$$

for $f\colon x \to y$.

The monoid $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f)$ represents the derived homotopy type in $\mathrm{ho}(\mathcal{V}\text{–Mon})$ *on fibrations between cofibrant-fibrant objects.*

2.2 Tensor model categories

First, we recall interaction between a monoidal structure and a model category structure.

Recall that the *pushout product axiom* for cofibrations in a model category with monoidal product $\oslash$ says that if $f\colon x \to y$ and $f'\colon x' \to y'$ are cofibrations, then the pushout map

$$(y \oslash x') \amalg_{x \oslash x'} (x \oslash y') \longrightarrow y \oslash y'$$

is a cofibration, and is a weak equivalence if either f or f' is.

The following definition is from Fausk and Isaksen [18, 12.1] (though see Remark 2.15).

Definition 2.14 A *tensor model category* is a model category $\mathcal{D}$ equipped with a monoidal product that

- satisfies the pushout product axiom for cofibrations,
- takes the product with an initial object in either variable to an initial object, and
- preserves weak equivalences when either of the inputs is cofibrant.

If $\mathcal{D}$ is further equipped with a symmetric tensor structure, $\mathcal{D}$ is a *symmetric tensor model category*.

Remark 2.15 Note that the second component makes this more restrictive than [18, 12.1]. Because the product with an initial object is always initial, the pushout product axiom implies that the monoidal product preserves cofibrant objects.

By analogy with the definition of a (lax) monoidal Quillen adjunction (Schwede and Shipley [40, 3.6]), we have the following.

Definition 2.16 Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are tensor model categories. A *tensor Quillen adjunction* is a Quillen adjoint pair of functors

$$L\colon \mathcal{D} \rightleftarrows \mathcal{D}' : R,$$

together with a lax monoidal structure on R, such that

- for any cofibrant objects $x, y \in \mathcal{D}$, the induced natural transformation $L(x \oslash y) \to L(x) \oslash' L(y)$ is a weak equivalence, and
- for some cofibrant replacement $\mathbb{I}_c$ of the unit $\mathbb{I}$ of $\mathcal{D}$, the induced map $L(\mathbb{I}_c) \to \mathbb{I}'$ is a weak equivalence.

We refer to this as a *symmetric tensor Quillen adjunction* if the functor R is lax symmetric monoidal, and a *tensor Quillen equivalence* if the underlying adjunction is a Quillen equivalence.

The definition below is based on Fausk and Isaksen [18, 12.2].

Definition 2.17 A *simplicial tensor model category* is a simplicial model category $\mathcal{D}$ equipped with a monoidal product such that

- this structure makes $\mathcal{D}$ into a tensor model category,

- there are choices of natural isomorphisms

$$K \otimes x \cong x \owedge (K \otimes \mathbb{I}),$$
$$K \otimes x \cong (K \otimes \mathbb{I}) \owedge x,$$

for $x \in \mathcal{D}$ and K a finite simplicial set which are compatible with the unit isomorphism, and

- the functor $\mathrm{Map}_{\mathcal{D}}(\mathbb{I}, -)$ is a right Quillen functor.

(Here $\otimes$ denotes the tensor of objects of $\mathcal{D}$ with simplicial sets from the simplicial model structure, and $\mathrm{Map}_{\mathcal{D}}$ denotes the simplicial mapping object.) In this case, we say that the monoidal structure on $\mathcal{D}$ is *compatible* with the simplicial model structure.

A *simplicial symmetric tensor model category* is a simplicial tensor model category such that the composite natural isomorphism

$$x \owedge (K \otimes \mathbb{I}) \cong K \otimes x \cong (K \otimes \mathbb{I}) \owedge x$$

is the natural symmetry isomorphism.

We will freely make use of phrases such as "simplicial (symmetric) tensor Quillen adjunction/equivalence" to indicate tensor Quillen adjunctions with an appropriate lift to a lax monoidal simplicial Quillen adjunction.

Remark 2.18 We note that several situations occur where the Quillen functors in question are each the identity functor, viewed as a Quillen functor between two distinct tensor model structures on the same monoidal category. In this circumstance, the extra axioms for a tensor Quillen equivalence or a simplicial symmetric Quillen equivalence are trivially satisfied.

Remark 2.19 The Yoneda embedding ensures that, for any finite K and L and any x, $(K \times L) \otimes x \cong K \otimes (L \otimes x)$. Compatibility then implies that this is isomorphic to $(K \otimes \mathbb{I}) \owedge (L \otimes \mathbb{I}) \owedge x$. These can be used to obtain well-behaved maps

$$\mathrm{Map}_{\mathcal{D}}(x, y) \times \mathrm{Map}_{\mathcal{D}}(x', y') \longrightarrow \mathrm{Map}_{\mathcal{D}}(x \owedge x', y \owedge y').$$

If the tensor structure is symmetric, this map is equivariant with respect to the symmetry isomorphisms.

Remark 2.20 The following are equivalent.

(1) The unit object $\mathbb{I}$ is cofibrant in $\mathcal{D}$.

(2) The functor $\mathrm{Map}_{\mathcal{D}}(\mathbb{I}, -)$, from $\mathcal{D}$ to simplicial sets, is a right Quillen functor.

(3) The functor $(-) \otimes \mathbb{I}$, from simplicial sets to $\mathcal{D}$, is a left Quillen functor.

(4) The functor $(-) \otimes \mathbb{I}$, from simplicial sets to $\mathcal{D}$, preserves cofibrations.

Evidently each implies the next. To complete the equivalence, we take the cofibration $\varnothing \rightarrowtail *$ and tensor with $\mathbb{I}$, which (again, checking the Yoneda embedding) is naturally isomorphic to the map from an initial object of $\mathcal{D}$ to $\mathbb{I}$.

It is unsatisfying to make the assumption that the unit is cofibrant, but it will ensure homotopical control on endomorphism operads. It may be dropped if we are willing to define operads without an object parametrizing 0–ary operations, but this significantly complicates the proof of Proposition 2.27.

The hypotheses of a simplicial tensor model category are designed to ensure that the monoidal structure can produce a reasonably-behaved multicategorical enrichment, and hence reasonably-behaved endomorphism operads.

For the sake of brevity in this paper we employ the following shorthand, with the implicit understanding that it demonstrates a prejudice towards simplicial sets.

Definition 2.21 An *operadic model category* is a simplicial symmetric tensor model category. An *operadic Quillen adjunction* is a simplicial symmetric tensor Quillen adjunction, and if the underlying adjunction is a Quillen equivalence we refer to it as an *operadic Quillen equivalence*.

We now relate these to operads in the ordinary sense.

Recall that a *symmetric sequence* is a collection of simplicial sets $\{X(n)\}_{n\geq 0}$ equipped with actions of the symmetric groups Σ_n. There is a model structure on symmetric sequences whose fibrations and weak equivalences are collections of equivariant maps $X(n) \to Y(n)$ which satisfy these properties levelwise (ignoring the action of the symmetric group). The category of symmetric sequences has a (nonsymmetric) monoidal structure $\circ$, the *composition product*, whose algebras are operads; eg see Markl, Shnider and Stasheff [29].

The main reason for introducing the concept of an operadic model category is the following proposition.

Proposition 2.22 *Let $\mathcal{V}$ be the category of symmetric sequences of simplicial sets. Then for an operadic model category $\mathcal{D}$, the definition*

$$\mathcal{V}\text{–}\mathrm{Map}_{\mathcal{D}}(x, y) = \{\mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, y)\}_n$$

makes $\mathcal{D}$ into a $\mathcal{V}$–enriched category satisfying the SM7 axiom.

Proof This is a straightforward consequence of the structure on $\mathcal{D}$, though it requires Remarks 2.15 and 2.20. □

Remark 2.23 In particular, for a map $f\colon x \to y$ in $\mathcal{D}$, the endomorphism operad $\mathcal{V}\text{–}\mathrm{End}_{\mathcal{D}}(f)$ is the symmetric sequence which, in degree n, is the pullback of the diagram

$$\mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, x) \longrightarrow \mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, y) \longleftarrow \mathrm{Map}_{\mathcal{D}}(y^{\oslash n}, y).$$

Remark 2.24 While operadic model categories have natural $\mathcal{V}$–enrichments, operadic Quillen adjunctions do not automatically yield $\mathcal{V}$–enriched adjunctions, except in a homotopical sense, unless both adjoints are strong monoidal.

2.3 Model structures on operads

The model structure on the category of symmetric sequences lifts to one on the category of operads in simplicial sets, with fibrations and weak equivalences defined levelwise; see Berger and Moerdijk [8, 3.3.1].

This extends to a simplicial model structure. This exact statement does not appear to be in the immediately available literature. However, it can be obtained using either one of the following two approaches.

- Rezk's thesis constructs a simplicial model structure on operads with weak equivalences and fibrations defined levelwise under an *equivariant* model structure [31, 3.2.11], extending a simplicial model structure on symmetric sequences. The method of proof extends to the Berger–Moerdijk model structure, with weak equivalences and fibrations defined to be ordinary *nonequivariant* weak equivalences and fibrations, by discarding some of the generating cofibrations and generating acyclic cofibrations. (This does not alter Rezk's [31, Proposition 3.1.5], the main technical tool for proving the result, which uses the existence of a functorial levelwise fibrant replacement for simplicial operads as in Schwede [36, B2].)

- Alternatively, we can use the fact that operads can be expressed algebraically. There is a functor which takes an $\mathbb{N}$–graded set $X = \{X_n\}$ and produces the free operad $\mathbb{O}(X)$ on X (which can be expressed in terms of rooted trees with nodes appropriately labelled by elements of X). The functor $\mathbb{O}$ is a monad on graded sets whose algebras are discrete operads. It also commutes with filtered colimits, which makes it a *multisorted theory* in the terminology of Rezk [33]. One can then apply [33, Theorem 7.1] to obtain the desired simplicial model structure on the category of simplicial $\mathbb{O}$–algebras, ie operads in simplicial sets.

2.4 Spaces of algebra structures

In this section, we assume that $\mathcal{D}$ is an operadic model category, viewed as a model category enriched in symmetric sequences of simplicial sets.

From this point forward, we will drop the enriching category from some of the notation as follows. For an object $x \in \mathcal{D}$, the endomorphism operad $\mathrm{End}_{\mathcal{D}}(x)$ is the symmetric sequence which, in degree n, is the simplicial set $\mathrm{Map}_{\mathcal{D}}(x^{\otimes n}, x)$. Similarly, for a map $f\colon x \to y$ in $\mathcal{D}$, we have the endomorphism operad $\mathrm{End}_{\mathcal{D}}(f)$.

Definition 2.25 For a cofibrant operad $\mathcal{O}$ and a cofibrant-fibrant object $x \in \mathcal{D}$, the *space of $\mathcal{O}$–algebra structures on x* is the space of operad maps

$$\mathrm{Map}_{\mathrm{operad}}(\mathcal{O}, \mathrm{End}_{\mathcal{D}}(x)).$$

For a map $\eta\colon x \to y$ between cofibrant-fibrant objects in $\mathcal{D}$, the *space of $\mathcal{O}$–algebra structures on η* is the space of operad maps

$$\mathrm{Map}_{\mathrm{operad}}(\mathcal{O}, \mathrm{End}_{\mathcal{D}}(\eta)).$$

Equivalently, this is the space of pairs of $\mathcal{O}$–algebra structures on x and y making η into a map of $\mathcal{O}$–algebras.

Corollary 2.26 *If $\mathcal{O}$ is a cofibrant operad, a weak equivalence $f\colon x \to y$ between cofibrant-fibrant objects in $\mathcal{D}$ determines an isomorphism in the homotopy category between the spaces of $\mathcal{O}$–algebra structures on x and y.*

Proof By Theorem 2.11, we find that the operads $\mathrm{End}_{\mathcal{D}}(x)$ and $\mathrm{End}_{\mathcal{D}}(y)$ are canonically equivalent in the homotopy category of operads, and the spaces of maps from $\mathcal{O}$ are equivalent. □

Proposition 2.27 *Let $f\colon \mathbb{I} \rightarrowtail \mathbb{I}_f$ be a fibrant replacement for the unit object of $\mathcal{D}$. Then the space of E_∞–algebra structures on $\mathbb{I}_f$ compatible with the multiplication on $\mathbb{I}$ is contractible.*

Proof The map $\mathbb{I} \rightarrowtail \mathbb{I}_f$ is an acyclic cofibration between cofibrant objects. The enrichment of the opposite category $\mathcal{D}^{\mathrm{op}}$ gives rise to a dual formulation of Corollary 2.8, and specifically implies that the map $\mathrm{End}_{\mathcal{D}}(f) \to \mathrm{End}_{\mathcal{D}}(\mathbb{I})$ is an acyclic fibration.

Let $\mathcal{E}$ be a cofibrant E_∞–operad (a cofibrant replacement for Com), and fix the map $\mathcal{E} \to \mathrm{Com} \to \mathrm{End}_{\mathcal{D}}(\mathbb{I})$ coming from $\mathbb{I}$ being the unit. Then the space of lifts in the diagram

$$\begin{array}{ccc} & \mathrm{End}_{\mathcal{D}}(f) & \longrightarrow & \mathrm{End}_{\mathcal{D}}(\mathbb{I}_f) \\ & {\nearrow}^{\cdots} & \downarrow{\scriptstyle \sim} & \\ \mathcal{E} & \longrightarrow & \mathrm{End}_{\mathcal{D}}(\mathbb{I}) & \end{array}$$

is contractible. However, via the map $\mathcal{E} \to \mathrm{End}_{\mathcal{D}}(\mathbb{I}_f)$, these lifts precisely parametrize E_∞–algebra structures on $\mathbb{I}_f$ which are compatible with the multiplication on $\mathbb{I}$. □

Finally, we note that endomorphism operads are invariant under certain Quillen equivalences.

Proposition 2.28 *Suppose that* $L\colon \mathcal{D} \rightleftarrows \mathcal{D}' : R$ *is an operadic Quillen adjunction. Then for any cofibrant-fibrant objects* $y \in \mathcal{D}$ *and* $x \in \mathcal{D}'$ *with an equivalence* $f\colon y \to Rx$, *there is a map in the homotopy category of operads from* $\mathrm{End}_{\mathcal{D}'}(x)$ *to* $\mathrm{End}_{\mathcal{D}}(y)$.

If, in addition, this adjunction is an operadic Quillen equivalence, this map is an isomorphism in the homotopy category of operads.

Proof Using Proposition 2.7, we may assume that the equivalence f is an acyclic fibration.

Since R has a simplicial lift which is lax symmetric monoidal, we obtain a natural map

$$\mathrm{End}_{\mathcal{D}'}(x) \longrightarrow \mathrm{End}_{\mathcal{D}}(Rx)$$

of operads. By Corollary 2.8, we have an acyclic fibration $\mathrm{End}_{\mathcal{D}}(f) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(Rx)$. The composite

$$\mathrm{End}_{\mathcal{D}'}(x) \longrightarrow \mathrm{End}_{\mathcal{D}}(Rx) \overset{\sim}{\twoheadleftarrow} \mathrm{End}_{\mathcal{D}}(f) \longrightarrow \mathrm{End}_{\mathcal{D}}(y)$$

provides the desired map in the homotopy category of operads.

Now we further assume that the adjunction is an operadic Quillen equivalence. Form the pullback

$$\mathcal{O} = \mathrm{End}_{\mathcal{D}}(f) \times_{\mathrm{End}_{\mathcal{D}}(Rx)} \mathrm{End}_{\mathcal{D}'}(x).$$

The map $\mathcal{O} \to \mathrm{End}_{\mathcal{D}'}(x)$ is a weak equivalence. To complete the proof it therefore suffices to show that the map $\mathcal{O} \to \mathrm{End}_{\mathcal{D}}(y)$ is a weak equivalence.

In degree n, $\mathcal{O}$ is the pullback of the diagram

$$\mathrm{Map}_{\mathcal{D}}(x^{\otimes n}, x) \twoheadrightarrow \mathrm{Map}_{\mathcal{D}}(x^{\otimes n}, Ry) \leftarrow \mathrm{Map}_{\mathcal{D}'}(y^{\otimes n}, y),$$

so it suffices to show that the right-hand map is an equivalence. However, this map is the composite

$$\mathrm{Map}_{\mathcal{D}'}(y^{\oslash n}, y) \longrightarrow \mathrm{Map}_{\mathcal{D}'}((Lx)^{\oslash n}, y) \longrightarrow \mathrm{Map}_{\mathcal{D}'}(L(x^{\oslash n}), y) \cong \mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, Ry).$$

The first of these maps is an equivalence because y is cofibrant-fibrant and Lx is cofibrant, while the second is an equivalence because $L(x^{\oslash n}) \to (Lx)^{\oslash n}$ is an equivalence in $\mathcal{D}'$ between cofibrant objects (by definition of a tensor Quillen adjunction). □

3 Algebra structures on rigid objects

In this section we assume that $\mathcal{D}$ is an operadic model category. Our goal is to prove a rigidity result (Theorem 3.5) allowing us to lift algebra structures, as mentioned in the introduction. In order for this to be ultimately applicable to pro-objects, we will first need to develop a theory which applies when the tensor structure carries something weaker than a right adjoint.

3.1 Weak function objects

Definition 3.1 A *weak function object* for the homotopy category $\mathrm{ho}(\mathcal{D})$ is a functor

$$F^{\mathrm{weak}}(-,-)\colon \mathrm{ho}(\mathcal{D})^{\mathrm{op}} \times \mathrm{ho}(\mathcal{D}) \longrightarrow \mathrm{ho}(\mathcal{D})$$

equipped with a natural transformation of functors

$$\mathbb{R}\,\mathrm{Map}_{\mathcal{D}}(x \oslash^{\mathbb{L}} y, z) \longrightarrow \mathbb{R}\,\mathrm{Map}_{\mathcal{D}}(x, F^{\mathrm{weak}}(y,z))$$

in the homotopy category of spaces. For specific x, y and z such that this map is an isomorphism in the homotopy category of spaces, we will say that the weak function object *provides an adjoint* for maps $x \oslash^{\mathbb{L}} y \to z$.

Example 3.2 Suppose the tensor model category $\mathcal{D}$ is closed, and use $F_{\mathcal{D}}(x,y)$ to denote the internal function object in $\mathcal{D}$. Then for any x cofibrant in $\mathcal{D}$, the functor $(-)\oslash x\colon \mathcal{D} \to \mathcal{D}$ is a left Quillen functor, the adjoint $F_{\mathcal{D}}(x,-)\colon \mathcal{D} \to \mathcal{D}$ is the corresponding right Quillen functor, and these determine an adjunction on the homotopy category. It follows that given arbitrary x and y in $\mathrm{ho}(\mathcal{D})$, if $F^{\mathrm{weak}}(x,y)$ is defined to be the image of $F_{\mathcal{D}}(x_c, y_f)$, where x_c and y_f are cofibrant and fibrant representatives of x and y respectively, then $\mathcal{D}$ has a weak function object that provides an adjoint for $x \oslash^{\mathbb{L}} y \to z$ for all $x, y, z \in \mathrm{ho}(\mathcal{D})$.

Remark 3.3 We have the following consequences of Definition 3.1.

- Substituting $x = \mathbb{I}$, we obtain a natural transformation
$$\mathbb{R}\,\mathrm{Map}_{\mathcal{D}}(y,z) \longrightarrow \mathbb{R}\,\mathrm{Map}_{\mathcal{D}}(\mathbb{I}, F^{\mathrm{weak}}(y,z)).$$
- Substituting $x = z$ and $y = \mathbb{I}$, the image of the natural isomorphism $x \wedge^{\mathbb{L}} \mathbb{I} \to x$ is a homotopy class of map $x \to F^{\mathrm{weak}}(\mathbb{I}, x)$. If this is a natural isomorphism, we refer to the weak function object as *unital*.
- Given a map $f\colon x \to x'$ between objects, the natural transformation of functors
$$F^{\mathrm{weak}}(x', -) \longrightarrow F^{\mathrm{weak}}(x, -)$$
will be referred to as the *map induced by* f and denoted by f^*. Similarly, the natural transformation
$$F^{\mathrm{weak}}(-, x) \longrightarrow F^{\mathrm{weak}}(-, x')$$
will be denoted by f_*.
- If the weak function object provides an adjoint for $F^{\mathrm{weak}}(y,z) \wedge^{\mathbb{L}} y \to z$, the identity self-map of $F^{\mathrm{weak}}(y,z)$ lifts to a natural evaluation map in the homotopy category of $\mathcal{D}$: $F^{\mathrm{weak}}(y,z) \wedge^{\mathbb{L}} y \to z$.

3.2 Rigid objects

Definition 3.4 Suppose that $\mathcal{D}$ has a weak function object. A map $\eta\colon x \to y$ in the homotopy category of $\mathcal{D}$ is *rigid* if the map
$$\eta^*\colon F^{\mathrm{weak}}(y,y) \longrightarrow F^{\mathrm{weak}}(x,y)$$
is a weak equivalence.

Theorem 3.5 *Suppose that $\eta\colon x \to y$ is a rigid map in $\mathrm{ho}(\mathcal{D})$. In addition, suppose that for any $n, m \geq 0$, the weak function object provides adjoints for*
$$(x^{\wedge^{\mathbb{L}} n} \wedge^{\mathbb{L}} y^{\wedge^{\mathbb{L}} m}) \wedge^{\mathbb{L}} x \longrightarrow y,$$
$$(x^{\wedge^{\mathbb{L}} n} \wedge^{\mathbb{L}} y^{\wedge^{\mathbb{L}} m}) \wedge^{\mathbb{L}} y \longrightarrow y.$$
Then the map of operads $\mathbb{R}\,\mathrm{End}_{\mathcal{D}}(\eta) \to \mathbb{R}\,\mathrm{End}_{\mathcal{D}}(x)$ is an equivalence.

In particular, if $\mathcal{O}$ is a cofibrant operad and x is equipped with a homotopy class of $\mathcal{O}$–algebra structure $\theta\colon \mathcal{O} \to \mathbb{R}\,\mathrm{End}_{\mathcal{D}}(x)$, the homotopy fiber over θ of the map
$$\mathbb{R}\,\mathrm{Map}_{\mathrm{operad}}(\mathcal{O}, \mathbb{R}\,\mathrm{End}_{\mathcal{D}}(\eta)) \longrightarrow \mathbb{R}\,\mathrm{Map}_{\mathrm{operad}}(\mathcal{O}, \mathbb{R}\,\mathrm{End}_{\mathcal{D}}(x))$$
is contractible.

Proof By Corollary 2.8 we can represent η by a fibration $\eta\colon x \twoheadrightarrow y$ between cofibrant-fibrant objects in $\mathcal{D}$. This implies that the iterated tensor powers $x^{\oslash n}$ and $y^{\oslash n}$ are also cofibrant by Remark 2.15 (with $n = 0$ true by assumption on $\mathcal{D}$).

We then apply the rigidity of η and the adjoints provided by the weak function object to find that in the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(x^{\oslash n} \oslash y^{\oslash (m+1)}, y) & \xrightarrow{(1\oslash\eta\oslash 1)^*} & \mathrm{Map}_{\mathcal{D}}(x^{\oslash (n+1)} \oslash y^{\oslash m}, y) \\ \sim\downarrow & & \downarrow\sim \\ \mathbb{R}\,\mathrm{Map}_{\mathcal{D}}(x^{\oslash n} \oslash y^{\oslash m}, F^{\mathrm{weak}}(y, y)) & \xrightarrow{\sim} & \mathbb{R}\,\mathrm{Map}_{\mathcal{D}}(x^{\oslash n} \oslash y^{\oslash m}, F^{\mathrm{weak}}(x, y)), \end{array}$$

the top map is a weak equivalence. In particular, we find by induction that

$$(\eta^{\oslash n})^*\colon \mathrm{Map}_{\mathcal{D}}(y^{\oslash n}, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, y)$$

is a weak equivalence for all n. (Moreover, the source and target of $(\eta^{\oslash n})^*$ represent derived function spaces.)

The endomorphism operad $\mathrm{End}_{\mathcal{D}}(\eta)$ is the symmetric sequence which, in degree n, is the pullback of the diagram

$$\mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, x) \twoheadrightarrow \mathrm{Map}_{\mathcal{D}}(x^{\oslash n}, y) \xleftarrow{\sim} \mathrm{Map}_{\mathcal{D}}(y^{\oslash n}, y).$$

In each degree $\mathrm{End}_{\mathcal{D}}(\eta)(n)$ is a homotopy pullback of the above diagram because one of the maps is a fibration. As the other map in this diagram is an equivalence and simplicial sets are right proper, we find that the "forgetful" map of operads $\mathrm{End}_{\mathcal{D}}(\eta) \to \mathrm{End}_{\mathcal{D}}(x)$ is a levelwise weak equivalence as desired.

For any $\mathcal{O}$–algebra structure $\theta\colon \mathcal{O} \to \mathrm{End}_{\mathcal{D}}(x)$, $\mathrm{End}_{\mathcal{D}}(\eta) \to \mathrm{End}_{\mathcal{D}}(x)$ is a weak equivalence. This implies that the homotopy fiber over θ is contractible, or equivalently that the space of homotopy lifts in the diagram

$$\begin{array}{ccc} & & \mathrm{End}_{\mathcal{D}}(\eta) \\ & \nearrow & \downarrow\sim \\ \mathcal{O} & \longrightarrow & \mathrm{End}_{\mathcal{D}}(x) \end{array}$$

is contractible as well. □

As a consequence of Proposition 2.27, we have the following.

Corollary 3.6 *Suppose $\eta\colon \mathbb{I} \to y$ is a rigid map in $\mathrm{ho}(\mathcal{D})$, and that for any $n \geq 0$ the weak function object provides adjoints for the maps*

$$y^{\oslash^{\mathbb{L}} n} \oslash^{\mathbb{L}} y \longrightarrow y,$$
$$y^{\oslash^{\mathbb{L}} n} \oslash^{\mathbb{L}} \mathbb{I} \longrightarrow y.$$

For any cofibrant E_∞–operad $\mathcal{E}$, the space of extensions to an action of $\mathcal{E}$ on y making η into an E_∞–algebra map is contractible.

4 Pro-objects

We first recall the basics on pro-objects in a category $\mathcal{C}$.

Definition 4.1 For a category $\mathcal{C}$, the *pro-category* pro–$\mathcal{C}$ is the category of cofiltered diagrams $X = \{x_\alpha\}_\alpha$ of objects of $\mathcal{C}$, with maps $X \to Y = \{y_\beta\}_\beta$ defined by

$$\mathrm{Hom}_{\text{pro–}\mathcal{C}}(X, Y) = \lim_\beta \operatorname*{colim}_\alpha \mathrm{Hom}_{\mathcal{C}}(x_\alpha, y_\beta).$$

For two cofiltered systems X and Y indexed by the same category, a *level map* $X \to Y$ is a natural transformation of diagrams; any map is isomorphic in the pro-category to a level map, by Artin and Mazur [2, Appendix 3.2].

A map $X \to Y$ of pro-objects satisfies a property *essentially levelwise* if it is isomorphic to a level map such that each component $x_\alpha \to y_\alpha$ satisfies this property.

Remark 4.2 For any cofiltered index category J, there exists a final map $I \to J$ where I is a cofinite directed set (for example, see Edwards and Hastings [15, 2.1.6]). This allows us to replace any pro-object by an isomorphic pro-object indexed on a cofinite directed set.

4.1 Model structures

We now recall the strict model structure on pro-objects from Isaksen [28].

Definition 4.3 [28, 3.1, 4.1, 4.2] Suppose $\mathcal{C}$ is a model category. A map $X \to Y$ in pro–$\mathcal{C}$ is:

- a *strict weak equivalence* if it is an essentially levelwise weak equivalence;
- a *strict cofibration* if it is an essentially levelwise cofibration;

- a *special fibration* if it is isomorphic to a level map $\{x_\alpha \to y_\alpha\}_\alpha$ indexed by a cofinite directed set such that, for all α, the relative matching map
$$x_\alpha \longrightarrow (\lim_{\beta<\alpha} x_\beta) \times_{\lim_{\beta<\alpha} y_\beta} y_\alpha$$
is a fibration;
- a *strict fibration* if it is a retract of a special fibration.

Remark 4.4 If I is cofinite directed, the category of I–diagrams admits an injective model structure (equivalently, a Reedy model structure) where weak equivalences and cofibrations are defined levelwise (see Hovey [23, 5.1.3] and Edwards and Hastings [15, Section 3.2]). In this structure, the fibrations are precisely those maps satisfying the condition in the definition of a special fibration, and fibrant objects are also levelwise fibrant.

By Remark 4.2, every pro-object X can be reindexed to an isomorphic pro-object X' indexed by a cofinite directed set. There is then a levelwise acyclic cofibration $X' \to X_f$ where X_f is an injective fibrant diagram, and hence represents a strict fibrant replacement; in addition, there is an injective fibration $X_c \to X'$ which is a levelwise weak equivalence, where X_c is levelwise cofibrant, which represents a strict cofibrant replacement.

Theorem 4.5 [28, 4.15] *If $\mathcal{C}$ is a proper model category, then the classes of strict weak equivalences, strict cofibrations and strict fibrations define a proper model structure on* pro–$\mathcal{C}$*.*

If $\mathcal{C}$ has a simplicial enrichment, we can extend this notion to the category pro–$\mathcal{C}$.

Definition 4.6 [28, Section 4.1] Let $\mathcal{C}$ be a simplicial model category. For objects X and Y in pro–$\mathcal{C}$, we define the mapping simplicial set by
$$\mathrm{Map}_{\text{pro–}\mathcal{C}}(X,Y) = \lim_\beta \operatorname*{colim}_\alpha \mathrm{Map}_{\mathcal{C}}(x_\alpha, y_\beta).$$
For $X \in$ pro–$\mathcal{C}$, the tensor and cotensor with a finite simplicial set K are defined levelwise, and for arbitrary K using limits and colimits in the pro-category.

Remark 4.7 As stated in the introduction, it is important to remember that limits and colimits of pro-objects cannot be formed levelwise (even for systems of level maps). In particular, for infinite complexes K the levelwise tensor and cotensor generally do not represent the tensor and cotensor in pro–$\mathcal{C}$.

Theorem 4.8 [28, 4.17] *If $\mathcal{C}$ is a proper simplicial model category, then the strict model structure on* pro–$\mathcal{C}$ *is also a simplicial model structure.*

Theorem 4.9 (Fausk and Isaksen [17, 6.4]) *Suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a Quillen adjunction between proper model categories. Then the induced adjunction of pro-categories $\{L\}\colon$ pro–$\mathcal{C} \rightleftarrows$ pro–$\mathcal{D} : \{R\}$ is a Quillen adjunction. If the original adjunction is a Quillen equivalence, then so is the adjunction on pro-categories.*

Corollary 4.10 *If $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a simplicial Quillen adjunction between proper simplicial model categories, then the induced Quillen adjunction $\{L\}\colon$ pro–$\mathcal{C} \rightleftarrows$ pro–$\mathcal{D} : \{R\}$ lifts to a simplicial Quillen adjunction between the pro-categories.*

Proof Applying $\lim_\beta \operatorname{colim}_\alpha$ to the natural isomorphism

$$\mathrm{Map}_{\mathcal{C}}(x_\alpha, R y_\beta) \cong \mathrm{Map}_{\mathcal{D}}(L x_\alpha, y_\beta)$$

extends the adjunction to a simplicial adjunction. □

Proposition 4.11 *Suppose $\mathcal{C}$ is a proper simplicial model category. For levelwise cofibrant X and levelwise fibrant Y in* pro–$\mathcal{C}$ *with strict fibrant replacement Y_f, the homotopically correct mapping simplicial set* $\mathrm{Map}_{\text{pro–}\mathcal{C}}(X, Y_f)$ *is a natural representative for the homotopy type*

$$\operatorname*{holim}_{\beta} \operatorname*{hocolim}_{\alpha} \mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(x_\alpha, y_\beta).$$

Proof Using the fact that X is strict cofibrant, Fausk and Isaksen [18, 5.3] show that $\mathrm{Map}_{\text{pro–}\mathcal{C}}(X, Y_f)$ is weakly equivalent to

$$\operatorname*{holim}_{\beta} \operatorname*{colim}_{\alpha} \mathrm{Map}_{\mathcal{C}}(x_\alpha, y_\beta).$$

Because X is levelwise cofibrant and Y is levelwise fibrant, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(x_\alpha, y_\beta)$ are representatives for the derived mapping spaces. Finally, in simplicial sets, filtered colimits are always representatives for homotopy colimits because filtered colimits preserve weak equivalences. □

4.2 Tensor structures

Definition 4.12 (Fausk and Isaksen [18, Section 11]) Suppose $\mathcal{C}$ has a monoidal operation $\otimes$ with unit $\mathbb{I}$. The *levelwise monoidal structure* on pro–$\mathcal{C}$ is defined so that for $X, Y \in$ pro–$\mathcal{C}$ indexed by I and J respectively, the tensor $X \otimes Y$ is the pro-object $\{x_\alpha \otimes y_\beta\}_{\alpha \times \beta}$ indexed by $I \times J$. The unit is the constant pro-object $\mathbb{I}$.

Remark 4.13 Note that this tensor structure on pro–$\mathcal{C}$ is almost never closed, even when the tensor structure on $\mathcal{C}$ is, as the levelwise tensor usually does not commute with colimits (including infinite coproducts) in either variable. However, the constant pro-object $\{\varnothing\}$ is initial in the pro-category, and is preserved by the levelwise tensor product if it is preserved by $\oslash$.

Proposition 4.14 [18, 12.7, 12.3] *If $\mathcal{C}$ is a proper tensor model category, then the strict model structure on pro–$\mathcal{C}$ is also a tensor model category under the levelwise tensor structure.*

If, in addition, $\mathcal{C}$ is an operadic model category, the levelwise tensor structure on pro–$\mathcal{C}$ makes pro–$\mathcal{C}$ into an operadic model category.

Proposition 4.15 *Suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a tensor Quillen adjunction between proper tensor model categories. Then the induced adjunction $\{L\}$: pro–$\mathcal{C} \rightleftarrows$ pro–$\mathcal{D}$:$\{R\}$ is a tensor Quillen adjunction, which is symmetric if the original Quillen adjunction is.*

Proof By Theorem 4.9, the pair $\{L\}$ and $\{R\}$ form a Quillen adjunction. For pro-objects X and Y, the maps $Rx_\alpha \oslash Ry_\beta \to R(x_\alpha \oslash y_\beta)$ assemble levelwise to a natural lax monoidal structure for the functor $\{R\}$ on pro-objects, and the induced natural transformations for $\{L\}$ are also computed levelwise. If X and Y are cofibrant objects of pro–$\mathcal{C}$, we may choose levelwise cofibrant models which make the conditions of Definition 2.16 immediate. □

Combining this with Corollary 4.10, we obtain the following.

Corollary 4.16 *If $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ is an operadic Quillen adjunction between proper operadic model categories, then the induced Quillen adjunction $\{L\}$: pro–$\mathcal{C} \rightleftarrows$ pro–$\mathcal{D}$:$\{R\}$ is an operadic Quillen adjunction.*

4.3 Function objects

For the remainder of Section 4, we will suppose that $\mathcal{C}$ is a proper operadic model category whose monoidal product is symmetric monoidal closed. Specifically, there is a cofibrant unit object $\mathbb{I}$, and for objects $x, y \in \mathcal{C}$ we have a product $x \oslash y$ and an internal function object $F_{\mathcal{C}}(x, y)$.

Definition 4.17 (Fausk [16, 9.14]) There is a functor

$$F_{\text{pro–}\mathcal{C}}\colon (\text{pro–}\mathcal{C})^{\text{op}} \times \text{pro–}\mathcal{C} \longrightarrow \text{pro–}\mathcal{C}$$

defined by

$$F_{\text{pro–}\mathcal{C}}(X,Y) = \{\operatorname*{colim}_{\alpha} F_{\mathcal{C}}(x_\alpha, y_\beta)\}_\beta,$$

equipped with a natural transformation

$$\mathrm{Map}_{\text{pro–}\mathcal{C}}(X \oslash Y, Z) \longrightarrow \mathrm{Map}_{\text{pro–}\mathcal{C}}(X, F_{\text{pro–}\mathcal{C}}(Y,Z))$$

given by the composite

$$\begin{aligned}\lim_{\gamma} \operatorname*{colim}_{\alpha\times\beta} \mathrm{Map}_{\mathcal{C}}(x_\alpha \oslash y_\beta, z_\gamma) &\xrightarrow{\cong} \lim_{\gamma} \operatorname*{colim}_{\alpha} \operatorname*{colim}_{\beta} \mathrm{Map}_{\mathcal{C}}(x_\alpha, F_{\mathcal{C}}(y_\beta, z_\gamma)) \\ &\longrightarrow \lim_{\gamma} \operatorname*{colim}_{\alpha} \mathrm{Map}_{\mathcal{C}}(x_\alpha, \operatorname*{colim}_{\beta} F_{\mathcal{C}}(y_\beta, z_\gamma)).\end{aligned}$$

Remark 4.18 In particular, the case $X = \{\mathbb{I}\}$ produces a natural isomorphism

$$\mathrm{Map}_{\text{pro–}\mathcal{C}}(\mathbb{I}, F_{\text{pro–}\mathcal{C}}(Y,Z)) \cong \mathrm{Map}_{\text{pro–}\mathcal{C}}(Y,Z).$$

Remark 4.19 The functor $F_{\text{pro–}\mathcal{C}}$ does not generally act as an internal function object, in large part due to the presence of the colimit in the definition.

4.4 Homotopical properties of function objects

We continue the assumptions of Section 4.3 on $\mathcal{C}$.

Proposition 4.20 *Suppose that filtered colimits preserve fibrations, represent homotopy colimits and commute with finite limits in $\mathcal{C}$. Then the function object $F_{\text{pro–}\mathcal{C}}(X,Y)$ satisfies the following properties.*

1. *For a fixed $Y \in$ pro–$\mathcal{C}$ and a cofiltered index category I, $F_{\text{pro–}\mathcal{C}}(-,Y)$ takes finite colimits in $\mathcal{C}^I$ to finite limits in pro–$\mathcal{C}$.*
2. *For a fixed $X \in$ pro–$\mathcal{C}$ and a cofiltered index category I, $F_{\text{pro–}\mathcal{C}}(X,-)$ takes finite limits in $\mathcal{C}^I$ to finite limits in pro–$\mathcal{C}$.*
3. *The function object satisfies an SM7 axiom: for any strict cofibration $i\colon A \rightarrowtail B$ and strict fibration $p\colon X \twoheadrightarrow Y$ in pro–$\mathcal{C}$, the induced map*

$$F_{\text{pro–}\mathcal{C}}(B,X) \longrightarrow F_{\text{pro–}\mathcal{C}}(B,Y) \times_{F_{\text{pro-}\mathcal{C}}(A,Y)} F_{\text{pro–}\mathcal{C}}(A,X)$$

is a fibration, which is a strict equivalence if either i or p is.

Proof **(1)** For a finite diagram $J \to \mathcal{C}^I$, the colimit as a diagram of pro-objects is computed by the colimit in $\mathcal{C}^I$, by Artin and Mazur [2, Appendix 4.2]. By assumption, the natural morphisms

$$\operatorname*{colim}_{\alpha} \lim_{j} F_{\mathcal{C}}(x^j_\alpha, y_\beta) \longrightarrow \lim_{j} \operatorname*{colim}_{\alpha} F_{\mathcal{C}}(x^j_\alpha, y_\beta)$$

are isomorphisms for all β, so we find that the natural map

$$F_{\text{pro–}\mathcal{C}}(\operatorname*{colim}_j X^j, Y) \longrightarrow \lim_j F_{\text{pro–}\mathcal{C}}(X^j, Y)$$

is an isomorphism.

(2) The proof of this item is identical to that of the previous one.

(3) We note that the statement is preserved by retracts in p, and so we may assume that $p\colon X \twoheadrightarrow Y$ is a special fibration. We can choose level representations for i and p with several properties:

- The map i is a levelwise cofibration $\{a_\alpha \rightarrowtail b_\alpha\}_\alpha$.
- The map i is a levelwise acyclic cofibration if i is a strict equivalence (Isaksen [28, 4.13]).
- The map p is indexed by a cofinite directed set.
- The maps from x_β to $M_\beta = y_\beta \times_{\lim_{\gamma<\beta} y_\gamma} (\lim_{\gamma<\beta} x_\gamma)$ defined by p are fibrations.
- The fibrations $x_\beta \twoheadrightarrow M_\beta$ are weak equivalences if p is a strict equivalence [28, 4.14].

The pushout product axiom in $\mathcal{C}$ is equivalent to the internal SM7 axiom. Hence for all α and β, we find that the map

$$F_{\mathcal{C}}(b_\alpha, x_\beta) \longrightarrow F_{\mathcal{C}}(b_\alpha, M_\beta) \times_{F_{\mathcal{C}}(a_\alpha, M_\beta)} F_{\mathcal{C}}(a_\alpha, x_\beta)$$

is a fibration, and is a weak equivalence if i or p is a strict equivalence. Using the fact that $F_{\mathcal{C}}$ preserves limits in the target variable, this says that the natural map

$$F_{\mathcal{C}}(b_\alpha, x_\beta) \longrightarrow Z_{\alpha,\beta} \times_{\lim\limits_{\gamma<\beta} Z_{\alpha,\gamma}} \big(\lim_{\gamma<\beta} F_{\mathcal{C}}(b_\alpha, x_\gamma)\big)$$

is a fibration which is trivial if i or p is, where

$$Z_{\alpha,\gamma} = F_{\mathcal{C}}(b_\alpha, y_\gamma) \times_{F_{\mathcal{C}}(a_\alpha, y_\gamma)} F_{\mathcal{C}}(a_\alpha, x_\gamma)$$

is the component of the fiber product in degree γ.

Taking colimits in α, which commutes with the fiber product and preserves fibrations by assumption, we obtain a level representation of the map

$$F_{\text{pro–}\mathcal{C}}(B, X) \longrightarrow F_{\text{pro–}\mathcal{C}}(B, Y) \times_{F_{\text{pro-}\mathcal{C}}(A,Y)} F_{\text{pro–}\mathcal{C}}(A, X)$$

by a special fibration. Since filtered colimits represent homotopy colimits, they preserve weak equivalences, and so this is a levelwise equivalence if i or p is a strict equivalence, as desired. □

Remark 4.21 This actually proves that the SM7 map of $F_{\text{pro–}\mathcal{C}}$ already provides a special fibration or special acyclic fibration if the original map p is a special fibration or special acyclic fibration.

Corollary 4.22 *Under the assumptions of Proposition 4.20, we have the following consequences.*

(1) *For fibrant $Y \in \text{pro–}\mathcal{C}$, $F_{\text{pro–}\mathcal{C}}(-, Y)$ preserves weak equivalences between cofibrant objects.*

(2) *For cofibrant $X \in \text{pro–}\mathcal{C}$, $F_{\text{pro–}\mathcal{C}}(X, -)$ preserves weak equivalences between fibrant objects.*

(3) *The functor $F_{\text{pro–}\mathcal{C}}(-,-)$ descends to a well-defined weak function object $F^{\text{weak}}(-,-)$ for the homotopy category of cofibrant-fibrant objects of $\text{pro–}\mathcal{C}$.*

Proof By Ken Brown's lemma (for example, see Hovey [23, 1.1.12]), to prove the first item it suffices to prove that $F_{\text{pro–}\mathcal{C}}(-, Y)$ takes acyclic cofibrations to weak equivalences. This follows by applying the SM7 property to an acyclic cofibration $X \to X'$ and the fibration $Y \to *$.

The second item follows exactly as in the previous case by applying the SM7 property to an acyclic fibration. The final item is then a direct consequence. □

The following proposition allows us to gain homotopical control on function objects from the associated pro-objects in the homotopy category.

Proposition 4.23 *Suppose that filtered colimits preserve fibrations, represent homotopy colimits and commute with finite limits in $\mathcal{C}$. Then given levelwise cofibrant X and levelwise fibrant Y in $\text{pro–}\mathcal{C}$ with fibrant replacement Y', the induced map $F_{\text{pro–}\mathcal{C}}(X, Y) \to F_{\text{pro–}\mathcal{C}}(X, Y')$ is a weak equivalence. The representative $F_{\text{pro–}\mathcal{C}}(X, Y')$ for the homotopically correct weak function object $F^{\text{weak}}(X, Y')$ is a representative for the homotopy type*

$$\left\{\operatorname*{hocolim}_{\alpha} \mathbb{R} F_{\mathcal{C}}(x_\alpha, y_\beta)\right\}_\beta.$$

Proof This argument closely follows Fausk and Isaksen [18, 5.3]. By assumption X is strict cofibrant, and by reindexing we may assume that Y is indexed by a cofinite directed set I and still levelwise fibrant. The index category I is a Reedy category, so we may choose a Reedy fibrant replacement $Y \to Y'$ which is a levelwise weak equivalence so that the maps $y'_\beta \to \lim_{\gamma<\beta} y'_\gamma$ are fibrations. In particular, y'_β is always fibrant.

The levelwise properties imply that the function objects $F_{\mathcal{C}}(x_\alpha, y_\beta)$ are representatives for the derived function objects, and that the maps $F_{\mathcal{C}}(x_\alpha, y_\beta) \to F_{\mathcal{C}}(x_\alpha, y'_\beta)$ are weak equivalences.

As in the proof of part (3) of Proposition 4.20, the homotopically correct function object $\{\operatorname{colim}_\alpha F_{\mathcal{C}}(x_\alpha, y'_\beta)\}_\beta$ is levelwise equivalent to $\{\operatorname{colim}_\alpha F_{\mathcal{C}}(x_\alpha, y_\beta)\}_\beta$. Since colimits represent homotopy colimits, we obtain the desired result. □

4.5 Pro-dualizable objects

Continuing the assumptions of Section 4.3, we now begin to study dualizability.

Definition 4.24 For an object $x \in \mathrm{ho}(\mathcal{C})$, the *dual* Dx is the function object $\mathbb{R} F_{\mathcal{C}}(x, \mathbb{I})$. The map $Dx \otimes^{\mathbb{L}} x \to \mathbb{I}$ is the *evaluation pairing*.

Given $y \in \mathrm{ho}(\mathcal{C})$, the adjoint to the map $Dx \otimes^{\mathbb{L}} x \otimes^{\mathbb{L}} y \to y$ is a natural transformation $Dx \otimes^{\mathbb{L}} y \to \mathbb{R} F_{\mathcal{C}}(x, y)$. The object $x \in \mathrm{ho}(\mathcal{C})$ is *dualizable* if this map is a natural isomorphism of functors on $\mathrm{ho}(\mathcal{C})$.

An object $X \in \mathrm{ho}(\text{pro–}\mathcal{C})$ is *pro-dualizable* if it is isomorphic in the homotopy category to a cofiltered diagram of objects whose images in the homotopy category are dualizable.

Remark 4.25 We follow Hovey and Strickland [27] in using the term *dualizable*, rather than the term *strongly dualizable* from Hovey, Palmieri and Strickland [25].

The following are immediate consequences of the definitions.

Proposition 4.26 *The unit $\mathbb{I}$ is dualizable. Dualizable objects are closed under the tensor in $\mathrm{ho}(\mathcal{C})$, and pro-dualizable objects are closed under the levelwise tensor in $\mathrm{ho}(\text{pro–}\mathcal{C})$.*

Suppose that the unit object $\mathbb{I}$ is *compact*, in the sense that the functor $\mathbb{R}\operatorname{Map}_{\mathcal{C}}(\mathbb{I}, -)$ commutes with filtered homotopy colimits. Then the natural equivalence

$$\mathbb{R}\operatorname{Map}_{\mathcal{C}}(x, y) \simeq \mathbb{R}\operatorname{Map}_{\mathcal{C}}(\mathbb{I}, Dx \otimes^{\mathbb{L}} y)$$

implies that $\mathbb{R}\operatorname{Map}_{\mathcal{C}}(x, -)$ commutes with filtered homotopy colimits. We then have the following result, which is similar in spirit to the earlier results by Bauer [5, B.3, (2)] and by Fausk [16, 9.15].

Proposition 4.27 *Suppose that filtered colimits preserve fibrations, represent homotopy colimits and commute with finite limits in $\mathcal{C}$. In addition, suppose that $\mathbb{R}\operatorname{Map}_{\mathcal{C}}(\mathbb{I}, -)$ commutes with filtered homotopy colimits. Let $X \in \mathrm{ho}(\text{pro–}\mathcal{C})$ be pro-dualizable. Then, for any Y and Z in $\mathrm{ho}(\text{pro–}\mathcal{C})$, the weak function object F^{weak} provides an adjoint to the map $X \otimes^{\mathbb{L}} Y \to Z$ (see Definition 3.1).*

Proof Without loss of generality, we can lift X to a pro-object represented by a diagram which is levelwise cofibrant and dualizable. Similarly, we choose lifts of Y to a strict cofibrant diagram and Z to a special fibrant diagram, which in particular is levelwise fibrant.

Combining Propositions 4.11 and 4.23, the natural transformation

$$\mathrm{Map}_{\mathrm{pro}\text{-}\mathcal{C}}(X \oslash^{\mathbb{L}} Y, Z) \longrightarrow \mathrm{Map}_{\mathrm{pro}\text{-}\mathcal{C}}(X, F^{\mathrm{weak}}(Y, Z))$$

is naturally represented by the map of homotopy types

$$\operatorname*{holim}_{\gamma} \operatorname*{hocolim}_{\alpha,\beta} \mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(x_\alpha \oslash y_\beta, z_\gamma) \longrightarrow \operatorname*{holim}_{\gamma} \operatorname*{hocolim}_{\alpha} \mathbb{R}\,\mathrm{Map}_{\mathcal{C}}\big(x_\alpha, \operatorname*{hocolim}_{\beta} \mathbb{R}\,F_{\mathcal{C}}(y_\beta, z_\gamma)\big).$$

As $\mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(x_\alpha, -)$ commutes with filtered homotopy colimits, this reduces to the adjunction $\mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(x_\alpha \oslash y_\beta, z_\gamma) \cong \mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(x_\alpha, F_{\mathcal{C}}(y_\beta, z_\gamma))$. □

Combining this with Theorem 3.5, we have the following result.

Theorem 4.28 *Suppose that filtered colimits preserve fibrations, represent homotopy colimits and commute with finite limits in $\mathcal{C}$. In addition, suppose that $\mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(\mathbb{I}, -)$ commutes with filtered homotopy colimits. Let $\eta\colon X \to Y$ in ho(pro–$\mathcal{C}$) be a rigid map between pro-dualizable objects. If X is an algebra over a cofibrant operad $\mathcal{O}$, then there exists an $\mathcal{O}$–algebra structure on Y, compatible with η, which is unique up to homotopy.*

5 Symmetric spectra and filtered colimits

In this section, we verify several conditions on model categories of interest in this paper. In particular, we show that the "base category" of symmetric spectra is a proper operadic model category, and hence, has an associated model category of pro-objects that is operadic. Also, we show that this base model structure satisfies several required assumptions from Section 4.

We write $\mathcal{S}p$ for the category of symmetric spectra in simplicial sets described in Hovey, Shipley and Smith [26]. For R a ring object in $\mathcal{S}p$, we write $\mathcal{S}p_R$ for the category of R–modules. We will follow Schwede [38] (which uses the term "absolute flat stable") in using the term *flat stable* model structure for what is called the R–model structure in Shipley [41].

The properties of filtered colimits preserving fibrations, preserving weak equivalences and commuting with finite limits are true in the category of simplicial sets, and are inherited by several categories based on diagrams of them.

Proposition 5.1 *Let R be a commutative ring object in $\mathcal{S}p$. Under the flat stable model structure, the category $\mathcal{S}p_R$ is a proper monoidal model category under $\wedge_R$ with a compatible simplicial enrichment and cofibrant unit. In this category, filtered colimits represent homotopy colimits, commute with finite limits and preserve fibrations. Mapping spaces out of R commute with filtered homotopy colimits.*

Remark 5.2 As in [41, 2.8], the identity functor is a Quillen equivalence between the ordinary stable model structure and the flat stable model structure. By Remark 2.18, the flat stable and ordinary stable model structures are essentially equivalent for considering operadic structures.

Proof By [41, 2.6, 2.7], the flat stable model structure makes $\mathcal{S}p_R$ a proper monoidal model category. The simplicial enrichment is tensored and cotensored, with the tensor compatible by definition.

The adjoint of the pushout product axiom implies that the internal function objects $F_R(-,-)$ obey an SM7 axiom in the flat stable model structure on R–modules. To conclude that the simplicial enrichment satisfies the SM7 axiom, it suffices to note that $\mathrm{Map}_R(X,Y)$ is the degree zero portion of the function spectrum $F_R(X,Y)$, and that the functor taking an R–module to its degree zero portion is a right Quillen functor (with adjoint $R \wedge (-)$).

As this model category is cofibrantly generated, and the generating cofibrations and acyclic cofibrations $A \rightarrowtail B$ have source and target which are compact, filtered colimits automatically preserve fibrations (cf Behrens and Davis [7, 5.3.1]).

Filtered colimits and finite limits in $\mathcal{S}p_R$ are formed levelwise in the category of pointed simplicial sets. In particular, filtered colimits commute with finite limits and preserve level equivalences.

Given a diagram $\{X_i\}_i$ in $\mathcal{S}p_R$ indexed by a cofinite directed set I, the homotopy colimit is the left derived functor of colimit, and is formed by taking the colimit of a cofibrant replacement $\{X'_i\}_i \to \{X_i\}_i$ in the projective model structure on I–diagrams. As the classes of cofibrations coincide in the projective model structures on I–diagrams for the flat level and flat stable model structures on $\mathcal{S}p_R$, and similarly for the classes of acyclic fibrations, the above cofibrant replacement is, objectwise, a level equivalence. The natural map $\mathrm{colim}_i X'_i \to \mathrm{colim}_i X_i$ is then a filtered colimit of level equivalences; it is therefore a level equivalence, and hence a stable equivalence.

The functor $\mathrm{Map}_R(R,-)$ is the zeroth space functor, which commutes with homotopy colimits. □

6 Towers of Moore spectra

In this section we will show that there are towers of Moore spectra admitting an E_∞ structure. We will deduce from this the existence of E_∞ structures on pro-spectrum lifts of the $K(n)$–local spheres, the telescopic ($T(n)$–local) spheres and the Morava E–theories $E(k,\Gamma)$.

Throughout this section we fix a prime p.

Theorem 6.1 (Hovey and Strickland [27, 4.22]) *For any integer $n \geq 1$, there is a tower $\{M_I\}_I$ of generalized Moore spectra of type n under $\mathbb{S}$ such that, for all finite spectra Z of type greater than or equal to n, the natural map*

$$Z \longrightarrow \{Z \wedge M_I\}_I$$

is an isomorphism of pro-objects in the homotopy category of spectra.

Any two such towers are isomorphic as pro-objects in the homotopy category.

We will refer to any such tower $\{M_I\}_I$ of generalized Moore spectra under $\mathbb{S}$ as a *Moore tower*.

Corollary 6.2 *For any Moore tower $\{M_I\}_I$, the unit map $\mathbb{S} \to \{M_I\}_I$ is rigid* (see 3.4) *in the homotopy category of pro-spectra.*

Proof By Proposition 4.23 it suffices to show that for any I, the natural map

$$\operatorname*{hocolim}_J F(M_J, M_I) \longrightarrow M_I$$

is an equivalence.

The dual of M_I is still finite of type n, and so the natural map

$$DM_I \longrightarrow \{DM_I \wedge M_J\}_J$$

becomes an isomorphism of pro-objects in the homotopy category. Taking duals, we find that

$$\{F(M_J, M_I)\}_J \longrightarrow M_I$$

becomes an isomorphism of ind-objects in the homotopy category. In particular, this map expresses the domain as being ind-constant in the homotopy category, and so the induced map

$$\operatorname*{hocolim}_J F(M_J, M_I) \longrightarrow M_I$$

is a weak equivalence, as desired. □

Theorem 6.3 *Any Moore tower admits the structure of an E_∞–algebra.*

For $n \geq 1$, let $K(n)$ and $T(n)$ denote a Morava K–theory of height n and the mapping telescope of a v_n–self-map of a type n complex. Then the $K(n)$–local and $T(n)$–local spheres lift to the E_∞–algebras $\{L_{K(n)}\mathbb{S} \wedge M_I\}_I$ and $\{L_{T(n)}\mathbb{S} \wedge M_I\}_I$ in the category of pro-spectra.

Proof As Moore towers are pro-dualizable, the first statement is obtained by application of Corollary 3.6 and Proposition 4.27.

By smashing a Moore tower with the constant pro-objects $L_{K(n)}\mathbb{S}$ and $L_{T(n)}\mathbb{S}$, each of which is an E_∞–algebra (since localizations of $\mathbb{S}$ are E_∞–algebras in spectra), and noting that the inverse limit is still the $K(n)$–local or $T(n)$–local sphere, we obtain the second statement. □

Since $E(k,\Gamma)$ is $K(n)$–local and an E_∞–algebra in spectra, the argument for the second part of the above theorem gives the following result.

Corollary 6.4 *Let $n \geq 1$, let k be any perfect field of characteristic p, and let Γ be any height n formal group law over k. Then $E(k,\Gamma)$ lifts to the E_∞–algebra $\{E(k,\Gamma) \wedge M_I\}_I$ in the category of pro-spectra, functorially in (k,Γ).*

7 Nilpotent completions

In this section, we roughly follow Bousfield [9, Section 5] in defining nilpotent resolutions, though we have been influenced by Carlsson [10] and Baker and Lazarev [4].

To recap assumptions, in this section the category $\mathcal{C}$ is

- an operadic model category,
- whose monoidal structure is closed,
- whose underlying model category is proper, and
- whose filtered colimits preserve fibrations, realize homotopy colimits, and commute with finite limits.

Finally, we now add the assumption that

- the underlying model category is stable.

As a result, $\mathrm{ho}(\mathcal{C})$ has the structure of a tensor triangulated category.

The main example in mind is the category of modules over a commutative symmetric ring spectrum.

Definition 7.1 (cf [9, 3.7]) Suppose E is an object in $\mathrm{ho}(\mathcal{C})$. The category $\mathrm{Nil}(E)$ of *E–nilpotent objects* is the smallest subcategory of $\mathrm{ho}(\mathcal{C})$ containing E which is closed under isomorphisms, cofiber sequences, retracts and tensoring with arbitrary objects of $\mathrm{ho}(\mathcal{C})$.

In other words, $\mathrm{Nil}(E)$ is the thick tensor ideal of $\mathrm{ho}(\mathcal{C})$ generated by E, or equivalently the thick subcategory generated by objects of the form $(E \oslash^{\mathbb{L}} x)$ for $x \in \mathrm{ho}(\mathcal{C})$.

Definition 7.2 (cf [9, 5.6]) For an element E in $\mathrm{ho}(\mathcal{C})$, an *E–nilpotent resolution* of $y \in \mathrm{ho}(\mathcal{C})$ is a tower $\{w_s\}_s$ of objects under y such that

(1) w_s is in $\mathrm{Nil}(E)$ for all $s \geq 0$, and
(2) for any E–nilpotent object z, the map $\mathrm{hocolim}_s\, \mathbb{R}F_{\mathcal{C}}(w_s, z) \to \mathbb{R}F_{\mathcal{C}}(y, z)$ is a weak equivalence.

Remark 7.3 The second condition is preserved by cofiber sequences, isomorphisms, and retracts in z, and so it suffices to show it for objects of the form $(E \oslash^{\mathbb{L}} x)$ for $x \in \mathrm{ho}(\mathcal{C})$.

Remark 7.4 Suppose that the category $\mathrm{ho}(\mathcal{C})$ has a collection of dualizable generators p_i. To check that a tower is an E–nilpotent resolution, it suffices to check that the maps

$$\left[p_i, \operatorname*{hocolim}_s \mathbb{R}F_{\mathcal{C}}(w_s, z)\right] \longrightarrow [p_i, \mathbb{R}F_{\mathcal{C}}(y, z)]$$

are isomorphisms, or equivalently that the maps

$$\operatorname*{colim}_s\, [w_s, Dp_i \oslash^{\mathbb{L}} z] \longrightarrow [y, Dp_i \oslash^{\mathbb{L}} z]$$

are isomorphisms. However, since z is E–nilpotent, so is $Dp_i \oslash^{\mathbb{L}} z$, and therefore it suffices to check that the map

$$\operatorname*{colim}_s\, [w_s, z] \longrightarrow [y, z]$$

is an isomorphism for all E–nilpotent z as in Bousfield's definition.

Given an E–nilpotent resolution $\{w_s\}_s$ of an object y, we can lift it to a map in pro–$\mathcal{C}$ from the constant pro-object y to a representing tower $\{w_s\}_s$. We will casually refer to a map of towers $\{y\} \to \{w_s\}_s$ in $\mathcal{C}$ as an E–nilpotent resolution if the domain is constant with value y and the image of the range in the homotopy category is an E–nilpotent resolution of y. We will view $\mathcal{C}$ as embedded in the category of towers in $\mathcal{C}$ so that we may abuse notation by writing this as a map $y \to \{w_s\}_s$.

Proposition 7.5 *If $\eta\colon y \to W$ and $\eta'\colon y \to W'$ are two E–nilpotent resolutions of y in $\mathcal{C}$, then the map $F^{\text{weak}}(W, W') \to F^{\text{weak}}(y, W')$ is a strict equivalence. In particular, the map η in* pro–$\mathcal{C}$ *is rigid.*

Proof We may assume without loss of generality that y is cofibrant and that the towers $W = \{w_s\}_s$ and $W' = \{w'_t\}_t$ are levelwise cofibrant-fibrant in $\mathcal{C}$. By Proposition 4.23, the map $F^{\text{weak}}(W, W') \to F^{\text{weak}}(y, W')$ is homotopy equivalent to a map of pro-objects

$$\big\{\operatorname*{hocolim}_s F_{\mathcal{C}}(w_s, w'_t)\}_t \longrightarrow \{F_{\mathcal{C}}(y, w'_t)\big\}_t.$$

As each w'_t is E–nilpotent, the maps $\operatorname{hocolim}_s F_{\mathcal{C}}(w_s, w'_t) \to F_{\mathcal{C}}(y, w'_t)$ are weak equivalences. Therefore, the associated map of towers $F^{\text{weak}}(W, W') \to F^{\text{weak}}(y, W')$ is a levelwise equivalence. □

Any two E–nilpotent resolutions are therefore pro-isomorphic in the homotopy category ho(pro–$\mathcal{C}$). We therefore will often casually refer to a map in the pro-category $y \to y^\wedge_E$, from the constant object y to an E–nilpotent resolution, as *the* E–nilpotent completion of y.

Definition 7.6 (cf Hovey and Strickland [27, 4.8]) A *μ–ring* is an object $E \in \operatorname{ho}(\mathcal{C})$ equipped with a map $\mathbb{I} \to E$ and a multiplication $E \oslash^{\mathbb{L}} E \to E$ which is left unital.

Proposition 7.7 *If the object E is a μ–ring in* ho($\mathcal{C}$) *and $y \in \mathcal{C}$, there exists an E–nilpotent resolution of y. If E and y are dualizable, then there exists a resolution which is pro-dualizable.*

Proof We apply the standard techniques to provide a canonical "Adams resolution" of y as follows. First form a fiber sequence $J \to \mathbb{I} \to E$ in ho($\mathcal{C}$). The maps

$$J^{\oslash^{\mathbb{L}}(n+1)} \longrightarrow \mathbb{I} \oslash^{\mathbb{L}} J^{\oslash^{\mathbb{L}} n} \cong J^{\oslash^{\mathbb{L}} n}$$

construct a tower of tensor powers of J. We then define $\mathbb{I}/J^n$ as the cofiber of the composite map $J^{\oslash^{\mathbb{L}} n} \to \mathbb{I}^{\oslash^{\mathbb{L}} n} \cong \mathbb{I}$.

For an arbitrary object y, to show that the tower

$$\{(\mathbb{I}/J^n) \oslash^{\mathbb{L}} y\}_n$$

is an E–nilpotent resolution of y, it suffices to show that for any object $x \in \operatorname{ho}(\mathcal{C})$ the map

$$\operatorname*{hocolim}_n \mathbb{R}F_{\mathcal{C}}((\mathbb{I}/J^n) \oslash^{\mathbb{L}} y, E \oslash^{\mathbb{L}} x) \longrightarrow \mathbb{R}F_{\mathcal{C}}(y, E \oslash^{\mathbb{L}} x)$$

is a weak equivalence. Taking fibers, it suffices to show that

$$\operatorname*{hocolim}_n \mathbb{R}F_{\mathcal{C}}(J^{\otimes^{\mathbb{L}} n} \otimes^{\mathbb{L}} y, E \otimes^{\mathbb{L}} x)$$

is trivial. However, the μ–ring structure on E implies that any map from $J^{\otimes^{\mathbb{L}} n} \otimes^{\mathbb{L}} y$ to $E \otimes^{\mathbb{L}} x$ automatically lifts to a map from $E \otimes^{\mathbb{L}} J^{\otimes^{\mathbb{L}} n} \otimes^{\mathbb{L}} y$, and thus restricts to the trivial map from $J^{\otimes^{\mathbb{L}}(n+1)} \otimes^{\mathbb{L}} y$.

Dualizable objects are closed under cofiber sequences and tensor products by Hovey, Palmieri and Strickland [25, 2.1.3], and so if E and y are dualizable this tower is pro-dualizable. □

Theorem 7.8 *Suppose that E is a dualizable μ–ring in* $\mathrm{ho}(\mathcal{C})$*, and $y \in \mathcal{C}$ is an algebra over a cofibrant operad $\mathcal{O}$. Then there exists a unique $\mathcal{O}$–algebra structure on the E–nilpotent completion $y^\wedge_E$ which is compatible with y.*

Proof This is obtained by applying Theorem 3.5, the hypotheses of which are verified by Propositions 7.7, 7.5 and 4.27. □

Remark 7.9 When Theorem 7.8 is applied to the category of modules over $E(k,\Gamma)$ (where $E(k,\Gamma)$ is any Morava E–theory, as defined in Section 1) with dualizable μ–ring given by the associated 2–periodic Morava K–theory, we recover the E_∞–structure on $E(k,\Gamma)$ in the category of pro-spectra. However, this construction does not respect the action of the extended Morava stabilizer group $G(k,\Gamma) = \mathrm{Aut}_{\mathcal{FG}}(k,\Gamma)$, the automorphism group of (k,Γ) in the category $\mathcal{FG}$. One could also apply this method to the smash product of a μ–ring with $E(k,\Gamma)$, $L_{E(n)}\mathbb{S}$, $L_{K(n)}\mathbb{S}$ or $L_{T(n)}\mathbb{S}$.

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