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Rational smoothness, cellular decompositions and GKM theory

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We introduce the notion of $\mathbb{Q}$–filtrable varieties: projective varieties with a torus action and a finite number of fixed points, such that the cells of the associated Białynicki-Birula decomposition are all rationally smooth. Our main results develop GKM theory in this setting. We also supply a method for building nice combinatorial bases on the equivariant cohomology of any $\mathbb{Q}$–filtrable GKM variety. Applications to the theory of group embeddings are provided.

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Introduction and statement of the main results

Let X be a smooth complex projective algebraic variety with a $\mathbb{C}^*$–action and finitely many fixed points $x_1, \dots, x_m$. The method of Białynicki-Birula [5] gives rise to a decomposition of X into locally closed subvarieties

$$W_i = \left\{ x \in X \;\middle|\; \lim_{t \to 0} tx = x_i \right\}.$$

Clearly, $X = \bigsqcup_i W_i$. The subvarieties W_i are called cells of the decomposition. [5, Theorem 4.3] asserts that all cells are isomorphic to affine spaces, that is, $W_i \simeq \mathbb{C}^{n_i}$ for all i. From this, one concludes that X has no cohomology in odd degrees. This method for breaking down a projective variety into pieces, also known as *BB–decomposition*, allows the computation of important topological invariants, eg Betti numbers. In this context, there is also an important connection between the BB–decomposition and Morse theory: Let S^1 be the maximal compact subgroup of $\mathbb{C}^*$ and consider the induced action of S^1 on X. By averaging if necessary, we may assume that the Fubini–Study form ω is a S^1–invariant Kähler form on X. With this assumption, the S^1–action preserves the symplectic structure on X defined by ω. Then there exists a moment map $f\colon X \to (\mathrm{Lie(S^1)})^* = \mathbb{R}$ for this action, and f is a non-degenerate Morse function whose critical set is precisely the set of fixed points of the $\mathbb{C}^*$–action on X (see Kirwan [26], and Chriss and Ginzburg [17, Proposition 2.4.22]). Additionally, the BB–decomposition coincides with the cell decomposition of X associated to f by means of Morse theory [26; 17, Corollary 2.4.24].

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It is worth emphasizing that many of the ideas of [5] extend to the singular case. In fact, the BB–decomposition makes sense even if X is singular, though, this time, the cells need not be so well-behaved.

Goresky, Kottwitz and MacPherson, in their seminal paper [21], developed a theory, nowadays called GKM theory, that makes it possible to describe the equivariant cohomology of certain *T–skeletal* varieties: projective algebraic varieties upon which an algebraic torus T acts with a finite number of fixed points and invariant curves. Cohomology, in this article, is considered with rational coefficients. Let X be a T–skeletal variety and denote by X^T the fixed point set. The main purpose of GKM theory is to identify the image of the functorial map

$$i^*\colon H_T^*(X) \to H_T^*(X^T),$$

assuming X has no cohomology in odd degrees (*equivariantly formal*). GKM theory asserts that if X is a *GKM variety*, ie T–skeletal and equivariantly formal, then the equivariant cohomology ring $H_T^*(X)$ can be identified with certain ring of piecewise polynomial functions $PP_T^*(X)$ (Theorem 2.5).

Mostly, GKM theory has been applied to smooth projective T–skeletal varieties, because they all have trivial cohomology in odd degrees (BB–decomposition). Furthermore, the GKM data consisting of the fixed points and invariant curves has been explicitly described for some interesting subclasses: flag varieties (Carrell [15] and Brion [9]), toric varieties (Brion [11], Vezzosi and Vistoli [41] and Uma [40]) and regular embeddings of reductive groups (Brion [10] and Uma [40]). Additionally, GKM theory has been applied to Schubert varieties (Carrell [15] and Brion [13; 9]). The latter ones, even though singular, are GKM varieties and their BB–cells (relative to an appropriate action of $\mathbb{C}^*$) are exactly the Bruhat cells.

Now let X be a complex algebraic variety of dimension n and $x \in X$. We say that X is *rationally smooth at* x if there exists a neighborhood U of x (in the complex topology) such that, for all $y \in U$, we have

$$\begin{aligned} H^m(X, X - \{y\}) &= (0) \quad \text{if } m \neq 2n, \\ H^{2n}(X, X - \{y\}) &= \mathbb{Q}. \end{aligned}$$

If X is rationally smooth at every $x \in X$, then X is called *rationally smooth*. Observe that this is precisely the requirement that X is a *rational cohomology manifold*. Such varieties satisfy Poincaré duality with rational coefficients McCrory [27]. See Brion [12] for an up-to-date discussion of rationally smooth singularities on complex algebraic varieties with torus action.

Let G be a connected reductive group. Recall that a normal irreducible projective variety X is called an *embedding* of G, or a *group embedding*, if X is a $G\times G$–variety containing an open orbit isomorphic to G. Let M be a reductive monoid with zero and unit group G. Then there exists a central one-parameter subgroup $\epsilon\colon \mathbb{C}^* \to G$, with image Z contained in the center of G, such that $\lim_{t\to 0} \epsilon(t) = 0$. Moreover, the quotient space

$$\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/Z$$

is a normal projective embedding of the quotient group G/Z. Embeddings of the form $\mathbb{P}_\epsilon(M)$ are called *standard group embeddings*. It is known that all normal projective embeddings of a connected reductive group are standard (Alexeev and Brion [1]). Using methods from the theory of algebraic monoids, Renner [34; 35] investigated those standard embeddings that are rationally smooth.

The purpose of this article is to establish GKM theory in the setting of *$\mathbb{Q}$–filtrable varieties*: projective varieties with a torus action having finitely many fixed points, such that the cells of an associated BB–decomposition are all rationally smooth, ie they are *rational cells*. In general, $\mathbb{Q}$–filtrable varieties have singularities. As an application of our theory, we show that rationally smooth standard embeddings are $\mathbb{Q}$–filtrable. Our results lay down the topological foundations for the study of rationally smooth standard embeddings via GKM theory.

This article is organized as follows. The first two sections briefly review GKM theory. In Section 3, we devote ourselves to the study of rational cells and state their main topological features (Theorem 3.10, Proposition 3.12 and Theorem 3.16). We also show that the singularities of rational cells are more general than those of orbifolds (Example 3.7). In Section 4 we introduce the notion $\mathbb{Q}$–filtrable varieties. Our main result in this section is given below.

Theorem 4.7 *Let X be a normal projective T–variety. Suppose that X is $\mathbb{Q}$–filtrable. Then:*

(a) *X admits a filtration into T–stable closed subvarieties X_i, $i = 0, \dots, m$, such that*

$$\varnothing = X_0 \subset X_1 \subset \cdots \subset X_{m-1} \subset X_m = X.$$

(b) *Each cell $C_i = X_i \setminus X_{i-1}$ is a rational cell, for $i = 1, \dots, m$.*

(c) *For each $i = 1, \dots, m$, the singular rational cohomology of X_i vanishes in odd degrees. In other words, each X_i is equivariantly formal.*

(d) *If, in addition, the T–action on X is T–skeletal, then each X_i is a GKM–variety.*

It is worth noting that $\mathbb{Q}$–filtrable spaces need not be rationally smooth. For instance, Schubert varieties admit a decomposition into affine cells but they are not always rationally smooth. In particular, the Schubert variety of codimension one in the Grassmannian of 2–planes in $\mathbb{C}^4$ does not satisfy Poincaré duality, and hence is not rationally smooth. References Arabia [2] and Brion [11] supply some criteria for rational smoothness of Schubert varieties. What is remarkable about $\mathbb{Q}$–filtrable varieties is that they are equivariantly formal.

In Section 6, after recalling Arabia's notion of equivariant Euler classes (Section 5), we construct free module generators on the equivariant cohomology of any $\mathbb{Q}$–filtrable GKM variety. Our findings extend the earlier works of Arabia [2], and Guillemin and Kogan [22]. The main result of Section 6 is the following.

Theorem 6.9 *Let X be a $\mathbb{Q}$–filtrable GKM–variety. Let $x_1 < x_2 < \cdots < x_m$ be the order relation on X^T compatible with the filtration of X given in Theorem 4.7. Then there exist unique classes $\theta_i \in H^*_T(X)$, $i = 1, \dots, m$, with the following properties:*

(i) $I_i(\theta_i) = 1$.

(ii) $I_j(\theta_i) = 0$ *for all* $j \neq i$.

(iii) *The restriction of* θ_i *to* $x_j \in X^T$ *is zero for all* $j < i$.

(iv) $\theta_i(x_i) = \mathrm{Eu}_T(i, C_i)$.

Moreover, the θ_i generate $H^*_T(X)$ freely as a module over $H^*_T(pt)$.

Here $I_i\colon H^*_T(X) \to H^*_T(pt)$ is the H^*_T–linear map obtained by integrating, along X_i, the pullback of a class in X to X_i, and $\mathrm{Eu}_T(i, C_i)$ stands for the equivariant Euler class. We should also point out that the filtration of X, together with the compatible total order of the fixed points, appearing in Theorem 4.7 and Theorem 6.9 respectively, depend on a particular choice of generic one-parameter subgroup (see Definition 4.6 and Section 6).

Although the class of $\mathbb{Q}$–filtrable varieties includes smooth projective T–skeletal varieties and Schubert varieties, its crucial attribute is that it also includes a large and interesting family of singular group embeddings, namely, rationally smooth standard embeddings. Indeed, in the last section of this article, we show that the notion of $\mathbb{Q}$–filtrable variety is well suited to the study of group embeddings and, in doing so, we provide our theory with its major set of fundamental examples. Our main result in this direction can be stated as follows.

Theorem 7.4 *Let $X = \mathbb{P}_\epsilon(M)$ be a standard group embedding. If X is rationally smooth, then X is $\mathbb{Q}$–filtrable and so it has no cohomology in odd degrees.*

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1 Equivariant cohomology and localization

Throughout this article, we work with complex algebraic varieties. Cohomology is always considered with rational coefficients.

1.1 The Borel construction

Let $T=(\mathbb{C}^*)^r$ be an algebraic torus and let X be a T–variety, that is, a complex algebraic variety with an algebraic action of T. Let $ET\to BT$ be a universal principal bundle for T. The *equivariant cohomology* of X (with rational coefficients) is defined to be

$$H^*_T(X):=H^*(X_T),$$

where $X_T=(X\times ET)/T$ is the total space associated to the fibration

$$X\hookrightarrow X_T\xrightarrow{p_X} BT.$$

This construction was introduced by Borel [7]. Here, BT is simply connected, the map p_X is induced by the canonical projection $ET\times X\to ET$, and T acts diagonally on $ET\times X$. Notice that $H^*_T(X)$ is, via p_X^*, an algebra over $H^*_T(pt)$. To simplify notation, we sometimes write H^*_T instead of $H^*_T(pt)$.

It can be shown that $H^*_T(X)$ is independent of the choice of universal T–bundle. See Borel [7] and Quillen [30] for more details.

Example 1.1 Let $T=(\mathbb{C}^*)^r$ be an algebraic torus. In this case, we have $BT=(\mathbb{C}\mathrm{P}^\infty)^r$, and consequently $H^*_T(pt)=H^*(BT)=\mathbb{Q}[x_1,\dots,x_r]$, where $\deg(x_i)=2$. A more intrinsic description of $H^*_T(pt)$ is given as follows. Denote by $\Xi(T)$ the character group of T. Any $\chi\in\Xi(T)$ defines a one-dimensional complex representation of T with space $\mathbb{C}_\chi$. Here T acts on $\mathbb{C}_\chi$ via $t\cdot z:=\chi(t)z$. Consider the associated complex line bundle

$$L(\chi):=(ET\times_T\mathbb{C}_\chi\to BT)$$

and its first Chern class $c(\chi) \in H^2(BT)$. Let S be the symmetric algebra over $\mathbb{Q}$ of the group $\Xi(T)$. Then S is a polynomial ring on r generators of degree 1, and the map $\chi \to c(\chi)$ extends to a ring isomorphism

$$c\colon S \to H_T^*(pt)$$

which doubles degrees: the *characteristic homomorphism* [7].

1.2 Localization theorem for torus actions

Let $S \subset H_T^*$ be the multiplicative system $H_T^* \setminus \{0\}$. For a given T–variety X, denote by X^T the fixed point set. The following is a classical theorem due to Borel [7]. See also Hsiang [25, Theorem III.1].

Theorem 1.2 *Let X be a T–variety. Suppose $H_T^*(X)$ is a finite H_T^*–module. Then the localized restriction homomorphism*

$$S^{-1}H_T^*(X) \to S^{-1}H_T^*(X^T) = H^*(X^T) \otimes_{\mathbb{Q}} (S^{-1}H_T^*)$$

is an isomorphism. □

2 GKM theory

GKM theory is a relatively recent tool that owes its name to the work of Goresky, Kottwitz and MacPherson [21]. This theory encompasses techniques that date back to the early works of Atiyah [3; 4], Segal [37], Borel [7] and Chang and Skjelbred [16].

2.1 Equivariant formality

Definition 2.1 Suppose an algebraic torus T acts on a (possibly singular) space X. Let $p_X\colon X_T \to BT$ be the fibration associated to the Borel construction. We say that X is *equivariantly formal* if the Serre spectral sequence

$$E_2^{p,q} = H^p(BT; H^q(X)) \Rightarrow H_T^{p+q}(X)$$

for this fibration degenerates at E_2.

In the literature on topological transformation groups, there are several definitions of equivariant formality for torus actions. Definition 2.1 is modelled after [7] and [21]. It states the only meaning of equivariant formality used throughout this article, and in most of the literature related to GKM theory. It is equivalent to the classical condition: *X is totally non-homologous to zero in X_T* (condition (b) of Theorem 2.2 below). For a comparison between Definition 2.1 and the other meanings of equivariant formality coming from equivariant rational homotopy theory, the interested reader could consult Scull [36], where these definitions are compared, and many of their discrepancies are underlined.

The following theorem characterizes equivariant formality in our setting. For a proof, see [21, Theorem 1.6.2], or [13, Lemma 1.2].

Theorem 2.2 *For a T–variety X, the following are equivalent.*

(a) *X is equivariantly formal.*

(b) *The edge homomorphism $H_T^*(X) \to H^*(X)$ is surjective; that is, the ordinary rational cohomology is given by extension of scalars,*

$$H^*(X) \simeq H_T^*(X) \otimes_{H_T^*} \mathbb{Q}.$$

(c) *$H_T^*(X, \mathbb{Q})$ is a free $H_T^*(pt)$–module.*

Each of them is implied by the following condition.

(d) *The singular rational cohomology of X vanishes in odd degrees.*

If X^T is finite, then (d) *is equivalent to* (a), (b) *and* (c). □

It follows that X is equivariantly formal if and only if there is an isomorphism of H_T^*–modules between $H_T^*(X)$ and $H^*(X) \otimes_{\mathbb{Q}} H_T^*$. Every smooth projective T–variety is equivariantly formal [21, Theorem 14.1 (7)].

A joint application of Theorem 1.2 and Theorem 2.2 leads to the following.

Corollary 2.3 *Let X be a T–variety with a finite number of fixed points. Then X is equivariantly formal if and only if $H_T^*(X)$ is a free H_T^*–module of rank $|X^T|$, the number of fixed points.* □

2.2 T–skeletal actions

Definition 2.4 Let X be a projective T–variety. Let μ: $T \times X \to X$ be the action map. We say that μ is a *T–skeletal action* if

(1) X^T is finite, and

(2) the number of one-dimensional orbits of T on X is finite.

In this context, X is called a *T–skeletal variety*. If a T–skeletal variety X is also equivariantly formal, then we say that X is a *GKM variety*.

Let X be a normal projective T–skeletal variety. Then X has an equivariant embedding into a projective space with a linear action of T (Sumihiro [39, Theorem 1]), and so the closure of any orbit of dimension one in X contains exactly two fixed points. Accordingly, it is possible to define a ring $PP^*_T(X)$ of *piecewise polynomial functions*. Indeed, let $R = \bigoplus_{x \in X^T} R_x$, where R_x is a copy of the polynomial algebra H^*_T. We then define $PP^*_T(X)$ as the subalgebra of R defined by

$$PP^*_T(X) = \left\{ (f_1, \dots, f_n) \in \bigoplus_{x \in X^T} R_x \;\middle|\; f_i \equiv f_j \mod \chi_{i,j} \right\},$$

where x_i and x_j are the two *distinct* fixed points in the closure of the one-dimensional T–orbit $\mathcal{C}_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $\mathcal{C}_{i,j}$. The character $\chi_{i,j}$ is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite).

Theorem 2.5 [16; 21] *Let X be a normal projective T–skeletal variety. Suppose that X is a GKM variety. Then the restriction mapping*

$$H^*_T(X) \to H^*_T(X^T) = \bigoplus_{x_i \in X^T} H^*_T$$

is injective, and its image is the subalgebra $PP^*_T(X)$. □

Theorem 2.2 characterizes normal projective GKM–varieties among all T–skeletal varieties.

Theorem 2.6 *Let X be a normal projective variety with a T–skeletal action*

$$\mu\colon T \times X \to X.$$

Then X is a GKM–variety if and only if X has no (rational) cohomology in odd degrees. □

We will show that the class of equivariantly formal spaces incorporates certain subclass of singular varieties, namely, $\mathbb{Q}$–filtrable varieties (Theorems 4.7 and 6.9). This subclass encompasses all rationally smooth standard embeddings of a reductive group (Theorem 7.4). As such, it is much larger than the subclass of smooth varieties.

3 Rational cells

This section is devoted to the study of our most important topological tool: rational cells.

Definition 3.1 Let X be an algebraic variety with an action of a torus T and a fixed point x. We say that x is an *attractive fixed point* if there exists a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ and a neighborhood U of x, such that $\lim_{t\to 0} \lambda(t)\cdot y = x$ for all points y in U.

There is an important characterization of attractive fixed points. A proof of the following result can be found in [12, Proposition A2].

Proposition 3.2 *For a torus T acting on a variety X with a fixed point x, the following conditions are equivalent:*

(i) *The weights of T in the Zariski tangent space $T_x(X)$ are contained in an open half-space.*

(ii) *There exists a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ such that, for all y in a neighborhood of x, we have $\lim_{t\to 0} \lambda(t)\, y = x$.*

If (ii) *holds, then the set*

$$X_x := \left\{ y \in X \;\middle|\; \lim_{t\to 0} \lambda(t)\, y = x \right\}$$

is the unique affine T–invariant open neighborhood of x in X. Moreover, X_x admits a closed T–equivariant embedding into T_xX. □

Lemma 3.3 *Let X be an irreducible affine variety with a T–action and an attractive fixed point $x_0 \in X$. Then X is rationally smooth at x_0 if and only if X is rationally smooth everywhere.*

Proof If X is rationally smooth everywhere, then it is rationally smooth at x_0. For the converse, we use Proposition 3.2(ii) and the affineness of X to guarantee the existence of a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ such that

$$X = \left\{ y \in X \;\middle|\; \lim_{t\to 0} \lambda(t)\, y = x_0 \right\}.$$

In symbols, $x_0 \in \overline{\mathbb{C}^* \cdot y}$, for any $y \in X$. Now consider the classical topology on X. We claim that any non-empty open T–stable subset of X containing x_0 is all of X. In effect, let U be a T–stable neighborhood of x_0. Then, for any $y \in X$, there exists $s_y \in \mathbb{C}^*$, such that $s_y \cdot y \in U$. Indeed, because x_0 is attractive, one can find a sequence $\{t_n\} \subset \mathbb{C}^*$ such that $t_n \cdot y$ converges to x_0. That is, there exists N with the property that $t_N \cdot y$ belongs to U. Setting $s_y = t_N$ yields $s_y \cdot y \in U$. However, U is T–stable, and therefore it contains the entire orbit $\mathbb{C}^* \cdot y$. In short, $y \in U$ or, equivalently, $U = X$.

Hence, the non-empty open T–stable subset of rationally smooth points of X is, *a fortiori*, equal to X. □

Definition 3.4 Let X be an irreducible affine variety with a T–action and an attractive fixed point $x_0 \in X$. If X is rationally smooth at x_0 (and thus everywhere), we refer to (X, x_0) as a *rational cell.*

It follows from Definition 3.4 and Proposition 3.2 that if (X, x_0) is a rational cell, then

$$X = \left\{ y \in X \;\middle|\; \lim_{t \to 0} \lambda(t)\, y = x_0 \right\},$$

for a suitable one-parameter subgroup λ. Notably, $\{x_0\}$ is the unique closed T–orbit in X.

Example 3.5 Certainly $\mathbb{C}^n$ is a rational cell with the usual $\mathbb{C}^*$–action by scalar multiplication. Here the origin is the unique attractive fixed point.

Example 3.6 Let $V = \{xy = z^2\} \subset \mathbb{C}^3$. The standard $\mathbb{C}^*$–action by scalar multiplication makes V a rational cell with $(0,0,0)$ as its attractive fixed point. This is clear once we observe that V is the quotient of $\mathbb{C}^2$ by the finite group with two elements, where the non-trivial element acts on $(s,t) \in \mathbb{C}^2$ via $(s,t) \mapsto (-s,-t)$. So [12, Proposition A1(iii)] implies that V is rationally smooth.

We should remark that not all rational cells are orbifolds, ie, the singularities of a rational cell need not always be of quotient type. We illustrate this fact with the following example.

Example 3.7 Let $V = V(p,q,r) \subset \mathbb{C}^3$ be the Brieskorn–Pham complex algebraic surface

$$z_1^p + z_2^q + z_3^r = 0,$$

where p, q and r are integers ≥ 2. It is well known that V has a normal isolated singularity at the origin. Now let L be the topological link of V at $(0,0,0)$, that is, L is the smooth, compact 3–manifold obtained by intersecting V with the unit sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Notably, by work of Milnor [28], the singularity type of V at $(0,0,0)$ can be completely determined from the topology of L. Milnor shows that L is diffeomorphic to a coset space of the form G/Γ, where G is a simply connected 3–dimensional Lie group and Γ is a discrete subgroup. In particular, the fundamental group $\pi_1(L)$ is isomorphic to this discrete subgroup $\Gamma \subset G$. There are three possibilities for G, according as the rational number

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$$

is positive, negative or zero. Furthermore, V has a quotient singularity at $(0,0,0)$ if and only if $\pi_1(L) \simeq \Gamma$ is finite or, equivalently, if and only if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1>0$ [28]. Keeping this in mind, we now proceed to construct examples of rational cells that are not orbifolds. First, observe that V admits a natural $\mathbb{C}^*$–action defined by

$$t\cdot(z_1,z_2,z_3) := (t^{qr}z_1, t^{pr}z_2, t^{pq}z_3).$$

Clearly the origin is the unique attractive fixed point of V. It turns out that V is a rational cell if and only if L is a rational homology sphere (eg Theorem 3.10(c)). Secondly, the work of Brieskorn provides useful criteria for determining when L is an integer or, more generally, a rational homology sphere (Dimca [19, Theorem 3.4.10]). For instance, in view of such criteria, L is an integer homology sphere whenever p, q and r are relatively prime. Therefore, choosing p,q and r relatively prime such that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\leq 0,$$

and considering the $\mathbb{C}^*$–action given above, yields a Brieskorn–Pham surface that is a rational cell with a non-quotient singularity at the origin. A classical example is the surface $V(2,3,7)$. Higher-dimensional analogues can also be constructed, by work of Brieskorn, Milnor and others (see [19] and the references therein).

Example 3.8 A normal variety is not necessarily rationally smooth. For instance, consider the hypersurface $H\subset\mathbb{C}^4$ defined by $\{xy=uv\}$. Because the singular locus of H, namely $\{(0,0,0,0)\}$, has codimension three, it follows that H is normal (Shafarevich [38, page 128, comments after Theorem II.5.1.3]). Nevertheless, H is not rationally smooth at the origin. To see this, let $T=(\mathbb{C}^*)^2$ act on H via $(t,s)\cdot(x,y,u,v)=(tx,ts^2y,su,st^2v)$. Then H has the origin as its unique attractive fixed point. Moreover, H contains four T–invariant curves (the four coordinate axes) passing through $(0,0,0,0)$. If H were rationally smooth at the origin, then, by a result of Brion Theorem 3.16, the dimension of H would equal the number of its T–invariant curves. This is a contradiction, since H is only three-dimensional.

Definition 3.9 Let Z be a rationally smooth complex projective variety. Let n be the (complex) dimension of Z. We say that Z is a *rational cohomology complex projective space* if there is a ring isomorphism

$$H^*(Z)\simeq\mathbb{Q}[t]/(t^{n+1}),$$

where $\deg(t)=2$.

Let (X,x) be a rational cell. Then, by Proposition 3.2, X admits a closed T–equivariant embedding into T_xX. Set $\dot{X}$ to be $X-\{x\}$. Choose an injective one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ as in Definition 3.4. Then all weights of the $\mathbb{C}^*$–action on $T_x X$ via λ are positive. Thus, the quotient

$$\mathbb{P}(X) := \dot{X}/\mathbb{C}^*$$

exists and is a projective variety [12]. Indeed, it is a closed subvariety of $\mathbb{P}(T_x X)$, a weighted projective space. The variety $\mathbb{P}(X)$ can be viewed as an algebraic version of the link of X at x.

Parts (a) and (d) of the following Theorem are due to Brion [12], and the idea of the proof of part (b) is due to Renner (personal communication).

Theorem 3.10 *Let (X, x_0) be a rational cell of dimension n. Then:*

(a) *X is contractible.*

(b) *$X - \{x_0\}$ is homeomorphic to $\mathbb{S}(X) \times \mathbb{R}^+$, where $\mathbb{S}(X) := X - \{x_0\}/\mathbb{R}^+$ is a compact topological space.*

(c) *$X - \{x_0\}$ deformation retracts to $\mathbb{S}(X)$. In addition, X is rationally smooth at x_0 if and only if $X - \{x_0\}$, and thus $\mathbb{S}(X)$, is a rational cohomology sphere.*

(d) *The space $\mathbb{P}(X) = X - \{x_0\}/\mathbb{C}^*$ is a rationally smooth complex projective variety of dimension $n-1$. Furthermore, X is rationally smooth if and only if $\mathbb{P}(X)$ is a rational cohomology complex projective space.*

Proof For part (a) simply notice that the action of $\mathbb{C}^*$ on X extends to a map $\mathbb{C} \times X \to X$ sending $0 \times X$ to x_0 and restricting to the identity $1 \times X \to X$. Since the proof of (d) can be found in [12, Lemma 1.3], it suffices to prove parts (b) and (c).

(b) From Proposition 3.2, we know that X admits a closed T–equivariant embedding into $T_{x_0} X \simeq \mathbb{C}^d$, which identifies x_0 with 0. Choosing a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ as in Definition 3.4 yields a $\mathbb{C}^*$–action on $\mathbb{C}^d$ with only positive weights $m_1, \dots, m_d$. Specifically, $\lambda \in \mathbb{C}^*$ acts on $\mathbb{C}^d$ via

$$\lambda \cdot (z_1, \dots, z_d) = (\lambda^{m_1} z_1, \dots, \lambda^{m_d} z_d).$$

Next, define an $\mathbb{R}^+$–equivariant map $N\colon \mathbb{C}^d \to \mathbb{R}$ by

$$N(z_1, \dots, z_d) = \sqrt{\textstyle\sum_{i=1}^d (z_i \overline{z_i})^{1/m_i}}.$$

Clearly, for $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^d$, the definition satisfies $N(\lambda \cdot z) = |\lambda| N(z)$ (here $\lambda \cdot z$ means $(\lambda^{m_1} z_1, \dots, \lambda^{m_d} z_d)$).

Since $\mathbb{R}^+$ acts freely on $X - \{0\} \subseteq \mathbb{C}^d - \{0\}$, the quotient map

$$X - \{0\} \to \mathbb{S}(X)$$

is a principal $\mathbb{R}^+$–fibration. We claim that this fibration is trivial, ie,

$$X - \{0\} \simeq \mathbb{S}(X) \times \mathbb{R}^+.$$

To prove the claim, we just need to provide a global section s. In fact, we can do so canonically. Let $s\colon \mathbb{S}(X) \to X - \{0\}$ be the map defined by

$$s([x]) = \frac{1}{N(x)} \cdot x.$$

This map is well defined (given that we are using the $\mathbb{C}^*$–action mentioned above) and not only defines a global section, but also a homeomorphism between $\mathbb{S}(X)$ and $X \cap N^{-1}(1)$, where $N^{-1}(1)$ is the "unit" sphere. Thus, $\mathbb{S}(X)$ is compact.

(c) The first claim follows immediately from part (b). As for the second assertion, remember that X is contractible. Thus, the long exact sequence of cohomology groups associated to the pair $(X, X - \{x_0\})$ splits into the short exact sequences

$$0 \longrightarrow H^i(X - \{x_0\}) \longrightarrow H^{i+1}(X, X - \{x_0\}) \longrightarrow 0,$$

whenever $i > 0$, and

$$0 \longrightarrow H^0(X) \longrightarrow H^0(X - \{x_0\}) \longrightarrow H^1(X, X - \{x_0\}) \longrightarrow 0.$$

Therefore, X is rationally smooth if and only if $X - \{x_0\}$ is a rational cohomology sphere of dimension $2n - 1$. □

Corollary 3.11 *Keeping the same notation as in Theorem 3.10, the rational cell X is homeomorphic to the open cone over $\mathbb{S}(X)$. Moreover, $\mathbb{P}(X)$ is equivariantly formal.*

Proof The first assertion follows from Theorem 3.10(c). As for the second, by Theorem 3.10 again, $\mathbb{P}(X)$ is a rational cohomology complex projective space and thus has no cohomology in odd degrees. We now apply Theorem 2.2 to conclude the proof. □

Proposition 3.12 *Let (X, x_0) be a rational cell of dimension n. Denote by X^+ its one point compactification. Then X^+ is simply connected and has the rational homotopy type of $\mathbb{S}^{2n}$, the Euclidean $2n$–sphere.*

Proof First, observe that X^+ is path-connected. As a consequence of Theorem 3.10, we can write X^+ as a union of two open cones D_0 and D_∞; namely, $D_0 = S \times [0,1)/S \times \{0\}$ and $D_\infty = S \times (\epsilon, \infty]/S \times \{\infty\}$, where S stands for $\mathbb{S}(X) = (X \setminus \{x_0\})/\mathbb{R}_+$, and ϵ is a positive number less than 1. Given that $X - \{x_0\}$ is path-connected, the intersection $D_0 \cap D_\infty = S \times (\epsilon, 1)$ is path-connected as well. In summary, X^+ can be written as the union of two contractible open subsets with path-connected intersection. Thus, by van Kampen's theorem, X^+ itself is simply connected. To finish the proof, we need to show that X^+ is a rational cohomology $2n$–sphere. This is a simple exercise, using the Mayer–Vietoris sequence of the cover $\{D_0, D_\infty\}$. $\square$

Lemma 3.13 (One-dimensional rational cells) *Let (X, x) be a rational cell of dimension one. Then:*

(1) *X is a cone over a topological circle.*

(2) *X is homeomorphic to $\mathbb{C}$.*

(3) *If, additionally, X is normal, then X is isomorphic to $\mathbb{C}$ as an algebraic variety.*

Proof Without loss of generality, we can assume that T acts faithfully on X. Thus, T is isomorphic to $\mathbb{C}^*$. Now assertions (1) and (2) can be proved as follows. Since X is one-dimensional, the singular locus is an invariant discrete set. Nonetheless, x_0 is the unique attractive fixed point, and $\mathbb{C}^*$ is *connected*, so the singular locus is either empty or consists of only one point, namely, x_0. As a result, $X \setminus \{x_0\}$ is smooth. Next notice that X has two $\mathbb{C}^*$–orbits: the attractive fixed point x_0, and a dense open orbit of the form $\mathbb{C}^*$. Hence, X is homeomorphic to $\mathbb{C}$ and it is a cone over the circle S^1.

Finally, if we also assume that X is normal and one-dimensional, then *a fortiori* X is smooth (Hartshorne [24]). This proves (3). $\square$

Lemma 3.14 *Let (X, x) be a rational cell. Suppose x is a smooth point. Then X is isomorphic to its tangent space at x.*

Proof By Proposition 3.2, we know that X admits an equivariant *closed* embedding into T_xX. If x is a smooth point, then both X and T_xX have the same dimension. For affine varieties this can only happen if $X = T_xX$. $\square$

We are now ready to state what we call the equivariant normalization theorem for rational cells. It is due to Brion [11, Proof of Theorem 18, implication (i) $\Rightarrow$ (ii)] and Arabia [2, Section 3.2.1].

Theorem 3.15 *Let (X, x) be a rational cell. Then there exists a T–module V and an equivariant finite surjective map $\pi\colon X \to V$ such that $\pi(x) = 0$ and $V^T = \{0\}$.* $\square$

We now specialize a result of Brion [12, Section 1.4, Corollary 2] to rational cells.

Theorem 3.16 *Let (X, x) be a rational cell. Suppose that the number of closed irreducible T–stable curves on X is finite. Let $n(X, x)$ be this number. Then $n(X, x) = \dim(X)$.* $\square$

4 Homology and Betti numbers of $\mathbb{Q}$–filtrable spaces

We define $\mathbb{Q}$–filtrable varieties, spaces that come equipped with a paving by rational cells, and show that they are equivariantly formal (Theorem 4.7).

4.1 The Białynicki-Birula decomposition

Let X be a projective algebraic variety with a $\mathbb{C}^*$–action and a finite number of fixed points $x_1, \dots, x_m$. Consider the associated BB–decomposition $X = \bigsqcup_i W_i$, where each cell is defined as follows:

$$W_i = \left\{ x \in X \;\middle|\; \lim_{t \to 0} t \cdot x = x_i \right\}.$$

In the present section, we show that rational cells are a good substitute for the notion of affine space in the topological study of singular varieties.

Remark 4.1 In general, the BB–decomposition of a projective variety is not a stratification; that is, it may happen that the closure of a cell is not the union of cells, even if we assume our variety to be smooth. For a justification of this claim, see Białynicki-Birula [6, Example 1].

Definition 4.2 Let X be a complex algebraic variety endowed with a $\mathbb{C}^*$–action and a finite number of fixed points. A *BB–decomposition* $\{W_i\}$ is said to be *filtrable* if there exists a finite increasing sequence $X_0 \subset X_1 \subset \cdots \subset X_m$ of closed subvarieties of X such that:

(a) $X_0 = \varnothing$, $X_m = X$.

(b) For each $j = 1, \dots, m$, the "stratum" $X_j \setminus X_{j-1}$ is a cell of the decomposition $\{W_i\}$.

The following result is due to Białynicki-Birula [6].

Theorem 4.3 *Let X be a normal projective variety with a torus action and a finite number of fixed points. Then the BB–decomposition is filtrable.* □

4.2 $\mathbb{Q}$–filtrable spaces

To begin with, let us introduce a few technical results.

Lemma 4.4 *Let X be a complex projective algebraic variety with a $\mathbb{C}^*$–action. Suppose X can be decomposed as the disjoint union*

$$X = Y \sqcup C,$$

where Y is a closed stable subvariety and C is an open rational cell containing a fixed point of X, say c_0, as its unique attractive fixed point. Denote by n the (complex) dimension of C. Then

$$H^k(X,Y) = \begin{cases} 0 & \text{if } k \neq 2n, \\ \mathbb{Q} & \text{if } k = 2n. \end{cases}$$

Consequently,

$$H^k(X) \simeq H^k(Y)$$

for all $k \neq 2n-1, 2n$; that is, attaching a complex n–dimensional rational cell produces no changes in cohomology, except in degrees $2n-1$ and $2n$.

Furthermore, if Y has no cohomology in odd degrees, then X has no odd cohomology either, and there is a short exact sequence of the form

$$0 \longrightarrow H_c^{2n}(C) \longrightarrow H^{2n}(X) \longrightarrow H^{2n}(Y) \longrightarrow 0.$$

Proof Let $H_c^*(-)$ denote cohomology with compact supports. It is well known that $H^*(X) = H_c^*(X)$ and $H^*(Y) = H_c^*(Y)$, because X and Y are complex projective varieties. Moreover, by Peters and Steenbrink [29, Corollary B.14], one has

$$H^*(X,Y) \simeq H_c^*(X-Y) = H_c^*(C).$$

Given that C is a rational cell, and a cone over a rational cohomology sphere of dimension $2n-1$ (Corollary 3.11), it follows easily that

$$H_c^k(C) = H^k(C, C-\{c_0\}) = \begin{cases} 0 & \text{if } k \neq 2n, \\ \mathbb{Q} & \text{if } k = 2n. \end{cases}$$

So the first claim is proved.

As for the second assertion, consider the long exact sequence of the pair (X,Y), namely,

$$\cdots \to H^{*-1}(Y) \to H^*(X,Y) \to H^*(X) \to H^*(Y) \to H^{*+1}(X,Y) \to \cdots .$$

By our previous remarks, this long exact sequence can be rewritten as

$$\cdots \to H^{*-1}(Y) \to H_c^*(C) \to H^*(X) \to H^*(Y) \to H_c^{*+1}(C) \to \cdots .$$

This gives

$$H^k(X) \simeq H^k(Y)$$

for $k \neq 2n-1, 2n$, as well as the exact sequence

$$0 \to H^{2n-1}(X) \to H^{2n-1}(Y) \to H_c^{2n}(C) = \mathbb{Q} \to H^{2n}(X) \to H^{2n}(Y) \to 0.$$

In other words, the passage from Y to X, by attaching C, only affects cohomology in degrees $2n-1$ and $2n$.

Finally, if Y has no cohomology in odd degrees, then the latter sequence splits further, yielding the identifications $H^i(X) = H^i(Y)$, whenever $i \neq 2n$, and a "lifting of generators" sequence:

$$0 \longrightarrow H_c^{2n}(C) \longrightarrow H^{2n}(X) \longrightarrow H^{2n}(Y) \longrightarrow 0.$$

The proof is now complete. □

Corollary 4.5 *Let X be a* normal *complex projective variety endowed with a $\mathbb{C}^*$–action and a finite number of fixed points. Suppose that each BB–cell is a rational cell. Then X has vanishing odd cohomology over the rationals, and the dimension of its cohomology group in degree $2k$ equals the number of rational cells of complex dimension k. Furthermore, X is equivariantly formal and $\chi(X) = |X^T|$.*

Proof Since the BB–decomposition on X is filtrable, the result follows from the previous lemma as we move up in the filtration by attaching one rational cell at a time. This process is systematic and preserves cohomology in lower and higher degrees at each step. □

Let T be an algebraic torus acting on a variety X. A one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ is called *generic* if $X^{\mathbb{C}^*} = X^T$, where $\mathbb{C}^*$ acts on X via λ. Generic one-parameter subgroups always exist. Note that the BB–cells of X, obtained using λ, are T–invariant.

Our results in this section suggest the following definition.

Definition 4.6 Let X be a projective variety equipped with a T–action. We say that X is $\mathbb{Q}$*–filtrable* if

(1) X is normal,

(2) the fixed point set X^T is finite, and

(3) there exists a generic one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ for which the associated BB–decomposition of X consists of *rational cells*.

Theorem 4.7 *Let X be a normal projective T–variety. Suppose that X is $\mathbb{Q}$–filtrable. Then:*

(a) *X admits a filtration into closed subvarieties X_i, $i = 0, \dots, m$, such that*

$$\varnothing = X_0 \subset X_1 \subset \cdots \subset X_{m-1} \subset X_m = X.$$

(b) *Each cell $C_i = X_i \setminus X_{i-1}$ is a rational cell, for $i = 1, \dots, m$.*

(c) *For each $i = 1, \dots, m$, the singular rational cohomology of X_i vanishes in odd degrees. In other words, each X_i is equivariantly formal.*

(d) *If, in addition, the T–action on X is T–skeletal, then each X_i is a GKM–variety.*

Proof Assertions (a) and (b) are a direct consequence of Definition 4.6 and Theorem 4.3. Applying Corollary 4.5 and Theorem 2.2 at each step of the filtration yields claim (c). For statement (d), we argue as follows. Notice that all the X_i have vanishing odd cohomology, as it is guaranteed by (c). Moreover, since the X_i are T–invariant and the T–action on X is T–skeletal, each X_i contains only a finite number of fixed points and T–invariant curves. In consequence, Theorem 2.6 applied to each X_i gives (d). □

Thus, we obtain the applicability of GKM theory at each step of the filtration, even though the various X_i are not necessarily rationally smooth. This approach is more flexible than the general approach (by comparing singular cohomology with intersection cohomology) used, for instance, in Renner [35, Theorem 3.7] or in Weber [42]. Such flexibility will become apparent from our results in Section 6, where we supply a method for constructing free module generators on the equivariant cohomology of $\mathbb{Q}$–filtrable GKM–varieties (Theorem 6.9). This method is based on the notion of equivariant Euler classes.

5 Equivariant Euler classes

We quickly review the theory of equivariant Euler classes. For a complete treatment of the subject, the reader is invited to consult [25; 2].

Let X be a T–variety with an isolated fixed point x. Let us first assume that (X, x) is a rational cell of complex dimension n. Recall that $\mathbb{S}(X) = [X - \{x\}]/\mathbb{R}^+$ is a rational cohomology sphere $\mathbb{S}^{2n-1}$ and that X is homeomorphic to the (open) cone over $\mathbb{S}(X)$ (Theorem 3.10 and Corollary 3.11). The Borel construction yields the fibration

$$\mathbb{S}(X) \hookrightarrow \mathbb{S}(X)_T \to BT.$$

Observe that the E_2–term of the corresponding Serre spectral sequence consists of only two rows, namely,

$$E_2^{p,q} = H^p(BT) \otimes H^q(\mathbb{S}(X)) \neq 0 \quad \text{only when } q = 0 \text{ and } q = 2n-1.$$

Let $\mathrm{Eu}_T(x, X) \in H^{2n}(BT)$ be the transgression of a generator of $H^{2n-1}(\mathbb{S}(X))$. We call $\mathrm{Eu}_T(x, X)$ the *equivariant Euler class of* X *at* x. It follows from [25, Theorem IV.6] that $\mathrm{Eu}_T(x, X)$ splits into a product of linear polynomials, namely

$$\mathrm{Eu}_T(x, X) = \omega_1^{k_1} \cdots \omega_s^{k_s},$$

where $\omega_i \in H^2(BT) \simeq \Xi(T) \otimes \mathbb{Q}$. Here $\Xi(T)$ stands for the character group of T, and the isomorphism is provided in Example 1.1.

Since X is a cone over $\mathbb{S}(X)$, $H_c^*(X) \simeq H^*(X, X - \{x\}) \simeq \mathbb{Q}$, where $H_c^*(-)$ denotes cohomology with compact supports. Using the Serre spectral sequence, one notices that these isomorphisms are also valid in equivariant cohomology:

$$H_{T,c}^*(X) \simeq H_T^*(X, X - \{x\}) \simeq H_T^*$$

Let $\mathcal{T}_X$ be the canonical module generator of $H_T^*(X, X - \{x\})$. This generator can be described by the commutative diagram

$$\begin{tikzcd}
H_T^*(X, X - \{x\}) \arrow[rr, "i^*"] \arrow[d, shift left, "\int_{[X]}"] & & H_T^*(X) \arrow[d, "\mathrm{res}"] \\
H_T^*(x) \arrow[u, shift left, "\Phi_X^*"] \arrow[rr, dashed, "\times(\mathrm{Eu}_T(x,X))"] & & H_T^*(x),
\end{tikzcd}$$

where Φ_X^* is an equivariant Thom isomorphism (recall that Φ_X^* is multiplication by $\mathcal{T}_X$). In other words, $\mathcal{T}_X$ is the unique class in $H_T^*(X, X - \{x\})$ whose restriction to $H_T^*(pt)$ coincides with $\mathrm{Eu}_T(x, X)$. The class $\mathcal{T}_X$ is commonly called the *Thom class of* X. Let us bear in mind that the map Φ_X^* raises degree by $2n$. Clearly, $H_T^*(X, X - \{x\}) \simeq H_c^*(X) \otimes H_T^*(pt)$ and so, $H_{T,c}^j(X) = 0$ for $j < 2n$. As for the integral appearing here, it corresponds to the inverse of Φ_X^*. The reason for this is that $(\Phi_X^*)^{-1}$ can be interpreted geometrically as a pushforward or as a morphism of "integration along the fibres" [2, Section 2].

Let $\mathcal{Q}_T$ be the quotient field of H_T^*. If $\mu \in H_{T,c}^*(X)$, then

$$\mathrm{Eu}_T(x, X) \wedge \int_{[X]} \mu = \mu_x,$$

where μ_x denotes restriction of the class μ to x. Hence, the identity

$$\frac{1}{\mathrm{Eu}_T(x,X)} = \frac{1}{\mu_x}\int_{[X]} \mu,$$

holds in $\mathcal{Q}_T$, for every non-zero μ in $H_T^*(X, X-\{x\})$.

More generally, let X be a T–variety with an isolated fixed point x. Suppose that X is rationally smooth at x. As pointed out in [2], we can replace X by a conical neighborhood U_x of x and define $\mathrm{Eu}_T(x,X) := \mathrm{Eu}_T(x,U_x)$. For instance, if x is also an attractive fixed point, we can let U_x be a rational cell (Proposition 3.2).

In case the isolated fixed point $x \in X$ is *not necessarily* a rationally smooth point, Arabia [2] has shown that we can still define an Euler class $\mathrm{Eu}_T(x,X)$. The key ingredient here is that, by Theorem 1.2, the map

$$i^*\colon H_T^*(X, X-\{x\}) \to H_T^*(x)$$

is an isomorphism modulo H_T^*–torsion. Therefore, the function that assigns to a non-torsion element $\mu \in H_T^*(X, X-\{x\})$ the fraction $\frac{1}{\mu_x}\int_X \mu \in \mathcal{Q}_T$ is *constant*.

Definition 5.1 Let X be a T–variety. Suppose that $x \in X^T$ is an isolated fixed point. The fraction

$$\frac{1}{\mathrm{Eu}_T(x,X)} := \frac{1}{\mu_x}\int_X \mu \in \mathcal{Q}_T,$$

where μ is any non-torsion element of $H_T^*(X, X-\{x\})$, is called the *inverse of the equivariant Euler class of X at x*. When this fraction is non-zero, we denote its inverse by $\mathrm{Eu}_T(x,X)$ and call it the *equivariant Euler class of X at x*.

If x is a rationally smooth isolated fixed point of a T–variety X, then the classical equivariant Euler class, given at the beginning of this section, agrees with the one presented in Definition 5.1, up to a non-zero rational number [2, Section 2]. Moreover, in this case, $\mathrm{Eu}_T(x,X)$ is a non-zero polynomial, and splits into a product of linear factors [2, Remark 2.3-2(b)].

The technical advantage of the modern approach to equivariant Euler classes (Definition 5.1) is that it allows to compute $\mathrm{Eu}_T(x,X)$ quite easily in case X has only finitely many T–invariant curves passing through the fixed point x; see eg [2, Section 3] and Corollary 5.6 below. From now on, equivariant Euler classes will be understood in the sense of Definition 5.1.

Example 5.2 When $X = \mathbb{C}^n$, $x = 0$, and the algebraic torus T acts linearly on $\mathbb{C}^n$, one proves

$$\mathrm{Eu}_T(0, \mathbb{C}^n) = \prod_{\alpha \in \mathcal{A}} \alpha,$$

where $\mathcal{A}$ is the collection of weights. Furthermore, if the weights in $\mathcal{A}$ are pairwise linearly independent, then the associated complex projective space $\mathbb{P}(\mathbb{C}^n_{\mathcal{A}})$ has exactly n T–fixed points: the lines $\mathbb{C}_{\alpha_i}$. One also verifies that

$$\mathrm{Eu}_T([\mathbb{C}_{\alpha_i}], \mathbb{P}(\mathbb{C}^n_{\mathcal{A}})) = \prod_{j \neq i} (\alpha_j - \alpha_i).$$

See [2, Remark 2.4.1-1].

Remark 5.3 The inverse of the equivariant Euler class coincides with the equivariant multiplicity at a nondegenerate fixed point [9, Section 4].

Proposition 5.4 (Atiyah–Bott localization formula [2]) *Let X be a complex projective variety. Suppose that a torus T acts on X with only a finite number of fixed points. Then*

$$\int_X \mu = \sum_{x \in X^T} \frac{\mu|_x}{\mathrm{Eu}_T(x, X)},$$

for any $\mu \in H_T^*(X)$. □

Let (X, x) be a rational cell. Then, by Proposition 3.2, X admits a closed T–equivariant embedding into its tangent space $T_x X$. Notice that there are only a finite number of codimension-one subtori $T_1, \dots, T_m$ of T for which $X^{T_j} \neq X^T$, since each one of them is contained in the kernel of a weight of T in $T_x X$.

Theorem 5.5 [2; 11] *Let (X, x) be a rational cell of dimension n. Let $\pi\colon X \to \mathbb{C}^n$ be the equivariant finite surjective map from Theorem 3.15. Then:*

(a) *The induced morphism in cohomology*

$$\pi^*\colon H_c^{2n}(\mathbb{C}^n) \longrightarrow H_c^{2n}(X)$$

is an isomorphism and satisfies $\int_Y \pi^*(\mu) = \deg(\pi) \int_{\mathbb{C}^n} \mu$, where $\deg(\pi)$ is the cardinality of a general fibre of π. This formula also holds in equivariant cohomology, in particular

$$\mathrm{Eu}_T(0, \mathbb{C}^n) = \deg(\pi) \cdot \mathrm{Eu}_T(x_0, X).$$

(b) $\mathrm{Eu}_T(X, x) = c \prod_{T_i} \mathrm{Eu}_T(X^{T_i}, x)$, *where c is a positive rational number, and the product runs over the finite number of codimension-one subtori T_i of T for which $X^{T_i} \neq X^T$.*

Proof We refer the reader to [2, Proposition 3.2.1-1] for the proof of part (a). Finally, part (b) follows from Remark 5.3 and [11, Theorem 18 (iii)]. □

Let (X, x) be a rational cell. At the beginning of this section it was shown that $\mathrm{Eu}_T(x, X)$ splits into a product of characters. The following result provides a geometric interpretation of this factorization.

Corollary 5.6 *Let (X, x) be a rational cell of dimension n. Suppose that X contains only a finite number of closed irreducible T–invariant curves $\mathcal{C}_i$, $i = 1, \dots, n$. Let χ_i be the character associated with the action of T on $\mathcal{C}_i$. Then*

$$\mathrm{Eu}_T(x, X) = c \cdot \chi_1 \cdots \chi_n,$$

where c is a positive rational number.

Proof There is only a finite number of codimension-one subtori T_i such that $X^{T_i} \neq X^T$. Notice that T acts on each X^{T_i} through its quotient $T/T_i \simeq \mathbb{C}^*$. Because x is an attractive fixed point of X, we can assume, without loss of generality, that x is an attractive fixed point of each X^{T_i}, for the induced action of $\mathbb{C}^* \simeq T/T_i$. It follows from [12, Section 1.4, Corollary 2] that $X^{T_i} = \mathcal{C}_i$. Moreover, by [12, Theorem 1.1], each X^{T_i} is rationally smooth at x. Hence, each X^{T_i} is a one-dimensional rational cell with attractive fixed point x (see Lemma 3.13 for a characterization of these cells). The result can now be deduced from Theorem 5.5 and Example 5.2. □

6 Local indices and module generators for the equivariant cohomology of $\mathbb{Q}$–filtrable GKM varieties

We supply a method for building canonical free module generators on the equivariant cohomology of any $\mathbb{Q}$–filtrable GKM–variety. Our findings here extend the work of Guillemin and Kogan [22] on the equivariant K–theory of orbifolds to the equivariant cohomology of a much larger class of singular varieties.

Let X be a $\mathbb{Q}$–filtrable GKM–variety. In other words, X is a normal projective T–variety with only a finite number of fixed points and T–invariant curves. Moreover, after choosing, once and for all, a generic one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ satisfying condition (c) of Definition 4.6, there exists a BB–decomposition of X as a disjoint union of rational cells, say $(C_1, x_1), \dots, (C_m, x_m)$, each one containing $x_i \in X^T$ as its unique attractive fixed point. (Observe that the fixed points need not be attractive in X, but they are so in their particular rational cells.) This decomposition induces a filtration of X,

$$\varnothing = X_0 \subset X_1 \subset X_2 \cdots \subset X_m = X,$$

by closed invariant subvarieties X_i, so that each difference $X_i \setminus X_{i-1}$ equals C_i, for $i = 1, \dots, m$. The key observation here is provided by Theorem 4.7. It states that every X_i is equivariantly formal and is made up of rational cells. In consequence, GKM theory can be applied to each X_i. We will refer to X_i as the i^{th} *filtered piece of* X, and m will be called the *length of the filtration*.

Denote by $x_1, \dots, x_m$ the fixed points of X. The filtration induces a total ordering of the fixed points, namely,

$$x_1 < x_2 < \cdots < x_m.$$

In the sequel, we refer to this ordering as *the order relation on* X^T *compatible with the filtration of* X. Keep in mind that both the filtration of X, as well as the compatible total order of the fixed points, depend on our fixed choice of λ given above.

Let (C_i, x_i) be a rational cell of X. From the previous section, we know that

$$H^*_{T,c}(C_i) \simeq H^*_T(C_i, C_i - \{x_i\}) \simeq H^*_T(x_i),$$

where the second isomorphism is provided by the Thom class $\mathcal{T}_i$, a well-known element of $H^*_T(C_i, C_i - \{x_i\})$. When restricted to $H^*_T(x_i)$, the Thom class $\mathcal{T}_i$ becomes a product of linear polynomials: the Euler class $\mathrm{Eu}_T(c_i, C_i)$.

In Section 4 we built non-equivariant short exact sequences of the form

$$0 \longrightarrow H^{2k}_c(C_i) \longrightarrow H^{2k}(X_i) \longrightarrow H^{2k}(X_{i-1}) \longrightarrow 0,$$

for every i. Since the spaces involved have zero cohomology in odd degrees, then these short exact sequences naturally generalize to the equivariant case, so we also have equivariant short exact sequences

$$0 \longrightarrow H^{2k}_{T,c}(C_i) \longrightarrow H^{2k}_T(X_i) \longrightarrow H^{2k}_T(X_{i-1}) \longrightarrow 0,$$

for each i. On the other hand, by equivariant formality, the singular equivariant cohomology of each X_i injects into $H^*_T(X_i^T) = \bigoplus_{j \le i} H^*_T(x_j)$.

In summary, for each i, we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^*_{T,c}(C_{i+1}) \arrow[r] \arrow[d] & H^*_T(X_{i+1}) \arrow[r] \arrow[d] & H^*_T(X_i) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^*_T(x_{i+1}) \arrow[r] & \bigoplus_{j \le i+1} H^*_T(x_j) \arrow[r] & \bigoplus_{j \le i} H^*_T(x_j) \arrow[r] & 0 \end{tikzcd} \tag{1}$$

where the vertical maps are all injective. Indeed, such maps correspond to the various restrictions to fixed point sets. We will use this diagram to build cohomology generators.

The next two lemmas are inspired by [23, Theorem 2.3 and Proposition 4.1], where Kac–Moody flag varieties are studied.

Lemma 6.1 *Let X be a $\mathbb{Q}$–filtrable variety. Then there exists a non-canonical isomorphism of H_T^*–modules*

$$H_T^*(X) \simeq \bigoplus_{x_i \in X^T} \mathrm{Eu}_T(C_i, x_i) H_T^*(pt),$$

which is compatible with restriction to the various i^{th} filtered pieces $X_i \subset X$.

Proof We argue by induction on the length of the filtration. The case $m = 1$ is simple, because it corresponds to $X = \{x_1\}$, a singleton. Assuming that we have proved the assertion for m, let us prove the case $m+1$. Substitute $i = m$ in the commutative diagram above. Then

$$H_T^*(X_{m+1}) = H_T^*(X) \simeq H_{T,c}^*(C_{m+1}) \oplus H_T^*(X_m).$$

By induction, $H_T^*(X_m) \simeq \prod_{i \leq m} \mathrm{Eu}_T(C_i, x_i) H_T^*(pt)$. So the claim for $m+1$ follows directly from the isomorphism $H_{T,c}^*(C_{m+1}) \simeq \mathrm{Eu}_T(C_{m+1}, x_{m+1}) H_T^*(pt)$. □

The isomorphism of the previous lemma is not canonical because the cellular decomposition of X depends on our particular choice of generic one-parameter subgroup and a compatible ordering of the fixed points.

Convention From now on, given a class $\mu \in H_T^*(X)$, we will denote by $\mu(x_i)$ its restriction to the fixed point x_i.

Lemma 6.2 *Let X be a projective T–variety. Assume that X is $\mathbb{Q}$–filtrable and let $x_1 < x_2 < \cdots < x_m$ be the order relation on X^T compatible with the filtration of X. For each i, let $\varphi_i \in H_T^*(X)$ be a class such that*

$$\varphi_i(x_j) = 0 \text{ for } j < i, \text{ and } \varphi_i(x_i) \text{ is a scalar multiple of } \mathrm{Eu}_T(i, C_i).$$

Then the classes $\{\varphi_i\}$ generate $H_T^*(X)$ freely as a module over $H_T^*(pt)$.

Proof Since X is equivariantly formal, we know that $H_T^*(X)$ injects into $H_T^*(X^T)$ and is a free H_T^*–module of rank $m = |X^T|$. First, we show that the φ_i are linearly independent. Arguing by contradiction, suppose there is a linear combination

$$\sum_{i=0}^{m} f_i \varphi_i = 0,$$

with $f_i \in H_T^*$, not all of them zero. Let k be the minimum of the set $\{i \mid f_i \neq 0\}$. Then we have

$$f_k \varphi_k + f_{k+1} \varphi_{k+1} + \cdots + f_m \varphi_m = 0,$$

where $f_k \neq 0$. Let us restrict this linear combination to x_k. Then

$$f_k \varphi_k(x_k) + f_{k+1} \varphi_{k+1}(x_k) + \cdots + f_m \varphi_m(x_k) = 0.$$

But $\varphi_\ell(x_k) = 0$ for all $\ell > k$. Thus we obtain

$$f_k \varphi(x_k) = 0.$$

However, $\varphi(x_k)$ is a non-zero multiple of the Euler class $\mathrm{Eu}_T(x_k, C_k)$ and, as such, it is non-zero. We conclude that f_k must be zero. This is a contradiction.

To conclude the proof, we need to show that the φ_i generate $H_T^*(X)$ as a module. But this is a routine exercise, using induction on the length of the filtration of X (the base case being trivial). The commutative diagram (1) then disposes of the inductive step. $\square$

As for the existence of classes satisfying Lemma 6.2, we will show that they can always be constructed on GKM–varieties. First, we need two technical lemmas.

Lemma 6.3 *Let X be a normal projective T–variety with finitely many fixed points. Choose a generic one-parameter subgroup and write X as $X = C \sqcup Y$, where*

$$C = \left\{ z \in X \;\middle|\; \lim_{t \to 0} tz = x \right\}$$

is the stable cell of $x \in X^T$, and Y is closed and T–stable. Then any closed irreducible T–stable curve that passes through x is contained in the Zariski closure of C.

Proof Let ℓ be a closed irreducible T–stable curve passing through x. Recall that ℓ is the closure of a one-dimensional orbit Tz. Moreover, $\ell = \overline{Tz}$ has two distinct fixed points, namely, x and a fixed point $y_{i(\ell)}$ contained *necessarily* in Y. We claim that $z \in C$. For otherwise, $\lim_{t \to 0} tz = y_{i(\ell)}$, which implies that z belongs to the stable subvariety of $y_{i(\ell)}$. Since Y is T–invariant and closed, $\ell = \overline{Tz} \subset Y$. That is, $x \in \partial \ell$ would belong to Y, which contradicts our original hypothesis. Thus $z \in C$.

The fact that C is also T–stable gives the inclusion $Tz \subset C$. We conclude that $\ell = \overline{Tz} \subset \overline{C}$. $\square$

Lemma 6.4 *Let X be a normal projective variety on which a torus acts with a finite number of fixed points and one-dimensional orbits. Suppose X is equivariantly formal and there is a generic one-parameter subgroup such that X can be written as a disjoint union $X = C \sqcup Y$, where*

$$C = \{z \in X \mid \lim_{t \to 0} tz = x\}$$

is a rational cell with unique attractive fixed point $x \in X^T$, and Y is closed and T–stable. Then the cohomology class $\tau \in \bigoplus_{w \in X^T} H_T^*(w)$, defined by

$$\tau(x) = \mathrm{Eu}_T(x, C) \quad \textit{and} \quad \tau(y) = 0 \quad \textit{for all } y \in Y^T,$$

belongs to the image of $H_T^*(X)$ in $H_T^*(X^T)$.

Proof The hypotheses imply that X is a GKM–variety. As a result, the equivariant cohomology of X can be described by the GKM-relations of Theorem 2.5. So, to prove the lemma, it is enough to verify that τ satisfies such relations.

Because τ restricts to zero at every fixed point except x, we need only show that

$$\tau(x) = \tau(x) - \tau(y_i) = \mathrm{Eu}_T(x, C)$$

is divisible by χ_i whenever the fixed points $x \in C$ and $y_i \in Y^T$ are joined by a T–curve ℓ_i in X, and T acts on ℓ_i through χ_i. Let p be the total number of ℓ_i.

By Lemma 6.3, the curve ℓ_i is contained in the Zariski closure $\overline{C}$ of C. In fact, $\ell_i \setminus \{y_i\} \subset C$. Also, it follows from Theorem 3.16 that $p = \dim(C)$. Thus, using Corollary 5.6, we conclude that $\mathrm{Eu}_T(x, C)$ is a non-zero multiple of the χ_i. In short, τ belongs to $H_T^*(X)$. □

It is noticeable that, in the previous lemmas, no assumption on the irreducibility of X has been made. Surely we allow for some flexibility in this matter, since the various filtered pieces X_i of a $\mathbb{Q}$–filtrable space X need not be irreducible.

Theorem 6.5 *Let X be a $\mathbb{Q}$–filtrable GKM–variety. Then cohomology generators $\{\varphi_i\}$ of $H_T^*(X)$ with the properties described in Lemma 6.2 exist.*

Proof We proceed by induction on m, the length of the filtration of X. If $m = 1$, then X is just a point, and the statement is clear, because we can simply choose $\varphi_1 = 1$. Now, assuming the statement holds for $\mathbb{Q}$–filtrable varieties X with a filtration of length m, let us prove it for those X with a filtration of length $m+1$. First, notice that we have a filtration

$$\varnothing = X_0 \subset X_1 \subset X_2 \cdots \subset X_m \subsetneq X_{m+1} = X.$$

By the inductive hypothesis, there are classes $\varphi_1, \dots, \varphi_m \in H_T^*(X_m)$, which satisfy the properties of Lemma 6.2 in $H_T^*(X_m)$. Now using the commutative diagram (1), we can lift them to classes $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_m$, which lie in $H_T^*(X_{m+1}) = H_T^*(X)$. A simple check shows that these lifted classes satisfy the conditions of Lemma 6.2 on X. In consequence, we just need to construct one extra class in $H_T^*(X)$, namely φ_{m+1}, with the sought-after qualities. Set $\varphi_{m+1}(x_{m+1}) = \mathrm{Eu}_T(x_{m+1}, C_{m+1})$ and $\varphi_{m+1}(x_j) = 0$ for all $j \leq m$. Lemma 6.4 guarantees that this class in fact belongs to $H_T^*(X)$. The inductive step is thus proved, concluding the argument. □

Definition 6.6 Let X be a $\mathbb{Q}$–filtrable T–variety. Fix an ordering of the fixed points, say $x_1 < x_2 < \cdots < x_m$. Given $\mu \in H_T^*(X)$, we define its *local index at* x_i, denoted $I_i(\mu)$, by the following formula:

$$I_i(\mu) = \int_{X_i} p_i^*(\mu),$$

where $p_i \colon X_i \to X$ denotes the inclusion of the i^{th} filtered piece into X. It follows from the definition that assigning local indices yields H_T^*–linear morphisms

$$I_i \colon H_T^*(X) \to H_T^*(pt).$$

Using the localization formula (Proposition 5.4), one can easily prove the following:

Lemma 6.7 *The local index of μ at x_i satisfies*

$$I_i(\mu) = \sum_{j \leq i} \frac{\mu(x_j)}{\mathrm{Eu}_T(x_j, X_i)},$$

where $\mu(x_j)$ denotes the restriction of μ to x_j. □

Corollary 6.8 *Let $x_i \in X^T$ be a fixed point. Suppose that $\mu \in H_T^*(X)$ is a cohomology class that satisfies $\mu(x_j) = 0$ for all $j < i$. Then*

$$\mu(x_i) = I_i(\mu)\mathrm{Eu}_T(x_i, X_i). \qquad \square$$

Our most important result in this section is the following generalization of the work of Guillemin and Kogan [22, Theorems 1.1 and 1.6] to $\mathbb{Q}$–filtrable GKM–varieties.

Theorem 6.9 *Let X be a $\mathbb{Q}$–filtrable GKM–variety. Let $x_1 < x_2 < \cdots < x_m$ be the order relation on X^T compatible with the filtration of X. Then, for each $i = 1, \dots, m$, there exists a unique class $\theta_i \in H_T^*(X)$ with the following properties:*

(i) $I_i(\theta_i) = 1$.

(ii) $I_j(\theta_i) = 0$ *for all* $j \neq i$.

(iii) *The restriction of* θ_i *to* $x_j \in X^T$ *is zero for all* $j < i$.

(iv) $\theta_i(x_i) = \mathrm{Eu}_T(i, C_i)$.

Moreover, the θ_i *generate* $H_T^*(X)$ *freely as a module over* $H_T^*(pt)$.

Proof By Theorem 6.5, choose a set of free generators $\{\varphi_i\}$ which satisfy the properties described in Lemma 6.2, together with the additional condition $\varphi_i(x_i) = \mathrm{Eu}_T(i, C_i)$.

Given i, notice that $I_j(\varphi_i) = 0$, for all $j < i$, and $I_i(\varphi_i) = 1$. We will show that we can modify these φ_i accordingly to obtain the generators θ_i. In fact, given $i \in \{1, \dots, m\}$, the only obstruction to setting $\theta_i = \varphi_i$ is that $I_j(\varphi_i)$ can be non-zero for some $j > i$.

Let $i \in \{1, \dots, m\}$. If $I_j(\varphi_i) = 0$ for all $j > i$, then let $\theta_i = \varphi_i$. Otherwise, proceed as follows. Let k_0 be the minimum of all $k > i$ such that $I_k(\varphi_i) \neq 0$. Define $\Psi_i = \varphi_i - I_{k_0}(\varphi_i)\varphi_{k_0}$. Let us compute the local indices of Ψ_i. Clearly, if $j < i$, we have $I_j(\Psi_i) = 0$. Also, if $j = i$, then $I_i(\Psi_i) = 1$. It is worth noticing that Ψ_i restricts to 0 at each x_j with $j < i$. Now if j satisfies $i < j \leq k_0$, then $I_j(\Psi_i) = 0$. So, arguing by induction, we can provide a class $\widetilde{\Psi}_i$ such that $I_j(\widetilde{\Psi}_i) = 0$ for all $j \neq i$, and $I_i(\widetilde{\Psi}_i) = 1$. Thus, set $\theta_i = \widetilde{\Psi}_i$. Proceeding systematically from $i = 1$ to $i = m$, we construct the family of classes θ_i satisfying the desired properties (i)–(iv).

Let us now prove uniqueness. Suppose there are classes $\{\theta_i\}$ and $\{\theta_i'\}$ satisfying all the properties of the theorem. Fix i and let $\tau = \theta_i - \theta_i'$. It is clear that τ is an element of $H_T^*(X)$ whose local index $I_j(\tau)$ is zero for all j. Suppose that τ is not zero. Then, since $H_T^*(X)$ injects into $H_T^*(X^T)$, there should be a k such that $\tau(x_k) \neq 0$. Take the minimum of all k's for which $\tau(x_k) \neq 0$. Denote this minimum by s. Then, by Corollary 6.8, one would have $\tau(x_s) = I_s(\tau)\mathrm{Eu}_T(x_s, X_s) = 0$. But this contradicts the fact that $\tau(x_s) \neq 0$. Therefore $\tau = 0$. Since i can be chosen arbitrarily, we conclude that $\theta_i = \theta_i'$ for all i.

Finally, notice that properties (iii) and (iv) together with Lemma 6.2 imply that the θ_i freely generate $H_T^*(X)$. We are done. □

7 Rational cells and standard group embeddings

Thus far, we have developed the theory of $\mathbb{Q}$–filtrable varieties. In this last section, we provide the theory with a large class of examples, namely, rationally smooth standard embeddings. We show that these varieties admit BB–decompositions into rational cells (Theorem 7.4). Thus, they are $\mathbb{Q}$–filtrable and satisfy Theorems 4.7 and 6.9.

First, let us set the stage. An affine algebraic monoid M is called *reductive* if it is irreducible, normal, and its unit group is a reductive algebraic group. See Renner [31] for many of the details. A reductive monoid is called *semisimple* if it has a zero element, and its unit group has a one-dimensional center.

Let M be a reductive monoid with zero. Denote by G its unit group and by T a maximal torus of G. Associated to M, there is a torus embedding $\overline{T} \subset M$ defined as follows

$$\overline{T} = \{x \in M \mid xt = tx, \text{ for all } t \in T\}.$$

Certainly, $T \subseteq \overline{T}$. Let $E(\overline{T})$ be the idempotent set of $\overline{T}$; that is,

$$E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}.$$

The *Renner monoid*, $\mathcal{R}$, is defined to be $\mathcal{R} := \overline{N_G(T)}/T$. It is a finite monoid whose group of units is W (the Weyl group) and contains $E(\overline{T})$ as idempotent set. In fact, any $x \in \mathcal{R}$ can be written as $x = fu$, where $f \in E(\overline{T})$ and $u \in W$. Recall that W is generated by reflections $\{s_\alpha\}_{\alpha\in\Phi}$, where Φ is the set of roots of G with respect to T.

Denote by $\mathcal{R}_k$ the set of elements of rank k in $\mathcal{R}$, that is,

$$\mathcal{R}_k = \{x \in \mathcal{R} \mid \dim Tx = k\}.$$

Definition 7.1 Let M be a reductive monoid with unit group G and zero element $0 \in M$. There exists a central one-parameter subgroup $\epsilon\colon \mathbb{C}^* \to G$ with image Z contained in the center of G, that converges to 0 Brion [14, Lemma 1.1.1]. Then $\mathbb{C}^*$ acts attractively on M via ϵ, and hence the quotient

$$\mathbb{P}_\epsilon(M) = [M \setminus \{0\}]/\mathbb{C}^*$$

is a normal projective variety. Notice also that $G \times G$ acts on $\mathbb{P}_\epsilon(M)$ via

$$G \times G \times \mathbb{P}_\epsilon(M) \to \mathbb{P}_\epsilon(M), \quad (g, h, [x]) \mapsto [gxh^{-1}].$$

Furthermore, $\mathbb{P}_\epsilon(M)$ is a normal projective embedding of the reductive group G/Z. In the sequel, $X = \mathbb{P}_\epsilon(M)$ will be called a *standard group embedding*.

When M is semisimple (in which case ϵ is essentially unique), we write $\mathbb{P}(M)$ for $\mathbb{P}_\epsilon(M)$. Indeed, for such a monoid, $Z \simeq \mathbb{C}^*$ is the connected center of the unit group G of M. Thus, a semisimple monoid with unit group G can be thought of as an affine cone over some projective embedding $\mathbb{P}(M)$ of the semisimple group $G_0 = G/Z$. For an up-to-date description of these and other embeddings, see [1].

Example 7.2 Let G_0 be a semisimple algebraic group over the complex numbers and let $\rho\colon G_0 \to \mathrm{End}(V)$ be a representation of G_0. Define Y_ρ to be the Zariski closure of $G = [\rho(G_0)]$ in $\mathbb{P}(\mathrm{End}(V))$, the projective space associated with $\mathrm{End}(V)$. Finally, let X_ρ be the normalization of Y_ρ. By definition, X_ρ is an standard group embedding of G. Notice that M_ρ, the Zariski closure of $\mathbb{C}^*\rho(G_0)$ in $\mathrm{End}(V)$, is a semisimple monoid whose group of units is $\mathbb{C}^*\rho(G_0)$. Rationally smooth standard embeddings of the form X_ρ, with ρ irreducible, have been classified combinatorially in [34].

Remark 7.3 Let $\mathbb{P}_\epsilon(M)$ be a standard group embedding. Associated to $\mathbb{P}_\epsilon(M)$, there is a standard torus embedding of T/Z, namely,

$$\mathbb{P}_\epsilon(\overline{T}) = [\overline{T} \setminus \{0\}]/\mathbb{C}^*.$$

By construction, $\mathbb{P}_\epsilon(\overline{T})$ is a normal projective torus embedding contained in $\mathbb{P}_\epsilon(M)$. Notably, by a result of Renner [34, Theorems 2.4 and 2.5], $\mathbb{P}_\epsilon(M)$ is rationally smooth if and only if $\mathbb{P}_\epsilon(\overline{T})$ is rationally smooth.

We now come to the main result of this section. It states that rationally smooth standard embeddings are equivariantly formal for the induced $T \times T$–action.

Theorem 7.4 *Let $X = \mathbb{P}_\epsilon(M)$ be a standard group embedding. If X is rationally smooth, then X is $\mathbb{Q}$–filtrable.*

Proof Renner has shown that X comes equipped with the following BB–decomposition:

$$X = \bigsqcup_{r \in \mathcal{R}_1} C_r,$$

where $\mathcal{R}_1 = X^{T\times T}$. See Renner [32, Theorem 3.4; 35, Theorem 4.3] for more details. Our strategy is to show that if X is rationally smooth, then each cell C_r is rationally smooth.

With this purpose in mind, we call the reader's attention to the fact that, in the terminology of [32], M is quasismooth [32, Definition 2.2] if and only if $M \setminus \{0\}$ is rationally smooth. The equivalence between these two notions follows from [32, Theorem 2.1] and [34, Theorems 2.1, 2.3, 2.4 and 2.5].

Next, by [32, Lemma 4.6 and Theorem 4.7], each C_r equals

$$U_1 \times C_r^* \times U_2,$$

where the U_i are affine spaces. Moreover, if we write $r \in \mathcal{R}_1$ as $r = ew$, with $e \in E_1(\overline{T})$ and $w \in W$, then $C_r^* = C_e^* w$. So it is enough to show that C_e^* is rationally smooth, for $e \in E_1(\overline{T})$.

By [32, Theorem 5.1], it follows that, if $X = \mathbb{P}_\epsilon(M)$ is rationally smooth, then

$$C_e^* = [f_e M(e)]/Z,$$

for some unique $f_e \in E(\overline{T})$, where $M(e) = M_e Z$ and M_e is rationally smooth [34, Theorem 2.5]. Furthermore, the proof of [32, Theorem 5.1] also implies that $[e]$ is the zero element of the rationally smooth, reductive, affine monoid $M(e)/Z$. Additionally,

$$C_e^* = \left\{ x \in M(e)/Z \;\middle|\; \lim_{s \to 0} sx = [e] \right\},$$

for some generic one-parameter subgroup. Using Lemma 7.5 below, one concludes that C_e^* is rationally smooth.

Finally, since X is normal, projective and admits a BB–decomposition into rational cells, we have compiled all the necessary data to conclude that X is $\mathbb{Q}$–filtrable. $\square$

Lemma 7.5 *Let M be a reductive monoid with zero. Suppose that zero 0 is a rationally smooth point of M. Let $f \in E(M)$ be an idempotent of M. Then $0 \in fM$ is a rationally smooth point of the closed subvariety fM.*

Proof By [14, Lemma 1.1.1], one can find a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$, with image S, such that $\lambda(0) = f$. Notice that

$$fM = \{x \in M \mid \lambda(t)x = x, \text{ for all } t \in \mathbb{C}^*\},$$

that is, fM is the fixed point set of the subtorus S of T. Thus, by [12, Theorem 1.1], one concludes that 0 is also a rationally smooth point of fM. $\square$

Next we provide a partial converse to Theorem 7.4.

Theorem 7.6 *Let $X = \mathbb{P}_\epsilon(M)$ be a standard embedding. Suppose that X contains a unique closed $G \times G$–orbit. If X is $\mathbb{Q}$–filtrable, then X is rationally smooth.*

Proof Since X contains a unique closed $G \times G$–orbit, it follows from [31, Chapter 7] that $W \times W$ acts transitively on $\mathcal{R}_1$, the set of representatives of the $T \times T$–fixed points of X. Because X is irreducible, there exists a unique cell, say C_σ, with $\sigma \in \mathcal{R}_1$, such that $X = \overline{C_\sigma}$. By assumption, C_σ is rationally smooth at σ, and so, X is rationally smooth at σ. We claim that X is rationally smooth at every $r \in \mathcal{R}_1 = X^{T \times T}$. Indeed, by the previous remarks, $r = w \cdot \sigma \cdot v$, for some $(w, v) \in W \times W$ and rational smoothness is a local property invariant under homeomorphisms. Now Lemma 7.7 below concludes the proof. $\square$

Lemma 7.7 *Let X be a projective T–variety with a finite number of fixed points $x_1, \dots, x_m$. Then X is rationally smooth at every $x \in X$ if and only if X is rationally smooth at every fixed point x_i.*

Proof One direction is clear. For the converse, pick a generic one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ such that $X^T = X^{\mathbb{C}^*}$. Let $x \in X$. Then, there exists $x_k \in X^T$ such that $x_k = \lim_{t \to 0} tx$ (BB–decomposition). Moreover, since X is rationally smooth at x_k, there exists a neighborhood V_k of x_k with the property that X is rationally smooth at every $y \in V_k$. By construction, there exists $s \in \mathbb{C}^*$ satisfying $sx \in V_k$. To see this, simply notice that we can find a sequence $\{s_n\} \subset \mathbb{C}^*$ for which $s_n \cdot x$ converges to x_k, ie, there is N such that $s_n \cdot x$ belongs to V_k, for all $n \geq N$. Now setting $s = s_N$ yields $s \cdot x \in V_k$. In other words, sx is a rationally smooth point of X. But the set of rationally smooth points is T–invariant. Hence, x is a rationally smooth point of X. Inasmuch as the point x was chosen arbitrarily, the argument is complete. □

In the author's thesis it was shown that all standard embeddings are $T \times T$–skeletal. Consequently, rationally smooth standard embeddings are also GKM–varieties. In a forthcoming paper [20], we find explicitly all the GKM–data (ie, fixed points, invariant curves and associated characters) of any rationally smooth standard embedding $\mathbb{P}_\epsilon(M)$, and describe $H^*_{T \times T}(\mathbb{P}_\epsilon(M))$ as a complete combinatorial invariant of M. The results will appear elsewhere.

We conclude by mentioning a few concrete examples to which our theory applies.

Example 7.8 Rationally smooth torus embeddings are exactly the simplicial toric varieties (Danilov [18]). Among them, those that are projective are also $\mathbb{Q}$–filtrable (by Theorem 7.4). In particular, the coarse moduli space of a toric Deligne–Mumford stack (Borisov, Chen and Smith [8]), when projective, is $\mathbb{Q}$–filtrable (cf [8, Proposition 3.7]).

Example 7.9 Let M be a semisimple monoid with zero and unit group G of the form $\mathbb{C}^* \times G_0$, where G_0 is a simple algebraic group of type A_2, C_2 or G_2. Then the associated standard embedding $\mathbb{P}(M)$ is always rationally smooth. This follows from Remark 7.3, since, in this context, the associated torus embedding $\mathbb{P}(\overline{T})$ is a simplicial toric surface.

Example 7.10 Let G be a semisimple algebraic group with Borel subgroup B and maximal torus $T \subset B$. An embedding of G is called *simple* if it contains a unique closed $G \times G$–orbit. Let X be such an embedding. Then, using the notation from Example 7.2, X is of the form $\mathbb{P}(M_{\rho_\lambda})$, for some irreducible representation ρ_λ of

G, with highest weight λ [34]. Moreover, the unique closed $G \times G$–orbit of X is the partial flag variety $G/P_J \times G/P_J^-$, where

$$J = \{s \in S \mid s(\lambda) = \lambda\}.$$

Here S is the set of simple involutions of W, the Weyl group of (G, T). Also, P_J is the standard parabolic subgroup associated to J, and P_J^- is the opposite parabolic subgroup. Renner has classified all rationally smooth simple embeddings combinatorially in terms of J and the Dynkin diagram for G. See [33, Corollary 3.5] for an exhaustive list of all possible J's that give rise to rationally smooth simple embeddings.

According to this list, if G is a semisimple group of adjoint type, then the choice $J = \varnothing$ yields the wonderful compactification of G.

In contrast, when G is a semisimple group of type A_n, with $n \geq 2$, the possibilities for J are as follows. Let $S = \{s_1, \dots, s_n\}$. The following subsets J of S produce rationally smooth embeddings of G:

(i) $J = \varnothing$

(ii) $J = \{s_1, \dots, s_i\}$, $1 \leq i < n$

(iii) $J = \{s_j, \dots, s_n\}$, $1 < j \leq n$

(iv) $J = \{s_1, \dots, s_i, s_j, \dots, s_n\}$, $1 \leq i$, $i \leq j - 3$ and $j \leq n$

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