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Distortion elements for surface homeomorphisms

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Let S be a compact orientable surface and f be an element of the group $\mathrm{Homeo}_0(S)$ of homeomorphisms of S isotopic to the identity. Denote by $\widetilde{f}$ a lift of f to the universal cover $\widetilde{S}$ of S. In this article, the following result is proved: If there exists a fundamental domain D of the covering $\widetilde{S} \to S$ such that

$$\lim_{n \to +\infty} \frac{1}{n} d_n \log(d_n) = 0,$$

where d_n is the diameter of $\widetilde{f}^n(D)$, then the homeomorphism f is a distortion element of the group $\mathrm{Homeo}_0(S)$.

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1 Introduction

Given a compact manifold M, we denote by $\mathrm{Diff}^r_0(M)$ the identity component of the group of C^r–diffeomorphisms of M. A way to understand this group is to try to describe its subgroups. In other words, given a group G, does there exist an injective group morphism from the group G to the group $\mathrm{Diff}^r_0(M)$? In case the answer is positive, one can try to describe the group morphisms from the group G to the group $\mathrm{Diff}^r_0(M)$ (ideally up to conjugacy, but this is often an unattainable goal).

The concept of distortion element allows one to obtain rigidity results on group morphisms from G to $\mathrm{Diff}^r_0(M)$. It will provide some very partial answers to these questions. Here is the definition. Remember that a group G is *finitely generated* if there exists a finite generating set $\mathcal{G}$: any element g in this group is a product of elements of $\mathcal{G}$ and their inverses, $g = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon}$, where the s_i are elements of $\mathcal{G}$ and the ϵ_i are equal to $+1$ or -1. The smallest integer n in such a decomposition is denoted by $l_{\mathcal{G}}(g)$. The map $l_{\mathcal{G}}$ is invariant under inverses and satisfies the triangle inequality $l_{\mathcal{G}}(gh) \leq l_{\mathcal{G}}(g) + l_{\mathcal{G}}(h)$. Therefore, for any element g in the group G, the sequence $(l_{\mathcal{G}}(g^n))_{n \geq 0}$ is sub-additive, so the sequence $(l_{\mathcal{G}}(g^n)/n)_n$ converges. When the limit of this sequence is zero, the element g is said to be *distorted* or a *distortion element* in the group G. Notice that this notion does not depend on the generating set $\mathcal{G}$. In

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other words, this concept is intrinsic to the group G. The notion extends to the case where the group G is not finitely generated. In this case, an element g of the group G is distorted if it belongs to a finitely generated subgroup of G in which it is distorted. The main interest of the notion of distortion is the following rigidity property: for any group morphism $\varphi\colon G \to H$, if an element g is distorted in the group G, then its image under φ is also distorted. Suppose that the group H does not contain any distortion element other than the identity element in H and that the group G contains a distortion element different from the identity. Then such a group morphism cannot be an embedding: the group G is not a subgroup of H.

Let us give now some simple examples of distortion elements.

(1) In any group G, the torsion elements, ie, those of finite order, are distorted.

(2) In free groups and free abelian groups, the only distorted element is the identity element.

(3) The simplest examples of groups which contain a distortion element which is not a torsion element are the Baumslag–Solitar groups. These groups have the following presentation:

$$BS(1,p) = \langle a, b \mid bab^{-1} = a^p \rangle$$

Then, for any integer n, the relation $b^n a b^{-n} = a^{p^n}$ holds. Taking $\mathcal{G} = \{a, b\}$ as a generating set for this group, we have $l_{\mathcal{G}}(a^{p^n}) \leq 2n+1$: the element a is distorted in the group $BS(1,p)$ if $|p| \geq 2$.

(4) A group G is said to be *nilpotent* if the sequence of subgroups $(G_n)_{n\geq 0}$ of G defined by $G_0 = G$ and $G_{n+1} = [G_n, G]$ (this is the subgroup generated by elements of the form $[g_n, g] = g_n g g_n^{-1} g^{-1}$, where $g_n \in G_n$ and $g \in G$) stabilizes and is equal to $\{1_G\}$ for a sufficiently large n. A typical example of a nilpotent group is the Heisenberg group, which is the group of upper triangular matrices whose diagonal entries are 1 and other entries are integers. In a nilpotent non-abelian group N, one can always find three distinct elements a, b and c different from the identity such that $[a,b] = c$ and the element c commutes with a and b. In this case, we have $c^{n^2} = [a^n, b^n]$ so that, in the subgroup generated by a and b (and also in N), the element c is distorted: $l_{\{a,b\}}(c^{n^2}) \leq 4n$.

(5) A general theorem by Lubotzky, Mozes and Raghunathan implies that there exist distortion elements (and even elements with a logarithmic growth) in some lattices of higher rank Lie groups, for instance in the group $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. In the case of the group $\mathrm{SL}_n(\mathbb{Z})$, one can even find a generating set consisting of distortion elements.

(6) In mapping class groups (see Farb, Lubotzky and Minsky [6]) and in the group of interval exchange transformations (see Novak [22]), any distorted element is a torsion element.

Let us consider now the case of diffeomorphisms groups. The following theorem has led to progress on Zimmer's Conjecture. Let us denote by S a compact surface without boundary endowed with a probability measure *area* with full support. Let us denote by $\mathrm{Diff}^1(S, area)$ the group of C^1–diffeomorphisms of the surface S that preserve the measure *area*. Then we have the following statement:

Theorem 1 (Polterovich [23], Franks and Handel [9]) *If the genus of the surface S is greater than one, any distortion element in the group $\mathrm{Diff}^1(S, area)$ is a torsion element.*

As nilpotent groups and $\mathrm{SL}_n(\mathbb{Z})$ have some non-torsion distortion elements, they are not subgroups of the group $\mathrm{Diff}^1(S, area)$. A natural question now is whether these theorems can be generalized in the case of more general diffeomorphisms or homeomorphisms groups (with no area-preservation hypothesis). Unfortunately, one may find lots of distorted elements in those cases. The most striking example of this phenomenon is the following theorem by Calegari and Freedman concerning the group of homeomorphisms of the d–dimensional sphere $\mathbb{S}^d$:

Theorem 2 (Calegari and Freedman [5]) *For any integer $d \geq 1$, every element in the group $\mathrm{Homeo}_0(\mathbb{S}^d)$ is distorted.*

In the case of a higher regularity, Avila proved in [2] that any element in $\mathrm{Diff}^\infty_0(\mathbb{S}^1)$ for which arbitrarily large iterates are arbitrarily close to the identity in the C^∞–topology (such an element will be said to be *recurrent*) is distorted in the group $\mathrm{Diff}^\infty_0(\mathbb{S}^1)$. We obtained the following result (see Militon [20]):

Theorem 3 *For any compact manifold M without boundary, any recurrent element in $\mathrm{Diff}^\infty_0(M)$ is distorted in this group.*

For instance, irrational rotations of the circle or of the 2–dimensional sphere or translations of the d–dimensional torus are distorted. More generally, take any manifold that admits a non-trivial C^∞ circle action. Then there exist non-trivial distortion elements in the group of C^∞–diffeomorphisms of this manifold. Notice that, thanks to the Anosov–Katok method (see Herman [13], and Fathi and Herman [7]), we can build recurrent elements in the case of the sphere or of the 2–dimensional torus that are not conjugate to a rotation.

Anyway, we could not hope for a result analogous to the theorem by Polterovich, Franks and Handel, at least in the C^1 category. Indeed, we will see that the Baumslag–Solitar group $BS(1,2)$ embeds in the group $\mathrm{Diff}^1_0(M)$ for any manifold M (this was indicated to me by Isabelle Liousse). Identify the circle $\mathbb{S}^1$ with $\mathbb{R} \cup \{\infty\}$. Then consider the (analytical) circle diffeomorphisms $a\colon x \mapsto x+1$ and $b\colon x \mapsto 2x$. The relation $bab^{-1} = a^2$ is satisfied and, therefore, the two elements a and b define an action of the group $BS(1,2)$ on the circle. By thickening the point at infinity (ie, by replacing the point at infinity with an interval), we obtain a compactly supported action of $BS(1,2)$ on $\mathbb{R}$. This last action can be made C^1. Finally, by a radial action, we have a compactly supported C^1 action of $BS(1,2)$ on $\mathbb{R}^d$. By identifying an open disc of a manifold with $\mathbb{R}^d$, we get an action of the Baumslag–Solitar group on any manifold. This gives some non-recurrent distortion elements in the group $\mathrm{Diff}^1_0(M)$ for any manifold M.

In the case of diffeomorphisms, it is difficult to approach a characterization of distortion element as there are many obstructions to being a distortion element (for instance, the differential cannot grow too fast along an orbit, the topological entropy of the diffeomorphism must vanish). On the contrary, in the groups of surface homeomorphisms, there is only one known obstruction to being a distortion element. We will describe it in the next section.

In this article, we will try to characterize geometrically the set of distortion elements in the group of homeomorphisms isotopic to the identity of a compact orientable surface. The theorem we obtain is a consequence of a result that is valid on any manifold and proved in the fourth section. This last result has a major drawback: it uses the fragmentation length, which is not well understood except in the case of spheres. Thus, we will try to connect this fragmentation length to a more geometric quantity: the diameter of the image of a fundamental domain under a lift of the given homeomorphism. It is not difficult to prove that the fragmentation length dominates this last quantity: this will be treated in the third section of this article. However, conversely, it is more difficult to show that this last quantity dominates the fragmentation length. In order to prove this, we will make a distinction between the case of surfaces with boundary (Section 5), which is the easiest, the case of the torus (Section 6) and the case of higher genus closed surfaces (Section 7). The last section contains examples of distortion elements in the group of homeomorphisms of the annulus for which the growth of the diameter of a fundamental domain is "fast".

2 Notation and results

Let M be a manifold, possibly with boundary. We denote by $\mathrm{Homeo}_0(M)$ (respectively $\mathrm{Homeo}_0(M, \partial M)$) the identity component of the group of compactly supported homeomorphisms of M (respectively of the group of homeomorphisms of M that pointwise fix a neighbourhood of the boundary ∂M of M).

Definition 2.1 Given two homeomorphisms f and g of M and a subset A of M, an *isotopy* between f and g relative to A is a continuous path of homeomorphisms $(f_t)_{t\in[0,1]}$ that pointwise fix A such that $f_0 = f$ and $f_1 = g$. An isotopy between f and g is an isotopy relative to the empty set. For a subset A of M, we denote by $\mathring{A}$ its interior and by $\overline{A}$ its closure.

In what follows, S denotes a compact orientable surface, possibly with boundary, different from the disc and from the sphere. We denote by $\Pi\colon \widetilde{S} \to S$ the universal cover of S. The surface $\widetilde{S}$ is seen as a subset of the Euclidean plane $\mathbb{R}^2$ or of the hyperbolic plane $\mathbb{H}^2$ so that the deck transformations are isometries for the Euclidean metric or the hyperbolic metric. We endow the surface $\widetilde{S}$ with this metric. We identify the fundamental group $\pi_1(S)$ of the surface S with the group of deck transformations of the covering $\Pi\colon \widetilde{S} \to S$. For any subset A of the hyperbolic plane $\mathbb{H}^2$ (respectively of the Euclidean plane $\mathbb{R}^2$), we denote by $\delta(A)$ the diameter of A for the hyperbolic distance (respectively the Euclidean distance).

Definition 2.2 For any homeomorphism f of S, a *lift* of f is a homeomorphism F of $\widetilde{S}$ that satisfies

$$\Pi \circ F = f \circ \Pi.$$

For any isotopy $(f_t)_{t\in[0,1]}$, a lift of $(f_t)_{t\in[0,1]}$ is a continuous path $(F_t)_{t\in[0,1]}$ of homeomorphisms of $\widetilde{S}$ such that, for any t, the homeomorphism F_t is a lift of the homeomorphism f_t.

For any homeomorphism f in $\mathrm{Homeo}_0(S)$, take an isotopy between the identity and f. Consider a lift of this isotopy which is equal to the identity for $t = 0$. We denote by $\tilde{f}$ the time 1 of this lift. If moreover the boundary of S is nonempty and the homeomorphism f belongs to $\mathrm{Homeo}_0(S, \partial S)$, the homeomorphism $\tilde{f}$ is obtained by lifting an isotopy relative to the boundary ∂S. If there exists a disc $\mathbb{D}^2$ embedded in the surface S that contains the support of the homeomorphism f, we require that the support of $\tilde{f}$ is contained in $\Pi^{-1}(\mathbb{D}^2)$.

Claim *Except in the cases of the groups* $\mathrm{Homeo}_0(\mathbb{T}^2)$ *and* $\mathrm{Homeo}_0([0,1]\times\mathbb{S}^1)$, *the homeomorphism* $\tilde{f}$ *is unique.*

Proof This is a consequence of a theorem by Hamstrom (see [11]): If S is a surface without boundary of genus greater than 1, then the topological space $\mathrm{Homeo}_0(S)$ is simply connected. Moreover, if S is a surface with nonempty boundary, the topological space $\mathrm{Homeo}_0(S, \partial S)$ is simply connected. Finally, let us prove the claim in the case of an element f in $\mathrm{Homeo}_0(S)$ for a surface S with nonempty boundary. The double

DS' of a surface S' with nonempty boundary is the surface obtained from $S' \times \{-1, 1\}$ by identifying $\partial S' \times \{-1\}$ with $\partial S' \times \{1\}$. Take two lifts F_1 and F_2 of f to $\widetilde{S}$ as above. Each homeomorphism F_i canonically induces a homeomorphism, called the double of F_i, on the double of $\widetilde{S}$. Observe that the surface $D\widetilde{S}$ is a covering space of the surface DS. Moreover, the double of F_1 and the double of F_2 are lifts of the double of the homeomorphism f. Hence the double of F_1 is equal to the double of F_2 by Hamstrom's Theorem. This proves the claim. □

Notice that, for any deck transformation $\gamma \in \pi_1(S)$, and any homeomorphism f in $\mathrm{Homeo}_0(S)$,

$$\gamma \circ \widetilde{f} = \widetilde{f} \circ \gamma.$$

Indeed, take a lift $\widetilde{f_t}$ of an isotopy between the identity and a homeomorphism f. Then, for any $\gamma \in \pi_1(S)$, the path $t \mapsto \widetilde{f_t} \circ \gamma \circ \widetilde{f_t}^{-1}$ is continuous with values in the discrete space of deck transformations: this path is constant.

Definition 2.3 We call a *fundamental domain* of $\widetilde{S}$ for the action of $\pi_1(S)$ any compact connected subset D of $\widetilde{S}$ that satisfies the following properties:

(1) $\Pi(D) = S$.

(2) For any deck transformation γ in $\pi_1(S)$ different from the identity, the interior of D is disjoint from the interior of $\gamma(D)$.

The main theorem of the present article is a partial converse to the following property (observed by Franks and Handel in [9, Lemma 6.1]):

Proposition 2.4 *Denote by D a fundamental domain of $\widetilde{S}$ for the action of $\pi_1(S)$. If a homeomorphism f in* $\mathrm{Homeo}_0(S)$ *(respectively in* $\mathrm{Homeo}_0(S, \partial S)$*) is a distortion element in* $\mathrm{Homeo}_0(S)$ *(respectively in* $\mathrm{Homeo}_0(S, \partial S)$*), then*

$$\lim_{n \to +\infty} \frac{\delta(\widetilde{f}^n(D))}{n} = 0.$$

Remark 2.5 In the case where the surface S under consideration is the torus $\mathbb{T}^2$ or the annulus $[0,1] \times \mathbb{S}^1$, the conclusion of this proposition is equivalent to saying that the rotation set of f is reduced to a single point (see Misiurewicz and Ziemian [21] for a definition of the rotation set of a homeomorphism of the torus isotopic to the identity; the definition is analogous in the case of the annulus). This proposition provides examples of non-distorted elements. For instance, consider the homeomorphism F of $\mathbb{R}^2$ commuting to integral translations defined by

$$(x, y) \mapsto \begin{cases} (x, y + x) & \text{if } 0 \le x \le \frac{1}{2}, \\ (x, y + 1 - x) & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

on the unit square. This homeomorphism is a lift of a homeomorphism f of the torus $\mathbb{T}^2$. Notice that the point $(0,0)$ is fixed under the homeomorphism F whereas $F^n(\frac{1}{2},0) = (\frac{1}{2},\frac{n}{2})$. Hence the conclusion of the proposition does not hold and the homeomorphism f is not distorted. Of course, the rotation set of the homeomorphism f is not reduced to a point: it is equal to $[0,\frac{1}{2}] \times \{0\}$.

Proof Let f be a distortion element in $\mathrm{Homeo}_0(S)$ (respectively in $\mathrm{Homeo}_0(S,\partial S)$). Denote by

$$\mathcal{G} = \{g_1, g_2, \dots, g_p\}$$

a finite subset of $\mathrm{Homeo}_0(S)$ (respectively of $\mathrm{Homeo}_0(S,\partial S)$) such that:

(1) The homeomorphism f belongs to the group generated by $\mathcal{G}$.

(2) The sequence $(l_{\mathcal{G}}(f^n)/n)_{n\geq 1}$ converges to 0.

Then we have a decomposition of the form $f^n = g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_{l_n}}$ where $l_n = l_{\mathcal{G}}(f^n)$. This implies the following equality: $I \circ \widetilde{f}^n = \widetilde{g}_{i_1} \circ \widetilde{g}_{i_2} \circ \cdots \circ \widetilde{g}_{i_{l_n}}$, where I is an isometry of $\widetilde{S}$. Let us take

$$\mu = \max_{1\leq i\leq p, \widetilde{x}\in\widetilde{S}} d(\widetilde{x}, \widetilde{g}_i(\widetilde{x})).$$

For any index i and any deck transformation γ in $\pi_1(S)$, $\gamma \circ \widetilde{g}_i = \widetilde{g}_i \circ \gamma$ and the distance d is invariant under deck transformations. Thus μ is finite. Then, for any two points $\widetilde{x}$ and $\widetilde{y}$ of the fundamental domain D, we have

$$\begin{aligned} d(\widetilde{f}^n(\widetilde{x}), \widetilde{f}^n(\widetilde{y})) &= d(I \circ \widetilde{f}^n(\widetilde{x}), I \circ \widetilde{f}^n(\widetilde{y})) \\ &\leq d(I \circ \widetilde{f}^n(\widetilde{x}), \widetilde{x}) + d(\widetilde{x}, \widetilde{y}) + d(I \circ \widetilde{f}^n(\widetilde{y}), \widetilde{y}) \\ &\leq l_n\mu + \delta(D) + l_n\mu. \end{aligned}$$

This implies the proposition, by sublinearity of the sequence $(l_n)_{n\geq 0}$. □

The main theorem of this article is the following:

Theorem 2.6 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(S)$ *or* $\mathrm{Homeo}_0(S,\partial S)$. *If*

$$\liminf_{n\to+\infty} \frac{\delta(\widetilde{f}^n(D)) \log(\delta(\widetilde{f}^n(D)))}{n} = 0,$$

then f is a distortion element in $\mathrm{Homeo}_0(S)$ *or* $\mathrm{Homeo}_0(S,\partial S)$, *respectively.*

Remark 2.7 The hypothesis

$$\lim_{n\to+\infty} \frac{\delta(\tilde{f}^n(D))\log(\delta(\tilde{f}^n(D)))}{n} = 0$$

is independent of the chosen fundamental domain D, as we will see in the next section. Thus, it is invariant under conjugation.

Definition 2.8 Let X and Y be topological spaces. A continuous map $f\colon X \to X$ is said to be semi-conjugate to a continuous map $g\colon Y \to Y$ if there exists an onto continuous map $h\colon X \to Y$ such that $hf = gh$.

Let us give some examples of homeomorphisms of the torus or of the annulus for which this theorem can be applied.

(1) Of course, the rotations of the annulus and the translations of the torus satisfy the hypothesis of this theorem. Actually, for any homeomorphism f of the torus (respectively of the annulus) that is semi-conjugate to a translation (respectively a rotation), the sequence $(\delta(\tilde{f}^n(D)))_n$ is bounded: the homeomorphism f is distorted.

(2) The homeomorphisms that are C^0–recurrent (in the sense that arbitrarily large powers of these homeomorphisms are arbitrarily close to the identity) satisfy the hypothesis of this theorem. In particular, the examples given after the statement of Theorem 3 satisfy the hypothesis of Theorem 2.6. However, we already knew that these homeomorphisms were distorted by Theorem 3.

(3) Tobias Jäger has built examples of homeomorphisms f in $\mathrm{Homeo}_0(\mathbb{T}^2)$ that are not semi-conjugate to a translation, and such that the sequence $(\delta(\tilde{f}^n(D)))_n$ is bounded (see [14, Proposition 2.1]). His examples are skew-products over a Denjoy counterexample. By Theorem 2.6, such homeomorphisms are distorted in $\mathrm{Homeo}_0(\mathbb{T}^2)$.

(4) In Section 8, for any sequence $(v_n)_n$ of positive numbers such that

$$\lim_{n\to+\infty} \frac{v_n}{n} = 0,$$

we will construct a simple example of homeomorphism f of the annulus such that, for any n,

$$v_n \leq \delta(\tilde{f}^n(D)) \leq v_n + 1 + \delta(D).$$

Theorem 2.6 can be applied in the case where $\lim_{n\to+\infty} v_n \log(v_n)/n = 0$.

(5) Recently, Koropecki and Tal built a C^∞ (area-preserving) diffeomorphism of the torus $\mathbb{T}^2$ such that every orbit is bounded and $\lim_{n\to+\infty}\delta(\widetilde{f}^n(D))/n = 0$ but the sequence $(\delta(\widetilde{f}^n(D)))_n$ is unbounded (see [17, Theorem 3]). The idea is to embed an open disc in the torus in a wild way (in particular, any lift of this disc to $\mathbb{R}^2$ is unbounded). Then consider a homeomorphism which is equal to the identity outside this disc and which is equal, in polar coordinates, to $(r,\theta)\mapsto(r,\theta+\phi(r))$ on the disc, where $\phi\colon [0,1]\to\mathbb{R}$ is a continuous map which sends 0 and 1 to 0. Of course, $\phi(r)$ has to converges sufficiently fast to 0 when r tends to 1 to ensure that the homeomorphism f is well-defined. If $\phi(r)$ converges sufficiently fast to 0 when r tends to 1, one can check that such a homeomorphism satisfies the hypothesis of Theorem 2.6. In the same article, with the same kind of construction, Koropecki and Tal built a homeomorphism g of the torus such that, for Lebesgue almost every point of $\mathbb{R}^2$, its forward and its backward orbit under $\widetilde{g}$ accumulate in every direction at infinity (see [17, Theorem 1]). We do not know whether such a homeomorphism is distorted.

The proof of Theorem 2.6 occupies the next five sections. For this proof, we need the following notion. Let M be a compact d–dimensional manifold. Denote by $\overline{B}(0,1)$ the closed unit ball of $\mathbb{R}^d$. A subset B of M will be called a closed ball if there exists an embedding $e\colon \mathbb{R}^d\to M$ such that $e(\overline{B}(0,1)) = B$. Let

$$H^d = \{(x_1,x_2,\dots,x_d)\in\mathbb{R}^N, x_1\geq 0\}.$$

We will call a closed half-ball of M the image of $\overline{B}(0,1)\cap H^d$ under an embedding $e\colon H^d\to M$ such that

$$e(\partial H^d) = e(H^d)\cap\partial M.$$

Let us fix a finite family $\mathcal{U}$ of closed balls or closed half-balls whose interiors cover M. Then, by the fragmentation lemma (see Bounemoura [4] or Fisher [8]), there exists a finite family $(f_i)_{1\leq i\leq n}$ of homeomorphisms in $\mathrm{Homeo}_0(M)$, each supported in one of the sets of $\mathcal{U}$, such that

$$f = f_1\circ f_2\circ\cdots\circ f_n.$$

We denote by $\mathrm{Frag}_{\mathcal{U}}(f)$ the minimal integer n in such a decomposition: it is the minimal number of factors necessary to write f as a composition of homeomorphisms that are each supported in one of the balls of $\mathcal{U}$.

Let us come back to the case of a compact surface S and denote by $\mathcal{U}$ a finite family of closed discs or closed half-discs whose interiors cover S. Denote by D a fundamental domain of $\widetilde{S}$ for the action of $\pi_1(S)$. We now describe the two steps of the proof of Theorem 2.6. The first step of the proof consists of checking that the quantity $\mathrm{Frag}_{\mathcal{U}}(f)$ is almost equal to $\delta(\widetilde{f}(D))$:

Theorem 2.9 *There exist two real constants $C > 0$ and C' such that, for any homeomorphism f in* $\mathrm{Homeo}_0(S)$,

$$\frac{1}{C}\delta(\tilde{f}(D)) - C' \leq \mathrm{Frag}_{\mathcal{U}}(f) \leq C\delta(\tilde{f}(D)) + C'.$$

There is a version of this theorem for the groups $\mathrm{Homeo}_0(S, \partial S)$ in case the surface S has nonempty boundary. Let us denote by S' a submanifold of S with the following properties: the surface S' is homeomorphic to S, is contained in the interior of S and is a deformation retract of S. We denote by $\mathcal{U}$ a family of closed balls of S whose interiors cover S'.

Theorem 2.10 *There exist two real constants $C > 0$ and C' such that, for any homeomorphism f in* $\mathrm{Homeo}_0(S, \partial S)$ *supported in S',*

$$\frac{1}{C}\delta(\tilde{f}(D)) - C' \leq \mathrm{Frag}_{\mathcal{U}}(f) \leq C\delta(\tilde{f}(D)) + C'.$$

It is not difficult to obtain the lower bound of the fragmentation length. This is treated in the next section. In the same section, we will also see that the quantity $\mathrm{Frag}_{\mathcal{U}}$ is essentially independent of the chosen cover $\mathcal{U}$. On the other hand, the argument for the upper bound is much more technical. For this bound, we distinguish three cases: the case of surfaces with boundary (Section 5), the case of the torus (Section 6) and the case of higher genus compact surfaces without boundary (Section 7). The proof seems to depend strongly on the fundamental group of the surface under consideration. In particular, it is easier in the case of surfaces with boundary whose fundamental groups are free. In the case of the torus, the proof is a little tricky. In the case of higher genus closed surfaces, the proof is more complex and uses Dehn's algorithm for small-cancellation groups (surface groups in this case).

Let us explain now the second step of the proof. Denote by M a compact manifold and by $\mathcal{U}$ a finite family of closed balls or half-balls whose interiors cover M. In Section 4, we will prove the following theorem. It asserts that, for a homeomorphism f in $\mathrm{Homeo}_0(M)$, if the sequence $\mathrm{Frag}_{\mathcal{U}}(f^n)$ does not grow too fast when $n \to +\infty$, then the homeomorphism f is a distortion element:

Theorem 2.11 *If*

$$\liminf_{n \to +\infty} \frac{\mathrm{Frag}_{\mathcal{U}}(f^n)\log(\mathrm{Frag}_{\mathcal{U}}(f^n))}{n} = 0,$$

then the homeomorphism f is a distortion element in $\mathrm{Homeo}_0(M)$.

Moreover, assume that the manifold M has nonempty boundary. Then, if $\mathcal{U}$ denotes a finite family of closed balls contained in the interior of M whose interiors cover the support of a homeomorphism f in $\mathrm{Homeo}_0(M, \partial M)$, this last theorem remains true in the group $\mathrm{Homeo}_0(M, \partial M)$. The proof of this theorem uses a technique due to Avila (see [2]).

Theorem 2.6 is clearly a consequence of these two theorems.

The following theorem shows that Proposition 2.4 is optimal. It will be proved in the last section.

Theorem 2.12 *Let $(v_n)_{n \geq 1}$ be a sequence of positive real numbers such that*

$$\lim_{n \to +\infty} \frac{v_n}{n} = 0.$$

Then there exists a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}/\mathbb{Z} \times [0,1], \mathbb{R}/\mathbb{Z} \times \{0,1\})$ *such that:*

(1) *For any $n \geq 1$, $\delta(\tilde{f}^n([0,1] \times [0,1])) \geq v_n$.*

(2) *The homeomorphism f is a distortion element in*

$$\mathrm{Homeo}_0(\mathbb{R}/\mathbb{Z} \times [0,1], \mathbb{R}/\mathbb{Z} \times \{0,1\}).$$

This theorem means that being a distortion element gives no information on the growth of the diameter of a fundamental domain other than the sublinearity of this growth. This theorem remains true for any surface S. To see this, it suffices to embed the annulus $\mathbb{R}/\mathbb{Z} \times [0,1]$ in the surface S.

3 Quasi-isometries

In this section, we prove the lower bound in Theorems 2.9 and 2.10. More precisely, we prove these theorems using the following propositions whose proofs will be discussed in Sections 5, 6 and 7.

Proposition 3.1 *There exists a finite cover $\mathcal{U}$ of S by closed discs and half-discs as well as real constants $C \geq 1$ and $C' \geq 0$ such that, for any homeomorphism f in* $\mathrm{Homeo}_0(S)$,

$$\mathrm{Frag}_{\mathcal{U}}(f) \leq C \, \mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)) + C'.$$

Here is a version of the previous proposition in the case of the group $\mathrm{Homeo}_0(S, \partial S)$.

Proposition 3.2 *Fix a subsurface with boundary S' of S that is contained in the interior of S, is a deformation retract of S and is homeomorphic to S. There exists a finite cover $\mathcal{U}$ of S' by closed discs contained in the interior of S as well as real constants $C \geq 1$ and $C' \geq 0$ such that, for any homeomorphism f in* $\mathrm{Homeo}_0(S, \partial S)$ *supported in S',*

$$\mathrm{Frag}_{\mathcal{U}}(f) \leq C \,\mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)) + C'.$$

In order to prove these theorems, we need some notation. As in the last section, let us denote by S a compact orientable surface.

Definition 3.3 Two maps $a, b\colon \mathrm{Homeo}_0(S) \to \mathbb{R}$ are *quasi-isometric* if and only if there exist real constants $C \geq 1$ and $C' \geq 0$ such that

$$\text{for any } f \in \mathrm{Homeo}_0(S), \quad \frac{1}{C} a(f) - C' \leq b(f) \leq C a(f) + C'.$$

More generally, an arbitrary number of maps $\mathrm{Homeo}_0(S) \to \mathbb{R}$ are said to be quasi-isometric if they are pairwise quasi-isometric.

Let us consider a fundamental domain D_0 of $\tilde{S}$ for the action of the group $\pi_1(S)$, which satisfies the following properties (see Figure 1):

(1) If the surface S of genus g is closed, the fundamental domain D_0 is a closed disc bounded by a $4g$–gone with geodesic edges.

(2) If the surface S has nonempty boundary, the fundamental domain D_0 is a closed disc bounded by a polygon with geodesic edges. We require that any edge of this polygon that is not contained in $\partial\tilde{S}$ connects two edges contained in $\partial\tilde{S}$.

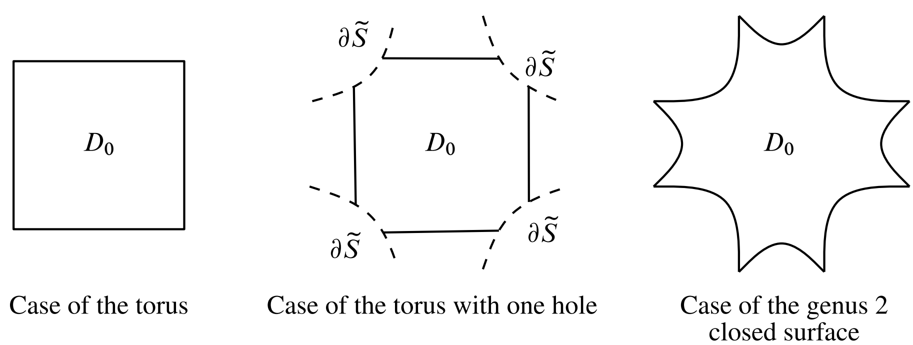

Figure 1: The fundamental domain D_0

Let $\mathcal{D} = \{\gamma(D_0) \mid \gamma \in \pi_1(S)\}$. For fundamental domains D and D' in $\mathcal{D}$, we denote by $d_{\mathcal{D}}(D, D') + 1$ the minimal number of fundamental domains met by a path that connects the interior of D to the interior of D'. The map $d_{\mathcal{D}}$ is a distance on $\mathcal{D}$. We now give an algebraic definition of this quantity. Denote by $\mathcal{G}$ the finite set of deck transformations in $\pi_1(S)$ that send D_0 to a polygon in $\mathcal{D}$ adjacent to D_0, ie, which shares an edge in common with D_0. Then the subset $\mathcal{G}$ is symmetric and is a generating set for $\pi_1(S)$. Notice that the map

$$\begin{aligned} d_{\mathcal{G}}\colon \pi_1(S)\times\pi_1(S) &\to \mathbb{R}, \\ (\varphi,\psi) &\mapsto l_{\mathcal{G}}(\varphi^{-1}\psi), \end{aligned}$$

is a distance on the group $\pi_1(S)$. Then, for any pair (φ,ψ) of deck transformations in the group $\pi_1(S)$, we have $l_{\mathcal{G}}(\varphi^{-1}\psi) = d_{\mathcal{D}}(\varphi(D_0),\psi(D_0))$. One can see it by noticing that $d_{\mathcal{D}}$ is invariant under the action of the group $\pi_1(S)$ and by proving by induction on $l_{\mathcal{G}}(\psi)$ that

$$l_{\mathcal{G}}(\psi) = d_{\mathcal{D}}(D_0,\psi(D_0)).$$

Given a compact subset A of $\widetilde{S}$, we call the *discrete diameter* of A the following quantity:

$$\operatorname{diam}_{\mathcal{D}}(A) = \max\{d_{\mathcal{D}}(D,D') \mid D\in\mathcal{D}, D'\in\mathcal{D}, D\cap A\neq\varnothing, D'\cap A\neq\varnothing\}$$

For a fundamental domain D_1 in $\mathcal{D}$, we call the *éloignement* of A with respect to D_1 the following quantity:

$$\operatorname{el}_{D_1}(A) = \max\{d_{\mathcal{D}}(D_1,D) \mid D\in\mathcal{D}, D\cap A\neq\varnothing\}$$

Notice that, in the case where $D_1\cap A\neq\varnothing$, we have

$$\operatorname{el}_{D_1}(A)\leq \operatorname{diam}_{\mathcal{D}}(A)\leq 2\operatorname{el}_{D_1}(A).$$

In this section, we prove the following statement, using Proposition 3.1:

Proposition 3.4 *Let $\mathcal{U}$ be a finite families of closed balls or half-balls whose interiors cover the surface S. Let D be a fundamental domain of $\widetilde{S}$ for the action of the fundamental group of S. Then the following maps $\operatorname{Homeo}_0(S)\to\mathbb{R}$ are quasi-isometric:*

(1) *The map* $\operatorname{Frag}_{\mathcal{U}}$
(2) *The map* $f\mapsto\delta(\widetilde{f}(D))$
(3) *The map* $f\mapsto\operatorname{diam}_{\mathcal{D}}(\widetilde{f}(D_0))$

Remark 3.5 The proposition implies the following properties. Let $\mathcal{U}$ and $\mathcal{U}'$ be two finite families of closed balls or half-balls whose interiors cover the surface S. Then the maps $\operatorname{Frag}_{\mathcal{U}}$ and $\operatorname{Frag}_{\mathcal{U}'}$ are quasi-isometric.

Take two fundamental domains D and D' of $\widetilde{S}$ for the action of the fundamental group of S. Then the maps $f \mapsto \delta(\widetilde{f}(D))$ and $f \mapsto \delta(\widetilde{f}(D'))$ are quasi-isometric.

When the boundary of the surface S is nonempty, we have an analogous proposition in the case of the group $\mathrm{Homeo}_0(S, \partial S)$. As in the last section, let us denote by S' a submanifold with boundary of S that is homeomorphic to S, contained in the interior of S, and a deformation retract of S, and by $\mathcal{U}$ a finite family of closed balls contained in the interior of S, the union of whose interiors contains S'. Finally, let us denote by $G_{S'}$ the group of homeomorphisms in $\mathrm{Homeo}_0(S, \partial S)$ that are supported in S'.

Proposition 3.6 *Let D be a fundamental domain of $\widetilde{S}$ for the action of the fundamental group of S. The following maps $G_{S'} \to \mathbb{R}$ are quasi-isometric:*

(1) *The map* $\mathrm{Frag}_{\mathcal{U}}$
(2) *The map* $f \mapsto \delta(\widetilde{f}(D))$
(3) *The map* $f \mapsto \mathrm{diam}_{\mathcal{D}}(\widetilde{f}(D_0))$

The proof of this proposition is similar to the proof of the previous one: that is why we will not provide it.

These two propositions directly imply Theorems 2.9 and 2.10.

Proof of Proposition 3.4 Let us prove first that, for any two fundamental domains D and D', the maps $f \mapsto \delta(\widetilde{f}(D))$ and $f \mapsto \delta(\widetilde{f}(D'))$ are quasi-isometric. Let

$$\{\gamma_1, \gamma_2, \dots, \gamma_p\} = \{\gamma \in \pi_1(S) \mid D' \cap \gamma(D) \neq \varnothing\}.$$

Notice that $D' \subset \bigcup_{i=1}^{p} \gamma_i(D)$ and the right-hand side is path-connected. Then

$$\widetilde{f}(D') \subset \bigcup_{i=1}^{p} \widetilde{f}(\gamma_i(D)).$$

The lemma below implies that $\delta(\widetilde{f}(D')) \leq p\delta(\widetilde{f}(D))$. As the fundamental domains D and D' play symmetric roles, this implies that the maps $f \mapsto \delta(\widetilde{f}(D))$ and $f \mapsto \delta(\widetilde{f}(D'))$ are quasi-isometric.

Lemma 3.7 *Let X be a path-connected metric space. Let $(A_i)_{1 \leq i \leq p}$ be a family of closed subsets of X such that $X = \bigcup_{i=1}^{p} A_i$. Then*

$$\delta(X) = \sup_{x \in X, y \in X} d(x, y) \leq p \max_{1 \leq i \leq p} \delta(A_i).$$

Proof Let x and y be two points in X. By path-connectedness of X, there exists an integer k between 1 and p, an injection $\sigma\colon \{1,\dots,k\} \to \{1,\dots,p\} \cap \mathbb{Z}$ and a sequence $(x_i)_{1\leq i\leq k+1}$ of points in X that satisfy the following properties:

(1) $x_1 = x$ and $x_{k+1} = y$.

(2) For any index i between 1 and k, the points x_i and x_{i+1} both belong to $A_{\sigma(i)}$.

Then:

$$d(x,y) \leq \sum_{i=1}^{k} d(x_i, x_{i+1}) \leq \sum_{i=1}^{k} \delta(A_{\sigma(i)}) \leq p \max_{1\leq i\leq p} \delta(A_i).$$

This last inequality implies the lemma. □

Let us show now that, for two finite families $\mathcal{U}$ and $\mathcal{U}'$ as in the statement of Proposition 3.4, the maps $\mathrm{Frag}_{\mathcal{U}}$ and $\mathrm{Frag}_{\mathcal{U}'}$ are quasi-isometric. The proof of this fact requires the following lemmas. Recall that we denoted by $\overline{B}(0,1)$ the unit closed ball of $\mathbb{R}^d$.

Lemma 3.8 *Let $\mathcal{V}$ be a neighbourhood of the identity in $\mathrm{Homeo}_0(\overline{B}(0,1), \partial\overline{B}(0,1))$. There exists an integer $N \geq 0$ such that any homeomorphism in*

$$\mathrm{Homeo}_0(\overline{B}(0,1), \partial\overline{B}(0,1))$$

can be written as a composition of at most N homeomorphisms in $\mathcal{V}$.

Lemma 3.9 *Let M be a compact manifold and $\{U_1, U_2, \dots, U_p\}$ be an open cover of M. There exist a neighbourhood $\mathcal{V}$ of the identity in $\mathrm{Homeo}_0(M)$ (respectively in $\mathrm{Homeo}_0(M,\partial M)$) and an integer $N' > 0$ such that the following property is satisfied. For any homeomorphism f in $\mathcal{V}$, there exist homeomorphisms $g_1,\dots,g_{N'}$ in $\mathrm{Homeo}_0(M)$ (respectively in $\mathrm{Homeo}_0(M,\partial M)$) such that:*

(1) *Each homeomorphism g_i is supported in one of the U_j.*

(2) $f = g_1 \circ g_2 \circ \cdots \circ g_{N'}$.

Lemma 3.8 is a consequence of Béguin, Crovisier, Le Roux and Patou [3, Lemma 5.2] (notice that the proof works in dimensions higher than 2). Lemma 3.9 is classical. It is a consequence of the proof of Theorem 1.2.3 in [4]. These two lemmas imply that, for an open cover of the disc $\mathbb{D}^2$, there exists an integer N such that any homeomorphism in $\mathrm{Homeo}_0(\mathbb{D}^2, \partial\mathbb{D}^2)$ can be written as a composition of at most N homeomorphisms, each supported in one of the open sets of the cover. Now, for an element U in $\mathcal{U}$, we denote by $U \cap \mathcal{U}'$ the cover of U given by the intersections of the elements of $\mathcal{U}'$ with U. The application of this last result to the ball U with the cover $U \cap \mathcal{U}'$ gives us a constant N_U. Let us denote by N the maximum of the N_U, where U varies over $\mathcal{U}$. We directly obtain that, for any homeomorphism f,

$$\mathrm{Frag}_{\mathcal{U}'}(f) \leq N\,\mathrm{Frag}_{\mathcal{U}}(f).$$

As the two covers $\mathcal{U}$ and $\mathcal{U}'$ play symmetric roles, the fact is proved. Notice that this fact is true in any dimension.

Using a quasi-isometry between the metric spaces $(\pi_1(S), d_S)$ and $\widetilde{S}$, we will prove the following lemma. It implies that the last two maps in the proposition are quasi-isometric.

Lemma 3.10 *There exist constants $C \geq 1$ and $C' \geq 0$ such that, for any compact subset A of $\widetilde{S}$,*

$$\frac{1}{C}\delta(A) - C' \leq \mathrm{diam}_{\mathcal{D}}(A) \leq C\delta(A) + C'.$$

Proof Let us fix a point x_0 in the interior of D_0. The map

$$\begin{aligned} q\colon \pi_1(S) &\to \widetilde{S}, \\ \gamma &\mapsto \gamma(x_0), \end{aligned}$$

is a quasi-isometry for the distance $d_{\mathcal{G}}$ and the distance on $\widetilde{S}$ (this is the Švarc–Milnor lemma; see de la Harpe [12] p. 87). We notice that, for a compact subset A of $\widetilde{S}$, the number $\mathrm{diam}_{\mathcal{D}}(A)$ is equal to the diameter of $q^{-1}(B)$ for the distance $d_{\mathcal{G}}$, where

$$B = \bigcup \{D \mid D \in \mathcal{D}, D \cap A \neq \varnothing\}.$$

We deduce that there exist constants $C_1 \geq 1$ and $C_1' \geq 0$ independent of A such that

$$\frac{1}{C_1}\delta(B) - C_1' \leq \mathrm{diam}_{\mathcal{D}}(A) \leq C_1\delta(B) + C_1'.$$

The inequalities $\delta(B) - 2\delta(D_0) \leq \delta(A) \leq \delta(B)$ complete the proof of the lemma. □

We now prove that, for any cover $\mathcal{U}$ as in the statement of Proposition 3.4, there exist constants $C \geq 1$ and $C' \geq 0$ such that, for any homeomorphism f in $\mathrm{Homeo}_0(S)$,

$$\frac{1}{C}\,\mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)) - C' \leq \mathrm{Frag}_{\mathcal{U}}(f).$$

Let us fix such a family $\mathcal{U}$. We will need the following lemma, which we will prove later:

Lemma 3.11 *There exists a constant $C > 0$ such that, for any compact subset A of $\widetilde{S}$ and any homeomorphism h supported in one of the sets in $\mathcal{U}$,*

$$\mathrm{diam}_{\mathcal{D}}(\widetilde{h}(A)) \geq \mathrm{diam}_{\mathcal{D}}(A) - C.$$

Take $k = \mathrm{Frag}_{\mathcal{U}}(f)$ and $f = g_1 \circ g_2 \circ \cdots \circ g_k$, where each homeomorphism g_i is supported in one of the elements of $\mathcal{U}$. Then $I \circ \widetilde{f} = \widetilde{g}_1 \circ \widetilde{g}_2 \circ \cdots \circ \widetilde{g}_k$, where I is a deck transformation (and an isometry). Lemma 3.11 and induction on j imply that

$$\text{for all } j \in [1,k] \cap \mathbb{Z}, \quad \mathrm{diam}_{\mathcal{D}}(\widetilde{g}_j^{-1} \circ \cdots \circ \widetilde{g}_1^{-1} \circ \widetilde{f}(D_0)) \geq \mathrm{diam}_{\mathcal{D}}(\widetilde{f}(D_0)) - jC,$$

as the homeomorphisms $\widetilde{g}_i$ commute with I. Hence

$$2 = \mathrm{diam}_{\mathcal{D}}(\widetilde{g}_k^{-1} \circ \cdots \circ \widetilde{g}_1^{-1} \circ \widetilde{f}(D_0)) \geq \mathrm{diam}_{\mathcal{D}}(\widetilde{f}(D_0)) - kC.$$

Therefore

$$\mathrm{Frag}_{\mathcal{U}}(f) \geq \frac{1}{C}\,\mathrm{diam}_{\mathcal{D}}(\widetilde{f}(D_0)) - \frac{2}{C}.$$

We obtain the wanted lower bound.

Proof of Lemma 3.11 For an element U in $\mathcal{U}$, we denote by $\widetilde{U}$ a lift of U, ie, a connected component of $\Pi^{-1}(U)$. Let $\mu(U) = \mathrm{diam}_{\mathcal{D}}(\widetilde{U})$. This quantity does not depend on the chosen lift $\widetilde{U}$. We denote by μ the maximum of the $\mu(U)$, for U in $\mathcal{U}$.

We denote by U_h an element in $\mathcal{U}$ which contains the support of h. Let us consider two fundamental domains D and D' which meet A and which satisfy the following relation:

$$d_{\mathcal{D}}(D, D') = \mathrm{diam}_{\mathcal{D}}(A).$$

Let us take a point x in $D \cap A$ and a point x' in $D' \cap A$. If the point x belongs to $\Pi^{-1}(U_h)$, we denote by $\widetilde{U_h}$ the lift of U_h which contains x. Then the point $\widetilde{h}(x)$ belongs to $\widetilde{U_h}$ and a fundamental domain $\widehat{D}$ which contains the point $\widetilde{h}(x)$ is at distance at most μ from D (for $d_{\mathcal{D}}$). Hence, in any case, there exists a fundamental domain $\widehat{D}$ which contains the point $\widetilde{h}(x)$ and is at distance at most μ from D. Similarly, there exists a fundamental domain $\widehat{D}'$ which contains the point $\widetilde{h}(x')$ and is at distance at most μ from D'. Therefore

$$d_{\mathcal{D}}(\widehat{D}, \widehat{D}') \geq d_{\mathcal{D}}(D, D') - 2\mu.$$

We deduce that $\mathrm{diam}_{\mathcal{D}}(\widetilde{h}(A)) \geq \mathrm{diam}_{\mathcal{D}}(A) - 2\mu$, which is what we wanted to prove. □

Thus, to complete the proof of Proposition 3.4, it suffices to prove Proposition 3.1. □

It suffices now to find a finite family $\mathcal{U}$ for which Proposition 3.1 or 3.2 holds. We will distinguish the following cases. A section is devoted to each of them.

(1) The surface S has nonempty boundary (Section 5).

(2) The surface S is the torus (Section 6).

(3) The surface S is closed of genus greater than one (Section 7).

The proof of Propositions 3.1 and 3.2, in each of these cases, consists in putting back the boundary of $\widetilde{f}(D_0)$ close to the boundary of ∂D_0 by using homeomorphisms that are each supported in the interior of one of the balls of a well-chosen cover $\mathcal{U}$. Most of the time, after composing with a homeomorphism supported in the interior of one of the balls of $\mathcal{U}$, the image of the fundamental domain D_0 will not meet faces that were not met before the composition. However, it will not be always possible, which explains the difficulty of parts of the proof. Then, we will have to ensure that, after composing by a uniformly bounded number of homeomorphisms supported in interiors of balls of $\mathcal{U}$, the image of the boundary of D_0 will be strictly closer to D_0 than before.

4 Distortion and fragmentation on manifolds

In this section, M denotes a compact d–dimensional manifold, possibly with boundary. Let us fix a finite family $\mathcal{U}$ of closed balls or half-balls of M whose interiors cover M. For any homeomorphism f in $\mathrm{Homeo}_0(M)$, we denote by $a_{\mathcal{U}}(f)$ the minimum of the quantities $l.\log(k)$, where there exists a finite set $\{f_i \mid 1 \leq i \leq k\}$ of k homeomorphisms in $\mathrm{Homeo}_0(M)$, each supported in one of the elements of $\mathcal{U}$, and a map $\nu\colon \{1,\dots,l\} \to \{1,\dots,k\}$ with

$$f = f_{\nu(1)} \circ f_{\nu(2)} \circ \cdots \circ f_{\nu(l)}.$$

The aim of this section is to prove the following proposition:

Proposition 4.1 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(M)$. *Then*

$$\liminf_{n \to +\infty} \frac{a_{\mathcal{U}}(f^n)}{n} = 0$$

if and only if the homeomorphism f is a distortion element in $\mathrm{Homeo}_0(M)$.

Let us give now an analogous statement in the case of the group $\mathrm{Homeo}_0(M, \partial M)$. Denote by M' a submanifold with boundary that is homeomorphic to M, contained in the interior of M and which is a deformation retract of M. We denote by $\mathcal{U}$ a family of closed balls of M whose interiors cover M'. For any homeomorphism f in $\mathrm{Homeo}_0(M, \partial M)$ supported in M', we define $a_{\mathcal{U}}(f)$ in the same way as before.

Proposition 4.2 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(M, \partial M)$ *supported in* M'. *Then*

$$\liminf_{n \to +\infty} \frac{a_{\mathcal{U}}(f^n)}{n} = 0$$

if and only if the homeomorphism f is a distortion element in $\mathrm{Homeo}_0(M, \partial M)$.

As $a_{\mathcal{U}}(f) \leq \mathrm{Frag}_{\mathcal{U}}(f). \log(\mathrm{Frag}_{\mathcal{U}}(f))$, these last propositions imply Theorem 2.11.

Proof of the "if" statement in Propositions 4.1 and 4.2 If the homeomorphism f is a distortion element, we denote by $\mathcal{G}$ the finite set that appears in the definition of a distortion element. Then we write each of the homeomorphisms in $\mathcal{G}$ as a product of homeomorphisms supported in one of the sets of $\mathcal{U}$. We denote by $\mathcal{G}'$ the (finite) set of homeomorphisms that appear in such a decomposition. Then the homeomorphism f^n is equal to a composition of l_n elements of $\mathcal{G}'$, where l_n is less than a constant independent of n times $l_{\mathcal{G}}(f^n)$. As the element f is distorted, $\lim_{n \to +\infty} l_n/n = 0$ and $a_{\mathcal{U}}(f^n) \leq \log(\mathrm{card}(\mathcal{G}'))l_n$. Therefore

$$\lim_{n \to +\infty} \frac{a_{\mathcal{U}}(f^n)}{n} = 0.$$

In the case of Proposition 4.2, there is only one new difficulty: the elements of $\mathcal{G}$ are not necessarily supported in the union of the balls of $\mathcal{U}$. Let us take a homeomorphism h in $\mathrm{Homeo}_0(M, \partial M)$ with the following properties: the homeomorphism h is equal to the identity on M' and sends the union of the supports of elements of $\mathcal{G}$ to the union of the interiors of the balls of $\mathcal{U}$. Then it suffices to consider the finite set $h\mathcal{G}h^{-1}$ instead of $\mathcal{G}$ in order to complete the proof. □

The full power of Propositions 4.1 and 4.2 will be used only for the proof of Theorem 2.12 (construction of the example). In order to prove Theorem 2.6, we just used Theorem 2.11, which is weaker.

Remark 4.3 Notice that, if $\mathcal{U}$ is the cover of the sphere by two neighbourhoods of the hemispheres, the map $\mathrm{Frag}_{\mathcal{U}}$ is bounded by 3 on the group $\mathrm{Homeo}_0(\mathbb{S}^d)$ of homeomorphisms of the n–dimensional sphere isotopic to the identity (see [5]). This is a consequence of the annulus theorem by Kirby (see [16]) and Quinn (see [24]). Thus, the following theorem by Calegari and Freedman (see [5]) is a consequence of Theorem 2.11:

Theorem 4.4 (Calegari and Freedman [5]) *Any homeomorphism in* $\mathrm{Homeo}_0(\mathbb{S}^d)$ *is a distortion element.*

The proof of Proposition 4.1 is based on the following lemma, whose proof uses a technique due to Avila (see [2]):

Lemma 4.5 *Let $(f_n)_{n\geq 1}$ be a sequence of homeomorphisms of $\mathbb{R}^d$ (respectively of H^d) supported in the open unit ball $B(0,1)$ (respectively in $B(0,1)\cap H^d$). There exists a finite set $\mathcal{G}$ of compactly supported homeomorphisms of $\mathbb{R}^d$ (respectively of H^d) such that:*

(1) *For any natural number n, the homeomorphism f_n belongs to the group generated by $\mathcal{G}$.*

(2) $l_{\mathcal{G}}(f_n) \leq 14.\log(n) + 14$.

This lemma is not true anymore in case of the C^r regularity, for $r \geq 1$. It crucially uses the following fact: given a sequence of homeomorphisms (h_n) supported in the unit ball $B(0,1)$, one can store all the information of this sequence in one homeomorphism. Let us explain now how to build such a homeomorphism. For any integer n, denote by g_n a homeomorphism that sends the unit ball to a ball B_n such that the balls B_n are pairwise disjoint and have a diameter that converges to 0. Then it suffices to consider the homeomorphism

$$\prod_{n=1}^{\infty} g_n h_n g_n^{-1}.$$

Such a construction is not possible in the case of a higher regularity.

Remark 4.6 There are two main differences between this lemma and the one stated by Avila:

(1) Avila's lemma deals with a sequence of diffeomorphisms that converges sufficiently fast (in the C^∞–topology) to the identity, whereas any sequence of homeomorphisms is considered here.

(2) The upper bound is logarithmic and not linear.

Remark 4.7 This lemma is optimal in the sense that, if the homeomorphisms f_n are pairwise distinct, the growth of $l_{\mathcal{G}}(f_n)$ is at least logarithmic. Indeed, if the generating set $\mathcal{G}$ contains k elements, there are at most $(k^{l+1}-1)/(k-1)$ homeomorphisms whose length with respect to $\mathcal{G}$ is less than or equal to l.

Before proving Lemma 4.5, let us see why this lemma implies Propositions 4.1 and 4.2.

End of the proof of Propositions 4.1 and 4.2 Suppose that

$$\liminf_{n\to+\infty} \frac{a_{\mathcal{U}}(f^n)}{n} = 0.$$

Consider an increasing map $\eta\colon \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to+\infty} \frac{a_{\mathcal{U}}(f^{\eta(n)})}{\eta(n)} = 0.$$

Let $\mathcal{U} = \{U_1, U_2, \dots, U_p\}$. For any integer i between 1 and p, denote by φ_i an embedding of $\mathbb{R}^d$ into M that sends the closed ball $\overline{B}(0,1)$ onto U_i if U_i is a closed ball, or an embedding of H^d into M that sends the closed half-ball $\overline{B}(0,1) \cap H^d$ onto U_i if U_i is a closed half-ball. For any natural number n, let l_n and k_n be two positive integers such that:

(1) $a_{\mathcal{U}}(f^{\eta(n)}) = l_n \log(k_n)$.

(2) There exists a sequence of homeomorphisms in $\mathrm{Homeo}_0(M)$,

$$(f_{1,n}, f_{2,n}, \dots, f_{k_n,n}),$$

each supported in one of the elements of $\mathcal{U}$, such that $f^{\eta(n)}$ is the composition of l_n homeomorphisms of this family.

Let us build an increasing one-to-one function $\sigma\colon \mathbb{N} \to \mathbb{N}$ that satisfies

$$\text{for all } n \in \mathbb{N}, \quad \frac{l_{\sigma(n)}(14\log(\sum_{i=1}^{n} k_{\sigma(i)}) + 14)}{\eta(\sigma(n))} \leq \frac{1}{n}.$$

Suppose that, for some $m \geq 0$, $\sigma(1), \sigma(2), \dots, \sigma(m)$ have been built. Then, as

$$\lim_{n\to+\infty} \frac{l_n \log(k_n)}{\eta(n)} = 0,$$

we have

$$\lim_{n\to+\infty} \frac{l_n(14\log(\sum_{i=1}^{m} k_{\sigma(i)} + k_n) + 14)}{\eta(n)} = 0.$$

Hence, we can find an integer $\sigma(m+1) > \sigma(m)$ such that

$$\frac{l_{\sigma(m+1)}(14\log(\sum_{i=1}^{m+1} k_{\sigma(i)}) + 14)}{\eta(\sigma(m+1))} \leq \frac{1}{m+1}.$$

This completes the construction of the map σ. Take a bijective map

$$\psi\colon \mathbb{N} \to \{(i, \sigma(j)) \in \mathbb{N} \times \mathbb{N} \mid i \leq k_{\sigma(j)}, j \in \mathbb{N}\}$$

such that, if $\psi(n_1) = (i_1, \sigma(j_1))$, $\psi(n_2) = (i_2, \sigma(j_2))$ and $\sigma(j_1) < \sigma(j_2)$, then $n_1 < n_2$. For instance, take the inverse of the bijective map

$$\{(i, \sigma(j)) \in \mathbb{N} \times \mathbb{N} \mid i \leq k_{\sigma(j)}, j \in \mathbb{N}\} \to \mathbb{N},$$
$$(i, \sigma(j)) \mapsto i + \sum_{j'<j} k_{\sigma(j')}.$$

Then $\psi^{-1}(i, \sigma(j)) \leq \sum_{l=1}^{j} k_{\sigma(l)}$. Denote by $\tau_{i,j}$ an integer between 1 and p such that $\mathrm{supp}(f_{i,j}) \subset U_{\tau_{i,j}}$. Then apply Lemma 4.5 to the sequence of homeomorphisms $\varphi_{\tau_{\psi(n)}}^{-1} \circ f_{\psi(n)} \circ \varphi_{\tau_{\psi(n)}}$, where the φ_i were defined at the beginning of the proof. Let us denote by $\mathcal{G}$ the finite set given by Lemma 4.5. Let $\mathcal{G}_i$ be the finite set of homeomorphisms supported in U_i of the form $\varphi_i \circ s \circ \varphi_i^{-1}$, where s is a homeomorphism in $\mathcal{G}$. Let $\mathcal{G}' = \bigcup_{i=1}^{p} \mathcal{G}_i$. By Lemma 4.5, for all $n \in \mathbb{N}$, $l_{\mathcal{G}'}(f_{\psi(n)}) \leq C \log(n) + C'$. Now the homeomorphism $f^{\eta(\sigma(n))}$ can be decomposed as $f^{\eta(\sigma(n))} = g_1 \circ g_2 \circ \cdots \circ g_{l_{\sigma(n)}}$, where each of the homeomorphisms g_i belongs to the set $\{f_{1,\sigma(n)}, f_{2,\sigma(n)}, \ldots, f_{k_{\sigma(n)},\sigma(n)}\}$. Thus

$$l_{\mathcal{G}'}(f^{\eta(\sigma(n))}) \leq l_{\sigma(n)}\Big(C \log\Big(\max_{1 \leq i \leq k_{\sigma(n)}} \psi^{-1}(i, \sigma(n))\Big) + C'\Big).$$

Therefore

$$\frac{l_{\mathcal{G}'}(f^{\eta(\sigma(n))})}{\eta(\sigma(n))} \leq \frac{l_{\sigma(n)}(C \log(\sum_{i=1}^{n} k_{\sigma(i)}) + C')}{\eta(\sigma(n))} \leq \frac{1}{n}$$

and the homeomorphism f is a distortion element in $\mathrm{Homeo}_0(M)$ (respectively in $\mathrm{Homeo}_0(M, \partial M)$). □

Now, let us prove Lemma 4.5. This will require two lemmas.

Let a and b be the generators of the free semigroup L_2 on two generators. For two compactly supported homeomorphisms f and h of $\mathbb{R}^d$, let $\eta_{f,h}$ be the semigroup morphism from L_2 to the group of homeomorphisms of $\mathbb{R}^d$ defined by $\eta_{f,h}(a) = f$ and $\eta_{f,h}(b) = h$.

Lemma 4.8 *There exist compactly supported homeomorphisms s_1 and s_2 of $\mathbb{R}^d$ such that*

$$\text{for all } m \in L_2,\ m' \in L_2, \quad m \neq m' \Rightarrow \eta_{s_1,s_2}(m)(B(0,2)) \cap \eta_{s_1,s_2}(m')(B(0,2)) = \varnothing$$

and the diameter of $\eta_{s_1,s_2}(m)(B(0,2))$ converges to 0 when the length of m tends to infinity.

Let us denote by $\mathrm{Homeo}_0(\mathbb{R}^d)$ the group of compactly supported homeomorphisms of $\mathbb{R}^d$.

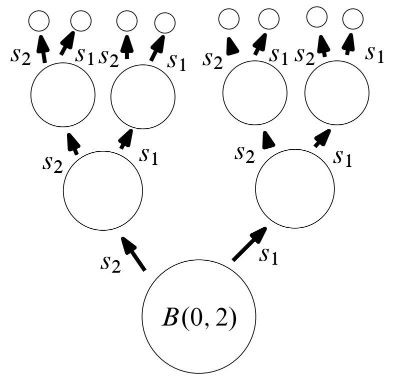

Figure 2: Lemma 4.8

Lemma 4.9 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(\mathbb{R}^d)$*. There exist two homeomorphisms h_1 and h_2 in* $\mathrm{Homeo}_0(\mathbb{R}^d)$ *such that $f = [h_1, h_2]$, where $[h_1, h_2] = h_1 \circ h_2 \circ h_1^{-1} \circ h_2^{-1}$.*

This lemma is classical and seems to appear for the first time in Anderson [1]. Let us prove it now.

Proof Denote by φ a homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^d)$ whose restriction to $B(0,2)$ is defined by

$$\begin{aligned} B(0,2) &\to \mathbb{R}^d, \\ x &\mapsto \frac{x}{2}. \end{aligned}$$

For any natural number n, let

$$A_n = \left\{ x \in \mathbb{R}^d \;\middle|\; \frac{1}{2^{n+1}} \leq \|x\| \leq \frac{1}{2^n} \right\}.$$

Let f be an element in $\mathrm{Homeo}_0(\mathbb{R}^N)$. As any element in $\mathrm{Homeo}_0(\mathbb{R}^N)$ is conjugate to an element supported in the interior of A_0, we may suppose that the homeomorphism f is supported in the interior of A_0. Then we define $h \in \mathrm{Homeo}_0(\mathbb{R}^d)$ by:

(1) $h = \mathrm{Id}$ outside $B(0,1)$

(2) for any natural number i, $h_{|A_i} = \varphi^i f \varphi^{-i}$

(3) $h(0) = 0$

Then $f = [h, \varphi]$. □

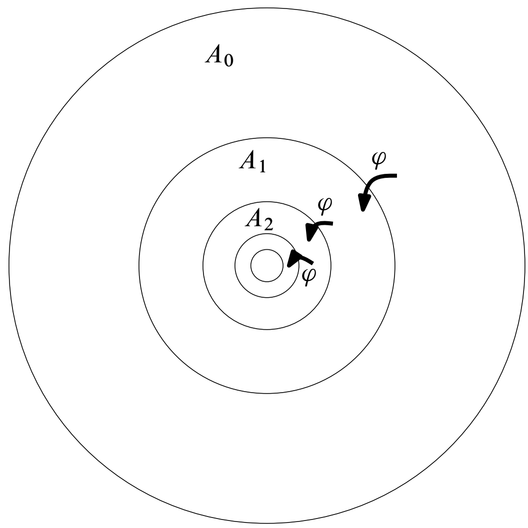

Figure 3: Proof of Lemma 4.9: Description of the homeomorphism φ

These two lemmas remain true when we replace $\mathbb{R}^d$ with H^d and $B(0,2)$ with $B(0,2) \cap H^d$.

Before proving Lemma 4.8, let us prove Lemma 4.5 with the help of these two lemmas.

Proof of Lemma 4.5 We prove the lemma in the case of homeomorphisms of $\mathbb{R}^d$. In the case of the half-space, the proof is similar. For an element m in L_2, let $l(m)$ be the length of m as a word in a and b. Let

$$\begin{aligned} \mathbb{N} &\to L_2, \\ n &\mapsto m_n, \end{aligned}$$

be a bijective map that satisfies $l(m_n) < l(m_{n'}) \Rightarrow n < n'$. This last condition implies that $l(m_n) = l \Leftrightarrow 2^l \leq n < 2^{l+1}$. In particular, for any natural number n, $l(m_n) \leq \log_2(n)$. Let s_1 and s_2 be the homeomorphisms in $\mathrm{Homeo}_0(\mathbb{R}^d)$ given by Lemma 4.8. Let s_3 be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^d)$ supported in the ball $B(0,2)$ that satisfies $s_3(B(0,1)) \cap B(0,1) = \varnothing$. We denote by B_n the closed ball $\eta_{s_1,s_2}(m_n)(B(0,1))$. By Lemma 4.9, there exist homeomorphisms $h_{n,1}$ and $h_{n,2}$ supported in $B(0,1)$ such that $f_n = [h_{n,1}, h_{n,2}]$.

Define the homeomorphism s_4 by

$$\begin{cases} \forall n \in \mathbb{N},\ s_{4|B_n} = \eta_{s_1,s_2}(m_n) \circ h_{n,1} \circ \eta_{s_1,s_2}(m_n)^{-1}, \\ \qquad s_4 = \mathrm{Id} \quad \text{on } \mathbb{R}^d - \bigcup_{n \in \mathbb{N}} B_n, \end{cases}$$

and the homeomorphism s_5 by

$$\begin{cases} \forall n \in \mathbb{N},\ s_{5|B_n} = \eta_{s_1,s_2}(m_n) \circ h_{n,2} \circ \eta_{s_1,s_2}(m_n)^{-1}, \\ \qquad\qquad s_5 = \mathrm{Id} \quad \text{on } \mathbb{R}^d - \bigcup_{n\in\mathbb{N}} B_n. \end{cases}$$

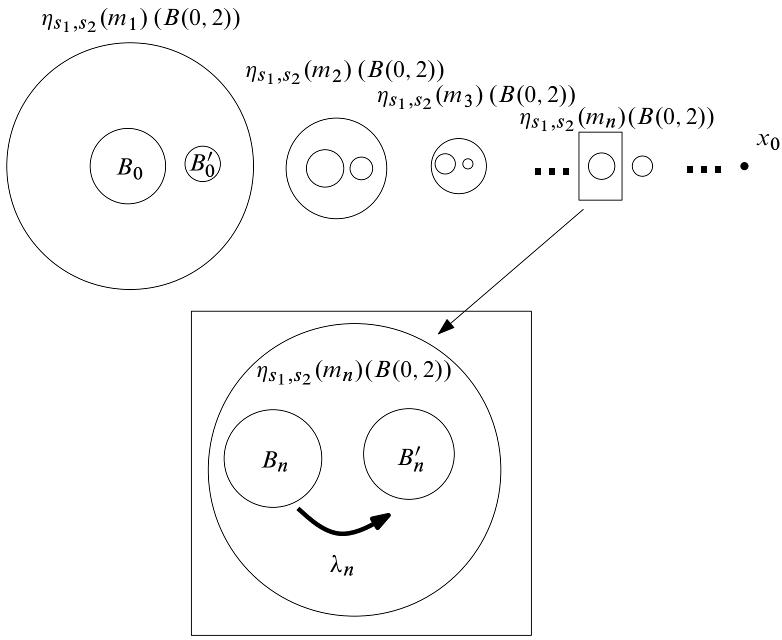

Figure 4: Notation in the proof of Lemma 4.5

Let $\mathcal{G} = \{s_i^{\epsilon} \mid i \in \{1,\dots,5\} \text{ and } \epsilon \in \{-1,1\}\}$. Let

$$\lambda_n = \eta_{s_1,s_2}(m_n) \circ s_3 \circ \eta_{s_1,s_2}(m_n)^{-1}, \quad B'_n = \lambda_n(B_n).$$

Notice that the balls B_n and B'_n are disjoint and contained in $\eta_{s_1,s_2}(m_n)(B(0,2))$. Notice also that:

$$s_4 \circ \lambda_n \circ s_4^{-1} \circ \lambda^{-1}_{n\ |\mathbb{R}^d-(B_n\cup B'_n)} = \mathrm{Id}$$
$$s_4 \circ \lambda_n \circ s_4^{-1} \circ \lambda^{-1}_{n\ |B_n} = \eta_{s_1,s_2}(m_n) \circ h_{n,1} \circ \eta_{s_1,s_2}(m_n)^{-1}$$
$$s_4 \circ \lambda_n \circ s_4^{-1} \circ \lambda^{-1}_{n\ |B'_n} = \lambda_n \circ \eta_{s_1,s_2}(m_n) \circ h^{-1}_{n,1} \circ \eta_{s_1,s_2}(m_n)^{-1} \circ \lambda_n^{-1}$$
$$s_5 \circ \lambda_n \circ s_5^{-1} \circ \lambda^{-1}_{n\ |\mathbb{R}^d-(B_n\cup B'_n)} = \mathrm{Id}$$
$$s_5 \circ \lambda_n \circ s_5^{-1} \circ \lambda^{-1}_{n\ |B_n} = \eta_{s_1,s_2}(m_n) \circ h_{n,2} \circ \eta_{s_1,s_2}(m_n)^{-1}$$
$$s_5 \circ \lambda_n \circ s_5^{-1} \circ \lambda^{-1}_{n\ |B'_n} = \lambda_n \circ \eta_{s_1,s_2}(m_n) \circ h^{-1}_{n,2} \circ \eta_{s_1,s_2}(m_n)^{-1} \circ \lambda_n^{-1}$$

$$s_4^{-1}s_5^{-1}\circ\lambda_n\circ s_5s_4\circ\lambda_{n\;|\mathbb{R}^d-(B_n\cup B_n')}^{-1}=\mathrm{Id}$$
$$s_4^{-1}s_5^{-1}\circ\lambda_n\circ s_5s_4\circ\lambda_{n\;|B_n}^{-1}=\eta_{s_1,s_2}(m_n)\circ h_{n,1}^{-1}h_{n,2}^{-1}\circ\eta_{s_1,s_2}(m_n)^{-1}$$
$$s_4^{-1}s_5^{-1}\circ\lambda_n\circ s_5s_4\circ\lambda_{n\;|B_n'}^{-1}=\lambda_n\circ\eta_{s_1,s_2}(m_n)\circ h_{n,2}h_{n,1}\circ\eta_{s_1,s_2}(m_n)^{-1}\circ\lambda_n^{-1}$$

Therefore, the homeomorphism $[s_4,\lambda_n][s_5,\lambda_n][s_4^{-1}s_5^{-1},\lambda_n]$ is equal to $\eta_{s_1,s_2}(m_n)\circ f_n\circ\eta_{s_1,s_2}(m_n)^{-1}$ on B_n and fixes the points outside B_n. Thus

$$f_n=\eta_{s_1,s_2}(m_n)^{-1}[s_4,\lambda_n][s_5,\lambda_n][s_4^{-1}s_5^{-1},\lambda_n]\eta_{s_1,s_2}(m_n).$$

Hence the homeomorphism f_n belongs to the group generated by $\mathcal{G}$ and

$$\begin{aligned}l_{\mathcal{G}}(f_n)&\leq 2l_{\mathcal{G}}(\eta_{s_1,s_2}(m_n))+6l_{\mathcal{G}}(\lambda_n)+8\\&\leq 2l_{\mathcal{G}}(\eta_{s_1,s_2}(m_n))+12l_{\mathcal{G}}(\eta_{s_1,s_2}(m_n))+14\\&\leq 14\log_2(n)+14.\end{aligned}$$
□

Proof of Lemma 4.8 First, let us prove the lemma in the case of homeomorphisms of $\mathbb{R}$. By perturbing two given homeomorphisms (as in Ghys [10]), one can find two compactly supported homeomorphisms $\hat{s}_1$ and $\hat{s}_2$ of $\mathbb{R}$ that satisfy the following property:

$$\text{for all } m\in L_2,\quad m'\in L_2,\ m\neq m'\Rightarrow \eta_{\hat{s}_1,\hat{s}_2}(m)(0)\neq\eta_{\hat{s}_1,\hat{s}_2}(m')(0)$$

Then, in the same way as in Denjoy's construction (see Katok and Hasselblatt [15] page 403), replace each point of the orbit of 0 under L_2 with an interval with positive length to obtain the wanted property. Thus, the proof is completed in the one-dimensional case. In the case of a higher dimension, denote by f and h the two homeomorphisms of $\mathbb{R}$ that we obtained in the one-dimensional case. Let $[-M,M]$ be an interval that contains the support of each of these homeomorphisms.

Let us look now at the case of $\mathbb{R}^d$. The homeomorphism

$$\begin{aligned}\mathbb{R}^d&\to\mathbb{R}^d,\\(x_1,x_2,\dots,x_d)&\mapsto(f(x_1),f(x_2),\dots,f(x_d)),\end{aligned}$$

preserves the cube $[-M,M]^d$. Let s_1 be a homeomorphism of $\mathbb{R}^d$ supported in $[-M-1,M+1]^d$ that is equal to the above homeomorphism on $[-M,M]^d$. Apply the same construction to the homeomorphism h to obtain a homeomorphism s_2. The ball centered on 0 in $\mathbb{R}^d$ of radius 2 is contained in the cube $[-2,2]^d$ and the diameter of the set

$$\eta_{s_1,s_2}(m)([-2,2]^d)=(\eta_{f,h}(m)([-2,2]))^d$$

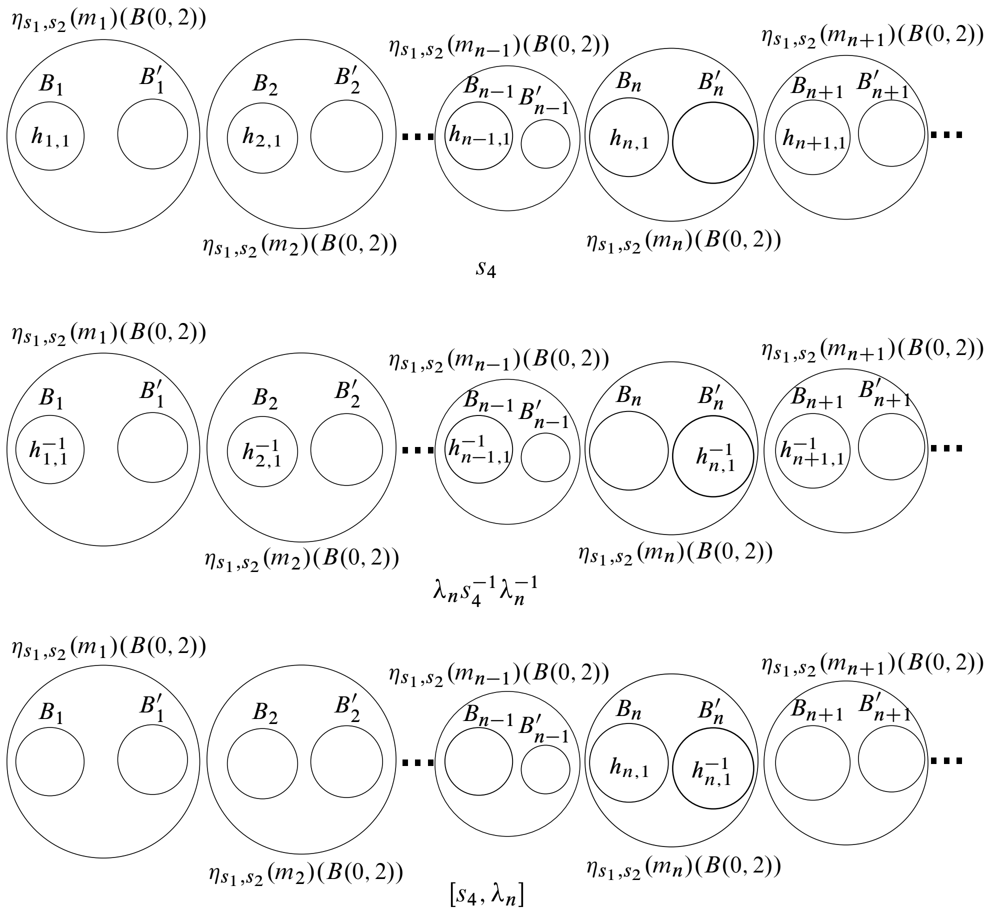

Figure 5a

converges to 0 when the length of the word m tends to infinity. The case of the half-spaces H^d is similar as long as compactly supported homeomorphisms that are equal to homeomorphisms of the form

$$\mathbb{R}_+ \times \mathbb{R}^{d-1} \to \mathbb{R}_+ \times \mathbb{R}^{d-1},$$
$$(t, x_1, x_2, \dots, x_{d-1}) \mapsto (\tfrac{t}{2}, f(x_1), f(x_2), \dots, f(x_{d-1})),$$

in a neighbourhood of 0 are used. □

5 Case of surfaces with boundary

Suppose that the boundary of the surface S is nonempty. Let us prove now Proposition 3.2. By considering a cover by half-discs, one can prove, with the same techniques as

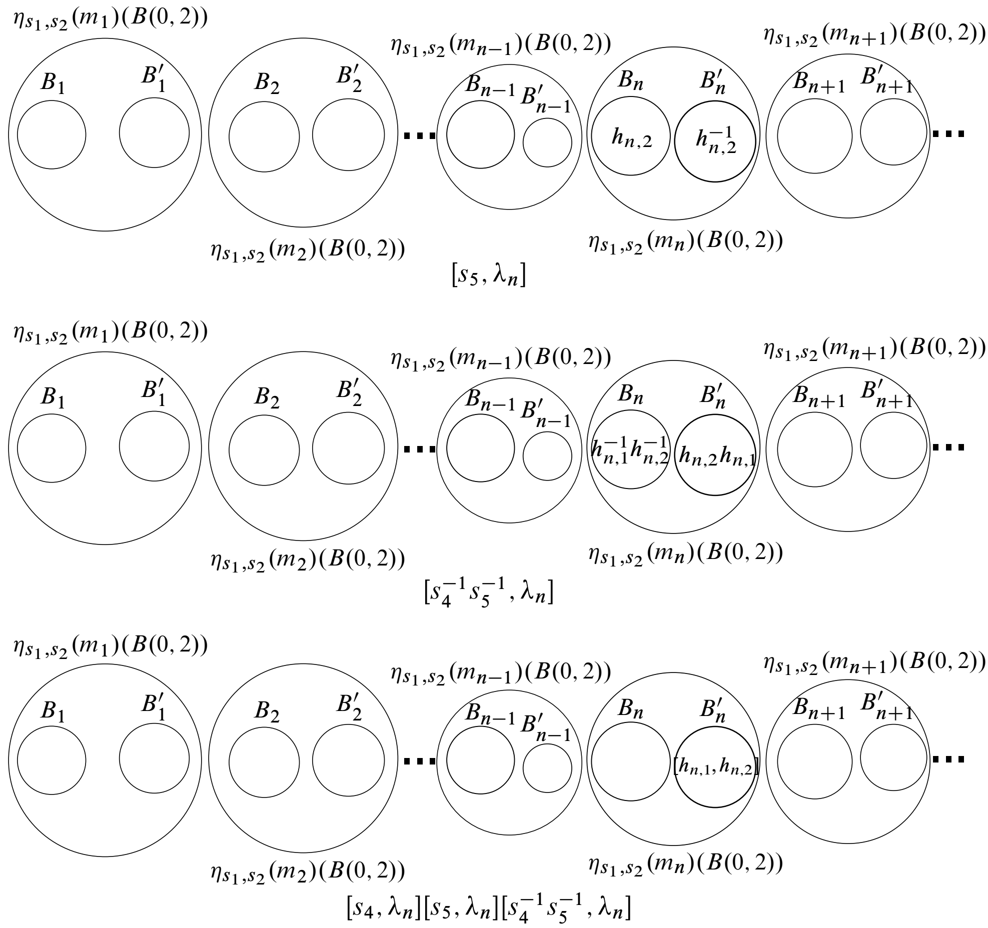

Figure 5b: The different homeomorphisms appearing in the proof of Lemma 4.5

below, Proposition 3.1 in the case where S has nonempty boundary: this case is left to the reader.

Recall that, in Section 3, we have chosen a "nice" polygonal fundamental domain D_0. Let B be the set of edges of the boundary ∂D_0 that are not contained in the boundary of $\widetilde{S}$ and let

$$A = \{\Pi(\beta) \mid \beta \in B\}.$$

For any edge α in A, let us consider a closed disc V_α with the following properties:

(1) The disc V_α does not meet the boundary of the surface S.

(2) The interior of V_α contains $\alpha \cap S'$.

(3) There exists a homeomorphism $\varphi_\alpha \colon V_\alpha \to \mathbb{D}^2$ that sends the set $\alpha \cap V_\alpha$ to the horizontal diameter of the unit disc $\mathbb{D}^2$.

Choose sufficiently thin discs V_α so that they are pairwise disjoint. Let U_1 be a closed disc that contains the union of the discs V_α. Let U_2 be a closed disc of S that satisfies the three following properties:

(1) The disc U_2 does not meet any edge in A, ie, it is contained in the interior of the fundamental domain D_0.

(2) The surface S' is contained in the interior of $\bigcup_{\alpha \in A} V_\alpha \cup U_2$.

(3) For any edge α in A, the set $U_2 \cap V_\alpha$ is homeomorphic to the disjoint union of two closed discs.

Let $\mathcal{U} = \{U_1, U_2\}$.

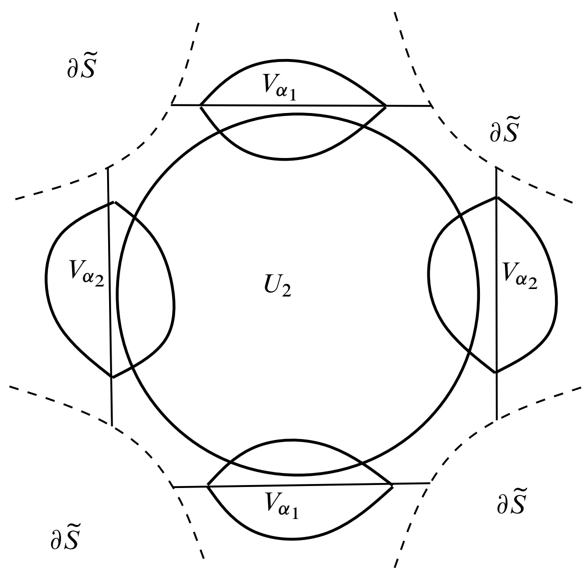

Figure 6: Notation in the case of surfaces with boundary

The proof of the inequality in the case of the group $\mathrm{Homeo}_0(S, \partial S)$ requires the following lemmas:

Lemma 5.1 *Let f be a homeomorphism in $\mathrm{Homeo}_0(S, \partial S)$ supported in the interior of $\bigcup V_\alpha \cup U_2$. Suppose that $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) \geq 2$. Then there exist homeomorphisms g_1, g_2 and g_3 in $\mathrm{Homeo}_0(S, \partial S)$ supported respectively in the interior of $\bigcup V_\alpha$, U_2 and $\bigcup V_\alpha$ such that the following property is satisfied:*

$$\mathrm{el}_{D_0}(\widetilde{g}_3 \circ \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\widetilde{f}(D_0)) - 1$$

Lemma 5.2 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(S, \partial S)$ *supported in the interior of* $\bigcup V_\alpha \cup U_2$. *If* $\mathrm{el}_{D_0}(\tilde{f}(D_0)) = 1$, *then* $\mathrm{Frag}_{\mathcal{U}}(f) \leq 6$.

End of the proof of Proposition 3.2 Let $k = \mathrm{el}_{D_0}(\tilde{f}(D_0))$. By Lemma 5.1, after composing the homeomorphism f with $3(k-1)$ homeomorphisms, each supported in one of the discs of $\mathcal{U}$, we obtain a homeomorphism f_1 supported in $\bigcup_{\alpha \in A} V_\alpha \cup U_2$ with $\mathrm{el}_{D_0}(\tilde{f_1}(D_0)) = 1$. Then, apply Lemma 5.2 to the homeomorphism f_1: $\mathrm{Frag}_{\mathcal{U}}(f_1) \leq 6$. Therefore $\mathrm{Frag}_{\mathcal{U}}(f) \leq 3(\mathrm{el}_{D_0}(\tilde{f}(D_0)) - 1) + 6$. However, as $D_0 \cap \tilde{f}(D_0) \neq \varnothing$ (the homeomorphism f pointwise fixes a neighbourhood of the boundary of S),

$$\mathrm{el}_{D_0}(\tilde{f}(D_0)) \leq \mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)).$$

Hence $\mathrm{Frag}_{\mathcal{U}}(f) \leq 3\,\mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)) + 3$. □

Notice that we indeed proved the following more precise proposition:

Proposition 5.3 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(S, \partial S)$ *supported in the interior of* $\bigcup_{\alpha \in A} V_\alpha \cup U_2$. *Then* $\mathrm{Frag}_{\mathcal{U}}(f) \leq 3\,\mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)) + 3$.

Proof of Lemma 5.1 Let us give an idea of the action of the homeomorphisms g_1, g_2 and g_3 that we will construct "by hand". If we look at the pieces of the disc $\tilde{f}(D_0)$ furthest from D_0, the homeomorphism g_1 repulses them back to the open set U_2, the homeomorphism g_2 repulses them outside the open set U_2 and the homeomorphism g_3 makes them exit from the fundamental domain of $\mathcal{D}$ in which these pieces were contained (see Figure 7). Let us give the precise construction of these homeomorphisms.

Let g_1 be a homeomorphism supported in $\bigcup_{\alpha \in A} V_\alpha$ such that:

(1) The homeomorphism g_1 pointwise fixes $\Pi(\partial D_0)$.
(2) For any edge α and any connected component C of $V_\alpha \cap f(\Pi(\partial D_0))$ that does not meet $\Pi(\partial D_0)$, $g_1(C) \subset \mathring{U}_2$.

One can build such a homeomorphism g_1 by taking the time 1 of the flow of a well-chosen vector field that vanishes on $\Pi(\partial D_0)$.

Let g_2 be a homeomorphism supported in U_2 that satisfies the following property: For any edge α in A and for any connected component C of $\mathring{U}_2 \cap g_1 \circ f(\Pi(\partial D_0))$ both of whose ends (ie, the points of the closure of C which do not belong to C) belong to the same connected component of $V_\alpha - \alpha$, the set $g_2(C)$ is contained in $\mathring{V}_\alpha$. Let us explain how such a homeomorphism g_2 can be built. We will need the following elementary lemma, which is a consequence of the Schönflies theorem:

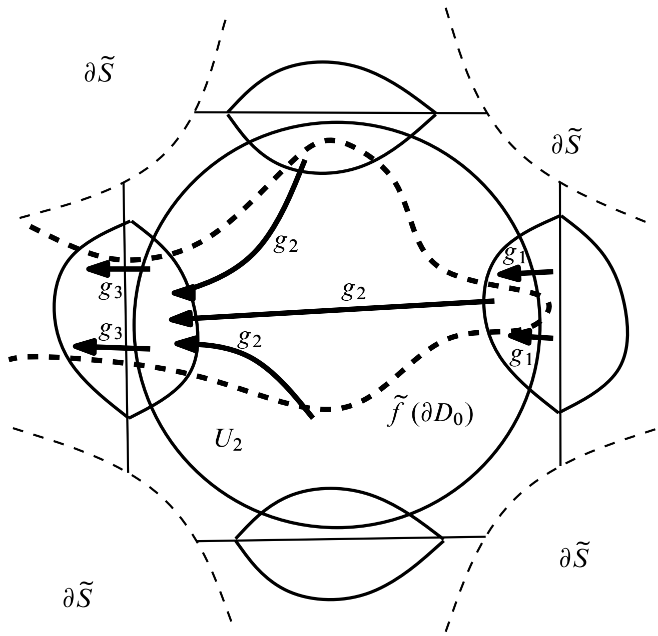

Figure 7: Illustration of the proof of Lemma 5.1

Lemma 5.4 *Let* $c_1\colon [0,1] \to \mathbb{D}^2$ *and* $c_2\colon [0,1] \to \mathbb{D}^2$ *be two injective curves that are equal in neighbourhoods of 0 and 1, and such that:*

(1) $c_1(0) = c_2(0) \in \partial\mathbb{D}^2$ *and* $c_1(1) = c_2(1) \in \partial\mathbb{D}^2$

(2) $c_1((0,1)) \subset \mathbb{D}^2 - \partial\mathbb{D}^2$ *and* $c_2((0,1)) \subset \mathbb{D}^2 - \partial\mathbb{D}^2$

Then, there exists a homeomorphism h *in* $\mathrm{Homeo}_0(\mathbb{D}^2, \partial\mathbb{D}^2)$ *such that for all* $t \in [0,1]$, $h(c_1(t)) = c_2(t)$.

Corollary 5.5 *Let* $(c_i)_{1\leq i\leq l}$ *and* $(c'_i)_{1\leq i\leq l}$ *be finite sequences of injective curves* $[0,1] \to \mathbb{D}^2$ *of the closed disc* $\mathbb{D}^2$ *such that:*

(1) *For any index* $1 \leq i \leq l$, *the maps* c_i *and* c'_i *are equal in a neighbourhood of* 0 *and of* 1.

(2) *The curves* c_i, *as well as the curves* c'_i, *are pairwise disjoint.*

(3) *For any index* i, *the points* $c_i(0)$ *and* $c_i(1)$ *belong to the boundary of the disc.*

(4) *For any index* i, *the sets* $c_i((0,1))$ *and* $c'_i((0,1))$ *are contained in* $\mathbb{D}^2 - \partial\mathbb{D}^2$.

Then there exists a homeomorphism h *in* $\mathrm{Homeo}_0(\mathbb{D}^2, \partial\mathbb{D}^2)$ *such that, for any index* $1 \leq i \leq l$, *for any* $t \in [0,1]$, $h(c_i(t)) = c'_i(t)$.

Proof of the corollary It suffices to use Lemma 5.4 and an induction argument. □

First, let us notice that only a finite number of connected components of

$$\mathring{U}_2 \cap g_1 \circ f(\Pi(\partial D_0))$$

are not contained in one of the open discs $\mathring{V}_\alpha$. We denote by $\mathcal{C}$ the set of such connected components with both ends in the same connected component of $V_\alpha - \alpha$, for some edge α in A. Let us fix now an edge α in A. Let C be a connected component in $\mathcal{C}$ whose ends both belong to V_α. We denote by $a_C\colon [0,1] \to U_2$ an injective path such that:

(1) The set $a_C((0,1))$ is contained in $\mathring{V}_\alpha \cap U_2$.
(2) The path a_C is equal to the path $\overline{C}$ in a neighbourhood of $a_C(0)$ and of $a_C(1)$.
(3) The path a_C does not meet the connected components of $g_1 \circ f(\Pi(\partial D_0)) \cap \mathring{U}_2$ that do not belong to $\mathcal{C}$.

The construction is made in such a way that the paths a_C are pairwise disjoint. Denote by Δ the closure of a connected component of

$$\mathring{U}_2 - \bigcup C,$$

where the union is taken over the connected components C of $g_1 \circ f(\Pi(\partial D_0)) \cap \mathring{U}_2$ that do not belong to $\mathcal{C}$. By a theorem by Kerekjarto (see Le Calvez and Yoccoz [18, page 246]), the set Δ is homeomorphic to a closed disc. Then, for each such disc Δ, we apply the last corollary in the disc Δ to the families of paths $(C)_{C \in \mathcal{C}, C \subset \Delta}$ and $(a_C)_{C \in \mathcal{C}, C \subset \Delta}$ to build the homeomorphism g_2 that we wanted.

Finally, let g_3 be a homeomorphism supported in $\bigcup_{\alpha \in A} V_\alpha$ that satisfies, for any edge α in A, the following properties:

(1) For any connected component C of $\mathring{V}_\alpha \cap g_2 \circ g_1 \circ f(\Pi(\partial D_0))$ whose ends both belong to the same connected component of $V_\alpha - \alpha$, $g_3(C) \cap \alpha = \varnothing$.
(2) The homeomorphism g_3 pointwise fixes any other connected component of $\mathring{V}_\alpha \cap g_2 \circ g_1 \circ f(\Pi(\partial D_0))$.

The construction of the homeomorphism g_3 is analogous to the construction of the homeomorphism g_2. In what follows, we will not give details anymore on this kind of construction.

We claim that homeomorphisms g_1, g_2 and g_3 that satisfy the above properties satisfy also the conclusion of Lemma 5.1. This is a consequence of the two following claims.

Claim 1 The set of fundamental domains in $\mathcal{D}$ which meet $\tilde{g}_3 \circ \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(D_0)$ is contained in the set of fundamental domains of $\mathcal{D}$ which meet $\tilde{f}(D_0)$.

If h is a homeomorphism in $\mathrm{Homeo}_0(S, \partial S)$, we say that a fundamental domain D in $\mathcal{D}$ is maximal for $\tilde{h}$ if it meets $\tilde{h}(D_0)$ and satisfies $d_{\mathcal{D}}(D, D_0) = \mathrm{el}_{D_0}(\tilde{h}(D_0))$.

Claim 2 The fundamental domains D in $\mathcal{D}$ that are maximal for $\tilde{f}$ do not meet $\tilde{g}_3 \circ \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(D_0)$.

Let us assume for the moment that these two claims are true and let us prove Lemma 5.1.

Claim 1 implies that

$$\mathrm{el}_{D_0}(\tilde{g}_3 \circ \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\tilde{f}(D_0)).$$

Suppose that we have an equality in the above inequality. Then there exists a fundamental domain D in $\mathcal{D}$ that is maximal for $\tilde{f}$ and that meets $\tilde{g}_3 \circ \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(D_0)$, a contradiction to Claim 2. This proves the lemma.

Now, let us prove Claim 1. Notice that, for any homeomorphism h in $\mathrm{Homeo}_0(S, \partial S)$, the set of fundamental domains in $\mathcal{D}$ met by $\tilde{h}(D_0)$ is equal to the set of fundamental domains in $\mathcal{D}$ met by $\tilde{h}(\partial D_0)$. Indeed, the interior of a fundamental domain cannot contain a fundamental domain.

As the homeomorphisms $\tilde{g}_1$ and $\tilde{g}_2$ both pointwise fix $\bigcup_{D \in \mathcal{D}} \partial D$, the set of elements of $\mathcal{D}$ met by the curve $\tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(\partial D_0)$ is equal to the set of elements of $\mathcal{D}$ met by $\tilde{f}(\partial D_0)$. Therefore, it suffices to prove the following inclusion:

$$\{D \in \mathcal{D} \mid \tilde{g}_3 \circ \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(\partial D_0) \cap D \neq \varnothing\} \subset \{D \in \mathcal{D} \mid \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(\partial D_0) \cap D \neq \varnothing\}$$

Let D be a fundamental domain that belongs to the left-hand set in the above inclusion. Let $\tilde{x}$ be a point in $\tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(\partial D_0)$ that satisfies $\tilde{g}_3(\tilde{x}) \in D$. If the point $\tilde{x}$ belongs to the fundamental domain D, then the fundamental domain D belongs to

$$\{D' \in \mathcal{D} \mid \tilde{g}_2 \circ \tilde{g}_1 \circ \tilde{f}(\partial D_0) \cap D' \neq \varnothing\}.$$

Let us suppose that the point $\tilde{x}$ does not belong to the fundamental domain D. As the homeomorphism g_3 is supported in $\bigcup_{\beta \in A} V_\beta$, there exists an edge α in A such that the point $\Pi(\tilde{x})$ belongs to the disc V_α. Let $\tilde{V}_\alpha$ be the lift of the disc V_α that contains $\tilde{x}$. By construction of the homeomorphism $\tilde{g}_3$, the point $\tilde{x}$ belongs to a connected component $\tilde{C}$ of

$$\tilde{g}_2 \circ \tilde{g}_1(\partial D_0) \cap \mathring{\tilde{V}}_\alpha$$

whose ends both belong to the interior $\mathring{D}'$ of a same fundamental domain D' in $\mathcal{D}$. Let us recall that the connected components that are not of this kind are fixed by the homeomorphism g_3. By the definition of $\widetilde{g}_3$, we have $\widetilde{g}_3(\widetilde{x}) \in \widetilde{g}_3(\widetilde{C}) \subset \mathring{D}'$ and, by hypothesis, $\widetilde{g}_3(\widetilde{x}) \in D$. Thus, $D' = D$ and, as the fundamental domain D' meets $\widetilde{C} \subset \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\partial D_0)$, the fundamental domain D belongs to the set

$$\{D \in \mathcal{D} \mid \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap D \neq \varnothing\}.$$

We now come to the proof of Claim 2. As in Section 3, let

$$\mathcal{G} = \{a_i \mid i \in \{1, \dots, P\}\} \cup \{a_i^{-1} \mid i \in \{1, \dots, P\}\}$$

be the generating set for the group $\pi_1(S)$, which consists of the deck transformations that send the fundamental domain D_0 to a fundamental domain in $\mathcal{D}$ adjacent to D_0. As, in the case under discussion, the surface S has nonempty boundary, the group $\pi_1(S)$ is the free group generated by $\{a_1, a_2, \dots, a_p\}$. Let $D_{\max}$ be a fundamental domain in $\mathcal{D}$ that is maximal for $\widetilde{f}$. By definition,

$$d_{\mathcal{D}}(D_{\max}, D_0) = \mathrm{el}_{D_0}(\widetilde{f}(D_0)).$$

Let us denote by γ the deck transformation that sends D_0 to $D_{\max}$. The element γ can be uniquely written as a reduced word in elements of $\mathcal{G}$: $\gamma = s_1 s_2 \cdots s_n$, where the s_i belong to the generating set $\mathcal{G}$ and $n = d_{\mathcal{D}}(D_{\mathrm{ex}}, D_0)$. Every fundamental domain in $\mathcal{D}$ adjacent to $D_{\max}$ is a domain of the form $\gamma(s(D_0))$, where s is an element in $\mathcal{G}$. If the element s is different from s_n^{-1}, then

$$d_{\mathcal{D}}(\gamma(s(D_0)), D_0) = l_{\mathcal{G}}(\gamma s) = n + 1 > n = \mathrm{el}_{D_0}(\widetilde{f}(\partial D_0)).$$

Thus, the only face adjacent to $D_{\max}$ that meets $\widetilde{f}(\partial D_0)$ is $\gamma \circ s_n^{-1}(D_0)$. We denote by $\widetilde{\alpha}$ the edge that is contained in the fundamental domains $\gamma \circ s_n^{-1}(D_0)$ and $D_{\max}$. The ends of any connected component of $\widetilde{f}(\partial D_0) \cap D_{\max}$ belong to $\widetilde{\alpha}$. Let $\widetilde{V}_{\widetilde{\alpha}}$ be the lift of $V_{\Pi(\widetilde{\alpha})}$ that contains $\widetilde{\alpha}$. We claim that

$$\widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap D_{\max} \subset \widetilde{V}_{\widetilde{\alpha}} \cup \widetilde{U}_2,$$

where $\widetilde{U}_2$ is the lift of U_2 that is contained in $D_{\max}$.

Let us prove this last claim. For a point $\widetilde{x}$ in $D_{\max} \cap \widetilde{f}(\partial D_0) \cap \Pi^{-1}(V_\beta) - \widetilde{V}_{\widetilde{\alpha}}$, where β is an edge in A, the connected component of $\widetilde{f}(\partial D_0) \cap \Pi^{-1}(\mathring{V}_\beta)$ that contains $\widetilde{x}$ does not meet the set $\Pi^{-1}(\beta)$. Hence the point $\widetilde{g}_1(\widetilde{x})$ belongs to U_2, by construction of g_1. Moreover, the homeomorphism $\widetilde{g}_1$ preserves the sets

$$\widetilde{U}_2 - \Big(\bigcup_{\beta \in A} \Pi^{-1}(V_\beta)\Big) \quad \text{and} \quad \widetilde{V}_{\widetilde{\alpha}}.$$

The claim is proved.

Notice also that

$$\widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap D_{\max} \subset \mathring{\widetilde{V}}_{\widetilde{\alpha}}.$$

Indeed, the ends of any connected component of $\widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap \mathring{\widetilde{U}}_2$ belong to $\widetilde{V}_{\widetilde{\alpha}}$.

Let us prove that

$$\widetilde{g}_3 \circ \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap D_{\max} = \varnothing.$$

Let C be a connected component of $\widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap \mathring{\widetilde{V}}_{\widetilde{\alpha}}$. As

$$\widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\partial D_0) \cap D_{\max} \subset \mathring{\widetilde{V}}_{\widetilde{\alpha}},$$

the ends of C do not belong to $\mathring{D}_{\max} \cap \widetilde{V}_{\widetilde{\alpha}}$ but to $\gamma \circ s_n^{-1}(\mathring{D}_0) \cap \widetilde{V}_{\widetilde{\alpha}}$, which is the other connected component of $\widetilde{V}_{\widetilde{\alpha}} - \widetilde{\alpha}$ (the ends of C do not belong to α because $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) = d_{\mathcal{D}}(D_{\max}, D_0) \geq 2$). By construction of the homeomorphism g_3,

$$\widetilde{g}_3(C) \subset \gamma \circ s_n^{-1}(\mathring{D}_0).$$

Thus, the set $\widetilde{g}_3(C)$ is disjoint from $D_{\max}$, which completes the proof of the second claim. □

Proof of Lemma 5.2 For any edge $\widetilde{\alpha}$ in B, we denote by $D_{\widetilde{\alpha}}$ the fundamental domain in $\mathcal{D}$ that satisfies $D_0 \cap D_{\widetilde{\alpha}} = \widetilde{\alpha}$. Let us fix an edge $\widetilde{\alpha}$ in B. As $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) = 1$, the curve $\widetilde{f}(\widetilde{\alpha})$ does not meet fundamental domains in $\mathcal{D}$ adjacent to $D_{\widetilde{\alpha}}$ and different from D_0: these fundamental domains are at distance 2 from D_0. Let us prove that, if $\widetilde{\beta}$ is an edge in B different from $\widetilde{\alpha}$, then $\widetilde{f}(\widetilde{\alpha}) \cap D_{\widetilde{\beta}} = \varnothing$. Otherwise, we would have $\widetilde{f}(D_{\widetilde{\alpha}}) \cap D_{\widetilde{\beta}} \neq \varnothing$, for an edge $\widetilde{\beta}$ different from $\widetilde{\alpha}$. Let us denote by s the deck transformation which sends D_0 to $D_{\widetilde{\alpha}}$. Then

$$2 = d_{\mathcal{D}}(D_{\widetilde{\alpha}}, D_{\widetilde{\beta}}) = d_{\mathcal{D}}(D_0, s^{-1}(D_{\widetilde{\beta}})).$$

Moreover $\widetilde{f}(s(D_0)) \cap D_{\widetilde{\beta}} \neq \varnothing$. Hence $\widetilde{f}(D_0) \cap s^{-1}(D_{\widetilde{\beta}}) \neq \varnothing$. It contradicts the hypothesis $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) = 1$. Thus, for any edge $\widetilde{\alpha}$ in B,

$$\widetilde{f}(\widetilde{\alpha}) \subset \mathring{D}_{\widetilde{\alpha}} \cup \mathring{D}_0 \cup \widetilde{\alpha}.$$

For an edge $\widetilde{\alpha}$ in B, we denote by $\widetilde{V}_{\widetilde{\alpha}}$ the lift of $V_{\Pi(\widetilde{\alpha})}$ which contains the edge $\widetilde{\alpha}$.

We now build homeomorphisms g_1 and g_2 supported respectively in $\bigcup_{\alpha \in A} V_\alpha$ and in U_2 such that

$$\text{for any } \widetilde{\alpha} \in B, \quad \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\widetilde{\alpha}) \subset \mathring{\widetilde{V}}_{\widetilde{\alpha}} \cup \widetilde{\alpha}.$$

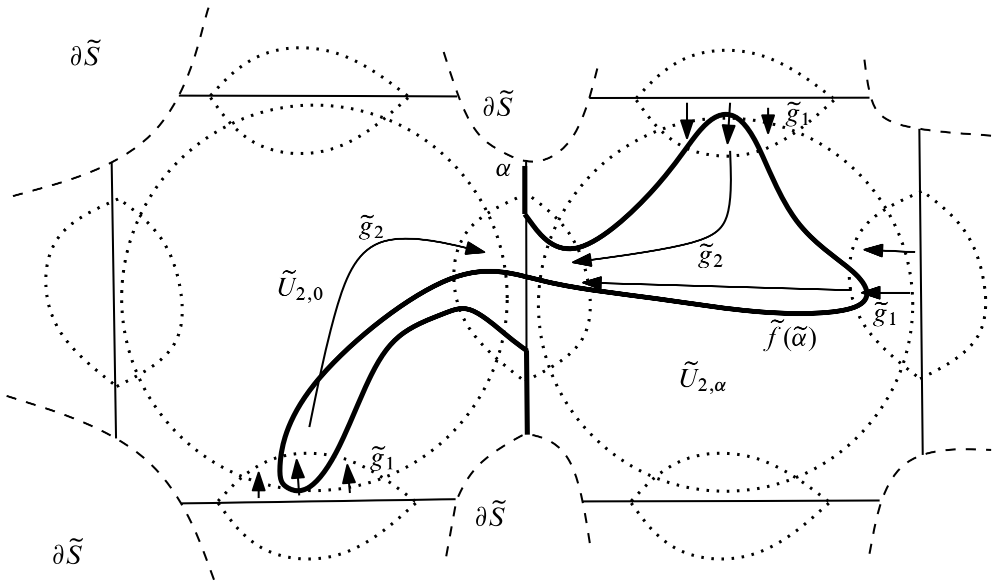

Figure 8: Proof of Lemma 5.2: The homeomorphisms g_1 and g_2

As in the proof of Lemma 5.1, we build homeomorphisms g_1 and g_2 that satisfy the following properties:

(1) The homeomorphism g_1 is supported in $\bigcup_{\alpha \in A} V_\alpha$ and pointwise fixes ∂D_0.

(2) For any edge α in A and any connected component C of $f(\Pi(\partial D_0)) \cap \mathring{V}_\alpha$ that does not meet α, we have $g_1(C) \subset \mathring{U}_2$.

(3) The homeomorphism g_2 is supported in U_2.

(4) For any edge α in A and any connected component C of $g_1 \circ f(\Pi(\partial D_0)) \cap \mathring{U}_2$ whose ends belong to the same connected component of $V_\alpha - \alpha$, $g_2(C) \subset \mathring{V}_\alpha$.

Let us denote by $\widetilde{U}_{2,0}$ the lift of the disc U_2 contained in D_0 and, for any edge $\widetilde{\alpha}$ in B, $\widetilde{U}_{2,\widetilde{\alpha}}$ the lift of the disc U_2 contained in $D_{\widetilde{\alpha}}$. As in the proof of Lemma 5.1, for any edge $\widetilde{\alpha}$ in B,

$$\widetilde{g}_1 \circ \widetilde{f}(\widetilde{\alpha}) \subset \mathring{\widetilde{U}}_{2,0} \cup \widetilde{V}_{\widetilde{\alpha}} \cup \mathring{\widetilde{U}}_{2,\widetilde{\alpha}} \quad \text{and} \quad \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}(\widetilde{\alpha}) \subset \mathring{\widetilde{V}}_{\widetilde{\alpha}}.$$

We will now build homeomorphisms g_3 and g_4 of S supported respectively in $\bigcup_{\alpha \in A} V_\alpha$ and U_2 such that, for any edge $\widetilde{\alpha}$ in B, the homeomorphism $\widetilde{g}_4 \circ \widetilde{g}_3 \circ \widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f}$ pointwise fixes $\partial \widetilde{V}_{\widetilde{\alpha}}$.

Let g_3 be a homeomorphism supported in $\bigcup_{\alpha \in A} V_\alpha$ that satisfies the following properties:

(1) The homeomorphism g_3 pointwise fixes $g_2 \circ g_1 \circ f(\alpha)$.

(2) For any connected component C of $g_2 \circ g_1 \circ f(\partial V_\alpha) \cap \mathring{V}_\alpha$: $g_3(C) \subset \mathring{U}_2$.

Then, the set $g_3 \circ g_2 \circ g_1 \circ f(\partial V_\alpha) \Delta \partial V_\alpha$ is contained in $\mathring{U}_2$.

We impose that the homeomorphism g_4 is supported in U_2 and satisfies the following property: The homeomorphism g_4 is equal to $(g_3 \circ g_2 \circ g_1 \circ f)^{-1}$ on the closed set $g_3 \circ g_2 \circ g_1 \circ f(\partial V_\alpha)$. Thus, as the homeomorphism $g_4 \circ g_3 \circ g_2 \circ g_1 \circ f$ pointwise fixes $\bigcup_{\alpha \in A} \partial V_\alpha$, the map $g_5\colon S \to S$, which is equal to $g_4 \circ g_3 \circ g_2 \circ g_1 \circ f$ on $\bigcup_{\alpha \in A} V_\alpha$ and to the identity outside this set, is a homeomorphism of S supported in $\bigcup_{\alpha \in A} V_\alpha$. Let $g_6 = (g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ f)^{-1}$. Then the homeomorphism g_6 is supported in U_2 and we have

$$f = g_1^{-1} \circ g_2^{-1} \circ g_3^{-1} \circ g_4^{-1} \circ g_5^{-1} \circ g_6^{-1}.$$

This implies that $\mathrm{Frag}_{\mathcal{U}}(f) \leq 6$. □

6 Case of the torus

In this section, we prove Proposition 3.1 in the case of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We set $D_0 = [0,1]^2$ and the covering Π is given by the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. We denote by A_0 (respectively A_1, B_0, B_1) the closed annulus $[-\frac{1}{4}, \frac{1}{2}] \times \mathbb{R}/\mathbb{Z} \subset \mathbb{T}^2$ (respectively $[\frac{1}{4}, 1] \times \mathbb{R}/\mathbb{Z}$, $\mathbb{R}/\mathbb{Z} \times [-\frac{1}{4}, \frac{1}{2}]$, $\mathbb{R}/\mathbb{Z} \times [\frac{1}{4}, 1]$). For any integer i, we denote by $\widetilde{A}_0^i$ (respectively $\widetilde{A}_1^i$, $\widetilde{B}_0^i$, $\widetilde{B}_1^i$) the band of the plane

$[i - \frac{1}{4}, i + \frac{1}{2}] \times \mathbb{R}$ (respectively $[i + \frac{1}{4}, i+1] \times \mathbb{R}$, $\mathbb{R} \times [i - \frac{1}{4}, i + \frac{1}{2}]$, $\mathbb{R} \times [i + \frac{1}{4}, i+1]$).

Finally, for $i \in \mathbb{Z}$ and $j \in \{0,1\}$, we denote by $\widetilde{\alpha}_j^i$ (respectively $\widetilde{\beta}_j^i$) the curve $\{i + \frac{j}{2}\} \times \mathbb{R}$ (respectively $\mathbb{R} \times \{i + \frac{j}{2}\}$). Let $\mathcal{U}$ be the cover of the torus $\mathbb{T}^2$ defined by

$$\mathcal{U} = \{I \times J \mid I, J \in \{[-\tfrac{1}{4}, \tfrac{1}{2}], [\tfrac{1}{4}, 1]\}\} = \{A_j \cap B_{j'} \mid j, j' \in \{0,1\}\}.$$

For a compact subset A of $\mathbb{R}^2$, we set

$$\mathrm{length}(A) = \mathrm{card}\{(i,j) \in \mathbb{Z} \times \{0,1\} \mid \widetilde{\alpha}_j^i \cap A \neq \varnothing\},$$
$$\mathrm{height}(A) = \mathrm{card}\{(i,j) \in \mathbb{Z} \times \{0,1\} \mid \widetilde{\beta}_j^i \cap A \neq \varnothing\}.$$

We claim that, for any compact path-connected subset A of $\mathbb{R}^2$,

$$\mathrm{length}(A) \leq 2\,\mathrm{diam}_{\mathcal{D}}(A) \quad \text{and} \quad \mathrm{height}(A) \leq 2\,\mathrm{diam}_{\mathcal{D}}(A).$$

Indeed, suppose that the set A meets the curves $\widetilde{\alpha}_0^{n_0}, \widetilde{\alpha}_0^{n_0+1}, \ldots, \widetilde{\alpha}_0^{n_0+N-1}$, for some $n_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. Then there exist fundamental domains D_1 and D_2 in $\mathcal{D}$ such that:

(1) The fundamental domain D_1 lies between the lines $\widetilde{\alpha}_0^{n_0-1}$ and $\widetilde{\alpha}_0^{n_0}$.
(2) The fundamental domain D_2 lies between the lines $\widetilde{\alpha}_0^{n_0+N-1}$ and $\widetilde{\alpha}_0^{n_0+N}$.
(3) The sets D_1 and D_2 meet A.

Then $N \leq d_{\mathcal{D}}(D_1, D_2) \leq \mathrm{diam}_{\mathcal{D}}(A)$. Similarly, if the set A meets the curves $\widetilde{\alpha}_1^{n_0}, \widetilde{\alpha}_1^{n_0+1}, \dots, \widetilde{\alpha}_1^{n_0+N-1}$ or the curves $\widetilde{\beta}_j^{n_0}, \widetilde{\beta}_j^{n_0+1}, \dots, \widetilde{\beta}_j^{n_0+N-1}$ for some $n_0 \in \mathbb{Z}$, $N \in \mathbb{N}$, $j \in \{0, 1\}$, then $N \leq \mathrm{diam}_{\mathcal{D}}(A)$. This proves the claim.

Let us fix a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{T}^2)$ and a lift $\widetilde{f}$ of f. Let $i_{\max,\alpha} \in \mathbb{Z}$ and $j_{\max,\alpha} \in \{0, 1\}$ (respectively $i_{\max,\beta}$ and $j_{\max,\beta}$) be the integers that satisfy

$$i_{\max,\alpha} + \tfrac{1}{2} j_{\max,\alpha} = \max\{i + \tfrac{1}{2} j \mid \widetilde{f}(D_0) \cap \widetilde{\alpha}_j^i \neq \varnothing\}$$
$$\text{(respectively} \quad i_{\max,\beta} + \tfrac{1}{2} j_{\max,\beta} = \max\{i + \tfrac{1}{2} j \mid \widetilde{f}(D_0) \cap \widetilde{\beta}_j^i \neq \varnothing\}).$$

Let (i_α, j_α) (respectively (i_β, j_β)) be the pair such that the interior of the band $\widetilde{A}_{j_\alpha}^{i_\alpha}$ (respectively $\widetilde{B}_{i_\alpha}^{i_\beta}$) contains the curve $\widetilde{\alpha}_{j_{\max,\alpha}}^{i_{\max,\alpha}} = \widetilde{\alpha}_{\max}$ (respectively $\widetilde{\beta}_{j_{\max,\beta}}^{i_{\max,\beta}} = \widetilde{\beta}_{\max}$). See Figure 9.

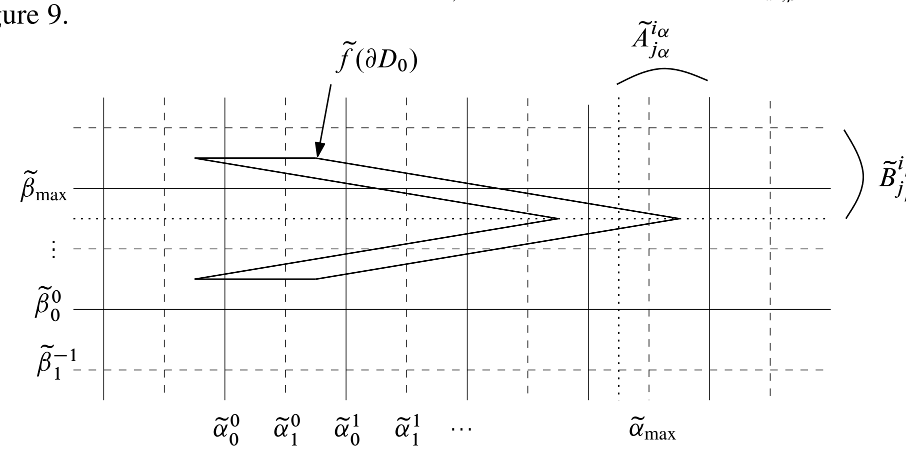

Figure 9: Notation in the case of the torus

Definition 6.1 The connected components of $\mathring{A}_{j_\alpha} \cap f(\Pi(\partial D_0))$ can be split into two classes:

(1) On the one hand, the connected components that are homeomorphic to $\mathbb{R}$ will be called the *regular connected components* of $\mathring{A}_{j_\alpha} \cap f(\Pi(\partial D_0))$.
(2) On the other hand, there exists at most one connected component homeomorphic to the union of two transverse straight lines in $\mathbb{R}^2$. This is the connected component that contains the point $f(0, 0)$. We will call it the *singular connected component* of $\mathring{A}_{j_\alpha} \cap f(\Pi(\partial D_0))$.

The connected components of $\mathring{B}_{j_\beta} \cap f(\Pi(\partial D_0))$ can be analogously split.

Suppose that either height$(\widetilde{f}(D_0)) > 5$ or length$(\widetilde{f}(D_0)) > 5$. We claim that one of the following cases occurs.

First case There exists a connected component $\widetilde{C}$ of $\Pi^{-1}(\mathring{A}_{j_\alpha}) \cap \widetilde{f}(\partial D_0)$ such that:

1. The ends of $\widetilde{C}$ belong to two different connected component of the boundary of $\Pi^{-1}(A_{j_\alpha})$.
2. height$(\widetilde{C}) \leq 5$.
3. $\Pi(\widetilde{C})$ is a regular connected component of $\mathring{A}_{j_\alpha} \cap f(\Pi(\partial D_0))$.

Second case There exists a connected component $\widetilde{C}$ of $\Pi^{-1}(\mathring{B}_{j_\beta}) \cap \widetilde{f}(\partial D_0)$ such that:

1. The ends of $\widetilde{C}$ belong to two different connected components of the boundary of $\Pi^{-1}(B_{j_\beta})$.
2. length$(\widetilde{C}) \leq 5$.
3. $\Pi(\widetilde{C})$ is a regular connected component of $\mathring{B}_{j_\beta} \cap f(\Pi(\partial D_0))$.

Let us prove this claim. Suppose first that the length of $\widetilde{f}(D_0)$ is greater than 5. Then there exists a regular connected component C of $\mathring{A}_{j_\alpha} \cap f(\Pi(\partial D_0))$ whose ends belong to different boundary components of A_{j_α}. Take a lift $\widetilde{C}$ of C contained in $\widetilde{f}(\partial D_0)$. If the first case does not occur for $\widetilde{C}$, the height of $\widetilde{C}$ is greater than 5. Therefore, there exists a connected component $\widetilde{C}'$ of $\Pi^{-1}(\mathring{B}_{j_\beta}) \cap \widetilde{C}$ whose ends belong to two different connected components of the boundary of $\Pi^{-1}(B_{j_\beta})$. In this case, the length of the component $\widetilde{C}'$ is at most 1: the second case occurs. Finally, suppose that the length of $\widetilde{f}(D_0)$ is smaller than or equal to 5 and the height of this component is greater than 5. Take a regular connected component of $\mathring{B}_{j_\beta} \cap f(\Pi(\partial D_0))$ whose ends belong to different connected components of ∂B_{j_β}. Then any lift of this connected component contained in $\widetilde{f}(\partial D_0)$ satisfies the properties of the second case.

The next lemmas will allow us to prove Proposition 3.1 in the case of the 2–dimensional torus.

Lemma 6.2 *In the first case above, there exists a homeomorphism h supported in A_{j_α} that satisfies the following properties:*

1. *If $p_2\colon \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the second coordinate, we have:*
$$\sup_{x\in\mathbb{R}^2} \left| p_2 \circ \widetilde{h}(x) - p_2(x) \right| < 4$$
2. height$(\widetilde{h} \circ \widetilde{f}(D_0)) \leq$ height$(\widetilde{f}(D_0))$

(3) $\mathrm{length}(\tilde{h} \circ \tilde{f}(D_0)) \leq \mathrm{length}(\tilde{f}(D_0)) - 1$

We have of course a symmetric statement in the second case.

Lemma 6.3 *There exists a constant $C' > 0$ such that, for any homeomorphism f in $\mathrm{Homeo}_0(\mathbb{T}^2)$ that satisfies the following properties,*

$$\mathrm{length}(\tilde{f}(D_0)) \leq 5 \quad \textit{and} \quad \mathrm{height}(\tilde{f}(D_0)) \leq 5,$$

we have $\mathrm{Frag}_{\mathcal{U}}(f) \leq C'$.

Proof of Proposition 3.1 in the case of the torus $\mathbb{T}^2$ Take any homeomorphism h supported in one of the A_j (respectively one of the B_j) with

$$\sup_{x \in \mathbb{R}^2} |p_2 \circ \tilde{h}(x) - p_2(x)| < 4$$

$$\Big(\text{respectively} \quad \sup_{x \in \mathbb{R}^2} |p_1 \circ \tilde{h}(x) - p_1(x)| < 4\Big).$$

Observe that $D_{A_j} = \tilde{A}_j^0 \cap [0,1] \times \mathbb{R}$ (respectively $D_{B_j} = \tilde{B}_j^0 \cap \mathbb{R} \times [0,1]$) is a fundamental domain for the covering map $\tilde{A}_j^0 \to A_j$ (respectively $\tilde{B}_j^0 \to B_j$). Let

$$\mathcal{D}_{A_j} = \{D_{A_j} + (0,k) \mid k \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{D}_{B_j} = \{D_{B_j} + (k,0) \mid k \in \mathbb{Z}\}.$$

Then $\mathrm{diam}_{\mathcal{D}_{A_j}}(\tilde{h}(D_{A_j})) \leq 7$ (respectively $\mathrm{diam}_{\mathcal{D}_{B_j}}(\tilde{h}(D_{B_j})) \leq 7$). Using Proposition 5.3 in the case of the annulus, we see that there exists a constant $C > 0$ such that, for any such homeomorphism h, we have

$$\mathrm{Frag}_{\mathcal{U}}(h) \leq C.$$

Using Lemma 6.2, we see that, after composing the homeomorphism f with at most

$$C(\max\{\mathrm{height}(\tilde{f}(D_0)) - 5, 0\} + \max\{\mathrm{length}(\tilde{f}(D_0)) - 5, 0\})$$

homeomorphisms supported in one of the discs of $\mathcal{U}$, we obtain a homeomorphism f_1 that satisfies the hypothesis of Lemma 6.3: $\mathrm{Frag}_{\mathcal{U}}(f_1) \leq C'$. Therefore

$$\mathrm{Frag}_{\mathcal{U}}(f) \leq 4C \,\mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)) + C'.$$

The proposition is proved in the case of the torus $\mathbb{T}^2$. □

Now, let us turn to the proof of the two above lemmas.

Proof of Lemma 6.2 Suppose that the first case occurs (the proof in the second case is symmetric). Let h be a homeomorphism supported in A_{j_α} that satisfies the following properties:

(1) For any regular connected component C of $f(\Pi(\partial D_0)) \cap \mathring{A}_{j_\alpha}$ that meets $\Pi(\widetilde{\alpha}_{\max})$ and both of whose ends belong to the same connected component of ∂A_{j_α}, $h(C) \cap \Pi(\widetilde{\alpha}_{\max}) = \varnothing$.

(2) The homeomorphism h fixes any regular connected component of $f(\Pi(\partial D_0)) \cap \mathring{A}_{j_\alpha}$ whose ends belong to different connected components of ∂A_{j_α}.

(3) The homeomorphism h fixes the projection of any connected component of $\widetilde{f}(\partial D_0) \cap \Pi^{-1}(\mathring{A}_{j_\alpha})$ that does not meet the set $\Pi^{-1}(\Pi(\widetilde{\alpha}_{\max}))$.

(4) For any connected component $\widetilde{C}$ of $\widetilde{f}(\partial D_0) \cap \Pi^{-1}(\mathring{A}_{j_\alpha})$, $p_2(\widetilde{h}(\widetilde{C})) \subset p_2(\widetilde{C})$.

(5) If the point $f(0,0)$ belongs to $\mathring{A}_{j_\alpha}$, we add the following condition. Let C_0 be the singular connected component of $f(\Pi(\partial D_0)) \cap \mathring{A}_{j_\alpha}$. If there exists a lift $\widetilde{C}_0$ of the component C_0 that meets the set $\widetilde{f}(\partial D_0) \cap \widetilde{\alpha}_{\max}$, we impose the following condition. Let us denote by C_1, C_2, C_3 and C_4 the connected components of $C_0 - \{f(0,0)\}$. Only three of these connected components admit a lift contained in $\widetilde{f}(D_0)$ that meets the interior of $\widetilde{A}_{j_\alpha}^{i_\alpha}$: for the last connected component, the two lifts of this one contained in $\widetilde{f}(D_0)$ are necessarily contained in the interior of $\widetilde{A}_{j_\alpha}^{i_\alpha - 1}$. We can suppose that these three connected components are C_1, C_2 and C_3. Let $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$ be respective lifts of C_1, C_2 and C_3 contained in $\widetilde{A}_{j_\alpha}^{i_\alpha}$. Then, for any integer i between 1 and 3, we add the following condition: $\widetilde{h}(\widetilde{C}_i) \cap \widetilde{\alpha}_{\max} = \varnothing$.

We claim that such a homeomorphism h satisfies the wanted properties. First, recall that there exists a connected component $\widetilde{C}$ of $\Pi^{-1}(\mathring{A}_{j_\alpha}) \cap \widetilde{f}(\partial D_0)$ whose ends belong to two different connected components of the boundary of $\Pi^{-1}(A_{j_\alpha})$ and whose height is less than or equal to 5 (and therefore $\sup p_2(\widetilde{C}) - \inf p_2(\widetilde{C}) \leq 3$). Recall also that the homeomorphism h pointwise fixes the projection of this connected component. Therefore

$$\sup_{x \in \mathbb{R}^2} |p_2 \circ \widetilde{h}(x) - p_2(x)| < 4.$$

The condition (4) on the second coordinate of the images under h of the connected components of the set $\mathring{A}_{j_\alpha} \cap f(\Pi(\partial D_0))$ implies that

$$\text{height}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \text{height}(\widetilde{f}(D_0)).$$

Finally, by construction, the set $\widetilde{h} \circ \widetilde{f}(D_0)$ does not meet the curve $\widetilde{\alpha}_{\max}$ anymore and meets only curves of the form $\widetilde{\alpha}_j^i$ already met by the set $\widetilde{f}(D_0)$. Thus

$$\text{length}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \text{length}(\widetilde{f}(D_0)) - 1.$$

Lemma 6.2 is proved. □

Proof of Lemma 6.3 During this proof, we will often use the following result, which is a direct consequence of Proposition 3.2 in the case of the annulus. There exists a constant $\lambda > 0$ such that, for any homeomorphism η in $\mathrm{Homeo}_0(\mathbb{T}^2)$ supported in $\mathring{A}_0$ or in $\mathring{A}_1$ that satisfies $\mathrm{height}(\widetilde{\eta}(D_0)) \leq 45$, we have $\mathrm{Frag}_{\mathcal{U}}(\eta) \leq \lambda$.

First, notice that the inequality $\mathrm{length}(\widetilde{f}(D_0)) \leq 5$ implies the inequality

$$\mathrm{length}(\widetilde{f}(\widetilde{\alpha}_0^0)) \leq 3.$$

Indeed, suppose that $\mathrm{length}(\widetilde{f}(\widetilde{\alpha}_0^0)) > 3$. Note that one of the edges of the square ∂D_0 is contained in $\widetilde{\alpha}_0^0$ and that the curve $\widetilde{f}(\widetilde{\alpha}_0^1)$ meets two curves among the $\widetilde{\alpha}_j^i$ that the curve $\widetilde{f}(\widetilde{\alpha}_0^0)$ does not meet. Therefore,

$$\mathrm{length}(\widetilde{f}(D_0)) \geq \mathrm{length}(\widetilde{f}(\widetilde{\alpha}_0^0)) + 2 > 5,$$

a contradiction. We denote by $n(\widetilde{f}(\widetilde{\alpha}_0^0))$ the number of connected components of $\bigcup_{i,j} \partial \widetilde{A}_j^i$ met by the path $\widetilde{f}(\widetilde{\alpha}_0^0)$. As the length of $\widetilde{f}(\widetilde{\alpha}_0^0)$ is less than or equal to 3, then $n(\widetilde{f}(\widetilde{\alpha}_0^0)) \leq 7$. We now prove that, after composing f with a homeomorphism whose fragmentation length with respect to $\mathcal{U}$ is less than or equal to 7λ if necessary, we can suppose that $n(\widetilde{f}(\widetilde{\alpha}_0^0)) = 0$.

Suppose that $n(\widetilde{f}(\widetilde{\alpha}_0^0)) > 0$. Consider $(i_0, j_0) \in \mathbb{Z} \times \{0, 1\}$ such that the band $\widetilde{A}_{j_0}^{i_0}$ is the leftmost band met by the set $\widetilde{f}(D_0)$. Then the set $\widetilde{f}(D_0)$ meets $\widetilde{A}_{j_0}^{i_0}$ but meets only one connected component of the boundary of $\widetilde{A}_{j_0}^{i_0}$ that we denote by c_{i_0,j_0}. Let $\widetilde{A}_{j_1}^{i_1}$ be the unique band among the $\widetilde{A}_j^i$ whose interior contains the curve c_{i_0,j_0}. Then $j_1 \neq j_0$.

First case The set $\widetilde{f}(D_0)$ meets the two connected components of the boundary of $\widetilde{A}_{i_1}^{j_1}$. Let h be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{T}^2)$ supported in the interior of the annulus A_{j_0} that satisfies the following properties:

(1) For any connected component $\widetilde{C}$ of $\widetilde{f}(\widetilde{\alpha}_0^0) \cap \Pi^{-1}(A_{j_0})$ whose projection is not contained in the interior of A_{j_1}, we have $h(\Pi(\widetilde{C})) \subset \mathring{A}_{j_1}$.

(2) The homeomorphism $\widetilde{h}$ pointwise fixes the other connected components of $\widetilde{f}(\widetilde{\alpha}_0^0) \cap \Pi^{-1}(A_{j_0})$.

(3) For any connected component $\widetilde{C}$ of $\widetilde{f}(\partial D_0) \cap \Pi^{-1}(\mathring{A}_{j_0})$, $p_2(\widetilde{h}(\widetilde{C})) \subset p_2(\widetilde{C})$.

(4) $\sup_{x \in \mathbb{R}^2} |p_2 \circ \widetilde{h}(x) - p_2(x)| < 3$.

Notice that this last condition is compatible with the first one. Indeed, as the height of $\widetilde{f}(D_0)$ is less than or equal to 5, then, for any connected component $\widetilde{C}$ of $\widetilde{f}(\partial D_0) \cap \Pi^{-1}(A_{j_0})$, we have $\mathrm{height}(\widetilde{C}) \leq 5$. Therefore, we can choose h so that the support of h is contained in a disjoint union of discs whose height is smaller than or equal to five.

For such a homeomorphism h, the following properties are satisfied:

$$\mathrm{Frag}_{\mathcal{U}}(h) \leq \lambda, \quad n(\widetilde{h} \circ \widetilde{f}(\widetilde{\alpha}_0^0)) < n(\widetilde{f}(\widetilde{\alpha}_0^0)), \quad \mathrm{height}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \mathrm{height}(\widetilde{f}(D_0)).$$

The second one comes from the fact that the set $\widetilde{h} \circ \widetilde{f}(\widetilde{\alpha}_0^0)$ no longer meets one of the connected components of the boundary of $\widetilde{A}_{j_1}^{i_1}$.

Second case Suppose that the set $\widetilde{f}(D_0)$ does not meet both boundary components of $\widetilde{A}_{i_1}^{j_1}$. Likewise, we build a homeomorphism in $\mathrm{Homeo}_0(\mathbb{T}^2)$ supported in $\mathring{A}_{j_1}$ such that the curve $\widetilde{h} \circ \widetilde{f}(\widetilde{\alpha}_0^0)$ does not meet the band $\widetilde{A}_{j_0}^{i_0}$ anymore and such that

$$\mathrm{Frag}_{\mathcal{U}}(h) \leq \lambda, \quad n(\widetilde{h} \circ \widetilde{f}(\widetilde{\alpha}_0^0)) < n(\widetilde{f}(\widetilde{\alpha}_0^0)), \quad \mathrm{height}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \mathrm{height}(\widetilde{f}(D_0)).$$

Thus, it suffices to prove the following property. There exists a constant $C > 0$ such that, if f is a homeomorphism in $\mathrm{Homeo}_0(\mathbb{T}^2)$ with $n(\widetilde{f}(\widetilde{\alpha}_0^0)) = 0$ and $\mathrm{height}(\widetilde{f}(D_0)) \leq 5$, then $\mathrm{Frag}_{\mathcal{U}}(f) \leq C$. Let us consider such a homeomorphism f.

First case ($f(\alpha_0) \not\subset A_0$) Let h be a homeomorphism supported in the annulus A_1 that preserves the horizontal foliation such that $h(f(\alpha_0)) \subset A_0$. The preservation of this foliation implies that $\mathrm{Frag}_{\mathcal{U}}(h) \leq \lambda$. We are led to the second case.

Second case ($f(\alpha_0) \subset A_0$) Let h be a homeomorphism supported in the annulus A_0 that is equal to the homeomorphism f in a neighbourhood of the curve α_0. As the height of $\widetilde{f}(D_0)$ is less than or equal to 5, we can suppose that $\mathrm{height}(\widetilde{h}(D_0)) \leq 5$. Thus $\mathrm{Frag}_{\mathcal{U}}(h) \leq \lambda$. Moreover $\mathrm{height}(\widetilde{h}^{-1} \circ \widetilde{f}(D_0)) \leq 15$. We have pointwise fixed α, which is one of the boundary components of A_1. By an analogous procedure, we can find a homeomorphism h' such that $h'^{-1} \circ h^{-1} \circ f$ pointwise fixes a neighbourhood of the boundary of A_1 and such that

$$\mathrm{Frag}_{\mathcal{U}}(h') \leq \lambda \quad \text{and} \quad \mathrm{height}(\widetilde{h}'^{-1} \circ \widetilde{h}^{-1} \circ \widetilde{f}(D_0)) \leq 45.$$

We denote by h_1 the homeomorphism supported in A_1 that is equal to $h'^{-1} \circ h^{-1} \circ f$ on A_1. The height of $\widetilde{h}_1(D_0)$ is less than or equal to 45 and that is why $\mathrm{Frag}_{\mathcal{U}}(h_1) \leq \lambda$. Moreover, the homeomorphism $h_2 = h_1^{-1} \circ h'^{-1} \circ h^{-1} \circ f$ is supported in A_0. The height of the image of D_0 under $\widetilde{h}_2$ is less than or equal to 45: $\mathrm{Frag}_{\mathcal{U}}(h_2) \leq \lambda$. Finally, $\mathrm{Frag}_{\mathcal{U}}(f) \leq 4\lambda$ in this case. □

7 Case of higher genus closed surfaces

In this section, we prove Proposition 3.1 for a closed surface S of genus $g \geq 2$. Let us begin by describing the cover $\mathcal{U}$ that we use in what follows. Let p be the point of S that is the image under Π of a vertex of the polygon ∂D_0. Let us denote by B the set of edges of the polygon ∂D_0 and by A the set of curves that are the images under Π of an edge in B. Let

$$\widetilde{A} = \{\gamma(\widetilde{\alpha}) \mid \widetilde{\alpha} \in B, \gamma \in \pi_1(S)\} = \Pi^{-1}(\Pi(B)).$$

We denote by U_0 a closed disc of S whose interior contains the point p and that satisfies the following property: if $\widetilde{U}_0$ is a lift of U_0 and $\widetilde{p}$ is a lift of the point p, then the disc $\widetilde{U}_0$ meets only edges in $\widetilde{A}$ for which one end is $\widetilde{p}$ and the boundary $\partial\widetilde{U}_0$ meets each of them in exactly one point. For any edge α in A, we denote by V_α a closed disc that does not contain the point p such that the following properties are satisfied:

(1) For any edge α in A, the set $V_\alpha \cup U_0$ is a neighbourhood of the edge α.

(2) For any edge α in A, the set $V_\alpha \cap U_0$ is the disjoint union of two closed discs.

(3) The discs V_α are pairwise disjoint.

We denote by U_1 a closed disc that contains the union of the V_α. Finally, we denote by U_2 a closed disc that does not meet any edge in A and that satisfies the following properties:

(1) For any edge α in A, the closed set $U_2 \cap V_\alpha$ is homeomorphic to the disjoint union of two closed discs.

(2) $\mathring{U}_2 \cup \mathring{U}_0 \cup \bigcup_{\alpha \in A} \mathring{V}_\alpha = S$.

(3) The closed set $(\bigcup_\alpha V_\alpha \cup U_2) \cap U_0$ is homeomorphic to an annulus for which one boundary component is ∂U_0.

Let $\mathcal{U} = \{U_0, U_1, U_2\}$.

Proposition 7.1 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(S)$. *Suppose that*

$$\mathrm{el}_{D_0}(\widetilde{f}(D_0)) \geq 4g.$$

Then there exists a homeomorphism h in $\mathrm{Homeo}_0(S)$ *that satisfies the following properties:*

(1) $\mathrm{Frag}_{\mathcal{U}}(h) \leq 8g + 3$

(2) $\mathrm{el}_{D_0}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\widetilde{f}(D_0)) - 1$

Remark 7.2 We did not try to obtain an optimal upper bound of the fragmentation length of a homeomorphism h with $\mathrm{el}_{D_0}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\widetilde{f}(D_0)) - 1$.

Lemma 7.3 *There exists a constant $C' > 0$ such that, for any homeomorphism f in* $\mathrm{Homeo}_0(S)$ *with* $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) \leq 4g$, *we have* $\mathrm{Frag}_{\mathcal{U}}(f) \leq C'$.

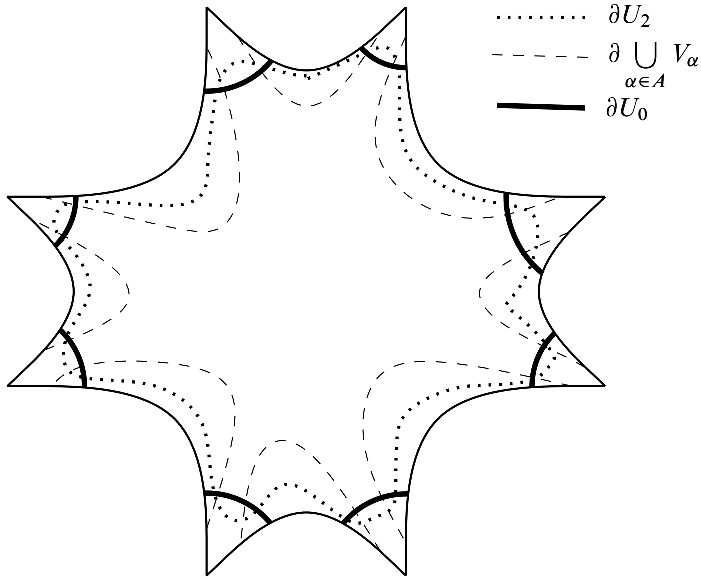

Figure 10: Notation in the case of higher genus closed surfaces

End of the proof of Proposition 3.1: Case of a higher genus closed surface Take any homeomorphism $f \in \mathrm{Homeo}_0(S)$. A classical result asserts that the homeomorphism $\tilde{f}$ has a fixed point. Indeed, suppose that it is not the case. Then associate to each point $\tilde{x}$ of $\tilde{S}$ the unit tangent vector at $\tilde{x}$ to the geodesic between the point $\tilde{x}$ and the point $\tilde{f}(\tilde{x})$, oriented from $\tilde{x}$ to $\tilde{f}(\tilde{x})$. This vector field on $\tilde{S}$ gives rise to a nowhere vanishing vector field on S, a contradiction.

Recall that the homeomorphism $\tilde{f}$ commutes with the deck transformations. Hence

$$\tilde{f}(D_0) \cap D_0 \neq \varnothing \quad \text{and} \quad \mathrm{el}_{D_0}(\tilde{f}(D_0)) \leq \mathrm{diam}_{\mathcal{D}}(\tilde{f}(D_0)).$$

Therefore, the two above lemmas allow us to complete the proof of Proposition 3.1 as in the case of surfaces with nonempty boundary. □

For the proof of Proposition 7.1, we will need some combinatorial lemmas concerning the group $\pi_1(S)$, which we state in the following subsection. The proofs of these lemmas will not be used elsewhere in the text: the reader can skip them if he wants.

7.1 Some combinatorial lemmas

7.1.1 Some definitions Recall that two fundamental domains D_1 and D_2 in $\mathcal{D}$ are *adjacent* if the intersection of D_1 with D_2 is an edge common to the polygons

∂D_1 and ∂D_2. Recall also that $\mathcal{G}$ is the generating set for $\pi_1(S)$ consisting of deck transformations that send the fundamental domain D_0 to a fundamental domain adjacent to D_0. By abuse of notation, for any word γ in elements of $\mathcal{G}$, we also denote by γ the corresponding element in the group $\pi_1(S)$.

Definition 7.4 We call a *geodesic word* a word γ in elements of $\mathcal{G} \subset \pi_1(S)$ such that the number of letters of the word γ is equal to $l_{\mathcal{G}}(\gamma)$.

We now describe a more geometric way to see the words whose letters are elements of $\mathcal{G}$.

Definition 7.5 We call a *path in $\mathcal{D}$ of origin D_0* any finite sequence $(D_0, D_1, \dots, D_p)$ of fundamental domains in $\mathcal{D}$ such that two consecutive fundamental domains in this sequence are adjacent. Such a path in $\mathcal{D}$ is said to be *geodesic* if for any index i, $d_{\mathcal{D}}(D_0, D_i) = i$.

Remark 7.6 Notice that there is a bijective map between words in the elements of $\mathcal{G}$ and the paths of origin D_0 in $\mathcal{D}$: to a word $l_1 \cdots l_p$, one can associate the path $(D_0, l_1(D_0), l_1 l_2(D_0), \dots, l_1 l_2 \cdots l_p(D_0))$. This last map is bijective and sends the geodesic words to geodesic paths in $\mathcal{D}$.

Definition 7.7 For a homeomorphism h in $\mathrm{Homeo}_0(S)$, we call a *maximal face* for h any fundamental domain in $\mathcal{D}$ at distance $\mathrm{el}_{D_0}(\widetilde{h}(D_0))$ from D_0.

We want to prove that, after composing h with a number independent of h of homeomorphisms supported in one of the discs in $\mathcal{U}$, the image of D_0 does not meet maximal faces for h anymore. There will be two different kinds of maximal faces for h. The first ones, which we call *non-exceptional*, are not problematic: after composing h with four homeomorphisms, each of them being supported in one of the discs of $\mathcal{U}$, the image of the fundamental domain D_0 will not meet these faces anymore.

Definition 7.8 A face D is called *non-exceptional* if it satisfies the following property: In the set of faces adjacent to D, there is only one element that is at distance $d_{\mathcal{D}}(D, D_0) - 1$ from D_0. The faces in $\mathcal{D}$ that do not satisfy this property are called *exceptional*.

In the case of exceptional faces, we will have to understand the relative arrangement of the nearby fundamental domains in $\mathcal{D}$.

Let us describe more precisely the crucial property used in this proof. Let us denote by D an exceptional face and by γ a geodesic word such that $\gamma(D_0) = D$. Let

$(D_0, D_1, \dots, D_N = D)$ be the geodesic path in $\mathcal{D}$ corresponding to the geodesic word γ. We will see later (see Lemmas 7.10 and 7.11) that the $2g-1$ last faces in this sequence share a vertex in common. The crucial property is the following: *if* $1 \leq k \leq 2g-2$*, for any geodesic path of the form* $(D_0, \dots, D_{N-k}, D'_{N-k+1}, \dots, D'_N)$*, where the face* D'_{N-k+1} *is different from the face* D_{N-k+1}*, then the faces* $D'_{N-k+1}, \dots, D'_N$ *are not exceptional* (see Lemma 7.18).

Remark 7.9 By replacing the face D_0 with any other fundamental domain D_1 in $\mathcal{D}$ and the generating set $\mathcal{G}$ with the generating set consisting of deck transformations that send D_1 to a face adjacent to D_1, we can define the notion of exceptional faces with respect to D_1. All the following statements dealing with exceptional faces (with respect to D_0) can be generalized to the case of an exceptional face with respect to any fundamental domain in $\mathcal{D}$. We implicitly use this remark during the proof of Lemma 7.20.

7.1.2 The set Λ

Let

$$\mathcal{G} = \left\{a_i^{\epsilon} \mid 1 \leq i \leq g \text{ and } \epsilon \in \{-1, 1\}\right\} \cup \left\{b_i^{\epsilon} \mid 1 \leq i \leq g \text{ and } \epsilon \in \{-1, 1\}\right\}$$

so that

$$\pi_1(S) = \left\langle (a_i)_{1 \leq i \leq g}, (b_i)_{1 \leq i \leq g} \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \right\rangle.$$

Let us denote by Λ the set of cyclic permutations of the words $[a_1, b_1] \cdots [a_g, b_g]$ and $[b_g, a_g] \cdots [b_1, a_1]$. In terms of paths in $\mathcal{D}$, these words correspond to a circle around one of the vertices of the polygon ∂D_0:

Lemma 7.10 *For any word* $\lambda_1 \cdots \lambda_{4g}$ *in* Λ*, the faces* $\lambda_1 \cdots \lambda_i(D_0)$*, for* $1 \leq i \leq 4g$*, share a point in common.*

Proof Let us denote by X the set of $4g$–tuples $(\delta_i)_{1 \leq i \leq 4g}$ of elements of $\mathcal{D}$ that satisfy the following properties:

(1) $\delta_{4g} = D_0$

(2) There exists a vertex $\widetilde{p}$ of D_0 such that the set of elements of $\mathcal{D}$ that contain the point $\widetilde{p}$ is $\{\delta_i \mid 1 \leq i \leq 4g\}$.

(3) Every counterclockwise oriented circle whose center is $\widetilde{p}$ and whose diameter is sufficiently small meets successively the fundamental domains $\delta_1, \dots, \delta_{4g}$. In particular, the faces δ_i and δ_{i+1} are adjacent.

The set X is naturally isomorphic to the set of vertices of the polygon ∂D_0. An element $a = (\delta_i)_{1 \leq i \leq 4g}$ in X is associated to a word $\varphi(a) = \lambda = \lambda_1 \cdots \lambda_{4g}$ in Λ defined in the following way: The letter λ_1 is the unique deck transformation in $\mathcal{G}$ that sends D_0 to δ_1. The second letter λ_2 is the unique deck transformation in $\mathcal{G}$ such that $\lambda_1 \lambda_2(D_0) = \delta_2$. Likewise, if we suppose that we have built the letters $\lambda_1, \dots, \lambda_i$ such that $\lambda_1 \cdots \lambda_i(D_0) = \delta_i$, the letter λ_{i+1} is defined by the relation $\lambda_1 \cdots \lambda_{i+1}(D_0) = \delta_{i+1}$. Finally, $\lambda_1 \cdots \lambda_{4g}(D_0) = D_0$ so the word $\lambda_1 \cdots \lambda_{4g}$ belongs to the set Λ.

Thus, we have built an injective map that, to any vertex $\tilde{p}$ of D_0, associates a word λ in Λ such that the fundamental domains $\lambda_1 \cdots \lambda_i(D_0)$, for $1 \leq i \leq 4g$, share the point $\tilde{p}$ in common. Notice that the word λ^{-1} also satisfies this property. Moreover, as the cardinality of the set Λ is $4g$ and as the cardinality of the set of vertices of the polygon ∂D_0 is $2g$, we obtain the following property: for a word λ in Λ, the fundamental domains $\lambda_1 \cdots \lambda_i(D_0)$, for $1 \leq i \leq 4g$, share a point in common. □

7.1.3 Geodesic words and exceptional faces The next lemma describes the shape of the geodesic words that send the face D_0 to an exceptional face.

Lemma 7.11 *Let D be an exceptional face different from D_0. For any geodesic word γ with $\gamma(D_0) = D$, one of the following properties holds:*

(1) *The $2g$ last letters of the word γ form a subword of a word of Λ.*

(2) *The $4g-1$ last letters of γ are the concatenation of two subwords λ_1 and λ_2 of words in Λ with the following properties:*

   (a) *The length of λ_1 is equal to $2g$ and the length of λ_2 is equal to $2g-1$.*

   (b) *If we denote by a the last letter of λ_1 and by b the first letter of λ_2, then the word ab is not contained in any word of Λ.*

In the second case above, denote by l the letter in $\mathcal{G}$ such that the word $\lambda_2 l$ is contained in some word in Λ. Then the word γl is not geodesic.

Moreover, there exists a geodesic word γ such that $\gamma(D_0) = D$ that satisfies the first property above. We denote by $l_1 \cdots l_{2g}$ its $2g$ last letters, where $l_1 \cdots l_{4g} \in \Lambda$. Then the $2g$ last letters of any geodesic word for which this first property holds are $l_1 \cdots l_{2g}$ or $l_{4g}^{-1} \cdots l_{2g+1}^{-1}$.

In the case $g = 2$, an example of a geodesic word associated to an exceptional face with the first property above is $[a_1, b_1] = [b_2, a_2]$ and an example of a geodesic word associated to an exceptional face with the second property above is

$$a_2^{-1} b_2^{-1} a_1 b_1^2 a_1^{-1} b_1^{-1} = a_2^{-1} b_2^{-1} a_1 b_1 a_1^{-1} [a_1, b_1] = b_2^{-1} a_2^{-1} b_1 [a_1, b_1].$$

The first property holds for this last word.

Proof Let us describe Dehn's algorithm, which we will use. Let m be a reduced word in elements of $\mathcal{G}$. At each step of the algorithm, we look for a subword f of m with length greater than $2g$ that is contained in a word $f.\lambda'$ of Λ (such a word f will be said to be *simplifiable*) and whose length is maximal among such words (it is said to be *maximal* in m). The word λ' will be called the *complementary word* of f. Then we replace in m the subword f with the word λ'^{-1} whose length is strictly smaller (the words in Λ have length $4g$) and we make if necessary the free group reductions to obtain a new reduced word. By a theorem by Dehn (see Lyndon and Schupp [19]), a reduced word represents the trivial element in $\pi_1(S)$ if and only if, after implementing a finite number of steps of this algorithm, we obtain the empty word.

Let us give some general facts on the group $\pi_1(S)$ that are immediate and are used below.

Fact 1 For any two letters a and b in $\mathcal{G}$, there exists at most one word in Λ whose two first letters are given by ab. The other words in Λ that contain the word ab are a cyclic permutation of this one.

Fact 2 For any letter a in $\mathcal{G}$, there exist exactly two words in Λ whose last letter (respectively first letter) is a. If b and c denote the penultimate letters (respectively the second letters) of these words, then the word $b^{-1}c$ is not contained in any word in Λ.

Fact 3 For any two letters a and b in $\mathcal{G}$ such that the word ab is contained in a word of Λ, let us denote by m_1 the word of Λ with first letter b, but whose last letter l_1 is different from a, and by m_2 the word in Λ whose last letter is a, but whose first letter l_2 is not b. Then $l_2^{-1}l_1^{-1}$ is not contained in any word in Λ.

We will use Fact 2 in the following situation: If, at a given step of Dehn's algorithm, we obtain a reduced word of the form $macm'$, where acm' is a subword of a word in Λ, ma is a simplifiable word and mac is not simplifiable, then, after replacing ma by the inverse of its complementary word, we obtain a word of the form $m''cm'$, where $m''c$ is not contained in any word in Λ. As for Fact 3, we will use it in the following situation: Suppose that, at a given step of Dehn's algorithm, we obtain a word of the form $mabm'$, where ab is a subword of a word in Λ, and ma and bm' are simplifiable. Suppose moreover that the words mab and abm' are not simplifiable (these are not subwords of words in Λ). Then after replacing the words ma and bm' with the inverse of their complementary words, we obtain a word of the form $nl_2^{-1}l_1^{-1}n'$, where the words $nl_2^{-1}l_1^{-1}$ and $l_2^{-1}l_1^{-1}n'$ are not contained in any word in Λ.

Let us come back to the proof of the lemma. As D is an exceptional face, there exist two geodesic words γ_1 and γ_2 with distinct last letters such that $\gamma_i(D_0) = D$ for $i = 1, 2$. We now prove that one of them satisfies necessarily the first property given by the lemma and both of them satisfy one of the properties stated in the lemma. Moreover, if both of them satisfy the first property of the lemma, there exists a word $l_1 \cdots l_{4g}$ in Λ such that the $2g$ last letters of γ_1 are $l_1 \cdots l_{2g}$ and the $2g$ last letters of γ_2 are $l_{4g}^{-1} \cdots l_{2g+1}^{-1}$. These two results imply all the claims of the lemma.

Then take two geodesic words γ_1 and γ_2 with distinct last letters such that $\gamma_i(D_0) = D$ for $i = 1, 2$. The word $\gamma_1 \gamma_2^{-1}$ is reduced but represents the trivial element in the group $\pi_1(S)$. We apply Dehn's algorithm to this word to prove the lemma. As the words γ_1 and γ_2 are geodesic, they do not contain simplifiable words. Let λ' be a simplifiable word that is maximal in $\gamma_1 \gamma_2^{-1}$. Let λ_3 be the complementary word of λ'. Then we have a decomposition of the word λ', $\lambda' = \lambda_1 \lambda_2$, with

$$\gamma_1 = \hat{\gamma}_1 \lambda_1, \quad \gamma_2 = \hat{\gamma}_2 \lambda_2^{-1}.$$

By the previous remark, the words λ_1 and λ_2 are nonempty. The words $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are geodesic. Moreover, as the words γ_1 and γ_2 are both geodesic, the words λ_1 and λ_2 are not simplifiable. Thus, if the length of λ' is $4g$, the words λ_1 and λ_2 both have length $2g$. We now prove the following fact.

Fact Such a word λ' necessarily has length greater than $4g - 2$.

Suppose first that the length of λ' is less than or equal to $4g - 3$ (ie, the length of λ_3 is greater than 2). After the first step of the algorithm, we obtain the word $\hat{\gamma}_1 \lambda_3^{-1} \hat{\gamma}_2^{-1}$, which is reduced as λ' is maximal. Moreover, the concatenation of the word λ_3^{-1} with the first letter of the word $\hat{\gamma}_2^{-1}$ is not contained in any word in Λ, and similarly for the concatenation of the last letter of the word $\hat{\gamma}_1$ with the word λ_3^{-1}: otherwise, by Fact 1, the word $\lambda_2 \hat{\gamma}_2^{-1}$ would not be reduced. Suppose by induction that, at a given step of the algorithm, we obtain a reduced word of the form

$$\tilde{\gamma}_1 \eta_1 \eta_2 \cdots \eta_k \tilde{\gamma}_2^{-1},$$

where $k \geq 1$, the words $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are geodesic and the words η_i are each contained in a word of Λ, have length smaller than $2g$ and satisfy the following properties:

(1) The words η_1 and η_k have length greater than 1 and, if they are both of length 2, then $k > 1$.

(2) For any index i between 1 and $k - 1$, the concatenation of the last letter of η_i with the first letter of η_{i+1} is not contained in any word in Λ.

(3) The concatenation of the word η_k with the first letter of the word $\widetilde{\gamma}_2^{-1}$ is not contained in any word in Λ and similarly for the concatenation of the last letter of the word $\widetilde{\gamma}_1$ with the word η_1.

Let us apply a new step of the algorithm. A simplifiable subword λ' of the above word is necessarily contained in one of the words $\widetilde{\gamma}_1\eta_1$ or $\eta_k\widetilde{\gamma}_2^{-1}$ by the second property above and by using the fact that each of the η_i has length smaller than $2g$. We may suppose, without loss of generality, that such a subword is contained in $\widetilde{\gamma}_1\eta_1$. By combining Fact 1 with the third property above, we obtain that the last letter a of the word $\lambda'=\lambda'_1 a$ is also the first letter of the word $\eta_1=a\eta'_1$. As the word $\widetilde{\gamma}_1$ is geodesic, it does not contain any simplifiable subword, so the word λ'_1, which it contains, has length $2g$. After applying the algorithm, we obtain the word

$$\widetilde{\gamma}'_1\widetilde{\lambda}^{-1}\eta'_1\eta_2\cdots\eta_k\widetilde{\gamma}_2^{-1},$$

where $\widetilde{\gamma}_1=\widetilde{\gamma}'_1\lambda'_1$ and $\widetilde{\lambda}$ is the complementary word of λ'. The obtained words $\widetilde{\gamma}'_1$ and $\widetilde{\gamma}_2$ are geodesic. The word $\widetilde{\lambda}$, of length $2g-1$, has length smaller than $2g$ and greater than 1. Moreover, if $k=1$, the length of η_1 is greater than 2 so the length of η'_1 is greater than 1. Fact 2 implies that the concatenation of the last letter of $\widetilde{\lambda}^{-1}$ with the first letter of η'_1 is not contained in any word in Λ. Finally, the third property is satisfied for this decomposition: denoting by l the last letter of $\widetilde{\gamma}'_1$, if the word $l\widetilde{\lambda}^{-1}$ were a subword of a word in Λ, then, by Fact 1, the first letter of the word λ' would be l^{-1}, which would contradict the fact that the word $\widetilde{\gamma}_1$ is reduced. At each step of the algorithm, the sum of the lengths of the geodesic words at the beginning and at the end of this decomposition strictly decreases. Therefore, after applying a finite number of steps of the algorithm, we obtain a word of the form

$$\widetilde{\gamma}_1\eta_1\eta_2\cdots\eta_k\widetilde{\gamma}_2^{-1},$$

where $k\geq 1$, which satisfies the three properties that we just described as well as the following property: The length of $\widetilde{\gamma}_1$ as well as the length of $\widetilde{\gamma}_2$ are less than $2g$. In this case, we can see that the considered word does not contain subwords of a word in Λ with length greater than $2g$, a contradiction.

Let us come back to the first step of the algorithm. Then the considered word λ' has length $4g-2$ or $4g-1$, if its length is not $4g$. Suppose now that the length of λ' is $4g-2$. We want to find a contradiction.

After the first step of the algorithm, we obtain a reduced word of the form $\widehat{\gamma}_1\lambda_3\widehat{\gamma}_2^{-1}$, where the length of $\lambda_3=ab$ is 2. As before, the concatenation of the last letter of $\widehat{\gamma}_1$ with the word λ_3 as well as the concatenation of the word λ_3 with the first letter of $\widehat{\gamma}_2^{-1}$ is not contained in any word of Λ. Without loss of generality, we may suppose that, during the second step of the algorithm, we choose a subword of a word in Λ of the form $b\widetilde{\lambda}_2$, where the word $\widetilde{\lambda}_2$ is the concatenation of the $2g$ first letters of the word $\widehat{\gamma}_2^{-1}$. Let us use notation from Fact 3. After applying a step of the algorithm, we obtain a word of the form $\widehat{\gamma}_1 a\eta_1\widetilde{\gamma}_2^{-1}$, where the length of η_1 is $2g-1$ and the first letter of η_1 is l_1^{-1}. While the subwords that were chosen during the algorithm do not meet $\widehat{\gamma}_1$, we obtain words of the form $\widehat{\gamma}_1 a\eta_1\eta_2\cdots\eta_k\widetilde{\gamma}_2^{-1}$, where the properties (1) and (2) are satisfied as well as property (3) for $\widetilde{\gamma}_2$ alone and where the first letter of η_1 is l_1^{-1}. After the first step for which we replace a subword which meets $\widehat{\gamma}_1$, we obtain a word of the form

$$\widetilde{\gamma}_1\eta_0\eta_1\cdots\eta_k\widetilde{\gamma}_2^{-1},$$

where the last letter of the word η_0 is l_2^{-1} and the first letter of η_1 is l_1^{-1}. Fact 3 implies the situation is the same as before, a contradiction.

Finally, in the case where the length of λ' is $4g-1$, one of the two geodesic words γ_1 or γ_2 satisfies necessarily the first property of the lemma. Similarly, after implementing the algorithm, we see that the second geodesic word satisfies the second property of the lemma. □

7.1.4 Faces of type (i, j)

Definition 7.12 For a natural number $l \geq 1$, we call a *face of type* $(0,l)$ any fundamental domain D in $\mathcal{D}$ that is at distance l from D_0 and that satisfies the following property: In the set of faces adjacent to D, only one element is at distance $l-1$ from D_0, ie, this face is not exceptional and is at distance l from D_0.

Remark 7.13 In the case where the fundamental domain D is a face of type $(0,l)$, the other faces adjacent to D are at distance $l+1$ from the fundamental domain D_0. Indeed, denote by m a word in elements of $\mathcal{G}$ and by λ a letter in $\mathcal{G}$. Then the elements $m\lambda$ and m of the group $\pi_1(S)$ do not have the same length $l_{\mathcal{G}}$ modulo 2, as the relations that define this group have even length.

Remark 7.14 By using the notion of geodesic word, another (equivalent) definition of faces of type $(0,l)$ can be given: a face of type $(0,l)$ is a fundamental domain D in $\mathcal{D}$ such that all the geodesic words γ with $\gamma(D_0) = D$ have the same last letter and their length is l.

Definition 7.15 For any integer k between 0 and l, we define by induction the set of faces of types (k,l). A *face of type* (k,l) is a fundamental domain D in $\mathcal{D}$ that is at distance $l-k$ from D_0 and that satisfies the following property: All the faces adjacent to D, except one, are faces of type $(k-1,l)$.

Remark 7.16 A face of type (k,l) is also a face of type $(0,l-k)$ (or even $(k-i,l-i)$, for $0 \leq i \leq k$).

Remark 7.17 An equivalent definition of faces of type (k,l) is the following. Let us consider a geodesic word γ' of length $l-k$ such that $\gamma'(D_0)=D$. The face D is a face of type (k,l) if and only if the following property holds. For any reduced word m with length less than or equal to k such that the word $\gamma' m$ is reduced, the face $\gamma' m(D_0)$ is not exceptional. This definition can also be interpreted in terms of geodesic paths in $\mathcal{D}$. Let us denote by $(D_0,\dots,D_{l-k})$ a geodesic path in $\mathcal{D}$. The fundamental domain D_{l-k} is a face of type (k,l) if and only if, for any geodesic extension of the form $(D_0,\dots,D_{l-k},D_{l-k+1},\dots,D_l)$ of this last path, the faces $D_{l-k},\dots,D_l$ are not exceptional.

Let us fix an exceptional face D. Let $l_1\cdots l_{4g}$ be a word in Λ and γ be a geodesic word whose $2g$ last letters are $l_1\cdots l_{2g}$ such that $\gamma(D_0)=D$. Let $\gamma=\gamma' l_1\cdots l_{2g}$ and, for $0\leq i\leq 2g$,

$$D_i^1=\gamma' l_1\cdots l_{2g-i}(D_0),\quad D_i^2=\gamma' l_{4g}^{-1}\cdots l_{2g+i+1}^{-1}(D_0).$$

Then $D_0^1=D_0^2=D$ and $D_{2g}^1=D_{2g}^2$. By Lemma 7.10, all the fundamental domains that we just defined meet in one point: they are the elements of the set of fundamental domains in $\mathcal{D}$ that contain this point.

The crucial property described above can be translated in the following way.

Lemma 7.18 *For any integers i between 1 and $2g-2$ and $j\in\{1,2\}$, the fundamental domains adjacent to D_i^j that are different from D_{i+1}^j and from D_{i-1}^j are faces of type $(i-1,d_{\mathcal{D}}(D_0,D))$.*

Remark 7.19 For any i and j, the face D_i^j is not a face of type $(i,d_{\mathcal{D}}(D,D_0))$ as the face D, which is exceptional, is at distance i from D. However, for any $1\leq i\leq 2g-2$ and j, the face D_i^j is a face of type $(i-1,d_{\mathcal{D}}(D,D_0)-1)$.

Proof of Lemma 7.18 The cases $j=1$ and $j=2$ are symmetric: suppose that $j=1$. Take an index $2\leq i'\leq 2g-1$ (think that $i'=2g-i$). By induction on the length of m, we prove that, for any reduced word m of length less than or equal to $2g-i'$ with a first letter distinct from $l_{i'+1}$ and from $l_{i'}^{-1}$:

(1) The word $\gamma' l_1 l_2\cdots l_{i'} m$ is geodesic.

(2) The fundamental domain $\gamma' l_1 l_2\cdots l_{i'} m(D_0)$ is not exceptional.

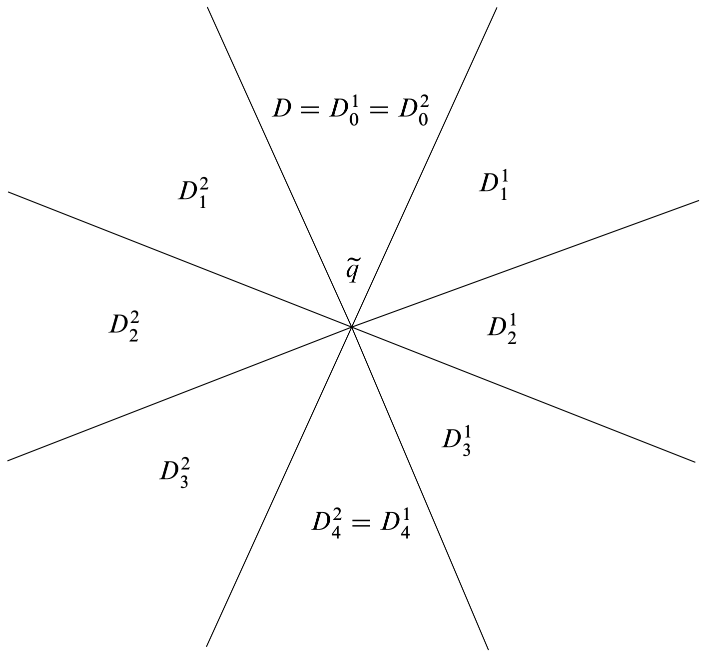

Figure 11: The D_i^j for a genus 2 surface

If the word m is empty, either the $2g-1$ last letters of the word $\gamma' l_1 l_2 \cdots l_{i'}$ are not contained in any word in Λ or $i' = 2g-1$ and the word $\gamma' l_1 l_2 \cdots l_{2g}$ is geodesic. In both cases, by Lemma 7.11, the face $\gamma' l_1 l_2 \cdots l_{i'}(D_0)$ is not exceptional.

Suppose that the property holds for a word m as above of length less than $2g-i'$. Let l be a letter in $\mathcal{G}$ different from the inverse of the last letter of m (or different from $l_{i'+1}$ and from $l_{i'}^{-1}$ if the word m is empty). As the fundamental domain $\gamma' l_1 l_2 \cdots l_{i'} m(D_0)$ is not an exceptional face, then

$$d_{\mathcal{D}}(\gamma' l_1 l_2 \cdots l_{i'} m l(D_0), D_0) = d_{\mathcal{D}}(\gamma' l_1 l_2 \cdots l_{i'} m(D_0), D_0) + 1$$

and the word $\gamma' l_1 l_2 \cdots l_{i'} m l$ is geodesic. Moreover, as the length of ml is less than or equal to $2g-i'$ and the word $\gamma' l_1 l_2 \cdots l_{2g}$ is geodesic, the face $\gamma' l_1 l_2 \cdots l_{i'} m l(D_0)$ is not exceptional. This completes the proof of Lemma 7.18. □

7.1.5 First letter of geodesic words

The next lemma is symmetric to Lemma 7.11.

Lemma 7.20 *Let D_1 be a fundamental domain in $\mathcal{D}$. Suppose that there exist two geodesic words with distinct first letters a and b such that*

$$\gamma_1(D_0) = \gamma_2(D_0) = D_1.$$

Then:

(1) *There exists a geodesic word γ such that $\gamma(D_0) = D_1$ whose $2g$ first letters $\lambda_1 \cdots \lambda_{2g}$ form a subword of a word $\lambda_1 \cdots \lambda_{4g}$ in Λ.*

(2) *The fundamental domains D_0, $a(D_0)$ and $b(D_0)$ share a point $\tilde{p}$ in common with the following property: The fundamental domains in $\mathcal{D}$ that contain the point $\tilde{p}$ are faces of the form $\lambda_1 \cdots \lambda_i(D_0)$ or $\lambda_{4g}^{-1} \cdots \lambda_{4g-i+1}^{-1}(D_0)$, with $0 \leq i \leq 2g$.*

Proof The generating set for the group $\pi_1(S)$ given by the deck transformations that send the fundamental domain D_1 on a fundamental domain in $\mathcal{D}$ adjacent to D_1 is $\gamma_1 \mathcal{G} \gamma_1^{-1}$. Notice that, under the hypothesis of the lemma, the fundamental domain D_0 is an exceptional face with respect to D_1. By Lemma 7.11, there exists a geodesic word in elements of $\gamma_1 \mathcal{G} \gamma_1^{-1}$ whose $2g$ last letters determine a word

$$(\gamma_1 \lambda_{2g}^{-1} \gamma_1^{-1})(\gamma_1 \lambda_{2g-1}^{-1} \gamma_1^{-1}) \cdots (\gamma_1 \lambda_1^{-1} \gamma_1^{-1}),$$

where $\lambda_1 \lambda_2 \cdots \lambda_{4g} \in \Lambda$, which sends the face D_1 to the face D_0. Thus, in the group $\pi_1(S)$,

$$\gamma_1^{-1} = \gamma_1 \eta^{-1} \lambda_{2g}^{-1} \lambda_{2g-1}^{-1} \cdots \lambda_1^{-1} \gamma_1^{-1},$$

where $\eta^{-1} \lambda_{2g}^{-1} \lambda_{2g-1}^{-1} \cdots \lambda_1^{-1}$ is a geodesic word in elements of $\mathcal{G}$. Let γ be the word $\lambda_1 \lambda_2 \cdots \lambda_{2g} \eta$. Then, in the group $\pi_1(S)$, $\gamma = \gamma_1$. Thus, the geodesic word γ satisfies the required properties. The second point of the lemma comes from the above argument and from Lemma 7.10. □

7.1.6 Image of a vertex of the polygon ∂D_0 For a homeomorphism h in $\mathrm{Homeo}_0(S)$, we denote by $l(h)$ the maximum of the quantities $d_{\mathcal{D}}(D, D_0)$, where D varies over the set of fundamental domains in $\mathcal{D}$ that contain the image under the homeomorphism $\tilde{h}$ of a vertex of the polygon ∂D_0. Let p be the image under Π of a vertex of the polygon ∂D_0.

Lemma 7.21 *Let h be a homeomorphism in* $\mathrm{Homeo}_0(S)$*. Suppose that $h(p) \notin h(\Pi(\partial D_0))$. There exists a unique fundamental domain D_1 in $\mathcal{D}$ whose interior contains the image under $\tilde{h}$ of a vertex $\tilde{p}$ of the polygon ∂D_0 such that*

$$d_{\mathcal{D}}(D_1, D_0) = l(h).$$

Moreover, the following properties hold:

(1) *There exists a word $\lambda_1 \lambda_2 \cdots \lambda_{4g}$ in Λ and a geodesic word γ such that $\gamma(D_0) = D_1$ and the $2g$ first letters of γ are $\lambda_1 \lambda_2 \cdots \lambda_{2g}$.*

(2) *The vertices of the polygon* ∂D_0 *are the points of the form*

$$\widetilde{p}_i = \lambda_i^{-1}\lambda_{i-1}^{-1}\cdots\lambda_1^{-1}(\widetilde{p}) \quad \text{or} \quad \widetilde{p}'_i = \lambda_{4g-i}\cdots\lambda_{4g}(\widetilde{p}),$$

with $0 \leq i \leq 2g-1$.

Proof Let us denote by $s(D_0)$ and $s'(D_0)$, where s and s' are deck transformations in $\mathcal{G}$, the faces that are adjacent to the face D_0 and that contain the point $\widetilde{p}$. Suppose that $d_{\mathcal{D}}(D_0, D_1) = l(h)$. If the relation $d_{\mathcal{D}}(s(D_0), D_1) = d_{\mathcal{D}}(D_0, D_1) + 1$ held, then we would have $d_{\mathcal{D}}(D_0, s^{-1}(D_1)) > l(h)$ and the vertex $s^{-1}(\widetilde{p})$ of ∂D_0 would satisfy

$$\widetilde{h}(s^{-1}(\widetilde{p})) = s^{-1}(\widetilde{h}(\widetilde{p})) \in s^{-1}(D_1),$$

which is not possible by definition of $l(h)$. Thus, necessarily,

$$d_{\mathcal{D}}(s(D_0), D_1) = d_{\mathcal{D}}(s'(D_0), D_1) = d_{\mathcal{D}}(D_0, D_1) - 1.$$

The face D_0 is exceptional with respect to D_1. By Lemma 7.11, there exists a word $\lambda_1\lambda_2\cdots\lambda_{4g}$ in Λ such that

$$\gamma = \lambda_1\lambda_2\cdots\lambda_{2g}\gamma' = \lambda_{4g}^{-1}\cdots\lambda_{2g+1}^{-1}\gamma' \quad \text{and} \quad \gamma(D_0) = D_1.$$

Moreover, by Lemma 7.10, the point $\widetilde{p}$ is common to the faces of the form

$$\lambda_1\lambda_2\cdots\lambda_i(D_0) \quad \text{and} \quad \lambda_{4g}^{-1}\lambda_{4g-1}\cdots\lambda_{4g-i+1}^{-1}(D_0)$$

with $0 \leq i \leq 2g$. Let i be an integer between 0 and $2g$. The point $\widetilde{p}$ is a vertex of the polygon $\lambda_1\lambda_2\cdots\lambda_i(D_0)$ so the point $\lambda_i^{-1}\lambda_{i-1}^{-1}\cdots\lambda_1^{-1}(\widetilde{p})$ belongs to the polygon ∂D_0. Therefore, we have $4g$ pairwise distinct points that are vertices of the polygon ∂D_0: we have obtained in this way all the vertices of the polygon ∂D_0. Moreover, if $i \geq 1$,

$$\widetilde{h}(\lambda_i^{-1}\lambda_{i-1}^{-1}\cdots\lambda_1^{-1}(\widetilde{p})) \in \lambda_{i+1}\lambda_{i+2}\cdots\lambda_{2g}\gamma'(D_0),$$
$$\widetilde{h}(\lambda_{4g-i+1}\lambda_{4g-i+2}\cdots\lambda_{4g}(\widetilde{p})) \in \lambda_{4g-i}^{-1}\lambda_{4g-i-1}^{-1}\cdots\lambda_{2g+1}^{-1}\gamma'(D_0),$$

so the image under the homeomorphism $\widetilde{h}$ of the vertices of the polygon ∂D_0 that are different from $\widetilde{p}$ belong to the interior of fundamental domains D in $\mathcal{D}$ strictly closer to D_0 than D_1. This implies the uniqueness of the face D_1. □

Now, let us start the proof of Proposition 7.1. Let f be a homeomorphism in $\mathrm{Homeo}_0(S)$ such that $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) \geq 4g$. The proof is decomposed into two parts. First we build a homeomorphism η_1 so that the set $\widetilde{\eta}_1 \circ \widetilde{f}(D_0)$ does not meet faces of type $(i, \mathrm{el}_{D_0}(\widetilde{f}(D_0)))$ for $0 \leq i \leq 2g-2$ anymore. Then we build a homeomorphism η_2 so that the set $\widetilde{\eta}_2 \circ \widetilde{\eta}_1 \circ \widetilde{f}(D_0)$ does not meet exceptional maximal faces for f.

In these constructions, we will ensure that the quantities $\mathrm{Frag}_{\mathcal{U}}(\eta_i)$ are bounded by a constant independent of the chosen homeomorphism f. Let us give more details now.

7.2 Pushing the image under f of D_0 away from faces of type $(i, \mathrm{el}_{D_0}(\tilde{f}(D_0)))$

Let us denote by p the image under the projection Π of any vertex of the polygon ∂D_0.

Lemma 7.22 *Let f be a homeomorphism in $\mathrm{Homeo}_0(S)$. Suppose that*

$$\mathrm{el}_{D_0}(\tilde{f}(D_0)) \geq 4g.$$

Then there exists a homeomorphism η in $\mathrm{Homeo}_0(S)$ such that:

1. $\mathrm{Frag}_{\mathcal{U}}(\eta) \leq 4(2g-1)+1$
2. $\eta \circ f(p) \notin \Pi(\partial D_0)$
3. $\mathrm{el}_{D_0}(\tilde{\eta} \circ \tilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\tilde{f}(D_0))$
4. *One of the following properties holds:*
   - (a) $\mathrm{el}_{D_0}(\tilde{\eta} \circ \tilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\tilde{f}(D_0)) - 1$
   - (b) *The set $\tilde{\eta} \circ \tilde{f}(D_0)$ does not meet any face of type $(i, \mathrm{el}_{D_0}(\tilde{f}(D_0)))$, for any index $0 \leq i \leq 2g-2$.*

Definition 7.23 There are two kinds of connected components of $f(\Pi(\partial D_0)) - \Pi(\partial D_0)$:

1. The connected components homeomorphic to $\mathbb{R}$, which will be called *regular*.
2. At most one connected component called *singular*, homeomorphic to the union of $2g$ pairwise transverse straight lines of the plane that meet in one point.

This last connected component is the one that contains the image under h of the vertex of $\Pi(\partial D_0)$ and will raise technical issues.

Proof of Lemma 7.22 Consider a little perturbation of the identity η_0 supported in the interior of one of the discs in $\mathcal{U}$ so that

$$\mathrm{el}_{D_0}(\tilde{\eta}_0 \circ \tilde{f}(D_0)) \leq \mu \quad \text{and} \quad \eta_0 \circ f(p) \notin \Pi(\partial D_0).$$

Notice that, if $\mathrm{el}_{D_0}(\tilde{\eta}_0 \circ \tilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\tilde{f}(D_0)) - 1$, then the lemma is proved with $\eta = \eta_0$. Suppose, by induction, that, for an integer $j \in [0, 2g-2]$, we have built a homeomorphism η_j in $\mathrm{Homeo}_0(S)$ such that:

(1) $\mathrm{Frag}_{\mathcal{U}}(\eta_j) \leq 4j+1$.

(2) $\mathrm{el}_{D_0}(\widetilde{\eta}_j \circ \widetilde{f}(D_0)) = \mathrm{el}_{D_0}(\widetilde{f}(D_0))$.

(3) The set $\widetilde{\eta}_j(\widetilde{f}(D_0))$ does not meet the faces of type (i,μ) for $0 \leq i < j$.

(4) The point $\eta_j \circ h(p)$ does not belong to $\Pi(\partial D_0)$.

We need the following lemma, which will be proved afterwards.

Lemma 7.24 *Let h be a homeomorphism in $\mathrm{Homeo}_0(S)$ and j be an integer in $[0, 2g-2]$. Suppose that:*

(1) *The set $\widetilde{h}(D_0)$ does not meet the faces of type $(i, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$, for $0 \leq i < j$.*

(2) *The point $h(p)$ does not belong to $\Pi(\partial D_0)$.*

Then there exists a homeomorphism η' in $\mathrm{Homeo}_0(S)$ such that:

(1) $\mathrm{Frag}_{\mathcal{U}}(\eta') \leq 4$.

(2) *Either $\mathrm{el}_{D_0}(\widetilde{\eta}' \circ \widetilde{h}(D_0)) < \mathrm{el}_{D_0}(\widetilde{h}(D_0))$ or the set $\widetilde{\eta}' \circ \widetilde{h}(D_0)$ does not meet the faces of type $(i, \mathrm{el}_{\widetilde{h}(D_0)})$, for any $0 \leq i \leq j$.*

The above lemma provides a homeomorphism η' so that either $\mathrm{el}_{D_0}(\widetilde{\eta}' \circ \widetilde{\eta}_j \circ \widetilde{f}(D_0)) \leq \mu - 1$ or the set $\widetilde{\eta}' \circ \widetilde{\eta}_j \circ \widetilde{f}(D_0)$ does not meet the faces of type (j,μ) either. Moreover, $\mathrm{Frag}_{\mathcal{U}}(\eta') \leq 4$. Hence it suffices to take $\eta_{j+1} = \eta' \circ \eta_j$. Lemma 7.22 is proved because, either $\mathrm{el}_{D_0}(\widetilde{\eta}_{j+1} \circ \widetilde{f}(D_0)) < \mathrm{el}_{D_0}(\widetilde{f}(D_0))$ and $\eta = \eta_{j+1}$ is appropriate, or one can repeat the process until the set $\widetilde{\eta}_j \circ \widetilde{f}(\partial D_0)$ does not meet faces of type $(k, \mathrm{el}_{D_0}(\widetilde{f}(D_0)))$ for any $0 \leq k \leq 2g-2$. □

We now prove Lemma 7.24. Let $\mu = \mathrm{el}_{D_0}(\widetilde{h}(D_0))$. The homeomorphism η' will be built by composing four homeomorphisms f_1, f_2, f_3 and f_4 each supported in the interior of one of the discs in $\mathcal{U}$. The homeomorphisms f_i for $1 \leq i \leq 3$ will satisfy the following Property P:

$$\{D \in \mathcal{D} \mid D \cap \widetilde{f}_i \cdots \widetilde{f}_1 \circ \widetilde{h}(D_0) \neq \varnothing\} = \{D \in \mathcal{D} \mid D \cap \widetilde{h}(D_0) \neq \varnothing\}.$$

The proof is divided into two cases.

7.2.1 Proof of Lemma 7.24: Easy case

Proof Suppose that the image under $\widetilde{h}$ of any vertex $\widetilde{p}$ of the polygon ∂D_0 does not belong to any face of type (j,μ).

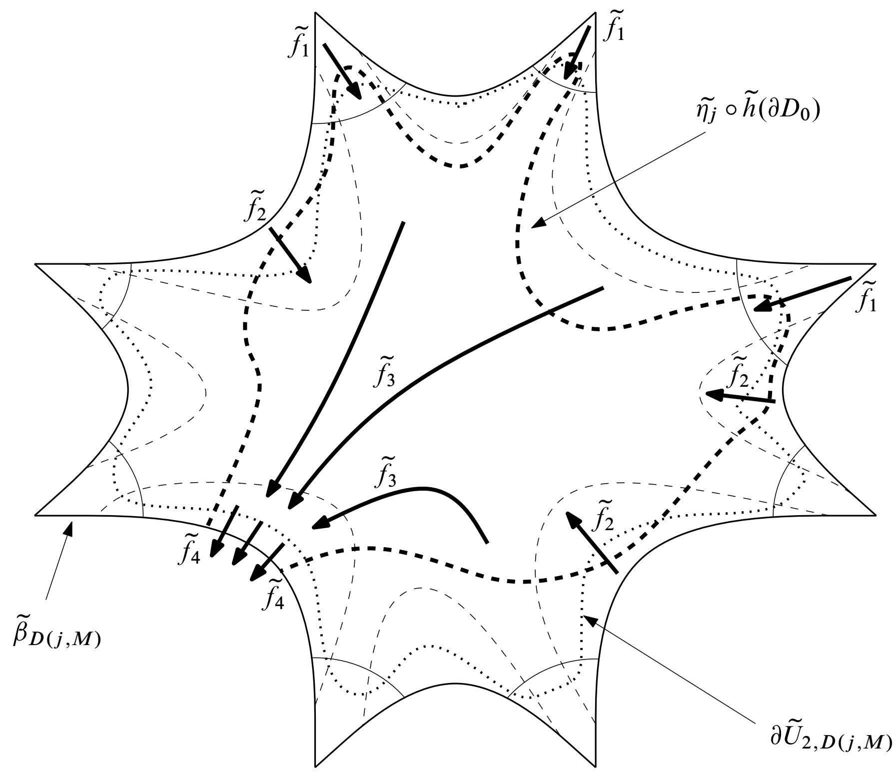

Figure 12: Idea of the proof of Lemma 7.22: the face $D(j,\mu)$

Let f_1 be a homeomorphism supported in the disc U_0 with the following properties:

(1) The homeomorphism f_1 globally preserves each edge in A.

(2) For any connected component C of $\mathring{U}_0 \cap h(\Pi(\partial D_0))$ that does not contain the point p, we have

$$f_1(C) \subset \bigcup_{\alpha \in A} \mathring{V}_\alpha \cup \mathring{U}_2.$$

To build such a homeomorphism f_1, it suffices to take the time 1 of the flow of a vector field with the following properties: the point p is a repulsive fixed point of the flow, the vector field is tangent to the edges of A and it is supported in the open disc $\mathring{U}_0$. As the homeomorphism $\widetilde{f}_1$ globally preserves each edge in $\widetilde{A}$, the homeomorphism f_1 satisfies Property P. Denote by $D(j,\mu)$ a face of type (j,μ). Recall that, by definition, if $j \geq 1$, all the faces adjacent to $D(j,\mu)$, except one, are of type $(j-1,\mu)$. Let $\widetilde{\beta}_{D(j,\mu)}$ be the edge common to both the face $D(j,\mu)$ and the unique face adjacent to $D(j,\mu)$ that is at distance $d_{\mathcal{D}}(D(j,\mu), D_0) - 1$ from the fundamental domain D_0.

Then, by hypothesis, the ends of any connected component of $\widetilde{h}(\partial D_0) \cap D(j,\mu)$ are contained in the interior $\widetilde{\beta}_{D(j,\mu)} - \partial\widetilde{\beta}_{D(j,\mu)}$ of the edge $\widetilde{\beta}_{D(j,\mu)}$. Let us denote by $\widetilde{U}_{2,D(j,\mu)}$ the lift of the disc U_2 contained in the fundamental domain $D(j,\mu)$. Then the construction of the homeomorphism f_1 implies

$$\widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap D(j,\mu) \subset \mathring{\widetilde{U}}_{2,D(j,\mu)} \cup \Pi^{-1}\Big(\bigcup_{\alpha \in A} \mathring{V}_\alpha\Big).$$

Let f_2 be a homeomorphism in $\mathrm{Homeo}_0(S)$ that is supported in the union of the discs V_α, where α varies over A, and which satisfies the following properties:

(1) The homeomorphism f_2 pointwise fixes all the edges in A.

(2) Take any edge α in A. Consider any connected component C of

$$f_1 \circ h(\Pi(\partial D_0)) \cap V_\alpha$$

that does not meet the edge α and whose ends are contained in U_2. Then $f_2(C) \subset \mathring{U}_2$.

Let $\widetilde{V}_{\widetilde{\beta}_{D(j,\mu)}}$ be the lift of the disc $V_{\Pi(\widetilde{\beta}_{D(j,\mu)})}$ that meets the edge $\widetilde{\beta}_{D(j,\mu)}$. As the homeomorphism $\widetilde{f_2}$ pointwise fixes $\Pi^{-1}(\Pi(\partial D_0))$, it satisfies Property P. Moreover, by construction of the homeomorphism f_2, we have, for any face $D(j,\mu)$ of type (j,μ),

$$\widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap D(j,\mu) \subset \mathring{\widetilde{V}}_{\widetilde{\beta}_{D(j,\mu)}} \cup \mathring{\widetilde{U}}_{2,D(j,\mu)}.$$

With the same method, we build a homeomorphism f_3 supported in the disc U_2 such that, for any face $D(j,\mu)$ of type (j,μ), we have

$$\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap D(j,\mu) \subset \mathring{\widetilde{V}}_{\widetilde{\beta}_{D(j,\mu)}}.$$

As this homeomorphism pointwise fixes $\Pi^{-1}(\Pi(\partial D_0))$, it also satisfies Property P.

Finally, let f_4 be a homeomorphism in $\mathrm{Homeo}_0(S)$ that is supported in the union of the discs $\mathring{V}_\alpha$, where α varies over the set A, and satisfies the following properties for any edge α in A:

(1) For any connected component C of $f_3 \circ f_2 \circ f_1 \circ h(\Pi(\partial D_0)) \cap \mathring{V}_\alpha$ whose ends belong to the same connected component of $V_\alpha - \alpha$, we have $f_4(C) \cap \alpha = \varnothing$.

(2) The homeomorphism f_4 pointwise fixes any other connected component of $f_3 \circ f_2 \circ f_1 \circ h(\Pi(\partial D_0)) \cap \mathring{V}_\alpha$.

We now prove that the homeomorphism $\eta' = f_4 \circ f_3 \circ f_2 \circ f_1$ satisfies the required property, namely that $\mathrm{el}_{D_0}(\widetilde{\eta'} \circ \widetilde{h}(D_0)) \leq \mathrm{el}_{D_0}(\widetilde{h}(D_0))$ and that the set $\widetilde{\eta'} \circ \widetilde{h}(D_0)$ does not meet the faces of type (i, μ) for $0 \leq i \leq j$. We will distinguish several pieces of the curve $\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0)$: the piece

$$\widetilde{k}_1 = \widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) - \Pi^{-1}(\cup_\alpha V_\alpha)$$

and the piece

$$\widetilde{k}_2 = \widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap \Pi^{-1}(\cup_\alpha V_\alpha).$$

In each of these cases, we prove that the image under f_4 of the chosen piece does not meet new faces (ie, which were not met by the curve $\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0)$) and does not meet faces of type (j, μ).

First case Take the closure $\widetilde{C}$ of a connected component of $\widetilde{k}_1$. Then $f_4(\widetilde{C}) = \widetilde{C}$ is contained in a face that belongs to the set

$$\{D \in \mathcal{D} \mid D \cap \widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(D_0) \neq \varnothing\} = \{D \in \mathcal{D} \mid D \cap \widetilde{h}(D_0) \neq \varnothing\}$$

and is not contained in a face of type (j, μ) because, for any face $D(j, \mu)$ of type (j, μ),

$$\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap D(j, \mu) \subset \mathring{\widetilde{V}}_{\widetilde{\beta}_{D(j,\mu)}}.$$

Second case Take a connected component $\widetilde{C}$ of $\widetilde{k}_2$ whose ends do not belong to the same connected component of $\Pi^{-1}(\cup_\alpha V_\alpha - \alpha)$. Then $\widetilde{f_4}(\widetilde{C}) = \widetilde{C}$ is contained in the union of the faces of the set $\{D \in \mathcal{D} \mid D \cap \widetilde{h}(D_0) \neq \varnothing\}$ and does not meet faces of type (j, μ).

Third case Take a connected component $\widetilde{C}$ of $\widetilde{k}_2$ whose ends all belong to the same connected component of $\Pi^{-1}(\cup_\alpha (V_\alpha - \alpha))$. Then the subset $\widetilde{f_4}(\widetilde{C})$ is contained in the interior of the fundamental domain in $\mathcal{D}$ that contains the ends of $\widetilde{C}$ and which, therefore, is not a face of type (j, μ). Indeed, for any face $D(j, \mu)$ of type (j, μ),

$$\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap D(j, \mu) \subset \mathring{\widetilde{V}}_{\widetilde{\beta}_{D(j,\mu)}}.$$

Moreover, such a face belongs to the set $\{D \in \mathcal{D} \mid D \cap \widetilde{h}(D_0) \neq \varnothing\}$. □

7.2.2 Proof of Lemma 7.24: Second case

Proof We suppose that the image under $\widetilde{h}$ of a vertex $\widetilde{p}$ of the polygon ∂D_0 belongs to a face D of type (j, μ). In this case, we need the following lemma.

Lemma 7.25 *Let h be a homeomorphism in* $\mathrm{Homeo}_0(S)$. *Take an integer j in* $[0, 2g-2]$. *Suppose that the following properties hold:*

(1) $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) \geq 4g$.
(2) *The point $h(p)$ does not belong to the set $\Pi(\partial D_0)$.*
(3) *The set $\widetilde{h}(D_0)$ does not meet faces of type $(i, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$ for $0 \leq i < j$.*
(4) *The image under $\widetilde{h}$ of a vertex $\widetilde{p}$ of the polygon ∂D_0 belongs to a face D_1 of type $(j, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$.*

In this case, the image under the homeomorphism $\widetilde{h}$ of any vertex of the polygon ∂D_0 different from $\widetilde{p}$ does not belong to a face of type $(j, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$. Moreover, the face D_0 is exceptional with respect to D_1 if $j \geq 1$.

Proof of Lemma 7.25 Suppose first that $j = 0$. Lemma 7.21 implies that the images under the homeomorphism $\widetilde{h}$ of the other vertices of the polygon ∂D_0 belong to fundamental domains in $\mathcal{D}$ strictly closer to D_0 than D_1. Suppose now that $j \geq 1$. We prove by contradiction that the face D_0 is exceptional with respect to D_1. Denote by $s(D_0)$, where s is a deck transformation in $\mathcal{G}$, a face adjacent to D_0 that contains the point $\widetilde{p}$. Notice that there are two such faces. Suppose by contradiction that $d_{\mathcal{D}}(s(D_0), D_1) = d_{\mathcal{D}}(D_0, D_1) + 1$. Then

$$d_{\mathcal{D}}(D_0, s^{-1}(D_1)) = d_{\mathcal{D}}(D_0, D_1) + 1, \quad \widetilde{h}(s^{-1}(\widetilde{p})) \in s^{-1}(D_1).$$

Let us prove that the fundamental domain $s^{-1}(D_1)$ is a face of type $(j-1, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$. As $\widetilde{h}(D_0) \cap s^{-1}(D_1) \neq \varnothing$, this contradicts the hypothesis of the lemma. Let γ be a geodesic word such that $\gamma(D_0) = D_1$. As $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) \geq 4g$, the length of the word γ is greater than or equal to $2g$. Moreover, as

$$d_{\mathcal{D}}(D_0, s^{-1}(D_1)) = d_{\mathcal{D}}(D_0, D_1) + 1,$$

the word $s^{-1}\gamma$ is geodesic. If we concatenate $i \in [0, j]$ letters $a_1, a_2, \dots, a_i$ on the right with γ so that the word $\gamma a_1 a_2 \cdots a_i$ is reduced, then the $2g$ last letters of the obtained word do not form a subword of a word in Λ, as the fundamental domain D_1 is a face of type $(j, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$. Therefore, if we concatenate $i \in [0, j-1]$ letters $a_1, a_2, \dots, a_i$ on the right with the geodesic word $s^{-1}\gamma$ so that the obtained word is reduced, the $2g-1$ last letters of the obtained word do not form a subword of a word in Λ. By Lemma 7.11, the faces $s^{-1}\gamma a_1 a_2 \cdots a_i(D_0)$ are not exceptional so the face $s^{-1}(D_1)$ is a face of type $(j-1, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$.

Thus, the face D_0 is exceptional with respect to D_1. Using Lemma 7.21, we see that the images under the homeomorphism $\widetilde{h}$ of the vertices of ∂D_0 distinct from $\widetilde{p}$ belong to fundamental domains in $\mathcal{D}$ strictly closer to D_0 than D_1. □

As the proof of Lemma 7.24 in this case is analogous to the proof in the first case we will just give details on what has to be changed in this case.

We denote by $\widetilde{C}_1$ the connected component of $\widetilde{h}(\partial D_0) \cap \mathring{D}$ that contains the point $\widetilde{h}(\widetilde{p})$. By Lemma 7.25, this is the unique connected component of $\widetilde{h}(\partial D_0) - \Pi^{-1}(\Pi(\partial D_0))$ with the following properties: it contains the image under the homeomorphism $\widetilde{h}$ of a vertex of the polygon ∂D_0 and it is contained in a face of type (j, μ). Notice that $\Pi(\widetilde{C}_1)$ is a subset of the singular component of $h(\Pi(D_0)) - \Pi(\partial D_0)$.

The constructions of the homeomorphisms f_1, f_2, f_3 and f_4 have to be slightly modified. Let f_1 be a homeomorphism supported in the disc U_0 with the following properties:

(1) The homeomorphism f_1 globally preserves each edge in A.

(2) For any connected component C of $\mathring{U}_0 \cap h(\Pi(\partial D_0))$ that does not contain the point p, we have
$$f_1(C) \subset \bigcup_{\alpha \in A} \mathring{V}_\alpha \cup \mathring{U}_2.$$

(3) The image of $\Pi(\widetilde{C}_1)$ under f_1 is contained in the open set
$$\bigcup_{\alpha \in A} \mathring{V}_\alpha \cup \mathring{U}_2.$$
Notice that this condition is not implied by the second one when $\Pi(\widetilde{C}_1)$ is contained in a connected component of $\mathring{U}_0 \cap h(\Pi(\partial D_0))$ that contains the point p.

As the set $\widetilde{C}_1$ is contained in a face of type (j, μ), the set $\overline{\Pi(\widetilde{C}_1)}$ does not contain the point p (otherwise the closed set $\overline{\widetilde{C}}_1$ would meet a face of type $(j-1, \mu)$ or a face at distance $\mu + 1$ from D_0, which contradicts the hypothesis on the homeomorphism h).

Let f_2 be a homeomorphism $\mathrm{Homeo}_0(S)$ that is supported in the union of the discs V_α, where α varies over A, and which satisfies the following properties:

(1) The homeomorphism f_2 pointwise fixes all the edges in A.

(2) Take any edge α in A. Consider any connected component C of
$$f_1 \circ h(\Pi(\partial D_0)) \cap V_\alpha$$
that does not meet the edge α and whose ends are contained in U_2. Then $f_2(C) \subset \mathring{U}_2$.

(3) $f_2 \circ f_1(\Pi(\widetilde{C}_1)) \subset \mathring{U}_2$.

The homeomorphisms f_1 and f_2 satisfy Property P. Moreover, for any face $D(j,\mu)$ of type (j,μ),

$$\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}(\partial D_0)\cap D(j,\mu)\subset \mathring{\widetilde{V}}_{\widetilde{\beta}_{D(j,\mu)}}\cup \mathring{\widetilde{U}}_{2,D(j,\mu)}.$$

With the same method, we build a homeomorphism f_3 supported in the disc U_2 such that, for any face $D(j,\mu)$ of type (j,μ), we have

$$\widetilde{f_3}\circ\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}(\partial D_0)\cap D(j,\mu)\subset \mathring{\widetilde{V}}_{\widetilde{\beta}_{D(j,\mu)}}.$$

Finally, let f_4 be a homeomorphism in $\mathrm{Homeo}_0(S)$ that is supported in the union of the discs $\mathring{V}_\alpha$, where α varies over the set A, and satisfies the following properties for any edge α in A:

(1) For any connected component C of $f_3\circ f_2\circ f_1\circ h(\Pi(\partial D_0))\cap \mathring{V}_\alpha$ whose ends belong to the same connected component of $V_\alpha-\alpha$, we have $f_4(C)\cap\alpha=\varnothing$.

(2) The homeomorphism f_4 pointwise fixes any other connected component of $f_3\circ f_2\circ f_1\circ h(\Pi(\partial D_0))\cap \mathring{V}_\alpha$ that is homeomorphic to $\mathbb{R}$.

(3) Denote by $\widetilde{C}_1'$ the connected component of $\widetilde{f_3}\circ\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}(\partial D_0)\cap\Pi^{-1}(\bigcup_\alpha \mathring{V}_\alpha)$ with the following properties: it contains the image under the homeomorphism $\widetilde{f_3}\circ\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}$ of a vertex of the polygon ∂D_0 and it meets a face of type (j,μ). Then $f_4(\Pi(\widetilde{C}_1'))\cap\alpha=\varnothing$.

Let

$$\widetilde{k}_1=\widetilde{f_3}\circ\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}(\partial D_0)-\Pi^{-1}\Big(\bigcup_\alpha V_\alpha\Big),\quad \widetilde{k}_2=\widetilde{f_3}\circ\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}(\partial D_0)\cap\Pi^{-1}\Big(\bigcup_\alpha V_\alpha\Big).$$

As in the first case, one can prove the following properties:

(1) If $\widetilde{C}$ is the closure of a connected component of $\widetilde{k}_1$. Then $f_4(\widetilde{C})=\widetilde{C}$ is contained in a face which belongs to the set $\{D\in\mathcal{D}\mid D\cap\widetilde{h}(D_0)\neq\varnothing\}$ and is not contained in a face of type (j,μ).

(2) Take any connected component $\widetilde{C}$ of $\widetilde{k}_2$ whose ends do not belong to the same connected component of $\Pi^{-1}(\bigcup_\alpha V_\alpha-\alpha)$ and such that the set $\widetilde{C}$ does not contain the image under the homeomorphism $\widetilde{f_3}\circ\widetilde{f_2}\circ\widetilde{f_1}\circ\widetilde{h}$ of a vertex of the polygon ∂D_0. Then $\widetilde{f_4}(\widetilde{C})=\widetilde{C}$ is contained in the union of the faces of the set $\{D\in\mathcal{D}\mid D\cap\widetilde{h}(D_0)\neq\varnothing\}$ and does not meet faces of type (j,μ).

(3) Take a connected component $\widetilde{C}$ of $\widetilde{k}_2$ whose ends all belong to the same connected component of $\Pi^{-1}(\bigcup_\alpha(V_\alpha-\alpha))$. Then the subset $\widetilde{f_4}(\widetilde{C})$ is contained in the interior of the fundamental domain in $\mathcal{D}$ that is not a face of type (j,μ). Moreover, such a face belongs to the set $\{D\in\mathcal{D}\mid D\cap\widetilde{h}(D_0)\neq\varnothing\}$.

Let us finally address the case where $\widetilde{C}$ is a connected component of $\widetilde{k}_2$ that contains the image under the homeomorphism $\widetilde{f}_3 \circ \widetilde{f}_2 \circ \widetilde{f}_1 \circ \widetilde{h}$ of a vertex of the polygon ∂D_0. Let $\widetilde{p}$ be the vertex of the polygon whose image under the homeomorphism $\widetilde{f}_3 \circ \widetilde{f}_2 \circ \widetilde{f}_1 \circ \widetilde{h}$ belongs to a face D_1 of type (j, μ). By Lemmas 7.21 and 7.25, there exists a geodesic word of the form $\lambda_1 \lambda_2 \cdots \lambda_{2g} \gamma$, where the word $\lambda_1 \lambda_2 \cdots \lambda_{4g}$ belongs to Λ, which sends the face D_0 to the face D_1. Let us denote by γ' the word γ without the last letter. By construction of the homeomorphism f_4 and by Lemma 7.21, the set $\widetilde{f}_4(\widetilde{C})$ is contained in the interior of the union of the following fundamental domains:

$$\begin{aligned} &\lambda_1 \cdots \lambda_{2g} \gamma'(D_0) \\ &\lambda_{i+1} \cdots \lambda_{2g} \gamma(D_0) \quad \text{if } 1 \leq i \leq 2g \\ &\lambda_{i+1} \cdots \lambda_{2g} \gamma'(D_0) \quad \text{if } 1 \leq i \leq 2g \\ &\lambda_{4g-i}^{-1} \cdots \lambda_{2g}^{-1} \gamma(D_0) \quad \text{if } 1 \leq i \leq 2g \\ &\lambda_{4g-i}^{-1} \cdots \lambda_{2g}^{-1} \gamma'(D_0) \quad \text{if } 1 \leq i \leq 2g \end{aligned}$$

These fundamental domains are each at distance less than or equal to $\mu - j - 1$ from D_0 and hence are not faces of type (i, μ) for $0 \leq i \leq j$. □

7.3 Pushing the image of D_0 into U_0

For a homeomorphism h in $\mathrm{Homeo}_0(S)$, we denote by $\mathcal{F}_h$ the union of the set of exceptional faces that are maximal for the homeomorphism h with the set of fundamental domains D in $\mathcal{D}$ such that:

1. The face D is at distance less than or equal to $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) - 1$ and greater than or equal to $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) - (2g-2)$ from D_0.
2. The face D shares a vertex in common with an exceptional face $D_{\max}$ that is maximal for h and with the two faces D_1 and D_2 in $\mathcal{D}$ with the following properties:
   (a) The faces D_1 and D_2 are distinct and adjacent to $D_{\max}$.
   (b) The faces D_1 and D_2 are at distance $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) - 1$ from the face D_0.

By Lemma 7.18, the faces D that belong to the set $\mathcal{F}_h$ satisfy the following property. Denote by $\widetilde{p}$ a vertex of the boundary of D that belongs to an exceptional maximal face and to two faces at distance $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) - 1$ from the face D_0. Then any face adjacent to D that does not contain the point $\widetilde{p}$ is a face of type $(i, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$, for some integer i between 0 and $2g-3$. This property implies that, for any face D in $\mathcal{F}_h$, there exists a unique exceptional face $D_{\max}$ such that the second property above holds. This exceptional maximal face will be called the *exceptional maximal face associated to* D.

Lemma 7.26 *Let h be a homeomorphism in* $\mathrm{Homeo}_0(S)$ *with the following properties:*

(1) $h(p) \notin \Pi(\partial D_0)$.

(2) $\mathrm{el}_{D_0}(\tilde{h}(D_0)) \geq 4g$.

(3) *The set* $\tilde{h}(D_0)$ *does not meet any face of type* $(i, \mathrm{el}_{D_0}(\tilde{h}(D_0)))$, *for any index* $0 \leq i \leq 2g-2$.

Then there exists a homeomorphism η in $\mathrm{Homeo}_0(S)$ *with the following properties:*

(1) *For any fundamental domain D in $\mathcal{F}_h$ that is at distance at most $2g-3$ from its associated exceptional maximal face,* $\tilde{\eta} \circ \tilde{h}(\partial D_0) \cap D \subset \Pi^{-1}(\mathring{U}_0)$.

(2) *For any fundamental domain D in $\mathcal{F}_h$ at distance $2g-2$ from its associated exceptional maximal face, any connected component of* $\tilde{\eta} \circ \tilde{h}(\partial D_0) \cap D$ *is either contained in* $\Pi^{-1}(\mathring{U}_0)$ *or in* $D - \Pi^{-1}(U_0)$.

(3) $\mathrm{Frag}_{\mathcal{U}}(\eta) \leq 4$.

(4) $\eta \circ h(p) \notin \Pi(\partial D_0)$.

(5) $\mathrm{el}_{D_0}(\tilde{\eta} \circ \tilde{h}(D_0)) \leq \mathrm{el}_{D_0}(\tilde{h}(D_0))$.

(6) *The set* $\tilde{\eta} \circ \tilde{h}(D_0)$ *does not meet any face of type* $(i, \mathrm{el}_{D_0}(\tilde{h}(D_0)))$, *for any index* $0 \leq i \leq 2g-2$.

For any homeomorphism h' in $\mathrm{Homeo}_0(S)$, let $\mathcal{E}_{h'}$ be the set of connected components $\tilde{C}$ of $\tilde{h}'(\partial D_0) - \Pi^{-1}(\Pi(\partial D_0))$ such that:

(1) The connected component $\tilde{C}$ contains the image under $\tilde{h}'$ of a vertex of the polygon ∂D_0.

(2) The face $D_{\tilde{C}}$ in $\mathcal{D}$ that contains $\tilde{C}$ belongs to $\mathcal{F}_h$.

For any edge $\tilde{\alpha}$ in $\tilde{A}$, we denote by $\tilde{V}_{\tilde{\alpha}}$ the lift of the disc $V_{\Pi(\tilde{\alpha})}$ that meets $\tilde{\alpha}$.

Definition 7.27 Two edges $\tilde{\alpha}$ and $\tilde{\beta}$ in $\tilde{A}$ are said to be *consecutive* if:

(1) They share a point in common.

(2) They are both contained in the same face $D^{\tilde{\alpha}\tilde{\beta}}$ in $\mathcal{D}$.

Given two consecutive edges $\tilde{\alpha}$ and $\tilde{\beta}$ in $\tilde{A}$, we denote by $\tilde{U}_0^{\tilde{\alpha}\tilde{\beta}}$ the lift of the disc U_0 that meets $\tilde{\alpha}$ and $\tilde{\beta}$ and by $\tilde{U}_2^{\tilde{\alpha}\tilde{\beta}}$ the lift of the disc U_2 that is contained in the face $D^{\tilde{\alpha}\tilde{\beta}}$.

We will first prove Lemma 7.26 under the additional hypothesis that the set $\mathcal{E}_h$ is empty. Then we explain the necessary modifications for the proof in the case where $\mathcal{E}_h \neq \varnothing$.

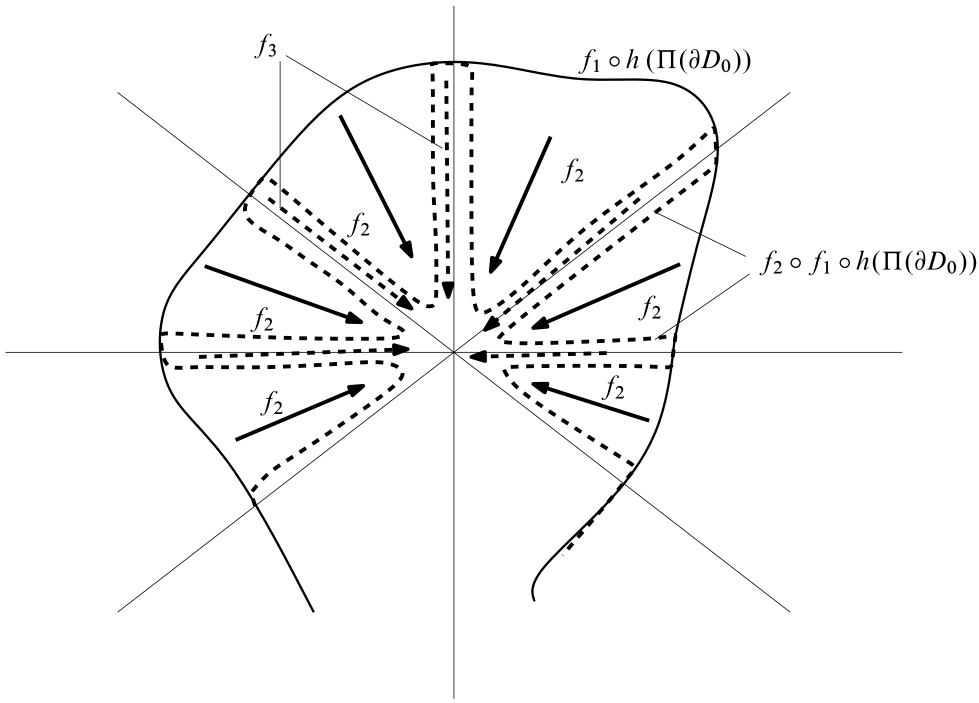

Figure 13: Illustration of the proof of Lemma 7.26

7.3.1 Proof of Lemma 7.26: Case $\mathcal{E}_h = \varnothing$

Proof By methods similar to those used to prove Lemma 7.22, we build a homeomorphism f_1 with the following properties:

(1) The homeomorphism f_1 is the composition of a homeomorphism supported in U_0 with a homeomorphism supported in the union of the V_α.

(2) The homeomorphism $\widetilde{f_1}$ globally preserves any edge in $\widetilde{A}$.

(3) Take any two consecutive edges $\widetilde{\alpha}$ and $\widetilde{\beta}$ and any connected component $\widetilde{C}$ of $\widetilde{h}(\partial D_0) - \Pi^{-1}(\Pi(\partial D_0))$ whose ends belong to $\widetilde{\alpha} \cup \widetilde{\beta}$ but are not endpoints of this path. Then the set $\widetilde{f_1}(\widetilde{C})$ is contained in the interior of the set

$$\widetilde{U}_0^{\widetilde{\alpha}\widetilde{\beta}} \cup \widetilde{V}_{\widetilde{\alpha}} \cup \widetilde{V}_{\widetilde{\beta}} \cup \widetilde{U}_2^{\widetilde{\alpha}\widetilde{\beta}}.$$

Moreover, if no end of $\widetilde{C}$ meets $\widetilde{E}$, where $\widetilde{E}$ is one of the sets $\widetilde{V}_{\widetilde{\alpha}}$, $\widetilde{V}_{\widetilde{\beta}}$, $\widetilde{U}_0^{\widetilde{\alpha}\widetilde{\beta}} \cup \widetilde{V}_{\widetilde{\alpha}}$ or $\widetilde{U}_0^{\widetilde{\alpha}\widetilde{\beta}} \cup \widetilde{V}_{\widetilde{\beta}}$, then $\widetilde{f_1}(\widetilde{C})$ does not meet $\widetilde{E}$.

Take a face D in $\mathcal{F}_h$. By Lemma 7.18, there exist only two edges $\widetilde{\alpha}_D$ and $\widetilde{\beta}_D$ in $\widetilde{A}$ such that:

(1) These edges are not contained in a face of type $(i, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$, for any $0 \leq i \leq 2g-2$.

(2) These edges are not contained in a face at distance $\mathrm{el}_{D_0}(\widetilde{h}(D_0))+1$ from D_0.

Moreover, these two edges are consecutive. By hypothesis, the ends of any connected component of $\widetilde{h}(\partial D_0) \cap D$ are contained in $\widetilde{\alpha}_D \cup \widetilde{\beta}_D$ and are not endpoints of this path. Hence the set $\widetilde{f}_1 \circ \widetilde{h}(\partial D_0) \cap D$ is contained in the interior of the set

$$\widetilde{U}_0^{\widetilde{\alpha}_D \widetilde{\beta}_D} \cup \widetilde{V}_{\widetilde{\alpha}_D} \cup \widetilde{V}_{\widetilde{\beta}_D} \cup \widetilde{U}_2^{\widetilde{\alpha}_D \widetilde{\beta}_D}.$$

We denote by $\mathcal{C}$ the set of lifts $\widetilde{C}$ of connected components of $f_1 \circ h(\Pi(\partial D_0)) - \Pi(\partial D_0)$ such that: all the ends of $\widetilde{C}$ belong either to the same edge in $\widetilde{A}$, or to two consecutive edges in $\widetilde{A}$.

We build a homeomorphism f_2 that is supported in U_2 with the following property. For any connected component $\widetilde{C}$ in $\mathcal{C}$ whose ends belong to the union of edges $\widetilde{\alpha} \cup \widetilde{\beta}$ but are not endpoints of this path, the set $\widetilde{f}_2(\widetilde{C})$ is contained in the interior of the set

$$\widetilde{V}_\alpha \cup \widetilde{V}_\beta \cup \widetilde{U}_0^{\widetilde{\alpha}\widetilde{\beta}}.$$

Moreover, if the ends of $\widetilde{C}$ do not meet a set $\widetilde{E}$ among $\widetilde{V}_\alpha$, $\widetilde{V}_\beta$, $\widetilde{U}_0^{\widetilde{\alpha}\widetilde{\beta}} \cup \widetilde{V}_{\widetilde{\alpha}}$ or $\widetilde{U}_0^{\widetilde{\alpha}\widetilde{\beta}} \cup \widetilde{V}_{\widetilde{\beta}}$, then $\widetilde{f}_2(\widetilde{C})$ is disjoint from $\widetilde{E}$. The construction implies that, for any fundamental domain D in $\mathcal{F}_h$ and any connected component $\widetilde{C}$ of $\widetilde{f}_1 \circ \widetilde{h}(\partial D_0) \cap D$, the set $\widetilde{f}_2(\widetilde{C})$ is contained in the interior of the set

$$\widetilde{U}_0^{\widetilde{\alpha}_D \widetilde{\beta}_D} \cup \widetilde{V}_{\widetilde{\alpha}_D} \cup \widetilde{V}_{\widetilde{\beta}_D}.$$

Also, if $\widetilde{C}$ doesn't meet a set $\widetilde{E}$ among $\widetilde{V}_{\widetilde{\alpha}_D}$, $\widetilde{V}_{\widetilde{\beta}_D}$, $\widetilde{U}_0^{\widetilde{\alpha}_D \widetilde{\beta}_D} \cup \widetilde{V}_{\widetilde{\alpha}_D}$ or $\widetilde{U}_0^{\widetilde{\alpha}_D \widetilde{\beta}_D} \cup \widetilde{V}_{\widetilde{\alpha}_D}$, then $\widetilde{f}_2(\widetilde{C})$ does not meet the set $\widetilde{E}$ either. As the homeomorphism $\widetilde{f}_2 \circ \widetilde{f}_1$ globally preserves any edge in $\widetilde{A}$,

$$\{D \in \mathcal{D} \mid \widetilde{f}_2 \circ \widetilde{f}_1 \circ \widetilde{h}(D_0) \cap D \neq \varnothing\} = \{D \in \mathcal{D} \mid \widetilde{h}(D_0) \cap D \neq \varnothing\}.$$

Let f_3 be a homeomorphism supported in the union of the V_α with the following properties:

(1) For any edge α in A and any connected component C of $f_2 \circ f_1 \circ h(\Pi(\partial D_0)) \cap V_\alpha$ whose ends belong to the same connected component of $\mathring{U}_0 \cap V_\alpha$, $f_3(C) \subset \mathring{U}_0$.

(2) For any edge $\widetilde{\alpha}$ in $\widetilde{A}$ and any connected component $\widetilde{C}$ of $\widetilde{f}_2 \circ \widetilde{f}_1 \circ \widetilde{h}(\partial D_0) \cap \widetilde{V}_{\widetilde{\alpha}}$ that does not meet the edge $\widetilde{\alpha}$, $\widetilde{f}_3(\widetilde{C}) \cap \widetilde{\alpha} = \varnothing$.

By the second property satisfied by the homeomorphism f_3,

$$\{D \in \mathcal{D} \mid \tilde{f}_3 \circ \tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{h}(D_0) \cap D \neq \varnothing\} \subset \{D \in \mathcal{D} \mid \tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{h}(D_0) \cap D \neq \varnothing\} \\ \subset \{D \in \mathcal{D} \mid \tilde{h}(D_0) \cap D \neq \varnothing\}.$$

Let D be a face in $\mathcal{F}_h$ at distance $i < 2g-2$ from an exceptional face that is maximal for h. We prove that, for any fundamental domain D' in $\mathcal{D}$ and any connected component $\tilde{C}$ of $D' \cap \tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{h}(\partial D_0)$,

$$\tilde{f}_3(\tilde{C}) \cap D \subset \mathring{\tilde{U}}_0^{\tilde{\alpha}_D \tilde{\beta}_D}.$$

If the face D' is not adjacent to D, as $\tilde{f}_3(\tilde{C})$ is contained in the set of faces adjacent to D', then $\tilde{f}_3(\tilde{C}) \cap D = \varnothing$. Otherwise, by Lemma 7.18, the face D' satisfies one of the following properties.

(1) The face D' is a face of type $(i-1, \mathrm{el}_{D_0}(\tilde{h}(D_0)))$.

(2) The face D' is at distance $\mathrm{el}_{D_0}(\tilde{h}(D_0)) + 1$ from the face D_0.

(3) The face D' belongs to $\mathcal{F}_h$.

In the first two cases, the face D' does not meet $\tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{h}(\partial D_0)$. Therefore, it suffices to consider the two following cases:

(1) The face D' belongs to $\mathcal{F}_h$ and is adjacent to D.

(2) $D' = D$.

In the first case, let $\tilde{\alpha} = D \cap D'$ and $\tilde{V}_{\tilde{\alpha}}$ be the lift of $V_{\Pi(\tilde{\alpha})}$ that meets $\tilde{\alpha}$. Notice that any point of $\tilde{C}$ that does not meet $\tilde{V}_{\tilde{\alpha}}$ has an image disjoint from D. Moreover, by construction of f_3, any connected component of $\tilde{C} \cap \tilde{V}_{\tilde{\alpha}}$ that does not meet $\tilde{\alpha}$ has an image under $\tilde{f}_3$ that does not meet the fundamental domain D. Let us denote by $\tilde{C}_1$ a connected component of $\tilde{C} \cap \tilde{V}_{\tilde{\alpha}}$ that meets $\tilde{\alpha}$ and denote by $\tilde{C}_1'$ the connected component of $\tilde{V}_{\tilde{\alpha}} \cap \tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{h}(\partial D_0)$ that contains $\tilde{C}_1$. The connected component $\tilde{C}_1'$ has necessarily both its ends contained in $\tilde{U}_{0,D}$ by the properties satisfied by $\tilde{f}_2 \circ \tilde{f}_1 \circ \tilde{h}$. Therefore, the set $\tilde{f}_3(\tilde{C}_1)$ is contained in the set $\tilde{f}_3(\tilde{C}_1')$, which is itself contained in $\tilde{U}_0^{\tilde{\alpha}_D \tilde{\beta}_D}$. This proves the above result in the first case. The second case is similar: the same kind of arguments implies that $\tilde{f}_3(\tilde{C}) \cap D \subset \tilde{U}_{0,D}$.

Take any face D in $\mathcal{F}_h$ at distance $2g-2$ from its associated exceptional maximal face. After interchanging $\tilde{\alpha}_D$ and $\tilde{\beta}_D$ if necessary, we can suppose that the edge $\tilde{\alpha}_D$ is equal to the intersection of the face D with a face that does not belong to $\mathcal{F}_h$. We will always make this assumption for such faces in what follows. Then, for any connected component $\widetilde{C}$ of $D \cap \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0)$,

$$\widetilde{f_3}(\widetilde{C}) \cap D \subset \mathring{\widetilde{U}}_0^{\widetilde{\alpha}_D \widetilde{\beta}_D} \cup \mathring{\widetilde{V}}_{\widetilde{\alpha}_D}.$$

Finally, let f_4 be a homeomorphism in $\mathrm{Homeo}_0(S)$ supported in the union of the V_α with the following properties:

(1) $f_4(U_0) \subset U_0$.

(2) Take any face D in $\mathcal{F}_h$ at distance $2g-2$ from its associated exceptional maximal face. For any connected component $\widetilde{C}$ of $\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap \widetilde{V}_{\widetilde{\alpha}_D}$, the following property is satisfied. Any connected component of $\widetilde{f_4}(\widetilde{C}) \cap D$ is contained either in $\Pi^{-1}(\mathring{U}_0)$ or in $D - \Pi^{-1}(U_0)$.

(3) For any edge α in A and any connected component C of $f_3 \circ f_2 \circ f_1 \circ h(\Pi(\partial D_0)) \cap V_\alpha$ that does not meet α, $f_4(C) = C$.

The compatibility of the second condition above with the third one is a consequence of the two remarks below:

(1) Let α be any edge in A and C be any connected component of $f_3 \circ f_2 \circ f_1 \circ h(\Pi(\partial D_0)) \cap V_\alpha$ whose ends are both contained in the same connected component of $V_\alpha \cap \mathring{U}_0$. Then the set C is contained in $\mathring{U}_0$.

(2) Let D be a face in $\mathcal{F}_h$ at distance $2g-2$ from an exceptional maximal face and suppose as above that $\widetilde{\alpha}_D$ is contained in a face that does not belong to $\mathcal{F}_h$. Let $\widetilde{C}$ be any connected component of $\widetilde{f_3} \circ \widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap \widetilde{V}_{\widetilde{\alpha}_D}$ that meets D. Then the set $\widetilde{C}$ is either contained in $\Pi^{-1}(\mathring{U}_0)$ or meets $\widetilde{\alpha}$. Indeed, if the connected component $\widetilde{C}$ did not satisfy any of these properties, then it would meet at least two connected components of $\Pi^{-1}(U_0) \cap D$, a contradiction.

The homeomorphism $\eta = f_4 \circ f_3 \circ f_2 \circ f_1$ satisfies the following properties. As the homeomorphism $f_4 \circ f_3$ is supported in U_1 and as $\mathrm{Frag}_{\mathcal{U}}(f_1) \leq 2$, we have $\mathrm{Frag}_{\mathcal{U}}(\eta) \leq 4$. For any face D in $\mathcal{F}_h$ at distance at most $2g-3$ from an exceptional maximal face,

$$\widetilde{\eta} \circ \widetilde{h}(\partial D_0) \cap D \subset \Pi^{-1}(\mathring{U}_0).$$

By construction of the homeomorphism f_4, for any face D in $\mathcal{F}_h$ at distance $2g-2$ from its associated exceptional maximal face, any connected component of $\widetilde{\eta} \circ \widetilde{h}(\partial D_0) \cap D$ is either contained in $\Pi^{-1}(\mathring{U}_0)$ or in $D - \Pi^{-1}(U_0)$. Moreover, by the third property satisfied by the homeomorphism f_4,

$$\{D \in \mathcal{D} \mid D \cap \widetilde{\eta} \circ \widetilde{h}(\partial D_0) \neq \varnothing\} \subset \{D \in \mathcal{D} \mid D \cap \widetilde{h}(\partial D_0) \neq \varnothing\}.$$

Hence $\mathrm{el}_{D_0}(\widetilde{h}(D_0)) \geq \mathrm{el}_{D_0}(\widetilde{\eta} \circ \widetilde{h}(D_0))$ and the set $\widetilde{\eta} \circ \widetilde{h}(D_0)$ does not meet faces of type $(i, \mathrm{el}_{D_0}(\widetilde{h}(D_0)))$ for any $0 \leq i \leq 2g-2$. It is easy to choose the f_i so that the property $\eta \circ h(p) \notin \Pi(\partial D_0)$ holds. □

7.3.2 Proof of Lemma 7.26: Case $\mathcal{E}_h \neq \varnothing$

Proof During this proof, we need the following lemma, which allows us to deal with the singular component:

Lemma 7.28 *Let h' be a homeomorphism in* $\mathrm{Homeo}_0(S)$ *that satisfies the hypothesis of Lemma 7.26. There exist words $\lambda_1 \cdots \lambda_{4g}$, $\lambda'_1 \cdots \lambda'_{4g}$ in Λ, a geodesic word of the form $\lambda_1 \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}$ and an integer $0 \leq i \leq 2g-2$ such that, for any connected component $\widetilde{C}$ in $\mathcal{E}_{h'}$, either*

there exists i', $1 \leq i' \leq 2g$ such that $D_{\widetilde{C}} = \lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1-i}(D_0)$,
or there exists i', $1 \leq i' \leq 2g$ such that $D_{\widetilde{C}} = \lambda^{-1}_{4g-i'} \cdots \lambda^{-1}_{2g+1} \gamma \lambda'_1 \cdots \lambda'_{2g-1-i}(D_0)$.

Moreover, in the first case above, the faces that are adjacent to $D_{\widetilde{C}}$ and are not faces of type $(j, \mathrm{el}_{D_0}(\widetilde{h}'(\partial D_0)))$ for any $0 \leq j \leq 2g-2$ are the faces

$$\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-2-i}(D_0) \quad \text{and} \quad \lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-i}(D_0).$$

In the second case above, the faces that are adjacent to $D_{\widetilde{C}}$ and are not faces of type $(j, \mathrm{el}_{D_0}(\widetilde{h}'(\partial D_0)))$ for any $0 \leq j \leq 2g-2$ are the faces

$$\lambda^{-1}_{4g-i'} \cdots \lambda^{-1}_{2g+1} \gamma \lambda'_1 \cdots \lambda'_{2g-2-i}(D_0) \quad \text{and} \quad \lambda^{-1}_{4g-i'} \cdots \lambda^{-1}_{2g+1} \gamma \lambda'_1 \cdots \lambda'_{2g-i}(D_0).$$

Proof Let us denote by $\widetilde{p}$ the vertex of the polygon ∂D_0 such that the point $\widetilde{h}'(\widetilde{p})$ belongs to a fundamental domain D_1 in $\mathcal{D}$ at distance $l(h')$ from D_0. Then, by Lemma 7.21, $D_1 = \lambda_1 \cdots \lambda_{2g} \gamma'(D_0)$, where $\lambda_1 \cdots \lambda_{2g}$ is a subword of length $2g$ of a word $\lambda_1 \cdots \lambda_{4g}$ in Λ and $\lambda_1 \cdots \lambda_{2g} \gamma'$ is a geodesic word. Suppose that the set $\mathcal{E}_{h'}$ is nonempty and fix an element $\widetilde{C}_0$ in $\mathcal{E}_{h'}$. Let $\widetilde{p}_0$ be the vertex of ∂D_0 whose image under the homeomorphism $\widetilde{h}'$ is contained in $\widetilde{C}_0$. By Lemma 7.21, after interchanging the roles of $\lambda_1 \cdots \lambda_{2g}$ and $\lambda^{-1}_{4g} \cdots \lambda^{-1}_{2g+1}$ if necessary, we can suppose that $\widetilde{p}_0 = \lambda^{-1}_{i'_0-1} \cdots \lambda^{-1}_1(\widetilde{p})$, where $1 \leq i'_0 \leq 2g$. Recall that the face

$$D_{\widetilde{C}_0} = \lambda_{i'_0} \cdots \lambda_{2g} \gamma'(D_0)$$

belongs to $\mathcal{F}_{h'}$. Therefore, there exists a subword $l_1 \cdots l_i$ of a word in Λ, with $0 \leq i \leq 2g-2$, such that the face $\lambda_{i'_0} \cdots \lambda_{2g} \gamma' l_1 \cdots l_i(D_0)$ is exceptional and maximal for h' and the word $\lambda_{i'_0} \cdots \lambda_{2g} \gamma' l_1 \cdots l_i$ is geodesic. By Lemma 7.11, the $2g-1$ last letters of the word $\lambda_{i'_0}\cdots\lambda_{2g}\gamma' l_1\cdots l_i$ are $\lambda'_1\cdots\lambda'_{2g-1}$, where $\lambda'_1\cdots\lambda'_{4g}$ is a word in Λ. Hence the word γ' can be written $\gamma'=\gamma\lambda'_1\cdots\lambda'_{2g-1-i}$.

Recall that, by Lemma 7.21, the vertices of the polygon ∂D_0 are the points of the form $\lambda_{i'-1}^{-1}\cdots\lambda_1^{-1}(\widetilde{p})$ or $\lambda_{4g-i'+1}\cdots\lambda_{4g}(\widetilde{p})$, where $1\leq i'\leq 2g$. Hence the faces in $\mathcal{D}$ that contain the image under $\widetilde{h}'$ of a vertex of the polygon ∂D_0 are the faces

$$\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-1-i}(D_0)\quad\text{and}\quad\lambda_{4g-i'}^{-1}\cdots\lambda_{2g+1}^{-1}\gamma\lambda'_1\cdots\lambda'_{2g-1-i}(D_0),$$

where $1\leq i'\leq 2g$. Each face of the form $D_{\widetilde{C}}$, where $\widetilde{C}$ belongs to $\mathcal{E}_{h'}$, is one of these faces.

Fix an element $\widetilde{C}$ of $\mathcal{E}_{h'}$. Let us look for the that which are adjacent to $D_{\widetilde{C}}$ and are not faces of type $(j,\mathrm{el}_{D_0}(\widetilde{h}(\partial D_0)))$ for any $0\leq j\leq 2g-2$. For ease of notation, suppose that

$$D_{\widetilde{C}}=\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-1-i}(D_0)$$

for some $1\leq i'\leq 2g$. Recall that the face $\lambda_{i'_0}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-1}(D_0)$ is exceptional and that the word $\lambda_{i'_0}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-1}$ is geodesic. Therefore, by Lemma 7.11, one of the following properties holds:

(1) The last letter of the word γ is λ'_{4g} and the penultimate letter of γ is different from λ'_{4g-1}.

(2) The concatenation of the last letter of the word γ and the letter λ'_1 is not contained in any word in Λ.

First case The last letter of the word γ is λ'_{4g}. For any reduced word w either of length less than $i-1$ or of length $i-1$ and different from $\lambda'_{2g-i}\cdots\lambda'_{2g-2}$, the concatenation of the $2g-1$ last letters of the word $\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-i-1}w$ is not a subword of a word in Λ. By Lemma 7.11, the face $\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-i-1}w(D_0)$ is not exceptional for such words w.

First subcase The face $\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-2}(D_0)$ is exceptional. This face is necessarily the exceptional maximal face D_{ex} associated to $D_{\widetilde{C}}$. The faces that share a vertex in common with the faces $D_{\widetilde{C}}$ and D_{ex} and are at distance less than or equal to $\mathrm{el}_{D_0}(\widetilde{h}'(D_0))$ from D_0 are the faces of the form

$$\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-1-k}(D_0)\quad\text{or}\quad\lambda_{i'}\cdots\lambda_{2g}\gamma(\lambda'_{4g})^{-1}\cdots(\lambda'_{2g+1+k})^{-1}(D_0),$$

where $0\leq k\leq 2g-1$. Among the faces above, only two of them are adjacent to $D_{\widetilde{C}}$:

$$\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-2-i}(D_0)\quad\text{and}\quad\lambda_{i'}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-i}(D_0).$$

By Lemma 7.18, these two faces are the only ones that are adjacent to $D_{\widetilde{C}}$ and that are not faces of type $(j, \mathrm{el}_{D_0}(\widetilde{h}'(D_0)))$ for any $0 \leq j \leq 2g-2$.

Second subcase The face $\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-2}(D_0)$ is not exceptional. Notice that the word $\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g}$ is not geodesic: the face $\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0)$ is exceptional. The face $\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0)$ is the exceptional maximal face associated to $D_{\widetilde{C}}$. Hence the faces that are adjacent to $D_{\widetilde{C}}$ and that are not faces of type $(j, \mathrm{el}_{D_0}(\widetilde{h}'(D_0)))$ for any $0 \leq j \leq 2g-2$ are

$$\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-2-i}(D_0) \quad \text{and} \quad \lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-i}(D_0).$$

Second case The concatenation of the last letter of the word γ with the letter λ'_1 is not contained in any word in Λ. Then either the face $\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0)$ or the face $\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g}(D_0)$ is the exceptional maximal face associated to $D_{\widetilde{C}}$. In either case, the faces that are adjacent to $D_{\widetilde{C}}$ and that are not faces of type $(j, \mathrm{el}_{D_0}(\widetilde{h}'(D_0)))$ for any $0 \leq j \leq 2g-2$ are

$$\lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-2-i}(D_0) \quad \text{and} \quad \lambda_{i'} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-i}(D_0). \qquad \square$$

We now prove Lemma 7.26 in the case $\mathcal{E}_h \neq \varnothing$. The proof is similar to the one in the first case: we will just indicate how to modify the proof in the case $\mathcal{E}_h = \varnothing$ to obtain a proof in the case $\mathcal{E}_h \neq \varnothing$.

The construction of the homeomorphism f_1 is identical to the construction in the first case.

The definition of the homeomorphism f_2 is also identical if we slightly change the definition of the set $\mathcal{C}$. Here we denote by $\mathcal{C}$ the union of $\mathcal{E}_{f_1 \circ h}$ with the set of lifts $\widetilde{C}$ of connected components of $f_1 \circ h(\Pi(\partial D_0)) - \Pi(\partial D_0)$ such that all the ends of $\widetilde{C}$ belong either to the same edge in $\widetilde{A}$, or to two consecutive edges in $\widetilde{A}$. Note that, by Lemma 7.28, for any elements $\widetilde{C}$ and $\widetilde{C}'$ in $\mathcal{E}_{f_1 \circ h}$, there exists a deck transformation $t_{\widetilde{C}\widetilde{C}'} \in \pi_1(S)$ such that

$$t_{\widetilde{C}\widetilde{C}'}(D_{\widetilde{C}}) = D_{\widetilde{C}'} \quad \text{and} \quad \{t_{\widetilde{C}\widetilde{C}'}(\widetilde{\alpha}_{D_{\widetilde{C}}}), t_{\widetilde{C}\widetilde{C}'}(\widetilde{\beta}_{D_{\widetilde{C}}})\} = \{\widetilde{\alpha}_{D_{\widetilde{C}'}}, \widetilde{\beta}_{D_{\widetilde{C}'}}\}.$$

This remark is a justification for the existence of the homeomorphism f_2 in this case.

The definition of the homeomorphism f_3 has to be slightly modified. Here f_3 denotes a homeomorphism supported in the union of the V_α with the following properties:

(1) For any edge α in A and any connected component C of $f_2 \circ f_1 \circ h(\Pi(\partial D_0)) \cap V_\alpha$ whose ends belong to the same connected component of $U_0 \cap V_\alpha$, $f_3(C) \subset \mathring{U}_0$.

(2) For any edge $\widetilde{\alpha}$ in $\widetilde{A}$ and any connected component $\widetilde{C}$ of $\widetilde{f}_2 \circ \widetilde{f}_1 \circ \widetilde{h}(\partial D_0) \cap \widetilde{V}_{\widetilde{\alpha}}$ that does not meet the edge $\widetilde{\alpha}$, $\widetilde{f}_3(\widetilde{C}) \cap \widetilde{\alpha} = \varnothing$.

(3) Let $\widetilde{\alpha}$ be an edge in $\widetilde{A}$ and $\widetilde{C}$ be a connected component of $\widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}(\partial D_0) \cap \widetilde{V_{\widetilde{\alpha}}}$. Suppose that the connected component $\widetilde{C}$ contains the image under the homeomorphism $\widetilde{f_2} \circ \widetilde{f_1} \circ \widetilde{h}$ of a vertex of the polygon ∂D_0 and is contained in the union of the faces in $\mathcal{F}_h$. Notice that all the ends of such a connected component belong to the same connected component of $\widetilde{V_{\widetilde{\alpha}}} \cap \Pi^{-1}(U_0)$. Then $f_3(\Pi(\widetilde{C})) \subset \mathring{U}_0$.

The definition of the homeomorphism f_4 and the rest of the proof are the same as in the case $\mathcal{E}_h = \varnothing$. □

7.4 End of the proof of Proposition 7.1

Proof Let $\mu = \mathrm{el}_{D_0}(\widetilde{f}(D_0))$. By Lemmas 7.22 and 7.26, after possibly composing the homeomorphism f with $8g+1$ homeomorphisms that are each supported in the interior of one of the discs of $\mathcal{U}$, we can suppose that the homeomorphism f satisfies the following properties:

(1) $f(p) \notin \Pi(\partial D_0)$.

(2) The set $\widetilde{f}(D_0)$ does not meet faces of type (i,μ), for any index $i \in [0, 2g-2]$.

(3) For any fundamental domain D in $\mathcal{F}_f$ at distance at most $2g-3$ from an exceptional maximal face, the set $\widetilde{f}(\partial D_0) \cap D$ is contained in the interior of $\widetilde{U}_{0,D}$, where $\widetilde{U}_{0,D}$ is the lift of U_0 with the following properties: it meets D, it meets an exceptional maximal face, it does not meet any face of type (j,μ) for any $0 \le j \le \mu$ and it meets only fundamental domains in $\mathcal{D}$ at distance less than or equal to μ from D_0.

(4) For any fundamental domain D in $\mathcal{F}_f$ at distance $2g-2$ from its exceptional maximal face, any connected component of $\widetilde{f}(\partial D_0) \cap D$ is contained either in $\Pi^{-1}(\mathring{U}_0)$ or in $D - \Pi^{-1}(U_0)$.

Definition 7.29 Two distinct connected components ξ_1 and ξ_2 of $U_0 - \Pi(\partial D_0)$ are said to be *adjacent* if $\overline{\xi}_1 \cap \overline{\xi}_2$ is homeomorphic to the interval $[0,1]$. Two connected components ξ_1 and ξ_2 of $U_0 - \Pi(\partial D_0)$ are said to be *almost adjacent* if there exists a connected component ξ of $U_0 - \Pi(\partial D_0)$ different from ξ_1 and from ξ_2 that is adjacent to ξ_1 and to ξ_2. Then such a connected component ξ is unique: we call it the *adjacency face* of ξ_1 and ξ_2.

We denote by $\mathcal{C}'$ the set of connected components of $f(\Pi(\partial D_0)) \cap \mathring{U}_0$ whose ends belong:

(1) either to the same connected component of $U_0 - \Pi(\partial D_0)$,

(2) or to the interior (in the sense interior of a manifold with boundary) of an arc of the form

$$\partial U_0 \cap \overline{\xi_1 \cup \xi_2},$$

where ξ_1 and ξ_2 are adjacent connected components of $U_0 - \Pi(\partial D_0)$,

(3) or to the interior of an arc of the form

$$\partial U_0 \cap \overline{\xi_1 \cup \xi \cup \xi_2},$$

where ξ_1 and ξ_2 are connected components of $U_0 - \Pi(\partial D_0)$ that are almost adjacent and whose adjacency face is ξ.

Suppose that the image under f of the vertex p of $\Pi(\partial D_0)$ is contained in $\mathring{U}_0$. We now look at the connected components of $\Pi^{-1}(U_0) \cap \tilde{f}(\partial D_0)$ that contain the image under the homeomorphism $\tilde{f}$ of a vertex of the polygon ∂D_0. Denote by $\tilde{p}$ the vertex of the polygon ∂D_0 whose image under $\tilde{f}$ belongs to a face D_1 in $\mathcal{D}$ such that $l(f) = d_{\mathcal{D}}(D_0, D_1)$. Such a point is unique by Lemma 7.21. Let us denote by $\tilde{U}^1_{0,0}$ the connected component of $\Pi^{-1}(U_0)$ that contains the point $\tilde{f}(\tilde{p})$. Let $D_{\max}$ be the face in $\mathcal{D}$ that realizes the maximum of $d_{\mathcal{D}}(D, D_0)$, where D varies over the faces in $\mathcal{D}$ that meet the disc $\tilde{U}^1_{0,0}$. The two faces that are adjacent to $D_{\max}$ and that meet the disc $\tilde{U}^1_{0,0}$ are closer to D_0 than $D_{\max}$. Hence the face $D_{\max}$ is exceptional. As in the proof of Lemma 7.28, one can prove that there exist words $\lambda_1 \cdots \lambda_{4g}$ and $\lambda'_1 \cdots \lambda'_{4g}$ in Λ and a word γ in the elements of $\mathcal{G}$ such that:

(1) $D_{\max} = \lambda_1 \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0)$.

(2) The word $\lambda_1 \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}$ is geodesic.

(3) The set of vertices of the polygon ∂D_0 is

$$\{\lambda_{i'}^{-1} \cdots \lambda_1^{-1}(\tilde{p}) \mid 0 \leq i' \leq 2g-1\} \cup \{\lambda_{4g-i'} \cdots \lambda_{4g}(\tilde{p}) \mid 0 \leq i' \leq 2g-1\}.$$

For any $0 \leq i' \leq 2g-1$, we denote by $\tilde{C}^1_{i'}$ (respectively $\tilde{C}^2_{i'}$) the connected component of the set $\tilde{f}(\partial D_0) \cap \Pi^{-1}(U_0)$ that contains the point

$$\tilde{p}^1_{i'} = \lambda_{i'}^{-1} \cdots \lambda_1^{-1} \tilde{f}(\tilde{p}) = \tilde{f}(\lambda_{i'}^{-1} \cdots \lambda_1^{-1}(\tilde{p}))$$

(respectively the point $\tilde{p}^2_{i'} = \tilde{f}(\lambda_{4g-i'} \cdots \lambda_{4g}(\tilde{p}))$). For any $l \in \{1, 2\}$, we denote by $\tilde{U}^l_{0,i'}$ the connected component of $\Pi^{-1}(U_0)$ that contains the connected component $\tilde{C}^l_{i'}$. Notice that $\tilde{U}^1_{0,i'} = \lambda_{i'}^{-1} \cdots \lambda_1^{-1}(\tilde{U}^1_{0,0})$ and that $\tilde{U}^2_{0,i'} = \lambda_{4g-i'} \cdots \lambda_{4g}(\tilde{U}^1_{0,0})$.

Fix now $0 \leq i' \leq 2g-1$. The faces in $\mathcal{D}$ that meet the disc $\widetilde{U}^1_{0,i'}$ are the faces of the form

$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_i(D_0) \quad \text{or} \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma (\lambda'_{4g})^{-1} \cdots (\lambda'_{2g+i+1})^{-1}(D_0),$$

where $1 \leq i \leq 2g$. As in Lemma 7.28, notice that one of the faces

$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0) \quad \text{or} \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g}(D_0)$$

is exceptional.

First case The two faces

$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0) \quad \text{and} \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g}(D_0)$$

are at distance less than μ from the face D_0. Then all the faces in $\mathcal{D}$ that meet the disc $\widetilde{U}^1_{0,i}$ are at distance less than μ from the face D_0.

Second case One of the faces

$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g-1}(D_0) \quad \text{and} \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1 \cdots \lambda'_{2g}(D_0)$$

is at distance greater than μ from the face D_0. Consider the set of faces in $\mathcal{D}$ that meet $\widetilde{U}^1_{0,i'}$ and are not faces of type (j,μ), for any $0 \leq j \leq 2g-2$. By Lemma 7.18, this set is contained in

$$\{D^1_{i',1}, D^1_{i',2}, D^1_{i',3}\},$$

where this last set consists either of the faces

$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma(D_0), \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma (\lambda'_{4g})^{-1}(D_0),$$
$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma (\lambda'_{4g})^{-1} (\lambda'_{4g-1})^{-1}(D_0)$$

or of the faces

$$\lambda_{i'+1} \cdots \lambda_{2g} \gamma(D_0), \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma \lambda'_1(D_0), \quad \lambda_{i'+1} \cdots \lambda_{2g} \gamma (\lambda'_{4g})^{-1}(D_0).$$

The connected component $\widetilde{C}^1_{i'}$ is contained in the interior of $D^1_{i',1} \cup D^1_{i',2} \cup D^1_{i',3}$. Suppose that one of the faces $D^1_{i',k}$ is at distance greater than or equal to μ from D_0. Then we claim that $i'=0$ and that two of the faces $D^1_{0,k}$ are faces of type $(0,\mu)$, the third one being at distance $\mu-1$ from D_0. Notice that this third face is necessarily one of the faces

$$\lambda_1 \cdots \lambda_{2g} \gamma(D_0) \quad \text{and} \quad \lambda_1 \cdots \lambda_{2g} \gamma (\lambda'_{4g})^{-1}(D_0)$$

and that it contains the set $\widetilde{C}^1_0$ in its interior.

Let us prove the claim now. Suppose that one of the faces $D^1_{i',k}$, with $1 \leq k \leq 3$, is exceptional and maximal for f. Then all the faces $D^1_{i',k}$, with $1 \leq k \leq 3$, either belong to $\mathcal{F}_h$ or are faces of type $(0,\mu)$ or are at distance greater than μ from the face D_0. Then the set $\widetilde{f}(\partial D_0)$ would be contained in the disc $\widetilde{U}^1_{0,i'}$, a contradiction. Moreover, it is not possible that all the $D^1_{i',k}$ are at distance greater than or equal to μ as the set $\widetilde{f}(\partial D_0)$ would not meet any of these faces. It remains only one possibility: two of the faces $D^1_{i',k}$ are faces of type $(0,\mu)$ and the third one is at distance $\mu-1$ from D_0. If $i'>0$, then the faces $D^1_{i'-1,k}$, with $1 \leq k \leq 3$, would be all at distance greater than or equal to μ from the face D_0, a contradiction.

Third case One of the faces

$$\lambda_{i'+1}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g-1}(D_0) \quad \text{and} \quad \lambda_{i'+1}\cdots\lambda_{2g}\gamma\lambda'_1\cdots\lambda'_{2g}(D_0)$$

is exceptional and maximal for the homeomorphism f. Consider the set of faces in $\mathcal{D}$ that meet $\widetilde{U}^1_{0,i'}$ and are not contained in a face in $\mathcal{F}_f$. This set is contained in

$$\{D^1_{i',1}, D^1_{i',2}, D^1_{i',3}\},$$

where this last set consists either of the faces

$$\lambda_{i'+1}\cdots\lambda_{2g}\gamma(D_0), \quad \lambda_{i'+1}\cdots\lambda_{2g}\gamma(\lambda'_{4g})^{-1}(D_0),$$
$$\lambda_{i'+1}\cdots\lambda_{2g}\gamma(\lambda'_{4g})^{-1}(\lambda'_{4g-1})^{-1}(D_0).$$

or of the faces

$$\lambda_{i'+1}\cdots\lambda_{2g}\gamma(D_0), \quad \lambda_{i'+1}\cdots\lambda_{2g}\gamma\lambda'_1(D_0), \quad \lambda_{i'+1}\cdots\lambda_{2g}\gamma(\lambda'_{4g})^{-1}(D_0).$$

In this case, the ends of the connected component $\widetilde{C}^1_{i'}$ is contained in the interior of $D^1_{i',1} \cup D^1_{i',2} \cup D^1_{i',3}$. Moreover, for any $1 \leq k \leq 3$, the face $D^1_{i',k}$ is at distance less than μ from D_0. Notice that

$$D^1_{i',1} = \lambda_1^{-1}\cdots\lambda_{i'}^{-1}(D^1_{0,1})$$

if the faces appearing in this equality are well-defined.

One can also define faces $D^2_{i',k}$ with similar properties. However, all the faces of the form $D^2_{i',k}$ are at distance less than μ from the face D_0.

Let us denote by L the subset of $\{1,2\} \times \{0,\ldots,2g-1\}$ consisting of pairs (l,i') such that the disc $\widetilde{U}^l_{0,i'}$ meets a face in $\mathcal{D}$ at distance greater than or equal to μ from the face D_0.

The faces $D^l_{i',k}$ can be chosen so that the following properties hold:

(1) Suppose that one of the faces $D^1_{0,k}$, with $1 \leq k \leq 3$, is at distance μ from the face D_0. Then $D^1_{0,1}$ is the face among the $D^1_{0,k}$ that is at distance $\mu - 1$ from the face D_0.

(2) For any elements (l_1, i'_1) and (l_2, i'_2) of L, there exists a deck transformation that sends the face $D^{l_1}_{i'_1,1}$ to the face $D^{l_2}_{i'_2,1}$.

In the case where the set L is nonempty, let $\xi_{\min}$ be the image under the projection Π of the interior of $\widetilde{U}^l_{0,i'} \cap D^l_{i',1}$, for some $(l, i') \in L$. This set is independent of the chosen pair $(l, i') \in L$.

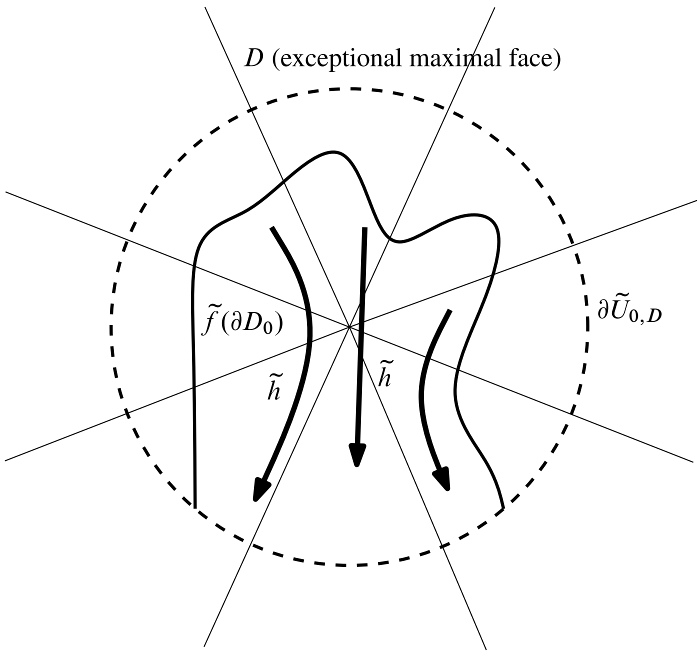

Figure 14: End of the proof of Proposition 7.1

Let h be a homeomorphism supported in $\mathring{U}_0$ with the following properties:

(1) Take any connected component C in $\mathcal{C}'$ whose ends belong to the same face or to two adjacent faces. Then $h(C)$ is contained in the interior of $\bigcup \overline{\xi}$, where the union is taken over the connected components ξ of $U_0 - \Pi(\partial D_0)$ that meet the ends of C.

(2) For any connected component C in $\mathcal{C}'$ whose ends belong to two almost adjacent connected components ξ_1 and ξ_2 of $U_0 - \Pi(\partial D_0)$ and to their adjacency face ξ, then $h(C) \subset \mathring{k}$, with $k = \overline{\xi_1} \cup \overline{\xi_2} \cup \overline{\xi}$.

(3) Suppose that the point $f(p)$ belongs to the interior of U_0, that the connected component of $f(\Pi(\partial D_0))$ that contains the image of the vertex of $\Pi(\partial D_0)$ does not belong to $\mathcal{C}'$ and that the set L is nonempty. Then the homeomorphism h satisfies the following properties:

   (a) $h(f(p)) \in \xi_{\min}$.

   (b) Take any pair (l,i') in L and any connected component $\widetilde{C}$ of $\widetilde{C}^l_{i'} - \widetilde{p}^l_{i'}$. Let $C = \Pi(\widetilde{C})$ and ξ be the connected component of $U_0 - \Pi(\partial D_0)$ that contains the end of C that meets ∂U_0. The connected component ξ is either almost adjacent or adjacent or equal to $\xi_{\min}$ by the discussion above. If the set ξ is either equal to $\xi_{\min}$ or adjacent to $\xi_{\min}$, then the set $h(C)$ is contained in the interior of the set $\overline{\xi} \cup \overline{\xi_{\min}}$. If the set ξ is almost adjacent to $\xi_{\min}$, denote by ξ' the adjacency face of ξ and $\xi_{\min}$. Then the set $h(C)$ is contained in the interior of the set $\overline{\xi} \cup \overline{\xi'} \cup \overline{\xi_{\min}}$.

By construction and by the discussion above, for any pair (l,i') in L, the set $\widetilde{h}(\widetilde{C}^l_{i'})$ is contained in the interior of $D^l_{i',1} \cup D^l_{i',2} \cup D^l_{i',3}$.

We claim that $\mathrm{el}_{D_0}(\widetilde{h} \circ \widetilde{f}(D_0)) \leq \mathrm{el}_{D_0}(\widetilde{f}(D_0)) - 1 = \mu - 1$. This completes the proof of Proposition 7.1.

First, for any point $\widetilde{y}$ in $\widetilde{f}(\partial D_0)$ that does not belong to $\Pi^{-1}(\mathring{U}_0)$, we have $\widetilde{h}(\widetilde{y}) = \widetilde{y}$ and the point $\widetilde{y}$ belongs neither to an exceptional maximal face nor to a face of type $(0,\mu)$ by the properties satisfied by f.

Let $\widetilde{C}$ be a connected component of $\widetilde{f}(\partial D_0) \cap \Pi^{-1}(U_0)$ that is contained in a lift of some connected component in $\mathcal{C}'$. Let D be an exceptional maximal face for f. Let us prove that $D \cap \widetilde{h}(\widetilde{C}) = \varnothing$. Suppose that the lift $\widetilde{U}_0$ of the disc U_0 that contains $\widetilde{C}$ does not meet D. Then, as the homeomorphism h is supported in U_0, this property holds. Suppose now that the disc $\widetilde{U}_0$ meets D. We now use notation from Lemma 7.18. The faces D^j_i, for $1 \leq i \leq 2g-2$ and $j \in \{1,2\}$, belong to $\mathcal{F}_f$. By the properties satisfied by the homeomorphism f, the connected component $\widetilde{C}$ necessarily has its ends contained in the union of the faces that meet $\widetilde{U}_0$ and do not belong to $\mathcal{F}_f$:

$$D^1_{2g-1}, \quad D^2_{2g-1} \quad \text{and} \quad D^1_{2g} = D^2_{2g}.$$

But the connected components $\Pi(\mathring{D}^1_{2g-1} \cap \widetilde{U}_0)$ and $\Pi(\mathring{D}^2_{2g-1} \cap \widetilde{U}_0)$ of $U_0 - \Pi(\partial D_0)$ are almost adjacent with adjacency face $\Pi(\mathring{D}^1_{2g} \cap \widetilde{U}_0)$. Hence the set $\widetilde{h}(\widetilde{C})$ is contained in the interior of the set $D^1_{2g-1} \cup D^2_{2g-1} \cup D^1_{2g}$. In particular, $\widetilde{h}(\widetilde{C}) \cap D = \varnothing$. Now, let D be a fundamental domain in $\mathcal{D}$ of type $(0,\mu)$. Let us prove that $\widetilde{h}(\widetilde{C}) \cap D = \varnothing$. By the properties satisfied by $\widetilde{f}$, the set $\widetilde{C}$ does not meet D. The set $\widetilde{h}(\widetilde{C})$ meets the face D only in the following case: the two ends of $\widetilde{C}$ belong to two distinct fundamental domains that are adjacent to D. However, these two fundamental domains would be at distance $\mu-1$ from D_0 (they cannot be at distance $\mu+1$ from D_0 by definition of μ), which would contradict the fact that the fundamental domain D is a face of type $(0,\mu)$.

Suppose now that the point $f(p)$ belongs to the interior of U_0 and that the connected component of $f(\Pi(\partial D_0))\cap U_0$ that contains the image of the vertex of $\Pi(\partial D_0)$ does not belong to $\mathcal{C}'$. For any pair $(l,i')\in\{1,2\}\times\{0,\dots,2g-1\}-L$, the set $\widetilde{h}(\widetilde{C}^l_{i'})$ is contained in the disc $\widetilde{U}^l_{0,i'}$ that does not meet any face at distance greater than or equal to μ from D_0. For any pair $(l,i')\in L$, the set $\widetilde{h}(\widetilde{C}^l_{i'})$ is contained in the interior of $D^l_{i',1}\cup D^l_{i',2}\cup D^l_{i',3}$. Moreover, the faces $D^l_{i',1}$, $D^l_{i',2}$ and $D^l_{i',3}$ are at distance less than μ from D_0, except possibly in the case $(l,i')=(1,0)$. In this last case, as the connected component $\widetilde{C}^1_0$ was contained in the interior of $D^1_{0,1}$, the set $\widetilde{h}(\widetilde{C}^1_0)$ is also contained in the interior of $D^1_{0,1}$ by construction of h. Now recall that the face $D^1_{0,1}$ is at distance $\mu-1$ from D_0 in this case. □

7.5 Proof of Lemma 7.3

Proof of Lemma 7.3 The proof of this lemma is analogous to the proof of Lemma 6.3. Let β and γ be simple closed curves of S that are homotopic and that are not homotopic to a point. Let us denote by α an edge in A and by α' a simple closed curve isotopic to α and disjoint from α. Let $S_{\alpha'}$ be the complement of an open tubular neighbourhood of α' and let S_α be the complement of an open tubular neighbourhood of α so that $\mathring{S}_{\alpha'}\cup\mathring{S}_\alpha=S$. Let f be a homeomorphism in $\mathrm{Homeo}_0(S)$ with $\mathrm{el}_{D_0}(\widetilde{f}(D_0))\leq 4g$. Throughout the proof, R denotes a positive constant that will be fixed later. We will use the following result, which is a consequence of Proposition 3.2 applied to neighbourhoods of S_α and of $S_{\alpha'}$: there exists $\lambda_R>0$ such that, for any homeomorphism h in $\mathrm{Homeo}_0(S_\alpha)$ or in $\mathrm{Homeo}_0(S_{\alpha'})$ with $\mathrm{el}_{D_0}(\widetilde{h}(D_0))\leq R$, we have $\mathrm{Frag}_{\mathcal{U}}(h)\leq\lambda_R$.

Let us give the idea of the proof. Let α_1 and α_2 (respectively α'_1 and α'_2) be the two connected components of the boundary of S_α (respectively of $S_{\alpha'}$). We will see that, after composing the homeomorphism f with at most $16g+1$ well-chosen homeomorphisms with fragmentation length (with respect to $\mathcal{U}$) less than or equal to λ_R, we obtain a homeomorphism f_2 that sends the curve α_1 to a curve contained in the interior of $\mathring{S}_{\alpha'}$. Then, after composing f_2 with a homeomorphism supported in $S_{\alpha'}$ that is equal to f_2^{-1} on a neighbourhood of $f_2(\alpha_1)$ and whose fragmentation length is bounded by λ_R, we obtain a homeomorphism f_3 that is equal to the identity on a neighbourhood of α_1 and is isotopic to the identity relative to α_1 (ie, the homeomorphism $\widetilde{f}_3$ is equal to the identity on a neighbourhood of $\Pi^{-1}(\alpha_1)$). By composing f_3 with at most two homeomorphisms supported in S_α or in $S_{\alpha'}$

and with fragmentation length bounded by λ_R, we obtain a homeomorphism f_5 that pointwise fixes a neighbourhood of the boundary of S_α and is isotopic to the identity relative to this boundary. Then the homeomorphism f_5 can be written as a product of a homeomorphism in $\mathrm{Homeo}_0(S_\alpha)$ and of a homeomorphism in $\mathrm{Homeo}_0(S_{\alpha'})$ with disjoint supports. The previous statement applied to these two homeomorphisms implies that the fragmentation length of f_5 is less than or equal to $2\lambda_R$. Of course, the constant R will have to be large enough so that this proof works.

Let us give now some details. For any two disjoint subsets A and B of $\widetilde{S}$, we denote by $\delta(A,B)$ the number of connected components of $\Pi^{-1}(\alpha_1 \cup \alpha_2 \cup \alpha_1' \cup \alpha_2')$ disjoint from A and from B that separate A and B. Let $\widetilde{\alpha}_1$ be a connected component of $\Pi^{-1}(\alpha_1)$ and let $\mu(f)$ be the maximum of $\delta(\widetilde{S}', \widetilde{\alpha}_1)$, where $\widetilde{S}'$ varies over all connected components of $\Pi^{-1}(S_\alpha)$ or of $\Pi^{-1}(S_{\alpha'})$ that meet $\widetilde{f}(\widetilde{\alpha}_1)$. As, by hypothesis, we have $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) \leq 4g$, then $\mu(f) \leq 16g$. Indeed, $\mu(f) \leq 4\,\mathrm{el}_{D_0}(\widetilde{f}(D_0))$ and the proof of this last fact is analogous to the proof of the claim $\mathrm{length}(\widetilde{f}(A)) \leq 2\,\mathrm{diam}_{\mathcal{D}}(\widetilde{f}(D_0))$ in Section 6. Notice that, if $\widetilde{S}'$ is a connected component of $\Pi^{-1}(S_\alpha)$ or of $\Pi^{-1}(S_{\alpha'})$ such that $\delta(\widetilde{S}', \widetilde{\alpha}_1) = \mu(f)$, then any connected component of $\widetilde{f}(\widetilde{\alpha}_1) \cap \widetilde{S}'$ has its ends in the same connected component of $\partial \widetilde{S}'$. Let $S' = \Pi(\widetilde{S}')$ and S'' be the surface S_α if $S' = S_{\alpha'}$, or the surface $S_{\alpha'}$ if $S' = S_\alpha$. Denote by h_1 a homeomorphism supported in S' with the following properties:

1. $\mathrm{el}_{D_0}(\widetilde{h}_1(D_0)) \leq 4g$.
2. For any connected component C of $f(\alpha_1) \cap S'$ whose ends are in the same connected component of $\partial S'$ and homotopic to a path on the boundary of S', $h_1(C) \subset \mathring{S}''$.

These two properties are compatible because $\mathrm{el}_{D_0}(\widetilde{f}(D_0)) \leq 4g$. Notice that we have $\mathrm{el}_{D_0}(\widetilde{h}_1 \circ \widetilde{f}(D_0)) \leq 8g$ and $\mathrm{Frag}_{\mathcal{U}}(h_1) \leq \lambda_R$ if $R \geq 4g$. Moreover, for any connected component $\widetilde{S}'$ of $\Pi^{-1}(S')$ with $d(\widetilde{\alpha}, \widetilde{S}') = \mu(f)$ and for any connected component $\widetilde{C}$ of $\widetilde{f}(\widetilde{\alpha}) \cap \widetilde{S}'$, $\widetilde{h}_1(\widetilde{C}) \subset \Pi^{-1}(\mathring{S}'')$. Now, let h_2 be a homeomorphism supported in S'' with the following properties:

1. $\mathrm{el}_{D_0}(\widetilde{h}_2(D_0)) \leq 8g$.
2. For any connected component C of $h_1 \circ f(\alpha_1) \cap S''$ whose ends are in the same connected component of $\partial S''$ and homotopic to a path on the boundary of S'', $h_2(C) \subset \mathring{S}'$.

These two properties are compatible because $\mathrm{el}_{D_0}(\widetilde{h}_1 \circ \widetilde{f}(\partial D_0)) \leq 8g$. Notice that we have

$$\mathrm{el}_{D_0}(\widetilde{h}_2 \circ \widetilde{h}_1 \circ \widetilde{f}(\partial D_0)) \leq 16g$$

and $\mathrm{Frag}_{\mathcal{U}}(h_2) \leq \lambda_R$ if $R \geq 16g$. Moreover, we have $\mu(h_2 \circ h_1 \circ f) \leq \mu(f) - 2$. We repeat this process at most $8g$ times so that, after composing the homeomorphism f with at most $16g$ homeomorphisms with fragmentation length less than or equal to λ_R (by taking $R \geq 2^{8g} 4g$), we obtain a homeomorphism f_1 with $\mu(f_1) = 0$ and that satisfies the following inequality:

$$\mathrm{el}_{D_0}(\widetilde{f_1}(D_0)) \leq 2^{8g+1} 4g$$

After composing if necessary the homeomorphism f_1 with a homeomorphism η_1 supported in S_α that pushes the curve $f_1(\alpha_1)$ into the interior of $S_{\alpha'}$ and such that

$$\mathrm{el}_{D_0}(\widetilde{\eta}_1(D_0)) \leq 2^{8g+1} 4g,$$

we obtain a homeomorphism f_2 with the following properties:

(1) The homeomorphism f_2 sends the curve α_1 to a curve contained in the interior of $S_{\alpha'}$.

(2) $\mathrm{el}_{D_0}(\widetilde{f_2}(D_0)) \leq 2^{8g+2} 4g$.

We then compose the homeomorphism f_2 with a homeomorphism η_2 supported in $S_{\alpha'}$ with the following properties:

(1) The homeomorphism $\widetilde{\eta}_2$ is equal to the homeomorphism $\widetilde{f}_2^{-1}$ on a neighbourhood of $\Pi^{-1}(\widetilde{f_2}(\widetilde{\alpha}_1))$.

(2) $\mathrm{el}_{D_0}(\widetilde{\eta}_2(D_0)) \leq 2^{8g+2} 4g$.

We obtain a homeomorphism f_3 that is equal to the identity on a neighbourhood of the curve α_1 and isotopic to the identity relative to this curve. Moreover

$$\mathrm{el}_{D_0}(\widetilde{f_3}(D_0)) \leq 2^{8g+3} 4g.$$

We compose this homeomorphism f_3 with a homeomorphism η_3 that pushes the curve $f_3(\alpha_2)$ into the interior of $S_{\alpha'}$ and that fixes the curve α_1 to obtain a homeomorphism f_4. As usual, we require that

$$\mathrm{el}_{D_0}(\widetilde{\eta}_3(D_0)) \leq 2^{8g+3} 4g.$$

Finally, compose the homeomorphism f_4 with a homeomorphism η_4 supported in $S_{\alpha'}$ to obtain a homeomorphism f_5 that pointwise fixes a neighbourhood of ∂S_α and that is isotopic to the identity relative to this neighbourhood. Of course, we also require that $\mathrm{el}_{D_0}(\widetilde{\eta}_4(D_0)) \leq 2^{8g+4} 4g$. Hence $\mathrm{el}_{D_0}(\widetilde{f_5}(D_0)) \leq 2^{8g+5} 4g$. The homeomorphism f_5 is the product of two homeomorphisms with disjoint supports and that are supported respectively in S_α and $S_{\alpha'}$. It suffices to take $R \geq 2^{8g+5} 4g$ to complete the proof of Lemma 7.3. □

8 Distortion elements with a fast orbit growth

In this section, we prove Theorem 2.12.

First notice that it suffices to prove Theorem 2.12 for sequences $(v_n)_{n\geq 1}$ with the following additional properties:

(1) The sequence $(v_n)_{n\geq 1}$ is strictly increasing.

(2) The sequence $(v_{n+1}-v_n)_{n\geq 1}$ is decreasing.

Let us prove this. Suppose we have proved Theorem 2.12 for strictly increasing sequences. If $(v_n)_{n\geq 1}$ is any sequence, it suffices to apply the theorem to the sequence $(\sup_{k\leq n} v_k + 1 - \frac{1}{2^n})_{n\geq 1}$ to deduce the general theorem. Suppose now that the theorem is proved only for sequences that satisfy the two properties above. Let us prove that it is true for any strictly increasing sequence. Let $(v_n)_{n\geq 1}$ be a strictly increasing sequence such that the sequence $(v_n/n)_n$ converges to 0. Let A be the convex hull in $\mathbb{R}^2$ of the set $\{(n,t) \mid n\geq 1 \text{ and } t\leq v_n\}$ and let $w_n = \sup\{t\in\mathbb{R} \mid (n,t)\in A\}$. The sequence $(w_n)_{n\geq 1}$ satisfies the two properties above and $\lim_{n\to+\infty} w_n/n = 0$. Then it suffices to apply the theorem to this sequence to prove it for the sequence $(v_n)_{n\geq 1}$.

In what follows, we suppose that $(v_n)_{n\geq 1}$ is a sequence that satisfies the hypothesis of Theorem 2.12 as well as the two above properties.

Let $\mathbb{A} = \mathbb{R}/\mathbb{Z}\times[-1,1]$ and let α be the curve $\{0\}\times[-1,1]\subset\mathbb{A}$. The homeomorphism f in $\mathrm{Homeo}_0(\mathbb{A},\partial\mathbb{A})$ that we are going to build will satisfy the following property:

$$\text{There exists } x\in\mathring{\mathbb{A}} \text{ such that } v_n + \frac{1}{2^n} \geq p_2(\tilde{f}^n(x)) - p_2(x) \geq v_n,$$

where $p_2\colon \mathbb{R}\times(-1,1)\to\mathbb{R}$ denotes the projection. As f is compactly supported, this guarantees that the property

$$\text{for all } n\geq 0, \quad \delta(\tilde{f}^n([0,1]\times[0,1]))\geq v_n,$$

holds. Now, let us consider the following embedding of $\mathbb{R}$ in $\mathring{\mathbb{A}}$:

$$L\colon \mathbb{R}\to\mathring{\mathbb{A}} = \mathbb{R}/\mathbb{Z}\times(-1,1),$$
$$x\mapsto (x \bmod 1, g(x)),$$

where g is a continuous strictly increasing function whose limit is $\frac{1}{2}$ as x tends to $+\infty$ and whose limit is $-\frac{1}{2}$ as x tends to $-\infty$. We identify a tubular neighbourhood T of $L(\mathbb{R})$ with the band $\mathbb{R}\times[-1,1]$, where the real line $\mathbb{R}$ is identified with the curve $L(\mathbb{R})$ via the map L so that, for any integer j, the path $\{j\}\times[-1,1]$ is contained in α. Let h be a homeomorphism of the line L, identified with $\mathbb{R}$, with the following properties:

(1) The map $x \mapsto h(x) - x$ is decreasing on the interval $[0, +\infty)$ and
$$\lim_{x \to +\infty} h(x) - x = 0.$$
(2) The homeomorphism h is equal to the identity on $(-\infty, -1]$.

(3) For any nonnegative integers i and n, $h^n(i) \notin \mathbb{Z}$.

(4) For any nonnegative integer n, $h^n(0) = v_n + (\epsilon_n/2^n)$, where ϵ_n is equal to 1 if v_n is an integer and vanishes otherwise.

The "ϵ_n" in the fourth property makes this property compatible with the third one. Let f be the homeomorphism defined on T by
$$\begin{aligned} f \colon \mathbb{R} \times [-1, 1] &\to \mathbb{R} \times [-1, 1], \\ (x, t) &\mapsto ((1 - |t|)h(x) + |t|x, t). \end{aligned}$$
This extends continuously to a homeomorphism in $\mathrm{Homeo}_0(\mathbb{A}, \partial\mathbb{A})$ that we denote by f by abuse. This extension is possible thanks to the first property satisfied by h that makes sure that the homeomorphism f is close to the identity when we are close to the circle $\mathbb{R}/\mathbb{Z} \times \{\frac{1}{2}\}$. The third property satisfied by h implies that, for any nonnegative integers i, j and n, the curve $f^n(\{i\} \times (-1, 1))$ is transverse to the curve $\{j\} \times (-1, 1)$. For any curve β in the annulus $\mathbb{A}$, let $l(\beta, \alpha)$ be the number of connected components of $\Pi^{-1}(\alpha)$ met by a lift of β. In order to prove that the homeomorphism f is a distortion element, the crucial proposition is the following:

Proposition 8.1 *Let l be a positive integer and let $\lambda_l = l(f^l(\alpha), \alpha)$. There exist two homeomorphisms g_1 and g_2 in* $\mathrm{Homeo}_0(\mathbb{A}, \partial\mathbb{A})$ *supported respectively in the complement of α and in a tubular neighbourhood of α such that*
$$l((g_2 \circ g_1)^{\lambda_l - 1}(f^l(\alpha)), \alpha) = 1.$$

First, let us see why this property implies Theorem 2.12.

Proof of Theorem 2.12 Let $\mathcal{U}$ be the open cover of $\mathbb{A}$ built at the beginning of Section 5. By Lemma 5.2, $\mathrm{Frag}_{\mathcal{U}}(g_1) \leq 6$ and $\mathrm{Frag}_{\mathcal{U}}(g_2) \leq 6$.

Remark 8.2 Looking closely at the proof of Lemma 5.2, we can see that the upper bound can be replaced with 3.

By Lemma 5.2, $\mathrm{Frag}_{\mathcal{U}}((g_2 \circ g_1)^{\lambda_l - 1} \circ f^l) \leq 6$. Recall that $a_l = a_{\mathcal{U}}(f^l)$ is the minimum of the $m.\log(k)$ where there exists a family $(h_i)_{1 \leq i \leq m}$ of homeomorphisms that are each supported in one of the open sets of $\mathcal{U}$ such that $f^l = h_1 \circ h_2 \circ \cdots \circ h_m$

and the cardinality of the set $\{h_p \mid 1 \leq p \leq m\}$ is k. So, for any positive integer l, $a_l \leq (12\lambda_l - 6)\log(18)$. But

$$\frac{\lambda_l}{l} = \frac{l(f^l(\alpha), \alpha)}{l} \leq \frac{v_l + \frac{1}{2^l}}{l},$$

where the left-hand side of the inequality converges to 0. Therefore, the sequence $(a_l/l)_{l>0}$ converges to 0. By Proposition 4.1, the homeomorphism f is a distortion element in $\mathrm{Homeo}_0(\mathbb{A}, \partial\mathbb{A})$. Notice that, here, the use of Proposition 4.1 is crucial as the hypothesis

$$\lim_{n\to+\infty} \frac{\mathrm{Frag}_{\mathcal{U}}(f^n)\log(\mathrm{Frag}_{\mathcal{U}}(f^n))}{n} = \lim_{n\to+\infty} \frac{\lambda_n \log(\lambda_n)}{n} = 0$$

of Theorem 2.11 does not necessarily hold. □

Proof of Proposition 8.1 Let $\lambda = \lambda_l = l(f^l(\alpha), \alpha)$. In what follows, everything will take place in the tubular neighbourhood T of the line L that is identified to $\mathbb{R}\times[-1,1]$. Therefore, we can "forget" the annulus $\mathbb{A}$. Let us give briefly the idea of the proof. As the curve $g(\{0\}\times(-1,1))$ has length λ with respect to α, we have no choice: in the product $(g_2 \circ g_1)^{\lambda-1}$, each factor must push this curve to the left and it must go across a curve of the form $\{i\}\times(-1,1)$ at each step (under the action of each factor $g_2 \circ g_1$). The curves $g(\{i\}\times(-1,1))$ are less dilated and must come back to their initial places in λ steps. Then we must "make them wait" so that they do not come back too fast: if they come back before the time λ, they go too far to the left, which we want to avoid. On Figure 15, we represented the action of $g_2 \circ g_1$ on $f^l(\alpha)$ on an example.

Let N be the minimal nonnegative integer such that

$$f^l(N,0) \in [N, N+1)\times\{0\} \subset \mathbb{R}\times[-1,1] \subset \mathbb{A}.$$

In the case of Figure 15, this integer is equal to 4. Let us take a real number ϵ in $(0, \frac{1}{2})$ such that, for any integer i in $[0, N]$, any connected component of

$$f^l(\alpha) \cap ([i-\epsilon, i+\epsilon]\times[-1,1] - f^l(\{i\}\times(-1,1))$$

joins both boundary components of $[i-\epsilon, i+\epsilon]\times(-1,1)$. The transversality property satisfied by f enables us to find such a real number ϵ. Let $\eta > 0$ such that, for any integer i in $[0, N]$, any connected component of

$$f^l(\alpha) \cap [i+\tfrac{\epsilon}{4}, i+1-\tfrac{\epsilon}{4}]\times[-1,1]$$

is contained in $[i+\frac{\epsilon}{4}, i+1-\frac{\epsilon}{4}]\times(-1+\eta, 1-\eta)$. Let us start with the construction of the homeomorphism g_2. Let g_2 be a homeomorphism with the following properties:

(1) The homeomorphism g_2 is supported in $\bigcup_{0 \leq i \leq N} (i-\epsilon, i+\epsilon) \times (-1,1)$.

(2) Denote by P_i the connected component of $[i-\epsilon, i+\epsilon] \times [-1,1] - g(\{i\} \times [-1,1])$ that contains $\{i-\epsilon\} \times [-1,1]$ and denote by K_i a topological closed disc contained in P_i that contains the connected components of

$$(f^l(\alpha) \cap [i-\epsilon, i+\tfrac{\epsilon}{2}] \times (-1,1)) - f^l(\{i\} \times [-1,1]).$$

Then for all i, $g_2(K_i) \subset [i-\epsilon, i-\frac{\epsilon}{2}] \times (-1+\eta, 1-\eta)$.

(3) The homeomorphism g_2 globally preserves each connected component of $g(\alpha) \cap [i-\epsilon, i+\epsilon] \times (-1,1)$.

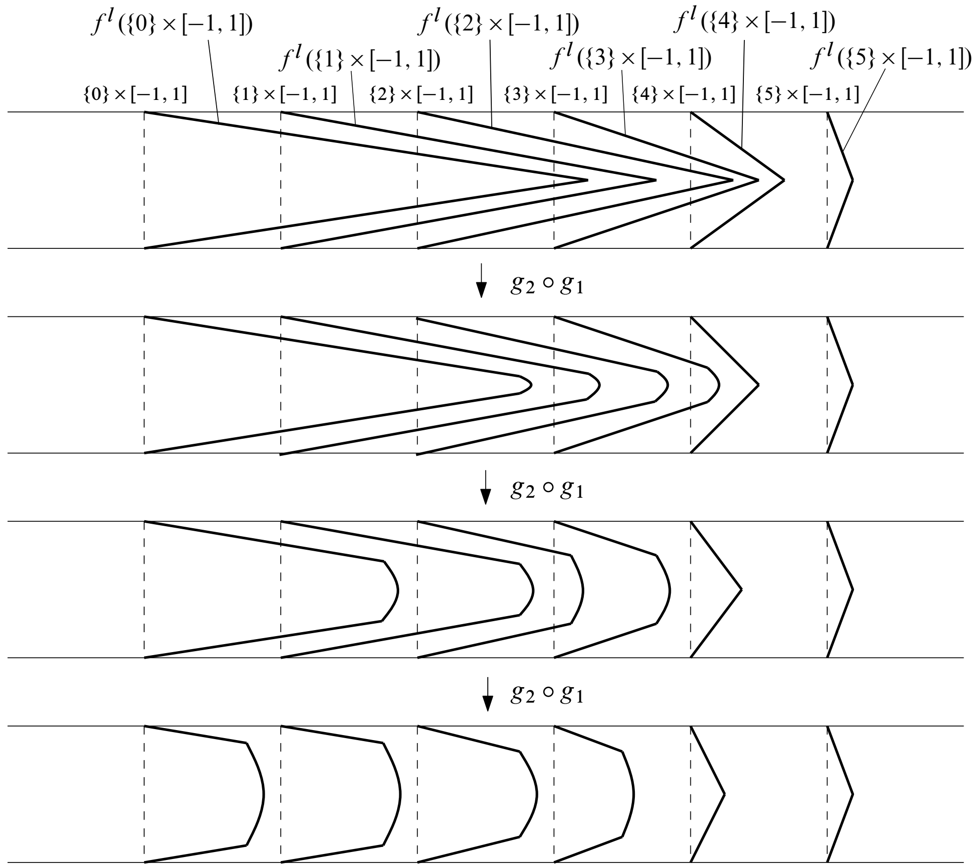

Figure 15: The action of $g_2 \circ g_1$

Before defining g_1, we first need to build a sequence of integers $(n_i)_{0 \leq i \leq N}$. For any integer i between 0 and N, let:

$$A_i = \left\{ j \in [0,N] \;\middle|\; \begin{array}{l} f^l(\{j\}\times[-1,1]) \cap \{i\}\times[-1,1] \neq \varnothing \\ f^l(\{j\}\times[-1,1]) \cap \{i+1\}\times[-1,1] = \varnothing \end{array} \right\}$$

Let

$$i_0 = \max\{i, \{i\}\times[-1,1) \cap f^l(\{0\}\times(-1,1)) \neq \varnothing\} = \lambda - 1.$$

The sets $A_0, A_1, \dots, A_{i_0-1}$ are all empty but we will see that, for any integer $N \geq m \geq i_0$, the set A_m is nonempty. In the case of Figure 15, the sets A_0, A_1 and A_2 are empty, $A_3 = \{0,1\}$ and $A_4 = \{2,3,4\}$. More generally, the family $(A_{i_0}, A_{i_0+1}, \dots, A_N)$ is a partition of $\{0,1,\dots,N\}$ such that: if $i_0 \leq m \leq m' \leq N$, then any integer in A_m is smaller than any integer in $A_{m'}$. Let us prove that if, for an integer i between 0 and $N-1$, the set A_i is nonempty, then the set A_{i+1} is nonempty. Notice that, for an integer j in the interval $[0,N]$,

$$l(f^l(\{j\}\times(-1,1)), \alpha) = \lfloor h^l(j) \rfloor - j + 1$$

by construction of f. As the map $x \mapsto h^l(x) - x$ is decreasing by construction of h, then the map

$$j \mapsto l(f^l(\{j\}\times(-1,1)), \alpha)$$

is decreasing on $[0,N]$. Let $j = \max(A_i)$. As

$$l(f^l(\{j+1\}\times(-1,1)), \alpha) \leq l(f^l(\{j\}\times(-1,1)), \alpha),$$

then the curve $f^l(\{j+1\}\times(-1,1))$ does not meet the curve $\{i+2\}\times[-1,1]$. The integer $j+1$ belongs to A_{i+1} which is nonempty. For any integer i between i_0 and N, let

$$A_i = \{j(i), j(i)+1, \dots, j(i+1)-1\}.$$

We define by induction a finite sequence of integers $(n_i)_{0 \leq i \leq N}$:

(1) If $i < i_0$, we set $n_i = 1$.

(2) Otherwise, assuming that the n_k, for $k < i$, have been defined, we set

$$n_i = \lambda - \sum_{k=j(i+1)-1}^{i-1} n_k.$$

The integer n_i will represent the number of iterations of $g_2 \circ g_1$ necessary for a compact set in a neighbourhood of $\{i+1\}\times(-1,1)$ to become disjoint from the set $(i, i+1)\times(-1,1)$. For any $0 \leq j \leq N$, let $i(j)$ be the unique integer such that $j \in A_{i(j)}$. After a number of iterations of $g_2 \circ g_1$ that is less than or equal to $n_{i(j)}$, the curve $f^l(\{j\}\times(-1,1))$ will become disjoint from $\{i(j)\}\times(-1,1)$. Then, after $n_{i(j)-1}$ iterations, it will cross the curve $\{i(j)-1\}\times(-1,1)$ and so on.... For instance, in the case of Figure 15, $n_0 = n_1 = n_2 = 1$, $n_3 = 2$ and $n_4 = 4$. Let us prove by induction that, for any integer $i \geq i_0$,

$$\sum_{k=j(i)}^{i-1} n_k < \lambda.$$

This will prove also that the integers n_i are positive. If $i = i_0 = \lambda - 1$, then, for $j < i_0$, the set A_j is empty and we have

$$\lambda - \sum_{k=0}^{i_0-1} n_k = \lambda - i_0 > 0.$$

The property holds for $i = i_0$. Suppose that the property holds for any integer k between i_0 and i given between 0 and $N-1$. Then

$$\sum_{k=j(i+1)}^{i} n_k = \lambda - \sum_{k=j(i+1)-1}^{i-1} n_k + \sum_{k=j(i+1)}^{i-1} n_k = \lambda - n_{j(i+1)-1} < \lambda$$

because $n_{j(i+1)-1} > 0$ by the induction hypothesis. The property is proved.

For any integer j between 0 and N, notice that, by construction, there is only one connected component of

$$g(\{j\} \times [-1,1]) \cap \bigcup_{0 \leq i \leq N} [i + \tfrac{\epsilon}{4}, i + 1 - \tfrac{\epsilon}{4}] \times (-1,1)$$

that does not join two distinct connected components of the boundary of

$$\bigcup_{0 \leq i \leq N} [i + \tfrac{\epsilon}{4}, i + 1 - \tfrac{\epsilon}{4}] \times (-1,1).$$

We denote this connected component by C_j. Notice that

$$C_j \subset [i(j) + \tfrac{\epsilon}{4}, i(j) + 1 - \tfrac{\epsilon}{4}] \times (-1,1).$$

Now, we can build an appropriate homeomorphism g_1. Let g_1 be a homeomorphism that is supported in

$$\bigcup_{0 \leq i \leq N} (i + \tfrac{\epsilon}{4}, i + 1 - \tfrac{\epsilon}{4}) \times [-1,1] \subset \mathbb{R} \times [-1,1] \subset \mathbb{A}$$

and that satisfies the following properties for any integer i between 0 and N:

(1) The homeomorphism g_1 globally preserves each connected component of $f^l(\alpha) \cap [i + \frac{\epsilon}{4}, i + 1 - \frac{\epsilon}{4}] \times [-1, 1]$ that joins the two boundary components of $[i + \frac{\epsilon}{4}, i + 1 - \frac{\epsilon}{4}] \times (-1, 1)$.

(2) For any integer j in A_i and any integer $r < \lambda - \sum_{k=j}^{i-1} n_k$,
$$g_1^r(C_j) \cap (i - \epsilon, i + \epsilon) \times [-1, 1] = C_j \cap (i - \epsilon, i + \epsilon) \times [-1, 1].$$

(3) For any integer j in A_i, the following inclusion holds:
$$g_1^{\lambda - \sum_{k=j}^{i-1} n_k}(C_j) \subset K_i$$
(notice that these properties are compatible as $\lambda - \sum_{k=j}^{i-1} n_k$ increases with j and, moreover,
$$\lambda - \sum_{k=j}^{i-1} n_k \leq n_i$$
by definition of n_i).

(4) The following inclusion holds:
$$g_1^{n_i}([i+\tfrac{\epsilon}{4}, i+1-\tfrac{\epsilon}{2}] \times (-1+\eta, 1-\eta)) \subset [i+\tfrac{\epsilon}{4}, i+\tfrac{\epsilon}{2}) \times (-1+\eta, 1-\eta) \cap K_i$$

(5) For any connected component C of $f^l(\alpha) \cap [i + \frac{\epsilon}{4}, i + 1 - \epsilon] \times (-1, 1)$ that joins the two boundary components of $[i + \frac{\epsilon}{4}, i + 1 - \epsilon] \times (-1, 1)$, we have
$$\text{for all } r < n_i, \quad g_1^r(C) \cap (i - \epsilon, i + \epsilon) \times [-1, 1] = C \cap (i - \epsilon, i + \epsilon) \times [-1, 1].$$

(6) For any integer $r < n_i$, the set $g_1^r([i + 1 - \epsilon, i + 1 - \frac{\epsilon}{4}] \times [-1, 1])$ does not meet the square$[i, i + \epsilon] \times [-1, 1]$.

The second and the third above properties give the speeds with which we push back the components C_j: the third property means that the piece C_j is pushed back in a K_i after time $\lambda - \sum_{k=j+1}^{i-1} n_k$ and the second condition implies that it cannot be pushed back before this time. The properties 4, 5 and 6 give the exact time necessary to pass through $[i, i + 1] \times (-1, 1)$.

Now, we prove that, for homeomorphisms g_1 and g_2 with the properties given above, we have
$$l((g_2 \circ g_1)^{\lambda - 1}(f^l(\alpha)), \alpha) = 1.$$

Let j be an integer between 0 and N and let $i = i(j)$. We denote by α_j the curve $\{j\} \times [-1, 1]$. Let us prove that, for any $j' \in [j - 1, i - 1]$ and any
$$\lambda - \sum_{k=j}^{j'} n_k > r \geq \lambda - \sum_{k=j}^{j'+1} n_k,$$

we have $l((g_2 \circ g_1)^r \circ f^l(\alpha_j), \alpha) = l(f^l(\alpha_j), \alpha) - (i - j' - 1)$. By the two first properties satisfied by g_1 and the third property satisfied by g_2, we have, for any positive integer r which is less than $\lambda - \sum_{k=j}^{i-1} n_k$,

$$(g_2 \circ g_1)^r (f^l(\alpha_j) \cap [0, i+\epsilon] \times [-1,1]) = f^l(\alpha_j) \cap [0, i+\epsilon] \times [-1,1],$$
$$(g_2 \circ g_1)^r (f^l(\alpha_j)) = g_1^r(f^l(\alpha_j)).$$

This implies the above property for $j' = i-1$. Therefore

$$g_1 \circ (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{i-1} n_k - 1} \circ f^l(\alpha_j) = g_1^{\lambda - \sum_{k=j}^{i-1} n_k}(f^l(\alpha_j)).$$

The third property satisfied by the homeomorphism g_1 implies that the intersection of the above set with $[i-\epsilon, +\infty) \times [-1,1]$ is contained in K_i. Therefore, the second property satisfied by the homeomorphism g_2 implies that

$$(g_2 \circ g_1)^{\lambda - \sum_{k=j}^{i-1} n_k} \circ f^l(\alpha_j) \subset [j, i - \tfrac{\epsilon}{2}] \times [-1+\eta, 1-\eta].$$

All of the extremal part of the curve has been put back in $[i-\epsilon, i-\frac{\epsilon}{2}] \times (-1,1)$. The remainder has not moved. Indeed

$$(g_2 \circ g_1)^{\lambda - \sum_{k=j}^{i-1} n_k}(f^l(\alpha_j) \cap [j, i-\epsilon] \times [-1,1]) = f^l(\alpha_j) \cap [j, i-\epsilon] \times [-1,1]$$
$$l((g_2 \circ g_1)^{\lambda - \sum_{k=j}^{i-1} n_k} \circ f^l(\alpha_j), \alpha) = i - j = l(f^l(\alpha_j), \alpha) - 1.$$

It suffices now to repeat this argument. Suppose that, for an integer j' between $j+1$ and $i-1$,

$$(g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k} \circ f^l(\alpha_j) \subset [j, j'+1-\tfrac{\epsilon}{2}] \times (-1+\eta, 1-\eta)$$
$$(g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k}(f^l(\alpha_j) \cap [j, j'+1-\epsilon] \times [-1,1])$$
$$= f^l(\alpha_j) \cap [j, j'+1-\epsilon] \times [-1,1]$$

We saw that this property holds for $j' = i-1$. Supposing that this property holds for an integer j', we prove now that it holds for the integer $j'-1$ and also that, under this hypothesis, for any integer r greater than $\lambda - \sum_{k=j}^{j'} n_k$ and smaller than $\lambda - \sum_{k=j}^{j'-1} n_k$,

$$l((g_2 \circ g_1)^r \circ f^l(\alpha_j), \alpha) = l(f^l(\alpha_j), \alpha) - (i - j').$$

By the fifth and the sixth properties satisfied by the homeomorphism g_1 and the third property satisfied by the homeomorphism g_2, for any integer $0 \leq r < n_{j'}$,

$$(g_2 \circ g_1)^r \circ (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k}(f^l(\alpha_j) \cap [0, j'+\epsilon] \times [-1,1]) \\ = f^l(\alpha_j) \cap [0, j'+\epsilon] \times [-1,1]$$

$$(g_2 \circ g_1)^r \circ (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k} \circ f^l(\alpha_j) = g_1^r (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k}(f^l(\alpha_j)).$$

Therefore

$$g_1 \circ (g_2 \circ g_1)^{n_{j'}-1} \circ (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k}(f^l(\alpha_j)) = g_1^{n_{j'}} (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k}(f^l(\alpha_j)),$$

so, by the fourth property satisfied by the homeomorphism g_1, the intersection of this set with the half-band $[j'+\epsilon, +\infty) \times [-1,1]$ is contained in the set $K_{j'}$. By the second property satisfied by the homeomorphism g_2:

$$(g_2 \circ g_1)^{n_{j'}} \circ (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{j'} n_k} \circ f^l(\alpha_j) \subset [j, j' - \tfrac{\epsilon}{2}] \times (-1+\eta, 1-\eta)$$

and, moreover,

$$(g_2 \circ g_1)^{n_{j'}} \circ (g_2 \circ g_1)^{\lambda - \sum_{k=j}^{i-1} n_k}(f^l(\alpha_j) \cap [j, j'-\epsilon] \times [-1,1]) \\ = f^l(\alpha_j) \cap [j, j'-\epsilon] \times [-1,1].$$

This completes the induction argument. One can prove, as before, that, for any $\lambda > r > \lambda - n_j$,

$$(g_2 \circ g_1)^r \circ f^l(\alpha_j) = g_1^{r-\lambda+n_j} (g_2 \circ g_1)^{\lambda - n_j}(f^l(\alpha_j)).$$

This implies that $l((g_2 \circ g_1)^{\lambda-1} \circ f^l(\alpha_j), \alpha) = 1$, which is what we wanted to prove. □

9 Generalization of the results

In this section, we will briefly generalize the results in two directions. First, we could look at other growth speeds of words than the linear speed. Moreover, we can also consider finite families of elements instead of looking at one element, and define a notion of distortion for this situation. The results are analogous to those we stated before. In what follows, let $(w_n)_{n \geq 0}$ be a sequence of positive real numbers that tends to $+\infty$. Let us start with a definition:

Definition 9.1 Let G be a group and g be an element of G. The element g is said to be $(w_n)_{n \geq 0}$–distorted in G if and only if there exists a finite set $\mathcal{G}$ in G such that:

(1) The element g belongs to the group generated by $\mathcal{G}$.

(2) The inferior limit of the sequence $(l_{\mathcal{G}}(g^n)/w_n)$ is 0.

This notion of distortion is interesting only if $\limsup_{n\to+\infty} w_n/n \neq +\infty$: otherwise, any element of G is $(w_n)_{n\geq 0}$–distorted. Moreover, this notion depends only on the equivalence class of $(w_n)_{n\geq 0}$ for the following equivalence relation:

$$(\omega_n) \equiv (\xi_n)$$
$$\iff \text{there exist } C>0, C'\geq 0 \text{ such that } \forall n\geq 0, \quad \frac{1}{C}\xi_n - C' \leq \omega_n \leq C\xi_n + C'.$$

Then, one can prove the following theorems:

Proposition 9.2 *Let D be a fundamental domain of $\widetilde{S}$ for the action of $\pi_1(S)$. If a homeomorphism f in* $\mathrm{Homeo}_0(S)$ *(respectively in* $\mathrm{Homeo}_0(S,\partial S)$*) is $(w_n)_{n\geq 0}$–distorted in* $\mathrm{Homeo}_0(S)$ *(respectively in* $\mathrm{Homeo}_0(S,\partial S)$*), then*

$$\liminf_{n\to+\infty} \frac{\delta(\widetilde{f}^n(D))}{w_n} = 0.$$

Theorem 9.3 *Let f be a homeomorphism in* $\mathrm{Homeo}_0(S)$ *or* $\mathrm{Homeo}_0(S,\partial S)$*. If*

$$\liminf_{n\to+\infty} \frac{\delta(\widetilde{f}^n(D))\log(\delta(\widetilde{f}^n(D)))}{w_n} = 0,$$

then f is $(w_n)_{n\geq 0}$–distorted in $\mathrm{Homeo}_0(S)$ *or* $\mathrm{Homeo}_0(S,\partial S)$*, respectively.*

Theorem 9.4 *Let $(v_n)_{n\geq 0}$ be a sequence of positive real numbers such that*

$$\liminf_{n\to+\infty} \frac{v_n}{w_n} = 0.$$

Then there exists a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}/\mathbb{Z}\times[0,1], \mathbb{R}/\mathbb{Z}\times\{0,1\})$ *such that:*

(1) *For any $n\geq 0$, $\delta(\widetilde{f}^n([0,1]\times[0,1])) \geq v_n$.*
(2) *The homeomorphism f is $(w_n)_{n\geq 0}$–distorted in*

$$\mathrm{Homeo}_0(\mathbb{R}/\mathbb{Z}\times[0,1], \mathbb{R}/\mathbb{Z}\times\{0,1\}).$$

For any positive integer k, we denote by $\mathbb{F}_k$ the free group on k generators. Let $a_1, a_2, \dots, a_k$ be the standard generators of this group and A be the set of these generators.

Definition 9.5 Let G be a group generated by a finite set $\mathcal{G}$. A k–tuple $(f_1, f_2, \dots, f_k)$ is said to be distorted if the map $\mathbb{F}_k \to G$, which sends the generator a_k to f_k, is not quasi-isometric for the distances d_A and $d_{\mathcal{G}}$. More generally, for any group G, a k–tuple $(f_1, f_2, \dots, f_k)$ is said to be distorted if there exists a subgroup of G that is finitely generated, that contains the elements f_i, and in which this k–tuple is distorted.

One can prove the following theorem for a compact surface S:

Theorem 9.6 *Let D be a fundamental domain of $\widetilde{S}$ for the action of $\pi_1(S)$. Let $(f_1, f_2, \dots, f_k)$ be a k–tuple of homeomorphisms of S. Suppose that there exists a sequence of words $(m_n)_{n\geq 0}$ on the f_i whose sequence of lengths $(l(m_n))_n$ tend to $+\infty$ such that*

$$\lim_{n\to+\infty} \frac{\delta(m_n(D))\log(\delta(m_n(D)))}{l(m_n)} = 0.$$

Then the k–tuple $(f_1, f_2, \dots, f_k)$ is distorted.

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