msp

Geometry & Topology 18 (2014) 491–520

Moments of the boundary hitting function for the geodesic flow on a hyperbolic manifold

MARTIN BRIDGEMAN
SER PEOW TAN

In this paper we consider geodesic flow on finite-volume hyperbolic manifolds with non-empty totally geodesic boundary. We analyse the time for the geodesic flow to hit the boundary and derive a formula for the moments of the associated random variable in terms of the orthospectrum. We show that the zeroth and first moments correspond to two cases of known identities for the orthospectrum. We also show that the second moment is given by the average time for the geodesic flow to hit the boundary. We further obtain an explicit formula in terms of the trilogarithm functions for the average time for the geodesic flow to hit the boundary in the surface case.

32Q45; 32G15

1 Introduction

Let X be a finite-volume hyperbolic manifold with non-empty totally geodesic boundary ∂X. An *orthogeodesic* for X is a proper geodesic arc (endpoints in ∂X) which is perpendicular to ∂X. These were first introduced by Basmajian in [1] in the study of totally geodesic submanifolds. We denote by $O_X = \{\alpha_i\}$ the collection of orthogeodesics of X and let l_i be the length of α_i. We note that O_X is countable as the elements correspond to a subset of the collection of closed geodesics of the double of X along its boundary. We call the set $L_X = \{l_i\}$ (with multiplicities) the *orthospectrum.*

In [1], Basmajian derived the following boundary orthospectrum identity:

$$\mathrm{Vol}(\partial X) = 2 \sum_{l \in L_X} V_{n-1}\big(\log\big(\coth \tfrac{l}{2}\big)\big), \tag{1}$$

where $V_n(r)$ is the volume of the ball of radius r in $\mathbb{H}^n$. The identity comes from considering the universal cover $\widetilde{X} \subseteq \mathbb{H}^n$ of X. Then $\partial \widetilde{X}$ is a countable collection of disjoint hyperbolic hyperplanes which are the lifts of the boundary components of ∂X. For each component C of $\partial \widetilde{X}$, we orthogonally project each of the other components of $\partial \widetilde{X}$ onto C to obtain a collection of disjoint disks on each component C of $\partial \widetilde{X}$. These disks form an equivariant family of disks that are full measure in $\partial \widetilde{X}$. They

Published: 29 January 2014 DOI: 10.2140/gt.2014.18.491

descend to a family of disjoint disks in ∂X of full measure. As each orthogeodesic lifts to a perpendicular between two components of $\partial\widetilde{X}$, each orthogeodesic corresponds to two disks (one at each end) in the family of disks in ∂X and this gives the above identity.

Using a decomposition of the unit tangent bundle, Bridgeman–Kahn (see [5]) derived the identity

$$\mathrm{Vol}(T_1(X)) = \sum_{l\in L_X} H_n(l), \tag{2}$$

where H_n is a smooth function depending only on the dimension n. As $\mathrm{Vol}(T_1(X)) = \mathrm{Vol}(X).V_{n-1}$, where V_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$, the above identity can also be written as

$$\mathrm{Vol}(X) = \sum_{l\in L_X} \bar{H}_n(l),$$

where $\bar{H}_n(l) = H_n(l)/V_{n-1}$. In the specific case of dimension two, the function H_n is given in terms of the Rogers dilogarithm (see Bridgeman [3]). In the papers [6; 7] Calegari gives an alternative derivation of the identity (2).

The motivation for this paper was to connect the above two identities in a natural framework. The connection is that they are the first two moments of the Liouville measure. A second motivation was to compute the average time it takes to hit the boundary under the geodesic flow. This can be put into the same framework and it turns out that consideration of the third moment gives a formula for the average time it takes to hit the boundary of M under the geodesic flow in terms of the orthospectrum. We also observe that for any two hyperbolic surfaces with geodesic boundary, the orthospectrum of one cannot dominate the other, which contrasts with the case for closed geodesics where it was shown that it is possible to deform a hyperbolic surface with boundary in such a way that the new surface has all closed geodesics with lengths strictly larger than the original surface (see Parlier [13], Papadopoulos and Théret [12] and Goldman, Margulis and Minsky [8]). It is conceivable that higher moments encode other important geometric invariants of the manifold.

Acknowledgements The authors thank Bill Goldman, Danny Calegari, Dick Canary, Ivan Izmestiev, Greg McShane, Hugo Parlier and Nick Vlamis for helpful conversations. They would also like to thank the referee for their very careful reading of the manuscript and the many helpful comments and suggestions which have greatly improved the paper. This work was partially supported by a grant from the Simons Foundation (number 266344 to Martin Bridgeman). Ser Peow Tan was partially supported by the National University of Singapore academic research grant R-146-000-156-112.

2 Moments of Liouville measure

We let $G(\mathbb{H}^n)$ be the space of oriented geodesics in $\mathbb{H}^n$. By identifying a geodesic with its endpoints on the sphere at infinity, the space $G(\mathbb{H}^n) \simeq (\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{Diagonal})$. The *Liouville measure* μ on $G(\mathbb{H}^n)$ is a Möbius invariant measure. In the upper half space model, we identify a geodesic with its endpoints $(x, y) \in \overline{\mathbb{R}}^{n-1} \times \overline{\mathbb{R}}^{n-1}$. Then the Liouville measure μ has the form

$$d\mu_{(x,y)} = \frac{2dV_x dV_y}{|x-y|^{2n-2}},$$

where $dV_x = dx_1 dx_2 \cdots dx_{n-1}$, for $x = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

If X is a hyperbolic n–manifold with totally geodesic boundary, we identify $\widetilde{X}$, the universal cover of X as a subset of $\mathbb{H}^n$ and $\Gamma \subseteq \mathrm{Isom}^+(\mathbb{H}^n)$ such that $X = \widetilde{X}/\Gamma$. Then $G(\widetilde{X}) \subseteq G(\mathbb{H}^n)$ is the set of oriented geodesics intersecting $\widetilde{X}$. We define $G(X) = G(\widetilde{X})/\Gamma$, the space of oriented geodesics in X. Then by invariance, the Liouville measure μ descends to a measure on $G(X)$ which we also denote by μ.

We define the measurable function $L\colon G(X) \to [0, \infty]$ by $L(g) := \mathrm{Length}(g)$, where the length is taken in X. This is the hitting length function for X. As the limit set L_Γ has measure zero, almost every geodesic hits the boundary of X and therefore for almost every geodesic g, $L(g)$ is finite and g is a proper geodesic arc.

We define the pushforward measure $M := L_*(\mu)$ on the real line. This measure is the distribution of lengths of geodesics in X. We define its k^{th} moment to be

$$M_k(X) = M(x^k) = \int_0^\infty x^k dM = \int_{G(X)} L^k(g)\, d\mu.$$

In general, the moments of a random variable give a set of measurements that describe distributional properties of the random variable such as the average value and variance. Using a decomposition of $G(X)$, we show that the moments $M_k(X)$ have formulae that extend the identities (1) and (2).

Theorem 2.1 *There exists smooth functions $F_{n,k}\colon \mathbb{R}_+ \to \mathbb{R}_+$ and constants $C_n > 0$ such that if X is a compact hyperbolic n–manifold with totally geodesic boundary $\partial X \neq \varnothing$, then:*

(1) *The moment $M_k(X)$ satisfies*

$$M_k(X) = \sum_{l \in L_X} F_{n,k}(l).$$

(2) $M_0(X) = C_n \,\mathrm{Vol}(\partial X)$ *and the identity for $M_0(X)$ is the identity of Basmajian given in identity (1).*

(3) $M_1(X) = \mathrm{Vol}(T_1(X))$ *and the identity for* $M_1(X)$ *is the identity of Bridgeman–Kahn given in identity* (2).

(4) $M_2(X) = 2\,\mathrm{Vol}(T_1(X))A(X)$ *where* $A(X)$ *is the average time for a vector in* $T_1(X)$ *to hit the boundary under geodesic flow. Therefore by the identity for* $M_2(X)$,

$$A(X) = \frac{1}{2\,\mathrm{Vol}(T_1(X))}\sum_{l\in L_X} F_{n,2}(l) = \sum_{l\in L_X} G_n(l).$$

In the surface case we obtain an explicit formula for the function G_2 and hence $A(X)$ in terms of polylogarithms. Furthermore, besides compact surfaces obtained as quotients of Fuchsian groups, the identity holds more generally for finite area surfaces, which we describe next.

If S is a finite area surface with totally geodesic boundary $\partial S \neq \varnothing$, then the boundary components are either closed geodesics or bi-infinite geodesics with cuspidal endpoints. We define a *boundary cusp* of S to be an ideal vertex of ∂S. We let C_S be the number of boundary cusps of S. Then we have the following explicit formula for $A(S)$:

Theorem 2.2 *Let* S *be a finite area hyperbolic surface with non-empty totally geodesic boundary. Then*

$$A(S) = \frac{1}{8\pi^2|\chi(S)|}\left(\sum_{l\in L_S} F\left(\mathrm{sech}^2 \tfrac{l}{2}\right) + 6\zeta(3)C_S\right),$$

where

$$F(a) = -12\zeta(3) - \frac{4\pi^2}{3}\log(1-a) + 6\log^2(1-a)\log(a) - 4\log(1-a)\log^2(a) - 8\log\left(\frac{a^2}{1-a}\right)\mathrm{Li}_2(a) + 24\,\mathrm{Li}_3(a) + 12\,\mathrm{Li}_3(1-a),$$

for $\mathrm{Li}_k(x)$ *the* k^{th} *polylogarithm function, and* ζ *the Riemann* ζ*–function.*

3 A natural fibering

We have the natural fiber bundle $p\colon T_1(\mathbb{H}^n) \to G(\mathbb{H}^n)$ such that v is tangent to the oriented geodesic $p(v)$. Let Ω be the volume measure on $T_1(\mathbb{H}^n)$. We parametrize $T_1(\mathbb{H}^n)$ as follows: We first choose a basepoint b_g on each geodesic g (say by taking the point closest to a fixed point p). Then a vector $v \in T_1(\mathbb{H}^n)$ is given by a triple

$(x, y, l) \in \overline{\mathbb{R}}^{n-1} \times \overline{\mathbb{R}}^{n-1} \times \mathbb{R}$ where v is tangent to the geodesic g with endpoints x, y (from x to y), and l_v is the signed length along g from the basepoint b_g. In terms of this parametrization the volume form Ω on $T_1(\mathbb{H}^n)$ is

$$d\Omega_v = \frac{2dV_x dV_y dl_v}{|x-y|^{2n-2}} = d\mu_{(x,y)}\, dl_v,$$

where μ is the Liouville measure (see Nicholls [11]). If X is a hyperbolic n–manifold with totally geodesic boundary and $\widetilde{X}$ the universal cover of X, then the fiber bundle p restricts to $T_1(\widetilde{X})$ to give the equivariant map $p\colon T_1(\widetilde{X}) \to G(\widetilde{X})$ which descends to a map $\overline{p}\colon T_1(X) \to G(X)$. We have the function $\overline{L}\colon T_1(X) \to [0, \infty]$ given by $\overline{L} = L \circ \overline{p}$. The measure $N = \overline{L}_*\Omega$ was introduced in Bridgeman [3] in order to derive the surface case of the identity (2). The measures M, N have a simple relation which we now describe.

For $\phi\colon [0, \infty) \to \mathbb{R}$ a smooth function with compact support then

$$\begin{aligned} N(\phi) = \Omega(\phi \circ \overline{L}) &= \int_{v \in T_1(X)} \phi(\overline{L}(v)) d\Omega_v \\ &= \int_{g \in G(X)} \int_{v \in p^{-1}(g)} \phi(L(g)) d\mu_g\, dl_v \\ &= \int_{g \in G(X)} \phi(L(g)) \left(\int_{v \in p^{-1}(g)} dl_v \right) d\mu_g \\ &= \int_{g \in G(X)} \phi(L(g)) L(g) d\mu_g = M(x\phi). \end{aligned}$$

It follows that the measures M, N satisfy $dN = x\,dM$. We define the k^{th} moment of N to be $N_k(X) = N(x^k)$. Then

$$N_k(X) = M_{k+1}(X). \tag{3}$$

Also as $N_0(X) = \operatorname{Vol}(T_1(X))$ it follows that $M_1(X) = \operatorname{Vol}(T_1(X))$.

Because of the above relation between the measures M and N, the results described in this paper can be given in terms of either. For the most part we will give our results in terms of the measure M and its moments, but when it is more natural to do so, we will consider the measure N.

We let $A(X)$ be the average time for a vector in $T_1(X)$ to hit the boundary under geodesic flow. Then as the time for v and $-v$ sum to $\overline{L}(v)$, $A(X)$ is half the average of the function $\overline{L}$. Then $A(X)$ is given by the first moment of measure N and as

$N_1(X) = M_2(X)$, we have the formula

$$A(X) = \frac{1}{2}\frac{M_2(X)}{\mathrm{Vol}(T_1(X))},$$

proving part (4) of Theorem 2.1.

4 Moments are finite

Before we derive summation formulae for the moments $M_k(X)$, we need to first show that they are finite. The proof that $M_0(X)$, $M_1(X)$ are finite will follow from explicit calculation, in particular, from the last section $M_1(X) = \mathrm{Vol}(T_1(X))$ which is finite by assumption. By identity (3), $M_k(X) = N_{k-1}(X)$, and therefore we need only show that $N_k(X)$ is finite for $k \geq 1$. To prove this, we show that the measure N on the real line decays exponentially, that is, there exist positive constants a, C such that $dN \leq Ce^{-at}dt$ for t large. Then $N_k(X)$ is finite as the measure $x^k e^{-at}dt$ is finite.

We first recall some background on Kleinian groups (see Maskit [10] for details). A *Kleinian group* Γ is a discrete subgroup of the isometries of $\mathbb{H}^n$. The *limit set* $L_\Gamma = \overline{\Gamma x} \cap \mathbb{S}^{n-1}_\infty$ is the accumulation set of an orbit of a point x on the boundary. It is easy to show that L_Γ is independent of x. The *convex hull* of Γ, denoted $H(\Gamma)$, is the smallest convex set containing all geodesics with endpoints in L_Γ. As $H(\Gamma)$ is invariant under Γ, the *convex core* is defined to be $C(\Gamma) = H(\Gamma)/\Gamma$. A Kleinian group is *convex cocompact* if $C(\Gamma)$ is compact. Also a group is *geometrically finite* if $N_\epsilon(C(\Gamma))$, the ϵ–neighborhood of the core, is finite-volume.

Let Γ be a convex cocompact Kleinian group with $N = \mathbb{H}^n/\Gamma$ and $X = H(\Gamma)/\Gamma$ its convex core. We let $\delta(\Gamma)$ be the Hausdorff dimension of the limit set L_Γ. Let g_t be the geodesic flow on $T_1(\mathbb{H}^n)$ and $\overline{g}_t$ be the quotient geodesic flow on $T_1(N)$. We define

$$B(t) = \{v \in T_1(X) \mid \overline{g}_t(v) \in T_1(X)\} = \overline{g}_{-t}(T_1(X)) \cap T_1(X).$$

The set $B(t)$ is the set of tangent vectors that remain in the convex core under time t flow. We now use a standard counting argument on orbits to bound the volume of the set $B(t)$ (see Nicholls [11] for background).

Lemma 4.1 *Given Γ a convex cocompact Kleinian group, then there exists constants A, T such that*

$$\mathrm{Vol}(B(t)) \leq Ae^{-(n-1-\delta(\Gamma))t}$$

for $t > T$. In particular if $L_\Gamma \neq \mathbb{S}^{n-1}_\infty$ then $\mathrm{Vol}(B(t))$ is exponentially decaying.

Proof We take $0 \in H(\Gamma)$ and consider its orbits under Γ. Then we let

$$O(r) = \{\gamma \in \Gamma \mid d(0, \gamma(0)) < r\} \quad \text{and} \quad N(r) = \#O(r).$$

By Sullivan [15], there exists constants A, r_0 such that $N(r) \leq Ae^{\delta(\Gamma)r}$ for $r > r_0$. We let D be the diameter of X. Given a unit tangent vector v, we denote its basepoint by $b(v)$. We let P be the Dirichlet domain centered at 0 for the action of Γ on $H(\Gamma)$. Then as the diameter of $X = H(\Gamma)/\Gamma$ is D, we have $P \subseteq B(0, D)$. We lift $B(t) \subseteq T_1(X)$ to a set $\widetilde{B}(t) \subseteq T_1(H(\Gamma))$ and choose the lift such that $\widetilde{B}(t) \subseteq T_1(P)$. Thus each vector in $\widetilde{B}(t)$ has basepoint within a distance D of 0. For $v \in \widetilde{B}(t)$, then by the triangle inequality, $g_t(v)$ has basepoint $b(g_t(v))$ such that $t - D < d(b(g_t(v)), 0) < t + D$. Again by the triangle inequality, $b(g_t(v))$ has nearest orbit $\gamma(0)$ such that $b(g_t(v)) \in B(\gamma(0), D)$, the ball of radius D around $\gamma(0)$. Also we have that $t - 2D < d(\gamma(0), 0) < t + 2D$. We use this to find a cover for the set $g_t(\widetilde{B}(t))$. We let $\mathcal{U}(t)$ be the collection of balls in $\mathbb{H}^n$ given by

$$\mathcal{U}(t) = \{B(\gamma(0), D) \mid t - 2D < d(0, \gamma(0)) < t + 2D)\}.$$

We further let $U(t)$ be the union of elements of $\mathcal{U}(t)$. Then for any $v \in \widetilde{B}(t)$, the basepoint $b(g_t(v))$ is contained in the set $U(t)$.

Let x be a distance at most D from 0. We want to bound the visual measure of $U(t)$ from x. Each $B(\gamma(0), D) \in \mathcal{U}(t)$ is a distance between $t - 3D, t + 3D$ from x. Let $T_1 > 3D$, and restrict to $t > T_1$. We denote by $P(\gamma(0), D)$ the radial projection of $B(\gamma(0), D) \in U(t)$ onto the $S(x, t)$, the sphere of radius t in hyperbolic space. Then the area of each $P(\gamma(0), D)$ is bounded above by a constant $C_1 > 0$. Therefore

$$\mathrm{Vis}_x(U(t)) \leq \frac{C_1 N(t + 2D)}{\mathrm{Vol}(S(x, t))}.$$

We have that $\mathrm{Vol}(S(x, t)) = S_n \sinh^{n-1}(t)$, where S_n is the volume of the standard Euclidean sphere of dimension $(n - 1)$. Thus for $t > T_1$, $\mathrm{Vol}(S(x, t)) \geq L_n e^{(n-1)t}$ for some $L_n > 0$.

Therefore

$$\mathrm{Vis}_x(U(t)) \leq \frac{C_1 N(t + 2D)}{L_n e^{(n-1)t}} \leq \frac{C_1 A e^{\delta(\Gamma)(t+2D)}}{L_n e^{(n-1)t}} \leq C e^{-((n-1)-\delta(\Gamma))t}$$

for some constant C.

In order to obtain the bound on $\mathrm{Vol}(B(t))$ we integrate the visual measure of $U(t)$ over $\widetilde{B}(t)$ to give

$$\mathrm{Vol}(B(t)) = \mathrm{Vol}(\widetilde{B}(t)) \leq \mathrm{Vol}(X) C e^{-((n-1)-\delta(\Gamma))t}$$

for $t > T$, giving our result for Γ convex cocompact. □

Corollary 4.2 *If X is a compact hyperbolic manifold with non-empty totally geodesic boundary then the moments $M_k(X)$ are finite for $k \geq 1$.*

Proof Recall that the function $\overline{L}\colon T_1(X) \to [0,\infty]$ is defined by letting $\overline{L}(v)$ be the length of the maximal geodesic arc in X tangent to v. We define

$$E(t) = \{v \in T_1(X) \mid \overline{L}(v) \in [t, t+1)\}.$$

Thus if $v \in E(t)$ then $\overline{L}(v) \geq t$ and v is tangent to a geodesic of length at least t in $T_1(X)$. Thus geodesic flow in either the positive or negative direction from v for time $t/2$ must remain in $T_1(X)$. Therefore either $\overline{g}_{t/2}(v) \in T_1(X)$ or $\overline{g}_{-t/2}(v) \in T_1(X)$. Thus $E(t) \subseteq B(t/2) \cup B(-t/2)$. Therefore by the above lemma, there are constants $a, K > 0$ such that $\mathrm{Vol}(E(t)) \leq Ke^{-at}$ for $t > 2T_0$. Therefore

$$N_k(X) \leq \sum_{n=0}^{\infty} (n+1)^k \,\mathrm{Vol}(E(n))$$

is finite for $k \geq 0$ by comparison with the series $\sum n^k e^{-an}$. Therefore $M_k(X)$ is finite for $k \geq 1$. □

5 Decomposition of the space of geodesics

As described in Section 2, we let X be a hyperbolic n–manifold with totally geodesic boundary, $\widetilde{X}$ the universal cover of X as a subset of $\mathbb{H}^n$ and $\Gamma \subseteq \mathrm{Isom}^+(\mathbb{H}^n)$ such that $X = \widetilde{X}/\Gamma$. We let $G(\widetilde{X}) \subseteq G(\mathbb{H}^n)$ be the set of geodesics intersecting $\widetilde{X}$ and $G(X) = G(\widetilde{X})/\Gamma$, the space of geodesics in X. Then by invariance, the Liouville measure μ descends to a measure on $G(X)$ which we also call μ.

The space $G(X)$ has a simple (full measure with respect to μ) decomposition via orthogeodesics. For α an orthogeodesic we define

$$F_\alpha = \{g \in G(X) \mid g \text{ is homotopic rel } \partial X \text{ to } \alpha\}.$$

The set of orthogeodesics is countable, so we index our orthogeodesics $O_X = \{\alpha_i\}$ and the sets $F_i = F_{\alpha_i}$. As the limit set L_Γ is measure zero, $\bigcup_i F_i$ gives a full measure partition of $G(X)$ with respect to μ. We note that as the stabilizer of a lift of an orthogeodesic with respect to Γ is the trivial set, the sets $F_i \subseteq \mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty$ are of the form $(D_1 \times D_2) \cup (D_2 \times D_1)$ where D_1, D_2 are disjoint round disks. Thus $G(X)$ decomposes (up to full measure) into a countable collection of elementary pieces indexed by the orthogeodesics.

Then

$$M_k(X) = M(x^k) = \int_0^\infty x^k\, dM = \int_{G(X)} L^k(g)\, d\mu = \sum_i \int_{F_i} L^k(g)\, d\mu,$$

where $L\colon G(X) \to \mathbb{R}$ is the length of a geodesic.

We now lift F_i to a set $\widetilde{F}_i \subset G(\mathbb{H}^n)$. In the upper half-space model, we choose two hyperplanes P, Q with orthogonal distance $l_i = \mathrm{Length}(\alpha_i)$. Then F_i lifts to the set of geodesics which intersect P and Q. In particular we can take P, Q to have boundary circles centered at 0 with radii $1, e^{l_i}$ respectively. Lifting the function L to $\widetilde{F}_i$ we see that it only depends on the endpoints (x, y) and the ortholength l_i. We denote it by $L(x, y, l_i)$. We have

$$\begin{aligned}\int_{F_i} L^k(g) d\mu &= \int_{\widetilde{F}_i} \frac{2L(x,y,l_i)^k\, dV_x dV_y}{|x-y|^{2n-2}} \\ &= 2\int_{|x|<1}\int_{|y|>e^{l_i}} \frac{2L(x,y,l_i)^k\, dV_x dV_y}{|x-y|^{2n-2}} = F_{n,k}(l_i).\end{aligned}$$

Therefore we have that

$$M_k(X) = \sum_i F_{n,k}(l_i)$$

and $F_{n,k}$ is given by the integral formula

$$F_{n,k}(t) = \int_{|x|<1}\int_{|y|>e^t} \frac{4L(x,y,t)^k\, dV_x dV_y}{|x-y|^{2n-2}}.$$

This gives the summation formula

$$M_k(X) = \sum_{l \in L_X} F_{n,k}(l)$$

for the moment and proves part (1) of Theorem 2.1.

By identity (3), $M_1(X) = N_0(X) = \mathrm{Vol}(T_1(X))$. Thus letting $E_i \subseteq T_1(X)$ given by $E_i = \overline{p}^{-1}(F_i)$ then the identity for $M_1(X)$ is given by

$$M_1(X) = \mathrm{Vol}(T_1(X)) = \sum_i \Omega(E_i).$$

In the paper [5], we showed that this is identity (2). This proves part (3) of Theorem 2.1.

6 Zero moment identity is Basmajian's identity

As $M_0(X) = \mu(G(X))$ the identity for $M_0(X)$ is

$$\mu(G(X)) = \sum_i \mu(F_i).$$

We now prove some properties of the Liouville measure μ needed to evaluate both sides of this identity and show that it gives Basmajian's identity (1).

Lemma 6.1 *Let μ be the Liouville measure on $G(\mathbb{H}^n)$ and P a hyperplane in $\mathbb{H}^n$. For $U \subseteq P$, let $G(U) \subseteq G(\mathbb{H}^n)$ be the set of geodesics intersecting U transversely. Then the measure μ_P on P defined by $\mu_P(U) = \mu(G(U))$ is a constant times volume measure on P. In particular there is a constant $K_n > 0$ depending only on dimension such that $\mu_P(U) = K_n \operatorname{Vol}(U)$.*

Proof In the case of $n = 2$ this is a standard property of μ (see Bonahon [2, Appendix A3]). In general we see that μ_P gives a Möbius invariant measure on the hyperbolic space P and therefore must be a multiple of volume measure. Thus $\mu_P(U) = K_n.\operatorname{Vol}(U)$, where K_n only depends on the dimension n. We calculate K_n in a later section. □

A Liouville measure preserving map We let P be a hyperplane in $\mathbb{H}^n$ and G_P the set of geodesics intersecting P transversely. Then $G_P = (D_- \times D_+) \cup (D_+ \times D_-)$, where D_-, D_+ are disjoint open disks in $\mathbb{S}^{n-1}_\infty$ with $\partial D_- = \partial D_+$. We let $g = (x(g), y(g)) \in G_P$ be the ordered pair of endpoints of g and $m(g) = g \cap P$ the point of intersection with P. We project the endpoints of g orthogonally onto P to obtain the ordered pair of points $(p(g), q(g)) \in P \times P$. We note that $m(g)$ is the midpoint of the hyperbolic geodesic arc in P joining $p(g), q(g)$. We now define a map $F\colon G_P \to G_P$ as follows; We first define $F\colon D_- \times D_+ \to D_- \times D_+$ and extend it to $D_+ \times D_-$ by conjugating with the map switching endpoints, that is, if $i(x, y) = (y, x)$ then $F(g) = i(F(i(g))$ for $g \in D_+ \times D_-$. If $g \in D_- \times D_+$ then we define $F(g) = h$ where h is the unique geodesic (in $D_- \times D_+$) such that $m(h) = q(g)$ and $q(h) = m(g)$ (see Figure 1). By construction, F is an involution and if u is an isometry of $\mathbb{H}^n$ fixing P, then F commutes with u. We note that F is not the action of an isometry on the space of geodesics.

Lemma 6.2 *The map $F\colon G_P \to G_P$ preserves Liouville measure.*

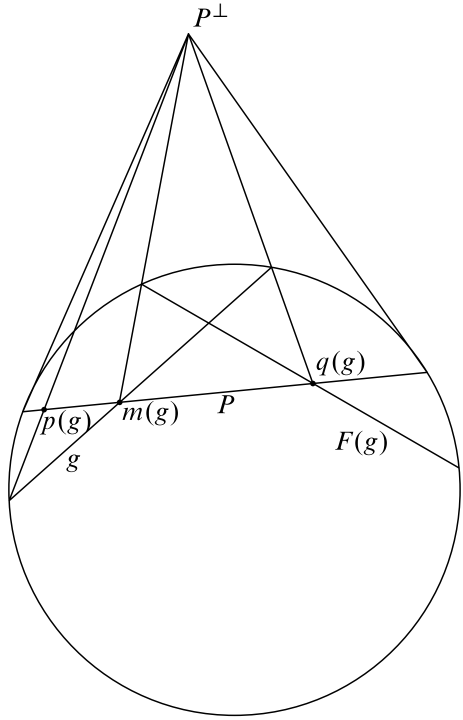

Figure 1: The involution F in the Klein model

Proof We show that $F_*(\mu) = \mu$ by showing the Radon–Nikodym derivative

$$\nu(x) = \frac{dF_*(\mu)}{d\mu}(x) = 1.$$

Alternately, $\nu(x)$ is the function such that for any ϕ smooth compactly supported function on G_P then

$$F_*(\mu)(\phi) = \int \phi(x) d(F_*(\mu)) = \int \phi(x)\nu(x) d\mu(x).$$

Therefore

$$F_*(\mu)(\phi) = \mu(\phi \circ F) = \int \phi(F(x)) d\mu(x) = \int \phi(x)\nu(x) d\mu(x).$$

As F is an involution $F(F(x)) = 1$, then by the change of variables formula we have

$$\begin{aligned}\int \phi(x) d\mu(x) = \int \phi(F(F(x)) d\mu(x) &= \int \phi(F(x))\nu(x) d\mu(x)\\ &= \int \phi(x)\nu(F^{-1}(x))\nu(x) d\mu(x).\end{aligned}$$

Therefore

$$\nu(x)\nu(F^{-1}(x)) = 1 \quad \text{or} \quad \nu(x)\nu(F(x)) = 1.$$

Similarly we have if u is a hyperbolic isometry then u preserves the Liouville measure and $u_*(\mu) = \mu$. If u also fixes P then F commutes with u and $F(u(x)) = u(F(x))$. Therefore by the change of variables formula again we have

$$\nu((u(x)) = \nu(x).$$

Thus combining the above, if for each $x \in G_P$ there exists an isometry u fixing P such that $u(x) = F(x)$, then $\nu(x) = \nu(F(x))$. But as $\nu(x)\nu(F(x)) = 1$ we then obtain $\nu(x) = 1$.

To find such an isometry, we note that for geodesic $x \in G_P$, and $y = F(x)$ we have the four points $p(x), m(x) = q(y), q(x) = m(y), p(y)$ all collinear on P. We choose the hyperplane P^* perpendicular to P which bisects the hyperbolic interval $[p(x), p(y)]$. Then refection in P^* fixes P and sends x to $y = F(x)$.

Thus $\nu(x) = 1$ for all $x \in G_P$ and therefore $F_*(\mu) = \mu$. □

Corollary 6.3 *If α is an orthogeodesic, then*

$$\mu(F_\alpha) = K_n V_{n-1}\left(\log\left(\coth \frac{l(\alpha)}{2}\right)\right).$$

Proof We consider disjoint hyperplanes P, Q with perpendicular distance equal the ortholength $l(\alpha)$. Then $\mu(F_\alpha) = \mu(S)$ where $S \subseteq G_P$ of geodesics which intersect Q. We let $G_P = (D_- \times D_+) \cup (D_+ \times D_-)$ where $\partial Q \subseteq D_+$. By Lemma 6.2, $\mu(S) = \mu(F(S))$. Let B be the orthogonal projection of Q onto P. Then by elementary hyperbolic geometry, B is a ball of radius $r = \log\left(\coth \frac{l(\alpha)}{2}\right)$. The set $F(S)$ is precisely the set of geodesics in G_P transversely intersecting B which we denote by $G(B)$. To see this, we note that if $g \in S$ then $F(g)$ intersects P in B giving $F(S) \subseteq G(B)$. Similarly if $g \in G(B)$ then $F(g)$ is in S. Thus, as F is an involution $F(S) = G(B)$.

Therefore by Lemma 6.1

$$\mu(F_\alpha) = \mu(S) = \mu(F(S)) = \mu(G(B)) = K_n \operatorname{Vol}(B) = K_n V_{n-1}\left(\log\left(\coth \frac{l(\alpha)}{2}\right)\right)$$

as required. □

We will now prove the second part of Theorem 2.1.

By the above Corollary 6.3 we obtain

$$\mu(G(X)) = \sum_{\alpha \in O_X} \mu(F_\alpha) = K_n \sum_{\alpha \in O_X} V_{n-1}\left(\log\left(\coth \frac{l(\alpha)}{2}\right)\right).$$

For each boundary component B_i of ∂X we let B_i' in $\mathbb{H}^n$ be a hyperplane which is a lift of B_i. We further take a fundamental domain D_i on B_i' for the action of Γ. We let C_i be the set of geodesics which intersect D_i transversely such that the geodesics point into $\widetilde{X}$ on B_i'. Let $G = \bigcup_i C_i$ then $G \subseteq G(\widetilde{X})$ and is a lift of $G(X)$ under the quotient map $G(\widetilde{X}) \to G(\widetilde{X})/\Gamma = G(X)$ (except for a set of measure zero). To see this note that for almost every $g \in G(X)$, g is a proper arc from a B_i to a B_j where the orientation of g is pointing into X at B_i and out at B_j. Therefore g has lift g' in C_i. Also for $i \neq j$ if $g \in C_i \cap C_j$ then g points inward on both B_i and B_j. Therefore $C_i \cap C_j = \varnothing$ for $i \neq j$. Also if $g_1, g_2 \in C_i$ are lifts of the same element of $G(X)$ then there is a $\gamma \in \Gamma$ with $g_2 = \gamma(g_1)$. As D_i is a fundamental domain for the action of Γ on B_i' then g_1, g_2 must have endpoints on the boundary of D_i which is measure zero.

Using this we can calculate $\mu(G(X))$. We have

$$\mu(G(X)) = \mu(G) = \sum_i \mu(C_i).$$

By the above Lemma 6.1, $\mu(C_i) = (K_n/2)\,\mathrm{Vol}(B_i)$ where the factor of two comes from C_i containing half the geodesics in the set $G(B_i)$ (those pointing into $\widetilde{X}$). Therefore

$$\mu(G(X)) = \frac{K_n}{2} \sum \mathrm{Vol}(B_i) = \frac{K_n}{2}\,\mathrm{Vol}(\partial X).$$

Thus

$$\frac{K_n}{2}\,\mathrm{Vol}(\partial X) = K_n \sum_{\alpha \in O_X} V_{n-1}\left(\log\left(\coth \frac{l(\alpha)}{2}\right)\right),$$

giving Basmajian's identity. □

7 Calculating K_n

To calculate the constant K_n, we derive the Lebesgue density of the measure μ_P in Lemma 6.1. Let p be at the origin of the Poincare model of $\mathbb{H}^n$ and let P be the horizontal hyperplane through p. Let $B = B_{n-1}(p, r)$ be a small $(n-1)$–dimensional ball in P about p. As before, we parameterize an oriented geodesic by its ordered pair of endpoints on the sphere at infinity. Therefore in terms of the Poincaré model we have $G(\mathbb{H}^n) \simeq (\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} - \text{Diagonal})$ and

$$d\mu = \frac{2dV_x dV_y}{|x-y|^{2n-2}},$$

where V is the volume measure on the unit sphere $\mathbb{S}^{n-1}$. If we consider the set of geodesics $G(B)$ then if $(x,y) \in G(B)$, we have $|x-y| \simeq 2$. Also we let θ_x be the angle the ray xp from x to p makes with the hyperplane P, we let $G(B)_x = \{y \in \mathbb{S}^{n-1} \mid (x,y) \in G(B)\}$. Then the set $G(B)_x$ is the image of B under stereographic projection from x onto $\mathbb{S}^{n-1}$. We let $\overline{x}$ be the antipodal point to x on $\mathbb{S}^{n-1}$ and T be the tangent plane to $\mathbb{S}^{n-1}$ at $\overline{x}$. We let $E(B)_x$ be the stereographic projection of B from x onto the plane T (see Figure 2). As B is small, $E(B)_x$ is approximately an ellipse of approximately the same area as $G(B)_x$. By elementary euclidean geometry, we see that $E(B)_x$ has one axis which lies in the plane spanned by the ray px and its projection to P which has length approximately $2r\sin(\theta_x)$ (shown in Figure 2). All other axes of $E(B)_x$ have length approximately $2r$ as they correspond to the projection of the $(n-2)$–dimensional plane in P that is orthogonal to ray xp. Therefore

$$\mathrm{Vol}(E(B)_x) \simeq 2^{n-1}\sin(\theta_x)\,\mathrm{Vol}(B^s_{n-1}(r)),$$

where $B^s_k(r)$ is a k–dimensional ball of radius r in the unit sphere.

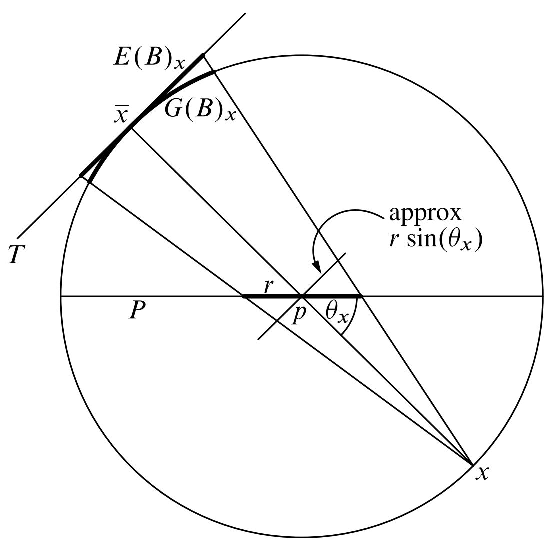

Figure 2: The set $G(B)_x$

As $G(B)_x$ and $E(B)_x$ are approximately the same area, we have

$$\mathrm{Vol}(G(B)_x) \simeq \mathrm{Vol}(E(B)_x) \simeq 2^{n-1}\sin(\theta_x)\,\mathrm{Vol}(B^s_{n-1}(r)).$$

Integrating we get

$$\mu(G(B)) = \int_{\mathbb{S}^{n-1}} 2\left(\int_{G(B)_x} \frac{dV_y}{|x-y|^{2n-2}}\right) dV_x \simeq \frac{\mathrm{Vol}(B^s_{n-1}(r))}{2^{n-2}} \int_{\mathbb{S}^{n-1}} \sin(\theta_x) dV_x.$$

Thus if A is the $(n-1)$–dimensional hyperbolic volume measure on P then

$$K_n = \frac{d\mu_P}{dA} = \frac{1}{2^{n-2}} \int_{\mathbb{S}^{n-1}} \sin(\theta_x) dV_x.$$

The set $S_t = \{x \mid \theta_x = t\}$ is an $(n-2)$–dimensional sphere of radius $|\cos(t)|$. Therefore, as $dV_x = d\theta dV_{S_t}$,

$$\begin{aligned} K_n &= \frac{\mathrm{Vol}(\mathbb{S}^{n-2})}{2^{n-2}} \int_0^{\pi} \sin(\theta)|\cos^{n-2}(\theta)| d\theta \\ &= \frac{\mathrm{Vol}(\mathbb{S}^{n-2})}{2^{n-1}} \int_0^{\pi/2} \sin(\theta)\cos^{n-2}(\theta) d\theta = \frac{\mathrm{Vol}(\mathbb{S}^{n-2})}{2^{n-1}(n-1)}. \end{aligned}$$

In terms of the Gamma function Γ we have

$$\mathrm{Vol}(\mathbb{S}^k) = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma(\frac{k+1}{2})}$$

giving

$$K_n = \frac{\pi^{\frac{n-1}{2}}}{2^{n-1}\Gamma(\frac{n+1}{2})}.$$

Note We have $K_2 = 1$.

8 Explicit integral formulae for $F_{n,k}$

In [3, Section 6] we derive a formula for L in the surface case. Using this we can write

$$F_{2,k}(l) = \frac{1}{2^{k-2}} \int_0^a \int_1^\infty \frac{\left(\log\left|\frac{y(y-a)(x-1)}{x(x-a)(y-1)}\right|\right)^k}{(y-x)^2} dx dy,$$

where $a = \mathrm{sech}^2(l/2)$.

In [5, Section 4] we consider the case $n > 2$ where we derive the explicit formula for L and reduce the integral formula to a triple integral via an elementary substitution. Using this we can also reduce the integral of $F_{n,k}$ to a triple integral of the form

$$F_{n,k}(l) = \frac{\mathrm{Vol}(\mathbb{S}^{n-2})\,\mathrm{Vol}(\mathbb{S}^{n-3})}{2^{k-2}} \int_0^1 \frac{r^{n-3}}{(1-r^2)^{\frac{n-2}{2}}}dr \int_{-1}^1 du \int_b^\infty \frac{\left(\log\left(\frac{(v^2-1)(u^2-b^2)}{(v^2-b^2)(u^2-1)}\right)\right)^k}{(v-u)^n} dv,$$

where $b = \sqrt{\frac{e^{2l}-r^2}{1-r^2}}$.

9 Moment generating function

The moment generating function of a random variable Y is the function $f_Y(t) = E[e^{tY}]$, where E is the expected value. We define the moment generating function for the measure M

$$\zeta_X^M(t) = M(e^{tx}) = \int_0^\infty e^{tx}\,dM = \int_{G(X)} e^{t\cdot L(g)}\,d\mu.$$

It follows from above that

$$\zeta_X^M(t) = \sum_{l\in L_X} F_n(t,l)$$

for some function F_n depending only on the dimension n. We similarly can define $\zeta_X^N(t) = N(e^{xt})$. Then it follows that the two functions are related by

$$\zeta_X^N(t) = \frac{d}{dt}\big(\zeta_X^M(t)\big).$$

A simple example, the ideal triangle We consider the case of X being an ideal triangle. In this case, since by Theorem 2.1(2) measure M has infinite mass, it is more natural to consider the measure N. In [4, Corollary 2] we show that

$$dN = \frac{12x^2}{\sinh^2 x}dx$$

and that

$$N_k(X) = \frac{3(k+2)!\zeta(k+2)}{2^{k-1}},$$

where ζ is the Riemann zeta function. In particular the average time to the boundary is

$$A(X) = \frac{N_1(X)}{2\,\mathrm{Vol}(T_1(X))} = \frac{9}{2\pi^2}\zeta(3).$$

It follows by integrating that

$$\zeta_X^N(t) = \int_0^\infty \frac{12x^2 e^{xt}}{\sinh^2 x} dx = 12\big(\zeta\big(2, 1 - \tfrac{t}{2}\big) + \tfrac{t}{2}\zeta\big(3, 1 - \tfrac{t}{2}\big)\big),$$

where $\zeta(s,t)$ is the Hurwitz zeta function

$$\zeta(s,t) = \sum_{k=0}^{\infty} \frac{1}{(k+t)^s}.$$

10 The surface case

For the surface case, the identities in Theorem 2.1 can be written in terms of polylogarithm functions.

Polylogarithms The k^{th} polylogarithm function Li_k is defined by the Taylor series

$$\mathrm{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^k}$$

for $|z| < 1$ and by analytic continuation to $\mathbb{C}$. In particular

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_1(z) = -\log(1-z).$$

Also

$$\mathrm{Li}_k'(z) = \frac{\mathrm{Li}_{k-1}(z)}{z} \qquad \text{giving} \qquad \mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}\, dt.$$

Also the functions Li_k are related to the Riemann ζ function by $\mathrm{Li}_k(1) = \zeta(k)$.

Below we describe some properties of the dilogarithm and trilogarithm function. They can all be found in the 1991 survey *Structural Properties of Polylogarithms* by L Lewin [9].

Dilogarithm The dilogarithm function $\mathrm{Li}_2(z)$ is given by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\, dt.$$

From the power series representation, it is easy to see that the dilogarithm function satisfies the functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \tfrac{1}{2}\,\mathrm{Li}_2(z^2).$$

Other functional relations of the dilogarithm can be best described by normalizing the dilogarithm function. The (extended) Rogers dilogarithm function (see Rogers [14]) is defined by

$$R(x)=\mathrm{Li}_2(x)+\tfrac{1}{2}\log|x|\log(1-x),\qquad x\le 1.$$

This function arises in calculating hyperbolic volume as the imaginary part of $R(z)$ is the volume of the hyperbolic tetrahedron with vertices having cross ratio z.

Also in terms of the Rogers function, various identities have nice form. Euler's reflection relations for the dilogarithm are given by

$$\begin{aligned} R(x)+R(1-x) &= R(1)=\frac{\pi^2}{6} && 0\le x\le 1,\\ R(-x)+R(-x^{-1}) &= 2R(-1)=-\frac{\pi^2}{6} && x>0. \end{aligned}\tag{4}$$

Also Landen's identity is

$$R\left(\frac{-x}{1-x}\right)=-R(x),\qquad 0<x<1,\tag{5}$$

and Abel's functional equation is

$$R(x)+R(y)=R(xy)+R\left(\frac{x(1-y)}{1-xy}\right)+R\left(\frac{y(1-x)}{1-xy}\right).\tag{6}$$

In [3], we showed that the orthospectra of a hyperbolic surface satisfies the following generalized orthospectrum identity.

Theorem 10.1 (Bridgeman [3, Length spectrum identity theorem]) *Let S be a finite area hyperbolic surface with non-empty totally geodesic boundary and C_S boundary cusps. Then*

$$\sum_{l\in L_S} R\big(\operatorname{sech}^2 \tfrac{l}{2}\big)=\frac{\pi^2(6|\chi(S)|-C_S)}{12}.$$

If S is a surface of finite type, we let $T(S)$ be the Teichmüller space of finite area hyperbolic surfaces on S with geodesic boundary components. The above formula can be considered as an identity on $T(S)$.

By Parlier [13], Papadopoulos and Théret [12] and Goldman, Margulis and Minsky [8], there exist distinct elements $X,Y\in T(S)$ such that $L_\alpha(X)<L_\alpha(Y)$ for all geodesic length functions $L_\alpha\colon T(S)\to[0,\infty)$. We have the following Corollary showing that this cannot happen for orthogedesics.

Corollary 10.2 *There does not exist $X, Y \in T(S)$, $X \neq Y$ such that $L_\alpha(X) < L_\alpha(Y)$ for all orthogeodesic length functions $L_\alpha\colon T(S) \to [0,\infty)$.*

Proof As the Rogers dilogarithm is strictly monotonic, it follows immediately from the identity above. □

We note that as the Basmajian identity depends on the boundary lengths, we cannot derive this fact using it.

Trilogarithm By definition, the trilogarithm function is given by

$$\mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\, dt.$$

The trilogarithm also satisfies a number of identities:

$$\mathrm{Li}_3(z) + \mathrm{Li}_3(-z) = \tfrac{1}{4}\mathrm{Li}_3(z^2) \tag{7}$$

$$\mathrm{Li}_3(-z) - \mathrm{Li}_3(-z^{-1}) = -\tfrac{1}{6}\log^3(z) - \tfrac{\pi^2}{6}\log(z) \tag{8}$$

$$\begin{aligned}\mathrm{Li}_3(z) + \mathrm{Li}_3(1-z) - \mathrm{Li}_3(1-z^{-1}) = \zeta(3) + \tfrac{1}{6}\log^3(z) + \tfrac{\pi^2}{6}\log(z)\\ - \tfrac{1}{2}\log^2(z)\log(1-z)\end{aligned} \tag{9}$$

If l_i is an ortholength of S, we define

$$a_i = \mathrm{sech}^2 \frac{l_i}{2}.$$

We will often use the spectrum $\{a_i\}$ instead of $\{l_i\}$. In the paper [3], we studied the measure N and derived the following:

Theorem 10.3 (Bridgeman [3, Main theorem]) *There exists a smooth function $\rho\colon \mathbb{R}_+ \times (0,1) \to \mathbb{R}_+$ such that if S is a finite area surface with totally geodesic boundary and C_S boundary cusps and orthospectrum $\{l_i\}$ then*

$$dN = \rho_S(x)dx = \left(\frac{4C_S x^2}{\sinh^2(x)} + \sum_{a_i} \rho(x,a_i)\right)dx,$$

where $a_i = \mathrm{sech}^2 \frac{l_i}{2}$. Furthermore if $\phi\colon [0,\infty) \to \mathbb{R}$ is a smooth function with compact support then

$$\int \phi(x)\rho(x,a)dx = \int_0^a \int_1^\infty \frac{4\phi(L_a(x,y))L_a(x,y)}{(y-x)^2} dx\,dy,$$

where

$$L_a(x,y) = \tfrac{1}{2}\log\left|\frac{y(y-a)(x-1)}{x(x-a)(y-1)}\right|.$$

The moment identities for surface If we apply the above Theorem to the function $\phi(x) = 1$ then we recover the identity in Theorem 10.1.

To find $M_k(S)$, we note that $M_k(S) = N_{k-1}(S)$. Therefore by the above, we let $\phi(x) = x^{k-1}$ to get

$$M_k(S) = N_{k-1}(S) = \int_0^\infty x^{k-1}\rho_S(x)dx$$
$$= \sum_{a_i}\left(\int_0^{a_i}\int_1^\infty \frac{4L_{a_i}^k(x,y)}{(y-x)^2}dxdy\right) + 4C_S\left(\int_0^\infty \frac{x^{k+1}dx}{\sinh^2(x)}\right).$$

We then define

$$F_k(a) = \int_0^a\int_1^\infty \frac{4L_a(x,y)^k}{(y-x)^2}dxdy = \frac{1}{2^{k-2}}\int_0^a\int_1^\infty \frac{\log\left|\frac{y(y-a)(x-1)}{x(x-a)(y-1)}\right|^k}{(y-x)^2}dxdy.$$

Integrating we have

$$\int_0^\infty \frac{x^{k+1}dx}{\sinh^2(x)} = \frac{(k+1)!\zeta(k+1)}{2^k}.$$

Therefore we obtain

$$M_k(S) = \left(\sum_{l\in L_S} F_k\left(\operatorname{sech}^2\frac{l}{2}\right) + C_S\frac{(k+1)!\zeta(k+1)}{2^{k-2}}\right).$$

We note that as $\zeta(1) = \infty$, if S has boundary cusps then $M_0(S)$ is not finite. This corresponds to the fact that $M_0(S) = \mathrm{Length}(\partial S)$ which is infinite in the case of boundary cusps.

Functions F_0, F_1 are given in terms of simple logarithms and dilogarithms respectively. An induction argument shows that F_k can be written as a sum of polylogarithm functions of order at most $k+1$. We will calculate an explicit formula for F_2 in terms of trilogarithms in the next section. This will give us the formula for the average hitting time $A(S)$ for geodesic flow described in Theorem 2.2.

Proposition 10.4 *The function F_2 is given by the formula*

$$F_2(a) = -12\zeta(3) - \tfrac{4\pi^2}{3}\log(1-a) + 6\log^2(1-a)\log(a) - 4\log(1-a)\log^2(a)$$
$$-8\log\left(\frac{a^2}{1-a}\right)\mathrm{Li}_2(a) + 24\,\mathrm{Li}_3(a) + 12\,\mathrm{Li}_3(1-a).$$

11 A somewhat brutal calculation

We will now obtain an explicit formula for the average hitting time in the surface case in terms of sums of polylogarithms evaluated at ortholengths. We let $F = F_2$ given by the above integral formula. Then

$$F(a) = \int_0^a \int_1^\infty \frac{\log \left| \frac{y(y-a)(x-1)}{x(x-a)(y-1)} \right|^2}{(y-x)^2} dx dy.$$

Using Mathematica to calculate the indefinite integral first, gives 7 858 polylogarithm terms which then need to be evaluated at the 4 limits to give a final total of approximately 30 000 terms. Also the terms must be grouped so that evaluation gives a finite limit. As this seems a daunting task, we do the calculation directly using hyperbolic relations to simplify as we go along. The calculation is somewhat tedious but the final answer surprisingly short.

For the reader who would rather skip the long and tedious calculation, evidence for its validity is given by Figure 3, which is a plot of the difference between the polylogarithm formula for $F(a)$ and its values using numerical integration. As can be seen from the plot, the difference is less than 10^{-6}, indicating they are the same function.

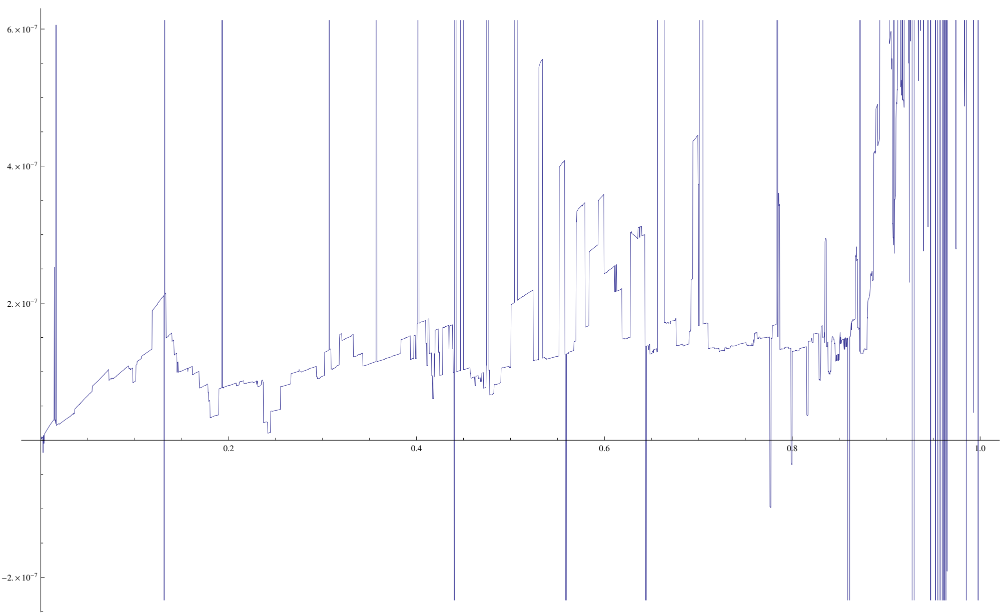

Figure 3: Difference between numerical integration of F and polylogarithm formula

We have that for $a \in (0,1)$

$$F(a) = \int_0^a \int_1^\infty \frac{\log\left(\frac{y(y-a)(y-1)}{x(x-a)(y-1)}\right)^2}{(y-x)^2} dydx.$$

Decomposing into cross-ratios, we have

$$\begin{aligned} F(a) &= \int_0^a \int_1^\infty \frac{\left(\log\left|\frac{y(x-1)}{x(y-1)}\right| + \log\left|\frac{y-a}{x-a}\right|\right)^2}{(y-x)^2} dydx \\ &= \int_0^a \int_1^\infty \frac{\log\left|\frac{y(x-1)}{x(y-1)}\right|^2 + 2\log\left|\frac{y(x-1)}{x(y-1)}\right| \log\left|\frac{y-a}{x-a}\right| + \log\left|\frac{y-a}{x-a}\right|^2}{(y-x)^2} dydx. \end{aligned}$$

Under the Möbius transformation $m(z) = a/z$, we let $X = m(x), Y = m(y)$, then by invariance of cross ratios,

$$\begin{aligned} \frac{y(x-1)}{x(y-1)} = \frac{(y-0)(x-1)}{(x-0)(y-1)} &= \frac{(m(y)-m(0))(m(x)-m(1))}{(m(x)-m(0))(m(y)-m(1))} \\ &= \frac{(Y-\infty)(X-a)}{(X-\infty)(Y-a)} = \frac{X-a}{Y-a}. \end{aligned}$$

Thus

$$\int_0^a \int_1^\infty \frac{\log\left|\frac{y(x-1)}{x(y-1)}\right|^2}{(y-x)^2} dydx = \int_\infty^1 \int_a^0 \frac{\log\left|\frac{X-a}{Y-a}\right|^2}{(Y-X)^2} dYdX = \int_0^a \int_1^\infty \frac{\log\left|\frac{y-a}{x-a}\right|^2}{(y-x)^2} dydx$$

and

$$F(a) = 2\int_0^a \int_1^\infty \frac{\log\left|\frac{y-a}{x-a}\right|^2}{(y-x)^2} dydx + 2\int_0^a \int_1^\infty \frac{\log\left|\frac{y(x-1)}{x(y-1)}\right| . \log\left|\frac{y-a}{x-a}\right|}{(y-x)^2} dydx.$$

We write this as $F(a) = 2I_1(a) + 2I_2(a)$, where I_1, I_2 are the above integrals.

In order to calculate the above integrals we will need the following integral identities:

$$\int \frac{\log(x)}{x-a} dx = \log(x)\log(1-x/a) + \mathrm{Li}_2(x/a) \tag{10}$$

$$\int \frac{\log(x)^2}{x-a} dx = \log(x)^2\log(1-x/a) + 2\log(x)\,\mathrm{Li}_2(x/a) - 2\,\mathrm{Li}_3(x/a) \tag{11}$$

$$\int \frac{\log(x)\log(x-a)}{x} dx = \tfrac{1}{2}\log(x)^2\log(a) - \log(x)\,\mathrm{Li}_2(x/a) + \mathrm{Li}_3(x/a) \tag{12}$$

11.1 Integral I_1

Lemma 11.1

$$I_1(a) = -\log(1-a)\log^2(a) + \log^3(1-a) - 4\log\left(\frac{a}{1-a}\right)\mathrm{Li}_2(a) - 6\,\mathrm{Li}_3\left(\frac{a}{a-1}\right)$$

Proof We decompose I_1 to obtain

$$I_1 = \int_0^a \int_1^\infty \frac{\log(y-a)^2 - 2\log(y-a)\log(a-x) + \log(a-x)^2}{(y-x)^2}\,dy\,dx = J_1 - 2J_2 + J_3.$$

Integral J_1 We have

$$J_1 = \int_1^\infty \log(y-a)^2\left(\int_0^a \frac{dx}{(y-x)^2}\right)dy = \int_1^\infty \log(y-a)^2\left(\frac{1}{y-a} - \frac{1}{y}\right)dy.$$

Then

$$\int \frac{\log(y-a)^2}{y-a}\,dy = \frac{\log(y-a)^3}{3}$$

Also by the integral equations above,

$$\int \frac{\log(y-a)^2}{y}\,dy = \log\left(\frac{y}{a}\right)\log(y-a)^2 + 2\log(y-a)\,\mathrm{Li}_2\left(1-\frac{y}{a}\right) - 2\,\mathrm{Li}_3\left(1-\frac{y}{a}\right).$$

Thus

$$\begin{aligned} J_1 = -\tfrac{1}{3}\log(1-a)^3 + \tfrac{\pi^2}{3}\log(a) + \tfrac{1}{3}\log(a)^3 + \log\left(\tfrac{1}{a}\right)\log(1-a)^2 \\ + 2\log(1-a)\,\mathrm{Li}_2\left(1-\tfrac{1}{a}\right) - 2\,\mathrm{Li}_3\left(1-\tfrac{1}{a}\right). \end{aligned}$$

Integral J_2 By parts we have

$$\begin{aligned} J_2 &= \int_0^a \log(a-x)\left(\int_1^\infty \frac{\log(y-a)}{(y-x)^2}\,dy\right)dx \\ &= \int_0^a \log(a-x)\left(\frac{\log(1-a)}{1-x} - \frac{\log(1-a)}{a-x} + \frac{\log(1-x)}{a-x}\right)dx. \end{aligned}$$

As above we have

$$\begin{aligned} \int_0^a \frac{\log(a-x)}{1-x}\,dx &= \left(-\log(a-x)\log\left(\frac{1-x}{1-a}\right) - \mathrm{Li}_2\left(\frac{a-x}{a-1}\right)\right)\Big|_0^a \\ &= -\log(a)\log(1-a) + \mathrm{Li}_2\left(\frac{a}{a-1}\right). \end{aligned}$$

Also

$$\int \frac{\log(a-x)}{a-x}dx = -\tfrac{1}{2}\log(a-x)^2,$$

$$\int \frac{\log(a-x)\log(1-x)}{a-x}dx = -\tfrac{1}{2}\log(a-x)^2\log(1-a) + \log(a-x)\operatorname{Li}_2\left(\frac{a-x}{a-1}\right) - \operatorname{Li}_3\left(\frac{a-x}{a-1}\right).$$

Combining, we get

$$\begin{aligned}\int_0^a \frac{\log(a-x)(\log(1-x)-\log(1-a))}{a-x}dx &= \log(a-x)\operatorname{Li}_2\left(\frac{a-x}{a-1}\right) - \operatorname{Li}_3\left(\frac{a-x}{a-1}\right)\Big|_0^a \\ &= -\log(a)\operatorname{Li}_2\left(\frac{a}{a-1}\right) + \operatorname{Li}_3\left(\frac{a}{a-1}\right).\end{aligned}$$

Thus

$$J_2 = -\log(a)\log(1-a)^2 - \log\left(\frac{a}{1-a}\right)\operatorname{Li}_2\left(\frac{a}{a-1}\right) + \operatorname{Li}_3\left(\frac{a}{a-1}\right).$$

Integral J_3 Similarly to J_1, we have

$$J_3 = \int_0^a \log(a-x)^2\left(\int_1^\infty \frac{1}{(y-x)^2}dy\right)dx = \int_0^a \frac{\log(a-x)^2}{1-x}dx.$$

Therefore

$$\begin{aligned}J_3 = &-\log(a-x)^2\log\left(\frac{1-x}{1-a}\right) - 2\log(a-x)\operatorname{Li}_2\left(\frac{a-x}{a-1}\right) + 2\operatorname{Li}_3\left(\frac{a-x}{a-1}\right)\Big|_0^a \\ &-\log(1-a)\log(a)^2 + 2\log(a)\operatorname{Li}_2\left(\frac{a}{a-1}\right) - 2\operatorname{Li}_3\left(\frac{a}{a-1}\right).\end{aligned}$$

Thus we have the formula for $I_1 = J_1 - 2J_2 + J_3$, giving

$$\begin{aligned}I_1 = \Big(&-\tfrac{1}{3}\log(1-a)^3 + \tfrac{\pi^2}{3}\log(a) + \tfrac{1}{3}\log(a)^3 - \log(a)\log(1-a)^2 \\ &+ 2\log(1-a)\operatorname{Li}_2\left(1-\frac{1}{a}\right) - 2\operatorname{Li}_3\left(1-\frac{1}{a}\right)\Big) \\ &-2\left(-\log(a)\log(1-a)^2 - \log\left(\frac{a}{1-a}\right)\operatorname{Li}_2\left(\frac{a}{a-1}\right) + \operatorname{Li}_3\left(\frac{a}{a-1}\right)\right) \\ &+\left(-\log(1-a)\log(a)^2 + 2\log(a)\operatorname{Li}_2\left(\frac{a}{a-1}\right) - 2\operatorname{Li}_3\left(\frac{a}{a-1}\right)\right).\end{aligned}$$

Using the identities (4), (8), we get

$$I_1(a) = -\log(1-a)\log\left(\frac{a}{1-a}\right)^2 + 4\log\left(\frac{a}{1-a}\right)\mathrm{Li}_2\left(\frac{a}{a-1}\right) - 6\,\mathrm{Li}_3\left(\frac{a}{a-1}\right).$$

Simplifying further we also get

$$I_1(a) = -\log(1-a)\log^2(a) + \log^3(1-a) - 4\log\left(\frac{a}{1-a}\right)\mathrm{Li}_2(a) - 6\,\mathrm{Li}_3\left(\frac{a}{a-1}\right)$$

as required. □

11.2 Integral I_2

Lemma 11.2

$$I_2(a) = 2\zeta(3) + \tfrac{2\pi^2}{3}\log(1-a) + \tfrac{1}{3}\log^3(1-a) - \log^2(1-a)\log(a) - \log^2(a)\log(1-a) \\ -4\log(a)\,\mathrm{Li}_2(a) + 4\,\mathrm{Li}_3(a) - 2\,\mathrm{Li}_3\left(\frac{-a}{1-a}\right) - 2\,\mathrm{Li}_3(1-a)$$

Proof We have

$$I_2 = \int_0^a \int_1^\infty \frac{\log\left|\frac{y(x-1)}{x(y-1)}\right| \log\left|\frac{y-a}{x-a}\right|}{(y-x)^2} dy\,dx.$$

Again we decompose into integrals

$$J_1 = \int_0^a\int_1^\infty \frac{\log\left(\frac{y}{y-1}\right)\log(y-a)}{(y-x)^2}dy\,dx, \quad J_2 = \int_0^a\int_1^\infty \frac{\log\left(\frac{1-x}{x}\right)\log(y-a)}{(y-x)^2}dy\,dx,$$

$$J_3 = \int_0^a\int_1^\infty \frac{\log\left(\frac{x}{1-x}\right)\log(a-x)}{(y-x)^2}dy\,dx, \quad J_4 = \int_0^a\int_1^\infty \frac{\log\left(\frac{y-1}{y}\right)\log(a-x)}{(y-x)^2}dy\,dx.$$

As before we can integrate to get

$$J_1 = \int_1^\infty \log\left(\frac{y}{y-1}\right)\log(y-a)\left(\frac{1}{y-a} - \frac{1}{y}\right)dy \quad \text{and} \quad J_3 = \int_0^a \frac{\log\left(\frac{x}{1-x}\right)\log(a-x)}{1-x}dx.$$

We integrate in J_2, J_3 the simple factor to get

$$\begin{aligned} J_2 &= \int_0^a \log\left(\frac{1-x}{x}\right)\left(\int_1^\infty \frac{\log(y-a)}{(y-x)^2}dy\right)dx \\ &= \int_0^a \log\left(\frac{1-x}{x}\right)\left(\frac{\log(1-x)-\log(1-a)}{a-x} + \frac{\log(1-a)}{1-x}\right)dx. \end{aligned}$$

Combining J_2, J_3 we get

$$J_2 + J_3 = \int_0^a \log\left(\frac{1-x}{x}\right)\left(\frac{\log\left(\frac{1-x}{1-a}\right)}{a-x} - \frac{\log\left(\frac{a-x}{1-a}\right)}{1-x}\right)dx.$$

The Rogers normalized dilogarithm is given by $R(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log|x|\log(1-x)$ for $x < 1$. We note that

$$R'(x) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log|x|}{1-x}\right).$$

Thus we let $R_1(x) = R(\frac{a-x}{1-x})$ and note that

$$R_1'(x) = \frac{1}{2}\left(\frac{\log\left(\frac{a-x}{1-a}\right)}{1-x} - \frac{\log\left(\frac{1-x}{1-a}\right)}{a-x}\right).$$

Thus

$$J_2 + J_3 = 2\int_0^a \log\left(\frac{1-x}{x}\right) R_1'(x)dx.$$

Considering J_4 we have similarly

$$\begin{aligned} J_4 &= \int_1^\infty \log\left(\frac{y-1}{y}\right)\left(\int_0^a \frac{\log(a-x)}{(y-x)^2}dx\right)dy \\ &= \int_1^\infty \log\left(\frac{y-1}{y}\right)\left(\frac{\log(y-a)-\log(y)+\log(a)}{y-a} - \frac{\log(a)}{y}\right)dy. \end{aligned}$$

We then have

$$J_1 + J_4 = \int_1^\infty \log\left(\frac{y-1}{y}\right)\left(\frac{\log\left(\frac{a}{y}\right)}{y-a} + \frac{\log\left(\frac{y-a}{a}\right)}{y}\right)dy.$$

We now let $R_2(y) = R(a/y)$, then

$$R_2'(y) = \frac{1}{2}\left(\frac{\log\left(\frac{a}{y}\right)}{y-a} + \frac{\log\left(\frac{y-a}{a}\right)}{y}\right),$$

giving

$$J_1 + J_4 = 2\int_1^\infty \log\left(\frac{y-1}{y}\right) R_2'(y)dy.$$

Letting $x = a/y$ we have

$$J_1 + J_4 = 2\int_a^0 \log\left(\frac{a-x}{a}\right) R'(x)dx = 2\log(a)R(a) - 2\int_0^a \log(a-x)R'(x)dx.$$

Similarly let $u=(a-x)/(1-x)$ then $x=(a-u)/(1-u)$ and

$$J_2+J_3=2\int_0^a \log\left(\frac{1-a}{a-u}\right)R'(u)\,du=2\log(1-a)R(a)-2\int_0^a \log(a-x)R'(x)\,dx,$$

giving

$$\begin{aligned}I_2&=2\log(a(1-a))R(a)-4\int_0^a \log(a-x)R'(x)\,dx\\&=2\log(a(1-a))R(a)+2\int_0^a \log(a-x)\left(\frac{\log(1-x)}{x}+\frac{\log(x)}{1-x}\right)dx.\end{aligned}$$

We note the formula

$$\begin{aligned}G(x,a)&=\int \frac{\log(a-x)\log(1-x)}{x}\,dx\\&=\log(1-x)\log\left(\frac{a-x}{a}\right)\log\left(\frac{x}{a}\right)+\log(a)\log(1-x)\log(x)\\&\quad+\tfrac12\log(a)\log^2(1-x)+\log(a(1-x))\,\mathrm{Li}_2(1-x)\\&\quad+\log\left(\frac{a-x}{a}\right)\mathrm{Li}_2\left(\frac{a-x}{a}\right)+\log\left(\frac{a-x}{a(1-x)}\right)\left(\mathrm{Li}_2\left(\frac{a-x}{1-x}\right)-\mathrm{Li}_2\left(\frac{a-x}{a(1-x)}\right)\right)\\&\quad-\mathrm{Li}_3(1-x)-\mathrm{Li}_3\left(\frac{a-x}{1-x}\right)-\mathrm{Li}_3\left(\frac{a-x}{a}\right)+\mathrm{Li}_3\left(\frac{a-x}{a(1-x)}\right).\end{aligned}$$

Thus taking limits we have

$$\begin{aligned}\int_0^a \frac{\log(a-x)\log(1-x)}{x}\,dx&=\zeta(3)-\tfrac{\pi^2}{6}\log(a)+\tfrac12\log(a)\log^2(1-a)\\&\quad+\log(1-a)\log^2(a)+\log(a(1-a))\,\mathrm{Li}_2(1-a)\\&\quad+\mathrm{Li}_3(a)-\mathrm{Li}_3(1-a).\end{aligned}$$

From the above

$$\begin{aligned}H(x,a)&=\int \frac{\log(a-x)\log(x)}{1-x}\,dx\\&=-G(1-a,1-x)\\&=-\log(x)\log\left(\frac{a-x}{1-a}\right)\log\left(\frac{1-x}{1-a}\right)-\log(1-a)\log(1-x)\log(x)\\&\quad-\tfrac12\log(1-a)\log^2(x)-\log((1-a)x)\,\mathrm{Li}_2(x)-\log\left(\frac{a-x}{1-a}\right)\mathrm{Li}_2\left(\frac{x-a}{1-a}\right)\end{aligned}$$

$$-\log\left(\frac{a-x}{(1-a)x}\right)\left(\mathrm{Li}_2\left(\frac{x-a}{x}\right)-\mathrm{Li}_2\left(\frac{x-a}{(1-a)x}\right)\right)$$
$$+\mathrm{Li}_3(x)+\mathrm{Li}_3\left(\frac{x-a}{x}\right)+\mathrm{Li}_3\left(\frac{x-a}{1-a}\right)-\mathrm{Li}_3\left(\frac{x-a}{(1-a)x}\right).$$

Thus taking limits we have

$$\int_0^a \frac{\log(a-x)\log(x)}{1-x}dx = \tfrac{\pi^2}{6}\log(1-a)-\tfrac{1}{3}\log^3(1-a)-\log(1-a)\log^2(a)$$
$$-\log((1-a)a)\,\mathrm{Li}_2(a)+\log\left(\frac{a}{1-a}\right)\mathrm{Li}_2\left(\frac{-a}{1-a}\right)$$
$$+\mathrm{Li}_3(a)-\mathrm{Li}_3\left(\frac{-a}{1-a}\right).$$

We now combine to obtain

$$I_2(a) = 2\zeta(3)+\tfrac{2\pi^2}{3}\log(1-a)+\tfrac{1}{3}\log^3(1-a)-\log^2(1-a)\log(a)-\log^2(a)\log(1-a)$$
$$-4\log(a)\,\mathrm{Li}_2(a)+4\,\mathrm{Li}_3(a)-2\,\mathrm{Li}_3\left(\frac{-a}{1-a}\right)-2\,\mathrm{Li}_3(1-a)$$

which completes the proof. □

Finally we combine I_1 and I_2 to get

$$F(a)=4\zeta(3)+\tfrac{4\pi^2}{3}\log(1-a)+\tfrac{8}{3}\log^3(1-a)-4\log(1-a)\log^2(a)-2\log^2(1-a)\log(a)$$
$$-8\log\left(\frac{a^2}{1-a}\right)\mathrm{Li}_2(a)+8\,\mathrm{Li}_3(a)-4\,\mathrm{Li}_3(1-a)-16\,\mathrm{Li}_3\left(\frac{-a}{1-a}\right).$$

Using the identity (9) we get

$$F(a) = -12\zeta(3)-\frac{4\pi^2}{3}\log(1-a)+6\log^2(1-a)\log(a)-4\log(1-a)\log^2(a)$$
$$-8\log\left(\frac{a^2}{1-a}\right)\mathrm{Li}_2(a)+24\,\mathrm{Li}_3(a)+12\,\mathrm{Li}_3(1-a).$$

The function has boundary values $F(0)=0$ and $F(1)=12\zeta(3)$ and is maximized at $a=0.754493$ with value 17.9804.

Figure 4 shows a graph of F.

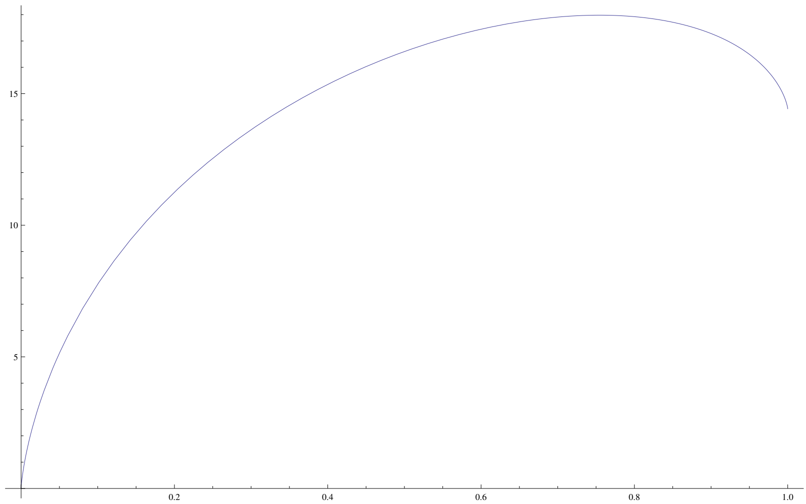

Figure 4: The function $F(x)$

References

[1] **A Basmajian**, *The orthogonal spectrum of a hyperbolic manifold*, Amer. J. Math. 115 (1993) 1139–1159 MR1246187

[2] **F Bonahon**, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. 92 (1988) 139–162 MR931208

[3] **M Bridgeman**, *Orthospectra of geodesic laminations and dilogarithm identities on moduli space*, Geom. Topol. 15 (2011) 707–733 MR2800364

[4] **M Bridgeman**, **D Dumas**, *Distribution of intersection lengths of a random geodesic with a geodesic lamination*, Ergodic Theory Dynam. Systems 27 (2007) 1055–1072 MR2342965

[5] **M Bridgeman**, **J Kahn**, *Hyperbolic volume of manifolds with geodesic boundary and orthospectra*, Geom. Funct. Anal. 20 (2010) 1210–1230 MR2746952

[6] **D Calegari**, *Chimneys, leopard spots and the identities of Basmajian and Bridgeman*, Algebr. Geom. Topol. 10 (2010) 1857–1863 MR2684984

[7] **D Calegari**, *Bridgeman's orthospectrum identity*, Topology Proc. 38 (2011) 173–179 MR2680162

[8] **W Goldman**, **G Margulis**, **Y Minsky**, *Complete flat Lorentz* 3*–manifolds and laminations on hyperbolic surfaces*, in preparation

[9] **L Lewin** (editor), *Structural properties of polylogarithms*, Mathematical Surveys and Monographs 37, Amer. Math. Soc. (1991) MR1148371

[10] **B Maskit**, *Kleinian groups*, Grundl. Math. Wissen. 287, Springer, Berlin (1988) MR959135

[11] **P J Nicholls**, *The ergodic theory of discrete groups*, London Mathematical Society Lecture Note Series 143, Cambridge Univ. Press (1989) MR1041575

[12] **A Papadopoulos**, **G Théret**, *Shortening all the simple closed geodesics on surfaces with boundary*, Proc. Amer. Math. Soc. 138 (2010) 1775–1784 MR2587462

[13] **H Parlier**, *Lengths of geodesics on Riemann surfaces with boundary*, Ann. Acad. Sci. Fenn. Math. 30 (2005) 227–236 MR2173363

[14] **L J Rogers**, *On function sum theorems connected with the series formula*, Proc. London Math. Soc. S2-4 (1907) 169 MR1576083

[15] **D Sullivan**, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979) 171–202 MR556586

Department of Mathematics, Boston College
Chestnut Hill, MA 02467, USA

Department of Mathematics, National University of Singapore
2 Science Drive, Singapore 117543, Singapore

bridgem@bc.edu, mattansp@nus.edu.sg

http://www2.bc.edu/martin-bridgeman/

Proposed: Danny Calegari
Seconded: Benson Farb, Jean-Pierre Otal

Received: 6 March 2013
Revised: 27 June 2013