Geometry & Topology 18 (2014) 327–377

# Chern–Simons line bundle on Teichmüller space

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Let $X$ be a non-compact geometrically finite hyperbolic 3–manifold without cusps of rank 1. The deformation space $\mathcal{H}$ of $X$ can be identified with the Teichmüller space $\mathcal{T}$ of the conformal boundary of $X$ as the graph of a section in $T^*\mathcal{T}$. We construct a Hermitian holomorphic line bundle $\mathcal{L}$ on $\mathcal{T}$, with curvature equal to a multiple of the Weil–Petersson symplectic form. This bundle has a canonical holomorphic section defined by $\exp(\frac{1}{\pi}\operatorname{Vol}_R(X) + 2\pi i\,\mathrm{CS}(X))$, where $\operatorname{Vol}_R(X)$ is the renormalized volume of $X$ and $\mathrm{CS}(X)$ is the Chern–Simons invariant of $X$. This section is parallel on $\mathcal{H}$ for the Hermitian connection modified by the $(1,0)$ component of the Liouville form on $T^*\mathcal{T}$. As applications, we deduce that $\mathcal{H}$ is Lagrangian in $T^*\mathcal{T}$, and that $\operatorname{Vol}_R(X)$ is a Kähler potential for the Weil–Petersson metric on $\mathcal{T}$ and on its quotient by a certain subgroup of the mapping class group. For the Schottky uniformisation, we use a formula of Zograf to construct an explicit isomorphism of holomorphic Hermitian line bundles between $\mathcal{L}^{-1}$ and the sixth power of the determinant line bundle.

32G15, 58J28

## 1 Introduction

In [4], S S Chern and J Simons defined secondary characteristic classes of connections on principal bundles, arising from Chern–Weil theory. Their work has been extensively developed to what is now called Chern–Simons theory, with many applications in geometry and topology, but also in theoretical physics. For a Riemannian oriented 3–manifold $X$, the Chern–Simons invariant $\mathrm{CS}(\omega, S)$ of the Levi-Civita connection form $\omega$ in an orthonormal frame $S$ is given by the integral of the 3–form on $X$,

$$\frac{1}{16\pi^2}\operatorname{Tr}(\omega\wedge d\omega + \tfrac{2}{3}\omega\wedge\omega\wedge\omega).$$

On closed 3–manifolds, the invariant $\mathrm{CS}(\omega)$ is independent of $S$ up to integers. By the Atiyah–Patodi–Singer theorem for the signature operator, the Chern–Simons invariant

Published: 29 January 2014 DOI: 10.2140/gt.2014.18.327

of the Levi-Civita connection is related to the eta invariant by the identity $3\eta \equiv 2\,\mathrm{CS}$ modulo $\mathbb{Z}$ (see for instance Yoshida [31]).

The theory has been extended to SU(2) flat connections on compact 3–manifolds with boundary by Ramadas, Singer and Weitsman [25], in which case $\mathrm{CS}(\omega)$ does depend on the boundary value of the section $S$. The Chern–Simons invariant $e^{2\pi i\,\mathrm{CS}(\cdot)}$ can be viewed as a section of a complex line bundle (with a Hermitian structure) over the moduli space of flat SU(2) connections on the boundary surface. They proved that this bundle is isomorphic to the determinant line bundle introduced by Quillen [24]. Some more systematic studies and extensions of the Chern–Simons bundle have been developed by Freed [7], and Kirk and Klassen [14]. One contribution of our present work is to give an explicit isomorphism between these Hermitian holomorphic line bundles in the Schottky setting.

An interesting field of applications of Chern–Simons theory is for hyperbolic 3–manifolds $X = \Gamma\backslash\mathbb{H}^3$, which possess a natural flat connection $\theta$ over a principal $\mathrm{PSL}_2(\mathbb{C})$–bundle. For closed manifolds, Yoshida [31] defined the $\mathrm{PSL}_2(\mathbb{C})$–Chern–Simons invariant as above by

$$\mathrm{CS}(\theta) = -\frac{1}{16\pi^2}\int_X S^*\big(\mathrm{Tr}(\theta\wedge d\theta + \tfrac{2}{3}\theta\wedge\theta\wedge\theta)\big),$$

where $S\colon X \to P$ are particular sections coming from the frame bundle over $X$. This is a complex number with imaginary part $-(1/2\pi^2)\,\mathrm{Vol}(X)$, and real part equal to the Chern–Simons invariant of the Levi-Civita connection on the frame bundle. Up to the contribution of a link in $X$, the function $F := \exp(\frac{2}{\pi}\,\mathrm{Vol}(X) + 4\pi i\,\mathrm{CS}(X))$ extends to a holomorphic function on a natural deformation space containing closed hyperbolic manifolds as a discrete set.

Our setting in this paper is that of 3–dimensional geometrically finite hyperbolic manifolds $X$ without rank-1 cusps, in particular *convex co-compact hyperbolic* manifolds, which are conformally compactifiable to a smooth manifold with boundary. Typical examples are quotients of $\mathbb{H}^3$ by quasi-Fuchsian or Schottky groups. The ends of $X$ are either funnels or rank-2 cusps. The funnels have a conformal boundary, which is a disjoint union of compact Riemann surfaces forming the *conformal boundary* $M$ of $X$. The deformation space of $X$ is essentially the deformation space of its conformal boundary, ie, Teichmüller space. Before defining a Chern–Simons invariant, it is natural to ask about a replacement of the volume in this case. For Einstein conformally compact manifolds, the notion of *renormalized volume* $\mathrm{Vol}_R(X)$ has been introduced by Henningson and Skenderis [12] in the physics literature and by Graham [8] in the mathematical literature. In the particular setting of hyperbolic 3–manifolds, this has been studied by Krasnov [15] and extended by Takhtajan and Teo [27], in relation with earlier work of Takhtajan and Zograf [28], to show that $\mathrm{Vol}_R$ is a Kähler potential for the Weil–Petersson metric in Schottky and quasi-Fuchsian settings. Krasnov and Schlenker [17] gave a more geometric proof of this, using the Schläfli formula on convex co-compact hyperbolic 3–manifolds to compute the variation of $\mathrm{Vol}_R$ in the deformation space

Before we introduce the Chern–Simons invariant in our setting, let us first recall the definition of $\mathrm{Vol}_R$ used by Krasnov and Schlenker [17]. A *hyperbolic funnel* is some collar $(0,\epsilon)_x \times M$ equipped with a metric

$$
\begin{aligned}
g &= \frac{dx^2+h(x)}{x^2}, \quad h(x)\in \mathcal{C}^\infty(M, S^2_+T^*M),\\
h(x) &= h_0\Big(\Big(\mathrm{Id}+\frac{x^2}{2}A\Big)\cdot, \Big(\mathrm{Id}+\frac{x^2}{2}A\Big)\cdot\Big),
\end{aligned}
\tag{1}
$$

where $M$ is a Riemann surface of genus at least 2 with a hyperbolic metric $h_0$, $A$ is an endomorphism of $TM$ satisfying $\mathrm{div}_{h_0} A = 0$, and $\mathrm{Tr}(A) = -\frac{1}{2}\,\mathrm{scal}_{h_0}$. The metric $g$ on the funnel is of constant sectional curvature $-1$, and every end of a convex co-compact hyperbolic manifold $X$ is isometric to such a hyperbolic funnel; see Fefferman and Graham [5], and Krasnov and Schlenker [17]. A couple $(h_0, A_0)$ can be considered as an element of $T^*_{h_0}\mathcal{T}$, if $A_0 = A - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{Id}$ is the trace-free part of the divergence-free tensor $A$. We therefore identify the cotangent bundle $T^*\mathcal{T}$ of $\mathcal{T}$ with the set of hyperbolic funnels modulo the action of the group $\mathcal{D}_0(M)$, acting trivially in the $x$ variable. Let $x$ be any smooth positive function on $X$ that extends the function $x$ defined in each funnel by (1), and is equal to 1 in each cusp end. The renormalized volume of $(X,g)$ is defined by

$$\mathrm{Vol}_R(X) := \mathrm{FP}_{\epsilon\to 0}\int_{x>\epsilon} \mathrm{dvol}_g,$$

where FP means finite part (ie, the coefficient of $\epsilon^0$ in the asymptotic expansion as $\epsilon\to 0$).

The tangent bundle to any 3–manifold is trivial. If $\omega$ is the so(3)–valued Levi-Civita connection 1–form on $X$ in an oriented orthonormal frame $S=(S_1,S_2,S_3)$, we define

$$\mathrm{CS}(g,S) := -\frac{1}{16\pi^2}\,\mathrm{FP}_{\epsilon\to 0}\int_{x>\epsilon}\mathrm{Tr}\big(\omega\wedge d\omega + \tfrac{2}{3}\omega\wedge\omega\wedge\omega\big). \tag{2}$$

We ask that $S$ be even to the first order at $\{x=0\}$ and also that, in each cusp end, $S$ be parallel in the direction of the vector field pointing towards to cusp point. Equipped with the conformal metric $\hat{g} := x^2 g$, the manifold $X$ extends to a smooth Riemannian manifold $\overline{X} = X\cup M$ with boundary $M$. The Chern–Simons invariant

$\mathrm{CS}(\hat{g},\hat{S})$ is therefore well defined if $\hat{S}=x^{-1}S$ is an orthonormal frame for $\hat{g}$. We define the $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons invariant $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g,S)$ on $(X,g)$ by the renormalized integral (2) where we replace $\omega$ by the complex-valued connection form $\theta:=\omega+iT$; here $T$ is the so(3)–valued 1–form defined by $T_{ij}(V):=g(V\times S_j,S_i)$ and $\times$ is the vector product with respect to the metric $g$. There exists a natural flat connection on a $\mathrm{PSL}_2(\mathbb{C})$ principal bundle $F^{\mathbb{C}}(X)$ over $X$ (which can be seen as a complexified frame bundle), with $\mathrm{sl}_2(\mathbb{C})$–valued connection 1–form $\Theta$, and we show that $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g,S)$ also equals the renormalized integral of the pull-back of the Chern–Simons form $-(1/4\pi^2)\operatorname{Tr}(\Theta\wedge d\Theta+\frac{2}{3}\Theta^3)$ of the flat connection $\Theta$; see Section 3. We first show:

**Proposition 1** *On a geometrically finite hyperbolic 3–manifold $(X,g)$ without rank-1 cusps, one has $\mathrm{CS}(g,S)=\mathrm{CS}(\hat{g},\hat{S})$, and*

$$\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g,S)=-\frac{i}{2\pi^2}\operatorname{Vol}_R(X)+\frac{i}{4\pi}\chi(M)+\mathrm{CS}(g,S), \tag{3}$$

*where $\chi(M)$ is the Euler characteristic of the conformal boundary $M$.*

The relation between $\mathrm{CS}(g,S)$ and $\mathrm{CS}(\hat{g},\hat{S})$ comes rather easily from the conformal change formula in the Chern–Simons form (the boundary term turns out to not contribute), while (3) is a generalization of a formula in Yoshida [31], but we give an independent easy proof. Similar identities to (3) can be found in the physics literature (see for instance Krasnov [16]).

Like the function $F$ of Yoshida, it is natural to consider the variation of $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g,S)$ in the set of convex co-compact hyperbolic 3–manifolds, especially since, in contrast with the finite volume case, there is a finite dimensional deformation space of smooth hyperbolic 3–manifolds, which essentially coincides with the Teichmüller space of their conformal boundaries. One of the problems, related to the work of Ramadas, Singer and Weitsman [25] is that $e^{2\pi i\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g,S)}$ depends on the choice of the frame $S$, since $X$ is not closed. This leads us to define a complex line bundle $\mathcal{L}$ over Teichmüller space $\mathcal{T}$ of Riemann surfaces of a fixed genus, in which $e^{2\pi i\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ and $e^{2\pi i\,\mathrm{CS}}$ are sections.

Let $\mathcal{T}$ be the Teichmüller space of a (not necessarily connected) oriented Riemann surface $M$ of genus $\boldsymbol{g}=(g_1,\dots,g_N)$, $g_j\geq 2$, defined as the space of hyperbolic metrics on $M$ modulo the group $\mathcal{D}_0(M)$ of diffeomorphisms isotopic to the identity. This is a complex simply connected manifold of complex dimension $3|\boldsymbol{g}|-3$, equipped with a natural Kähler metric called the Weil–Petersson metric (see Section 7.1). The *mapping class group* Mod of isotopy classes of orientation-preserving diffeomorphisms of $M$ acts properly discontinuously on $\mathcal{T}$. Let $(X,g)$ be a geometrically finite hyperbolic 3–manifold without cusp of rank 1, with conformal boundary $M$. By Marden [18, Theorem 3.1], there is a smooth map $\Phi$ from $\mathcal{T}$ to the set of geometrically finite hyperbolic metrics on $X$ (up to diffeomorphisms of $X$ homotopic to identity) such that the conformal boundary of $\Phi(h)$ is $(M,h)$ for any $h \in \mathcal{T}$. The subgroup $\mathrm{Mod}_X$ of Mod consisting of elements that extend to diffeomorphisms on $\overline{X}$ homotopic to the identity acts freely, properly discontinuously on $\mathcal{T}$ and the quotient is a complex manifold of dimension $3|\boldsymbol{g}|-3$. The map $\Phi$ is invariant under the action of $\mathrm{Mod}_X$ and the deformation space $\mathcal{T}_X$ of $X$ is identified with a quotient of the Teichmüller space $\mathcal{T}_X = \mathcal{T}/\mathrm{Mod}_X$; see [18, Theorem 3.1].

**Theorem 2** *Let $(X,g)$ be a geometrically finite hyperbolic 3–manifold without rank-1 cusps, and with conformal boundary $M$. There exists a holomorphic Hermitian line bundle $\mathcal{L}$ over $\mathcal{T}$ equipped with a Hermitian connection $\nabla^{\mathcal{L}}$, with curvature given by $\frac{i}{8\pi}$ times the Weil–Petersson symplectic form $\omega_{\mathrm{WP}}$ on $\mathcal{T}$. The bundle $\mathcal{L}$ with its connection descend to $\mathcal{T}_X$ and if $g_h = \Phi(h)$ is the geometrically finite hyperbolic metric with conformal boundary $h \in \mathcal{T}$, then $h \to e^{2\pi i\,\mathrm{CS}(g_h,\cdot)}$ is a global section of $\mathcal{L}$.*

The line bundle is defined using the cocycle that appears in the Chern–Simons action under gauge transformations; this is explained in Section 7.3. We remark that the computation of the curvature of $\mathcal{L}$ reduces to the computation of the curvature of the vertical tangent bundle in a fibration related to the universal Teichmüller curve over $\mathcal{T}$, and we show that the fiberwise integral of the first Pontrjagin form of this bundle is given by the Weil–Petersson form, which is similar to a result of Wolpert [30]. An analogous line bundle, but in a more general setting, has been recently studied by Bunke [3].

Since funnels can be identified to elements in $T^*\mathcal{T}$, the map $\Phi$ described above induces a section $\sigma$ of the bundle $T^*\mathcal{T}$ (which descends to $T^*\mathcal{T}_X$) by assigning to $h \in \mathcal{T}$ the funnels of $\Phi(h)$. The image of $\sigma$

$$\mathcal{H} := \{\sigma(h) \in T^*\mathcal{T}_X, h \in \mathcal{T}_X\}$$

identifies the set of geometrically finite hyperbolic metrics on $X$ as a graph in $T^*\mathcal{T}_X$.

Let us still denote by $(\mathcal{L}, \nabla^{\mathcal{L}})$ the Chern–Simons line bundle pulled-back to $T^*\mathcal{T}$ by the projection $\pi\colon T^*\mathcal{T} \to \mathcal{T}$, and define a modified connection

$$\nabla^{\mu} := \nabla^{\mathcal{L}} + \frac{2}{\pi}\mu^{1,0} \tag{4}$$

on $\mathcal{L}$ over $T^*\mathcal{T}$, where $\mu^{1,0}$ is the $(1,0)$ part of the Liouville 1–form $\mu$ on $T^*\mathcal{T}$. As before, the connection descends to $T^*\mathcal{T}_X$, and notice that it is not Hermitian (since

$\mu^{1,0}$ is not purely imaginary) but $\nabla^\mu$ and $\nabla^{\mathcal{L}}$ induce the same holomorphic structure on $\mathcal{L}$.

By Theorem 2 and Proposition 1, $e^{2\pi i\,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ is a section of $\mathcal{L}$ on $\mathcal{T}_X$; its pull-back by $\pi$ also gives a section of $\mathcal{L}$ on $\mathcal{H}$, which we still denote $e^{2\pi i\,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$.

**Theorem 3** *Let $V \in T\mathcal{H}$ be a vector field tangent to $\mathcal{H}$. Then $\nabla^\mu_V e^{2\pi i\,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}} = 0$, ie, $\nabla^\mu$ is flat on $\mathcal{H} \subset T^*\mathcal{T}_X$.*

The curvature of $\nabla^\mu$ vanishes on $\mathcal{H}$ by Theorem 3 while the curvature of $\nabla^{\mathcal{L}}$ is $(i/8\pi)\omega_{\mathrm{WP}}$ (by Theorem 2). By considering the real and imaginary parts of these curvature identities, we obtain as a direct corollary:

**Corollary 4** *The manifold $\mathcal{H}$ is Lagrangian in $T^*\mathcal{T}_X$ for the Liouville symplectic form $\mu$ and $d\,\mathrm{Vol}_R = -\frac{1}{4}\mu$ on $\mathcal{H}$. The renormalized volume is a Kähler potential for the Weil–Petersson metric on $\mathcal{T}_X$:*

$$\bar\partial\partial(\mathrm{Vol}_R \circ \sigma) = \frac{i}{16}\,\omega_{\mathrm{WP}}$$

Our final result relates the Chern–Simons line bundle $\mathcal{L}$ to the Quillen determinant line bundle $\det\partial$ of $\partial$ on functions in the particular case of Schottky hyperbolic manifolds. If $M$ is a connected surface of genus $g \ge 2$, one can realize any complex structure on $M$ as a quotient of an open set $\Omega_\Gamma \subset \mathbb{C}$ by a Schottky group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$; using a marking $\alpha_1, \dots, \alpha_g$ of $\pi_1(M)$ and a certain normalization, there is complex manifold $\mathfrak{S}$, called the Schottky space, of such groups. This is isomorphic to $\mathcal{T}_X$, where $X := \Gamma\backslash\mathbb{H}^3$ is the solid torus bounding $M$ in which the curves $\alpha_j$ are contractible. The Chern–Simons line bundle $\mathcal{L}$ can then be defined on $\mathfrak{S}$. The Quillen determinant bundle $\det\partial$ is equipped with its Quillen metric and a natural holomorphic structure induced by $\mathfrak{S}$ (see Section 9.2), therefore inducing a Hermitian connection compatible with the holomorphic structure. Moreover, there is a canonical section of $\det\partial = \Lambda^g(\operatorname{coker}\partial)$ given by $\varphi := \varphi_1 \wedge \cdots \wedge \varphi_g$, where the $\varphi_j$ are holomorphic $1$–forms on $M$ normalized by the marking through the requirement $\int_{\alpha_j} \varphi_k = \delta_{jk}$. Using a formula of Zograf [32], we show:

**Theorem 5** *There is an explicit isometric isomorphism of holomorphic Hermitian line bundles between the inverse $\mathcal{L}^{-1}$ of the Chern–Simons line bundle and the $6^{\text{th}}$ power $(\det\partial)^{\otimes 6}$ of the determinant line bundle $\det\partial$, given by*

$$(F\varphi)^{\otimes 6} \mapsto e^{-2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}.$$

*Here $\varphi$ is the canonical section of* $\det \bar\partial$ *defined above,* $c_{\boldsymbol{g}}$ *is a constant, and* $F$ *is a holomorphic function on* $\mathfrak{S}$ *that is given, on the open set where the product converges absolutely, by*

$$F(\Gamma) = c_{\boldsymbol{g}} \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_\gamma^{1+m}),$$

*where* $q_\gamma$ *is the multiplier of* $\gamma \in \Gamma$*, and* $\{\gamma\}$ *runs over all distinct primitive conjugacy classes in* $\Gamma \in \mathfrak{S}$ *except the identity.*

## Novelties and perspectives

Our main contribution in this work is to introduce the Chern–Simons theory and its line bundle over Teichmüller space in relation with Kleinian groups. The strength of this construction appears through a variety of applications to Teichmüller theory in essentially a general setting, all at once and self-contained. For example, the property of the renormalized volume of being a Kähler potential for the Weil–Petersson metric, previously known in the particular cases of Schottky and quasi-Fuchsian groups [15; 27; 28; 17], follows directly from the Chern–Simons approach for all geometrically finite Kleinian groups without cusps of rank 1 (for instance, the proof in [17] is based on an explicit computation at the Fuchsian locus and does not seem to be extendable to general groups). In fact, finding Kähler potentials for the Weil–Petersson metric and canonical holomorphic sections of the line bundle associated to Kleinian cobordisms of Bers and Marden is not only a generalisation of the quasi-Fuchsian and Schottky cases. Indeed, the Chern–Simons bundle $\mathcal{L}$ is a prequantum line bundle and the canonical holomorphic sections $e^{2\pi i \,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ (or large powers of it) could be used as quantizations of these cobordisms. We hope to pursue this idea of geometric quantization of $\mathcal{T}$ elsewhere.

The existence of a non-explicit isomorphism between the Chern–Simons bundle on the (compact) moduli space of $\mathrm{SU}(2)$ flat connections and the determinant line bundle was discovered in [25]. In contrast, in our non-compact $\mathrm{PSL}_2(\mathbb{C})$ setting we find an *explicit* isomorphism, involving a formula of Zograf on Schottky space, which as far as we know is the first of its kind; it would be natural to generalize it to all convex co-compact groups.

More generally, we expect that the results of this paper extend to all geometrically finite hyperbolic 3–manifolds but several technical difficulties appear when we perform our analysis to cusps of rank 1.

**Organization of the paper** In order to simplify the presentation as much as possible, we discuss the case of convex co-compact manifolds in the main body of the paper, and add an appendix including the case of rank-2 cusps. In several parts of the paper, we also consider the more general setting of asymptotically hyperbolic manifolds, which only have asymptotic constant curvature near infinity. The paper splits in two parts: In Sections 2–6, we introduce Chern–Simons invariants associated to the Levi-Civita connection and to a certain complexification thereof on asymptotically hyperbolic 3–manifolds with totally geodesic boundary, and we study their relationship with the renormalized volume in the case of convex co-compact hyperbolic metrics. In the second part, Sections 7–9, we define the Chern–Simons line bundle over $\mathcal{T}$, its connections, we compute the variation of the Chern–Simons invariants, and derive the implications on the Weil–Petersson metric and on the determinant line bundle.

# 2 Asymptotically hyperbolic manifolds

Let $(X, g)$ be an 3–dimensional *asymptotically hyperbolic* manifold, ie, $X$ is the interior of a compact smooth 3–manifold with boundary $\overline{X}$, and there exists a smooth boundary-defining function $x$ such that near the boundary $\{x = 0\}$ the Riemannian metric $g$ has the form

$$g = \frac{dx^2 + h(x)}{x^2}$$

in a product decomposition $[0, \epsilon)_x \times M \hookrightarrow \overline{X}$ near the boundary $M = \partial\overline{X}$, for some smooth one-parameter family $h(x)$ of metrics on $M$. A boundary-defining function $x$ inducing this product decomposition satisfies $|dx|_{x^2 g} = 1$ near $\partial\overline{X}$, and is called a *geodesic boundary defining function*. When $\partial_x h(x)_{|x=0} = 0$, the boundary $M$ is totally geodesic for the metric $\hat{g} := x^2 g$ and we shall say that $g$ has *totally geodesic boundary*. This condition is shown in Guillarmou [9] to be invariant with respect to the choice of $x$. Examples of asymptotically hyperbolic manifolds with totally geodesic boundary are the hyperbolic space $\mathbb{H}^3$, or more generally convex co-compact hyperbolic manifolds (cf (6)). The *conformal boundary* of $(X, g)$ is the compact manifold $M = \partial\overline{X}$ equipped with the conformal class $\{h_0\}$ of $h_0 := h(0) = x^2 g|_{TM}$.

## 2.1 Convex co-compact hyperbolic quotients

Let $X$ be an oriented complete hyperbolic 3–manifold, equipped with its constant curvature metric $g$. The universal cover $\widetilde{X}$ is isometric to the 3–dimensional hyperbolic space $\mathbb{H}^3$, and the deck transformation group is conjugated via this isometry to a Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ (we recall below that $\mathrm{PSL}_2(\mathbb{C})$ can be viewed as the group of orientation-preserving isometries of $\mathbb{H}^3$). In this way we get a representation of the fundamental group

$$\rho\colon \pi_1(X) \to \mathrm{PSL}_2(\mathbb{C}) \tag{5}$$

with image $\Gamma$, well-defined up to conjugation.

We say that $X$ is *convex co-compact hyperbolic* if it is isometric to $\Gamma\backslash\mathbb{H}^3$ for some discrete group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ with no elliptic, nor parabolic transformations, such that $\Gamma$ admits a fundamental domain in $\mathbb{H}^3$ with a finite number of sides. Then the manifold $X$ has a smooth compactification into a manifold $\overline{X}$, with boundary $M$ that is a disjoint union of compact Riemann surfaces. The boundary can be realized as the quotient $\Gamma\backslash\Omega(\Gamma)$, where $\Omega(\Gamma) \subset S^2$ is the domain of discontinuity of the convex co-compact subgroup $\Gamma$, acting as conformal transformations on the sphere $S^2$. Each connected component of $M$ has a projective structure induced by the group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$. It is proved in [5; 17] that the constant sectional curvature condition implies the following structure for the metric near infinity: there exists a product decomposition $[0,\epsilon)_x \times M$ of $\overline{X}$ near $M$, induced by the choice of a geodesic boundary-defining function $x$ of $M$, a metric $h_0$ on $M$ and a symmetric endomorphism $A$ of $TM$ such that the metric $g$ is of the form

$$g = \frac{dx^2 + h(x)}{x^2}, \quad h(x) = h_0\Big(\Big(1 + \frac{x^2}{2}A\Big)\cdot, \Big(1 + \frac{x^2}{2}A\Big)\cdot\Big), \tag{6}$$

and moreover $A$ satisfies

$$\mathrm{Tr}(A) = -\tfrac{1}{2}\,\mathrm{scal}_{h_0}, \quad d^{\nabla^*}A = 0. \tag{7}$$

## 2.2 Tangent, cotangent and frame bundles

There exists a smooth vector bundle over $\overline{X}$ spanned over $\mathcal{C}^\infty(\overline{X})$ by smooth vector fields vanishing on the boundary $\partial\overline{X}$ (those are locally spanned near $\partial\overline{X}$ by $x\partial_x$ and $x\partial_y$ if $x$ is a boundary defining function and $y_1, y_2$ are coordinates on the boundary), we denote this bundle by ${}^0T\overline{X}$. Its dual is denoted ${}^0T^*\overline{X}$ and is locally spanned over $\mathcal{C}^\infty(\overline{X})$ by the forms $dx/x$ and $dy/x$. An asymptotically hyperbolic metric can be also defined to be a smooth section of the bundle of positive definite symmetric tensors $S^2_+({}^0T^*\overline{X})$ such that $|dx/x|_g = 1$ at $\partial\overline{X}$. The (orthonormal) frame bundle $F_0(X)$ for an asymptotically hyperbolic metric $g$ is a SO(3)–principal bundle and its sections are triples of smooth $g$–orthonormal vector fields in ${}^0T\overline{X}$. It is clearly canonically isomorphic to the (orthonormal) frame bundle $F(X)$ of the compactified metric $\hat{g} := x^2 g$ if $x$ is a boundary defining function. A smooth frame $S \in F_0(X)$ is said to be *even to first order* if, in local coordinates $(y_1, y_2, x)$ near $\partial\overline{X}$ induced by any geodesic defining function $x$, the vector fields forming $S$ are of the form $x(u_1\partial_{y_1} + u_2\partial_{y_2} + u_3\partial_x)$, where the $u_j$ are such that $\partial_x u_j|_M = 0$, or equivalently $[\partial_x, \hat{S}]|_M = 0$ if $\hat{S} := x^{-1}S$ is the related frame for $\hat{g} := x^2 g$. In general, we refer the reader to Mazzeo and Melrose [20], and Mazzeo [19] for more details about the 0–structures and bundles.

### 2.3 Orientation convention

For an oriented asymptotically hyperbolic manifold, the orientation of the boundary at infinity $M$ is defined by the requirement that $(\partial_x, Y_1, Y_2)$ is a positive frame on $X$ if and only if $(Y_1, Y_2)$ is a positive frame on $M$. With this convention, Stokes' formula gives

$$\int_{\overline{X}} d\alpha = -\int_M \alpha$$

for every $\alpha \in \mathcal{C}^\infty(\overline{X}, \Lambda^2 \overline{X})$.

### 2.4 Renormalized integrals

Let $\omega \in x^{-N}\mathcal{C}^\infty(\overline{X}, \Lambda^3\overline{X}) + \mathcal{C}^\infty(\overline{X}, \Lambda^3\overline{X})$ for some $N \in \mathbb{R}^+$. The 0–integral (or renormalized integral) of $\omega$ on $X$ is defined by

$$\int_X^0 w := \mathrm{FP}_{\epsilon\to 0} \int_{x>\epsilon} \omega,$$

where FP denotes the finite part, ie, the coefficient of $\epsilon^0$ in the expansion of the integral at $\epsilon = 0$. This is independent of the choice of function $x$ when $N$ is not an integer or $N > -1$ but it depends a priori on the choice of $x$ when $N$ is a negative integer. In the present paper, we shall always fix the geodesic boundary defining function $x$ so that the induced metric $h_0 = x^2 g|_{TM}$ is the unique hyperbolic metric in its conformal class. More generally, one can define renormalized integrals of polyhomogeneous forms but this will not be used here. We refer the reader to Albin [1] and Guillarmou [10] for detailed discussions on this topic. An example which has been introduced by Henningson and Skenderis [12] and Graham [8] for asymptotically hyperbolic Einstein manifolds is the renormalized volume defined by

$$\mathrm{Vol}_R(X) := \int_X^0 \mathrm{dvol}_g,$$

where $\mathrm{dvol}_g$ is the volume form on $(X, g)$.

## 3 The bundle of infinitesimal Killing vector fields for hyperbolic manifolds

The hyperbolic 3–space $\mathbb{H}^3$ can be viewed as a subset of quaternions

$$\mathbb{H}^3 \simeq \{y_1 + iy_2 + y_3 j : y_3 > 0, y_1, y_2 \in \mathbb{R}\}, \quad g_{\mathbb{H}^3} = \frac{dy^2}{y_3^2}.$$

The action of

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C})$$

on $\zeta = y_1 + i y_2 + y_3 j \in \mathbb{H}^3$ is given by $\gamma.\zeta = (a\zeta + b)(c\zeta + d)^{-1}$. This action identifies $\mathrm{PSL}_2(\mathbb{C})$ with the group of oriented isometries of $\mathbb{H}^3$, which is diffeomorphic to the frame bundle $F(\mathbb{H}^3) = \mathbb{H}^3 \times \mathrm{SO}(3)$ of $\mathbb{H}^3$ via the map

$$\Phi\colon \mathrm{PSL}_2(\mathbb{C}) \to F(\mathbb{H}^3), \quad \gamma \mapsto (\gamma.j, \gamma_*(\partial_{y_1}, \partial_{y_2}, \partial_{y_3})).$$

There exists a natural embedding

$$\tilde{q}\colon F(\mathbb{H}^3) \to \mathbb{H}^3 \times \mathrm{PSL}_2(\mathbb{C}), \quad (m, V_m) \mapsto (m, \Phi^{-1}(m, V_m)), \tag{8}$$

which is equivariant with respect to the right action of $\mathrm{SO}(3)$. If $X = \Gamma\backslash\mathbb{H}^3$ is an oriented hyperbolic quotient, $\tilde{q}$ descends to a bundle map

$$q\colon F(X) \to \mathbb{H}^3 \times_\Gamma \mathrm{PSL}_2(\mathbb{C}) =: F^{\mathbb{C}}(X), \tag{9}$$

where $F^{\mathbb{C}}(X)$ is a principal bundle over $X$ with fiber $\mathrm{PSL}_2(\mathbb{C})$. The trivial flat connection on the product $\mathbb{H}^3 \times \mathrm{PSL}_2(\mathbb{C})$ also descends to a flat connection on $F^{\mathbb{C}}(X)$, denoted $\theta$ (ie, an $\mathrm{sl}_2(\mathbb{C})$–valued 1–form on $F^{\mathbb{C}}(X)$), with holonomy representation conjugated to $\rho$, where $\rho$ is defined in (5).

Let $\nabla$ be the Levi-Civita connection on $TX$ with respect to the hyperbolic metric $g$, and let

$$T \in \Lambda^1(X, \mathrm{End}(TX)), \quad T_V W := -V \times W, \tag{10}$$

where $\times$ is the vector product with respect to the metric $g$.

**Proposition 6** *The vector bundle $E(X)$ associated to the principal bundle $F^{\mathbb{C}}(X)$ with respect to the adjoint representation is isomorphic, as a complex bundle, to the complexified tangent bundle $T_{\mathbb{C}}X$. The connection induced by $\theta$ is $D := \nabla + iT$.*

**Proof** The associated bundle with respect to the adjoint representation is given by

$$E(X) = \mathbb{H}^3 \times_\Gamma \mathrm{PSL}_2(\mathbb{C}) \times_{\mathrm{PSL}_2(\mathbb{C})} \mathrm{sl}(2,\mathbb{C}) = \mathbb{H}^3 \times_\Gamma \mathrm{sl}(2,\mathbb{C}) = (\mathbb{H}^3 \times \mathrm{sl}_2(\mathbb{C}))/\sim,$$

where the equivalence relation is $[m, h] \sim [\gamma m, \gamma h \gamma^{-1}]$ for all $\gamma \in \Gamma$. We also have $TX = \Gamma\backslash T\mathbb{H}^3$, where the action of $\mathrm{PSL}_2(\mathbb{C})$ on $T\mathbb{H}^3$ is given by $\gamma.(m, v_m) := (\gamma m, \gamma_*(v_m))$. For every vector field $u$ on $X$ define its *canonical lift* $s_u$ to $T_{\mathbb{C}}X$ by

$$s_u := u + \frac{i}{2}\,\mathrm{curl}(u),$$

where we recall that the curl operator corresponds to $*d$ when identifying vector fields and 1–forms through the metric. The map $u \mapsto s_u$ is thus a first-order differential operator. Note that the sign in front of curl is different from the one used in Hodgson and Kerckhoff [13]. For every $h \in \mathrm{sl}_2(\mathbb{C})$ let $\kappa_h$ be the Killing field on $\mathbb{H}^3$ corresponding to the infinitesimal isometry $h$.

**Lemma 7** *We have* $\kappa_{ih} = -\frac{1}{2}\operatorname{curl}(\kappa_h)$, *thus* $s_{\kappa_h} = \kappa_h - i\kappa_{ih}$ *and* $s_{\kappa_{ih}} = i s_{\kappa_h}$.

**Proof** We proceed by direct verification on a basis of $\mathrm{sl}_2(\mathbb{C})$, using the explicit formula for $\kappa_h$ at $q \in \mathbb{H}^3$:

$$\kappa_h = b + aq + qa - qcq, \quad \text{where } h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{sl}_2(\mathbb{C})$$

and $T_q\mathbb{H}^3$ was identified with $\mathbb{C} \oplus j\mathbb{R}$. □

Define a vector bundle morphism

$$\Psi\colon \mathcal{C}^\infty(\mathbb{H}^3, E(\mathbb{H}^3)) \to \mathcal{C}^\infty(\mathbb{H}^3, T_{\mathbb{C}}\mathbb{H}^3),$$
$$(m,h) \mapsto s_{\kappa_h}(m) = \kappa_h(m) + \tfrac{i}{2}\operatorname{curl}(\kappa_h)(m).$$

This map is injective, for a Killing field that vanishes at a point together with its curl must vanish identically. By dimensional reasons, $\Psi$ must be a bundle isomorphism. Moreover, $\Psi$ is $\mathrm{PSL}_2(\mathbb{C})$–equivariant in the sense that for all $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ we have $\Psi(\gamma m, \mathrm{ad}_\gamma h) = \gamma_* \Psi(m,h)$ (this is clear for the real part by definition, while for the imaginary part we use the fact that $\gamma$ is an isometry to commute it across curl), hence $\Psi$ descends to the $\Gamma$–quotient as an isomorphism $\mathcal{C}^\infty(X, E(X)) \to \mathcal{C}^\infty(X, T_{\mathbb{C}}X)$. By Lemma 7, this isomorphism is compatible with the complex structures. It remains to identify the push-forward $D$ of the flat connection from $E(X)$ to $T_{\mathbb{C}}X$ under this map. It is enough to prove that on $\mathbb{H}^3$ we have $D = \nabla + iT$, since both terms are $\mathrm{PSL}_2(\mathbb{C})$–invariant.

**Lemma 8** *Let* $\kappa$ *be a Killing vector field on an oriented* 3*–manifold of constant sectional curvature* $\epsilon$. *Then the field* $v := -\frac{1}{2}\operatorname{curl}(\kappa)$ *is also Killing, and satisfies, for every vector* $U$,

$$\nabla_U \kappa = U \times v, \quad \nabla_U v = \epsilon U \times \kappa.$$

**Proof** Directly from the Koszul formula, the Levi-Civita covariant derivative of a Killing vector field $\kappa$ satisfies $\langle \nabla_U \kappa, V\rangle = \frac{1}{2} d\kappa(U,V)$ (identifying $\kappa$ with a 1–form), which in dimension 3 implies

$$\nabla_U \kappa = -\tfrac{1}{2} U \times \operatorname{curl}(\kappa) = U \times v.$$

Now let $(U_1, U_2, U_3)$ be a radially parallel orthonormal frame near a point $p$, so that $\nabla_{U_i} U_j = 0$ at $p$. On one hand, by assumption on the sectional curvatures, one has $\langle R_{U_1 U_2}\kappa, U_3\rangle = 0$, where $R$ is the curvature tensor of the metric. On the other hand, at the point $p$ we have

$$\begin{aligned}\langle R_{U_1 U_2}\kappa, U_3\rangle &= U_1\langle \nabla_{U_2}\kappa, U_3\rangle - U_2\langle \nabla_{U_1}\kappa, U_3\rangle \\ &= U_1\langle v \times U_2, U_3\rangle - U_2\langle v \times U_1, U_3\rangle \\ &= \langle \nabla_{U_1} v, U_1\rangle + \langle \nabla_{U_2} v, U_2\rangle.\end{aligned}$$

Similarly, $\langle \nabla_{U_2} v, U_2\rangle + \langle \nabla_{U_3} v, U_3\rangle = 0$ and $\langle \nabla_{U_3} v, U_3\rangle + \langle \nabla_{U_1} v, U_1\rangle = 0$ so we deduce that $\langle \nabla_{U_j} v, U_j\rangle = 0$ at $p$. So $\nabla v$ is skew-symmetric at the (arbitrary) point $p$, or equivalently $v$ is Killing.

Since $\nabla_{U_i}\kappa = U_i \times v$ we see that

$$\begin{aligned} v = \sum \langle v, U_i\rangle U_i &= \langle v, U_2 \times U_3\rangle U_1 + \langle v, U_3 \times U_1\rangle U_2 + \langle v, U_1 \times U_2\rangle U_3 \\ &= \langle U_3 \times v, U_2\rangle U_1 + \langle U_1 \times v, U_3\rangle U_2 + \langle U_2 \times v, U_1\rangle U_3 \\ &= \langle \nabla_{U_3}\kappa, U_2\rangle U_1 + \langle \nabla_{U_1}\kappa, U_3\rangle U_2 + \langle \nabla_{U_2}\kappa, U_1\rangle U_3,\end{aligned}$$

hence at the point $p$ where $U_j$ are parallel and commute, using $\langle \nabla_{U_2}\kappa, U_2\rangle = 0$,

$$\langle \nabla_{U_2} v, U_1\rangle = \langle \nabla_{U_2}\nabla_{U_3}\kappa, U_2\rangle = \langle R_{U_2 U_3}\kappa, U_2\rangle = \epsilon\langle \kappa, U_3\rangle = \epsilon\langle U_2 \times \kappa, U_1\rangle.$$

Similarly, $\langle \nabla_{U_2} v, U_3\rangle = \epsilon\langle U_2 \times \kappa, U_3\rangle$. Together with $\langle \nabla_{U_2} v, U_2\rangle = 0$ proved above, we deduce $\nabla_{U_2} v = U_2 \times \kappa$. This identity clearly holds for any $U$ in place of $U_2$. $\square$

For every $h \in \mathrm{sl}_2(\mathbb{C})$, the section $S_h\colon m \mapsto (m, h)$ is by definition a flat section in $E(\mathbb{H}^3)$, so (again by definition) $D_U\Psi(S_h) = 0$ for every vector $U$. Using the above lemma, we also have $(\nabla_U + iT(U))(\kappa_h + \frac{i}{2}\operatorname{curl}(\kappa_h)) = 0$. Thus the connections $\nabla + iT$ and $D$ have the same parallel (generating) sections, hence they coincide. $\square$

## 4 Chern–Simons forms and invariants

Let $Z$ be a manifold, $n \in \mathbb{N}^*$ and $\theta \in \Lambda^1(Z, M_n(\mathbb{C}))$ a matrix-valued 1–form, and set $\Omega := d\theta + \theta \wedge \theta$. Define

$$\mathrm{cs}(\theta) := \mathrm{Tr}(\theta \wedge d\theta + \tfrac{2}{3}\theta^3) = \mathrm{Tr}(\theta \wedge \Omega - \tfrac{1}{3}\theta^3).$$

(Notation: If $\alpha_j$ are $M_n(\mathbb{C})$–valued forms of degree $d_j$ on $Z$, $j = 1, \dots, k$, their exterior product is defined by its action on vectors $V_1, \dots, V_N$, $N := \sum_{j=1}^k d_j$ as follows:

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(V_1, \ldots, V_N) := \frac{1}{d_1! \cdots d_k!} \sum_{\sigma \in \Sigma(N)} \epsilon(\sigma) \alpha_1(V_{\sigma(1)}, \ldots, V_{\sigma(d_1)}) \ldots \alpha_k(V_{\sigma(N-d_k+1)}, \ldots, V_{\sigma(N)}),$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma$.

## 4.1 Properties of Chern–Simons forms

**Relation to Chern–Weil forms** An easy computation shows that

$$d(\mathrm{cs}(\theta)) = \mathrm{Tr}(\Omega \wedge \Omega).$$

**Variation** If $\theta^t$ is a 1–parameter family of 1–forms and $\dot{\theta} = \partial_t \theta^t|_{t=0}$, the variation of cs is computed using the trace identity:

$$\partial_t \,\mathrm{cs}(\theta^t)|_{t=0} = \mathrm{Tr}(\dot{\theta} \wedge d\theta + \theta \wedge d\dot{\theta} + 2\dot{\theta} \wedge \theta^2) = d\,\mathrm{Tr}(\dot{\theta} \wedge \theta) + 2\,\mathrm{Tr}(\dot{\theta} \wedge \Omega) \tag{11}$$

**Pull-back** If $\Phi\colon Z' \to Z$ is a smooth map, we have $\mathrm{cs}(\Phi^*\theta) = \Phi^* \mathrm{cs}(\theta)$.

**Action of representations** If $\theta$ takes values in a linear Lie algebra $\mathfrak{g} \subset M_n(\mathbb{C})$ and $\rho$ is a representation of $\mathfrak{g}$ in $M_m(\mathbb{C})$ such that there exists some $\mu_\rho \in \mathbb{C}$ with $\mathrm{Tr}(\rho(a)\rho(b)) = \mu\,\mathrm{Tr}(ab)$ for every $a, b \in \mathfrak{g}$, then

$$\mathrm{cs}(\rho(\theta)) = \mu_\rho\,\mathrm{cs}(\theta). \tag{12}$$

**Gauge transformation** If $a\colon Z \to M_n(\mathbb{C})$ is a smooth map and $\gamma := a^{-1}\theta a + a^{-1} da$, then

$$\mathrm{cs}(\gamma) = \mathrm{cs}(\theta) + d\,\mathrm{Tr}(\theta \wedge daa^{-1}) - \tfrac{1}{3}\mathrm{Tr}((a^{-1}da)^3). \tag{13}$$

The particular cases of $\theta$ considered below are connection 1–forms either in a principal bundle or in a trivialization of a vector bundle.

## 4.2 The $\mathrm{PSL}_2(\mathbb{C})$ invariant

Let $\theta$ be the flat connection in the $\mathrm{PSL}_2(\mathbb{C})$ bundle $F^{\mathbb{C}}(X)$ of an oriented hyperbolic 3–manifold $X$, as defined in the previous section. Let $S\colon X \to F(X)$ be a section in the orthonormal frame bundle (recall that oriented 3–manifolds are parallelizable), so $q \circ S$ is a section in $F^{\mathbb{C}}(X)$, where $q$ is the natural map from $F(X)$ to $F^{\mathbb{C}}(X)$

defined in (9). The $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons form $\mathrm{cs}(\theta, S)$ is by definition the complex valued 3–form $(q \circ S)^* \mathrm{cs}(\theta)$ on $X$.

For $X$ compact, the $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons invariant of $\theta$ with respect to $S$ is defined by

$$\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(\theta, S) := -\frac{1}{4\pi^2} \int_X \mathrm{cs}(\theta, S).$$

The normalization coefficient in front of $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}$ is so chosen because

$$-\frac{1}{4\pi^2} \int_K \mathrm{Tr}(-\tfrac{1}{3}(\omega_{MC})^3) = 1,$$

where $\omega_{MC}$ is the $\mathrm{sl}_2(\mathbb{C})$–valued Maurer–Cartan 1–form on $\mathrm{PSL}_2(\mathbb{C})$, and $K$ is the (compact) stabilizer of $j \in \mathbb{H}^3$. Because $\mathrm{PSL}_2(\mathbb{C}) = K \times \mathbb{H}^3$ is homotopy equivalent to $K$, this identity implies that $\mathrm{Tr}((\omega_{\mathrm{MC}})^3)/12\pi^2$ is an integer cohomology class on $\mathrm{PSL}_2(\mathbb{C})$. Thus for $X$ closed, (13) implies that $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(\theta, S)$ is independent of $S$ modulo $\mathbb{Z}$.

**Definition 9** Let $(X, g)$ be a convex co-compact hyperbolic 3–manifold and $S$ be a section in the orthonormal frame bundle which is even to first order. The $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons invariant of $g$ with respect to $S$ is defined by

$$\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g, S) := -\frac{1}{4\pi^2} \int_X^0 \mathrm{cs}(\theta, S) = -\frac{1}{4\pi^2} \mathrm{FP}_{\epsilon \to 0} \int_{x > \epsilon} (q \circ S)^* \mathrm{cs}(\theta),$$

where $\theta$ is the flat connection in the $\mathrm{PSL}_2(\mathbb{C})$ bundle $F^{\mathbb{C}}(X)$ induced by $g$.

This invariant is our main object of study in the present paper.

We can express $\mathrm{cs}((q \circ S)^* \theta)$ in terms of the Riemannian connection of $g$ as follows: Let

$$h_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad h_2 := \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad h_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

be a complex basis in $\mathrm{sl}_2(\mathbb{C})$. The corresponding Killing vector fields on $\mathbb{H}^3$ evaluated at $j$ take the values

$$\kappa_{h_k} = 2\partial_{y_k}, \quad \kappa_{ih_k} = 0$$

for $k = 1, 2, 3$. If $U_k$ is the section over $X$ in the bundle $F^{\mathbb{C}}(X) \times_{\mathrm{ad}} \mathrm{sl}_2(\mathbb{C})$ corresponding to the vector $h_k$ in the trivialization $q \circ S$, the above relations show that the complex vector field corresponding to $U_k$ by the isomorphism from Proposition 6 is just $2S_k$. Thus

$$\mathrm{ad}((q \circ S)^* \theta) = \omega + iT,$$

where $\omega$ is the so(3)–valued connection 1–form of the Levi-Civita covariant derivative $\nabla$ in the frame $S$, and $T$ denotes the so(3) valued 1–form $T$ from Equation (10) in the basis $S$:

$$\omega_{ij}(Y) := g(\nabla_Y S_j, S_i), \quad T_{ij}(Y) := g(Y \times S_j, S_i). \tag{14}$$

**Lemma 10** *The* $\mathrm{PSL}_2(\mathbb{C})$ *Chern–Simons form of a hyperbolic metric on a* 3*–manifold satisfies*

$$\mathrm{cs}((q \circ S)^*\theta) = \tfrac{1}{4}\,\mathrm{cs}(\omega + iT).$$

**Proof** We use the following identity, valid for every $u, v \in \mathrm{sl}_2(\mathbb{C})$:

$$\mathrm{Tr}^{\mathrm{M}_2(\mathbb{C})}(uv) = \tfrac{1}{4}\,\mathrm{Tr}^{\mathrm{M}_3(\mathbb{C})}(\mathrm{ad}_u \mathrm{ad}_v)$$

The lemma follows from the above discussion and Equation (12). □

By this lemma, the $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons form of $g$ in the trivialization $S$ is given by $\mathrm{cs}(\omega + iT)$; thus, the $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons invariant is also given by

$$\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g, S) = -\frac{1}{16\pi^2}\int_X^0 \mathrm{cs}(\omega + iT).$$

**Proposition 11** *The* $\mathrm{PSL}_2(\mathbb{C})$ *Chern–Simons form on a hyperbolic* 3*–manifold, pulled back by a section* $S \in F_0(X)$*, has the following real and imaginary parts:*

$$\mathrm{cs}(\theta, S) = 2i\,\mathrm{dvol}_g + \tfrac{i}{4} d(\mathrm{Tr}(T \wedge \omega)) + \tfrac{1}{4}\,\mathrm{Tr}(\omega \wedge d\omega + \tfrac{2}{3}\omega^3)$$

**Proof** Since the connection $D$ is flat, it follows that $\mathrm{cs}(\omega + iT) = -\frac{1}{3}\,\mathrm{Tr}((\omega + iT)^3)$. By the above lemma we can write (using the cyclicity of the trace)

$$\begin{aligned} -12\,\mathrm{cs}(\theta, S) &= \mathrm{Tr}^{\mathrm{M}_3(\mathbb{C})}(\omega + iT)^3) \\ &= \big(\mathrm{Tr}^{\mathrm{M}_3(\mathbb{C})}(\omega^3 - 3T^2 \wedge \omega) + i\,\mathrm{Tr}(3\omega^2 \wedge T - T^3)\big). \end{aligned} \tag{15}$$

The vanishing of the $D$–curvature implies $d\omega + \omega^2 - T^2 = 0$ and $dT + T \wedge \omega + \omega \wedge T = 0$. Taking the exterior product of the first identity with $\omega$ and taking the trace, we deduce that

$$\mathrm{Tr}(\omega^3 - 3T^2 \wedge \omega) = -3\,\mathrm{Tr}(\omega \wedge d\omega + \tfrac{2}{3}\omega^3). \tag{16}$$

Similarly, since both $T \wedge (d\omega + \omega^2 - T^2)$ and $\omega \wedge (dT + T \wedge \omega + \omega \wedge T)$ are $0$, one can take their trace and take the difference to deduce

$$0 = \mathrm{Tr}(T \wedge d\omega - dT \wedge \omega - T^3 - \omega^2 \wedge T) = -d(\mathrm{Tr}(T \wedge \omega)) - \mathrm{Tr}(T^3) - \mathrm{Tr}(\omega^2 \wedge T),$$

and then

$$\mathrm{Tr}(3\omega^2 \wedge T - T^3) = -4\,\mathrm{Tr}(T^3) - 3d(\mathrm{Tr}(T \wedge \omega)). \tag{17}$$

We also easily see that $\mathrm{Tr}(T^3) = 6\,\mathrm{dvol}_g$ and therefore combining (17), (16) and (15), we have proved the proposition. □

## 4.3 The SO(3) Chern–Simons invariant

Let $(X, g)$ be an oriented Riemannian manifold of dimension $3$. Denote by $\mathrm{cs}(g)$ the Chern–Simons form of the Levi-Civita connection $1$–form of $g$ on the orthonormal frame bundle $F(X)$. Note that $S^* \mathrm{cs}(g) = \mathrm{cs}(\omega)$, where $\omega$ is the connection $1$–form in the trivialization $S$. Again, recall that oriented $3$–manifolds are parallelizable, thus orthonormal frames $S$ do exist.

If $X$ is compact, for every section $S\colon X \to F(X)$ in the orthonormal frame bundle we define the Chern–Simons invariant

$$\mathrm{CS}(g, S) := -\frac{1}{16\pi^2} \int_X S^* \mathrm{cs}(g).$$

When $a\colon X \to \mathrm{SO}(3)$ is a compactly supported map (ie, $a \cong 1$ outside a compact), we have

$$\frac{1}{16\pi^2} \int_X \mathrm{Tr}(-\tfrac{1}{3}(a^{-1}da)^3) = -\deg(a) \in \mathbb{Z}, \tag{18}$$

so when $X$ is closed, using (13) we see that the Chern–Simons invariant is independent of $S$ modulo $\mathbb{Z}$.

We aim to define a SO(3) Chern–Simons invariant associated to the Levi-Civita connection $\nabla^g$ on an asymptotically hyperbolic $3$–manifold $(X, g)$ with totally geodesic boundary. First, we fix a geodesic boundary defining function $x$ and we set $\hat{g} := x^2 g$. Let $\hat{S}\colon X \to F(X, \hat{g})$ be a smooth section of the orthonormal frame bundle associated to the metric $\hat{g}$. We say that $\hat{S}$ is *even to first order* if $\mathcal{L}_{\partial_x} \hat{S}|_M = 0$, where $\mathcal{L}$ denotes the Lie derivative; note that this coincides with the local definition from Section 2.2. We define, starting from $\hat{S}$, a section $S := x\hat{S}$ in the frame bundle $F_0(X)$ associated to $g$.

**Definition 12** The SO(3) Chern–Simons invariant of an asymptotically hyperbolic metric $g$ with totally geodesic boundary, with respect to an even to first order trivialization $S$ of $F_0(X)$, is

$$\mathrm{CS}(g, S) := -\frac{1}{16\pi^2} \int_X^0 S^* \mathrm{cs}(g).$$

# 5 Comparison between the asymptotically hyperbolic and the compact Chern–Simons SO(3) invariants

For any pair of conformal metrics $\widehat{g} = x^2 g$, we can relate $\mathrm{cs}(g, S)$ to $\mathrm{cs}(\widehat{g}, \widehat{S})$ as follows. We denote by $\omega, \widehat{\omega}$ the connection 1–forms of $g, \widehat{g}$ in the trivialization $S$, respectively $\widehat{S} = x^{-1} S$. For every $Y \in TX$ that means

$$\widehat{\omega}_{ij}(Y) = \widehat{g}(\nabla^{\widehat{g}}_Y \widehat{S}_j, \widehat{S}_i), \quad \omega_{ij}(Y) = g(\nabla^g_Y S_j, S_i).$$

**Lemma 13** *The connection forms of the conformal metrics $g$ and $\widehat{g} = x^2 g$ satisfy $\widehat{\omega} = \omega + \alpha$, where*

$$\alpha_{ij}(Y) := \widehat{g}(Y, \widehat{S}_i)\widehat{S}_j(a) - \widehat{g}(Y, \widehat{S}_j)\widehat{S}_i(a) = g(Y, S_i)S_j(a) - g(Y, S_j)S_i(a) \tag{19}$$

*with $a := \log(x)$.*

**Proof** The lemma is proved by an easy computation using Koszul's formula for the Riemannian connection in a frame. □

Let $g^t = x^{2t} g$, where $t \in [0, 1]$. Then $S^t := x^{-t} S$ defines a section of the frame bundle $F^t(X)$ of $g^t$. Consider $\omega^t$ to be the connection form of $g^t$ in the basis $S^t$, and write $\alpha^t = \omega^t - \omega$. Notice from (19) that $\alpha^t = t\alpha$ is linear in $t$, so we compute the variation of $\mathrm{cs}(\omega^t)$ using (11):

$$\partial_t \,\mathrm{cs}(\omega^t)|_{t=0} = d\,\mathrm{Tr}(\alpha \wedge \omega) + 2\,\mathrm{Tr}(\alpha \wedge \Omega),$$

where $\Omega = d\omega + \omega \wedge \omega$ is the curvature of $\omega$, and $\partial_t \omega^t = \partial_t \alpha^t = \alpha$, valid for all $t$.

**Lemma 14** *We have* $\mathrm{Tr}(\alpha \wedge \Omega) \equiv 0$.

**Proof** At points where $\nabla a = 0$ this is clear. At other points, take an orthonormal basis $(X_1, X_2, X_3)$ of $TX$ for $g$ such that $X_3$ is proportional to $\mathrm{grad}(a)$. Since $\alpha \wedge \Omega$ is a tensor, we can compute its trace in the basis $X_j$ instead of $S_j$:

$$\begin{aligned}\mathrm{Tr}(\alpha \wedge \Omega)(X_1, X_2, X_3) &= \sum_{i,j} \alpha_{ij}(X_1)\Omega_{ji}(X_2, X_3) - \alpha_{ij}(X_2)\Omega_{ji}(X_1, X_3) \\ &= 2(\langle R_{X_2 X_3} X_1, \mathrm{grad}(a)\rangle - \langle R_{X_1 X_3} X_2, \mathrm{grad}(a)\rangle)\end{aligned}$$

and this vanishes using the symmetry of the Riemannian curvature together with the fact that $X_3$ and $\mathrm{grad}(a)$ are collinear. □

We deduce that $\partial_t \operatorname{cs}(\omega^t)|_{t=0} = d\operatorname{Tr}(\alpha\wedge\omega)$. Similarly, one has $\operatorname{Tr}(\alpha\wedge\Omega^t)=0$ if $\Omega^t = d\omega^t + \omega^t\wedge\omega^t$. Then, using (11) and $\partial_s(\alpha^{t+s})_{s=0}=\alpha$, we get

$$\partial_t \operatorname{cs}(\omega^t) = \partial_s \operatorname{cs}(\omega^{t+s})|_{s=0} = d\operatorname{Tr}(\alpha\wedge\omega^t) + 2\operatorname{Tr}(\alpha\wedge\Omega^t) = d\operatorname{Tr}(\alpha\wedge(\omega+t\alpha)).$$

Since $\operatorname{Tr}(\alpha\wedge\alpha)=0$ by cyclicity of the trace, we find

$$\operatorname{cs}(\hat{g},\hat{S}) = \operatorname{cs}(g,S) + d\operatorname{Tr}(\alpha\wedge\omega). \tag{20}$$

**Proposition 15** *Let $g$ be an asymptotically hyperbolic metric on $X$ with totally geodesic boundary, let $x$ be a smooth geodesic boundary defining function and set $\hat{g} := x^2 g$. Let $\hat{S}$ be an even to first order section in $F(X)$ with respect to $\hat{g}$, and let $S = x\hat{S}$ be the corresponding section in $F_0(X)$. Then the* $\mathrm{SO}(3)$ *Chern–Simons invariants of $g$ and $\hat{g}$ with respect to $S$ and $\hat{S}$ coincide:*

$$\mathrm{CS}(g,S) = \mathrm{CS}(\hat{g},\hat{S})$$

**Proof** By integrating over $X$ and using Stokes, we get

$$16\pi^2\,\mathrm{CS}(\hat{g},\hat{S}) = 16\pi^2\,\mathrm{CS}(g,S) - \mathrm{FP}_{\epsilon\to 0}\int_{x=\epsilon}\operatorname{Tr}(\alpha\wedge\omega). \tag{21}$$

The proof is finished by showing that the trace $\operatorname{Tr}(\alpha\wedge\omega)$ is odd in $x$ to order $O(x^4)$, so

$$\mathrm{FP}_{\epsilon\to 0}\int_{x=\epsilon}\operatorname{Tr}(\alpha\wedge\omega) = 0.$$

For this, note that $x\alpha$ is smooth in $x$ and has an even expansion at $x=0$ in powers of $x$ up to $O(x^3)$ by assumption on the section $S$, while $x\omega$ is smooth in $x$ but a priori not even. Setting $a := \log x$ we write for $Y_1, Y_2$ vector fields on $\partial\overline{X}$ (thus orthogonal to $\nabla a$) and $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$:

$$\begin{aligned}
\operatorname{Tr}(\alpha\wedge\omega)(Y_1,Y_2) &= \sum_{1\le i,j\le 3}\alpha_{ij}(Y_1)\omega_{ji}(Y_2) - \alpha_{ij}(Y_2)\omega_{ji}(Y_1)\\
&= \sum_{1\le j\le 3} S_j(a)\langle\nabla_{Y_2}S_j, Y_1\rangle - S_j(a)\langle\nabla_{Y_1}S_j, Y_2\rangle\\
&\qquad\qquad - \langle S_j, Y_1\rangle\langle\nabla_{Y_2}S_j,\nabla a\rangle + \langle S_j, Y_2\rangle\langle\nabla_{Y_1}S_j,\nabla a\rangle\\
&= 2\langle\nabla_{Y_2}\nabla a, Y_1\rangle - 2\langle\nabla_{Y_1}\nabla a, Y_2\rangle\\
&\qquad\qquad - 2\sum_{1\le j\le 3} Y_2(S_j(a))\langle S_j, Y_1\rangle - Y_1(S_j(a))\langle S_j, Y_2\rangle\\
&= \sum_{1\le j\le 3} -2Y_2(S_j(a))\langle S_j, Y_1\rangle + 2Y_1(S_j(a))\langle S_j, Y_2\rangle
\end{aligned}$$

and this is odd in $x$ to order $O(x^2)$ since $da = -dx/x$, $\hat{S}_j = x^{-1}S_j$ is even to first order, and the metric has totally geodesic boundary (ie, $x^2 g$ is even to order $O(x^3)$). □

## 6 The $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{SO}(3)$ invariants in constant curvature

In this section we establish the relation between the $\mathrm{PSL}_2(\mathbb{C})$ and the $\mathrm{SO}(3)$ Chern–Simons invariants. This was known in the compact case and in the finite volume case since the work of Yoshida [31].

**Proposition 16** *Let $(X,g) = \Gamma\backslash\mathbb{H}^3$ be a convex co-compact hyperbolic 3–manifold with $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ and let $\theta$ be the associated flat connection on the bundle $F^{\mathbb{C}}(X) = \mathbb{H}^3 \times_\Gamma \mathrm{PSL}_2(\mathbb{C})$. Let $S\colon X \to F_0(X)$ be an even section of $F_0(X)$. Then*

$$\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(\theta, S) = -\frac{i}{2\pi^2}\,\mathrm{Vol}_R(X) + \frac{i}{2\pi}\chi(M) + \mathrm{CS}(g,S).$$

**Proof** Using Proposition 11 and Stokes' formula, we have

$$\begin{aligned}\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(\theta, S) &= \mathrm{FP}_{\epsilon\to 0}\int_{\{x>\epsilon\}} -\frac{i}{2\pi^2}\,\mathrm{dvol}_{\mathbb{H}^3} + \frac{i}{16\pi^2}\,d(\mathrm{Tr}(\omega\wedge T)) - \frac{1}{16\pi^2}\,\mathrm{cs}(\omega)\\ &= -\frac{i}{2\pi^2}\,\mathrm{Vol}_R(X) - \mathrm{FP}_{\epsilon\to 0}\,\frac{i}{16\pi^2}\int_{x=\epsilon}\mathrm{Tr}(\omega\wedge T) + \mathrm{CS}(g,S).\end{aligned}$$

The conclusion follows from Lemma 17. □

**Lemma 17** *We have*

$$\mathrm{FP}_{\epsilon\to 0}\int_{x=\epsilon}\mathrm{Tr}(T\wedge\omega) = 2\int_M \mathrm{scal}_{h_0}\,\mathrm{dvol}_{h_0} = 8\pi\chi(M). \tag{22}$$

**Proof** Let $U_j := x^{-1}S_j$ denote the orthonormal frame for the compact metric $\hat{g} = x^2 g$, $(S^1,S^2,S^3)$ the dual basis to $S$, $\hat{\omega}_{ij}(Y) := \hat{g}(\nabla^{\hat{g}}_Y U_i, U_j)$ the Levi-Civita connection 1–form of $\hat{g}$ in the frame $U$, and $U^j = xS^j$ the dual co-frame. Let $Y_1, Y_2$ be a local orthonormal frame on $M$ for $h_0$ of eigenvectors for the map $A$ defined on $TM$ by (6), extended on $X$ constantly in $x$ near $M$.

We split $\omega = \hat{\omega} - \alpha$ and we first compute

$$\begin{aligned}\mathrm{Tr}(T\wedge\alpha)(Y_1,Y_2) = \sum_{i,j}&\langle Y_1\times S_i, S_j\rangle(\langle Y_2,S_j\rangle S_i(a) - \langle Y_2,S_i\rangle S_j(a))\\ &- \langle Y_2\times S_i, S_j\rangle(\langle Y_1,S_j\rangle S_i(a) - \langle Y_1,S_i\rangle S_j(a))\\ = {}&-4\langle Y_1\times Y_2, \nabla a\rangle.\end{aligned}$$

Define $\widetilde{Y}_j := (1+\frac{1}{2}x^2A)^{-1}Y_j$. Then $x\widetilde{Y}_1, x\widetilde{Y}_2, \nabla a$ form an orthonormal frame near $x=0$ and $x\widetilde{Y}_1 \times x\widetilde{Y}_2 = \nabla a$. Thus,

$$x^2\left(1+\frac{x^2\lambda_1}{2}\right)^{-1}\left(1+\frac{x^2\lambda_2}{2}\right)^{-1}\langle Y_1\times Y_2, \nabla a\rangle = 1,$$

which shows that

$$\mathrm{FP}_{\epsilon\to 0}\operatorname{Tr}(T\wedge\alpha) = 2\operatorname{tr}(A)\ \mathrm{dvol}_{h_0}.$$

Let us now compute the form $\operatorname{Tr}(\widehat{\omega}\wedge T)$ on the hypersurface $x=\epsilon$. Notice that

$$T_{12} = S^3, \quad T_{23} = S^1, \quad T_{31} = S^2$$

so $xT$ is smooth in $x$, and we easily see that

$$\tfrac{1}{2}\operatorname{Tr}(\widehat{\omega}\wedge T) = S^1\wedge\widehat{\omega}_{23} + S^2\wedge\widehat{\omega}_{31} + S^3\wedge\widehat{\omega}_{12}.$$

Using the Koszul formula and the evenness of $g$ and $S$, for a vector $Y\in TM$ independent of $x$ the term $\widehat{\omega}_{ji}$ can be decomposed in the form

$$\begin{aligned} 2\widehat{\omega}_{ji}(Y) &= U_i(\widehat{g}(Y,U_j)) - U_j(\widehat{g}(Y,U_i)) - \widehat{g}([U_i,U_j],Y) + \text{even function of } x \\ &= dY^{\#}(U_i,U_j) + \text{even function of } x \end{aligned}$$

so the odd component is tensorial in $U_j$. Therefore we can compute $\mathrm{FP}_{\epsilon\to 0}\operatorname{Tr}(\widehat{\omega}\wedge T)$ using the orthonormal frame $\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3 := \partial_x$:

$$\begin{aligned} \operatorname{Tr}(T\wedge\widehat{\omega})(Y_1,Y_2) &= x^{-1}\sum_{i,j}\langle Y_1\times\widetilde{Y}_i,\widetilde{Y}_j\rangle\langle\nabla_{Y_2}\widetilde{Y}_i,\widetilde{Y}_j\rangle - \langle Y_2\times\widetilde{Y}_i,\widetilde{Y}_j\rangle\langle\nabla_{Y_1}\widetilde{Y}_i,\widetilde{Y}_j\rangle \\ &= x^{-1}\sum_i\langle\nabla_{Y_2}\widetilde{Y}_i, Y_1\times\widetilde{Y}_i\rangle - \langle\nabla_{Y_1}\widetilde{Y}_i, Y_2\times\widetilde{Y}_i\rangle \end{aligned}$$

(here the vector product is with respect to $\widehat{g}$). Since $\widetilde{Y}_j - Y_j$ is of order $x^2$, the finite part is unchanged if we replace $Y_1, Y_2$ by $\widetilde{Y}_1, \widetilde{Y}_2$ in the above, thus getting

$$x^{-1}\sum_i\langle\nabla_{\widetilde{Y}_2}\widetilde{Y}_i, \widetilde{Y}_1\times\widetilde{Y}_i\rangle - \langle\nabla_{\widetilde{Y}_1}\widetilde{Y}_i, \widetilde{Y}_2\times\widetilde{Y}_i\rangle.$$

For $k=1,2$ the coefficient of $x$ in $\langle\nabla_{\widetilde{Y}_k}\widetilde{Y}_k,\partial_x\rangle = -\langle\nabla_{\widetilde{Y}_k}\partial_x,\widetilde{Y}_k\rangle$ is $-\lambda_k$. We thus get

$$\mathrm{FP}_{\epsilon\to 0}\operatorname{Tr}(T\wedge\widehat{\omega}) = -2\operatorname{tr}(A)\ \mathrm{dvol}_{h_0}.$$

Together with the identity $2\operatorname{tr}(A) = -\operatorname{scal}_{h_0}$ and Gauss–Bonnet this ends the proof. □

# 7 The Chern–Simons line bundle and its connection

## 7.1 Tangent space of Teichmüller space as the set of hyperbolic funnels

In this subsection, we shall see that the tangent space $T\mathcal{T}$ of Teichmüller space of Riemann surfaces of genus $\boldsymbol{g}$ can be identified with ends of hyperbolic 3–manifolds of funnel type. For Teichmüller space definition and conventions, we follow the book by Tromba [29].

The Teichmüller space $\mathcal{T}$ is defined here as the quotient $\mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma)$, where $\mathcal{M}_{-1}(\Sigma)$ is the set of smooth metrics with Gaussian curvature $-1$ on a fixed smooth surface $\Sigma$ of genus $\boldsymbol{g}$, and $\mathcal{D}_0(\Sigma)$ is the group of orientation-preserving smooth diffeomorphisms of $\Sigma$ that are isotopic to the identity. Here $M$ is not necessarily connected and $\boldsymbol{g} \in (\mathbb{N} \setminus \{0,1\})^N$, where $N = \pi_0(M)$.

First, we shall identify each point of $T\mathcal{T}$ with an isometry class of 3–dimensional hyperbolic ends, with conformal infinity given by the base point.

**Definition 18** A *hyperbolic funnel* is a couple $(M, g)$, where $M$ is a Riemann surface (not necessarily connected) equipped with a metric $h_0$ of Gaussian curvature $-1$ and $g$ is a metric on the product $M \times (0, \epsilon)_x$ for some small $\epsilon > 0$, which is of the form

$$g = \frac{dx^2 + h(x)}{x^2}, \quad h(x) := h_0 + x^2 h_2 + \tfrac{1}{4} x^4 h_2 \circ h_2, \tag{23}$$

where $h_2$ is a symmetric tensor satisfying

$$\mathrm{Tr}_{h_0}(h_2) = 1, \quad \mathrm{div}_{h_0}(h_2) = 0.$$

It is shown in Fefferman and Graham [5] that $M \times (0, \epsilon)$ equipped with such a metric $g$ is a (non-complete) hyperbolic manifold if $\epsilon > 0$ is chosen small enough, and conversely every end of a convex co-compact hyperbolic manifold with conformal infinity $(M, \{h_0\})$ and $\mathrm{genus}(M) > 1$ is isometric to a unique funnel (23) with $h_0$ the hyperbolic metric representing the conformal class $\{h_0\}$. There is an action of the group $\mathcal{D}(M)$ of diffeomorphisms of $M$ on the space of funnels, simply given by

$$\psi^*(M, g) := \left(M, \frac{dx^2 + \psi^* h(x)}{x^2}\right)$$

for all $\psi \in \mathcal{D}(M)$, where $\psi^* h(x)$ is the pull-back of the metric $h(x)$ on $M$. Notice also that a funnel induces a representation of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$ up to conjugation.

The tangent space $T\mathcal{M}_{-1}(M)$ has a natural inner product, the $L^2$–metric, defined as follows (see [29, Section 2.6]): let $h_0 \in \mathcal{M}_{-1}(M)$, and $h, k \in T_{h_0}\mathcal{M}_{-1}(M)$. Since

$\mathcal{M}_{-1}(M)$ is a Fréchet submanifold in the space of symmetric tensors on $M$, it follows that $T\mathcal{M}_{-1}(M) \subset \mathcal{C}^{\infty}(M, S^2T^*M)$; define

$$\langle h, k\rangle := \int_M \langle h, k\rangle_{h_0} \, \mathrm{dvol}_{h_0} . \tag{24}$$

This scalar product is $\mathcal{D}(M)$–invariant.

For any $h_0 \in \mathcal{M}_{-1}(M)$ consider the vector space

$$V_{h_0} := \{h \in \mathcal{C}^{\infty}(M, S^2T^*M) : \mathrm{Tr}_{h_0}(h) = 0, \mathrm{div}_{h_0}(h) = 0\},$$

ie, the set of transverse traceless symmetric tensors with respect to $h_0$. This is a real vector space of finite dimension that is precisely the orthogonal complement in $T_{h_0}\mathcal{M}_{-1}(M)$ of the orbit of $\mathcal{D}_0(M)$ with respect to the $L^2(M, h_0)$ inner product. When $h_0$ varies, these spaces form a locally trivial vector bundle $V$ over $\mathcal{M}_{-1}(M)$ of rank $6|\boldsymbol{g}| - 6$ (assuming that the genera of the connected components $M_j$ are strictly larger than 1), which we think of as the horizontal tangent bundle in the principal Riemannian fibration $\mathcal{M}_{-1}(M) \to \mathcal{T}$. The group $\mathcal{D}(M)$ acts isometrically on this bundle by pull-back of tensors, and the restriction of this action to the subgroup $\mathcal{D}_0(M)$ is free. The quotient of $V$ by $\mathcal{D}_0(M)$ is identified [29, Section 2.4] with the tangent bundle $T\mathcal{T}$ of the Teichmüller space of genus $\boldsymbol{g}$. Thus, Teichmüller space inherits a Riemannian metric called the *Weil–Petersson metric*. Explicitly, on vectors in $T_{[h_0]}\mathcal{T}$ described by trace-free, divergence-free symmetric tensors $h, k$ with respect to a representative $h_0 \in [h_0]$, the Weil–Petersson metric is defined by

$$\langle h, k\rangle_{\mathrm{WP}} := \int_M \langle h, k\rangle_{h_0} \, \mathrm{dvol}_{h_0} . \tag{25}$$

The following is a direct consequence of the discussion above:

**Lemma 19** *There is a canonical bijection $\Psi$ from the total space of the horizontal tangent bundle $V \to \mathcal{M}_{-1}(M)$ to the set $F_{\boldsymbol{g}}$ of hyperbolic funnels of genus $\boldsymbol{g}$, defined explicitly by*

$$\Psi\colon (h_0, h_2^0) \mapsto \left(M, \frac{dx^2 + h(x)}{x^2}\right), \tag{26}$$

$$h(x) = h_0 + x^2 h_2 + \frac{x^4}{4} h_2 \circ h_2, \quad h_2 = h_2^0 + \frac{h_0}{2} .$$

*This bijection commutes with the action of $\mathcal{D}_0(M)$ on both sides and hence descends to a bijection from $T\mathcal{T}$ to the space of $\mathcal{D}_0(M)$–equivalence classes of hyperbolic funnels.*

Any divergence-free traceless tensor $k = u\,dy_1^2 - u\,dy_2^2 - 2v\,dy_1dy_2$ with respect to a metric $h_0$ is the real part of a quadratic holomorphic differential (QHD in short):

$$\tfrac{1}{2}k = \mathrm{Re}(k^{0,1}) \quad \text{with } k^{0,1} := \tfrac{1}{2}(u+iv)\,dz^2$$

in local complex coordinates $z = y_1 + iy_2$. The complex structure $J$ on Teichmüller space is then given by multiplication by $-i$ on QHD, which on the level of transverse traceless tensors means

$$Jk := v\,dy_1^2 - v\,dy_2^2 + 2u\,dy_1dy_2 \tag{27}$$

or setting $K$ to be the symmetric endomorphism of $TM$ defined by $k(\cdot,\cdot) = h_0(K\cdot,\cdot)$,

$$JK = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u & -v \\ -v & -u \end{pmatrix} = \begin{pmatrix} v & u \\ u & -v \end{pmatrix}.$$

The space $T^{0,1}\mathcal{T}$ is then defined to be the subspace of complexified tangent space $T_{\mathbb{C}}\mathcal{T}$ spanned by the elements $k + iJk$ with $k \in T\mathcal{T}$, and $T^{1,0}\mathcal{T}$ is spanned by the $k - iJk$. Notice also that, with the notations used just above, one has

$$k^{0,1} = \tfrac{1}{2}(k + iJk) \in T^{0,1}\mathcal{T}, \quad k^{1,0} = \overline{k^{0,1}} = \tfrac{1}{2}(k - iJk) \in T^{1,0}\mathcal{T}. \tag{28}$$

The Weil–Petersson metric on $T\mathcal{T}$ induces an isomorphism $\Phi$ between $T\mathcal{T}$ and $T^*\mathcal{T}$. It is also a Hermitian metric for the complex structure $J$, in the sense that $\langle Jh, Jk\rangle_{\mathrm{WP}} = \langle h, k\rangle_{\mathrm{WP}}$ for all $h, k \in T\mathcal{T}$, and the associated symplectic form is $\omega_{\mathrm{WP}}(\cdot,\cdot) := \langle J\cdot, \cdot\rangle_{\mathrm{WP}}$. By convention, the metric $\langle\cdot,\cdot\rangle_{\mathrm{WP}}$ on $T\mathcal{T}$ is extended to be bilinear on $T_{\mathbb{C}}\mathcal{T}$, so that $\langle k^{0,1}, h^{0,1}\rangle_{\mathrm{WP}} = \langle k^{1,0}, h^{1,0}\rangle_{\mathrm{WP}} = 0$ for all $h, k \in T\mathcal{T}$ and $\langle k^{0,1}, k^{1,0}\rangle_{\mathrm{WP}} \geq 0$ for all $k \in T\mathcal{T}$.

On $T^*\mathcal{T}$, there is a natural symplectic form, obtained by taking the exterior derivative $d\mu$ of the Liouville 1–form $\mu$ defined for $h_0 \in \mathcal{T}$, $k^* \in T^*_{h_0}\mathcal{T}$ by

$$\mu_{(h_0,k^*)} := k^*.\,d\pi$$

if $\pi\colon T^*\mathcal{T} \to \mathcal{T}$ is the natural projection. Since $T^*\mathcal{T}$ has a complex structure induced naturally by that of $\mathcal{T}$ we can also define the $(0,1)$ component $\mu^{1,0}$ of the Liouville measure. The Liouville form $\mu$ and $\mu^{1,0}$ pull-back to natural form on $T\mathcal{T}$ through $\Phi$, satisfying

$$\Phi^*\mu_{(h_0,k)}(\dot{h}_0, \dot{k}) = \langle k, \dot{h}_0\rangle_{\mathrm{WP}}, \quad \Phi^*\mu^{1,0}_{(h_0,k)}(\dot{h}_0, \dot{k}) = \langle k^{0,1}, \dot{h}_0^{1,0}\rangle_{\mathrm{WP}}$$

for $(h_0, k) \in T\mathcal{T}$, and $(\dot{h}_0, \dot{k}) \in TT_{h_0}\mathcal{T} = T_{h_0}\mathcal{T} \oplus T_{h_0}\mathcal{T}$. Notice that $d\mu^{1,0}$ is a $(1,1)$ type form on $T^*\mathcal{T}$.

## 7.2 The cocycle

In order to define the Chern–Simons line bundle $\mathcal{L}$ over $\mathcal{T}$ in a way similar to Freed [7], and Ramadas, Singer and Weitsman [25], we need to define a certain cocycle. The natural bundle turns out to be the $\mathrm{SO}(3)$ Chern–Simons line bundle associated to a 3–manifold bounding a given surface.

We fix $M$ a compact Riemann surface (possibly with several connected components) admitting hyperbolic metrics. We will consider Riemannian compact 3–manifolds with totally geodesic boundary $(M, h_0)$ and will denote them $(\overline{X}, \hat{g})$. For any such $\overline{X}$, the restriction of $T\overline{X}$ to $M$ is isometric to the orthogonal direct sum $TM \oplus \mathbb{R}$. The orthonormal frame bundle $F(X)$ associated to $\hat{g}$ on $\overline{X}$ is trivial since the tangent bundle to any 3–manifold is trivial. We can identify the restriction of $F(X)$ to $M$ with the restriction to $M$ of the orthonormal frame bundle $F(TM \oplus \mathbb{R})$ of $TM \oplus \mathbb{R}$ for the product metric $dx^2 + h_0$. For a fixed metric $(\overline{X}, \hat{g})$ consider the map $c^X \colon \mathcal{C}^\infty(M, F(X)) \times \mathcal{C}^\infty(\overline{X}, \mathrm{SO}(3)) \to \mathbb{C}$ defined by

$$\begin{aligned}(29)\quad & c^X(\hat{S}, \tilde{a}) \\ &= \exp\left(2\pi i \int_M \frac{1}{16\pi^2}\operatorname{Tr}(\hat{\omega} \wedge da\, a^{-1}) + 2\pi i \int_X \frac{1}{48\pi^2}\operatorname{Tr}((\tilde{a}^{-1} d\tilde{a})^3)\right)\end{aligned}$$

where $\hat{\omega}$ is the connection form of the Levi-Civita connection of $\hat{g}$ along $M$ in the frame $\hat{S}$, and $a = \tilde{a}|_M$.

**Lemma 20** *Let $(M, h_0)$ be a closed oriented surface, $\hat{S}$ an orthonormal frame on $M$ in the bundle $TM \oplus \mathbb{R}$, and $a \in \mathcal{C}^\infty(M, \mathrm{SO}(3))$. Then there exists a compact manifold $\overline{X}$ bounding $M$ such that $a$ extends to an $\tilde{a} \in \mathcal{C}^\infty(\overline{X}, \mathrm{SO}(3))$ on $\overline{X}$. Moreover, $c^X(\hat{S}, \tilde{a})$ defined in (29) depends only on $h_0$, $\hat{S}$ and $a$.*

**Proof** Assume for the moment that there exists some $\overline{X}$ bounding $M$ such that $a$ extends to $\overline{X}$ (we prove this below). We must prove that $c^X(\hat{S}, a)$ does not depend on the choice of $\overline{X}$ bounding $M$, of the metric $\hat{g}$, and on the extension of $a$ from $M$ to $\overline{X}$. The independence of $\hat{\omega}|_{TM}$ with respect to $\hat{g}$ is a consequence of Koszul's formula and of the evenness to first order of $\hat{g}$ near $M$, which is another way of saying that $M$ is totally geodesic. If now $\tilde{a}_j \in \mathcal{C}^\infty(\overline{X}_j, \mathrm{SO}(3))$, $j = 1, 2$ are two extensions of $a$, then one can glue them to a continuous, piecewise smooth $\tilde{a}$ defined on the closed 3–manifold $Z := \overline{X}_1 \sqcup -\overline{X}_2$ glued along $M = \partial\overline{X}$, and we know by (18) that

$$\exp\left(-2\pi i \int_Z \frac{1}{48\pi^2}\operatorname{Tr}((\tilde{a}^{-1} d\tilde{a})^3)\right) = 1,$$

which implies the independence of $c^X$ with respect to the choice of $\tilde{a}$.

To show that $\overline{X}$ exists, we will construct it by induction on the genus. If $M$ is a sphere, then $a$ extends to a 3–disk because $\pi_2(\mathrm{SO}(3)) = 0$ ($\pi_2$ of any Lie group is trivial). Let $a\colon M \to \mathrm{SO}(3)$, and denote by $a_*$ the induced map from $\pi_1(M)$ to $\mathbb{Z}/2\mathbb{Z}$. Since the target is Abelian, this map factors through the Abelianization $H_1(M,\mathbb{Z})$. Assume the genus is at least 1 and cut $M$ along some simple closed curve $\gamma$ to decompose it into a surface of genus 1 and one of genus $g-1$, both with boundary $\gamma$. Since $\gamma$ is null in homology (because it bounds), $a_{|\gamma}$ is contractible, hence it extends to a disk $D$ with boundary $\gamma$. Thus $a$ is now defined on two closed surfaces: a torus $T$ and a surface $M'$ of genus $g-1$ which meet along $D$. By the induction hypothesis, $a$ extends on a handlebody whose boundary is $M'$. If we can extend it also to a solid torus of boundary $T$, then by gluing along $D$ we have defined $a$ on a handlebody bounded by $M$.

Thus the proof will be ended by proving the case of genus 1, ie, when $M$ is a torus. View $M$ as the quotient $\mathbb{C}/\Gamma$, where $\Gamma$ is the lattice generated by 1 and $\tau$ with $\mathrm{Im}(\tau) > 0$. For $z \in \Gamma^*$ let $c_z$ denote the closed curve in $M$ obtained by projecting the segment joining 0 to $z$. We claim that $a$ is contractible along at least one of the curves $c_1$, $c_\tau$ or $c_{1+\tau}$. Indeed, in homology we have $[c_1]+[c_\tau]=[c_{1+\tau}]$, hence the values of $a_*$ on the corresponding homology classes cannot be all 1; if $a_*[c_z]=0$ it means that $a$ is contractible along $c_z$. Consider a solid torus of boundary $M$ in which $c_z$ bounds a disk on which $a$ extends. Again since $\pi_2(\mathrm{SO}(3)) = 0$ we can extend $a$ to the whole solid torus. □

Denote by $\mathcal{C}^\infty_X(M, F(X))$ and $\mathcal{C}^\infty_X(M,\mathrm{SO}(3))$ the subsets of $\mathcal{C}^\infty(M, F(X))$ and $\mathcal{C}^\infty(M,\mathrm{SO}(3))$ made of sections on $M$ which extend smoothly to $\overline{X}$. Since $F(X)$ is trivial, $\mathcal{C}^\infty(\overline{X}, F(X))$ is non-empty and $\mathcal{C}^\infty_X(M, F(X))$ consists of restrictions to $M$ of elements in $\mathcal{C}^\infty(\overline{X}, F(X))$. Although $\mathcal{C}^\infty(\overline{X}, F(X))$ depends on the choice of metric $\hat{g}$, it is clear that the space $\mathcal{C}^\infty_X(M, F(X))$ does not depend on $\hat{g}$ as long as $\hat{g} = dx^2 + h_0 + O(x^2)$, but it depends on $\overline{X}$. The space $\mathcal{C}^\infty(\overline{X},\mathrm{SO}(3))$ is also clearly non-empty and $\mathcal{C}^\infty_X(M,\mathrm{SO}(3))$ consists of restrictions to $M$ of elements in $\mathcal{C}^\infty(\overline{X},\mathrm{SO}(3))$.

Let $\tilde{a}$ be any smooth extension of $a$ on some filling $\overline{X}$. Such a filling exists by the above lemma, and the value of the cocycle $c^X(\hat{S},\tilde{a})$ is independent of the filling $\overline{X}$ and of the extension $\tilde{a}$. It follows that $c^X$ descends to a map

$$c\colon \mathcal{C}^\infty(M, F(TM \oplus \mathbb{R})) \times \mathcal{C}^\infty(M,\mathrm{SO}(3)) \to \mathbb{C}.$$

For a fixed $\overline{X}$ we get a map $c^X\colon \mathcal{C}^\infty(M, F(X)) \times \mathcal{C}^\infty_X(M,\mathrm{SO}(3)) \to \mathbb{C}$. As we shall see below, when $\hat{S}, a, b$ all extend to the same $\overline{X}$, this map satisfies the cocycle relation

$$c^X(\hat{S}, ab) = c^X(\hat{S}, a)\, c^X(\hat{S}a, b) \tag{30}$$

(we keep the notation $c = c^X$ when we work with a fixed $X$).

### 7.3 The Chern–Simons line bundle $\mathcal{L}$

We follow the presentation given in the lecture notes of Baseilhac [2], but adapted to our setting. We take a smooth map $\Phi\colon \mathcal{M}_{-1}(M) \to \mathcal{C}^\infty(\overline{X}, S^2_+ T^*\overline{X})$ such that $\hat{g} = \Phi(h_0)$ satisfies $\hat{g} = dx^2 + h_0 + O(x^2)$ in a collar neighbourhood $[0,\epsilon)_x \times M$ near $M = \partial\overline{X}$ (this is equivalent to $M$ being totally geodesic in $\overline{X}$).

**Definition 21** The complex line $L^X_{h_0}$ over $h_0 \in \mathcal{M}_{-1}(M)$ is defined for a choice of extension $X$ by

$$L^X_{h_0} := \{f\colon \mathcal{C}^\infty_X(M, F(X)) \to \mathbb{C} : \forall a \in \mathcal{C}^\infty_X(M, \mathrm{SO}(3)),\ f(\hat{S}a) = c(\hat{S}, a) f(\hat{S})\}.$$

We define the Chern–Simons line bundle (as a set) over $\mathcal{M}_{-1}(M)$ by

$$\mathcal{L}^X := \bigsqcup_{h_0 \in \mathcal{M}_{-1}(M)} L^X_{h_0}.$$

Here $F(X)$ is the orthonormal frame bundle with respect to the metric $\hat{g} = \Phi(h_0)$ on $\overline{X}$.

Using the gauge transformation law (13), we deduce:

**Lemma 22** *For any metric $\hat{g}$ on $\overline{X}$ with $\hat{g}|_{TM} = h_0$ and $M$ totally geodesic for $\hat{g}$, the map $\hat{S} \mapsto e^{2\pi i\, \mathrm{CS}(\hat{g}, \hat{S})}$ is an element of the fiber $L^X_{h_0}$ over $h_0$.*

This fact directly implies the cocycle condition (30).

Since two frames in $\mathcal{C}^\infty(\overline{X}, F(X))$ are related by a gauge transformation

$$\tilde{a} \in \mathcal{C}^\infty(\overline{X}, \mathrm{SO}(3)),$$

an element in $L^X_{h_0}$ is determined by its value on any frame extendible to $\overline{X}$ by the condition $f(\hat{S}a) = c(\hat{S}, a) f(\hat{S})$; therefore, the dimension of the $\mathbb{C}$–vector space $L^X_{h_0}$ is $1$. We define the smooth structure on $\mathcal{L}$ through global trivializations as follows: let $\hat{S}$ be a smooth positively oriented frame (not a priori orthonormal) on $\overline{X}$ and let $\hat{S}_{h_0}$ be the orthonormal frame obtained from $\hat{S}$ by the Gram–Schmidt process with respect to the metric $dx^2 + h_0$ near the boundary $\partial\overline{X}$ and define a global trivialization by

$$\mathcal{L}^X \to \mathcal{M}_{-1}(M) \times \mathbb{C}, \quad (h_0, f) \mapsto (h_0, f(\hat{S}_{h_0})).$$

Changes of trivializations corresponding to different choices of $\hat{S}$ are smooth on $\mathcal{M}_{-1}(M)$, thus we get a structure of smooth line bundle on $\mathcal{L}^X$ over $\mathcal{M}_{-1}(M)$.

If $X_1$ and $X_2$ are two fillings of $M$, let $Z = X_1 \cup (-X_2)$ be the oriented closed manifold obtained by gluing $X_1$ and $X_2$ (the latter with its reverse orientation) along $M$, then any frame on $Z$ restricted to $M$ is extendible both to $X_1$ and $X_2$ (recall that the frame bundle on $Z$ is trivialisable since $Z$ is of dimension 3). Let $\widehat{S}_0$ be such a frame on $Z$, orthonormal along $M$ with respect to $dx^2 + h_0$. We denote also by $\widehat{S}_0$ its restriction to $X_1$ and to $X_2$. We define an isomorphism

$$\Psi\colon L_{h_0}^{X_1} \to L_{h_0}^{X_2}, \quad (h_0, f^{X_1}) \mapsto (h_0, f^{X_2})$$

where $f^{X_2}$ is uniquely determined by the requirement $f^{X_2}(\widehat{S}_0) := f^{X_1}(\widehat{S}_0)$.

More precisely, this amounts to setting

$$f^{X_2}(\widehat{S}_0 a) := c(\widehat{S}_0, a) f^{X_1}(\widehat{S}_0)$$

for all $a \in \mathcal{C}^\infty_{X_2}(M, \mathrm{SO}(3))$. Since every $\widehat{S} \in \mathcal{C}^\infty_{X_2}(M, F(X_2))$ can be written as $\widehat{S}_0 a$ for some $a \in \mathcal{C}^\infty_{X_2}(M, \mathrm{SO}(3))$, $f^{X_2}$ does define an element in $L_{h_0}^{X_2}$. We claim that this isomorphism is independent of the choice of $\widehat{S}_0$ extendible to $X_1$ and $X_2$. Indeed, for any other frame $\widehat{S}_0' \in \mathcal{C}^\infty(Z, F(Z))$ on $Z = X_1 \cup (-X_2)$, there exists $a_0 \in \mathcal{C}^\infty(Z, \mathrm{SO}(3))$ with $\widehat{S}_0' a_0 = \widehat{S}_0$. If $\Psi'\colon L_{h_0}^{X_1} \to L_{h_0}^{X_2}$ is the associated map defined as above by using $\widehat{S}_0'$ instead of $\widehat{S}_0$, for all $a \in \mathcal{C}^\infty_{X_2}(M, \mathrm{SO}(3))$ we have

$$\begin{aligned}\Psi'(f^{X_1})(\widehat{S}_0 a) = \Psi'(\widehat{S}_0' a_0 a) = c(\widehat{S}_0', a_0 a) f^{X_1}(\widehat{S}_0') &= c(S_0, a) c(\widehat{S}_0', a_0) f^{X_1}(\widehat{S}_0') \\ &= c(\widehat{S}_0, a) f^{X_1}(\widehat{S}_0) = \Psi(f^{X_1})(\widehat{S}_0 a),\end{aligned}$$

where we used Lemma 20 and the cocycle formula (30). This shows that $\Psi' = \Psi$ and hence the line bundle $\mathcal{L}^X$ is well defined (up to canonical isomorphisms) independently from the choice of filling $X$.

## 7.4 Action by diffeomorphisms

The mapping class group Mod is the set of isotopy classes of orientation preserving diffeomorphisms of $M = \partial \overline{X}$, it acts on $\mathcal{T}$ properly discontinuously. By Marden [18, Theorem 3.1], the subgroup $\mathrm{Mod}_X$ of Mod arising from elements that extend to diffeomorphisms of $\overline{X}$ homotopic to the identity on $\overline{X}$, acts freely on $\mathcal{T}$ and the quotient $\mathcal{T}_X := \mathcal{T} / \mathrm{Mod}_X$ is a complex manifold of dimension $3|\boldsymbol{g}| - 3$. Moreover the Weil–Petersson metric descends to $\mathcal{T}_X$.

Every diffeomorphism $\psi\colon \overline{X} \to \overline{X}$ induces an isomorphism

$$L_{h_0}^X \to L_{\psi^* h_0}^X$$

defined by $f \mapsto f_\psi := (\widehat{S} \mapsto f(\psi_* \widehat{S}))$. In particular since any $\psi \in \mathcal{D}_0(M)$ can be extended on $\overline{X}$ as a diffeomorphism homotopic to the identity in $\overline{X}$ and the map $L^X_{h_0} \to L^X_{\psi^* h_0}$ does not depend on the extension, the bundle $\mathcal{L}^X$ descends to Teichmüller space $\mathcal{T}$ as a complex line bundle. We shall work mainly with Teichmüller space but all constructions are $\mathrm{Mod}_X$ invariants and descend to $\mathcal{T}_X$.

We define the pull-back bundle $\pi^*\mathcal{L}$ on $T\mathcal{T}$ if $\pi\colon T\mathcal{T} \to \mathcal{T}$ is the projection on the base and we shall use the notation $\mathcal{L}$ instead of $\pi^*\mathcal{L}$.

## 7.5 Hermitian metric on $\mathcal{L}$

Since the cocycle is of absolute value $1$, there exists on $\mathcal{L}$ a canonical Hermitian metric, denoted $\langle \cdot, \cdot \rangle_{\mathrm{CS}}$, given simply by

$$\langle f_1, f_2 \rangle_{\mathrm{CS}} := f_1(\widehat{S})\overline{f_2(\widehat{S})} \tag{31}$$

if $f_1, f_2$ are two sections of $\mathcal{L}$ and $\widehat{S} \in \mathcal{C}^\infty(M, F(X))$.

## 7.6 The connections on $\mathcal{L}$

We define $2$ different connections on $\mathcal{L}$. We start with a Hermitian connection coming from the base $\mathcal{T}$. For any $\hat{g}$–orthonormal frame $\widehat{S}$ on $X$ we define $\widehat{S}^t$ to be the parallel transport of $\widehat{S}$ in the $t$ direction with respect to the metric $\widehat{G} := dt^2 + \hat{g}^t$ ($\widehat{S}^t$ is a $\hat{g}^t$–orthonormal frame because $\partial_t$ is a geodesic vector field for $\widehat{G}$).

**Definition 23** Let $h_0^t \in \mathcal{M}_{-1}(M)$ for $t \in \mathbb{R}$ be a curve of hyperbolic metrics on $M$ extended evenly to first order to a metric $\hat{g}^t$ on $X$, with $h_0^0 =: h_0$. For any section $f$ of $\mathcal{L}$, we define for $\dot{h}_0 = \partial_t h_0^t|_{t=0} \in T_{h_0}\mathcal{M}_{-1}(M)$

$$(\nabla^{\mathcal{L}}_{\dot{h}_0} f)(\widehat{S}) := \partial_t f(h_0^t, \widehat{S}^t)|_{t=0} - 2\pi i f(\widehat{S}) \int_M \frac{1}{16\pi^2} \operatorname{Tr}(\dot{\hat{\omega}} \wedge \hat{\omega})$$

where $\dot{\hat{\omega}} = \partial_t \hat{\omega}^t|_{t=0}$ and $\hat{\omega}^t$ is the Levi-Civita connection $1$–form ($\mathrm{so}(3)$–valued) of the metric $\hat{g}^t$ in the frame $\widehat{S}^t$.

One can check that the frame $\widehat{S}^t$ constructed above is even to first order. We leave this verification to the reader; it follows from the Koszul formula by writing the parallel transport equation $\nabla_{\partial_t} \widehat{S}^t = 0$ as a system of ODEs and using the evenness of $\hat{g}^t$ and $\widehat{S}^0$.

**Lemma 24** *The differential operation $\nabla^{\mathcal{L}}$ in Definition 23 is a connection on $\mathcal{L}$.*

**Proof** One needs to check that $\nabla^{\mathcal{L}}_{h_0} f$ belongs to $\mathcal{L}$ if $f$ is a section of $\mathcal{L}$, ie, it changes by the cocycle under gauge transformations. For this, let $a\colon M \to \mathrm{SO}(3)$ and consider the $\hat{g}^t$–orthonormal frame $\hat{S}^t a$ with components $V_i = \sum_{j=1}^3 \hat{S}^t_j a_{ji}$. Since $a$ is independent of $t$, from the Leibniz rule we compute $\nabla_{\partial_t} V_i = 0$. Thus the parallel transport of $\hat{S}a$ in the direction of $\partial_t$ is given by $(\hat{S}a)^t = \hat{S}^t a$. Moreover, directly from the Equation (29) of the cocycle,

$$\partial_t c(\hat{S}^t, a)|_{t=0} = \frac{2\pi i}{16\pi^2} c(\hat{S}, a) \int_M \mathrm{Tr}(\dot{\hat{\omega}} \wedge da a^{-1}). \tag{32}$$

Furthermore, the connection form in the frame $\hat{S}^t a$ is just $\hat{\omega}^t_{\hat{S}a} = a^{-1}\hat{\omega}^t a + a^{-1}\, da$, thus

$$\partial_t \hat{\omega}^t_{\hat{S}a}|_{t=0} = a^{-1}\dot{\hat{\omega}} a.$$

Using $f(\hat{S}^t a) = c(\hat{S}^t, a) f(\hat{S}^t)$ for sections of $\mathcal{L}$, then Definition 23 gives

$$\begin{aligned}
&(\nabla^{\mathcal{L}}_{h_0} f)(\hat{S}a)\\
&= c(\hat{S}, a)\Big(\partial_t f(h_0^t, \hat{S}^t)|_{t=0} - 2\pi i f(\hat{S}) \int_M \frac{1}{16\pi^2} \mathrm{Tr}(a^{-1}\dot{\hat{\omega}} a \wedge (a^{-1}\hat{\omega} a + a^{-1} da))\Big)\\
&\qquad + f(\hat{S})\partial_t c(\hat{S}^t, a)|_{|t=0},
\end{aligned}$$

which by (32) implies

$$(\nabla^{\mathcal{L}}_{h_0} f)(\hat{S}a) = c(\hat{S}, a)(\nabla^{\mathcal{L}}_{h_0} f)(\hat{S}).$$

This shows that the connection maps indeed into sections of $\mathcal{L}$. □

This connection is $\mathcal{D}(\overline{X})$–invariant (recall that $\mathcal{D}(\overline{X})$ acts on $\mathcal{L}$ over $\mathcal{M}_{-1}(M)$), thus we get a connection in the Chern–Simons bundle over $\mathcal{T}$ and any of its quotients by a subgroup of the mapping class group acting freely on $\mathcal{T}$ whose elements can be realized as diffeomorphism of $\overline{X}$ for some given $\overline{X}$ bounding $M$.

A straightforward application of Koszul formula shows:

**Lemma 25** *Let $S \in \mathcal{C}^\infty(\overline{X}, F_0(X))$ be an even to first order orthonormal frame on $X$ with respect to an even to first order AH metric $g$, and let $g^t$ be a curve of even to first order AH metrics with $g^0 = g$. Write the metric $g^t$ near the boundary under the funnel form (23)*

$$g^t = \frac{dx^2 + h^t(x)}{x^2}.$$

*Then the parallel transported frame $S^t$ of $S$ in the $t$ direction with respect to the metric $G = dt^2 + g^t$ is equal to $x\hat{S}^t$, where $x$ is a geodesic boundary defining function for*

*$g$ and $\widehat{S}^t$ is the parallel transported frame of $\widehat{S} := x^{-1}S$ for $\widehat{G}^t = dt^2 + x^2 g^t$ in the $t$–direction.*

## 7.7 The curvature of $\nabla^{\mathcal{L}}$

Consider the trivial fibration $\mathcal{M}_{-1}(M) \times M \to \mathcal{M}_{-1}(M)$ with fiber type $M$, with metric $h$ along the fiber above $h \in \mathcal{M}_{-1}(M)$. The action of the group $\mathcal{D}(M)$ on $\mathcal{M}_{-1}(M)$ extends isometrically to the fibers, thus by quotienting through the free action of $\mathcal{D}_0(M)$ we obtain the so-called universal curve $\mathcal{F} \to \mathcal{T}$ with fiber type $M$, which is a Riemannian submersion over $\mathcal{T}$. In the proof below we shall consider the restriction of the fibration $\mathcal{M}_{-1}(M) \times M \to \mathcal{M}_{-1}(M)$ above the image of a local section in $\mathcal{M}_{-1}(M) \to \mathcal{T}$. The resulting trivial fibration is canonically diffeomorphic to an open set in $\mathcal{F}$ but not isometric, although the identification is an isometry along the fibers.

**Proposition 26** *The curvature of $\nabla^{\mathcal{L}}$ equals $(i/8\pi)\omega_{\mathrm{WP}}$, where $\omega_{\mathrm{WP}}$ denotes the Weil–Petersson symplectic form on $\mathcal{T}$, $\omega_{\mathrm{WP}}(U, V) = \langle JU, V\rangle_{\mathrm{WP}}$.*

**Proof** Let $f$ be a local section in $\mathcal{L} \to \mathcal{U} \subset \mathcal{T}$ constructed as follows: first, choose a local section $s\colon \mathcal{U} \subset \mathcal{T} \to \mathcal{M}_{-1}(M)$ in the principal fibration $\mathcal{M}_{-1}(M) \to \mathcal{T}$, ie, a smooth family of hyperbolic metrics $\mathcal{U} \ni [h] \mapsto h$, which by projection give a local parametrization of $\mathcal{T}$. By restricting the metric of $\mathcal{M}_{-1}(M) \times M$ to $s(\mathcal{U}) \times M =: \mathcal{M}^{\mathcal{U}}$, we obtain a metric on $\mathcal{M}^{\mathcal{U}}$ with respect to which $s(\mathcal{U})$ and $M$ are orthogonal, the projection on $\mathcal{U}$ is a Riemannian submersion on the Weil–Petersson metric (24) on $s(\mathcal{U})$, and the metric on the fiber $\{h\} \times M$ is $h = s([h])$. Next, extend each metric $h \in s(\mathcal{U})$ to a metric $g_{[h]}$ on a fixed compact manifold $\overline{X}$ with boundary $M$, so that for each $[h] \in \mathcal{U}$, $g_{[h]}$ restricts to $h$ on $M$, has totally geodesic boundary, and depends smoothly on $[h]$. We get in this way a metric $G$ on $\mathcal{X}^{\mathcal{U}} := s(\mathcal{U}) \times \overline{X}$ with respect to which $s(\mathcal{U})$ and $\overline{X}$ are orthogonal, the projection on $\mathcal{U}$ is a Riemannian submersion, and the metric on the fiber $\{h\} \times \overline{X}$ is $g_{[h]}$. Define

$$f\colon \mathcal{U} \to \mathcal{L}, \qquad f([h]) := e^{2\pi i \,\mathrm{CS}(g_{[h]}, \cdot)}.$$

Let $\mathbb{R} \ni t \mapsto h^t$ be a smooth curve in $s(\mathcal{U})$ parametrized by arc-length and $\dot{h}$ its tangent vector at $t = 0$. By the variation formula (11), the covariant derivative of the section $f$ in the direction $[\dot{h}]$ is

$$(\nabla^{\mathcal{L}}_{[\dot{h}]} f)(S) = -\frac{2\pi i}{16\pi^2} f(S) \int_{\overline{X}} 2\,\mathrm{Tr}(\dot{\omega} \wedge \Omega),$$

where $\Omega$ is the curvature tensor of $g_{[h]}$ on $\overline{X}$, and $\omega$ is the connection 1–form, in any orthonormal frame $S$ for $h^0$, parallel transported in the direction of $\partial_t$ with respect to the metric $G = dt^2 + g^t$, where $g^t := g([h^t])$. Therefore the connection 1–form $\alpha \in \Lambda^1(\mathcal{U})$ of $\nabla^{\mathcal{L}}$ in the trivialization $f$ is given by

$$\alpha([\dot{h}]) = \frac{1}{4\pi i} \int_{\overline{X}} \sum_{i,j=1}^{3} \dot{\omega}_{ij} \wedge \Omega_{ji}$$

(we note that this does not depend on $S$ anymore). Let $R^G$ be the curvature tensor of $G$, and $R^V$ the curvature of the vertical connection $\nabla^V := \Pi_{T\overline{X}} \circ \nabla^G$. As a side note, we remark that this vertical connection is independent on the choice of metric on the horizontal distribution, so we could have chosen in the definition of $G$ any other metric, for instance the one induced from $\mathcal{T}$ via the projection. We compute

$$\begin{aligned} \partial_t \omega_{ij}(Y) &= \langle R^G_{\partial_t, Y} S_j, S_i \rangle = \langle R^V_{\partial_t, Y} S_j, S_i \rangle, \\ \Omega_{ji}(Y_2, Y_3) &= \langle R^G_{Y_2, Y_3} S_i, S_j \rangle = \langle R^V_{Y_2, Y_3} S_i, S_j \rangle, \end{aligned}$$

where the scalar products are with respect to $G$. This implies

$$\alpha([\dot{h}]) = \frac{1}{8\pi i} \int_{\overline{X}} \partial_t \lrcorner \operatorname{Tr}((R^V)^2).$$

The Chern–Simons form of the connection 1–form $\omega^V$ of $\nabla^V$ in a vertical frame $S$ is a transgression for the Chern–Weil form $\operatorname{Tr}((R^V)^2)$:

$$d \operatorname{cs}(\omega^V) = \operatorname{Tr}((R^V)^2).$$

Writing $d = d^{\overline{X}} + d^{\mathbb{R}}$ and using Stokes, we get

$$\alpha([\dot{h}]) = \frac{1}{8\pi i} \left( \int_M \partial_t \lrcorner \operatorname{cs}(\omega^V) \right) + \frac{1}{8\pi i} \partial_t \left( \int_{\overline{X}} \operatorname{cs}(\omega^V) \right) \Big|_{t=0}.$$

Thus the connection 1–form of $\nabla^{\mathcal{L}}$ over $s(\mathcal{U})$ satisfies

$$\alpha = \frac{1}{8\pi i} \int_{\mathcal{M}^{\mathcal{U}}/s(\mathcal{U})} \operatorname{cs}(\omega^V) + \frac{1}{8\pi i}\, d \int_{\mathcal{X}^{\mathcal{U}}/s(\mathcal{U})} \operatorname{cs}(\omega^V).$$

The second contribution is an exact form; the curvature of $\nabla^{\mathcal{L}}$ is therefore the horizontal exterior differential

$$R^{\mathrm{SO}(3)} = d\alpha = \frac{1}{8\pi i} \int_{\mathcal{M}^{\mathcal{U}}/s(\mathcal{U})} d^H \operatorname{cs}(\omega^V).$$

By Stokes, we can add inside the integral the vertical exterior differential, thus:

$$R^{\mathrm{SO}(3)} = \frac{1}{8\pi i}\int_{\mathcal{M}^{\mathcal{U}}/s(\mathcal{U})} d\,\mathrm{cs}(\omega^V) \tag{33}$$
$$= \frac{1}{8\pi i}\int_{\mathcal{M}^{\mathcal{U}}/s(\mathcal{U})} \mathrm{Tr}((R^V)^2) = \frac{1}{8\pi i}\int_{\mathcal{M}^{\mathcal{U}}/s(\mathcal{U})} \mathrm{Tr}(R^2)$$

Here $R$ is the curvature of the vertical tangent bundle of $\mathcal{M}^{\mathcal{U}} \to s(\mathcal{U})$ with respect to the natural connection induced by the vertical metric and the horizontal distribution. Notice that the vertical tangent bundle of the fibration $\mathcal{X}^{\mathcal{U}} \to s(\mathcal{U})$ splits orthogonally along $\mathcal{M}^{\mathcal{U}} \to s(\mathcal{U})$ into a flat real line bundle corresponding to the normal bundle to $M \subset \overline{X}$, and the tangent bundle to the fibers of $\mathcal{M}^{\mathcal{U}}$. Thus in the above Chern–Weil integral we can eliminate the normal bundle to $M$ in $\overline{X}$, which justifies the last equality in (33).

Next, we compute explicitly this integral along the fibers of the universal curve in terms of the Weil–Petersson form on $\mathcal{T}$. Take a 2–parameter family $h^{t,s}$ in $\mathcal{M}_{-1}(M)$ and let $\dot{H}^t, \dot{H}^s \in \mathrm{End}(TM)$ be defined by

$$\partial_t h^{t,s}|_{t=s=0} = h(\dot{H}^t\cdot,\cdot), \quad \partial_s h^{t,s}|_{t=s=0} = h(\dot{H}^s\cdot,\cdot),$$

where $h := h^{t,s}|_{t=s=0}$.

Let $X_1, X_2$ be a local frame on $M$, orthogonal at some point $p \in M$ with respect to the metric $h$, and $R$ the curvature of the connection on $TM$ over $\mathbb{R}^2 \times M$.

**Lemma 27** *At the point $p \in M$ where the frame $X_j$ is orthonormal, we have*

$$R_{\partial_s\partial_t}X_j = -\tfrac{1}{4}[\dot{H}^s, \dot{H}^t]X_j, \quad \langle R_{X_1,X_2}X_2, X_1\rangle = -1$$

*and if we choose the family $h$ such that $\dot{H}^t \in V_h$, the space of transverse traceless symmetric 2–tensors, then $R_{\partial_t X_j} = 0$.*

**Proof** We first compute from Koszul's formula

$$\langle \nabla_{\partial_t}X_i, X_j\rangle = \tfrac{1}{2}\partial_t\langle X_i, X_j\rangle = \tfrac{1}{2}\langle \dot{H}^t(X_i), X_j\rangle,$$

so $\nabla_{\partial_t}X_i = \frac{1}{2}\dot{H}^t(X_i)$ and similarly $\nabla_{\partial_s}X_i = \frac{1}{2}\dot{H}^s(X_i)$. Next we compute

$$\begin{aligned}
&\langle R_{\partial_s\partial_t}X_i, X_j\rangle \\
&\quad= \langle \nabla_{\partial_s}\nabla_{\partial_t}X_i, X_j\rangle - \langle \nabla_{\partial_t}\nabla_{\partial_s}X_i, X_j\rangle \\
&\quad= \partial_s\langle \nabla_{\partial_t}X_i, X_j\rangle - \langle \nabla_{\partial_t}X_i, \nabla_{\partial_s}X_j\rangle - \partial_t\langle \nabla_{\partial_s}X_i, X_j\rangle + \langle \nabla_{\partial_s}X_i, \nabla_{\partial_t}X_j\rangle \\
&\quad= \tfrac{1}{2}\partial_s\partial_t\langle X_i, X_j\rangle - \tfrac{1}{4}\langle \dot{H}^t(X_i), \dot{H}^s(X_j)\rangle - \tfrac{1}{2}\partial_t\partial_s\langle X_i, X_j\rangle + \tfrac{1}{4}\langle \dot{H}^s(X_i), \dot{H}^t(X_j)\rangle
\end{aligned}$$

which proves the first identity of the lemma. The second identity is simply the fact that metric along the fibers has curvature $-1$. For the third, assume that $\dot{H}^t$ is transverse traceless. At a fixed point $p\in M$ choose a holomorphic coordinate $z=x_1+ix_2$ for $h$ such that $h=|dz^2|+O(|z|^2)$, and choose $X_1=\partial_{x_1}$, $X_2=\partial_{x_2}$. Using that $\nabla_{X_i}X_j=0$ at $p$, we compute at that point

$$\begin{aligned}\langle\nabla_{\partial_t}\nabla_{X_1}X_1,X_2\rangle&=\partial_t\langle\nabla_{X_1}X_1,X_2\rangle=\partial_t(X_1\langle X_1,X_2\rangle-\tfrac{1}{2}X_2\langle X_1,X_1\rangle)\\&=\partial_{x_1}\dot{H}^t_{12}-\tfrac{1}{2}\partial_{x_2}\dot{H}^t_{11},\\ \langle\nabla_{X_1}\nabla_{\partial_t}X_1,X_2\rangle&=X_1\langle\nabla_{\partial_t}X_1,X_2\rangle=\tfrac{1}{2}\partial_{x_1}\dot{H}^t_{12}\end{aligned}$$

which implies at $p$

$$\langle R_{\partial_t,X_1}X_1,X_2\rangle=\tfrac{1}{2}(\partial_{x_1}\dot{H}^t_{12}-\partial_{x_2}\dot{H}^t_{11}).$$

This last quantity vanishes by the Cauchy–Riemann equations when we expand $\dot{H}^t_{ij}$ using $\dot{H}^t=\mathrm{Re}(f(z)\,dz^2)$ for some holomorphic function $f$. □

Lemma 27 implies for the trace of the curvature at $p\in M$,

$$\begin{aligned}\mathrm{Tr}(R^2)(\partial_s,\partial_t,X_1,X_2)&=2\,\mathrm{Tr}(R_{\partial_s,\partial_t}R_{X_1,X_2})=4\langle R_{\partial_s,\partial_t}X_1,X_2\rangle\langle R_{X_1,X_2}X_2,X_1\rangle\\&=\langle[\dot{H}^s,\dot{H}^t]X_1,X_2\rangle\\&=-\langle\dot{H}^s\dot{H}^tJX_2,X_2\rangle-\langle\dot{H}^t\dot{H}^sX_1,JX_1\rangle=-\mathrm{Tr}(J\dot{H}^s\dot{H}^t).\end{aligned}$$

Since $\dot{H}^t$ is transverse traceless, the Weil–Petersson inner product of the vectors $\partial_t$, $J\partial_s\in T_h\mathcal{T}$ is just the $L^2$ product $\int_M\mathrm{Tr}(\dot{H}^tJ\dot{H}^s)\,\mathrm{dvol}_h$. The proof is finished by applying (33). □

The identity (33) expressing the curvature of the Chern–Simons bundle as the fiberwise integral of the Pontrjagin form $\mathrm{Tr}(R^2)$ was proved for arbitrary surface fibrations by U Bunke [3] in the context of smooth cohomology.

Since the curvature of the connection $\nabla^{\mathcal{L}}$ is a $(1,1)$ form, we get the:

**Corollary 28** *The complex line bundle $\mathcal{L}$ on $\mathcal{T}$ has a holomorphic structure induced by the connection $\nabla^{\mathcal{L}}$, such that the $\overline{\partial}$ operator is the $(0,1)$–component of $\nabla^{\mathcal{L}}$.*

# 8 Variation of the Chern–Simons invariant and curvature of $\mathcal{L}$

In this section we study the covariant derivative of the Chern–Simons $\mathrm{CS}(\theta)$ invariant viewed as a section in the pull-back of Chern–Simons bundle to $T\mathcal{T}$.

By Proposition 16, Proposition 15 and Lemma 22, if $g^t$ is a curve of convex co-compact hyperbolic 3–manifolds, then the invariant

$$e^{2\pi i\,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g^t,\cdot)}$$

can be seen as a section of the line bundle $\mathcal{L}$ over a curve $h_0^t \in \mathcal{M}_{-1}(M)$ induced by the conformal infinities of $g^t$.

**Theorem 29** *Let $(X, g^t)$, $t \in (-\epsilon, \epsilon)$, be a smooth curve of convex co-compact hyperbolic 3–manifolds with conformal infinity a Riemann surface $M$, and such that $g$ is isometric near $M$ to the funnel $(0,\epsilon)_x \times M$*

$$g^t = \frac{dx^2 + h^t(x)}{x^2}, \quad h(x) = h_0^t + x^2 h_2^t + \tfrac{1}{4} x^4 h_2^t \circ h_2^t, \tag{34}$$

*with $(h_0^t, h_2^t - \frac{1}{2}h_0^t) \in T\mathcal{T}$. Let $S \in \mathcal{C}^\infty(\overline{X}, F_0(\overline{X}))$ be an orthonormal frame for $g^0$ and let $S^t$ be the parallel transport of $S$ in the $t$ direction with respect to the metric $G = dt^2 + g^t$ on $X \times (-\epsilon, \epsilon)$. Then, setting $\dot{h}_0 := \partial_t h_0^t|_{t=0}$ and $h_2 := (h_2^t - \frac{1}{2}h_0^t)|_{t=0}$ so that $\dot{h}_0, h_2 \in T_{h_0}\mathcal{T}$, one has*

$$\partial_t \,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g^t, S^t)|_{t=0} = \frac{1}{16\pi^2}\int_M \mathrm{Tr}(\dot{\widehat{\omega}} \wedge \widehat{\omega}) + \frac{i}{8\pi^2}\langle (\mathrm{Id} - iJ)\dot{h}_0, h_2\rangle_{\mathrm{WP}}.$$

Notice, by Lemma 25, that $\widehat{S}^t := x^{-1}S^t$ is parallel for $\widehat{G}^t = dt^2 + x^2 g^t$ and thus Theorem 29 is sufficient to compute the covariant derivative of $e^{2\pi i\,\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ with respect to $\nabla^{\mathcal{L}}$ in the direction of conformal infinities of hyperbolic metrics on $X$.

Before giving the proof, let us give as an application the variation formula for the renormalized volume.

**Corollary 30** *Let $X^t := (X, g^t)$ be a smooth curve of convex co-compact hyperbolic 3–manifolds like in Theorem 29. Then*

$$\partial_t(\mathrm{Vol}_R(X^t))|_{t=0} = -\tfrac{1}{4}\langle \dot{h}_0, h_2\rangle_{\mathrm{WP}}.$$

**Proof** It suffices to combine Theorem 29 with Proposition 16 and consider the imaginary part in the variation formula of $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}$. □

This formula was proved by Krasnov and Schlenker [17], using the Schläfli formula, in order to show that the renormalized volume is a Kähler potential for the Weil–Petersson metric on Teichmüller space. The Chern–Simons approach thus provides another proof.

**Proof of Theorem 29** Let $T \in \Lambda^1(X, \mathrm{End}(TX))$ be defined by $T_U(V) := -U \times V$, where $\times$ denotes the vector product with respect to the Riemannian metric. Clearly $T$ is anti-symmetric. We consider a 1–parameter family of metrics on $X$ hyperbolic outside a compact set,

$$g^t = \frac{dx^2 + h^t(x)}{x^2},$$

and we define a Riemannian metric on $\mathbb{R} \times X$ by $G = dt^2 + g^t$. Recall that for every fixed $t$, the metrics $h^t(x)$ and $h_0^t := h^t(0)$ on $M$ are related by (6).

For a given section $S$ in the orthonormal frame bundle for $g_0$, we define $S^t$ as the parallel transport in the $t$ direction of $S$, more precisely $\nabla_{\partial_t} S_j^t = 0$ for $j = 1, 2, 3$. Here and in what follows, $\nabla$ denotes the Riemannian connection for $G$. Since the integral curves of $\partial_t$ are geodesics, it follows that $S^t$ is an orthonormal frame for $g^t$.

Consider the connections $D^t = \nabla^{g^t} + iT^t$ on $T_{\mathbb{C}}X$ corresponding to the metric $g^t$. In the trivialization given by the section $S^t$, the connection form is $\theta^t = \omega^t + iT^t$. It is a $\mathrm{so}(3) \otimes \mathbb{C}$–valued 1–form with real and imaginary parts

$$\omega_{ij}^t(Y) = \langle \nabla_Y^{g^t} S_j^t, S_i^t \rangle_{g^t}, \quad T_{ij}^t(Y) = \langle Y \times^t S_i^t, S_j^t \rangle_{g^t}.$$

We first compute the variation with respect to $t$ of the Chern–Simons form of $\theta^t$ on $X$. In what follows, we will drop the superscript $t$ when we evaluate at $t = 0$ and we shall use a dot to denote the $t$–derivative at $t = 0$. Substituting in (11) for $\theta = \omega + iT = \hat{\omega} - \alpha + iT$ like in (19), with $\hat{\omega}$ the connection form of the Levi-Civita connection of the conformally compactified metric $\hat{g} = x^2 g$, we get

$$\begin{aligned}
(35)\quad & \partial_t \, \mathrm{cs}(\theta^t)|_{t=0} \\
&= d(\mathrm{Tr}(\dot{\omega} \wedge \omega) - \mathrm{Tr}(\dot{T} \wedge T) + i(\mathrm{Tr}(\dot{\omega} \wedge T) + \mathrm{Tr}(\dot{T} \wedge \omega))) + 2\,\mathrm{Tr}(\dot{\theta} \wedge \Omega^\theta) \\
&= d\,\mathrm{Tr}(\dot{\hat{\omega}} \wedge \hat{\omega}) + d[\mathrm{Tr}(\dot{\alpha} \wedge \hat{\omega}) + \mathrm{Tr}(\dot{\hat{\omega}} \wedge \alpha)] + i\,d[\mathrm{Tr}(\dot{\omega} \wedge T) + \mathrm{Tr}(\dot{T} \wedge \omega)] \\
&\qquad + d[\mathrm{Tr}(\dot{\alpha} \wedge \alpha) - \mathrm{Tr}(\dot{T} \wedge T)] + 2\,\mathrm{Tr}(\dot{\theta} \wedge \Omega^\theta).
\end{aligned}$$

Observe that if $g^t$ is a variation through hyperbolic metrics on $X$ then $\Omega^\theta$ vanishes. We claim that in the variation formula for the Chern–Simons invariant of $\theta^t$, the finite parts corresponding to the terms $\mathrm{Tr}(\dot{\alpha} \wedge \alpha)$ and $\mathrm{Tr}(\dot{T} \wedge T)$ vanish. We start with the term $\mathrm{Tr}(T \wedge \dot{T})$:

**Lemma 31** *We have*

$$\mathrm{FP}_{\epsilon=0} \int_{x=\epsilon} \mathrm{Tr}(T \wedge \dot{T}) = 0.$$

**Proof** Let $Y_1, Y_2$ be vector fields on $M$, independent of $t$, then using that $\nabla^G_{\partial_t} S_j = 0$, we have $\dot{T}_{ij}(Y_k) = \langle \nabla^G_{\partial_t} Y_k \times S_i, S_j \rangle$ so

$$\begin{aligned}
\operatorname{Tr}(T \wedge \dot{T})(Y_1, Y_2)& \\
&= \sum_{i,j} \langle Y_1 \times S_i, S_j \rangle \langle \nabla^G_{\partial_t} Y_2 \times S_i, S_j \rangle - \langle Y_2 \times S_i, S_j \rangle \langle \nabla^G_{\partial_t} Y_1 \times S_i, S_j \rangle \\
&= \langle Y_1, \nabla_{\partial_t} Y_2 \rangle - \langle Y_2, \nabla_{\partial_t} Y_1 \rangle
\end{aligned}$$

which is zero because by Koszul, $\langle Y_1, \nabla_{\partial_t} Y_2 \rangle = \frac{1}{2}(L_{\partial_t} G)(Y_1, Y_2)$ is symmetric in $Y_1, Y_2$. □

**Lemma 32** *For $\epsilon > 0$ sufficiently small, we have $\operatorname{Tr}(\dot{\alpha} \wedge \alpha)|_{x=\epsilon} = 0$.*

**Proof** Let $Y_1, Y_2$ be tangent vector fields to $M$, independent of $t$. Notice that for $S_i$ parallel with respect to $\nabla^G_{\partial_t}$ then $\widehat{S}_i = x^{-1} S_i$ is parallel with respect to $\nabla^{\widehat{G}}_{\partial_t}$, where $\widehat{G} = dt^2 + \widehat{g}^t$. Then since $\partial_x$ is also killed by $\nabla^{\widehat{G}}_{\partial_t}$,

$$\begin{aligned}
&x \operatorname{Tr}(\dot{\alpha} \wedge \alpha)(Y_1, Y_2) \\
&\quad = \partial_t [\widehat{g}^t(Y_1, \widehat{S}^t_i) \widehat{S}^t_j(x) - \widehat{g}^t(Y_1, \widehat{S}^t_j) \widehat{S}^t_i(x)]|_{t=0} \big( \widehat{g}(Y_2, \widehat{S}_j) \widehat{S}_i(x) - \widehat{g}(Y_2, \widehat{S}_i) \widehat{S}_j(x) \big) \\
&\qquad\qquad - \operatorname{Sym}(Y_1 \to Y_2) \\
&\quad = -2\widehat{G}(\nabla^{\widehat{G}}_{\partial_t} Y_1, Y_2) + 2\widehat{G}(\nabla^{\widehat{G}}_{\partial_t} Y_2, Y_1) = -2\widehat{G}(\nabla^{\widehat{G}}_{Y_1} \partial_t, Y_2) + 2\widehat{G}(\nabla^{\widehat{G}}_{Y_2} \partial_t, Y_1) \\
&\quad = 2\widehat{G}(\nabla^{\widehat{G}}_{Y_1} Y_2, \partial_t) - 2\widehat{G}(\nabla^{\widehat{G}}_{Y_2} Y_1, \partial_t) = 2\widehat{G}([Y_1, Y_2], \partial_t) = 0
\end{aligned}$$

and this finishes the proof. □

We now consider the term $\operatorname{Tr}(\dot{\widehat{\omega}} \wedge \alpha) + \operatorname{Tr}(\dot{\alpha} \wedge \widehat{\omega})$.

**Lemma 33** *Let $\dot{H}_0$ and $A$ be the symmetric endomorphism on $TM$ defined by $\dot{h}_0(\cdot, \cdot) = h_0(\dot{H}_0 \cdot, \cdot)$ and $h_2(\cdot, \cdot) = h_0(A \cdot, \cdot)$. We have the following identity*

$$\operatorname{FP}_{\epsilon \to 0} \big( \operatorname{Tr}(\dot{\widehat{\omega}} \wedge \alpha) + \operatorname{Tr}(\dot{\alpha} \wedge \widehat{\omega}) \big)|_{x=\epsilon} = 2 \int_M \operatorname{Tr}(J \dot{H}_0 A) \, \operatorname{dvol}_{h_0},$$

*where $J$ is the complex structure on $TM$.*

**Proof** First, from the proof of Proposition 15, we know that $\operatorname{FP}_{\epsilon \to 0} \operatorname{Tr}(\alpha \wedge \widehat{\omega})|_{x=\epsilon} = 0$, and therefore

$$\operatorname{FP}_{\epsilon \to 0} \big( \operatorname{Tr}(\dot{\widehat{\omega}} \wedge \alpha) + \operatorname{Tr}(\dot{\alpha} \wedge \widehat{\omega}) \big)|_{x=\epsilon} = 2 \operatorname{FP}_{\epsilon \to 0} \operatorname{Tr}(\dot{\widehat{\omega}} \wedge \alpha).$$

Now, for $Y_1, Y_2$ tangent to $M$ and independent of $t$, we can use that $\nabla^{\widehat{G}}_{\partial_t} \widehat{S}_i = 0$ and

$$\widehat{\omega}_{ij}(Y) = \widehat{g}(\nabla^{\widehat{g}}_Y \widehat{S}_j, \widehat{S}_i) = \widehat{G}(\nabla^{\widehat{G}}_Y \widehat{S}_j, \widehat{S}_i)$$

to deduce

$$\dot{\widehat{\omega}}_{ij}(Y) = \partial_t \langle \nabla_Y \widehat{S}_j, \widehat{S}_i\rangle = \langle \nabla_{\partial_t} \nabla_Y \widehat{S}_j, \widehat{S}_i\rangle = \langle R_{\partial_t Y} \widehat{S}_j, \widehat{S}_i\rangle,$$

where $\widehat{R}$ is the curvature tensor of $\widehat{G}$, therefore

$$\begin{aligned}
\operatorname{Tr}(\dot{\widehat{\omega}} \wedge \alpha)(Y_1, Y_2) &= \sum_{i,j} \langle R_{\partial_t Y_1} \widehat{S}_j, \widehat{S}_i\rangle (\langle Y_2, S_j\rangle S_i(a) - \langle Y_2, S_i\rangle S_j(a)) \\
&\qquad\qquad - \langle \widehat{R}_{\partial_t Y_2} \widehat{S}_j, \widehat{S}_i\rangle (\langle Y_1, S_j\rangle S_i(a) - \langle Y_1, S_i\rangle S_j(a)) \\
&= 2(\langle \widehat{R}_{\partial_t Y_2} Y_1, x^{-1}\partial_x\rangle - \langle \widehat{R}_{\partial_t Y_1} Y_2, x^{-1}\partial_x\rangle) \\
&= 2x^{-1} \langle \widehat{R}_{\partial_t, \partial_x} Y_1, Y_2\rangle
\end{aligned}$$

by Bianchi. Since we are interested in the finite part, we can modify $Y_1, Y_2$ by a term of order $x^2$ without changing the result, and we will take $\widetilde{Y}_i^t = (1 - \frac{1}{2}x^2 A^t) Y_i$, where the endomorphism $A^t$ of $TM$ is defined by $h_2^t(\cdot,\cdot) = h_0^t(A^t\cdot, \cdot)$. Then

$$\begin{aligned}
\widehat{G}(\widehat{R}_{\partial_t, \partial_x} Y_1, Y_2) &= -\partial_x \big(\widehat{g}(\nabla^{\widehat{G}}_{\partial_t} \widetilde{Y}_1^t, \widetilde{Y}_2^t)\big)\big|_{t=0} + O(x^2) \\
&= -\tfrac{1}{2}\partial_x \big(\partial_t (\widehat{g}^t(\widetilde{Y}_1^t, \widetilde{Y}_2^t)) \\
&\qquad + \widehat{g}^t([\partial_t, \widetilde{Y}_1^t], \widetilde{Y}_2^t)|_{t=0} - \widehat{g}^t([\partial_t, \widetilde{Y}_2^t], \widetilde{Y}_1)|_{t=0}\big) + O(x^2).
\end{aligned}$$

The term $\partial_t(\widehat{g}^t(\widetilde{Y}_1^t, \widetilde{Y}_2^t))|_{t=0}$ is easily seen to be a $\dot{h}_0(Y_1, Y_2) + O(x^3)$ by using that $\widehat{g}^t = dx^2 + h_0^t + x^2 h_0^t(A^t\cdot, \cdot) + O(x^4)$, while the other two terms are

$$\begin{aligned}
&\widehat{g}^t([\partial_t, \widetilde{Y}_1^t], \widetilde{Y}_2^t)|_{t=0} - \widehat{g}^t([\partial_t, \widetilde{Y}_2^t], \widetilde{Y}_1^t)|_{t=0} \\
&\qquad = \tfrac{1}{2}x^2 h_0(\dot{A}Y_1, Y_2) - \tfrac{1}{2}x^2 h_0(\dot{A}Y_2, Y_1) + O(x^4) \\
&\qquad = \tfrac{1}{2}x^2 h_0((\dot{A} - \dot{A}^T)Y_1, Y_2) + O(x^4),
\end{aligned}$$

but since $A^t$ is symmetric with respect to $h_0^t$, we deduce by differentiating at $t = 0$ that $\dot{A} - \dot{A}^T = (\dot{H}_0 A)^T - \dot{H}_0 A$ and therefore

$$\begin{aligned}
&\widehat{g}^t([\partial_t, \widetilde{Y}_1^t], \widetilde{Y}_2^t)|_{t=0} - \widehat{g}^t([\partial_t, \widetilde{Y}_2^t], \widetilde{Y}_1^t)|_{t=0} \\
&\qquad = \tfrac{1}{2}x^2 h_0(\dot{H}_0 A Y_1, J Y_1) + \tfrac{1}{2}x^2 h_0(\dot{H}_0 A Y_2, J Y_2) + O(x^4) \\
&\qquad = -\tfrac{1}{2}x^2 \operatorname{Tr}(J \dot{H}_0 A) + O(x^4).
\end{aligned}$$

We conclude that the limit of $\frac{2}{x}\widehat{G}(\widehat{R}_{\partial_t, \partial_x} Y_1, Y_2)$ as $x \to 0$ is given by $\operatorname{Tr}(J\dot{H}_0 A)$. □

Next, we reduce the sum $\operatorname{Tr}(\dot{T} \wedge \omega) + \operatorname{Tr}(\dot{\omega} \wedge T)$ as follows:

**Lemma 34** *We have the following identity:*

$$\mathrm{FP}_{\epsilon=0}\int_{x=\epsilon}\mathrm{Tr}(\dot{T}\wedge\omega)+\mathrm{Tr}(\dot{\omega}\wedge T)=2\,\mathrm{FP}_{\epsilon=0}\int_{x=\epsilon}\mathrm{Tr}(\dot{\omega}\wedge T)$$

**Proof** It suffices to use (22) to deduce that

$$\partial_t\,\mathrm{FP}_{\epsilon=0}\int_{x=\epsilon}\mathrm{Tr}(\omega\wedge T)=8\pi\,\partial_t(\chi(M))=0. \qquad \square$$

**Proposition 35** *Let $\dot{H}_0$ be the endomorphism on $TM$ defined by*

$$\dot{h}_0(\cdot,\cdot)=h_0(\dot{H}_0\cdot,\cdot).$$

*Then near $x=0$ we have*

$$\mathrm{Tr}(\dot{\omega}\wedge T)=[-x^{-2}\,\mathrm{Tr}(\dot{H}_0)+\mathrm{Tr}(\dot{A})-\tfrac{1}{2}\,\mathrm{Tr}(A)\,\mathrm{Tr}(\dot{H}_0)+\mathrm{Tr}(\dot{H}_0A)]\;\mathrm{dvol}_{h_0}+O(x^2).$$

**Proof** Notice that for every $Y$ tangent to $X$ we have $\omega_{ij}^t(Y)=\omega_{ij}(Y)$, as a simple consequence of the Koszul formula. For a vector field $Y$ on $X$ extended on $\mathbb{R}\times X$ to be constant with respect to the flow of $\partial_t$ we compute

$$(\partial_t\omega_{ij})(Y)|_{t=0}=\partial_t\langle\nabla_Y S_j^t,S_i^t\rangle|_{t=0}=\langle\nabla_{\partial_t}\nabla_Y S_j^t,S_i^t\rangle|_{t=0}=\langle R_{\partial_t,Y}S_j,S_i\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the metric $G$. In the last equality we have used the fact that $S_i^t$ is parallel in the direction of $\partial_t$ and the vanishing of the bracket $[\partial_t,Y]$. By the symmetry of the Riemannian curvature tensor, we rewrite the last term as $-\langle R_{S_i,S_j}\partial_t,Y\rangle$. It follows that

$$\begin{aligned}\text{(36)}\quad &\mathrm{Tr}(\dot{\omega}\wedge T)(Y_1,Y_2)\\ &=\sum_{i,j=1}^{3}-\langle R_{S_iS_j}\partial_t,Y_1\rangle\langle Y_2\times S_j,S_i\rangle+\langle R_{S_iS_j}\partial_t,Y_2\rangle\langle Y_1\times S_j,S_i\rangle\\ &=\sum_{j=1}^{3}-\langle R_{Y_2\times S_j,S_j}\partial_t,Y_1\rangle+\langle R_{Y_1\times S_j,S_j}\partial_t,Y_2\rangle=E(Y_1,Y_2)-E(Y_2,Y_1),\end{aligned}$$

where we have defined

$$E(Y,Z):=\sum_{j=1}^{3}\langle R^G_{Y\times S_j,S_j}\partial_t,Z\rangle.$$

For every vector field $Y$ on $M$, define a vector field $\widetilde{Y}^t$ on a neighborhood of $M$ in $X$ by

$$\widetilde{Y}^t=\Big(1+\frac{x^2}{2}A^t\Big)^{-1}Y,$$

where $h_2^t = h_0^t(A^t\cdot,\cdot)$. From (6) we see that for any orthonormal frame $Y_1, Y_2$ on $M$ for $h_0$, the frame $\widetilde{Y}_1^t, \widetilde{Y}_2^t$ at $t=0$ is also orthonormal on $X$. The complex structure $J$ on $\{t\}\times\{x\}\times M$ satisfies $J\widetilde{Y}^t = x\partial_x \times^t \widetilde{Y}^t$, so in particular $J\widetilde{Y} = \widetilde{JY}$ at $t=0$.

**Lemma 36** *Let $Y, Z$ be vector fields on $M$. Then near $x=0$ we have the expansion*

$$E(J\widetilde{Y},\widetilde{Z}) = x^{-2}\dot{h}_0(Y,Z) - \tfrac{1}{2}(h_0(\dot{A}Y,Z)+h_0(Y,\dot{A}Z)) - \dot{h}_0(AY,Z) + O(x^2). \tag{37}$$

**Proof** The expression defining $E$ is independent of the orthonormal frame $S_j$ for $g$, thus we can compute it using the frame $x\widetilde{Y}, xJ\widetilde{Y}, x\partial_x$ (all these are at $t=0$):

$$E(J\widetilde{Y},\widetilde{Z}) = 2\langle R_{\widetilde{Y},x\partial_x}\partial_t, \widetilde{Z}\rangle.$$

Note the following identities:

$$\begin{aligned} \widetilde{Y}^t &= Y - \frac{x^2}{2}A^tY + O(x^4),\\ [x\partial_x, x\widetilde{Y}^t] &= x\widetilde{Y}^t - x^3A^tY + O(x^5),\\ \nabla^G_{x\partial_x} x\widetilde{Z}^t &= O(x^4). \end{aligned} \tag{38}$$

Also, note that $\nabla^G_{\partial_x}\partial_t = 0$. Using these facts, we get

$$\begin{aligned} \langle\nabla^G_{x\widetilde{Y}}\partial_t, x\widetilde{Z}\rangle &= \tfrac{1}{2}(L_{\partial_t}G)(x\widetilde{Y}^t, x\widetilde{Z}^t)|_{t=0} \\ &= \tfrac{1}{2}\Big(\Big(\partial_t h_0^t\Big(\Big(1+\frac{x^2}{2}A^t\Big)\cdot, \Big(1+\frac{x^2}{2}A^t\Big)\cdot\Big)\Big)(\widetilde{Y},\widetilde{Z})\Big)\Big|_{t=0} + O(x^4) \\ &= \tfrac{1}{2}\dot{h}_0(Y,Z) + \frac{x^2}{4}(h_0(\dot{A}Y,Z)+h_0(Y,\dot{A}Z)) + O(x^4), \end{aligned} \tag{39}$$

therefore by using (38),

$$\begin{aligned} &\langle R_{x\partial_x, x\widetilde{Y}}\partial_t, x\widetilde{Z}\rangle \\ &\quad= x\partial_x\langle\nabla^G_{x\widetilde{Y}}\partial_t, x\widetilde{Z}\rangle - \langle\nabla^G_{[x\partial_x, x\widetilde{Y}]}\partial_t, x\widetilde{Z}\rangle \\ &\quad= (x\partial_x - 1)\langle\nabla^G_{x\widetilde{Y}}\partial_t, x\widetilde{Z}\rangle + x^2\langle\nabla^G_{x\widetilde{AY}}\partial_t, x\widetilde{Z}\rangle \\ &\quad= -\tfrac{1}{2}\dot{h}_0(Y,Z) + \frac{x^2}{4}(h_0(\dot{A}Y,Z)+h_0(Y,\dot{A}Z)) + \frac{x^2}{2}\dot{h}_0(AY,Z) + O(x^4) \end{aligned}$$

(in the last step we have used (39) for $AY$ in the place of $Y$). Using the tensoriality of the curvature to get out the factors of $x$, we proved the lemma. □

Let us now write (all what follows is at $t=0$)

$$Y = \widetilde{Y} + \frac{x^2}{2}AY + O(x^4) = \widetilde{Y} + \frac{x^2}{2}\widetilde{AY} + O(x^4).$$

By linearity we get

$$E(Y,Z) = E(\widetilde{Y},\widetilde{Z}) + \frac{x^2}{2}(E(\widetilde{AY},Z) + E(Y,\widetilde{AZ})) + O(x^2). \tag{40}$$

Assume now that $Y_j$ have been chosen at a given point on $M$ as (orthonormal) eigenvectors of $A$ for $h_0$ of eigenvalue $\lambda_j$, with $JY_1 = Y_2$. Then from (40) we get

$$\begin{aligned}
E(Y_2,Y_1) &= \Big(1 + \frac{x^2}{2}(\lambda_1+\lambda_2)\Big)E(\widetilde{Y}_2,\widetilde{Y}_1),\\
E(Y_1,Y_2) &= \Big(1 + \frac{x^2}{2}(\lambda_1+\lambda_2)\Big)E(\widetilde{Y}_1,\widetilde{Y}_2),
\end{aligned}$$

therefore from (36) and Lemma 36

$$\begin{aligned}
\operatorname{Tr}(\dot\omega\wedge T)(Y_1,Y_2) &= E(Y_1,Y_2) - E(Y_2,Y_1)\\
&= \Big(1+\frac{x^2}{2}\operatorname{Tr}(A)\Big)(E(\widetilde{Y}_1,\widetilde{Y}_2) - E(\widetilde{Y}_2,\widetilde{Y}_1))\\
&= -\Big(1+\frac{x^2}{2}\operatorname{Tr}(A)\Big)(E(J\widetilde{Y}_1,\widetilde{Y}_1) + E(J\widetilde{Y}_2,\widetilde{Y}_2))\\
&= \Big(1+\frac{x^2}{2}\operatorname{Tr}(A)\Big)(-x^{-2}\operatorname{Tr}(\dot H^0) + \operatorname{Tr}(\dot A) + \operatorname{Tr}(\dot H_0 A) + O(x^2))\\
&= -x^{-2}\operatorname{Tr}(\dot H_0) + \operatorname{Tr}(\dot A) - \tfrac12\operatorname{Tr}(A)\operatorname{Tr}(\dot H_0) + \operatorname{Tr}(\dot H_0 A) + O(x^2)
\end{aligned}$$

which is the claim of Proposition 35. □

We are now in position to finish the proof of Theorem 29. Since we consider a family of hyperbolic metrics $g^t$, we have $\operatorname{Tr}(A^t) = -\frac12 \operatorname{scal}_{h_0^t}$ by (7), so by Gauss–Bonnet the following integral is constant in $t$:

$$\int_M \operatorname{Tr}(A^t)\ \mathrm{dvol}_{h_0^t} = 2\pi\chi(M). \tag{41}$$

Using $\partial_t\,\mathrm{dvol}_{h_0^t}|_{t=0} = \frac12\operatorname{Tr}(\dot h_0)\ \mathrm{dvol}_{h_0}$, we deduce by differentiating (41) that

$$\int_M (\operatorname{Tr}(\dot A) + \tfrac12\operatorname{Tr}(A)\operatorname{Tr}(\dot H_0))\ \mathrm{dvol}_{h_0} = 0,$$

so

$$\int_{x=\epsilon}\operatorname{Tr}(\dot\omega\wedge T) = -\epsilon^{-2}\partial_t\operatorname{Vol}(M,h_0) + \int_M (2\operatorname{Tr}(\dot A) + \operatorname{Tr}(\dot H_0 A))\ \mathrm{dvol}_{h_0} + O(\epsilon^2).$$

This achieves the proof of Theorem 29. □

# 9 Chern–Simons line bundle and determinant line bundle

Ramadas, Singer and Weitsman [25] introduced the Chern–Simons line bundle on the moduli space $\mathcal{A}^s_F/\mathcal{G}$ of irreducible flat SU(2) connections up to gauge, they showed that it has a natural connection whose curvature is (up to a factor of $i$) the standard symplectic form, and a natural Hermitian structure. Quillen [24] defined the determinant line bundle over the space $\{\overline{\partial}_A : A \in \mathcal{A}^s_F\}$ of d-bar operators for a given complex structure on the surface $M$: he showed that it descends to $\mathcal{A}^s_F/\mathcal{G}$ as a Hermitian line bundle with a natural connection and with curvature the standard symplectic form (up to a factor of $i$). Ramadas, Singer and Weitsman proved that these bundles are isomorphic as Hermitian line bundles with connection over $\mathcal{A}^s_F/\mathcal{G}$. Moreover their curvature form is of $(1,1)$ type with respect to the natural complex structure on $\mathcal{A}^s_F/\mathcal{G}$ and therefore the line bundle admits a holomorphic structure. In what follows, we shall construct, in particular cases, a similar isomorphism using our Chern–Simons invariant and the determinant of the Laplacian.

## 9.1 The submanifold $\mathcal{H}$ of hyperbolic 3–manifolds

Let us be more precise and first make the following assumption: if $\mathcal{T}$ is Teichmüller space for a given oriented surface $M$ of genus $g$ (possibly not connected) and $h_0 \in \mathcal{T}$, we assume that we fix a convex co-compact 3–manifold $X$ with conformal boundary $(M, h_0)$.

**Proposition 37** *There exists a neighborhood $\mathcal{U} \subset \mathcal{T}$ of $h_0$ and a smooth map $F\colon \mathcal{U} \to \mathcal{C}^\infty(\overline{X}, S^2_+({}^0T^*\overline{X}))$ such that $F(h)$ is hyperbolic convex-cocompact with conformal boundary $(M, h)$ for all $h \in \mathcal{U}$.*

**Proof** The proof is written for instance in Moroianu and Schlenker [23]. A quasiconformal approach can be found for instance in Marden [18]. □

This map induces by Lemma 19 a local section in the tangent bundle of $\mathcal{T}$. By Mostow rigidity [18, Theorem 2.12] and Marden [18, Theorem 3.1], this section is unique and extends to a global smooth section $\sigma\colon \mathcal{T} \to T\mathcal{T}$. The graph

$$\mathcal{H} := \{(h, \sigma(h)) \in T\mathcal{T} : h \in \mathcal{T}\} \tag{42}$$

is then a smooth submanifold of $T\mathcal{T}$ of dimension $\dim \mathcal{T}$. By uniqueness, the subgroup of modular transformations of $M$ consisting of classes of diffeomorphisms that extend to $\overline{X}$ leaves this section invariant, therefore $\sigma$ descends to any quotient of $\mathcal{T}$ by such a subgroup. For instance, this applies to the deformation space of a given convex co-compact hyperbolic 3 manifold $X = \Gamma\backslash\mathbb{H}^3$, which is the quotient $\mathcal{T}_X := \mathcal{T}/\operatorname{Mod}_X$ of $\mathcal{T}$ by the subgroup $\operatorname{Mod}_X$ defined in Section 4.

Let us introduce a new connection on the pull-back of $\mathcal{L}$ to $T\mathcal{T}$ (still denoted $\mathcal{L}$), for which the $\mathrm{PSL}_2(\mathbb{C})$ Chern–Simons section is flat along the deformation space of hyperbolic metrics on $X$.

**Definition 38** We define the connection $\nabla^{\mu}$ on $\mathcal{L}$ by

$$\nabla^{\mu} = \nabla^{\mathcal{L}} + \frac{1}{2\pi}\mu^{1,0},$$

where $\mu$ is the Liouville 1–form $T\mathcal{T}$ (obtained from $T^*\mathcal{T}$ and the duality isomorphism $T\mathcal{T} \to T^*\mathcal{T}$ induced by $\langle \cdot, \cdot \rangle_{\mathrm{WP}}$). The connection $\nabla^{\mathcal{L}}$ is understood as the pull-back of the $\nabla^{\mathcal{L}}$ connection on $\mathcal{T}$ by $\pi\colon T\mathcal{T} \to \mathcal{T}$.

This connection is *not* Hermitian with respect to $\langle \cdot, \cdot \rangle_{\mathrm{CS}}$ since the form $\mu^{1,0}$ is not purely imaginary. The Chern–Simons line bundle $\mathcal{L}$ equipped with the connection $\nabla^{\mu}$ has curvature

$$\Omega_{\nabla^{\mu}} = \frac{i}{8\pi}\omega_{\mathrm{WP}} + \frac{1}{2\pi}\bar{\partial}\mu^{1,0} \tag{43}$$

with real part $\mathrm{Re}(\Omega_{\nabla^{\mu}}) = \frac{1}{4\pi}d\mu$, where here and below, $\omega_{\mathrm{WP}}$ is understood as $\pi^*\omega_{\mathrm{WP}}$ if $\pi = T\mathcal{T} \to \mathcal{T}$ is the projection on the basis.

**Theorem 39** *The Chern–Simons invariant $e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ restricted to the submanifold $\mathcal{H}$ in (42) is a parallel section of $\mathcal{L}|_{\mathcal{H}}$ for the connection $\nabla^{\mu}$. As a result, $\mathcal{H}$ is a Lagrangian submanifold of $T\mathcal{T}$ for the standard symplectic Liouville form $d\mu$ on $T\mathcal{T}$ obtained from pull-back by the duality isomorphism $T\mathcal{T} \to T^*\mathcal{T}$ induced by $\langle \cdot, \cdot \rangle_{\mathrm{WP}}$.*

**Proof** This is a direct consequence of the variation formula in Theorem 29 and the definition of the connection $\nabla^{\mu}$. □

It is proved by Krasnov [15] for Schottky cases and more generally by Takhtajan and Teo [27] for Kleinian groups of class A (see also Krasnov and Schlenker [17] for quasi-Fuchsian cases), that

$$\bar{\partial}\partial(\mathrm{Vol}_R) = \frac{i}{16}\,\omega_{\mathrm{WP}}$$

using previous work of Takhtajan and Zograf [28] on the Liouville functional. Here we use our convention for the Weil–Petersson metric. The theorem above generalizes these results providing a unified treatment:

**Corollary 40** *For $h_0$ in an open set $U \subset \mathcal{T}$, let $X_h = (X, g_h)$ be a smooth family of convex co-compact hyperbolic 3–manifolds with conformal infinity parametrized smoothly by $h \in U$; then*

$$\bar{\partial}\partial\,\mathrm{Vol}_R(X_h) = \frac{i}{16}\,\omega_{\mathrm{WP}}.$$

**Proof** Let $\sigma\colon U \subset \mathcal{T} \to T\mathcal{T}$ be the section $h \to (h, \sigma(h))$ parametrizing the submanifold $\mathcal{H}$. We consider $\mathrm{Vol}_R(X_h)$ as a function on $U$. By Corollary 30, we have for $\dot{h} \in T_h\mathcal{T}$

$$\partial \,\mathrm{Vol}_R(X_h).\dot{h} = -\tfrac{1}{4}\sigma^*\mu^{1,0}(\dot{h}).$$

From the vanishing of the curvature $\Omega_{\mathcal{L}}$ on $\mathcal{H}$ and the formula (43), we obtain for any $\dot{h}, \dot{\ell} \in T_h\mathcal{T}$

$$d\mu^{1,0}_{\sigma(h)}(d\sigma.\dot{h}, d\sigma.\dot{\ell}) = -\tfrac{i}{4}\omega_{\mathrm{WP}}(\dot{h}, \dot{\ell})$$

and since $\sigma^* d\mu_h^{1,0}(\dot{h}, \dot{\ell}) = d(\sigma^*\mu^{1,0})(\dot{h}, \dot{\ell}) = -4\bar{\partial}\partial\,\mathrm{Vol}_R(X_h)(\dot{h}, \dot{\ell})$, the proof is finished. □

## 9.2 An isomorphism with the determinant line bundle

Finally, we construct an explicit isomorphism of Hermitian line bundles between $\mathcal{L}_{\mathcal{H}}$ and the determinant line bundle in the particular cases of Schottky manifolds.

Let $M$ be a marked Riemann surface of genus $g$, ie, a surface with a distinguished set of generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of $\pi_1(M, x_0)$ for some $x_0 \in M$. With respect to this marking, if a complex structure is given on $M$, there is a basis $\varphi_1, \dots, \varphi_g$ of holomorphic 1–forms such that $\int_{\alpha_j} \varphi_i = \delta_{ij}$ and this defines the period matrix $(\tau_{ij}) = (\int_{\beta_j} \varphi_i)$ whose imaginary part is positive definite since $2\,\mathrm{Im}\,\tau_{ij} = \langle \varphi_i, \varphi_j \rangle$. Schottky groups are free groups generated by $L_1, \dots, L_g \in \mathrm{PSL}_2(\mathbb{C})$ that map circles $C_1 \dots, C_g \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to other circles $C_{-1}, \dots, C_{-g} \in \widehat{\mathbb{C}}$ (with orientation reversed). Each element $\gamma \in \Gamma$ is conjugated in $\mathrm{PSL}_2(\mathbb{C})$ to $z \to q_\gamma z$ for some $q_\gamma \in \mathbb{C}$ with $|q_\gamma| < 1$, called the multiplier of $\gamma$. The quotient of the discontinuity set $\Omega_\Gamma$ of $\Gamma$ by $\Gamma$ is a closed Riemann surface and every closed Riemann surface of genus $g$ can be represented in this manner by a result of Koebe; see Ford [6]. The Schottky group is marked if each $C_k$ is homotopic to $\alpha_k$ in the quotient $\Gamma\backslash\Omega_\Gamma$. The marked group is unique up to a global conjugation in $\mathrm{PSL}_2(\mathbb{C})$ and a normalization condition (by assigning the 2 fixed points of $L_1$ and one of $L_2$) can be set to fix it. One then obtains the Schottky space $\mathfrak{S}$ that covers the moduli space (ie, the set of isomorphism classes of compact Riemann surface of genus $g$) but is covered by Teichmüller space $\mathcal{T}$ whose points are isomorphism classes of marked compact Riemann surfaces.

Since any Schottky group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ acts as isometries on $\mathbb{H}^3$ as a convex cocompact group, there is a canonical hyperbolic 3–manifold $\Gamma\backslash\mathbb{H}^3$ with conformal infinity given by $\Gamma\backslash\Omega_\Gamma$. This manifold denoted $X$ is a handlebody with conformal boundary $M$. Let $\mathcal{D}_X(M)$ the group of diffeomorphisms of $M$ which extend to $\overline{X}$ factored by the group $\mathcal{D}_0(M)$ of diffeomorphisms of $M$ homotopic to the identity.

The Chern–Simons line bundle defined on $\mathcal{T}$ above is acted upon by $\mathcal{D}_X(M)$, thus it descends to the Schottky space $\mathfrak{S}$, which is a quotient of $\mathcal{T}$ by a subgroup of $\mathcal{D}_X(M)$; we denote it $\mathcal{L}_{\mathfrak{S}}$. The connection on $\mathcal{L}$ over $\mathcal{T}$ defined in Section 7.6 is $\mathcal{D}_X(M)$–invariant, hence it descends to $\mathfrak{S}$. The Liouville form on $T\mathcal{T}$ is $\mathcal{D}(M)$–invariant and thus also descends to $T\mathfrak{S}$; then, the connection $\nabla^{\mathcal{L}}$ descends to $T\mathfrak{S}$, we denote it by $\nabla^{\mathfrak{S}}$. Again, we can define the Lagrangian submanifold $\mathcal{H} \subset T\mathfrak{S}$ consisting of those funnels which extend to Schottky 3–manifolds. The operator $\partial_\Gamma \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M, \Lambda^{1,0}M)$ for a given complex structure induced by $\Gamma$ on $M$ is Fredholm on Sobolev spaces and, considered as a family of operators parametrized by points $\Gamma \in \mathfrak{S}_{\boldsymbol{g}}$, one can define its determinant line bundle $\det(\partial)$ of $\partial$, as in Quillen [24], to be at $\Gamma$ the line[1]

$$\det(\partial_\Gamma) := \Lambda^{\boldsymbol{g}}(\operatorname{coker} \partial_\Gamma)$$

when $\boldsymbol{g} \in \mathbb{N}$ and

$$\operatorname{coker} \partial_\Gamma = \ker(\bar{\partial}_\Gamma \colon \mathcal{C}^\infty(M, \Lambda^{1,0}) \to \mathcal{C}^\infty(M, \Lambda^2(M))) =: H^{0,1}(\Gamma \backslash \Omega_\Gamma)$$

is the vector space of holomorphic 1–forms on $M \simeq \Gamma \backslash \Omega_\Gamma$. The line bundle $\det(\partial)$ over $\mathfrak{S}$ is a holomorphic line bundle with a holomorphic canonical section

$$\varphi := \varphi_1 \wedge \cdots \wedge \varphi_{\boldsymbol{g}} \tag{44}$$

and is equipped with a Hermitian norm, called the Quillen metric, defined as follows: for each Riemann surface $\Gamma \backslash \Omega_\Gamma$ with $\Gamma \in \mathfrak{S}_{\boldsymbol{g}}$, let $h_0$ be the associated hyperbolic metric obtained by uniformisation and define $\det' \Delta_{h_0}$ to be the determinant of its Laplacian, as defined in Ray and Singer [26], then the Hermitian metric on $\det(\partial)$ is given at $\Gamma \in \mathfrak{S}$ by

$$\|\varphi\|_Q^2 := \frac{\|\varphi\|_{h_0}^2}{\det' \Delta_{h_0}} = \frac{\det \operatorname{Im} \tau}{\det' \Delta_{h_0}}, \tag{45}$$

where $\|\cdot\|_{h_0}$ is the Hermitian product on $\Lambda^{\boldsymbol{g}}(\operatorname{coker} \partial_\Gamma)$ induced by the metric $h_0$ on differential forms on $M$. We denote by $\nabla^{\det}$ the unique Hermitian connection associated to the holomorphic structure on $\det \partial$ and the Hermitian norm $\|\varphi\|_Q$.

To state the isomorphism between powers of Chern–Simons line bundle and a power of the determinant line bundle, we will use a formula proved by Zograf [32] and generalized by McIntyre and Takhtajan [21]:

[1] We have ignored the kernel of $\partial$ since it is only made of constants with norm given essentially by the Euler characteristic of $M$ by Gauss–Bonnet, thus not depending at all on the complex structure on $M$.

**Theorem 41** (Zograf) *There is a holomorphic function $F(\Gamma)\colon \mathfrak{S}_{\boldsymbol{g}} \to \mathbb{C}$ such that*

$$\frac{\det' \Delta_{h_0}}{\det \operatorname{Im} \tau} = c_{\boldsymbol{g}} \exp\left(\frac{\operatorname{Vol}_R(X)}{3\pi}\right) |F(\Gamma)|^2, \tag{46}$$

*where $c_{\boldsymbol{g}}$ is a constant depending only on $\boldsymbol{g}$ where $X = \Gamma\backslash\mathbb{H}^3$ when we see $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ as a group of isometries of $\mathbb{H}^3$, and $h_0$ is the hyperbolic metric on $\Gamma\backslash\Omega_\Gamma \simeq \partial\overline{X}$. For points in $\mathfrak{S}$ corresponding to Schottky groups $\Gamma$ with dimension of limit set $\delta_\Gamma < 1$, the function $F(\Gamma)$ is given by the following absolutely convergent product:*

$$F(\Gamma) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_\gamma^{1+m}), \tag{47}$$

*where $q_\gamma$ is the multiplier of $\gamma \in \Gamma$, and $\{\gamma\}$ runs over all distinct primitive conjugacy classes in $\Gamma$ excluding the identity.*

**Remark 42** The formula (46) was in fact given in terms of Liouville action $S$ instead of renormalized volume, but it has been shown that $S = -4 \operatorname{Vol}_R(X) + c_{\boldsymbol{g}}$ for some constant $c_{\boldsymbol{g}}$ depending only on $\boldsymbol{g}$, by Krasnov [15] for Schottky manifolds and by Takhtajan and Teo [27] for quasi-Fuchsian manifolds.

We therefore deduce from this last theorem and our construction the following:

**Theorem 43** *On the Schottky space $\mathfrak{S}$, the bundle $\mathcal{L}_{\mathfrak{S}}^{-1}$ is isomorphic to $(\det \partial)^{\otimes 6}$ when equipped with their connections and Hermitian products induced by those of $(\mathcal{L}_{\mathfrak{S}}, \nabla^{\mathfrak{S}}, \|\cdot\|_{\mathcal{L}})$ and $(\det \partial, \nabla^{\det}, \|\cdot\|_Q)$. There is an explicit isometric isomorphism of holomorphic Hermitian line bundles given by*

$$(\sqrt{c_{\boldsymbol{g}}}\, F\varphi)^{\otimes 6} \mapsto e^{-2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}},$$

*where $F$ and $c_{\boldsymbol{g}}$ are the holomorphic functions and constants of Theorem 41 and $\varphi$ is the canonical section of $\det \partial$ defined in (44).*

**Proof** By a result of Zograf [32], the function $F$ extends analytically to $\mathfrak{S}$. The section $e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}} \otimes (\sqrt{c_{\boldsymbol{g}}}\, F\varphi)^{\otimes 6}$ is holomorphic and has non-zero constant norm in the Hermitian line bundle $\mathcal{L}_{\mathfrak{S}} \otimes (\det \partial)^{\otimes 6}$. But any holomorphic section of constant norm in a Hermitian line bundle must be parallel with respect to the Chern connection (ie, the unique connection compatible with the Hermitian metric and whose $(0,1)$–component is $\overline{\partial}$). Hence the bundle is flat with respect to the Chern connection, and the parallel section provides an isomorphism with the trivial line bundle in the category of holomorphic Hermitian bundles. □

**Remark 44** The Hermitian bundles $\mathcal{L}_{\mathfrak{S}}^{-1}$ and $(\det\partial)^{\otimes 6}$ have been shown to have the same curvature, thus they are locally isomorphic. This would not be enough to deduce anything globally since the Schottky moduli space is not simply connected. Our proof rests on the construction of the holomorphic Chern–Simons section in $\mathcal{L}_{\mathfrak{S}}^{-1}$, which happens to have the same norm as the determinant section, corrected by the function $F$. Thus, the global existence of $F$ is needed for our argument. In turn, our construction shows that the lift of $F$ from its domain of absolute convergence to $\mathcal{T}$ admits a global analytic extension, but this extension might not be invariant under the Schottky modular group, so we cannot re-prove by our methods Zograf's result on the extension of $F$ to $\mathfrak{S}$.

**Remark 45** In our previous work with Park [11], we proved that

$$F(\Gamma) = |F(\Gamma)| \exp\Big(-\frac{\pi i}{2}\,\eta(A)\Big)$$

when $\delta(\Gamma) < 1$, where $\eta(A)$ is the eta invariant of the signature operator $A = *d + d*$ on odd-dimensional forms on the Schottky 3–manifold $\Gamma\backslash\mathbb{H}^3$.

**Remark 46** Using the result of McIntyre and Takhajan [21], and McIntyre and Teo [22], a similar result with different powers of the bundles is easily obtained in the Schottky and quasi-Fuchsian cases if one replaces the bundle $\det\partial$ by the determinant line bundle $\det\Lambda_n$ of the vector space of holomorphic $n$–differentials on $M$.

# Appendix: Chern–Simons invariants of 3–manifolds with funnels and cusps of rank 2

In this appendix we show how to extend the results of this paper to include 3–manifolds of finite geometry with funnels and/or rank-2 cusps. We will concentrate on the cusps since funnels have already been treated.

By definition, a *cusp of maximal rank* is a half-complete warped product $(a,\infty)\times M$ with metric $dt^2 + e^{-2t}h$, where $h$ is a flat metric on $M$. Here $M$ will be of dimension 2. After a linear change of variables in $t$, we can thus assume that $M$ is isometric to a flat torus with a closed simple geodesic of length 1.

By changing variables $x := e^{-t} \in (0, e^{-a})$, the cusp metric becomes

$$\frac{dx^2}{x^2} + x^2 h = x^2\Big(\frac{dx^2}{x^4} + h\Big).$$

Thus a cusp is conformal to a half-infinite cylinder $dy^2 + h$ where $y := x^{-1} = e^t \in [e^a, \infty)$, the conformal factor being $x = y^{-1}$. The function $x$ can be used to glue to the cusps a copy of $M$ at $x = 0$, thus compactifying $X$. Thus if we choose $\rho\colon X \to (0, \infty)$ to be a function that agrees with $x$ on funnels and with $y$ on cusps, it follows that $X$ is conformal to a manifold with boundary (corresponding to the funnels) and flat half-infinite cylindrical ends (corresponding to each cusp):

$$g = \rho^{-2}\hat{g}, \quad \hat{g} = d\rho^2 + h(\rho),$$

where on the cusps, $h(\rho) = h$ is flat and independent of $\rho$.

Let $\hat{S}$ be a orthonormal frame for $\hat{g}$ which is parallel in the $y$ direction in the cusp. Then both the connection 1–form $\hat{\omega}$ and the curvature form $\hat{\Omega}$ vanish when contracted with $\partial_y$. It follows that the Chern–Simons form $\mathrm{cs}(\hat{g}, \hat{S})$ vanishes identically on the cusp, thus the $\mathrm{SO}(3)$ Chern–Simons invariant for $\hat{g}$ is well-defined and moreover it coincides with the invariant of the compact manifold with boundary obtained by chopping off the cylindrical ends.

The line bundle $\mathcal{L}$ is constructed now over the set of constant-curvature metrics on $M$, namely hyperbolic on the funnel ends and flat on the cusp ends. In the definition of the cocycle $c^X(\hat{S}, a)$ notice that the second term vanishes identically on the cusp, since we work with frames $S$ parallel in the direction of $y$, which implies that $\partial_y \tilde{a} = 0$, or in other words $\tilde{a}$ is independent of $y$. The definition of the $\mathrm{SO}(3)$ connection is unchanged if we include now in $M$ the flat components corresponding to the cusps. Its curvature is computed in terms of a fiberwise integral of the Pontrjagin form by following verbatim the proof of Proposition 26. However in Lemma 27 the curvature of the torus fibers vanishes, thus the cusps do not contribute to the curvature and so the curvature of $\nabla^{\mathcal{L}}$ is $i/(8\pi)$ times the Weil–Petersson symplectic form of the Teichmüller space corresponding to the funnels, ie, it does not "see" the cusps.

We define now the $\mathrm{SO}(3)$ invariant of the hyperbolic metric $g$. Using (20) with the roles of $g, \hat{g}$ reversed and (19) we see that in the cusps, the Chern–Simons form $\mathrm{cs}(g, S)$ of $g$ equals $d\,\mathrm{Tr}(\hat{\alpha} \wedge \hat{\omega})$, where

$$\hat{\alpha}_{ij}(Y) = y^{-1}[\hat{g}(Y, \hat{S}_j) S_i(y) - \hat{g}(Y, S_i) S_j(y)].$$

Now $\hat{\omega}_{ij}$ is constant in $y$ in the sense that $\mathcal{L}_{\partial_y}\hat{\omega}_{ij} = 0$, while $\hat{\alpha}$ is of homogeneity $-1$. It follows that $\mathrm{cs}(g, S)$ decreases like $y^{-2}$ as $y \to \infty$, thus it is integrable without regularization. Moreover the form $\mathrm{Tr}(\hat{\alpha} \wedge \hat{\omega})$ from (20) is homogeneous in $y$ of degree $-1$, hence Proposition 15 continues to hold in the setting of this appendix.

To define the $\mathrm{PSL}_2(\mathbb{C})$ invariant we use Proposition 11. We note that the volume of the cusps is finite, the $\mathrm{SO}(3)$ Chern–Simons form was proved above to be integrable in the cusp, and we claim that the remaining term $\mathrm{Tr}(T \wedge \omega)$ decreases in the cusp like $y^{-1}$. Indeed, we have seen above that $\omega = \hat{\omega} + \hat{\alpha}$ is of homogeneity $0$ and $-1$, while $T = y^{-1}\hat{T}$ is of homogeneity $-1$. Therefore $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}$ does not involve regularization in the cusps, while Proposition 16 continues to hold. Note that the Euler characteristic of a torus is $0$, so it is irrelevant whether the tori closing the cusps are included or not in the formula from Proposition 16 when we allow cusps.

The variation formula for $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}$ (Theorem 29) continues to hold as in the case without cusps because in (35) the cusp terms involved (other than the first one, which is the connection $1$–form) do not have contributions of degree $0$ in $y$. This is obvious if one takes into account that $\alpha$ and $T$ are of homogeneity $-1$, while $\hat{\omega}$ is of homogeneity $0$. Hence the variation of the regularized volume of a hyperbolic manifold with funnels and cusps is given by Corollary 30 (and only depends on local data on the funnels).

Finally, the correspondence between hyperbolic metrics on $X$ and the conformal infinity in the funnels continues to hold in the presence of cusps [18].

These hyperbolic metrics with cusps and funnels form therefore a Lagrangian submanifold in $T\mathcal{T}$, and their renormalized volume is a Kähler potential for the Teichmüller space corresponding to the funnels (see Corollary 40).

### Acknowledgements

The subject of this paper arose from our joint work [11] with J Park, to whom we owe the idea of connecting the determinant and Chern–Simons line bundles in this context. We thank him and also S Baseilhac, U Bunke, D Cheptea, C Ciobotaru, K Krasnov, J Marché, G Massuyeau and J-M Schlenker for useful discussions. C G was supported partly by the grant ANR-09-JCJC-0099-01 and ANR 10-BLAN 0105, while S M was supported by the ANCS grant PN-II-RU-TE-2011-3-0053. This work was done in part while S M was a visiting researcher for CNRS at the DMA of the ENS Paris and a visitor at IHES; we thank these institutions for their support.

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Proposed: Jean-Pierre Otal  Received: 11 October 2011
Seconded: Walter Neumann, Yasha Eliashberg  Revised: 19 January 2013