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Logarithmic structures on topological K–theory spectra

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We study a modified version of Rognes' logarithmic structures on structured ring spectra. In our setup, we obtain canonical logarithmic structures on connective K–theory spectra which approximate the respective periodic spectra. The inclusion of the p–complete Adams summand into the p–complete connective complex K–theory spectrum is compatible with these logarithmic structures. The vanishing of appropriate logarithmic topological André–Quillen homology groups confirms that the inclusion of the Adams summand should be viewed as a tamely ramified extension of ring spectra.

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1 Introduction

A *pre-log structure* on a commutative ring A is a commutative monoid M together with a monoid map $\alpha\colon M \to (A, \cdot)$ into the multiplicative monoid of A. It is a *log structure* if the map $\alpha^{-1}(A^\times) \to A^\times$ from the submonoid $\alpha^{-1}(A^\times) \subseteq M$ of elements mapping to the units $A^\times$ of A is an isomorphism. The trivial log structure $A^\times \hookrightarrow (A, \cdot)$ is the easiest example. A *log ring* is a commutative ring with a log structure. This notion is the affine version of the *log schemes* studied in algebraic geometry. Log schemes are useful because they for example enlarge the range of smooth and étale maps.

We consider a very basic example of interest to us: An integral domain A may be viewed as a log ring (A, M) with $M = A \setminus 0$. The localization map to its fraction field $A \to K = A[M^{-1}]$ admits a factorization

$$(A, A^\times) \to (A, M) \to (K, K^\times) \tag{1-1}$$

in log rings, and we may view (A, M) as an intermediate localization of A which is "milder" than K. In contrast, $A \to K$ does in general not factor in a nontrivial way as a map of commutative rings.

We switch to the topological K–theory spectra of algebraic topology. Consider the inclusion $\ell_p \to ku_p$ of the p–complete Adams summand into the p–complete connective complex K–theory spectrum ku_p. On homotopy groups, it induces a map

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$\mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u]$ sending v_1 to u^{p-1}. Since $p-1$ is invertible in $\mathbb{Z}_p$, this map of homotopy groups behaves like a tamely ramified extension if we interpret u and v_1 as uniformizers. As observed by Hesselholt and explained by Ausoni [1, Section 10.4], computations of the topological Hochschild homology THH of ku_p and ℓ_p provide a much deeper reason for why $\ell_p \to ku_p$ should be viewed as a tamely ramified extension on the level of structured ring spectra: On certain relative THH terms, $\ell_p \to ku_p$ shows the same behavior as tamely ramified extensions of discrete valuation rings whose THH is studied by Hesselholt and Madsen [8, Section 2]. This is also supported by the THH localization sequences for ℓ_p and ku_p established by Blumberg and Mandell [6].

In order to explain this and other phenomena arising in connection with the THH and algebraic K–theory of structured ring spectra, Rognes [14] introduced a notion of *log ring spectra*. This is a homotopical generalization of the log rings defined above in which ring spectra play the role of commutative rings. One question that remained open in Rognes' work was how to extend $\ell_p \to ku_p$ to a *formally log étale* map of log ring spectra, ie a map whose *log topological André–Quillen homology* vanishes. In the algebraic setup, the vanishing of the corresponding module of *log Kähler differentials* detects tame ramification. So being formally log étale is one reasonable candidate for a definition of a tamely ramified extension of ring spectra, and $\ell_p \to ku_p$ should be formally log étale with respect to suitable log structures.

The aim of the present paper is to resolve the above issue by modifying Rognes' definition to what we call *graded* log ring spectra. There are canonical graded log structures on connective K–theory spectra like ku_p and ℓ_p turning them into graded log ring spectra. Generalizing the algebraic example (1-1), the latter provide intermediate objects between connective and periodic K–theory spectra equipped with their trivial graded log structures. The localizations of these intermediate graded log ring spectra are the respective periodic K–theory spectra. Moreover, $\ell_p \to ku_p$ extends to a map of graded log ring spectra which is *formally graded log étale*, that is, a map whose *graded log topological André–Quillen homology* vanishes.

1.1 Logarithmic ring spectra

Structured ring spectra provide a homotopical generalization of commutative rings which allows one to transfer many concepts from algebra to homotopy theory, including for example algebraic K–theory, Galois theory and Morita theory. There are several equivalent definitions of structured ring spectra. In the present paper, we will work with the category of *commutative symmetric ring spectra* $\mathcal{C}\mathrm{Sp}^{\Sigma}$; see Hovey, Shipley and Smith [10], Mandell, May, Schwede and Shipley [12] and Schwede [20]. The objects are symmetric spectra which are commutative monoids with respect to the smash product of symmetric spectra.

To generalize the notions of pre-log and log structures, we need to know what the "underlying multiplicative monoid" of a commutative symmetric ring spectrum A is. If we were working in the more classical framework of E_∞ ring spectra, this would be the underlying multiplicative E_∞ space of an E_∞ spectrum. When dealing with (strictly) commutative symmetric ring spectra, it is more useful to model E_∞ spaces by *commutative $\mathcal{I}$–space monoids*; see the author and Schlichtkrull [17]. By definition, a commutative $\mathcal{I}$–space monoid M is a space valued functor on the category of finite sets and injections $\mathcal{I}$ together with appropriate multiplication maps. These multiplications turn M into a commutative monoid with respect to a convolution product on this functor category. The category of commutative $\mathcal{I}$–space monoids $\mathcal{CS}^{\mathcal{I}}$ admits a model structure making it Quillen equivalent to the category of E_∞ spaces. Moreover, there is a Quillen adjunction

$$\mathbb{S}^{\mathcal{I}}[-]\colon \mathcal{CS}^{\mathcal{I}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{I}} \tag{1-2}$$

whose right adjoint models the underlying multiplicative monoid.

An *$\mathcal{I}$–space pre-log structure* on a commutative symmetric ring spectrum A is then a commutative $\mathcal{I}$–space monoid M together with a map $\alpha\colon M \to \Omega^{\mathcal{I}}(A)$ in $\mathcal{CS}^{\mathcal{I}}$; Rognes [14, Definition 7.1]. The units of A are the subobject $\mathrm{GL}_1^{\mathcal{I}}(A)$ of $\Omega^{\mathcal{I}}(A)$ given by the invertible path components. In analogy with the algebraic definition, an $\mathcal{I}$–space pre-log structure (M,α) is an *$\mathcal{I}$–space log structure* if $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{I}}(A)) \to \mathrm{GL}_1^{\mathcal{I}}(A)$ is a weak equivalence of commutative $\mathcal{I}$–space monoids.

Although this is an obvious and useful generalization of the algebraic definition, we would like to emphasize one aspect which is not optimal: If $i\colon A \to B$ is a map of commutative rings, then the pullback of

$$B^{\times} \to (B,\,\cdot) \leftarrow (A,\,\cdot) \tag{1-3}$$

defines a log structure $i_*(B^{\times})$ on A. For example, the log ring (A,M) in (1-1) arises from $A \to K$ in this way. Let ku be the connective complex K–theory spectrum. Since the Bott class u becomes invertible in $\pi_*(KU)$, the periodic KU has more units in its homotopy groups, and one may hope that the map $i\colon ku \to KU$ into the periodic spectrum induces an interesting $\mathcal{I}$–space log structure as in (1-3). However, the pullback of

$$\mathrm{GL}_1^{\mathcal{I}}(KU) \to \Omega^{\mathcal{I}}(KU) \leftarrow \Omega^{\mathcal{I}}(ku) \tag{1-4}$$

only provides the trivial log structure on ku. The problem is that $\mathrm{GL}_1^{\mathcal{I}}(ku)$ and $\mathrm{GL}_1^{\mathcal{I}}(KU)$ are equivalent: $\mathrm{GL}_1^{\mathcal{I}}(KU)$ only detects the units in $\pi_0(KU) \cong \mathbb{Z}$ and ignores that the graded ring $\pi_*(KU) = \mathbb{Z}[u^{\pm 1}]$ has more units than $\pi_*(ku) = \mathbb{Z}[u]$.

1.2 Graded E_∞–spaces

To overcome the difficulty outlined in the previous example, it is desirable to have a notion of the units of a ring spectrum A which takes all units in the graded ring $\pi_*(A)$ into account. Such *graded* units have been defined by the author in joint work with Schlichtkrull [17].

The key idea behind the graded units is to replace the category $\mathcal{I}$ used above by a more elaborate indexing category. The appropriate choice turns out to be the category $\mathcal{J} = \Sigma^{-1}\Sigma$ given by Quillen's localization construction on the category of finite sets and bijections Σ. This $\mathcal{J}$ is a symmetric monoidal category whose classifying space $B\mathcal{J}$ has the homotopy type of QS^0. The same constructions as in the case of $\mathcal{I}$–spaces lead to a model category of commutative $\mathcal{J}$–space monoids $\mathcal{CS}^{\mathcal{J}}$. We show in [17] that $\mathcal{CS}^{\mathcal{J}}$ is Quillen equivalent to the category of E_∞ spaces over $B\mathcal{J}$. So commutative $\mathcal{J}$–space monoids correspond to "commutative monoids over the underlying additive monoid of the sphere spectrum", just as $\mathbb{Z}$–graded monoids in algebra can be defined as commutative monoids over the additive monoid of $\mathbb{Z}$. This is why we think of commutative $\mathcal{J}$–space monoids as graded E_∞ spaces.

The reason for why $\mathcal{J}$ is useful for studying units is a beautiful connection to the combinatorics of symmetric spectra [17, Section 4.21]. It gives rise to a Quillen adjunction

$$\mathbb{S}^{\mathcal{J}}[-]\colon \mathcal{CS}^{\mathcal{J}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{J}}. \tag{1-5}$$

If A is a commutative symmetric ring spectrum, we think of $\Omega^{\mathcal{J}}(A)$ as the underlying graded multiplicative monoid of A. The point about $\Omega^{\mathcal{J}}(A)$ is that it is built from all spaces $\Omega^{m_2}(A_{m_1})$, while the $\mathcal{I}$–space version only uses the spaces $\Omega^m(A_m)$. This makes it possible to define a commutative $\mathcal{J}$–space monoid $\mathrm{GL}_1^{\mathcal{J}}(A) \subseteq \Omega^{\mathcal{J}}(A)$ of *graded units* of A from which we can recover all units in the graded ring $\pi_*(A)$.

1.3 Graded logarithmic ring spectra

Given the previous discussion, we define a *graded pre-log structure* on a commutative symmetric ring spectrum A to be a commutative $\mathcal{J}$–space monoid M together with a map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$. It is a log structure if $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \to \mathrm{GL}_1^{\mathcal{J}}(A)$ is a weak equivalence in $\mathcal{CS}^{\mathcal{J}}$. The pullback

$$\mathrm{GL}_1^{\mathcal{J}}(KU) \to \Omega^{\mathcal{J}}(KU) \leftarrow \Omega^{\mathcal{J}}(ku) \tag{1-6}$$

provides an interesting nontrivial graded log structure $i_*\mathrm{GL}_1^{\mathcal{J}}(KU)$ on ku. In analogy with the algebraic situation (1-1), the $(ku, i_*\mathrm{GL}_1^{\mathcal{J}}(KU))$ is part of a factorization

$$(ku, \mathrm{GL}_1^{\mathcal{J}}(ku)) \to (ku, i_*\mathrm{GL}_1^{\mathcal{J}}(KU)) \to (KU, \mathrm{GL}_1^{\mathcal{J}}(KU)). \tag{1-7}$$

Let (M,α) be a graded log structure on A. A group completion $M \to M^{\mathrm{gp}}$ for commutative $\mathcal{J}$–space monoids is constructed by the author in [16]. Together with the adjoint of $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ under the adjunction (1-5), the group completion induces maps

$$\mathbb{S}^{\mathcal{J}}[M^{\mathrm{gp}}] \leftarrow \mathbb{S}^{\mathcal{J}}[M] \to A$$

in $\mathcal{C}\mathrm{Sp}^{\Sigma}$. The *localization* $A[M^{-1}]$ of (A,M) is the pushout of this diagram in commutative symmetric ring spectra. As one may hope from the algebraic example, the next theorem allows us to interpret $(ku, i_*\mathrm{GL}_1^{\mathcal{J}}(KU))$ as an approximation to KU. It is the special case of a more general theorem about log structures on connective covers of periodic ring spectra.

Theorem 1.4 *The map* $(ku, i_*\mathrm{GL}_1^{\mathcal{J}}(KU)) \to (KU, \mathrm{GL}_1^{\mathcal{J}}(KU))$ *induces a stable equivalence between the localization* $ku[(i_*\mathrm{GL}_1^{\mathcal{J}}(KU))^{-1}]$ *and the periodic spectrum* KU*. A similar statement holds for the* p*–complete and* p*–local connective complex K–theory spectra* ku_p *and* $ku_{(p)}$ *and their Adams summands* ℓ_p *and* ℓ*.*

The proof of the theorem depends heavily on our analysis of the group completion for commutative $\mathcal{J}$–space monoids in [16]. The various $\mathcal{I}$–space and operadic pre-log and log structures considered by Rognes [14] do not have this property.

1.5 Formally log étale extensions

Let $(f, f^{\flat})\colon (R,P,\rho) \to (A,M,\alpha)$ be a map of log rings in the algebraic setup. If X is an A–module, a log derivation of $(f, f^{\flat})$ with values in X is a pair (D,δ) where $D\colon A \to X$ is an ordinary R–linear derivation of A and $\delta\colon M \to X$ is a monoid map such that $\delta f^{\flat} = 0$ and $D(\alpha(m)) = \alpha(m)\delta(m)$; see Kato [11], Ogus [13]. The resulting set of log derivations $\mathrm{Der}_{(R,P)}((A,M),X)$ is corepresented by the A–module of *log Kähler differentials* $\Omega^1_{(A,M)/(R,P)}$. This module is isomorphic to the quotient of $\Omega^1_{A/R} \oplus (A \otimes M^{\mathrm{gp}})$ obtained by imposing the relations $d(\alpha(m)) = \alpha(m) \otimes m$ and $1 \otimes f^{\flat}(p) = 0$ for monoid elements $m \in M$ and $p \in P$. Writing $a\,\mathrm{d}\log m$ for $a \otimes m$, the first relation shows that $\mathrm{d}\log m$ has the properties of a logarithmic differential. This is the source of the term “log” in this theory. The relation also shows that generators of the form $d\alpha(m)$ in $\Omega^1_{A/R}$ become once divisible by $\alpha(m)$ when passing to $\Omega^1_{(A,M)/(R,P)}$. So it is a milder localization of $\Omega^1_{A/R}$ than $\Omega^1_{A[M^{-1}]/R}$.

In the context of ring spectra, Basterra [5] has shown that R–linear derivations of A are corepresented by the *topological André–Quillen homology* $\mathrm{TAQ}^R(A)$. Building on Basterra's result, Rognes has shown that log derivations of log ring spectra are corepresented by a *log topological André–Quillen homology*. In the present paper, we construct the corresponding *graded log topological André–Quillen homology* $\mathrm{TAQ}^{(R,P)}(A,M)$. It corepresents graded log derivations. The definition and the analysis of the graded log TAQ rely on the equivalence between the homotopy category of grouplike commutative $\mathcal{J}$–space monoids and a suitable homotopy category of augmented connective spectra established in [16].

A map $(R,P) \to (A,M)$ of graded log ring spectra is *formally graded log étale* if the graded log topological André–Quillen homology $\mathrm{TAQ}^{(R,P)}(A,M)$ is contractible.

Theorem 1.6 *The inclusion $\ell_p \to ku_p$ extends to map of graded log ring spectra*

$$(\ell_p, i_*\mathrm{GL}_1^{\mathcal{J}}(L_p)) \to (ku_p, i_*\mathrm{GL}_1^{\mathcal{J}}(KU_p))$$

and this map is formally graded log étale. The same hold in the p–local case.

For the proof, we replace these log structures by "smaller" pre-log structures and analyze them using group completions. As with Theorem 1.4, the $\mathcal{I}$–space and operadic log structures on ℓ_p and ku_p considered by Rognes do not have this property.

It is also possible to transfer Rognes' notion of *logarithmic topological Hochschild homology* to our graded context. Applying this to the graded log structures discussed above provides interesting homotopy cofiber sequences relating logarithmic and ordinary THH. This is studied in detail by Rognes, Sagave and Schlichtkrull in [15].

1.7 Organization

In Section 2 we give background about symmetric spectra and commutative $\mathcal{J}$–space monoids. Section 3 contains basic terminology about graded log structures. In Section 4 we study various pre-log and log structures generated by homotopy classes and obtain Theorem 1.4 as a special case of Theorem 4.4 proven there. The graded log topological André–Quillen homology is constructed in Section 5. Section 6 features the proof of Theorem 1.6. The final Section 7 contains auxiliary results about commutative $\mathcal{J}$–space monoids.

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2 Symmetric ring spectra and structured diagram spaces

The category of symmetric spectra Sp^{Σ} [10] is a stable model category whose homotopy category is equivalent to the stable homotopy category of algebraic topology. The smash product of symmetric spectra $\wedge$ induces the smash product on the homotopy category. A *commutative symmetric ring spectrum* is a commutative monoid in $(\mathrm{Sp}^{\Sigma}, \wedge)$. We write $\mathcal{C}\mathrm{Sp}^{\Sigma}$ for the category of commutative symmetric ring spectra. It admits a *positive stable model structure* [12] making it Quillen equivalent to the category of E_∞ spectra. So all homotopy types of E_∞ spectra are represented by commutative symmetric ring spectra, and in many cases it is even possible to write down explicit models in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ [20].

2.1 Commutative $\mathcal{I}$–space monoids

We will now recall from [17, Section 3] how one can use structured diagram spaces to model the "underlying multiplicative E_∞ spaces" of commutative symmetric ring spectra.

Definition 2.2 Let $\mathcal{I}$ be the category with objects the sets $\boldsymbol{m} = \{1, \dots, m\}$ for $m \geq 0$ and morphisms the injective maps. The ordered concatenation $\sqcup$ of ordered sets turns $\mathcal{I}$ into a symmetric monoidal category with strict unit and associativity. The monoidal unit is the empty set $\mathbf{0}$. The symmetry isomorphism is the shuffle $\chi_{m,n}\colon \boldsymbol{m} \sqcup \boldsymbol{n} \to \boldsymbol{n} \sqcup \boldsymbol{m}$ moving the first m elements past the last n elements.

An *$\mathcal{I}$–space* is a functor from $\mathcal{I}$ to the category of unpointed simplicial sets $\mathcal{S}$. The resulting functor category $\mathcal{S}^{\mathcal{I}}$ of $\mathcal{I}$–spaces inherits a symmetric monoidal product $\boxtimes$ from $\mathcal{I}$ and $\mathcal{S}$: For $\mathcal{I}$–spaces X and Y, the product $X \boxtimes Y$ is the left Kan extension of their object-wise cartesian product along $-\sqcup -\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$. So

$$(X \boxtimes Y)(\boldsymbol{n}) = \operatorname*{colim}_{\boldsymbol{k} \sqcup \boldsymbol{l} \to \boldsymbol{n}} X(\boldsymbol{k}) \times Y(\boldsymbol{l}).$$

The monoidal unit for $\boxtimes$ is the $\mathcal{I}$–space $U^{\mathcal{I}} = \mathcal{I}(\mathbf{0}, -)$.

Definition 2.3 A commutative monoid in $(\mathcal{S}^{\mathcal{I}}, \boxtimes, U^{\mathcal{I}})$ is called a *commutative $\mathcal{I}$–space monoid* and $\mathcal{C}\mathcal{S}^{\mathcal{I}}$ denotes the category of commutative $\mathcal{I}$–space monoids.

More explicitly, a commutative $\mathcal{I}$–space monoid M is an $\mathcal{I}$–space M together with multiplications $M(\boldsymbol{k}) \times M(\boldsymbol{l}) \to M(\boldsymbol{k} \sqcup \boldsymbol{l})$ and a unit map $* \to M(\mathbf{0})$ satisfying appropriate coherence conditions.

While strictly commutative simplicial monoids fail to model all homotopy types of E_∞ spaces, the additional symmetry of $\mathcal{I}$–spaces and the use of a *positive* model structure ensure that E_∞ spaces admit strictly commutative models in $\mathcal{I}$–spaces:

Theorem 2.4 [17, Theorem 1.2] *The category of commutative $\mathcal{I}$–space monoids $\mathcal{CS}^{\mathcal{I}}$ admits a positive $\mathcal{I}$–model structure making it Quillen equivalent to the category of E_∞ spaces.*

The weak equivalences in this *positive $\mathcal{I}$–model structure* are the *$\mathcal{I}$–equivalences*, ie the maps $f\colon M \to N$ which induce weak equivalences $f_{h\mathcal{I}}\colon M_{h\mathcal{I}} \to N_{h\mathcal{I}}$ on homotopy colimits. Here

$$M_{h\mathcal{I}} = \mathrm{hocolim}_{\mathcal{I}} M = \mathrm{diag}\Bigl([s] \mapsto \coprod_{\boldsymbol{k_0} \leftarrow \dots \leftarrow \boldsymbol{k_s}} M(\boldsymbol{k_s})\Bigr)$$

is the usual Bousfield–Kan homotopy colimit. A commutative $\mathcal{I}$–space monoid A is fibrant in this model structure if every morphism $\boldsymbol{k} \to \boldsymbol{l}$ in $\mathcal{I}$ with $k \geq 1$ induces a weak equivalence of Kan-complexes $M(\boldsymbol{k}) \to M(\boldsymbol{l})$. The positivity condition $k \geq 1$ ensures that we do not represent the common homotopy type of the $M(\boldsymbol{k})$ by a commutative simplicial monoid.

Commutative $\mathcal{I}$–space monoids are relevant in connection with symmetric ring spectra because there is a Quillen adjunction

$$\mathbb{S}^{\mathcal{I}}[-]\colon \mathcal{CS}^{\mathcal{I}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{I}}$$

with respect to the positive model structures [17, Proposition 3.19]. The right adjoint $\Omega^{\mathcal{I}}$ is on objects given by $\Omega^{\mathcal{I}}(A)(\boldsymbol{m}) = \Omega^m(A_m)$, and the multiplication

$$\Omega^k(A_k) \times \Omega^l(A_l) \to \Omega^{k+l}(A_{k+l})$$

sends (f, g) to the composite

$$S^{k+l} \cong S^k \wedge S^l \xrightarrow{f \wedge g} A_k \wedge A_l \longrightarrow A_{k+l}. \tag{2-1}$$

For a positive fibrant A in $\mathcal{C}\mathrm{Sp}^{\Sigma}$, the $\Omega^{\mathcal{I}}(A)$ models the underlying multiplicative E_∞ space of A. The *units* $\mathrm{GL}_1^{\mathcal{I}}(A)$ of A is the sub commutative $\mathcal{I}$–space monoid of $\Omega^{\mathcal{I}}(A)$ consisting of those path components that map to units in the commutative monoid $\pi_0(\Omega^{\mathcal{I}}(A)_{h\mathcal{I}}) \cong \pi_0(A)$.

2.5 Commutative $\mathcal{J}$–space monoids

As explained in the introduction, the units $\mathrm{GL}_1^{\mathcal{I}}(A)$ and other equivalent approaches used in the literature have the undesirable feature that they do not detect the difference between a periodic spectrum and its connective cover. We now recall from [17, Section 4] how we can overcome this by using a more subtle indexing category for structured diagram spaces.

Definition 2.6 Let $\mathcal{J}$ be the category whose objects $(\boldsymbol{m_1}, \boldsymbol{m_2})$ are pairs of objects in $\mathcal{I}$. There are no morphisms $(\boldsymbol{m_1}, \boldsymbol{m_2}) \to (\boldsymbol{n_1}, \boldsymbol{n_2})$ unless $m_2 - m_1 = n_2 - n_1$, and in this case a morphism $(\boldsymbol{m_1}, \boldsymbol{m_2}) \to (\boldsymbol{n_1}, \boldsymbol{n_2})$ is a triple $(\beta_1, \beta_2, \sigma)$ with the $\beta_i \colon \boldsymbol{m_i} \to \boldsymbol{n_i}$ injections and $\sigma \colon \boldsymbol{n_1} \setminus \beta_1(\boldsymbol{m_1}) \to \boldsymbol{n_2} \setminus \beta_2(\boldsymbol{m_2})$ a bijection identifying the complements of β_1 and β_2. The composition of

$$(\boldsymbol{l_1}, \boldsymbol{l_2}) \xrightarrow{(\alpha_1,\alpha_2,\rho)} (\boldsymbol{m_1}, \boldsymbol{m_2}) \xrightarrow{(\beta_1,\beta_2,\sigma)} (\boldsymbol{n_1}, \boldsymbol{n_2})$$

is $\beta_i\alpha_i$ in the first two entries and the map $\sigma \cup \beta_2\rho\beta_1^{-1}$ in the third entry.

This category $\mathcal{J}$ is equivalent to Quillen's localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections Σ. By the Barratt–Priddy–Quillen Theorem (see eg Segal [22]), the classifying space $B\mathcal{J}$ has the homotopy type of $QS^0 \simeq \Omega^\infty\Sigma^\infty S^0$.

Concatenation in both entries makes $\mathcal{J}$ a symmetric monoidal category with monoidal unit $(\boldsymbol{0}, \boldsymbol{0})$. As in the case of $\mathcal{I}$–spaces, we obtain a symmetric monoidal category of $\mathcal{J}$–spaces $(\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}})$ with unit $U^{\mathcal{J}} = \mathcal{J}((\boldsymbol{0}, \boldsymbol{0}), -)$.

Definition 2.7 A commutative monoid in $(\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}})$ is called a *commutative $\mathcal{J}$–space monoid* and $\mathcal{CS}^{\mathcal{J}}$ denotes the category of commutative $\mathcal{J}$–space monoids.

One can define a *positive $\mathcal{J}$–model structure* on $\mathcal{CS}^{\mathcal{J}}$ in which the weak equivalences are the $\mathcal{J}$–equivalences, ie the maps $f \colon M \to N$ inducing weak equivalences $f_{h\mathcal{J}} \colon M_{h\mathcal{J}} \to N_{h\mathcal{J}}$ on homotopy colimits over $\mathcal{J}$ [17, Proposition 4.10]. The *positive $\mathcal{J}$–fibrant* objects are those M for which all morphisms $(\boldsymbol{m_1}, \boldsymbol{m_2}) \to (\boldsymbol{n_1}, \boldsymbol{n_2})$ with $m_1 \geq 1$ induce weak equivalences of Kan complexes $M(\boldsymbol{m_1}, \boldsymbol{m_2}) \to M(\boldsymbol{n_1}, \boldsymbol{n_2})$. We will frequently use that this model structure is both left and right proper.

Theorem 2.8 [17, Theorem 1.7] *With respect to the positive $\mathcal{J}$–model structure, $\mathcal{CS}^{\mathcal{J}}$ is Quillen equivalent to the category of E_∞–spaces over $B\mathcal{J}$.*

As explained in the introduction, this allows us to interpret commutative $\mathcal{J}$–space monoids as *graded E_∞ spaces*. The point in using this specific category $\mathcal{J}$ is that there is a Quillen adjunction

$$\mathbb{S}^{\mathcal{J}}[-] \colon \mathcal{CS}^{\mathcal{J}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} : \Omega^{\mathcal{J}} \tag{2-2}$$

with respect to the positive model structures. On objects, the right adjoint is given by $\Omega^{\mathcal{J}}(A)(\boldsymbol{m_1}, \boldsymbol{m_2}) = \Omega^{m_2}(A_{m_1})$, and the multiplication on $\Omega^{\mathcal{J}}(A)$ is defined similarly as in (2-1). The structure maps of $\Omega^{\mathcal{J}}(A)$ depend on the bijections in the definition of the morphisms in $\mathcal{J}$; see [17, Section 4.21] and [16, Section 2.9].

For a positive fibrant A in $\mathcal{C}\mathrm{Sp}^{\Sigma}$, we therefore view $\Omega^{\mathcal{J}}(A)$ as the *underlying graded multiplicative* E_∞ *space* of A. This terminology is justified as follows: Every point $f\colon S^{m_2} \to A_{m_1}$ in $\Omega^{\mathcal{J}}(A)(\boldsymbol{m_1}, \boldsymbol{m_2})$ represents a homotopy class in $\pi_{m_2-m_1}(A)$, and one can recover the underlying graded multiplicative monoid of $\pi_*(A)$ together with its sign action of $\{\pm 1\}$ from $\Omega^{\mathcal{J}}(A)$ [17, Section 4.21]. (The sign action is inherent to commutative $\mathcal{J}$–space monoids because $\pi_1(B\mathcal{J}) \cong \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$.)

Definition 2.9 The *graded units* $\mathrm{GL}_1^{\mathcal{J}}(A)$ of a positive fibrant commutative symmetric ring spectrum A is the sub commutative $\mathcal{J}$–space monoid of $\Omega^{\mathcal{J}}(A)$ given by the path components that map to units in the graded ring $\pi_*(A)$.

We collect some results about $\mathcal{J}$–spaces needed below. One of the benefits of working with strictly commutative monoids in $\mathcal{S}^{\mathcal{I}}$ or $\mathcal{S}^{\mathcal{J}}$ rather than with (augmented) E_∞ spaces is that coproducts and pushouts admit an explicit construction, similarly as in commutative rings or commutative ring spectra: The coproduct of M and N in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ is $M \boxtimes N$, and the pushout of $M \leftarrow P \to N$ is the coequalizer $M \boxtimes_P N$ of $M \boxtimes P \boxtimes N \rightrightarrows M \boxtimes N$.

We will often have to ensure that the coproduct $M \boxtimes N$ is homotopy invariant in a suitable sense. For this, a cofibrancy condition is necessary.

Definition 2.10 Let $\partial(\mathcal{J} \downarrow (\boldsymbol{n_1}, \boldsymbol{n_2}))$ be the full subcategory on the objects in $\mathcal{J} \downarrow (\boldsymbol{n_1}, \boldsymbol{n_2})$ which are not isomorphisms. A $\mathcal{J}$–space X is *flat* if the latching map

$$L_{(\boldsymbol{n_1},\boldsymbol{n_2})}X = \operatorname*{colim}_{\substack{(\boldsymbol{m_1},\boldsymbol{m_2})\to(\boldsymbol{n_1},\boldsymbol{n_2})\\ \in\partial(\mathcal{J}\downarrow(\boldsymbol{n_1},\boldsymbol{n_2}))}} X(\boldsymbol{m_1}, \boldsymbol{m_2}) \to X(\boldsymbol{n_1}, \boldsymbol{n_2})$$

is a cofibration of simplicial sets for every object $(\boldsymbol{n_1}, \boldsymbol{n_2})$ of $\mathcal{J}$. A commutative $\mathcal{J}$–space monoid M is flat if its underlying $\mathcal{J}$–space is.

This is the counterpart to the notion of flat (or S–cofibrant) symmetric spectra [20]. Cofibrant objects in the positive $\mathcal{J}$–model structure on $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ are flat [17, Proposition 4.28]. The free $\mathcal{J}$–space $\mathcal{J}((\boldsymbol{m_1}, \boldsymbol{m_2}), -)$ with $\mathcal{J}((\boldsymbol{m_1}, \boldsymbol{m_2}), -)_{h\mathcal{J}} \simeq *$ is flat. If X is a flat $\mathcal{J}$–space, then $X \boxtimes -$ preserves $\mathcal{J}$–equivalences between all objects [17, Proposition 8.2]. This is useful because $X \boxtimes Y$ captures the homotopy type of the left derived product as soon as *one of* X or Y is flat.

As explained for the case of $\mathcal{I}$–spaces in [18, Section 2.24], the homotopy colimit functor $(-)_{h\mathcal{J}}\colon \mathcal{S}^{\mathcal{J}} \to \mathcal{S}$ is a monoidal (but not symmetric monoidal) functor. This means in particular that there is a natural monoidal structure map

$$X_{h\mathcal{J}} \times Y_{h\mathcal{J}} \to (X \boxtimes Y)_{h\mathcal{J}} \tag{2-3}$$

for $\mathcal{J}$–spaces X and Y. Under a flatness hypothesis, we can use this map to analyze the homotopy type of $(X \boxtimes Y)_{h\mathcal{J}}$:

Lemma 2.11 *If one of X or Y is flat, then the map (2-3) is a weak equivalence.*

Proof The same proof as in the case of $\mathcal{I}$–spaces [18, Lemma 2.25] applies. □

2.12 Group completions of commutative $\mathcal{J}$–space monoids

We need a group completion functor for commutative $\mathcal{J}$–space monoids in order to study the graded log structures we will define using $\mathcal{CS}^{\mathcal{J}}$.

In the case of $\mathcal{I}$–spaces, it is easy to construct group completions because the usual bar construction for simplicial or topological monoids lifts to commutative $\mathcal{I}$–space monoids [18, Section 4]. In $\mathcal{J}$–spaces, the situation is different: The monoidal unit $U^{\mathcal{J}}$ is concentrated in $\mathcal{J}$–space degree 0, ie $U^{\mathcal{J}}(\boldsymbol{m_1}, \boldsymbol{m_2}) = \varnothing$ unless $m_2 - m_1 = 0$. So $U^{\mathcal{J}}$ is not a zero object in $\mathcal{CS}^{\mathcal{J}}$ because it is not terminal, and there is no two sided bar construction $B^{\boxtimes}(U^{\mathcal{J}}, M, U^{\mathcal{J}})$ for general M.

We show in [16] how one can overcome this difficulty by constructing group completions via a localization of the positive $\mathcal{J}$–model structure on $\mathcal{CS}^{\mathcal{J}}$. To explain some of the details, we recall from [16, Section 3] that there is a functor

$$\gamma\colon \mathcal{CS}^{\mathcal{J}} \to \Gamma^{\mathrm{op}}\text{–}\mathcal{S} \tag{2-4}$$

from $\mathcal{CS}^{\mathcal{J}}$ into Segal's category of Γ–spaces. It satisfies $\gamma(M)(1^+) = M_{h\mathcal{J}}$ and takes values in *special* Γ–spaces. Applying it for example to the terminal commutative $\mathcal{J}$–space monoid $*$ defines a Γ–space $b\mathcal{J} = \gamma(*)$ that provides an infinite delooping of the classifying space $B\mathcal{J} = (*)_{h\mathcal{J}}$. We say that a commutative $\mathcal{J}$–space monoid M is *grouplike* if the commutative monoid $\pi_0(M_{h\mathcal{J}})$ is a group. So M is grouplike if and only if $\gamma(M)$ is *very special*.

Theorem 2.13 [16, Theorem 5.5] *The category $\mathcal{CS}^{\mathcal{J}}$ admits a* group completion model structure. *The cofibrations are those of the positive $\mathcal{J}$–model structure. A map $M \to N$ is a weak equivalence if $\gamma(M) \to \gamma(N)$ is a stable equivalence of Γ–spaces. The fibrant objects are the positive $\mathcal{J}$–fibrant objects that are grouplike.*

In particular, a fibrant replacement functor for this model structure defines a functorial group completion $M \rightarrowtail M^{\mathrm{gp}}$ for commutative $\mathcal{J}$–space monoids. Weak equivalences between grouplike objects are $\mathcal{J}$–equivalences.

We recall an example for a group completion that will become relevant later:

Example 2.14 [16, Example 5.8] Let

$$M = \coprod_{n \geq 0} (\mathcal{J}((\boldsymbol{m_1}, \boldsymbol{m_2}), -))^{\boxtimes n} / \Sigma_n \tag{2-5}$$

be the free commutative $\mathcal{J}$–space monoid on a point in degree $(\boldsymbol{m_1}, \boldsymbol{m_2})$. We set $m = m_2 - m_1$ and assume $m \neq 0$ and $m_1 > 0$. Then M is concentrated in $\mathcal{J}$–space degrees mk for $k \in \mathbb{N}_0$, ie $M(\boldsymbol{n_1}, \boldsymbol{n_2}) = \varnothing$ unless $n_2 - n_1$ is of the form mk. The same argument as in [18, Example 3.7] shows that $M_{h\mathcal{J}} \simeq \coprod_{n \geq 0} B\Sigma_n$, and it follows from the Barratt–Priddy–Quillen Theorem that the group completion $(M)_{h\mathcal{J}} \to (M^{\mathrm{gp}})_{h\mathcal{J}}$ is homotopic to $\coprod_{n \geq 0} B\Sigma_n \to QS^0$.

The augmentation $M_{h\mathcal{J}} \to (*)_{h\mathcal{J}} \cong B\mathcal{J}$ associated with M maps the generator $\mathrm{id}_{(\boldsymbol{m_1}, \boldsymbol{m_2})}$ to the component of $m \in \pi_0(B\mathcal{J}) \cong \mathbb{Z}$. It follows that the augmentation $QS^0 \simeq (M^{\mathrm{gp}})_{h\mathcal{J}} \to B\mathcal{J} \simeq QS^0$ is multiplication with m.

3 Symmetric ring spectra with logarithmic structures

In the first part of this section we introduce the basic definitions, examples, and constructions for the theory of graded topological logarithmic structures. In many cases these are the straightforward modifications of the corresponding notions for topological logarithmic structures studied by Rognes [14, Section 7]. The difference is that we replace the commutative $\mathcal{I}$–space monoids (modeling E_∞ spaces) used in Rognes' $\mathcal{I}$–space log structures theory by commutative $\mathcal{J}$–space monoids (modeling *graded* E_∞ spaces). Both Rognes' and our notions may be viewed as homotopical generalizations of the affine version of the corresponding concepts in logarithmic geometry as for example described by Kato [11, Section 1] and Ogus [13].

In the second part of this section, we give a first result in which our graded log structures show a desirable behavior which is not shared by $\mathcal{I}$–space pre-log structures: We prove that the localization of the free graded pre-log structure on a ring spectrum inverts the homotopy class of its generator in the homotopy groups of the ring spectrum.

3.1 Pre-log structures, log structures and logification

Throughout this section, we let A be a positive fibrant commutative symmetric ring spectrum. Using the adjunction $(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ relating commutative $\mathcal{J}$–space monoids and commutative symmetric ring spectra, we can state the following definition:

Definition 3.2 A *graded pre-log structure* (M, α) on A is a commutative $\mathcal{J}$–space monoid M together with a map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ in $\mathcal{CS}^{\mathcal{J}}$. If (M, α) is a graded pre-log structure on A, the triple (A, M, α) is a *graded pre-log ring spectrum*. We write (A, M) for (A, M, α) if α is understood from the context.

A map $(f, f^\flat)\colon (A, M, \alpha) \to (B, N, \beta)$ of graded pre-log ring spectra consists of a map $f\colon A \to B$ of commutative symmetric ring spectra and a map $f^\flat\colon M \to N$ of commutative $\mathcal{J}$–space monoids such that $\beta f^\flat = (\Omega^{\mathcal{J}}(f))\alpha$. We call $(f, f^\flat)$ a *weak equivalence* if f is a stable equivalence of symmetric spectra and $f^\flat$ is a $\mathcal{J}$–equivalence.

Remark 3.3 We explained in the introduction that the notion of pre-log and log structures defined here differs from Rognes' notion in [14]. We therefore use the attribute "$\mathcal{I}$–space" for the various notions of pre-log and log structures considered in [14] to distinguish them from their graded counterparts studied in the present paper. When there is no risk of confusion with Rognes' definition, we occasionally drop the "graded" in our notion to ease notation.

Example 3.4 Let $x\colon S^{n_2} \to A_{n_1}$ be a basepoint preserving map. It represents a homotopy class $[x] \in \pi_{n_2-n_1}(A)$, and since A is positive fibrant every homotopy class in $\pi_n(A)$ can be represented by such a map if $n_1 \geq 1$ and $n = n_2 - n_1$. Since $\Omega^{\mathcal{J}}(A)(\boldsymbol{n_1}, \boldsymbol{n_2}) = \Omega^{n_2}(A_{n_1})$, we may view x as a point in $\Omega^{\mathcal{J}}(A)(\boldsymbol{n_1}, \boldsymbol{n_2})$.

By adjunction, x induces a map

$$\alpha\colon C(x) = \coprod_{i\geq 0} (\mathcal{J}((\boldsymbol{n_1}, \boldsymbol{n_2}), -))^{\boxtimes i}/\Sigma_i \to \Omega^{\mathcal{J}}(A) \tag{3-1}$$

from the free commutative $\mathcal{J}$–space monoid on a point in degree $(\boldsymbol{n_1}, \boldsymbol{n_2})$ to $\Omega^{\mathcal{J}}(A)$. This map defines the *free graded pre-log structure* on A. We write $(A, C(x))$ for the resulting graded pre-log ring spectrum.

Example 3.5 If M is a commutative $\mathcal{J}$–space monoid, the adjunction unit of $(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ induces the *canonical graded pre-log structure* $M \to \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{\mathrm{fib}})$ on a fibrant replacement $\mathbb{S}^{\mathcal{J}}[M]^{\mathrm{fib}}$ of $\mathbb{S}^{\mathcal{J}}[M]$.

Example 3.6 Let (B, N, β) be a graded pre-log ring spectrum and let $f\colon A \to B$ a map of commutative symmetric ring spectra. The pullback diagram

$$\begin{tikzcd} f_*N \arrow[r, "f_*\beta"] \arrow[d] & \Omega^{\mathcal{J}}(A) \arrow[d] \\ N \arrow[r, "\beta"] & \Omega^{\mathcal{J}}(B) \end{tikzcd}$$

provides a graded pre-log structure $(f_*N, f_*\beta)$ on A. Following the terminology of [14, Definition 7.26], we call $(f_*N, f_*\beta)$ the *graded direct image pre-log structure* induced by f and (N, β). It comes with a canonical map $(A, f_*N) \to (B, N)$. Because $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ is right proper, assuming that β is a positive $\mathcal{J}$–fibration or that f is a positive fibration in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ ensures that this preserves weak equivalences. (The attribute "direct image" is taken from the algebraic notion of log rings where the variance refers to the associated affine log schemes [11].)

Definition 3.7 Let (M, α) be a graded pre-log structure on a positive fibrant commutative symmetric ring spectrum A and consider the pullback diagram:

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \arrow[r, "\widetilde{\alpha}"] \arrow[d] & \mathrm{GL}_1^{\mathcal{J}}(A) \arrow[d] \\ M \arrow[r, "\alpha"] & \Omega^{\mathcal{J}}(A) \end{tikzcd} \tag{3-2}$$

Then (M, α) is a *graded log structure* if $\widetilde{\alpha}$ is a $\mathcal{J}$–equivalence. A *graded log ring spectrum* (A, M) is a graded pre-log ring spectrum with a graded log-structure.

In the definition, our assumption that A is positive fibrant ensures that $\Omega^{\mathcal{J}}(A)$ and $\mathrm{GL}_1^{\mathcal{J}}(A)$ capture the desired homotopy types. Since $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ is an inclusion of path components, it is a positive $\mathcal{J}$–fibration. Hence (3-2) is homotopy cartesian because $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ is right proper, and condition of (A, M) being a log ring spectrum is homotopy invariant.

Example 3.8 The inclusion $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ provides the *trivial graded log structure* on A.

Example 3.9 Let (B, N) be a graded log ring spectrum and let $f\colon A \to B$ be a map of positive fibrant commutative symmetric ring spectra. Forming the base change of $\beta\colon N \to \Omega^{\mathcal{J}}(B)$ along

$$\mathrm{GL}_1^{\mathcal{J}}(A) \to \mathrm{GL}_1^{\mathcal{J}}(B) \to \Omega^{\mathcal{J}}(B) \quad \text{and} \quad \mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(B)$$

shows that the graded direct image pre-log structure f_*N of Example 3.6 is a log structure if β is a positive $\mathcal{J}$–fibration or f is a positive fibration in $\mathcal{C}\mathrm{Sp}^{\Sigma}$.

Example 3.10 Combining the last two examples, any map $f\colon A \to B$ of positive fibrant objects in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ gives rise to the graded log structure $f_*(\mathrm{GL}_1^{\mathcal{J}}(B)) \to \Omega^{\mathcal{J}}(A)$ on A. We call this the *graded direct image log structure* induced by f. This applies for instance to the map of complex K–theory spectra $i\colon ku \to KU$ which exhibits the connective ku as the connective cover of the periodic KU. We will study the properties of this log structure in Section 4. As mentioned in the introduction, the $\mathcal{I}$–space counterpart of this construction only provides the trivial log structure since both $\Omega^{\mathcal{I}}(ku) \to \Omega^{\mathcal{I}}(KU)$ and $\mathrm{GL}_1^{\mathcal{I}}(ku) \to \mathrm{GL}_1^{\mathcal{I}}(KU)$ are $\mathcal{I}$–equivalences.

There is a logification process turning pre-log structures into log structures:

Construction 3.11 If (M,α) is a graded pre-log structure on A, we choose a (functorial) factorization

$$\alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \rightarrowtail G \xrightarrow{\sim} \mathrm{GL}_1^{\mathcal{J}}(A) \tag{3-3}$$

of $\widetilde{\alpha}$ and define M^a to be the pushout in $\mathcal{CS}^{\mathcal{J}}$ displayed on the left-hand side of the diagram:

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \arrow[r, tail] \arrow[d] & G \arrow[r, two heads, "\sim"] \arrow[d] & \mathrm{GL}_1^{\mathcal{J}}(A) \arrow[d] \\ M \arrow[r] \arrow[rr, bend right] & M^a \arrow[r, dashed, "\alpha^a"] & \Omega^{\mathcal{J}}(A) \end{tikzcd}$$

Hence $M^a = M \boxtimes_{\alpha^{-1}\mathrm{GL}_1^{\mathcal{J}}(A)} G$, and the universal property of the pushout provides the map $\alpha^a\colon M^a \to \Omega^{\mathcal{J}}(A)$. This construction preserves weak equivalences: Left properness of $\mathcal{CS}^{\mathcal{J}}$ ensures that the pushout coincides with the homotopy pushout.

The pre-log structure (M^a,α^a) is the *associated graded log structure* of (A,M), and the canonical map $(M,\alpha) \to (M^a,\alpha^a)$ is the *logification* of (M,α).

This terminology is justified by the following lemma.

Lemma 3.12 *The associated graded log structure (M^a,α) is a log-structure. If (A,M) is a graded log ring spectrum, then the logification is a weak equivalence.*

Proof For brevity we write $W = \Omega^{\mathcal{J}}(A)$. Let $W^{\times} = \mathrm{GL}_1^{\mathcal{J}}(A)$ be union of invertible path components, and let $\widehat{W}$ be the complement of $W^{\times}$, ie the $\mathcal{J}$–space given by the components of W which are not invertible. For a map $N \to W$ in $\mathcal{CS}^{\mathcal{J}}$, we let $N = \widetilde{N} \amalg \widehat{N}$ be the decomposition of the underlying $\mathcal{J}$–space of N into the part $\widetilde{N} = N \times_W W^{\times}$ that maps to the units and the part $\widehat{N} = N \times_W \widehat{W}$ that maps to the nonunits. Then $\alpha^{-1}\mathrm{GL}_1^{\mathcal{J}}(A) = \widetilde{M}$, and there are isomorphisms

$$M^a = M \boxtimes_{\widetilde{M}} G \cong (\widetilde{M} \amalg \widehat{M}) \boxtimes_{\widetilde{M}} G \cong G \amalg (\widehat{M} \boxtimes_{\widetilde{M}} G).$$

Since G maps to the units and $\widehat{M} \boxtimes_{\widetilde{M}} G$ maps to the nonunits, this shows $\widetilde{M}^a \cong G$. So (M^a,α^a) is a log structure.

The second assertion is clear because the cofibration $\alpha^{-1}\mathrm{GL}_1^{\mathcal{J}}(A) \rightarrowtail G$ is a $\mathcal{J}$–equivalence if (M,A) is a log ring spectrum. □

The proof of the previous lemma also proves the following statement.

Lemma 3.13 *Let (A,M) be graded pre-log structure. If $M \to \Omega^{\mathcal{J}}(A)$ factors through $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$, then (A,M^a) is weakly equivalent to the trivial graded log structure.*

3.14 Log structures and localization

In this paragraph we consider a positive fibrant commutative symmetric ring spectrum A with a graded pre-log structure (M,α).

As discussed in Section 2.12, we can form the group completion $M \rightarrowtail M^{\mathrm{gp}}$ of M. Combining this map with $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ and using the left adjoint $\mathbb{S}^{\mathcal{J}}[-]$ of $\Omega^{\mathcal{J}}$, we obtain the following diagram in $\mathcal{C}\mathrm{Sp}^{\Sigma}$:

$$\mathbb{S}^{\mathcal{J}}[M^{\mathrm{gp}}] \longleftarrow\!\!\prec \mathbb{S}^{\mathcal{J}}[M] \longrightarrow A \tag{3-4}$$

Definition 3.15 The pushout $A[M^{-1}] = \mathbb{S}^{\mathcal{J}}[M^{\mathrm{gp}}] \wedge_{\mathbb{S}^{\mathcal{J}}[M]} A$ of the diagram (3-4) in the category of commutative symmetric ring spectra is the *localization* of the graded pre-log ring spectrum (A,M).

Since the group completion is defined as a functorial fibrant replacement, this is functorial. If M is cofibrant, the fact that $M \to M^{\mathrm{gp}}$ is a cofibration ensures that the localization sends weak equivalences of pre-log ring spectra to stable equivalences. To ensure this desirable property, we implicitly form a cofibrant replacement of M whenever we consider $A[M^{-1}]$ for a noncofibrant M.

The localization of (A,M) does only depend on its logification:

Lemma 3.16 *If M is cofibrant, then the logification $(A,M) \to (A,M^a)$ induces a stable equivalence $A[M^{-1}] \to A[(M^a)^{-1}]$.*

Proof By definition, the right-hand square in

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{J}}[M] \arrow[r] \arrow[d] & \mathbb{S}^{\mathcal{J}}[M^a] \arrow[r] \arrow[d] & A \arrow[d] \\
\mathbb{S}^{\mathcal{J}}[M^{\mathrm{gp}}] \arrow[r] & \mathbb{S}^{\mathcal{J}}[(M^a)^{\mathrm{gp}}] \arrow[r] & A[(M^a)^{-1}]
\end{tikzcd}$$

is a pushout. Let Q be the pushout of $M^{\mathrm{gp}} \leftarrow M \to M^a$. Then because M is cofibrant, so are Q and $(M^a)^{\mathrm{gp}}$. Since $\mathbb{S}^{\mathcal{J}}[-]$ sends $\mathcal{J}$–equivalences between cofibrant objects to stable equivalences, the diagram indicates that it is enough to show that the canonical map $Q \to (M^a)^{\mathrm{gp}}$ is a $\mathcal{J}$–equivalence. For this we consider the iterated pushout:

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \arrow[r, tail] \arrow[d] & G \arrow[d] \\ M \arrow[r, tail] \arrow[d] & M^a \arrow[d] \\ M^{\mathrm{gp}} \arrow[r, tail] & Q \end{tikzcd}$$

Both $M \to M^{\mathrm{gp}}$ and $M^a \to (M^a)^{\mathrm{gp}}$ are acyclic cofibrations in the group completion model structure. By cobase change this holds for $M^a \to Q$, and two out of three shows that $Q \to (M^a)^{\mathrm{gp}}$ is a weak equivalence in the group completion model structure. We need to show that Q is grouplike in order to see that it is a $\mathcal{J}$–equivalence. By definition, Q is grouplike if $Q_{h\mathcal{J}}$ is. The canonical map

$$\pi_0((M^{\mathrm{gp}} \boxtimes G)_{h\mathcal{J}}) \to \pi_0((M^{\mathrm{gp}} \boxtimes_{\alpha^{-1}\mathrm{GL}_1^{\mathcal{J}}(A)} G)_{h\mathcal{J}}) \cong \pi_0(Q_{h\mathcal{J}}) \tag{3-5}$$

is clearly surjective. Because M^{gp} is flat, Lemma 2.11 provides natural isomorphism $\pi_0((M^{\mathrm{gp}})_{h\mathcal{J}}) \times \pi_0(G_{h\mathcal{J}}) \cong \pi_0((M^{\mathrm{gp}} \boxtimes G)_{h\mathcal{J}})$. Since M^{gp} and G are grouplike, the domain of the surjection (3-5) is a group, and so is $\pi_0(Q_{h\mathcal{J}})$. □

We recall from [20] that the process of inverting an element in a graded ring can be lifted from homotopy groups to symmetric ring spectra:

Proposition 3.17 *Let $x\colon S^{n_2} \to A_{n_1}$ represent a homotopy class $[x] \in \pi_{n_2-n_1}(A)$. Then x gives rise to a positive fibrant commutative symmetric ring spectrum $A[1/x]$ and a positive cofibration $i\colon A \to A[1/x]$ in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ which induces an isomorphism $\pi_*(A)[1/[x]] \to \pi_*(A[1/x])$. This construction is natural in A.*

Proof An explicit representative for $A \to A[1/x]$ is constructed in [20, I. Corollary 4.69]. This map has the desired properties apart from being a cofibration with fibrant codomain. Forming a fibrant replacement of the codomain and factoring the map from A into the fibrant replacement by a cofibration followed by an acyclic fibration provides the desired cofibration $i\colon A \to A[1/x]$ with fibrant codomain. □

The following universal property of this construction is also discussed in [20]:

Corollary 3.18 *Let $A \to B$ be a map into a positive fibrant commutative symmetric ring spectrum B which sends $[x]$ to a unit in $\pi_*(B)$. Then $A \to B$ extends over $A \to A[1/x]$.*

One can use the statement of Proposition 3.19 and the argument given in the last part of its proof to see that two possible extensions $A[1/x] \to B$ in the corollary are homotopic relative A.

Proof of Corollary 3.18 Inverting x and its image in B we obtain a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A[1/x] \arrow[r] & B[1/x] \end{tikzcd}$$

in which the map $B \to B[1/x]$ is a stable equivalence. So we get a map $A[1/x] \to B$ in the homotopy category of A–algebras. Since $A[1/x]$ is cofibrant as an A–algebra and B is fibrant, we can realize it as a map of A–algebras. □

In many interesting cases, we can now identify the localizations of the free graded pre-log structures introduced in Example 3.4:

Proposition 3.19 *Let A be a positive fibrant commutative symmetric ring spectrum, and let $x\colon S^{n_2} \to A_{n_1}$ represent a homotopy class $[x] \in \pi_*(A)$ of degree $n = n_2 - n_1$. Then the localization $A[(C(x))^{-1}]$ of the free graded pre-log structure $C(x)$ associated with x is stably equivalent to the commutative symmetric ring spectrum $A[1/x]$.*

Remark 3.20 The proposition exhibits one of the key features of graded log structures: The map x also generates a free $\mathcal{I}$–space pre-log structure on S^n [14, Example 7.18]. However, the localization of this $\mathcal{I}$–space pre-log structure is a homotopy pushout of connective ring spectra, and will not give $A[1/x]$ if $n > 0$. In contrast, our graded setup shows a behavior that one would naively expect from algebra: If A is a commutative ring, then an element $a \in A$ gives rise to a map $\mathbb{N}_0 \to (A, \cdot)$ from the free commutative monoid on one generator into the multiplicative monoid of A. This determines a ring map $\mathbb{Z}[x] \cong \mathbb{Z}[\mathbb{N}_0] \to A$, and the pushout of $\mathbb{Z}[(\mathbb{Z},+)] = \mathbb{Z}[x^{\pm 1}] \leftarrow \mathbb{Z}[x] \to A$ is the ring $A[1/a]$.

Proof of Proposition 3.19 Since $[x]$ is a unit in $\pi_*(A[1/x])$, we can extend the composite $C(x) \to \Omega^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A[1/x])$ over $C(x) \rightarrowtail C(x)^{\mathrm{gp}}$. The adjunction $(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ turns this into maps in $\mathcal{C}\mathrm{Sp}^{\Sigma}$, and the universal property of the pushout provides a map $A[C(x)^{-1}] \to A[1/x]$. Factoring this map as

$$A[C(x)^{-1}] \underset{\simeq}{\overset{j}{\rightarrowtail}} P \twoheadrightarrow A[1/x]$$

gives a fibrant replacement P of $A[C(x)^{-1}]$, and it is enough to show that the map of A–algebras $P \to A[1/x]$ is a stable equivalence.

We first show that the map induced by the composite $A \to A[C(x)^{-1}] \to P$ sends the homotopy class of x to a unit in $\pi_*(P)$: The image of $[x]$ is invertible in $\pi_*(P)$ if and only if the image of x in $\pi_0((\Omega^{\mathcal{J}} P)_{h\mathcal{J}})$ represents a unit. The latter condition is satisfied since by construction, the image of x in $\pi_0((\Omega^{\mathcal{J}} P)_{h\mathcal{J}})$ lies in the image of the homomorphism $\pi_0((C(x)^{\mathrm{gp}})_{h\mathcal{J}}) \to \pi_0((\Omega^{\mathcal{J}} P)_{h\mathcal{J}})$ whose domain is a group. Hence Corollary 3.18 provides a map $A[1/x] \to P$ of A–algebras.

The composite $A[1/x] \to P \to A[1/x]$ is a stable equivalence: The induced map of homotopy groups $\pi_*(A)[1/[x]] \cong \pi_*(A[1/x]) \to \pi_*(A[1/x]) \cong \pi_*(A)[1/[x]]$ is a map under $\pi_*(A)$ and hence an isomorphism. So it remains to show that the composite $P \to A[1/x] \to P$ is a stable equivalence. Let us write k for this map. We use the acyclic cofibration $j\colon A[C(x)^{-1}] \to P$ defined above and claim that j and kj are homotopic as maps of A–algebras. By adjunction, these maps correspond to two possible extensions of $C(x) \to \mathrm{GL}_1^{\mathcal{J}}(P)$ along $C(x) \to C(x)^{\mathrm{gp}}$. Applying the homotopy uniqueness of lifts in model categories (Hirschhorn [9, Proposition 7.6.13]) to this extension problem in the group completion model structure shows that the extensions are homotopic, and it follows that j and kj are homotopic. So k is a stable equivalence because j is. □

4 Direct image log structures generated by homotopy classes

In this section we compare various ways of how classes in the homotopy groups of connective ring spectra give rise to graded pre-log and log structures and apply this to the case of K–theory spectra.

4.1 Log structures on connective ring spectra

As before we let A be a positive fibrant commutative symmetric ring spectrum. In addition, we now assume that A is connective and fix a map $x\colon S^{n_2} \to A_{n_1}$ with $n_1 \geq 1$ that represents a nontrivial homotopy class $[x] \in \pi_*(A)$ of even positive degree $n = n_2 - n_1$ such that the localization map $i\colon A \to A[1/x]$ of Proposition 3.17 exhibits A as the connective cover of $A[1/x]$. This means that $i\colon A \to A[1/x]$ induces an isomorphism on stable homotopy groups of nonnegative degrees. The connective complex K–theory spectrum and the Bott class are the motivating example for this setup.

Since A is positive fibrant, the existence of such a representing map x for the relevant homotopy class is always ensured. Our assumptions in particular imply that A represents a nontrivial homotopy type and that $[x]$ is not already a unit in $\pi_*(A)$. So x has a chance to generate a nontrivial log structure.

We have already introduced the free graded pre-log structure $C(x)$ on A (Example 3.4) and the graded direct image log structure $i_*\mathrm{GL}_1^{\mathcal{J}}A[1/x]$ induced by $i\colon A \to A[1/x]$ (Example 3.9). To clarify their properties and relationship, we consider another less obvious but highly useful graded pre-log structure arising in this setup:

Construction 4.2 The following diagram summarizes the various steps to be described next. They will lead to a graded pre-log structure $D(x)$ on A.

$$\begin{tikzcd} C(x) \arrow[r, tail] \arrow[d] \arrow[rrr, bend left=10] & D(x) \arrow[r, two heads, "\sim"] & D'(x) \arrow[r] \arrow[dll] & \Omega^{\mathcal{J}}(A) \arrow[d] \\ C(x)^{\mathrm{gp}} \arrow[rr] & & \mathrm{GL}_1^{\mathcal{J}}(A[1/x]) \arrow[r] & \Omega^{\mathcal{J}}(A[1/x]) \end{tikzcd} \tag{4-1}$$

To obtain $D(x)$, we first observe that the composite

$$C(x) \to \Omega^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A[1/x])$$

factors through $\mathrm{GL}_1^{\mathcal{J}}(A[1/x]) \to \Omega^{\mathcal{J}}(A[1/x])$ because $[x]$ becomes invertible in $\pi_*(A[1/x])$. We then factor the resulting map $C(x) \to \mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ in the group completion model structure as an acyclic cofibration $C(x) \to C(x)^{\mathrm{gp}}$ followed by a fibration $C(x)^{\mathrm{gp}} \to \mathrm{GL}_1^{\mathcal{J}}(A[1/x])$. Since $\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ is positive $\mathcal{J}$–fibrant and grouplike, so is $C(x)^{\mathrm{gp}}$, and $C(x) \to C(x)^{\mathrm{gp}}$ is a possible choice for the group completion of $C(x)$.

Next we form the pullback $D'(x)$ of $C(x)^{\mathrm{gp}} \twoheadrightarrow \Omega^{\mathcal{J}}(A[1/x]) \leftarrow \Omega^{\mathcal{J}}(A)$ to obtain a pre-log structure $D'(x)$ on $\Omega^{\mathcal{J}}(A)$. A cofibrant replacement of $D'(x)$ relative to $C(x)$ defines the desired pre-log structure $D(x)$ on $\Omega^{\mathcal{J}}(A)$.

We call $D(x)$ the *graded direct image pre-log structure* associated with x. There will be no confusion with the more general direct image pre-log structure discussed in Example 3.6, although $D'(x)$ may of course be viewed as the direct image pre-log structure induced by $i\colon A \to A[1/x]$ and $C(x)^{\mathrm{gp}} \to \Omega^{\mathcal{J}}(A[1/x])$.

Remark 4.3 In the construction, one could also extend $C(x) \to \mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ over a given group completion $C(x) \rightarrowtail C(x)^{\mathrm{gp}}$ of $C(x)$ by means of the lifting axiom in the group completion model structure. Defining $C(x)^{\mathrm{gp}}$ as we did has the advantage of being functorial and making both $D'(x)$ and $D(x)$ homotopy invariant.

The universal property of the coproduct $\boxtimes$ in $\mathcal{CS}^{\mathcal{J}}$, the pre-log structure $C(x) \to \Omega^{\mathcal{J}}(A)$ and the trivial log structure $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ together give rise to a pre-log ring spectrum $(A, C(x) \boxtimes \mathrm{GL}_1^{\mathcal{J}}(A))$. The diagram of commutative $\mathcal{J}$–space monoids (4-1) now induces the following commutative diagram of graded pre-log ring spectra:

$$\begin{tikzcd}
(A, C(x)) \arrow[r] \arrow[d] & (A, D(x)) \arrow[d] \arrow[dr] & \\
(A, C(x) \boxtimes \mathrm{GL}_1^{\mathcal{J}}(A)) \arrow[r] & (A, i_* \mathrm{GL}_1^{\mathcal{J}}(A[1/x]) \arrow[r] & (A[1/x], \mathrm{GL}_1^{\mathcal{J}}(A[1/x])
\end{tikzcd} \tag{4-2}$$

The next theorem summarizes the properties of these pre-log ring spectra.

Theorem 4.4 *Let A be a connective positive fibrant commutative symmetric ring spectrum and let $x\colon S^{n_2} \to A_{n_1}$ represent a nontrivial homotopy class $[x] \in \pi_*(A)$ of even positive degree $n = n_2 - n_1$ such that the localization map $i\colon A \to A[1/x]$ exhibits A as the connective cover of $A[1/x]$.*

Then the graded pre-log structures in the bottom line of (4-2) are log structures, the two vertical maps induce weak equivalences after logification, and the localization of all four graded pre-log and log structures on A is stably equivalent to $A[1/x]$.

In other words, x induces two different graded log structures $C(x) \boxtimes \mathrm{GL}_1^{\mathcal{J}}(A)$ and $i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ on A which arise as the logification of "smaller" graded pre-log structures $C(x)$ and $D(x)$. Both these log structures approximate $A[1/x]$ in that their localizations are equivalent to $A[1/x]$. However, we will see that $i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ is the more useful one: It is more canonical in that it does not depend on the choice of the representative x, and we employ it for our results about formally graded log étale extensions in Section 6.

The reason for introducing $D(x)$ is that it is a convenient presentation of the log-structure $i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ that is easier to work with. This will become clear in the proof Theorem 6.1 and in the proof of Theorem 4.4 at the end of this paragraph.

Remark 4.5 The analogy with the situation for discrete valuation rings allows for the following geometric interpretation of the previous theorem: The localization $A \to A[1/x]$ corresponds to an open immersion $j\colon \mathrm{spec}(A[1/x]) \to \mathrm{spec}(A)$. The maps in (4-2) give rise to various factorizations

$$\mathrm{spec}(A[1/x]) \to \mathrm{spec}(A, M) \to \mathrm{spec}(A)$$

of j in the category of log schemes that provide relative compactifications of j.

In the evolving subject of derived algebraic geometry, one often considers sheaves of *connective* E_∞ spectra. The above suggests that graded log structures might be useful for treating sheaves of *periodic* E_∞ ring spectra.

In order to prove the theorem, we begin with giving a more explicit description of the homotopy type of the commutative $\mathcal{J}$–space monoid $D(x)$.

Lemma 4.6 *The space $D(x)_{h\mathcal{J}}$ is weakly equivalent to $(QS^0)_{\geq 0}$, and the map $C(x)_{h\mathcal{J}} \to D(x)_{h\mathcal{J}}$ is homotopic to the map $\coprod_{k\geq 0} B\Sigma_k \to (QS^0)_{\geq 0}$ obtained by restricting to the nonnegative path components in the group completion QS^0 of $C(x)_{h\mathcal{J}}$.*

Proof It is enough to consider $D'(x)$. The space $D'(x)(\boldsymbol{m_1},\boldsymbol{m_2})$ is the pullback of

$$(C(x)^{\mathrm{gp}})(\boldsymbol{m_1},\boldsymbol{m_2}) \to \Omega^{\mathcal{J}}(A[1/x])(\boldsymbol{m_1},\boldsymbol{m_2}) \leftarrow \Omega^{\mathcal{J}}(A)(\boldsymbol{m_1},\boldsymbol{m_2}).$$

If $m_2 - m_1$ is negative, the pullback is empty because the image of a point in $D'(x)(\boldsymbol{m_1},\boldsymbol{m_2})$ in $\Omega^{\mathcal{J}}(A)(\boldsymbol{m_1},\boldsymbol{m_2})$ would represent a power of an inverse of $[x]$ in $\pi_*(A)$. But our assumptions imply that $[x]$ is not a unit in $\pi_*(A)$.

Since $\pi_*(A) \to \pi_*(A[1/x])$ is an isomorphism in nonnegative degrees, the map $\Omega^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A[1/x])$ is a weak equivalence when evaluated on $(\boldsymbol{m_1},\boldsymbol{m_2})$ with $m_1 \geq 1$ and $m_2 - m_1 \geq 0$. The same holds for the base change $D'(x) \to C(x)^{\mathrm{gp}}$ because $C(x)^{\mathrm{gp}} \to \Omega^{\mathcal{J}}(A[1/x])$ is a positive fibration by construction. Since the full subcategory of $\mathcal{J}$ on the objects $(\boldsymbol{m_1},\boldsymbol{m_2})$ with $m_1 \geq 1$ is a homotopy cofinal subcategory of $\mathcal{J}$ [17, Corollary 5.9], the description of the group completion $C(x)^{\mathrm{gp}}$ of $C(x)$ in Example 2.14 proves the claim. □

The following lemma is the key step towards Theorem 4.4:

Lemma 4.7 *The map $(A, D(x)) \to (A, i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x]))$ in (4-2) induces a weak equivalence $(A, D(x)^a) \to (A, i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x]))$.*

Proof It is enough to show the statement for $(A, D'(x)) \to (A, i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x]))$.

We write $\alpha\colon D'(x) \to \Omega^{\mathcal{J}}(A)$ for the structure map of $(A, D'(x))$. In $\mathcal{J}$–space degree 0, its restriction $D'(x)_0 \to \Omega^{\mathcal{J}}(A)$ factors through $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ since the zero component of $D'(x)_{h\mathcal{J}}$ maps into the component of the unit of $(\Omega^{\mathcal{J}}(A))_{h\mathcal{J}}$. $\mathrm{GL}_1^{\mathcal{J}}(A)$ is concentrated in $\mathcal{J}$–space degree 0 because A is connective. It follows that the pullback $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A))$ is isomorphic to $D'(x)_0$.

Let G be the replacement of $\mathrm{GL}_1^{\mathcal{J}}(A)$ used in the logification of Construction 3.11. We have to show that the following map is a $\mathcal{J}$–equivalence:

$$D'(x) \boxtimes_{D'(x)_0} G \to i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$$

Composing the canonical maps $\mathcal{J}((\boldsymbol{n_1},\boldsymbol{n_2}),-)^{\boxtimes k}\to C(x)$ with $C(x)\to D'(x)$ and inducing up to a map of $D'(x)_0$–modules provides a map of $D'(x)_0$–modules

$$\coprod_{k\geq 0} D'(x)_0\boxtimes(\mathcal{J}((\boldsymbol{n_1},\boldsymbol{n_2}),-))^{\boxtimes k}\to D'(x). \tag{4-3}$$

Lemma 4.6 and Lemma 2.11 show that this map is a $\mathcal{J}$–equivalence. If we view $G\boxtimes_{D'(x)_0}-$ as a functor from $D'(x)_0$–modules to G–modules, we can apply it to (4-3). This reduces the claim to showing that the composite map

$$\coprod_{k\geq 0} G\boxtimes(\mathcal{J}((\boldsymbol{n_1},\boldsymbol{n_2}),-))^{\boxtimes k}\to i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x]) \tag{4-4}$$

is a $\mathcal{J}$–equivalence. Since A is connective, G is concentrated in $\mathcal{J}$–space degree 0, and the map $G\to i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])_0\simeq\mathrm{GL}_1^{\mathcal{J}}(A[1/x])_0$ into the $\mathcal{J}$–space degree 0 part of $i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ is a $\mathcal{J}$–equivalence. The $i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ is the part of $\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$ sitting in nonnegative $\mathcal{J}$–space degrees. Since multiplication with the element in $\mathrm{GL}_1^{\mathcal{J}}(A[1/x])(\boldsymbol{m_1},\boldsymbol{m_2})$ represented by x induces an equivalence between the different $\mathcal{J}$–space degree parts of $i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x])$, it follows from Lemma 2.11 that the map (4-4) is a $\mathcal{J}$–equivalence. □

Remark 4.8 The proof of the previous lemma suggests that we may think of $D(x)$ as a kind of polynomial algebra on x, although it is certainly not free. The key feature of $D(x)$ is that the group completion in its construction ensures that the components corresponding to the powers of x are equivalent. Obviously, this property is not shared by the free object $C(x)$. This distinction between the homotopical counterparts "free" and "polynomial" algebras does not occur in algebra and is one reasons for why not all pre-log structures in (4-2) have algebraic precursors.

Lemma 4.9 *The map $(A,C(x))\to(A,C(x)\boxtimes\mathrm{GL}_1^{\mathcal{J}}(A))$ induces a weak equivalence $(A,C(x)^a)\to(A,C(x)\boxtimes\mathrm{GL}_1^{\mathcal{J}}(A))$.*

Proof Let $\alpha\colon C(x)\to\Omega^{\mathcal{J}}(A)$ the structure map of $(A,C(x))$. A similar argument as in Lemma 4.7 shows $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A))\cong C(x)_0$. Since $C(x)_0$ is the monoidal unit, $C(x)^a=C(x)\boxtimes_{C(x)_0}G\cong C(x)\boxtimes G$ is $\mathcal{J}$–equivalent to $C(x)\boxtimes\mathrm{GL}_1^{\mathcal{J}}(A)$. □

Lemma 4.10 *The localization $A[D(x)^{-1}]$ of $(A,D(x))$ is stably equivalent to the commutative symmetric ring spectrum $A[1/x]$.*

Proof Since the localization of $(A,C(x))$ is stably equivalent to $A[1/x]$, then by Proposition 3.19, it is enough to show that $(A,C(x))\to(A,D(x))$ induces a weak equivalence on localizations. Arguing as in the proof of Lemma 3.16, this reduces to showing that the pushout Q of $C(x)^{\mathrm{gp}} \leftarrow C(x) \to D(x)$ is grouplike. We know that there is a surjection $\pi_0(C(x)^{\mathrm{gp}}_{h\mathcal{J}}) \times \pi_0(D(x)_{h\mathcal{J}}) \to \pi_0(Q_{h\mathcal{J}})$. The latter map induces a surjection $\pi_0(C(x)^{\mathrm{gp}}_{h\mathcal{J}}) \otimes_{\pi_0(C(x)_{h\mathcal{J}})} \pi_0(D(x)_{h\mathcal{J}}) \to \pi_0(Q_{h\mathcal{J}})$, whose domain is a group because $\pi_0((C(x) \to D(x))_{h\mathcal{J}})$ is an isomorphism by Lemma 4.6. □

Proof of Theorem 4.4 Proposition 3.19 and Lemma 4.10 show that the localizations of $(A, C(x))$ and $(A, D(x))$ are stably equivalent to $A[1/x]$. Lemma 4.7 and Lemma 4.9 show that the vertical maps in (4-2) are equivalences after logification. Since the two logifications are log-ring spectra (A, M) with cofibrant M by construction, Lemma 3.16 implies that the localizations of $(A, C(x) \boxtimes \mathrm{GL}_1^{\mathcal{J}}(A))$ and $(A, i_*\mathrm{GL}_1^{\mathcal{J}}(A[1/x]))$ are equivalent to $A[1/x]$. □

Remark 4.11 One may wonder if a homotopy class $[x] \in \pi_0(A)$ of degree 0 gives rise to graded pre-log structures with the same properties as in Theorem 4.4. For such $[x]$, we can still pick a representing map x and form $D(x)$ as in Construction 4.2. However, the proof of Lemma 4.7 does not apply to this $D(x)$ because its components are not equivalent. Nevertheless, a similar argument as in Lemma 4.7 shows that the localization of $(A, D(x))$ is stably equivalent to $A[1/x]$.

4.12 Log structures on K–theory spectra

The connective complex K–theory spectrum ku and its p–local or p–complete counterparts $ku_{(p)}$ and ku_p at a prime p are E_∞ ring spectra; see eg Elmendorf, Kriz, Mandell and May [7] or [20]. They can hence be represented by positive fibrant commutative symmetric ring spectra. The Bott class $u \in \pi_*(ku) \cong \mathbb{Z}[u]$ (respectively $u \in \pi_*(ku_{(p)}) \cong \mathbb{Z}_{(p)}[u]$ or $u \in \pi_*(ku_p) \cong \mathbb{Z}_p[u]$) is a homotopy class of degree 2, and the corresponding periodic theories KU, $KU_{(p)}$, and KU_p can be obtained by inverting the Bott class. Theorem 4.4 implies that the periodic spectra give rise to direct image log structures on the connective spectra whose localization is the periodic theory.

If p is odd, the same applies to the p–local Adams summand ℓ of $ku_{(p)}$ and the p–complete Adams summand ℓ_p of ku_p. Baker and Richter [4, Corollary 1.4] have shown that ℓ admits a unique E_∞ structure and that the resulting E_∞ structure on ℓ_p coincides with the one considered, eg in [1, Section 2.1]. Therefore, these spectra may be represented by positive fibrant commutative symmetric ring spectra. This time, $v_1 \in \pi_*(\ell) \cong \mathbb{Z}_{(p)}[v_1]$ (resp. $v_1 \in \pi_*(\ell_p) \cong \mathbb{Z}_p[v_1]$) is a homotopy class of degree $2p-2$. Inverting v_1 gives the periodic theories L and L_p, and we may apply Theorem 4.4 as above.

According to Ausoni [1, Section 2.1] and Baker and Richter [3, Remark 9.4], the inclusions $\ell_p \to ku_p$ and $\ell \to ku_{(p)}$ can be represented by maps of E_∞ ring spectra and therefore by maps of commutative symmetric ring spectra. Moreover, passing to fibrant replacements we may assume that these maps are positive fibrations of positive fibrant commutative symmetric ring spectra $\iota_{(p)}\colon \ell \to ku_{(p)}$ and $\iota_p\colon \ell_p \to ku_p$. Under these assumptions, the fact that the induced map of homotopy groups send v_1 to u^{p-1} implies the following proposition.

Proposition 4.13 *The homotopy classes $u \in \pi_2(ku_p)$ and $v_1 \in \pi_{2p-2}(\ell_p)$ admit representatives $u\colon S^3 \to (ku_p)_1$ and $v_1\colon S^{3(p-1)} \to (\ell_p)_{p-1}$ such that ι_p induces the following commutative diagram of pre-log ring spectra:*

$$
\begin{tikzcd}
(\ell_p, C(v_1)) \arrow[r] \arrow[d] & (\ell_p, D(v_1)) \arrow[r] \arrow[d] & (\ell_p, i_* \mathrm{GL}_1^{\mathcal{J}}(L_p)) \arrow[r] \arrow[d] & (L_p, \mathrm{GL}_1^{\mathcal{J}}(L_p)) \arrow[d] \\
(ku_p, C(u)) \arrow[r] & (ku_p, D(u)) \arrow[r] & (ku_p, i_* \mathrm{GL}_1^{\mathcal{J}}(KU_p)) \arrow[r] & (KU_p, \mathrm{GL}_1^{\mathcal{J}}(KU_p))
\end{tikzcd}
\tag{4-5}
$$

The same holds in the p–local case.

To ease notation, we have started to use the same symbols for the homotopy classes and their representatives.

Proof Two arbitrary representatives $v_1'\colon S^{3(p-1)} \to (\ell_p)_{p-1}$ and $u\colon S^3 \to (ku_p)_1$ of v_1 and u may be viewed as points in $\Omega^{\mathcal{J}}(ku_p)(\mathbf{1},\mathbf{3})$ and $\Omega^{\mathcal{J}}(\ell_p)(\boldsymbol{p}-\mathbf{1}, \mathbf{3}(\boldsymbol{p}-\mathbf{1}))$. They have the property that the image of v_1' under $(\iota_p\colon \ell_p \to ku_p)_*$ lies in the same component of $\Omega^{\mathcal{J}}(ku_p)(\boldsymbol{p}-\mathbf{1}, \mathbf{3}(\boldsymbol{p}-\mathbf{1}))$ as u^{p-1}. Since we assumed $\iota_p\colon \ell_p \to ku_p$ to be a positive fibration, we can use the path lifting property of the Kan fibration

$$\Omega^{\mathcal{J}}(\ell_p)(\boldsymbol{p}-\mathbf{1}, \mathbf{3}(\boldsymbol{p}-\mathbf{1})) \to \Omega^{\mathcal{J}}(ku_p)(\boldsymbol{p}-\mathbf{1}, \mathbf{3}(\boldsymbol{p}-\mathbf{1}))$$

to get a representative $v_1\colon S^{3(p-1)} \to (\ell_p)_{p-1}$ hitting u^{p-1}. With $KU_p = ku_p[1/u]$ and $L_p = \ell_p[1/v_1]$ as models for the periodic spectra, Corollary 3.18 shows that $\ell_p \to ku_p$ extends to a commutative square:

$$
\begin{tikzcd}
\ell_p \arrow[r] \arrow[d] & L_p \arrow[d] \\
ku_p \arrow[r] & KU_p
\end{tikzcd}
$$

So the right-hand square in (4-5) commutes. The relation $(\iota_p)_*(v_1) = u^{p-1}$ provides the commutativity of the outer square in the diagram:

$$
\begin{tikzcd}
C(v_1) \arrow[r, tail, "\sim"] \arrow[d] & C(v_1)^{\mathrm{gp}} \arrow[r, two heads] \arrow[d, dashed] & \mathrm{GL}_1^{\mathcal{J}}(L_p) \arrow[r] \arrow[d] & \Omega^{\mathcal{J}}(L_p) \arrow[d] \\
C(u) \arrow[r, tail, "\sim"] & C(u)^{\mathrm{gp}} \arrow[r, two heads] & \mathrm{GL}_1^{\mathcal{J}}(KU_p) \arrow[r] & \Omega^{\mathcal{J}}(KU_p)
\end{tikzcd}
$$

in which the left-hand vertical map sends the generator in $C(v_1)$ to the $(p-1)$–fold power of the generator in $C(u)$. The indicated acyclic cofibrations and fibrations in the diagram refer to the group completion model structure, and the lifting axiom in this model structure provides the dotted map. Passing to the pullbacks defining the direct image pre-log and log structures gives the commutativity of the diagram in the statement of the lemma. The p–local case works analogously. □

Remark 4.14 The previous proposition exhibits another advantage of graded log structures: As explained in [14, Remark 7.19], the free $\mathcal{I}$–space pre-log structures on ℓ and $ku_{(p)}$ do not allow to extend $\iota_{(p)}$ to a map of pre-log ring spectra. In [14], Rognes develops a theory of *based* $\mathcal{I}$–space log structures in order to make this possible. Because of the proposition, we do not need to address a based version of graded log structures for this purpose.

The map $(\ell_p, D(v_1)) \to (ku_p, D(u))$ of Proposition 4.13, utilizing the adjunction $(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$, induces a commutative square in $\mathcal{C}\mathrm{Sp}^{\Sigma}$:

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{J}}[D(v_1)] & \longrightarrow & \ell_p \\ \downarrow & & \downarrow \\ \mathbb{S}^{\mathcal{J}}[D(u)] & \longrightarrow & ku_p \end{array} \tag{4-6}$$

The following proposition will become crucial in Section 6:

Proposition 4.15 *The square (4-6) is a homotopy cocartesian square of commutative symmetric ring spectra, and the same holds in the p–local case.*

As explained in [14, Example 12.16 and Example 12.17], the counterpart of this statement for the based $\mathcal{I}$–space pre-log structures on these spectra does not hold. It will also become clear from the proof that this does not hold if we replace $D(v_1)$ and $D(u)$ by the free graded log structures $C(v_1)$ and $C(u)$ on these spectra.

Proof Without loss of generality, we assume that $D(v_1) \to D(u)$ is a cofibration of commutative $\mathcal{J}$–space monoids. In the sequel, we view $D(u)$ as a $D(v_1)$–module via $D(v_1) \to D(u)$ and consider the $D(v_1)$–module

$$E = \coprod_{0 \le i \le p-2} \mathcal{J}((\mathbf{1}, \mathbf{3}), -)^{\boxtimes i} \boxtimes D(v_1).$$

Composing the canonical maps $\mathcal{J}((\mathbf{1}, \mathbf{3}), -)^{\boxtimes i} \to C(u)$ with $C(u) \to D(u)$, we obtain an induced map of $D(v_1)$–modules $E \to D(u)$. Since $D(u)$ is positive fibrant, the choice of a factorization $E \overset{\sim}{\rightarrowtail} E' \twoheadrightarrow D(u)$ into an acyclic cofibration followed by a fibration in the positive $\mathcal{J}$–model structure on $D(v_1)$–modules provides a fibrant replacement E' of E. Applying the functor $\mathbb{S}^{\mathcal{J}}[-]\wedge_{\mathbb{S}^{\mathcal{J}}[D(v_1)]}\ell_p$ from $D(v_1)$–modules to ℓ_p–modules, we obtain a sequence of maps

$$\begin{aligned}(4\text{-}7)\quad \mathbb{S}^{\mathcal{J}}[E]\wedge_{\mathbb{S}^{\mathcal{J}}[D(v_1)]}\ell_p \to \mathbb{S}^{\mathcal{J}}[E']\wedge_{\mathbb{S}^{\mathcal{J}}[D(v_1)]}\ell_p \\ \to \mathbb{S}^{\mathcal{J}}[D(u)]\wedge_{\mathbb{S}^{\mathcal{J}}[D(v_1)]}\ell_p \to ku_p.\end{aligned}$$

Our cofibrancy assumptions imply that the smash products coincide with the derived smash products. The claim of the proposition is that the last map in (4-7) is a stable equivalence, and it is enough to show that the first two maps and the composition of all three maps in (4-7) are stable equivalences. The first map is a stable equivalence because $E\to E'$ is an acyclic cofibration, and these are preserved by $\mathbb{S}^{\mathcal{J}}[-]\wedge_{\mathbb{S}^{\mathcal{J}}[D(v_1)]}\ell_p$. Using [17, Lemma 14.3], the composite map can be identified with the map

$$\coprod_{0\le i\le p-2}\Sigma^{2i}\ell_p \simeq \coprod_{0\le i\le p-2}\ell_p\wedge (F_1S^3)^{\wedge i}\to ku_p$$

induced by multiplication with iterated powers of the map $F_1S^3\to ku_p$ from the free symmetric spectrum $F_1S^3\simeq\Sigma^2\mathbb{S}$ specified by u. It is a stable equivalence since $\pi_*(\iota_p)$ sends v_1 to u^{p-1}.

It remains to verify that the middle map in (4-7) is a stable equivalence. By construction and Lemma 4.6, the map $Q_{\ge 0}S^0\simeq D(v_1)_{h\mathcal{J}}\to D(u)_{h\mathcal{J}}\simeq Q_{\ge 0}S^0$ is multiplication with $(p-1)$. Since the multiplication with the image of the generator $\mathrm{id}_{(\mathbf{3},\mathbf{1})}$ of $C(u)$ in $D(u)$ is a weak equivalence, it follows that $(E')_{h\mathcal{J}}\to D(u)_{h\mathcal{J}}$ is a π_0–isomorphism and a $\pi_i(-)\otimes\mathbb{Z}_{(p)}$–isomorphism for $i\ge 1$. Using the homotopy fiber sequence of [17, Lemma 4.2] and the fact that E' and $D(u)$ are positive fibrant, a five lemma argument implies that for all objects $(\boldsymbol{n_1},\boldsymbol{n_2})$ of $\mathcal{J}$ with $n_1\ge 1$,

$$E'(\boldsymbol{n_1},\boldsymbol{n_2})\to D(u)(\boldsymbol{n_1},\boldsymbol{n_2}) \tag{4-8}$$

is a π_0–isomorphism and a $\pi_i(-)\otimes\mathbb{Z}_{(p)}$–isomorphism for $i\ge 1$. Since the map $D(u)_{h\mathcal{J}}\to B\mathcal{J}$ is a map of associative simplicial monoids, the nonempty homotopy cofibers have the homotopy types of simplicial monoids. Hence the components of the $D(u)(\boldsymbol{n_1},\boldsymbol{n_2})$ for $n_1\ge 1$ are simple spaces. The same argument applies to $D(v_1)_{h\mathcal{J}}\to B\mathcal{J}$, and the commutative diagram

$$\begin{tikzcd}
D(v_1)_{h\mathcal{J}} \arrow[d] & (\mathcal{J}((\mathbf{1},\mathbf{3}),-)^{\boxtimes i})_{h\mathcal{J}}\times D(v_1)_{h\mathcal{J}} \arrow[l, "\mathrm{pr}", "\sim"'] \arrow[r, "\sim"] \arrow[d] & (\mathcal{J}((\mathbf{1},\mathbf{3}),-)^{\boxtimes i}\boxtimes D(v_1))_{h\mathcal{J}} \arrow[d] \\
B\mathcal{J} & (\mathcal{J}((\mathbf{1},\mathbf{3}),-)^{\boxtimes i})_{h\mathcal{J}}\times B\mathcal{J} \arrow[l, "\mathrm{pr}", "\sim"'] \arrow[r, "\sim"] & B\mathcal{J}
\end{tikzcd}$$

implies that the components of $E'(\boldsymbol{n_1},\boldsymbol{n_2})$ for $n_1 \geq 1$ are simple spaces. It follows that the map (4-8) is a map of spaces with nilpotent path components and hence a $H_*(-;\mathbb{Z}_{(p)})$–equivalence.

The cofibrancy assumptions on $D(v_1)$ and $D(u)$ imply that the Σ_{n_2}–actions on $E'(\boldsymbol{n_1},\boldsymbol{n_2})$ and $D(u)(\boldsymbol{n_1},\boldsymbol{n_2})$ are free for all $n_2 \geq 1$ (see [15, Appendix A] for details). Hence the explicit description of $\mathbb{S}^{\mathcal{J}}[-]$ in [17, (4.5)] and an application of the Cartan–Leray spectral sequence show that $\mathbb{S}^{\mathcal{J}}[E'] \to \mathbb{S}^{\mathcal{J}}[D(u)]$ is a $H_*(-;\mathbb{Z}_{(p)})$–isomorphism in positive levels. Since the assembly map $X \wedge H\mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}[X]$ is a π_*–isomorphism for symmetric spectra X [20, II. Section 6], it follows that the map

$$\mathbb{S}^{\mathcal{J}}[E'] \wedge H\mathbb{Z}_{(p)} \to \mathbb{S}^{\mathcal{J}}[D(u)] \wedge H\mathbb{Z}_{(p)}$$

is a stable equivalence. Our cofibrancy assumptions on $D(u)$ and $D(v_1)$ imply that the symmetric spectra $\mathbb{S}^{\mathcal{J}}[E']$ and $\mathbb{S}^{\mathcal{J}}[D(u)]$ are flat. Hence the above smash products have the homotopy types of the derived smash products, and it follows that $\mathbb{S}^{\mathcal{J}}[E'] \to \mathbb{S}^{\mathcal{J}}[D(u)]$ is a $H\mathbb{Z}_{(p)}$–local equivalence. Because E' and $D(u)$ are concentrated in nonnegative $\mathcal{J}$–space degrees, a cell induction argument shows that the spectra $\mathbb{S}^{\mathcal{J}}[E']$ and $\mathbb{S}^{\mathcal{J}}[D(u)]$ are connective. It follows that $\mathbb{S}^{\mathcal{J}}[E'] \to \mathbb{S}^{\mathcal{J}}[D(u)]$ is a p–local equivalence. After cobase change along $\mathbb{S}^{\mathcal{J}}[D(v_1)] \to \ell_p$, it becomes a p–local equivalence of ℓ_p–modules and hence a stable equivalence. The same arguments apply in the p–local case. □

Remark 4.16 The p–complete complex K–theory spectrum ku_p admits other interesting graded pre-log structures. Let $p\colon S^1 \to (ku_p)_1$ and $u\colon S^3 \to (ku_p)_1$ represent the homotopy classes $p \in \pi_0(ku_p)$ and $u \in \pi_2(ku_p)$. As explained in Remark 4.11, we get a pre-log structure $D(p)$ even though p has degree 0. Using that $\boxtimes$ is the coproduct in $\mathcal{CS}^{\mathcal{J}}$, we obtain a pre-log structure $D(p) \boxtimes D(u) \to \Omega^{\mathcal{J}}(ku_p)$ on ku_p. The resulting pre-log ring spectrum $(ku_p, D(p) \boxtimes D(u))$ has a canonical map to $ku_p[1/p, 1/u]$ with its trivial log structure. In view of Rognes' discussion in [14, Section 1.9], $(ku_p, D(p) \boxtimes D(u))$ is a candidate for a hypothetical object known as the *fraction field of topological K–theory*. Its existence is supported by algebraic K–theory computations of Ausoni and Rognes [2]. This example emphasizes that it is potentially interesting to develop an algebraic K–theory of graded log ring spectra. We intend to pursue this in a later paper.

5 Logarithmic topological André–Quillen homology

In this section we define graded log derivations and graded log topological André–Quillen homology. To a large extent, this works analogous to the log TAQ for $\mathcal{I}$–space log ring spectra developed by Rognes in [14, Section 10 and Section 11]. Thus our presentation may also serve as a review of Rognes constructions. However, there are also some differences: The fact that commutative $\mathcal{J}$–space monoids do not admit a zero object requires some extra care when dealing with square zero extensions, derivations, and their corepresenting objects. To make this section more readable, we have deferred the proof of the necessary technical results about commutative $\mathcal{J}$–space monoids to Section 7.

5.1 Logarithmic derivations

Throughout this section let A be a positive fibrant commutative symmetric ring spectrum and let X be a left A–module spectrum.

Because A is commutative, we may also view X as a right A–module. The *square zero extension* of A by X is the commutative A–algebra $A \vee X$ with multiplication

$$(A \vee X) \wedge (A \vee X) \cong (A \wedge A) \vee (A \wedge X) \vee (X \wedge A) \vee (X \wedge X) \to A \vee X$$

induced by $A \wedge A \to A$, the (left and right) A–module structure on X, and the trivial map $X \wedge X \to *$. The map $X \to *$ induces an augmentation $A \vee X \to A$.

Since $A \vee X$ is clearly not fibrant and we will often consider maps into it, we fix a notation for a fibrant replacement:

Definition 5.2 We write $A \vee_f X$ for the positive fibrant replacement of $A \vee X$ in commutative symmetric ring spectra which is defined by the factorization

$$A \vee X \xrightarrow{\sim} A \vee_f X \twoheadrightarrow A .$$

Its augmentation is denoted by $\varepsilon_X\colon A \vee_f X \to A$.

Recall that $U^{\mathcal{J}} = \mathcal{J}((\mathbf{0},\mathbf{0}), -)$ denotes the monoidal unit of $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$. It is the initial commutative $\mathcal{J}$–space monoid.

Definition 5.3 Let the commutative $\mathcal{J}$–space monoid $(1+X)^{\mathcal{J}}$ be a cofibrant replacement of the pullback of

$$U^{\mathcal{J}} \longrightarrow \mathrm{GL}_1^{\mathcal{J}}(A) \xleftarrow{(\varepsilon_X)_*} \mathrm{GL}_1^{\mathcal{J}}(A \vee_f X) .$$

The notation is chosen in analogy to the subgroup of elements of the form $1+x$ in the units of the square zero extension of an ordinary ring by a module X. Since $(1+X)^{\mathcal{J}}$ is augmented over $U^{\mathcal{J}}$, it is concentrated in $\mathcal{J}$–space degree 0 because $U^{\mathcal{J}}$ is. In other words, $(1+X)^{\mathcal{J}}(\mathbf{m_1},\mathbf{m_2}) = \varnothing$ unless $m_2 - m_1 = 0$. Although this may look wrong at first sight, the following results show that $(1+X)^{\mathcal{J}}$ captures the desired portion of the units of the square zero extension.

Appealing again to the situation in ordinary algebra, we recall that the units of a square zero decompose into the units of the ring and the additive group of the module. The following two results about $(1+X)^{\mathcal{J}}$ may be viewed as a homotopical counterpart of this.

Since $\boxtimes$ is the coproduct in $\mathcal{CS}^{\mathcal{J}}$, we get a canonical map

$$\mathrm{GL}_1^{\mathcal{J}}(A) \boxtimes (1+X)^{\mathcal{J}} \longrightarrow \mathrm{GL}_1^{\mathcal{J}}(A \vee_f X).$$

Lemma 5.4 *This map is a $\mathcal{J}$–equivalence.*

Proof Applying $(-)_{h\mathcal{J}}$, the induced map fits into the commutative square:

$$\begin{array}{ccccc} ((1+X)^{\mathcal{J}})_{h\mathcal{J}} & \longrightarrow & (\mathrm{GL}_1^{\mathcal{J}}(A) \boxtimes (1+X)^{\mathcal{J}})_{h\mathcal{J}} & \longrightarrow & \mathrm{GL}_1^{\mathcal{J}}(A)_{h\mathcal{J}} \\ \downarrow & & \downarrow & & \downarrow \\ ((1+X)^{\mathcal{J}})_{h\mathcal{J}} & \longrightarrow & \mathrm{GL}_1^{\mathcal{J}}(A \vee_f X)_{h\mathcal{J}} & \longrightarrow & \mathrm{GL}_1^{\mathcal{J}}(A)_{h\mathcal{J}} \end{array}$$

Lemma 2.11 implies that the upper sequence is a homotopy fiber sequence and by [17, Corollary 11.4] the lower sequence is a homotopy fiber sequence as well. Hence the claim follows from the long exact sequence of homotopy groups since all spaces involved are grouplike simplicial monoids. □

The underlying spectrum of a module spectrum plays the role of the additive group of the module spectrum. We will prove the following proposition about its relation to $(1+X)^{\mathcal{J}}$ in Section 7:

Proposition 5.5 *The spectrum associated with the Γ–space $\gamma((1+X)^{\mathcal{J}})$ is stably equivalent to the connective cover of the underlying spectrum of the A–module X.*

We now explain how to form square zero extensions in pre-log ring spectra:

Construction 5.6 Let (M,α) be a graded pre-log structure on A. The universal property of the coproduct $\boxtimes$ in $\mathcal{CS}^{\mathcal{J}}$ and the maps

$$\alpha\colon M \to \Omega^{\mathcal{J}}(A \vee_f X) \quad \text{and} \quad (1+X)^{\mathcal{J}} \to \mathrm{GL}_1^{\mathcal{J}}(A \vee_f X) \to \Omega^{\mathcal{J}}(A \vee_f X)$$

induce a pre-log structure $M \boxtimes (1+X)^{\mathcal{J}} \to \Omega^{\mathcal{J}}(A \vee_f X)$ on $A \vee_f X$. Using the augmentation $(1+X)^{\mathcal{J}} \to U^{\mathcal{J}}$ and the isomorphism $M \boxtimes U^{\mathcal{J}} \cong M$, we obtain a map $M \boxtimes (1+X)^{\mathcal{J}} \to M$ such that the outer square in

$$\begin{tikzcd}
M \boxtimes (1+X)^{\mathcal{J}} \arrow[rr] \arrow[dd] \arrow[dr, tail, "\sim"] & & \Omega^{\mathcal{J}}(A \vee_f X) \arrow[dd, two heads] \\
& (M+X)^{\mathcal{J}} \arrow[ur, dashed] \arrow[dl, two heads] & \\
M \arrow[rr] & & \Omega^{\mathcal{J}}(A)
\end{tikzcd} \tag{5-1}$$

commutes. We define $(M+X)^{\mathcal{J}}$ by the indicated factorization in the positive $\mathcal{J}$–model structure. Since $A \vee_f X \to A$ is a fibration by assumption, the dotted arrow exists and defines the pre-log ring spectrum $(A \vee_f X, (M+X)^{\mathcal{J}})$. Having ensured that $(M+X)^{\mathcal{J}} \to M$ is a fibration will turn out useful when considering maps into $(A \vee_f X, (M+X)^{\mathcal{J}})$.

The diagram induces a sequence of maps $(A, M) \to (A \vee_f X, (M+X)^{\mathcal{J}}) \to (A, M)$ making $(A, (M+X)^{\mathcal{J}})$ a graded pre-log ring spectrum under and over (A, M). Using this, we can state the homotopical counterpart of the algebraic notion of log derivations outlined in the introduction:

Definition 5.7 Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra and let X be an A–module. A *graded log derivation* of (A, M) over (R, P) with values in X is a map

$$(d, d^\flat)\colon (A, M) \to (A \vee_f X, (M+X)^{\mathcal{J}})$$

under (R, P) and over (A, M). That is, $(d, d^\flat)$ is a map such that the diagram

$$\begin{tikzcd}
(R, P) \arrow[r] \arrow[d] & (A \vee_f X, (M+X)^{\mathcal{J}}) \arrow[d] \\
(A, M) \arrow[r] \arrow[ur, dashed, "{(d,d^\flat)}"] & (A, M)
\end{tikzcd}$$

commutes.

We will use this notion to motivate and justify the definition of logarithmic topological André–Quillen homology below.

For later use we note two more properties of the induced pre-log structure on the square zero extension:

Lemma 5.8 *The logification $(M,\alpha) \to (M^a,\alpha^a)$ induces a weak equivalence*

$$(A \vee_f X, ((M+X)^{\mathcal{J}})^a) \to (A \vee_f X, (M^a + X)^{\mathcal{J}}). \tag{5-2}$$

If (A, M) is a graded log ring spectrum, then so is $(A \vee_f X, (M+X)^{\mathcal{J}})$.

Proof It follows from the lifting axioms that $(M,\alpha) \to (M^a,\alpha^a)$ extends to the map of pre-log ring spectra (5-2). Since the log condition and the logification are invariant under weak equivalences, it is enough to check the claim for the pre-log structures defined by $M \boxtimes (1+X)^{\mathcal{J}}$ and $M^a \boxtimes (1+X)^{\mathcal{J}}$.

Let (M,α) be a pre-log structure. In the notation of Lemma 3.12, M decomposes as the part $\widetilde{M}$ that maps to the units $\mathrm{GL}_1^{\mathcal{J}}(A)$ of A and its complement $\widehat{M}$. Since $(1+X)^{\mathcal{J}}$ is grouplike, it is easy to see that

$$(\widetilde{M} \boxtimes (1+X)^{\mathcal{J}}) \amalg (\widehat{M} \boxtimes (1+X)^{\mathcal{J}}) \to \Omega^{\mathcal{J}}(A \vee_f X)$$

is the corresponding decomposition of $M \boxtimes (1+X)^{\mathcal{J}}$ over $\Omega^{\mathcal{J}}(A \vee_f X)$. Using that $(1+X)^{\mathcal{J}}$ is flat and that the $\boxtimes$–product with a flat $\mathcal{J}$–space preserves $\mathcal{J}$–equivalences ([17, Proposition 8.2]), Lemma 5.4 shows the second claim of the lemma.

Defining G as in the construction of the logification of (M,α), the same arguments show that the factorization

$$\widetilde{M} \boxtimes (1+X)^{\mathcal{J}} \rightarrowtail G \boxtimes (1+X)^{\mathcal{J}} \xrightarrow{\sim} \mathrm{GL}_1^{\mathcal{J}}(A \vee_f X)$$

may be used to form the pushout giving the logification of $(M+X)^{\mathcal{J}}$. With this choice for the cofibrant replacement, the pushout is $M^a \boxtimes (1+X)^{\mathcal{J}}$. □

Lemma 5.9 *The commutative square*

$$\begin{array}{ccc} (M+X)^{\mathcal{J}} & \longrightarrow & (M^a+X)^{\mathcal{J}} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^a \end{array}$$

induced by the logification is a homotopy pullback.

Proof This follows by combining the description of $(M^a+X)^{\mathcal{J}}$ in the last lemma, Lemma 2.11, and the fact that homotopy pullbacks in $\mathcal{J}$–spaces are detected by $(-)_{h\mathcal{J}}$ (see [17, Corollary 11.4]). □

5.10 Spaces of maps between graded pre-log ring spectra

When working with mapping spaces, one often has to ensure a cofibrant domain and a fibrant codomain. To implement this in the case at hand, we use the following "projective" model category structure on graded pre-log ring spectra whose proof easily follows from the fact that $(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ is a Quillen adjunction:

Lemma 5.11 *The category of graded pre-log ring spectra admits a model structure in which $(f, f^\flat)\colon (A, M) \to (B, N)$ is a weak equivalence (or fibration) if f is a stable equivalence (or stable positive fibration) in $\mathcal{C}\mathrm{Sp}^\Sigma$ and $f^\flat$ is a $\mathcal{J}$–equivalence (or positive $\mathcal{J}$–fibration) in $\mathcal{CS}^{\mathcal{J}}$.* □

So the weak equivalences are those of Definition 3.2, and (A, M) is cofibrant if and only if M is cofibrant and $\mathbb{S}^{\mathcal{J}}[M] \to A$ is a cofibration.

Both commutative symmetric ring spectra and commutative $\mathcal{J}$–space monoids are simplicial model categories. We denote their mapping spaces by $\mathrm{Map}_{\mathcal{C}\mathrm{Sp}^\Sigma}$ and $\mathrm{Map}_{\mathcal{CS}^{\mathcal{J}}}$. These mapping space can be defined using the tensor with the cosimplicial simplicial set $[n] \mapsto \Delta^n$, and this tensor is in turn defined via the coproduct over Δ^n_k and the realization. In particular, the left adjoint $\mathbb{S}^{\mathcal{J}}[-]$ commutes with the tensor and induces a map of spaces $\mathrm{Map}_{\mathcal{CS}^{\mathcal{J}}}(M, N) \to \mathrm{Map}_{\mathcal{C}\mathrm{Sp}^\Sigma}(\mathbb{S}^{\mathcal{J}}[M], \mathbb{S}^{\mathcal{J}}[N])$.

If (A, M) and (B, N) are graded pre-log ring spectra, then their structure maps induce the diagram:

$$\mathrm{Map}_{\mathcal{CS}^{\mathcal{J}}}(A, B) \to \mathrm{Map}_{\mathcal{CS}^{\mathcal{J}}}(\mathbb{S}^{\mathcal{J}}[M], B) \leftarrow \mathrm{Map}_{\mathcal{CS}^{\mathcal{J}}}(M, N) \tag{5-3}$$

Definition 5.12 The space of maps $\mathrm{Map}_{\mathcal{P}}((A, M), (B, N))$ between pre-log ring spectra (A, M) and (B, N) is the pullback of diagram (5-3).

Corollary 5.13 *If (A, M) is cofibrant and (B, N) is fibrant, then the pullback of (5-3) captures the homotopy type of its homotopy pullback.* □

Hence the mapping space $\mathrm{Map}_{\mathcal{P}}$ is invariant under weak equivalences in both variables if the objects are sufficiently cofibrant and fibrant.

If $\mathcal{C}$ is a model category and $E \to G$ is a map in $\mathcal{C}$, then there is a canonical model structure on the category $\mathcal{C}^E_G$ of objects under E and over G: Objects in $\mathcal{C}^E_G$ are factorizations $E \to F \to G$, and a map from $E \to F \to G$ to $E \to F' \to G$ is a weak equivalence, cofibration, or fibration if and only if its projection to $F \to F'$ is. If $\mathcal{C}$ is in addition a simplicial model category, taking iterated pullbacks of diagrams induced by the augmentation and the coaugmentation of the domain and the codomain defines mapping spaces for $\mathcal{C}^E_G$.

Definition 5.14 Let $(R,P)\to(A,M)$ be a cofibration of graded pre-log ring spectra and let X be a fibrant A–module. The *space of graded log derivations* of (A,M) over (R,P) with values in X is the mapping space:

$$\mathrm{Der}_{(R,P)}((A,M),X)=\mathrm{Map}^{(R,P)}_{(A,M)}((A,M),(A\vee_f X,(M+X)^{\mathcal{J}}))$$

This is homotopically meaningful because $(A\vee_f X,(M+X)^{\mathcal{J}})$ is fibrant.

5.15 Construction of graded log TAQ

We will now explain how the space $\mathrm{Der}_{(R,P)}((A,M),X)$ can be corepresented by an A–module.

We start by decomposing this space as a homotopy pullback. For this we will write $\mathrm{Map}^P_M(-,-)$ for the mapping space in $(\mathcal{C}\mathcal{S}^{\mathcal{J}})^P_M$ and $\mathrm{Map}^R_A(-,-)$ for the mapping space in $(\mathcal{C}\mathrm{Sp}^{\Sigma})^R_A$. Then the commutative squares

$$\begin{tikzcd} \mathbb{S}^{\mathcal{J}}[(M+X)^{\mathcal{J}}] \arrow[r] \arrow[d] & A\vee_f X \arrow[d] \\ \mathbb{S}^{\mathcal{J}}[M] \arrow[r] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{S}^{\mathcal{J}}[P] \arrow[r] \arrow[d] & R \arrow[d] \\ \mathbb{S}^{\mathcal{J}}[M] \arrow[r] & A \end{tikzcd}$$

induced by the maps $(A\vee_f X,(M+X)^{\mathcal{J}})\to(A,M)$ and $(R,P)\to(A,M)$ give rise to a commutative square:

$$\begin{tikzcd} \mathrm{Der}_{(R,P)}((A,M),X) \arrow[r] \arrow[d] & \mathrm{Map}^R_A(A,A\vee_f X) \arrow[d] \\ \mathrm{Map}^P_M(M,(M+X)^{\mathcal{J}}) \arrow[r] & \mathrm{Map}^{\mathbb{S}^{\mathcal{J}}[P]}_A(\mathbb{S}^{\mathcal{J}}[M],A\vee_f X) \end{tikzcd} \tag{5-4}$$

The description of the mapping space in (5-3) implies the following lemma.

Lemma 5.16 *The square (5-4) is homotopy cartesian.* □

By definition, the upper right-hand corner of (5-4) is the space $\mathrm{Der}_R(A,X)$ of R–algebra derivations of A with values in X. The corepresenting object has been constructed by Basterra [5] and is known as the *topological André–Quillen homology* $\mathrm{TAQ}^R(A)$ of $R\to A$. It has been extensively studied in the literature.

We briefly recall the construction of TAQ from [5]: Using the left derived product over R, the map $R\to A$ gives rise to an augmented A–algebra $A\wedge^{\mathbb{L}}_R A$. The augmentation ideal I_A may be viewed as a functor from augmented A–algebras to the category nonunital A–algebras $\mathcal{N}_A$. With suitable model structures, it participates as the right adjoint in a Quillen equivalence. Evaluating its right derived functor $I_A^{\mathbb{R}}$ to $A \wedge_R^{\mathbb{L}} A$ provides a nonunital A–algebra. Applying the left derived functor of the indecomposables $Q_A \colon \mathcal{N}_A \to A\text{-}\mathrm{Mod}$ to this nonunital A–algebra defines the A–module $\mathrm{TAQ}^R(A)$. The fact that Basterra uses S–modules in the sense of [7] rather than symmetric spectra in her description may be dealt with by either mimicking her approach in symmetric spectra, or using Schwede's equivalence between symmetric spectra and S–modules [21] to go back and forth between the two setups.

As indicated above, we shall mostly need the following result about TAQ:

Proposition 5.17 [5, Proposition 3.2] *The space* $\mathrm{Der}_R(A, X)$ *is naturally weakly equivalent to the space of maps of* A*–module spectra* $\mathrm{Map}_{A\text{-}\mathrm{Mod}}(\mathrm{TAQ}^R(A), X)$*.*

To study the lower right-hand corner of (5-4), we first observe that there is a dotted arrow making the following diagram commutative:

$$\begin{tikzcd}
 & \mathbb{S}^{\mathcal{J}}[M] \vee X \arrow[r] \arrow[d, tail, "\sim"'] & A \vee X \arrow[d, tail, "\sim"] \\
\mathbb{S}^{\mathcal{J}}[P] \arrow[r] \arrow[d] & \mathbb{S}^{\mathcal{J}}[M] \vee_f X \arrow[r, dashed] \arrow[d, two heads] & A \vee_f X \arrow[d, two heads] \\
\mathbb{S}^{\mathcal{J}}[M] \arrow[r, "="] & \mathbb{S}^{\mathcal{J}}[M] \arrow[r] & A
\end{tikzcd}$$

The fact that the lower right-hand square is homotopy cartesian easily implies that there is a weak equivalence

$$\mathrm{Der}_{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M], X) \to \mathrm{Map}_A^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M], A \vee_f X).$$

So the previous proposition implies the following corollary.

Corollary 5.18 *The space* $\mathrm{Map}_A^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M], A \vee_f X)$ *is naturally weakly equivalent to mapping space of* A*–module spectra* $\mathrm{Map}_{A\text{-}\mathrm{Mod}}(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathrm{TAQ}^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]), X)$*.*

The lower left-hand corner $\mathrm{Map}_M^P(M, (M + X)^{\mathcal{J}})$ of (5-4) may be interpreted as the space of commutative $\mathcal{J}$–space monoid derivations, ie as the space P–algebra derivations of M with values in the grouplike commutative $\mathcal{J}$–space monoid $(1+X)^{\mathcal{J}}$. We will prove the following result in Section 7:

Proposition 5.19 *The space* $\mathrm{Map}_M^P(M, (M + X)^{\mathcal{J}})$ *is naturally weakly equivalent to the space of maps of* A*–module spectra* $\mathrm{Map}_{A\text{-}\mathrm{Mod}}(A \wedge (\gamma(M)/\gamma(P)), X)$*.*

In the proposition, $\gamma(M)/\gamma(P)$ is the homotopy cofiber of the map of Γ–spaces induced by $P \to M$, and $A \wedge (\gamma(M)/\gamma(P))$ is the A–module spectrum obtained from the spectrum associated with $\gamma(M)/\gamma(P)$ by extension of scalars along $\mathbb{S} \to A$.

As functors in X, the mapping spaces in all but the upper left-hand corner of the square (5-4) are corepresented by A–modules. Hence we obtain the following maps between the corepresenting objects:

$$A \wedge (\gamma(M)/\gamma(P)) \leftarrow A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathrm{TAQ}^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]) \to \mathrm{TAQ}^{R}(A) \tag{5-5}$$

Definition 5.20 Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra. The *graded log topological André–Quillen homology* $\mathrm{TAQ}^{(R,P)}(A, M)$ of (A, M) over (R, P) is the A–module given by the homotopy pushout of (5-5).

We call this *log* TAQ rather than *pre-log* TAQ because it is invariant under logification. We prove this in Corollary 6.7 below. The following result is the main motivation behind the definition of $\mathrm{TAQ}^{(R,P)}(A, M)$:

Proposition 5.21 *Let $(R, P) \to (A, M)$ be a cofibration of graded pre-log ring spectra, let (A, M) be fibrant, and let X be a fibrant A–module. There is a weak equivalence between $\mathrm{Map}_{A-\mathrm{Mod}}(\mathrm{TAQ}^{(R,P)}(A, M), X)$ and $\mathrm{Der}_{(R,P)}((A, M), X)$.*

Proof Since $\mathrm{Map}_{A-\mathrm{Mod}}(-, X)$ maps homotopy pushouts to homotopy pullbacks, this follows from the definition of graded log TAQ, Proposition 5.17, Corollary 5.18, and Proposition 5.19. □

Definition 5.22 A map of graded pre-log ring spectra $(R, P) \to (A, M)$ is *formally graded log étale* if the A–module $\mathrm{TAQ}^{(R,P)}(A, M)$ is contractible.

Examples of formally graded log étale extensions will be given in the next section.

6 Log étale extensions of K–theory spectra

Let p be an odd prime. Recall from Proposition 4.13 that the inclusion of the p–complete Adams summand into the p–complete connective K–theory spectrum $\ell_p \to ku_p$ can be extended to a map of graded log ring spectra

$$(\ell_p, i_*\mathrm{GL}_1^{\mathcal{J}}(L_p)) \to (ku_p, i_*\mathrm{GL}_1^{\mathcal{J}}(KU_p)).$$

The graded log-structures are the direct image log structures of the respective periodic theories. The following theorem is one of the main results of this paper. As discussed in the introduction, it confirms that $\ell_p \to ku_p$ should be viewed as a tamely ramified extension of ring spectra.

Theorem 6.1 *The map $(\ell_p, i_*\mathrm{GL}_1^{\mathcal{J}}(L_p)) \to (ku_p, i_*\mathrm{GL}_1^{\mathcal{J}}(KU_p))$ is formally graded log étale. That is, the graded log topological André–Quillen homology spectrum*

$$\mathrm{TAQ}^{(\ell_p, i_*\mathrm{GL}_1^{\mathcal{J}}(L_p))}(ku_p, i_*\mathrm{GL}_1^{\mathcal{J}}(KU_p))$$

is contractible. The same holds in the p–local case.

The proof of Theorem 6.1 will be given at the end of this section. The first step towards its proof is the following graded analogue of [14, Lemma 11.25]. It gives a useful criterion for showing that a map is formally graded log étale:

Lemma 6.2 *Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra, and let C be the homotopy pushout of $R \leftarrow \mathbb{S}^{\mathcal{J}}[P] \to \mathbb{S}^{\mathcal{J}}[M]$. Then there is a homotopy cofiber sequence*

$$A \wedge (\gamma(M)/\gamma(P)) \to \mathrm{TAQ}^{(R,P)}(A, M) \to \mathrm{TAQ}^{C}(A) \tag{6-1}$$

of A–module spectra.

Proof The proof is completely analogous to the proof of [14, Lemma 11.25]: The defining homotopy pushout (5-5) shows that the homotopy cofiber of the left map in (6-1) is equivalent to the homotopy cofiber of

$$A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathrm{TAQ}^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]) \to \mathrm{TAQ}^{R}(A).$$

Flat base change and the transitivity sequence for TAQ [5, Section 4] allow to identify this homotopy cofiber with $\mathrm{TAQ}^{C}(A)$. □

Corollary 6.3 *The map $(R, P) \to (A, M)$ is formally graded log étale if the map $\gamma(P) \to \gamma(A)$ is an A–homology equivalence and $\mathrm{TAQ}^{C}(A)$ is contractible.* □

The next aim is to show that graded log TAQ is invariant under logification. For this we begin with the following lemma:

Lemma 6.4 *Let (A, M) be a pre-log ring spectrum and let (B, N) be a log ring spectrum. The logification $(A, M) \to (A, M^a)$ induces a weak equivalence of mapping spaces $\mathrm{Map}_{\mathcal{P}}((A, M^a), (B, N)) \to \mathrm{Map}_{\mathcal{P}}((A, M), (B, N))$.*

Proof We may assume that $N \to \Omega^{\mathcal{J}}(B)$ is a positive fibration. By an adjunction argument [9, Lemma 9.3.6], it is enough to show that the pushout product map of

$$((A, M) \to (A, M^a)) \otimes (\partial\Delta^n \to \Delta^n) \tag{6-2}$$

has the left lifting property with respect to $(B, N) \to *$.

Let $K \to L$ be a cofibration in $\mathcal{CS}^{\mathcal{J}}$ with L grouplike. The log condition on (B, N) implies that $N \to \Omega^{\mathcal{J}}(B)$ has the right lifting property with respect to $K \to L$. Using the adjunction $(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$, this shows that $(B, N) \to *$ has the right lifting property with respect to $(\mathbb{S}^{\mathcal{J}}[L], K) \to (\mathbb{S}^{\mathcal{J}}[L], L)$. The logification $(A, M) \to (A, M^a)$ can be viewed as the map from the pushout to the lower right corner in the diagram:

$$\begin{array}{ccc} (\mathbb{S}^{\mathcal{J}}[G], \alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A))) & \longrightarrow & (A, M) \\ \downarrow & & \downarrow \\ (\mathbb{S}^{\mathcal{J}}[G], G) & \longrightarrow & (A, M^a) \end{array}$$

This shows that the pushout product map of (6-2) can be obtained as the cobase change of the pushout product map of

$$\Big((\mathbb{S}^{\mathcal{J}}[G], \alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A))) \to (\mathbb{S}^{\mathcal{J}}[G], G)\Big) \otimes (\partial\Delta^n \to \Delta^n) \tag{6-3}$$

along the map induced by $\big((\mathbb{S}^{\mathcal{J}}[G], \alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A))) \to (A, M)\big) \otimes \Delta^n$. So it is enough to show that this pushout product map has the lifting property with respect to $(B, N) \to *$. One can check that it is the map of pre-log ring spectra $(\mathbb{S}^{\mathcal{J}}[L], K) \to (\mathbb{S}^{\mathcal{J}}[L], L)$ associated with the map

$$K = \alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \otimes \Delta^n \boxtimes_{\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \otimes \partial\Delta^n} G \otimes \partial\Delta^n \to G \otimes \Delta^n = L.$$

Since $\mathcal{CS}^{\mathcal{J}}$ is a simplicial model category, $K \to L$ is a cofibration. The $G \otimes \Delta^n$ is grouplike because G is. So the claim follows by the right lifting property for $(B, N) \to *$ established above. □

Remark 6.5 The argument in the proof of the last lemma can be used show that the (prefibrant) log ring spectra may be identified with the local objects of a left Bousfield localization of the category pre-log ring spectra. The fibrant replacement in the localization is then a model for the logification. We omit the details because this *log model structure* is not needed in the present paper.

We now look at the square

$$\begin{array}{ccc} (R, P) & \longrightarrow & (R, P^a) \\ \downarrow & & \downarrow \\ (A, M) & \longrightarrow & (A, M^a) \end{array} \tag{6-4}$$

induced by the logification of (R, P) and (A, M). The following lemma is the graded analogue of [14, Lemma 11.9]. In the lemma, we implicitly assume that the vertical arrows in (6-4) are cofibrations of fibrant objects. (This can always be achieved up to weak equivalence.)

Lemma 6.6 *Let X be a fibrant A–module. The maps in the square (6-4) induce weak equivalences:*

$$\begin{tikzcd}
\mathrm{Der}_{(R,P)}((A,M),X) \arrow[r, "\sim"] & \mathrm{Map}^{(R,P)}_{(A,M^a)}((A,M),(A\vee_f X,(M^a+X)^{\mathcal{J}})) \\
\mathrm{Der}_{(R,P^a)}((A,M^a),X) \arrow[r, "\sim"] & \mathrm{Der}_{(R,P)}((A,M^a),X) \arrow[u, "\sim"']
\end{tikzcd}$$

Proof The upper horizontal map is a weak equivalence by the homotopy pullback established in Lemma 5.9. The two other maps are weak equivalences by Lemma 6.4. □

Proposition 5.21 and the last lemma now easily imply the counterpart of [14, Corollary 11.23]:

Corollary 6.7 *The maps in (6-4) induce a zig-zag of stable equivalences of A–modules between*

$$\mathrm{TAQ}^{(R,P)}(A,M), \quad \mathrm{TAQ}^{(R,P)}(A,M^a) \quad \textit{and} \quad \mathrm{TAQ}^{(R,P^a)}(A,M^a). \qquad \square$$

We have now developed all tools for the proof of the main theorem of this section:

Proof of Theorem 6.1 By Corollary 6.7, Proposition 4.13, and Theorem 4.4 it is enough to show that the map of pre-log ring spectra

$$(\ell_p, D(v_1)) \to (ku_p, D(u))$$

is formally graded log étale. We use the criterion of Corollary 6.3. Proposition 4.15 implies that the map from the homotopy pushout C of

$$\ell_p \leftarrow \mathbb{S}^{\mathcal{J}}[D(v_1)] \to \mathbb{S}^{\mathcal{J}}[D(u)]$$

to ku_p is a stable equivalence. Hence $\mathrm{TAQ}^C(ku_p)$ is contractible. Using Lemma 4.6, it is easy to see that both $\gamma(D(v_1))$ and $\gamma(D(u))$ have the homotopy type of the sphere spectrum, and that the map $\gamma(D(v_1)) \to \gamma(D(u))$ is the multiplication by $p-1$. Since ku_p is p–complete, the induced map $A\wedge\gamma(D(v_1)) \to A\wedge\gamma(D(u))$ is a stable equivalence. The same arguments apply in the p–local case. □

7 Commutative $\mathcal{J}$–space monoid derivations

In this section we give the proof of Proposition 5.5 and Proposition 5.19. These relate the space of commutative $\mathcal{J}$–space monoid derivations $\mathrm{Map}^{\boldsymbol{P}}_{M}(M,(M+X)^{\mathcal{J}})$ with the A–module spectrum X used to define $(M+X)^{\mathcal{J}}$.

Let A be a positive fibrant in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ and let X be an A–module. Similarly as the commutative $\mathcal{J}$–space monoid $(1+X)^{\mathcal{J}}$ introduced in Definition 5.3, we obtain commutative $\mathcal{I}$–space monoid $(1+X)^{\mathcal{I}}$ as cofibrant replacement of the pullback of

$$U^{\mathcal{I}} \longrightarrow \mathrm{GL}_1^{\mathcal{I}}(A) \xleftarrow{(\varepsilon_X)_*} \mathrm{GL}_1^{\mathcal{I}}(A \vee_f X).$$

As indicated in Section 2.12 and explained in detail in [16, Section 3], the commutative $\mathcal{J}$–space monoid $(1+X)^{\mathcal{J}}$ gives rise to a Γ–space $\gamma((1+X)^{\mathcal{J}})$. This functor γ is in turn motivated by a functor $\gamma\colon \mathcal{C}\mathcal{S}^{\mathcal{I}} \to \Gamma^{\mathrm{op}}\text{–}\mathcal{S}$ which is (under a different name) considered by Schlichtkrull in [19] and the author and Schlichtkrull in [18]. The latter functor provides a Γ–space $\gamma((1+X)^{\mathcal{I}})$ associated with $(1+X)^{\mathcal{I}}$.

Lemma 7.1 *The Γ–spaces $\gamma((1+X)^{\mathcal{I}})$ and $\gamma((1+X)^{\mathcal{J}})$ are level equivalent.*

Proof The functors γ send $\mathcal{I}$- and $\mathcal{J}$–equivalences to level equivalences of Γ–spaces. Hence it is enough to show the claim for the actual pullbacks used to define $(1+X)^{\mathcal{I}}$ and $(1+X)^{\mathcal{J}}$. In this proof, we denote them also by $(1+X)^{\mathcal{I}}$ and $(1+X)^{\mathcal{J}}$.

By [16, Lemma 2.12], the strong symmetric monoidal functor $\Delta\colon \mathcal{I} \to \mathcal{J}$ sending $\boldsymbol{m}$ to $(\boldsymbol{m},\boldsymbol{m})$ has the property $\Delta^*\mathrm{GL}_1^{\mathcal{J}}(A) \cong \mathrm{GL}_1^{\mathcal{I}}(A)$. We obtain a commutative diagram of commutative $\mathcal{I}$–space monoids:

$$\begin{array}{ccccc}
\Delta^* U^{\mathcal{J}} & \longrightarrow & \Delta^*\,\mathrm{GL}_1^{\mathcal{J}}(A) & \longleftarrow & \Delta^*\,\mathrm{GL}_1^{\mathcal{J}}(A \vee_f X) \\
\uparrow & & \uparrow & & \uparrow \\
U^{\mathcal{I}} & \longrightarrow & \mathrm{GL}_1^{\mathcal{I}}(A) & \longleftarrow & \mathrm{GL}_1^{\mathcal{I}}(A \vee_f X)
\end{array}$$

Using the natural transformation $(\Delta^*(-))_{h\mathcal{I}} \to (-)_{h\mathcal{J}}$ induced by Δ, the fact that Δ^* commutes with pullbacks and [16, Lemma 3.16], it follows that we obtain a map $c_X\colon \gamma((1+X)^{\mathcal{I}}) \to \gamma((1+X)^{\mathcal{J}})$. To see that it is a level equivalence, it is enough to show that its evaluation at 1^+ is a weak equivalence since both Γ–spaces are special. The natural transformation $(\Delta^*(-))_{h\mathcal{I}} \to (-)_{h\mathcal{J}}$ provides a commutative square:

$$\begin{array}{ccccc}
\big(\mathrm{GL}_1^{\mathcal{I}}(A \vee_f X)\big)_{h\mathcal{I}} & \xrightarrow{\cong} & \big(\Delta^*\,\mathrm{GL}_1^{\mathcal{J}}(A \vee_f X)\big)_{h\mathcal{I}} & \longrightarrow & \big(\mathrm{GL}_1^{\mathcal{J}}(A \vee_f X)\big)_{h\mathcal{J}} \\
\downarrow & & \downarrow & & \downarrow \\
\big(\mathrm{GL}_1^{\mathcal{I}}(A)\big)_{h\mathcal{I}} & \xrightarrow{\cong} & \big(\Delta^*\,\mathrm{GL}_1^{\mathcal{J}}(A)\big)_{h\mathcal{I}} & \longrightarrow & \big(\mathrm{GL}_1^{\mathcal{J}}(A)\big)_{h\mathcal{J}}
\end{array} \tag{7-1}$$

Since both $U^{\mathcal{I}}_{h\mathcal{I}}$ and $U^{\mathcal{J}}_{h\mathcal{J}}$ are contractible, $c_X(1^+)$ will be a weak equivalence if the right-hand square in (7-1) is homotopy cartesian. This holds because the map of horizontal homotopy fibers in the outer square is a weak equivalence by [16, Proposition 4.1]. □

Proof of Proposition 5.5 It is shown in [14, Lemma 11.2] that the Γ–space associated with the commutative $\mathcal{I}$–space monoid $(1+X)^{\mathcal{I}}$ is stably equivalent to the connective cover of the underlying spectrum of X. Using this, the $\mathcal{J}$–space version follows from Lemma 7.1. □

We start to work towards the proof of Proposition 5.19. Recall from Construction 5.6 that $(M+X)^{\mathcal{J}}$ is fibrant replacement of $M \boxtimes (1+X)^{\mathcal{J}}$ relative to M. In particular, it is an object under P and over M. We will again use the functor $\gamma\colon \mathcal{CS}^{\mathcal{J}} \to \Gamma^{\mathrm{op}}\text{–}\mathcal{S}$ of (2-4) and write $(\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}$ for a level fibrant replacement of the (very special) Γ–space associated with $(1+X)^{\mathcal{J}}$. Since $\Gamma^{\mathrm{op}}\text{–}\mathcal{S}$ has a zero object, $(1+X)^{\mathcal{J}}$ gives rise to a second Γ–space

$$\gamma(P) \to \gamma(M) \times (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}} \xrightarrow{\mathrm{pr}} \gamma(M)$$

under $\gamma(P)$ and over $\gamma(M)$.

Lemma 7.2 *With respect to the maps from $\gamma(P)$ and to $\gamma(M)$ specified above, $\gamma(M+X)^{\mathcal{J}}$ and $\gamma(M)\times(\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}$ are level equivalent in $\gamma(P)\downarrow\Gamma^{\mathrm{op}}\text{–}\mathcal{S}\downarrow\gamma(M)$.*

Proof We write $N = (1+X)^{\mathcal{J}}$. A levelwise application of the map (2-3) provides a map of Γ–spaces $\gamma(M)\times\gamma(N) \to \gamma(M \boxtimes N)$. (See the proof of [16, Lemma 7.22] for more details about this map.) By Lemma 2.11, the map is a level equivalence since N is flat by construction. Using that N is augmented over $U^{\mathcal{J}}$, we define $\gamma(N)'$ by the factorization $\gamma(N) \overset{\sim}{\rightarrowtail} \gamma(N)' \twoheadrightarrow \gamma(U^{\mathcal{J}})$ and observe that since $\gamma(U^{\mathcal{J}})$ is contractible, the fiber $\gamma(N)^{\mathrm{fib}}$ of $\gamma(N)' \to \gamma(U^{\mathcal{J}})$ is a possible choice for a fibrant replacement of $\gamma(N)$. We obtain a commutative diagram:

$$\begin{tikzcd}
\gamma(M)\times\gamma(N)^{\mathrm{fib}} \arrow[r, "\sim"] \arrow[d] & \gamma(M)\times\gamma(N)' \arrow[d] & \gamma(M)\times\gamma(N) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & \gamma(M\boxtimes N) \arrow[d] \\
\gamma(M) \arrow[r, "\sim"] & \gamma(M)\times\gamma(U^{\mathcal{J}}) \arrow[r, "="] & \gamma(M)\times\gamma(U^{\mathcal{J}}) \arrow[r, "\sim"] & \gamma(M)
\end{tikzcd}$$

The resulting equivalence between $\gamma(M)\times\gamma(N)^{\mathrm{fib}}$ and $\gamma(M\boxtimes N)$ lies over $\gamma(M)$ because the bottom composition is the identity. It is easy to see that we get maps under $\gamma(M)$ and hence under $\gamma(P)$. The claim follows using the $\mathcal{J}$–equivalence $M\boxtimes(1+X)^{\mathcal{J}} \to (M+X)^{\mathcal{J}}$. □

Writing $\gamma(M)/\gamma(P)$ for a cofibrant model of the homotopy cofiber of the map of Γ–spaces $\gamma(P) \to \gamma(M)$, we can now use the relation between commutative $\mathcal{J}$–space monoids and Γ–spaces established in [16, Section 7] to express commutative $\mathcal{J}$–space monoid derivations by mapping spaces in Γ^{op}–$\mathcal{S}$:

Lemma 7.3 *There is a natural weak equivalence between the mapping spaces*

$$\mathrm{Map}_M^P(M, (M+X)^{\mathcal{J}}) \quad \textit{and} \quad \mathrm{Map}_{\Gamma^{\mathrm{op}}\text{–}\mathcal{S}}\big(\gamma(M)/\gamma(P), (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big).$$

Proof We denote the mapping spaces in the comma category $\gamma(P) \downarrow \Gamma^{\mathrm{op}}\text{–}\mathcal{S} \downarrow \gamma(M)$ by $\mathrm{Map}_{\gamma(M)}^{\gamma(P)}(-,-)$, and we let

$$\gamma(P) \rightarrowtail \gamma(M)^{\mathrm{cof}} \xrightarrow{\sim} \gamma(M)$$

be a cofibrant replacement of $\gamma(M)$ in this comma category.

Since Quillen equivalences induce weak equivalences between the homotopy types of mapping spaces in the respective categories, the Quillen equivalences of [16, Corollary 7.12] and Lemma 7.2 show that the space $\mathrm{Map}_M^P(M, (M+X)^{\mathcal{J}})$ is weakly equivalent to the mapping space $\mathrm{Map}_{\gamma(M)}^{\gamma(P)}\big(\gamma(M)^{\mathrm{cof}}, \gamma(M) \times (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big)$ in the model category $(\Gamma^{\mathrm{op}}\text{–}\mathcal{S})_{\mathrm{pre}}/b\mathcal{J}$ defined in [16, Section 7]. The iterated pullbacks that are used to define mapping spaces in comma categories show that the mapping spaces in $\big((\Gamma^{\mathrm{op}}\text{–}\mathcal{S})_{\mathrm{pre}}/b\mathcal{J})\big) \downarrow \gamma(M)$ coincide with the mapping spaces in the category $(\Gamma^{\mathrm{op}}\text{–}\mathcal{S})_{\mathrm{pre}} \downarrow \gamma(M)$. A similar argument shows that the space of maps into $\gamma(M) \times (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}$ in the category $\gamma(P) \downarrow (\Gamma^{\mathrm{op}}\text{–}\mathcal{S})_{\mathrm{pre}} \downarrow \gamma(M)$ coincides with the space of maps into $(\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}$ in the category $\gamma(P) \downarrow (\Gamma^{\mathrm{op}}\text{–}\mathcal{S})_{\mathrm{pre}}$. Since $(1+X)^{\mathcal{J}}$ is grouplike, the Γ–space $\gamma(1+X)^{\mathcal{J}}$ is very special and its level fibrant replacement $(\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}$ is fibrant in the stable model structure on Γ–spaces. Since the latter model structure can be constructed as a left Bousfield localization of $(\Gamma^{\mathrm{op}}\text{–}\mathcal{S})_{\mathrm{pre}}$, spaces of maps into $(\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}$ coincide in these two model structures. So we have built a weak equivalence

$$\mathrm{Map}_{\gamma(M)}^{\gamma(P)}\big(\gamma(M)^{\mathrm{cof}}, \gamma(M) \times (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big) \simeq \mathrm{Map}^{\gamma(P)}\big(\gamma(M)^{\mathrm{cof}}, (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big).$$

Applying the functor $\mathrm{Map}_{\Gamma^{\mathrm{op}}\text{–}\mathcal{S}}(-, (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}})$ to the pushout

$$\begin{array}{ccc} \gamma(P) & \rightarrowtail & \gamma(M)^{\mathrm{cof}} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \gamma(M)/\gamma(P) \end{array}$$

shows that the space $\mathrm{Map}^{\gamma(P)}\big(\gamma(M)^{\mathrm{cof}}, (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big)$ is weakly equivalent to $\mathrm{Map}_{\Gamma^{\mathrm{op}}\text{–}\mathcal{S}}\big(\gamma(M)/\gamma(P), (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big)$. □

Using Lemma 7.3 and Proposition 5.5, we can prove the remaining proposition:

Proof of Proposition 5.19 By Lemma 7.3, the space of commutative $\mathcal{J}$–space derivations is weakly equivalent to $\mathrm{Map}_{\Gamma^{\mathrm{op}}-\mathcal{S}}\big(\gamma(M)/\gamma(P), (\gamma(1+X)^{\mathcal{J}})^{\mathrm{fib}}\big)$. We have shown in Proposition 5.5 that the Γ–space $\gamma(1+X)^{\mathcal{J}}$ models the connective cover of the underlying spectrum of X. Keeping the notation $\gamma(M)/\gamma(P)$ for the (symmetric) spectrum associated with $\gamma(M)/\gamma(P)$ [12], this spectrum being connective implies that the mapping space in question is equivalent to $\mathrm{Map}_{\mathrm{Sp}^{\Sigma}}(\gamma(M)/\gamma(P), X)$. The desired equivalence to $\mathrm{Map}_{A-\mathrm{Mod}}(A \wedge (\gamma(M)/\gamma(P)), X)$ follows by extension of scalars along $\mathbb{S} \to A$. □

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