msp

Geometry & Topology 18 (2014) 633–668

A weakly second-order differential structure on rectifiable metric measure spaces

SHOUHEI HONDA

We give a definition of angles on the Gromov–Hausdorff limit space of a sequence of complete n–dimensional Riemannian manifolds with a lower Ricci curvature bound. We apply this to prove there is a weakly second-order differential structure on these spaces and prove that they admit a unique Levi-Civita connection, allowing us to define the Hessian of a twice differentiable function.

53C20; 53C21

1 Introduction

Let X be a metric space. We say that a map γ from $[0,l]$ to X is a *minimal geodesic* if γ is an isometric embedding. Let γ_1 and γ_2 be minimal geodesics on X beginning at a point $x \in X$. Define the angle $\angle\dot{\gamma}_1\dot{\gamma}_2 \in [0,\pi]$ between γ_1 and γ_2 at x by

$$\cos\angle\dot{\gamma}_1\dot{\gamma}_2 = \lim_{t\to 0}\frac{2t^2 - \overline{\gamma_1(t),\gamma_2(t)}^2}{2t^2}$$

if the limit exists, where $\overline{x,y}$ is the distance between x and y.

This notion of an angle is crucial in the study of metric spaces. For example, on Alexandrov spaces (or $\mathrm{CAT}(\kappa)$–spaces), the angle between every two minimal geodesics beginning at a common point always exists. The existence follows directly from a monotonicity property induced by Toponogov's comparison inequality. Roughly speaking, the monotonicity property is closely related to a lower (or upper) bound on the sectional curvature of the space. See for instance a fundamental work about Alexandrov spaces [2] by Burago, Gromov and Perelman. Note that in general, angles are *not* well defined.

Now we consider the following question:

Question Is the angle between two given minimal geodesics beginning at a common point on a metric (measure) space with a lower Ricci curvature bound well defined?

Published: 20 March 2014 DOI: 10.2140/gt.2014.18.633

Since the angle on Alexandrov space is well defined, the answer to the question is affirmative under a lower bound on the sectional curvature. There are many important works about lower Ricci curvature bounds on metric measure spaces. See for instance Ohta [26], Sturm [34; 35], Lott and Villani [25] and Villani [36]. Note that a typical example of them is the Gromov–Hausdorff limit space of a sequence of Riemannian manifolds with a lower Ricci curvature bound.

In this paper we prove that on limits of manifolds with lower Ricci curvature bounds, the answer is *almost positive*, in a way which we will soon make more precise in Theorem 1.2. First we observe that the following recent very interesting result of Colding and Naber implies that in general, the answer to the question above is *negative*:

Theorem 1.1 [10, Theorems 1.2 and 1.3] *For every $n \geq 3$, there exists a pointed proper metric space (Y, p) with the following properties:*

(i) *(Y, p) is a noncollapsing Gromov–Hausdorff limit space of a sequence of pointed n–dimensional complete Riemannian manifolds with a lower Ricci curvature bound.*

(ii) *All points of Y are regular points. Moreover, Y is a uniform Reifenberg space.*

(iii) *For every two minimal geodesics γ_1, γ_2 beginning at p and every $\theta \in [0, \pi]$, there exists a sequence $t_i \to 0$ such that*

$$\cos\theta = \lim_{i\to\infty} \frac{2t_i^2 - \overline{\gamma_1(t_i), \gamma_2(t_i)}^2}{2t_i^2}.$$

Note that (Y, p) as above has some nice properties. See [10, Theorem 1.2] by Colding and Naber for the details. This important example implies that even on a metric space with a lower Ricci curvature bound and nice properties, in general, angles are not well defined.

In order to give the first main theorem of this paper, let (M_∞, m_∞) be the Gromov–Hausdorff limit space of a sequence of pointed complete n–dimensional Riemannian manifolds $\{(M_i, m_i)\}_{i<\infty}$ with $\mathrm{Ric}_{M_i} \geq -(n-1)$. See [5; 6; 7; 8] for the wonderful structure theory of M_∞ developed by Cheeger and Colding. The following is the first main result of this paper:

Theorem 1.2 *Let $p, q \in M_\infty \setminus \{m_\infty\}$ with $m_\infty \notin C_p \cup C_q$. Then the angle $\angle p m_\infty q \in [0, \pi]$ of $p m_\infty q$ is well defined. In fact, we have*

$$\cos \angle p m_\infty q = \lim_{t\to 0} \frac{2t^2 - \overline{\gamma_p(t), \gamma_q(t)}^2}{2t^2}$$

for any minimal geodesics γ_p from m_∞ to p and γ_q from m_∞ to q, where C_p is the cut locus of p defined by $C_p = \{x \in M_\infty : \overline{p,x} + \overline{x,z} > \overline{p,z}$ for every $z \in M_\infty \setminus \{x\}\}$.

Theorem 1.2 implies that:

Corollary 1.3 *The angle between any pair of minimal geodesics, $\gamma_i \colon [0, l_i] \to M_\infty$ beginning at $x \in M_\infty$ is well defined as long as they can be extended minimally through x, namely, there exists $\epsilon > 0$ such that $\gamma_i \colon [-\epsilon, \epsilon] \to M_\infty$ is minimal.*

In [16, Theorem 3.2], the cut locus is shown to have measure zero with respect to any limit measure υ on M_∞.

Corollary 1.4 *For every $p, q \in M_\infty$, the angle $\angle pxq$ is well defined for υ–almost every $x \in M_\infty$.*

See Theorem 4.4 for the proof of Theorem 1.2. See also [11, Corollary A.4] by Colding and Naber.

We will also discuss some Hölder continuity of angles (Corollary 4.8) and show the existence of the *weakly $C^{1,\alpha}$–structure* on M_∞ in some sense (Corollary 4.9) for some $\alpha = \alpha(n) < 1$.

The second main result of this paper is the following:

Theorem 1.5 *M_∞ has a weakly second-order differential structure.*

See Definition 3.16 for the precise definition of a weakly second-order differential structure on metric (measure) spaces. Note that this second-order differential structure is *better* than the $C^{1,\alpha}$–structure above in some sense. In fact, for instance, we can give a suitable definition of twice differentiable functions on a space having a weakly second-order differential structure (Definition 3.20). We will show that all eigenfunctions with respect to the Dirichlet problem on M_∞ are weakly twice differentiable (Corollary 4.20).

On the other hand, Cheeger defined a notion of a weak Riemannian metric in [3, Section 4], which is known to be well defined on M_∞ by [8, Section 7] by Cheeger and Colding. We will review the definition of this notion in Theorems 2.2 and 2.3. We will show that the Riemannian metric of M_∞ is *Lipschitz* in some sense with respect to a weakly second-order differential structure as in Theorem 1.5. See Theorem 4.17. As corollaries, we will show that the *Levi-Civita connection on M_∞* exists and is unique (Theorem 3.26), and study the *Hessian of a twice differentiable function* (Proposition 3.27).

For example, let (Z, z) be the noncollapsing Gromov–Hausdorff limit of a sequence of pointed complete n–dimensional Einstein manifolds $\{(\widehat{M}_i, \widehat{m}_i)\}_i$ with $\mathrm{Ric}_{\widehat{M}_i} = H(n-1)$, where H is a fixed real number. Then in [6] Cheeger and Colding showed that the regular set $\mathcal{R}$ of Z is open and a smooth Riemannian manifold. We see that the Levi-Civita connection given in this paper coincides with that defined by the smooth structure of $\mathcal{R}$. See Theorems 2.3, 3.26, 4.17 and [6, Theorem 7.3] for the details.

Next we give a remark about Theorem 1.5. For that, we now recall a celebrated work on *(measurable) differentiable structures* on metric measure spaces by Cheeger. In [3], Cheeger showed that a metric measure space satisfying the Poincaré inequality and the doubling condition has a differentiable structure in some sense. For instance, we can find very interesting examples of them in [24] by Laakso and [29] by Pansu. See also [22] by Keith. Note that M_∞ with a limit measure is a typical example of these spaces. It is important that we can discuss the once differentiability for functions on such metric measure spaces. In fact, it is shown that all Lipschitz functions on such spaces are differentiable almost everywhere in some sense, as in Rademacher's Theorem [32]. On the other hand, in general, it seems that it is not easy to give a suitable definition of a second-order differential structure on metric measure spaces. However, in several situations, as of that of Alexandrov spaces, we can consider such a second-order differential structure (see for instance Burago, Gromov and Perelman [2], Otsu [27], Otsu and Shioya [28] and Perelman [30; 31]). The notion of a weakly second-order differential structure on metric measure spaces given in this paper gives a framework for such structures that includes limit spaces of Riemannian manifolds with a lower Ricci curvature bound.

Finally we introduce fundamental tools used in the proofs of Theorems 1.2 and 1.5. In the proof of Theorem 1.2, we will essentially use the proof of Cheeger and Colding's splitting theorem [5, Theorem 6.64] and several fundamental properties of the convergence of the differentials of Lipschitz functions with respect to the measured Gromov–Hausdorff topology given in [17] by the author. In the proof of Theorem 1.5, we will essentially use several fundamental properties of the convergence of spectral structures with respect to the measured Gromov–Hausdorff topology given in [23] by Kuwae and Shioya and also use several results given in [17] again.

As a continuation of this paper, in [15] we will prove a Bochner-type inequality on M_∞ that involves the Hessian defined in Section 3, discuss a weak L^2–convergence of Hessians with respect to the Gromov–Hausdorff topology and give a relationship between the Laplacian defined by using the twice differential structure in Section 3 and the Dirichlet Laplacian defined by Cheeger and Colding in [8]. In particular, we will show that in noncollapsing setting, these Laplacians coincide on a dense subspace in L^2. These are strong motivations for this work.

The organization of this paper is as follows:

In Section 2, we will introduce several fundamental notions on metric measure spaces and on limit spaces of Riemannian manifolds. In Section 3, we will give the definition of the weakly second-order differential structure on metric measure spaces and study several properties. In Section 4, we will give the proofs of Theorems 1.2 and 1.5.

Acknowledgments

The author would like to express his appreciation to Tobias Holck Colding and Aaron Naber for helpful comments. He is grateful to Kazuhiro Kuwae for giving valuable suggestions. He wishes to thank Ayato Mitsuishi, Koichi Nagano, Takashi Shioya and Takao Yamaguchi, for giving valuable comments at the informal geometry seminar at Tsukuba University. He also thanks the referee for valuable suggestion and comments. He was supported by Grant-in-Aid for Research Activity Start-up 22840027 from JSPS and Grant-in-Aid for Young Scientists (B) 24740046.

2 Preliminaries

In this section, we will introduce several fundamental notions on metric measure spaces and on limit spaces of Riemannian manifolds. Let X be a metric space. For $R > 0$, $x \in X$, we set $B_R(x) = \{y \in X : \overline{x, y} < R\}$ and $\overline{B}_R(x) = \{y \in X : \overline{x, y} \le R\}$.

2.1 Metric measure spaces

We say that X is *proper* if every bounded closed subset of X is compact. We say that X is a *geodesic space* if for every $x, y \in X$, there exists a minimal geodesic γ from x to y. Let υ be a Radon measure on X. In this paper, we say that (X, υ) is a *metric measure space* if X is a proper geodesic space and if $\upsilon(B_r(x)) > 0$ for every $x \in X$ and every $r > 0$. We now recall the notion of *rectifiability* for metric measure spaces defined by Cheeger and Colding in [8]:

Definition 2.1 [8, Definition 5.3] Let (X, υ) be a metric measure space. We say that *X is weakly υ–rectifiable (or (X, υ) is weakly rectifiable)* if there exists a positive integer m, a collection of Borel subsets $\{C_i^l\}_{1 \le l \le m, i \in \mathbb{N}}$ of X, and a collection of bi-Lipschitz embedding maps $\{\phi_i^l \colon C_i^l \to \mathbb{R}^l\}_{l,i}$ with the following properties (i) and (ii):

(i) $\upsilon\Big(X \setminus \bigcup_{l,i} C_i^l\Big) = 0$.

(ii) υ is Ahlfors l–regular at every $x \in C_i^l$, ie, there exist $C \geq 1$ and $r > 0$ such that $C^{-1} \leq \upsilon(B_t(x))/t^l \leq C$ for every $0 < t < r$.

Moreover we say that X *is υ–rectifiable (or* (X, υ) *is rectifiable)* if the following condition holds:

(iii) For every l, every $x \in \bigcup_{i \in \mathbb{N}} C_i^l$ and every $0 < \delta < 1$, there exists i such that $x \in C_i^l$ and the map ϕ_i^l is $(1 \pm \delta)$–bi-Lipschitz to its image $\phi_i^l(C_i^l)$.

Our third condition is a strong additional condition not usually required in the definition of rectifiable spaces. See [8, Definition 5.3] for the standard definition by Cheeger and Colding. This third condition is in [8, *iii*), page 60] and holds on all limit spaces of Riemannian manifolds we are studying in this paper.

In this paper, we say that a family $\{(C_i^l, \phi_i^l)\}_{l,i}$ as in Definition 2.1 is a *(weakly) rectifiable coordinate system of* (X, υ) if X is (weakly) υ–rectifiable. See also [22] by Keith. It is important that the cotangent bundle on a rectifiable metric measure space exists in some sense. We now give several fundamental properties of the cotangent bundle:

Theorem 2.2 (Cheeger [3], Cheeger and Colding [8]) *Let (X, υ) be a rectifiable metric measure space. Then, there exist a topological space T^*X and a Borel map $\pi\colon T^*X \to X$ with the following properties:*

(i) $\upsilon(X \setminus \pi(T^*X)) = 0$.

(ii) $\pi^{-1}(w)(= T^*_w X)$ *is a finite-dimensional real vector space with canonical inner product $\langle \cdot, \cdot \rangle_w$ for every $w \in \pi(T^*X)$ ($|v|(w) = \sqrt{\langle v, v \rangle_w}$).*

(iii) *For every Lipschitz function f on X, there exist a Borel subset V of X, and a Borel map df (called the differential of f) from V to T^*X such that $\upsilon(X \setminus V) = 0$ and that $\pi \circ df(w) = w$, $|df|(w) = \operatorname{Lip} f(w) = Lip f(w)$ for every $w \in V$, where*

(a) $$\operatorname{Lip} f(x) = \lim_{r \to 0}\Big(\sup_{y \in B_r(x) \setminus \{x\}} (|f(x) - f(y)|/\overline{x, y})\Big),$$

(b) $$Lip f(x) = \liminf_{r \to 0}\Big(\sup_{y \in \partial B_r(x)} (|f(x) - f(y)|/\overline{x, y})\Big).$$

We now give a short review of the construction of the cotangent bundle T^*X as in Theorem 2.2: Let $\{(C_i^l, \phi_i^l)\}_{l,i}$ be a rectifiable coordinate system of (X, υ). By the classical Rademacher's Theorem and Definition 2.1, without loss of generality, we can assume that the following properties hold:

(i) Every
$$\phi_i^l \circ (\phi_j^l)^{-1} \colon \phi_j^l(C_i^l \cap C_j^l) \to \phi_i^l(C_i^l \cap C_j^l)$$
is differentiable at every $w \in \phi_j^l(C_i^l \cap C_j^l)$ (see Section 3.1 for the notion of differentiability for a Lipschitz function defined on a Borel subset of Euclidean space).

(ii) For every i, l, $x \in C_i^l$ and every $(a_1, \dots, a_l), (b_1, \dots, b_l) \in \mathbb{R}^l$,

(a) $\mathrm{Lip}\big(\sum_j a_j \phi_{i,j}^l\big)(x) = \mathit{Lip}\big(\sum_j a_j \phi_{i,j}^l\big)(x)$,

(b) $\mathrm{Lip}\big(\sum_j a_j \phi_{i,j}^l\big)(x) = 0$ holds if and only if $(a_1, \dots, a_l) = 0$ holds.

(c) $\mathrm{Lip}\big(\sum_j (a_j + b_j)\phi_{i,j}^l\big)(x)^2 + \mathrm{Lip}\big(\sum_j (a_j - b_j)\phi_{i,j}^l\big)(x)^2$
$$= 2\,\mathrm{Lip}\big(\textstyle\sum_j a_j \phi_{i,j}^l\big)(x)^2 + 2\,\mathrm{Lip}\big(\textstyle\sum_j b_j \phi_{i,j}^l\big)(x)^2.$$

(iii) For every Lipschitz function f on X, we have $\mathrm{Lip}\, f(x) = \mathit{Lip}\, f(x)$ for almost every $x \in X$.

For points $(x, u), (y, v) \in \bigsqcup_{i,l} (\phi_i^l(C_i^l) \times \mathbb{R}^l)$, we define
$$(x, u) \sim (y, v) \quad \text{if} \quad x = \phi_i^l \circ (\phi_j^l)^{-1}(y) \text{ and } u = J(\phi_i^l \circ (\phi_j^l)^{-1})(y)^t v$$
for some i, j, l, where $J(f)$ is the Jacobi matrix of a function f. We set
$$T^*X = \Big(\bigsqcup_{i,l} (\phi_i^l(C_i^l) \times \mathbb{R}^l)\Big)/\sim$$
and define a map π by $\pi(x, u) = (\phi_i^l)^{-1}(x)$ if $x \in \phi_i^l(C_i^l)$. By the condition (b) above, for every $x \in \pi(T^*X)$ with $x \in C_i^l$, $|a|_x = \mathrm{Lip}\big(\sum_j a_j \phi_{i,j}^l\big)(x)$ is a norm on $\mathbb{R}^l$. By the condition (c) above, which follows from the υ–rectifiable condition (iii), we see that the norm comes from an inner product $\langle \cdot, \cdot \rangle_x$ on $\mathbb{R}^l$. Then it is easy to check that $(T^*X, \pi, \langle \cdot, \cdot \rangle_x)$ satisfies the conditions as in Theorem 2.2.

See [8, Section 6] by Cheeger and Colding for the details and [3, pages 458–459] by Cheeger for a more general case. We set $T^*A = \pi^{-1}(A)$ for every subset A of X.

2.2 Limit spaces of Riemannian manifolds with a lower Ricci curvature bound

We recall the definition of *Gromov–Hausdorff convergence*. Let $\{(X_i, x_i)\}_{1 \le i \le \infty}$ be a sequence of pointed proper geodesic spaces. We say that (X_i, x_i) *Gromov–Hausdorff converges to* (X_∞, x_∞) if there exist sequences of positive numbers $\epsilon_i \to 0$, $R_i \to \infty$ and of maps $\phi_i \colon B_{R_i}(x_i) \to B_{R_i}(x_\infty)$ (called *an* ϵ_i*–almost isometry*) with the following three properties:

(i) $|\overline{x,y} - \overline{\phi_i(x),\phi_i(y)}| < \epsilon_i$ for every $x, y \in B_{R_i}(x_i)$.

(ii) $B_{R_i}(x_\infty) \subset B_{\epsilon_i}(\mathrm{Image}(\phi_i))$.

(iii) $\phi_i(x_i) \to x_\infty$ (denote it by $x_i \to x_\infty$ for the sake of simplicity).

See [14] by Gromov. We denote this by $(X_i, x_i) \to (X_\infty, x_\infty)$ for brevity.

Moreover, we now give the definition of *measured Gromov–Hausdorff convergence*. For a sequence $\{\upsilon_i\}_{1 \le i \le \infty}$ of Borel measures υ_i on X_i, we say that υ_∞ *is the limit measure of* $\{\upsilon_i\}_i$ if $\upsilon_i(B_r(y_i)) \to \upsilon_\infty(B_r(y_\infty))$ for every $r > 0$ and every sequence $\{y_i\}_i$ of points $y_i \in X_i$ with $\phi_i(y_i) \to y_\infty$ (denote this by $y_i \to y_\infty$). See [13] by Fukaya for the original definition and see also [6] by Cheeger and Colding for this version. We denote this by $(X_i, x_i, \upsilon_i) \to (X_\infty, x_\infty, \upsilon_\infty)$ for brevity.

Let $n \in \mathbb{N}$, $K \in \mathbb{R}$ and let (M_∞, m_∞) be a pointed proper geodesic space. We say that (M_∞, m_∞) *is an* (n, K)*–Ricci limit space (of* $\{(M_i, m_i)\}_i$*)* if there exist sequences of real numbers $K_i \to K$ and of pointed complete n–dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \ge K_i(n-1)$ such that $(M_i, m_i) \to (M_\infty, m_\infty)$. We call an $(n, -1)$–Ricci limit space a *Ricci limit space* for brevity. Moreover we say that a Radon measure υ on M_∞ is the *limit measure of* $\{(M_i, m_i)\}_i$ if υ is the limit measure of $\{\mathrm{vol} / \mathrm{vol}\, B_1(m_i)\}_i$. Then we say that $(M_\infty, m_\infty, \upsilon)$ is the Ricci limit space of $\{(M_i, m_i, \mathrm{vol} / \mathrm{vol}\, B_1(m_i))\}_i$.

Assume that $(M_\infty, m_\infty, \upsilon)$ is the Ricci limit space of $\{(M_i, m_i, \mathrm{vol} / \mathrm{vol}\, B_1(m_i))\}_i$. Cheeger and Colding have proven that the $(1, p)$–Sobolev space $H_{1,p}(U)$ on every open subset U of M_∞ is well defined for every $1 < p < \infty$ and that for every $f \in H_{1,p}(U)$, the differential $df(x) \in T^*_x M_\infty$ is well defined for almost every $x \in U$. See [3, Theorems 4.14 and 4.47] by Cheeger for the detail.

Cheeger and Colding proved the existence of rectifiable coordinate system, defined as in Definition 2.1, constructed from harmonic functions:

Theorem 2.3 [8, Theorem 3.3, 5.5 and 5.7] *There exists a rectifiable coordinate system* $\{(C^l_i, \phi^l_i)\}_{l,i}$ *of* (M_∞, υ) *such that the following property holds: There exists a subsequence* $\{k(j)\}_j \subset \mathbb{N}$ *such that for every* l, i, *there exist* $x_\infty \in M_\infty$, $r > 0$ *with* $C^l_i \subset B_r(x_\infty)$, *a sequence* $\{x_{k(j)}\}_j$ *of* $x_{k(j)} \in M_{k(j)}$ *with* $x_{k(j)} \to x_\infty$, *a sequence* $\{f_{k(j),s}\}_{j,s}$ *of harmonic functions* $f_{k(j),s}$ *on* $B_r(x_{k(j)})$ *such that* $\sup_{j,s} \mathbf{Lip} f_{k(j),s} < \infty$, $f_{k(j),s} \to \phi^l_{i,s}$ *on* C^l_i *as* $j \to \infty$ *for every* s, *where* $\mathbf{Lip} f$ *is the Lipschitz constant of* f *and* $\phi^l_i = (\phi^l_{i,1}, \ldots, \phi^l_{i,k})$.

See Definition 2.5 for the definition of the pointwise convergence of functions $f_i \to f_\infty$ with respect to the Gromov–Hausdorff topology. More recently the author proved the existence of rectifiable coordinate system constructed from distance functions:

Theorem 2.4 [17, Theorem 3.1] *There exists a rectifiable coordinate system*

$$\{(C_i^l, \phi_i^l)\}_{1 \le l \le n, i < \infty}$$

of (M_∞, υ) *such that every* $\phi_{i,s}^l$ *is the distance function from a point in* M_∞*.*

In Section 4, roughly speaking, we will show the following:

(i) A rectifiable coordinate system as in Theorem 2.4 implies a *weakly* $C^{1,\alpha}$*–structure of* M_∞ for some $\alpha = \alpha(n) < 1$.

(ii) A rectifiable coordinate system as in Theorem 2.3 implies a *weakly second-order differential structure of* M_∞.

See Corollary 4.9 and Theorem 4.17 for the precise statements.

Definition 2.5 Given functions $f_i \colon B_R(m_i) \to \mathbb{R}$ and $x_\infty \in B_R(m_\infty)$, we say that f_i *converges to* f_∞ *at* x_∞ if for any sequence $x_i \in B_R(m_i)$ such that $x_i \to x_\infty$ we have $f_i(x_i) \to f_\infty(x_\infty)$. We denote this by $f_i \to f_\infty$ at x_∞. If this holds for all $x_\infty \in B_R(m_\infty)$ we say $f_i \to f_\infty$ on $B_R(m_\infty)$.

Finally, we introduce the definition of a convergence of the differentials of Lipschitz functions with respect to the measured Gromov–Hausdorff topology given in [17].

Definition 2.6 [17, Definitions 1.1 or 4.4] Given Lipschitz functions $f_i \colon B_R(m_i) \to \mathbb{R}$ and $x_\infty \in B_R(m_\infty)$, we say that df_i *converges to* df_∞ *at* x_∞ if

$$\sup_i \mathbf{Lip} f_i < \infty$$

for every $\epsilon > 0$, and for every $z_i \to z_\infty$, there exists $r > 0$ such that

$$\limsup_{i \to \infty} \left| \frac{1}{\operatorname{vol} B_t(x_i)} \int_{B_t(x_i)} \langle dr_{z_i}, df_i \rangle \, d\mathrm{vol} - \frac{1}{\upsilon(B_t(x_\infty))} \int_{B_t(x_\infty)} \langle dr_{z_\infty}, df_\infty \rangle \, d\upsilon \right| < \epsilon,$$

$$\limsup_{i \to \infty} \frac{1}{\operatorname{vol} B_t(x_i)} \int_{B_t(x_i)} |df_i|^2 \, d\mathrm{vol} \le \frac{1}{\upsilon(B_t(x_\infty))} \int_{B_t(x_\infty)} |df_\infty|^2 \, d\upsilon + \epsilon$$

for every $0 < t < r$ and every $x_i \to x_\infty$, where r_z is the distance function from z. We denote this by $df_i \to df_\infty$ at x_∞. If this holds for all $x_\infty \in B_R(m_\infty)$ we say $df_i \to df_\infty$ on $B_R(m_\infty)$.

We write $(f_i, df_i) \to (f_\infty, df_\infty)$ at x_∞ if $f_i \to f_\infty$ and $df_i \to df_\infty$ at x_∞.

Remark 2.7 In [15], we will see that df_i converges to df_∞ on $B_R(m_\infty)$ in the sense of Definition 2.6 if and only if df_i L^p–converges strongly to df_∞ on $B_R(m_\infty)$ for every $1 < p < \infty$.

We end this subsection by giving three fundamental properties of this convergence, which will be used essentially in Section 4:

(i) If $x_i \to x_\infty (x_i \in M_i)$, then $(r_{x_i}, dr_{x_i}) \to (r_{x_\infty}, dr_{x_\infty})$ on M_∞.

(ii) Let $\{f_i\}_{i \le \infty}$ be a sequence of Lipschitz functions f_i on $B_R(m_i)$ with

$$\sup_i \mathbf{Lip} f_i < \infty.$$

Assume that f_i is a C^2–function for every $i < \infty$, $f_i \to f_\infty$ on $B_R(m_\infty)$ and that

$$\sup_{i<\infty} \frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} (\Delta f_i)^2 \, d\mathrm{vol} < \infty.$$

Then we have $df_i \to df_\infty$ on $B_R(m_\infty)$.

(iii) Let $k \in \mathbb{N}$, $\{F_i\}_{1 \le i \le \infty} \subset C^0(\mathbb{R}^k)$ and let $\{f_i^l, g_i^l\}_{1 \le i \le \infty, 1 \le l \le k}$ be a collection of Lipschitz functions f_i^l, g_i^l on $B_R(m_i)$ with $\sup_{l,i}(\mathbf{Lip}\, f_i^l + \mathbf{Lip}\, g_i^l) < \infty$. Assume that both of the following properties hold:

(a) F_i converges to F_∞ with respect to the compact uniform topology on $\mathbb{R}^k$.

(b) $df_i^l \to df_\infty^l$ and $dg_i^l \to dg_\infty^l$ at almost every $\alpha \in B_R(m_\infty)$ for every $1 \le l \le k$.

Then we have

$$\lim_{i\to\infty} \frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} F_i(\langle df_i^1, dg_i^1\rangle, \dots, \langle df_i^k, dg_i^k\rangle) \, d\mathrm{vol}$$
$$= \frac{1}{\upsilon(B_R(m_\infty))} \int_{B_R(m_\infty)} F_\infty(\langle df_\infty^1, dg_\infty^1\rangle, \dots, \langle df_\infty^k, dg_\infty^k\rangle) \, d\upsilon.$$

See [17, Proposition 4.8, Corollaries 4.4 and 4.5] for the proofs.

3 Weak Hölder continuity and weak Lipschitz continuity

In this section, we will give several new notions for metric measure spaces and their fundamental properties. Note that the proofs of these properties are elementary; however, with the theory of convergence of Riemannian manifolds, they perform crucial roles in the analysis of Ricci limit spaces in Section 4.

We start this section by giving the following definition:

Definition 3.1 Let A be a Borel subset of a metric measure space (X, υ), Y a metric space, f a Borel map from A to Y, and $0 < \alpha \leq 1$. We say that:

(i) f is *weakly α–Hölder continuous on A* if there exists a countable family $\{A_i\}_i$ of Borel subsets A_i of A such that $\upsilon(A \setminus \bigcup_i A_i) = 0$ and that every $f|_{A_i}$ is α–Hölder continuous.

(ii) f is *weakly Lipschitz on A* if f is weakly 1–Hölder continuous on A.

Remark 3.2 Let M be an n–dimensional Riemannian manifold, f a function on M and A an open subset of M. Then it is easy to check that the following conditions are equivalent:

(i) f is differentiable at almost every $x \in A$.

(ii) f is weakly Lipschitz on A.

3.1 Weakly twice differentiable functions on a Borel subset of $\mathbb{R}^k$

Let A be a Borel subset of $\mathbb{R}^k$, f a Lipschitz function on A and $y \in \mathrm{Leb}\, A$, where $\mathrm{Leb}\, A = \{a \in A : \lim_{r\to 0} H^k(A \cap B_r(a))/H^k(B_r(a)) = 1\}$. Then we say that f *is differentiable at y* if there exists a Lipschitz function $\hat{f}$ on $\mathbb{R}^k$ such that $\hat{f}|_A = f$ and that $\hat{f}$ is differentiable at y. Note that if f is differentiable at y, then a vector $(\partial \hat{f}/\partial x_1(y), \dots, \partial \hat{f}/\partial x_n(y))$ does not depend on the choice of such $\hat{f}$. Thus we denote the vector by $J(f)(y) = (\partial f/\partial x_1(y), \dots, \partial f/\partial x_n(y))$. Let $F = (f_1, \dots, f_m)$ be a Lipschitz map from A to $\mathbb{R}^m$. We say that F *is differentiable at y* if every f_i is differentiable at y. Note that by Rademacher's Theorem [32], F is differentiable at almost every $x \in A$. Let us denote the Jacobi matrix of F at x by $J(F)(x) = (\partial f_i/\partial x_j(x))_{ij}$ if F is differentiable at x.

Definition 3.3 Let $\omega = \sum_{i_1<\cdots<i_p} f_{i_1,\dots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ be a p–form on A and $0 < \alpha \leq 1$. We say that:

(i) ω is a *Borel p–form on A* if every $f_{i_1,\dots,i_p}$ is a Borel function.

(ii) ω is *weakly α–Hölder continuous on A* if every $f_{i_1,\dots,i_p}$ is weakly α–Hölder continuous on A.

(iii) ω is *weakly Lipschitz on A* if every $f_{i_1,\dots,i_p}$ is weakly Lipschitz on A.

For two Borel p–forms $\{\omega_i\}_{i=1,2}$ on A, we say that ω_1 *is equivalent to* ω_2 if $\omega_1(x) = \omega_2(x)$ for almost every $x \in A$. Le us denote the equivalent class of ω by $[\omega]$, the set of equivalent classes by $\Gamma_{\mathrm{Bor}}(\bigwedge^p T^*A)$, and the set of equivalent classes represented by a weakly α–Hölder continuous p–form by $\Gamma_\alpha(\bigwedge^p T^*A)$. We often write $\omega = [\omega]$ for brevity.

Let ω be a weakly Lipschitz p–form on A. Define a Borel $(p+1)$–form $d\omega$ on A by

$$d\omega = \sum_{i_1<\cdots<i_p} (\partial f^j_{i_1,\ldots,i_p}/\partial x_l)\, dx_l \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $\omega = \sum_{i_1<\cdots<i_p} f^j_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Note that if ω_1 is equivalent to ω_2, then $d\omega_1$ is equivalent to $d\omega_2$. Therefore d is well defined as a linear map from $\Gamma_1(\bigwedge^p T^*A)$ to $\Gamma_{\mathrm{Bor}}(\bigwedge^{p+1} T^*A)$. Note that the following product rule holds: $d(\eta \wedge \omega) = d\eta \wedge \omega + (-1)^p \eta \wedge d\omega \in \Gamma_{\mathrm{Bor}}(\bigwedge^{p+q} T^*A)$ for every $\eta \in \Gamma_1(\bigwedge^p T^*A)$ and every $\omega \in \Gamma_1(\bigwedge^q T^*A)$.

The next lemma is standard, however, we give a proof for convenience:

Lemma 3.4 *Let F be a Lipschitz function on $\mathbb{R}^k$. Then, there exists a sequence $\{F_i\}_i \subset C^\infty(\mathbb{R}^k)$ such that $F_i \to F$ in $L^\infty(\mathbb{R}^k)$ and that $J(F_i)(x) \to J(F)(x)$ for almost every $x \in \mathbb{R}^k$.*

Proof Let ρ be a nonnegative smooth function on $\mathbb{R}^k$ with $\mathrm{supp}(\rho) \subset B_1(0_k)$ and

$$\int_{\mathbb{R}^k} \rho(x)\, dH^k = 1,$$

where H^k is the k–dimensional Hausdorff measure. For every $\epsilon > 0$, define smooth functions ρ_ϵ and F_ϵ on $\mathbb{R}^k$ by $\rho_\epsilon(x) = \epsilon^{-k}\rho(x/\epsilon)$ and

$$F_\epsilon(x) = \int_{\mathbb{R}^k} \rho_\epsilon(x-y) F(y)\, dH^k.$$

Let $L \geq 1$ with $\sup \rho + \mathbf{Lip} F \leq L$. For every $x \in \mathbb{R}^k$, we have

$$\begin{aligned} |F_\epsilon(x) - F(x)| &\leq \int_{\mathbb{R}^k} \rho_\epsilon(x-y)|F(y)-F(x)|\, dH^k \\ &= \int_{B_\epsilon(x)} \rho_\epsilon(x-y)|F(y)-F(x)|\, dH^k \\ &\leq L\epsilon \int_{B_\epsilon(x)} \rho_\epsilon(x-y)\, dH^k = L\epsilon \int_{B_\epsilon(0_k)} \rho_\epsilon(y)\, dH^k \xrightarrow{\epsilon \to 0} 0. \end{aligned}$$

Therefore we have the first assertion. For every $x \in \mathbb{R}^k$ and every $h \in \mathbb{R}$, by the dominated convergence theorem, we have:

$$\begin{aligned} \frac{F_\epsilon(x+he_i) - F_\epsilon(x)}{h} &= \int_{\mathbb{R}^k} \rho_\epsilon(y) \left(\frac{F(x+he_i-y) - F(x-y)}{h} \right) dH^k \\ &= \int_{B_\epsilon(0_k)} \rho_\epsilon(y) \left(\frac{F(x+he_i-y) - F(x-y)}{h} \right) dH^k \end{aligned}$$

$$\xrightarrow{h\to 0} \int_{B_\epsilon(0_k)} \rho_\epsilon(y)\,\frac{\partial F}{\partial x_i}(x-y)\,dH^k$$
$$=\int_{B_\epsilon(0_k)} \rho_\epsilon(x-y)\frac{\partial F}{\partial x_i}(y)\,dH^k = \int_{\mathbb{R}^k} \rho_\epsilon(x-y)\frac{\partial F}{\partial x_i}(y)\,dH^k$$

By Lusin's Theorem, for every $\delta>0$ and every $R>0$, there exists a Borel subset A_δ^R of $B_R(0_k)$ such that $H^k(B_R(0_k)\setminus A_\delta^R)<\delta$ and that $J(F)|_{A_\delta^R}$ is continuous. Thus, for every $x\in \operatorname{Leb} A_\delta^R$, we have:

$$\begin{aligned}
\left|\frac{\partial F_\epsilon}{\partial x_i}(x)-\frac{\partial F}{\partial x_i}(x)\right| &\le \int_{\mathbb{R}^k} \rho_\epsilon(x-y)\left|\frac{\partial F}{\partial x_i}(y)-\frac{\partial F}{\partial x_i}(x)\right| dH^k\\
&= \int_{B_\epsilon(x)} \rho_\epsilon(x-y)\left|\frac{\partial F}{\partial x_i}(y)-\frac{\partial F}{\partial x_i}(x)\right| dH^k\\
&= \int_{B_\epsilon(x)\cap A_\delta^R} \rho_\epsilon(x-y)\left|\frac{\partial F}{\partial x_i}(y)-\frac{\partial F}{\partial x_i}(x)\right| dH^k\\
&\qquad + \int_{B_\epsilon(x)\setminus A_\delta^R} \rho_\epsilon(x-y)\left|\frac{\partial F}{\partial x_i}(y)-\frac{\partial F}{\partial x_i}(x)\right| dH^k\\
&\le \sup_{y\in B_\epsilon(x)\cap A_\delta^R} \left|J(F)(y)-J(F)(x)\right| + 2L\epsilon^{-k} H^k(B_\epsilon(x)\setminus A_\delta^R)\\
&\xrightarrow{\epsilon\to 0} 0
\end{aligned}$$

Since δ and R are arbitrary, we have the second assertion. □

Let $G=(G_1,\dots,G_k)$ be a bi-Lipschitz embedding from A to $\mathbb{R}^k$. For every $\omega\in\Gamma_{\mathrm{Bor}}(\bigwedge^p T^*G(A))$, define

$$G^*\omega=\sum f_{i_1,\dots,i_p}\circ G\,(\partial G_{i_1}/\partial x_{j_1})\cdots(\partial G_{i_p}/\partial x_{j_p})dx_{j_1}\wedge\cdots\wedge dx_{j_p}\in\Gamma_{\mathrm{Bor}}\big(\textstyle\bigwedge^p T^*A\big),$$

where $\omega=\sum f_{i_1,\dots,i_p}dx_{i_1}\wedge\cdots\wedge dx_{i_p}$. Note that if $J(G)$ is weakly Lipschitz on A, then $G^*\omega\in\Gamma_1(\bigwedge^p T^*A)$ for every $\omega\in\Gamma_1(\bigwedge^p T^*G(A))$.

Proposition 3.5 *Let $\omega=\sum f_{i_1,\dots,i_p}dx_{i_1}\wedge\cdots\wedge dx_{i_p}\in\Gamma_1(\bigwedge^p T^*G(A))$. Assume that $J(G)$ is weakly Lipschitz on A. Then we have*

$$d(G^*\omega)=G^*(d\omega)\in\Gamma_{\mathrm{Bor}}\big(\textstyle\bigwedge^{p+1} T^*A\big).$$

Proof Without loss of generality, we can assume that G and every $f_{i_1,\dots,i_p}$ are Lipschitz on A. By Lemma 3.4, there exist sequences of smooth maps $\{G^j\}_j$ from $\mathbb{R}^k$ to $\mathbb{R}^k$, and of smooth functions $\{f^j_{i_1,\dots,i_p}\}_j$ on $\mathbb{R}^k$ such that the following properties hold:

(i) $G^j \to G$ and $f^j_{i_1,\dots,i_p} \to f_{i_1,\dots,i_p}$ in $L^\infty(A)$.

(ii) $J(G^j)(x) \to J(G)(x)$ and $J(f^j_{i_1,\dots,i_p})(x) \to J(f_{i_1,\dots,i_p})(x)$ for almost every $x \in A$.

Since

$$d\Big((G^j)^*\Big(\sum f^j_{i_1,\dots,i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}\Big)\Big)(x) = (G^j)^*\Big(d\Big(\sum f^j_{i_1,\dots,i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}\Big)\Big)(x)$$

for every $x \in \mathbb{R}^k$ and every j, by letting $j \to \infty$, this completes the proof. □

Note that in the same way as Definition 3.3, we can give definitions of a Borel vector (tensor) field on A, of its equivalence, of its weak α–Hölder continuity, and so on. Denote the set of equivalent classes of Borel vector fields by $\Gamma_{\mathrm{Bor}}(TA)$ and the set of equivalent classes represented by a weakly Lipschitz vector field by $\Gamma_1(TA)$.

For every weakly Lipschitz function f on A and every $X \in \Gamma_{\mathrm{Bor}}(TA)$, define a Borel function $X(f) = \sum X_i \partial f/\partial x_i$ on A, where $X = \sum X_i \partial/\partial x_i$. For every $X \in \Gamma_{\mathrm{Bor}}(TA)$, define $G_* X = \sum X(G_i)\partial/\partial x_i \in \Gamma_{\mathrm{Bor}}(TG(A))$. For every $X, Y \in \Gamma_1(TA)$, define $[X, Y] \in \Gamma_{\mathrm{Bor}}(TA)$ by

$$[X, Y] = \sum_{i,j} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \frac{\partial}{\partial x_j},$$

where $X = \sum X_i \partial/\partial x_i$, $Y = \sum Y_i \partial/\partial x_i$.

Proposition 3.6 *Let $X, Y \in \Gamma_1(TA)$. Assume that $J(G)$ is weakly Lipschitz on A. Then we have $[G_* X, G_* Y] = G_*[X, Y] \in \Gamma_{\mathrm{Bor}}(TG(A))$.*

Proof The proposition follows from an argument similar to that of the proof of Proposition 3.5. □

Definition 3.7 (Weakly twice differentiable function) Let f be a Borel function on A. We say that f *is weakly twice differentiable on* A if f is weakly Lipschitz function on A and $df \in \Gamma_1(T^*A)$ ($= \Gamma_1(\bigwedge^1 T^*A)$).

Note that the following is *not* trivial.

Proposition 3.8 *Let f be a weakly twice differentiable function on A. Then we have $d(df) = 0 \in \Gamma_{\mathrm{Bor}}(\bigwedge^2 T^*A)$.*

Proof Let $\widehat{A} = \{x \in \mathrm{Leb}\, A : \partial f/\partial x_1(x) = \cdots = \partial f/\partial x_k(x) = 0\}$. Note that $d(d(f|_{\widehat{A}})) = 0 \in \Gamma_{\mathrm{Bor}}(\bigwedge^2 T^*\widehat{A})$ because $d(f|_{\widehat{A}}) = 0 \in \Gamma_1(T^*\widehat{A})$. Let

$$A_i = \{x \in \mathrm{Leb}\, A \setminus \widehat{A} : \{df(x)\} \cup \{dx_j(x)\}_{j \neq i} \text{ is a base of } T_x^*\mathbb{R}^k\}.$$

By [17, Theorem 3.4], there exists a countable collection $\{A_i^m\}_{1 \le i \le k, m \in \mathbb{N}}$ of Borel subsets A_i^m of A_i such that $H^k\big((A \setminus \widehat{A}) \setminus \bigcup_{i,m} A_i^m\big) = 0$ and that every map

$$\Phi_i^m = (f, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$$

is a bi-Lipschitz embedding from A_i^m to $\mathbb{R}^k$. By the assumption, we see that every $\langle df, dx_j \rangle$ is weakly Lipschitz on A. Therefore, $J(\Phi_i^m)$, $J((\Phi_i^m)^{-1})$ are weakly Lipschitz on A_i^m, $\Phi_i^m(A_i^m)$, respectively. Since $(\Phi_i^m)^* dx_1 = df$ and $d(dx_1) = 0$, the proposition follows directly from Proposition 3.5. □

Let f be a weakly twice differentiable function on A. Put

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x).$$

Note that Proposition 3.8 implies that for every i, j we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

for almost every $x \in A$.

For $\omega = \sum f_{i_1,\dots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \Gamma_1(\bigwedge^p T^*A)$, we say that ω *is weakly twice differentiable on* A if every $f_{i_1,\dots,i_p}$ is weakly twice differentiable on A. Similarly, we can give definitions of weak twice differentiability for vector (tensor) fields on A, for maps from A to $\mathbb{R}^m$ and so on.

Corollary 3.9 *Let ω be a weakly twice differentiable p–form on A. Then we have $d(d\omega) = 0 \in \Gamma_{\mathrm{Bor}}(\bigwedge^{p+2} T^*A)$.*

Proof Since

$$d(d\omega) = \sum d(df_{i_1,\dots,i_p}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

the corollary follows directly from Proposition 3.8. □

3.2 Riemannian metric on a Borel subset of $\mathbb{R}^k$

Let A be a Borel subset of $\mathbb{R}^k$. In this subsection, we will study *Riemannian metrics on* A in the following sense:

Definition 3.10 (Riemannian metric) Let $g=\{g_a\}_{a\in A}$ be a family of inner products g_a on $T_a\mathbb{R}^k$. We say that:

(i) *g is a Borel Riemannian metric on A* if every $g_{ij}(a)=g_a(\partial/\partial x_i, \partial/\partial x_j)$ is a Borel function on A.

(ii) *g is a weakly α–Hölder continuous Riemannian metric on A* if every g_{ij} is a weakly α–Hölder continuous function on A.

(iii) *g is a weakly Lipschitz Riemannian metric on A* if every g_{ij} is a weakly Lipschitz function on A.

Note that for a Borel Riemannian metric g on A, g is weakly α–Hölder continuous if and only if $g \in \Gamma_\alpha(T^*A\otimes T^*A)$. For two Borel Riemannian metrics $g, \hat{g}$ on A, we say that *g is equivalent to $\hat{g}$* if g_{ij} is equivalent to $\hat{g}_{ij}$. Let us denote by $\mathrm{Riem}_{\mathrm{Bor}}(A)(\subset \Gamma_{\mathrm{Bor}}(T^*A\otimes T^*A))$ the set of equivalent classes of Borel Riemannian metrics and by $\mathrm{Riem}_\alpha(A)$ the set of equivalent classes represented by a weakly α–Hölder continuous Riemannian metric.

For a weakly Lipschitz function f on A, $g\in \mathrm{Riem}_{\mathrm{Bor}}(A)$, $X=\sum X_i\partial/\partial x_i \in \Gamma_{\mathrm{Bor}}(TA)$ and $\omega=\sum \omega_i dx_i\in\Gamma_{\mathrm{Bor}}(T^*A)$, define $X^*\in\Gamma_{\mathrm{Bor}}(T^*A)$, $\omega^*\in\Gamma_{\mathrm{Bor}}(TA)$ and $\nabla^g f\in\Gamma_{\mathrm{Bor}}(TA)$ by $X^*=\sum g_{ij}X_i dx_j$, $\omega^*=\sum g^{ij}\omega_i\partial/\partial x_j$ and $\nabla^g f=(df)^*$, respectively, where g^{ij} is the ij^{th} term of the inverse of the matrix defined by g_{ij}.

Proposition 3.11 (Levi-Civita connection) *Let $g\in\mathrm{Riem}_1(A)$. Then there exists a Levi-Civita connection ∇^g on A defined uniquely in the following sense:*

(i) *∇^g is a map from $\Gamma_{\mathrm{Bor}}(TA)\times\Gamma_1(TA)$ to $\Gamma_{\mathrm{Bor}}(TA)$ ($\nabla^g_X Y := \nabla^g(X,Y)$).*

(ii) *$\nabla^g_X(Y+Z)=\nabla^g_X Y+\nabla^g_X Z$ for every $X\in\Gamma_{\mathrm{Bor}}(TA)$ and every $Y,Z\in\Gamma_1(TA)$.*

(iii) *$\nabla^g_{fX+hY}Z=f\nabla^g_X Z+h\nabla^g_Y Z$ for every $X,Y\in\Gamma_{\mathrm{Bor}}(TA)$, every $Z\in\Gamma_1(TA)$ and every pair of Borel functions f,h on A.*

(iv) *$\nabla^g_X(fY)=X(f)Y+f\nabla^g_X Y$ for every $X\in\Gamma_{\mathrm{Bor}}(TA)$, every $Y\in\Gamma_1(TA)$ and every weakly Lipschitz function f on A.*

(v) *$\nabla^g_X Y-\nabla^g_Y X=[X,Y]$ for every $X,Y\in\Gamma_1(TA)$.*

(vi) *$Xg(Y,Z)=g(\nabla^g_X Y,Z)+g(Y,\nabla^g_X Z)$ for every $X\in\Gamma_{\mathrm{Bor}}(TA)$ and every $Y,Z\in\Gamma_1(TA)$.*

Proof Let

$$\Gamma^m_{i,j}=\frac{1}{2}\sum_l g^{ml}\left(\frac{\partial g_{jl}}{\partial x_i}+\frac{\partial g_{il}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_l}\right),$$

$$\nabla^g_X Y=\sum_{i,j}\left(X_i\frac{\partial Y_j}{\partial x_i}\frac{\partial}{\partial x_j}+X_iY_j\Gamma^m_{i,j}\frac{\partial}{\partial x_m}\right),$$

where $X = \sum X_i \partial/\partial x_i$ and $Y = \sum Y_i \partial/\partial x_i$. It is easy to check that the properties above hold for this ∇^g. Therefore we have the existence.

Next, we check the uniqueness. Let ∇^1 and ∇^2 be Levi-Civita connections on A. Fix $X = \sum X_i \partial/\partial x_i \in \Gamma_1(TA)$, $Y = \sum Y_i \partial/\partial x_i \in \Gamma_1(TA)$. Since

$$2g(\nabla^l_X Y, Z)$$
$$= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

for every $Z \in \Gamma_1(TA)$ and every $l = 1,2$, we see that $g(\nabla^1_X Y - \nabla^2_X Y, Z) = 0$ for every $Z \in \Gamma_1(TA)$. Put $\nabla^1_X Y - \nabla^2_X Y = \sum h_i \partial/\partial x_i$. By Lusin's Theorem, there exists a sequence of compact subsets $\{A_j\}_j$ of A such that $H^k(A \setminus A_j) \to 0$ as $j \to \infty$ and that $h_i|_{A_j}$ is continuous for every i, j. We now make the following elementary claim:

Claim 3.12 *Let K be a bounded Borel subset of $\mathbb{R}^k$, h a continuous function on K and $\epsilon > 0$. Then there exist a Borel subset K_ϵ of K and a Lipschitz function h_ϵ on $\mathbb{R}^k$ such that $|h(x) - h_\epsilon(x)| < \epsilon$ for every $x \in K_\epsilon$ and that $H^k(K \setminus K_\epsilon) < \epsilon$.*

The proof is as follows. For every $x \in \mathrm{Leb}\, K$, there exists $r_x > 0$ such that

$$H^k(B_r(x) \cap K)/H^k(B_r(x)) \geq 1 - \epsilon$$

for every $0 < r < r_x$, and that $|h(x) - h(y)| < \epsilon$ for every $y \in K \cap B_{r_x}(x)$. By the standard covering lemma (see for instance Chapter 1 in Simon [33]), there exists a countable pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_i$ such that $x_i \in \mathrm{Leb}\, K$, $r_i < r_{x_i}/5$ and that $\mathrm{Leb}\, K \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)$ for every N. Fix N such that $\sum_{j=N+1}^\infty H^k(B_{r_i}(x_i)) < \epsilon/5^k$. Then we have

$$H^k\Big(\mathrm{Leb}\, K \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i)\Big) < \epsilon.$$

Define a Lipschitz function f on $\bigcup_{i=1}^N B_{r_i}(x_i)$ by $f|_{B_{r_i}(x_i)} \equiv h(x_i)$. Let $K_\epsilon = K \cap \bigcup_{i=1}^N B_{r_i}(x_i)$ and let h_ϵ be a Lipschitz function on $\mathbb{R}^k$ with $h_\epsilon|_{K_\epsilon} = f$. Then we have $|h_\epsilon(x) - h(x)| < \epsilon$ for every $x \in K_\epsilon$. Thus we have Claim 3.12.

Therefore there exist collections of Borel subsets $\{A_{j,k}\}_k$ of A_j, and of Lipschitz functions $\{h_{i,j,k}\}_{i,j,k}$ on $\mathbb{R}^k$ such that $H^k(A_j \setminus A_{j,k}) < 2^{-k}$ and $|h_i(x) - h_{i,j,k}(x)| < 2^{-k}$ for every $x \in A_{j,k}$. Let $\widehat{A}_j = \bigcap_{m=1}^\infty \bigcup_{k=m} A_{j,k}$. Taking $Z = \sum_i h_{i,j,k} \partial/\partial x_i$, we have

$$g\Big(\sum_i h_i \partial/\partial x_i, \sum_i h_{i,j,k} \partial/\partial x_i\Big) = 0$$

on $A_{j,k}$. Letting $k \to \infty$ we have

$$g\Big(\sum_i h_i \partial/\partial x_i, \sum_i h_i \partial/\partial x_i\Big) = 0$$

and thus $h_i(x) \equiv 0$ on $\hat{A}_j$. Since $H^k(A_j \setminus \hat{A}_j) = 0$, we have $\nabla^1_X Y = \nabla^2_X Y$. Therefore for every $\hat{X} = \sum \hat{X}_i \partial/\partial x_i \in \Gamma_{\mathrm{Bor}}(TA)$, we have

$$\nabla^1_{\hat{X}} Y = \sum \hat{X}_i \nabla^1_{\partial/\partial x_i} Y = \sum \hat{X}_i \nabla^2_{\partial/\partial x_i} Y = \nabla^2_{\hat{X}} Y.$$

Thus we have the uniqueness. □

Let G be a bi-Lipschitz embedding from A to $\mathbb{R}^k$ and $g \in \mathrm{Riem}_1(A)$. Assume that G is weakly twice differentiable on A, ie, $J(G)$ is weakly Lipschitz on A. Note that $J(G^{-1})$ is weakly Lipschitz on $G(A)$ and that a Riemannian metric $G_* g$ on $G(A)$ defined by $G_* g(\partial/\partial x_i, \partial/\partial x_j) = g((G^{-1})_*(\partial/\partial x_i), (G^{-1})_*(\partial/\partial x_j))$ is weakly Lipschitz on $G(A)$.

Corollary 3.13 *With the same notation as above, we have* $G_*(\nabla^g_X Y) = \nabla^{G_* g}_{G_* X} G_* Y$ *for every* $X \in \Gamma_{\mathrm{Bor}}(TA)$ *and every* $Y \in \Gamma_1(TA)$*.*

Proof It is easy to check that if we define a map T from $\Gamma_{\mathrm{Bor}}(TG(A)) \times \Gamma_1(TG(A))$ to $\Gamma_{\mathrm{Bor}}(TG(A))$ by $T(X,Y) = G_*(\nabla^g_{(G^{-1})_* X}(G^{-1})_* Y)$, then T satisfies the properties of the Levi-Civita connection of $G_* g$ on $G(A)$. Thus the corollary follows from the uniqueness of the Levi-Civita connection. □

Definition 3.14 Let f be a weakly twice differentiable function on A, $\omega \in \Gamma_1(T^*A)$ and $X \in \Gamma_1(TA)$. Define:

(i) A Borel tensor field $\nabla^g \omega \in \Gamma_{\mathrm{Bor}}(T^*A \otimes T^*A)$ of type $(0,2)$ on A by

$$\nabla^g \omega = \sum_{i,j} g\left(\nabla^g_{\frac{\partial}{\partial x_i}} \omega^*, \frac{\partial}{\partial x_j}\right) dx_i \otimes dx_j.$$

(ii) *The Hessian* Hess^g_f *of* f by $\mathrm{Hess}^g_f = \nabla^g df$.

(iii) *The divergence* $\mathrm{div}^g\, X$ *of* X by

$$\mathrm{div}^g\, X = \text{trace of } \nabla^g X^* = \sum_i g\left(\nabla^g_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_i}\right).$$

(iv) *The Laplacian* $\Delta^g f$ *of* f by $\Delta^g f = -\mathrm{div}^g\, \nabla^g f$.

We end this subsection by giving several properties of them:

Corollary 3.15 *With the same notation as in Definition 3.14, we have the following:*

(i) $\nabla^g \omega = G^*(\nabla^{G_* g}(G^{-1})^* \omega)$

(ii) $\mathrm{Hess}^g_f = G^*(\mathrm{Hess}^{G_* g}_{f \circ G^{-1}})$

(iii) $\mathrm{div}^g\ X \circ G^{-1} = \mathrm{div}^{G_* g}\ G_* X$

(iv) $\mathrm{Hess}^g_f(x)$ *is symmetric for almost every* $x \in A$

(v) $\mathrm{div}^g\,(h(\nabla^g f)) = -h\Delta^g f + g(\nabla^g f, \nabla^g h)$ *for every weakly twice differentiable function* h *on* A

(vi) $\Delta^g(fh) = h\Delta^g f - 2g(\nabla^g f, \nabla^g h) + f\Delta^g h$ *for every weakly twice differentiable function* h *on* A

Proof (i), (ii) and (iii) all follow directly from Corollary 3.13. Since

$$\mathrm{Hess}^g_f = \sum_{i,j}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \left(\nabla^g_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)(f)\right) dx_i \otimes dx_j,$$

(iv) follows from Proposition 3.8. On the other hand, by simple calculations, we have (v) and (vi). □

3.3 Weakly second-order differential structure on weakly rectifiable metric measure spaces

In this subsection, we will discuss some weak twice differentiability on weakly rectifiable metric measure spaces.

Definition 3.16 (Weakly second-order differential structure) Let (X, υ) be a metric measure space and $0 < \alpha \le 1$. We say that:

(i) (X, υ) *has a weakly* $C^{1,\alpha}$*–structure* if there exists a weakly rectifiable coordinate system $\{(C^l_i, \phi^l_i)\}_{l,i}$ of (X, υ) such that every Jacobi matrix map $J(\Phi^l_{ij})$ of $\Phi^l_{ij} = \phi^l_j \circ (\phi^l_i)^{-1}$ from $\phi^l_i(C^l_i \cap C^l_j)$ to $\phi^l_j(C^l_i \cap C^l_j)$ is weakly α–Hölder continuous.

(ii) (X, υ) *has a weakly second-order differential structure* if (X, υ) has a weakly $C^{1,1}$–structure.

Definition 3.17 Let $0 < \hat{\alpha} \le \alpha \le 1$, let (X, υ) be a metric measure space having a weakly $C^{1,\alpha}$–structure with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$, a Borel subset A of X and $\omega = \{\omega^l_i\}_{l,i}$ a family of Borel p–forms ω^l_i on $\phi^l_i(C^l_i \cap A)$. We say that:

(i) ω is a *Borel p–form on A* if $(\Phi^l_{ij})^*\omega^l_j = \omega^l_i$ on $\phi^l_i(C^l_i \cap C^l_j \cap A)$ for every i, l, j with $\upsilon(C^l_i \cap C^l_j \cap A) > 0$.

(ii) ω is a *weakly $\hat{\alpha}$–Hölder continuous p–form on A* if ω is a Borel p–form on A, and $\omega^l_i \in \Gamma_{\hat{\alpha}}(\bigwedge^p T^*\phi^l_i(C^l_i \cap A))$.

Write $\omega|_{C^l_i \cap A} = \omega^l_i$.

Note that ω can be identified as a Borel section from A to a L^∞–vector bundle $\bigwedge^p T^*X$ on X. See [4, Section 4] by Cheeger or [8, Section 6] by Cheeger and Colding for the details. Note that in the same way as in Definition 3.17, we can give definitions of Borel vector (tensor) field on A, its Hölder continuity, its equivalence, and so on. For instance, denote the set of equivalent classes of Borel sections $s\colon A \to T^*X \otimes T^*X$ by $\Gamma_{\mathrm{Bor}}(T^*A \otimes T^*A)$. Similarly, define $\Gamma_{\mathrm{Bor}}(T^*A)$, $\Gamma_{\mathrm{Bor}}(\bigwedge^p T^*A)$, and so on.

Assume that (X, υ) has a weakly second-order differential structure with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$. Let A be a Borel subset of X. As in the previous section, denote the set of equivalent classes of Borel vector fields on A represented by a weakly Lipschitz vector field on A by $\Gamma_1(TA, \{(C^l_i, \phi^l_i)\}_{l,i})$. Similarly define $\Gamma_1(T^*A \otimes T^*A, \{(C^l_i, \phi^l_i)\}_{l,i})$ and so on. We often write

$$\Gamma_1\big(\textstyle\bigwedge^p T^*A\big) = \Gamma_1\big(\textstyle\bigwedge^p T^*A, \{(C^l_i, \phi^l_i)\}_{l,i}\big),$$
$$\Gamma_1(T^*A \otimes T^*A) = \Gamma_1(T^*A \otimes T^*A, \{(C^l_i, \phi^l_i)\}_{l,i})$$

and so on, for brevity.

Proposition 3.18 *Let $\omega \in \Gamma_1(\bigwedge^p T^*A)$. Then there exists $d\omega \in \Gamma_{\mathrm{Bor}}(\bigwedge^{p+1} T^*A)$ defined uniquely such that $d\omega|_{C^l_i \cap A} = d(\omega|_{\phi^l_i(C^l_i \cap A)})$.*

Proof This is a direct consequence of Proposition 3.5. □

Proposition 3.19 *Let $V, W \in \Gamma_1(TA)$. Then there exists $[V, W] \in \Gamma_{\mathrm{Bor}}(TA)$ defined uniquely such that $[V, W]|_{C^l_i \cap A} = [V_{C^l_i \cap A}, W_{C^l_i \cap A}]$.*

Proof This is a direct consequence of Proposition 3.6. □

Definition 3.20 (Weakly twice differentiable function) We say that a Borel function f on A is *weakly twice differentiable on A with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$* if every $f \circ (\phi^l_i)^{-1}$ is weakly twice differentiable on $\phi^l_i(C^l_i \cap A)$.

The following is a direct consequence of Proposition 3.8:

Corollary 3.21 *Let f be a weakly twice differentiable function on A with respect to $\{(C_i^l, \phi_i^l)\}_{l,i}$. Then we have $d(df) = 0 \in \Gamma_{\mathrm{Bor}}(\bigwedge^2 T^*A)$.*

We say that $g \in \Gamma_{\mathrm{Bor}}(T^*A \otimes T^*A)$ is a *Borel Riemannian metric on A* if g is symmetric and positive definite. For a Borel Riemannian metric g on A, a weakly Lipschitz function f on A, $X \in \Gamma_{\mathrm{Bor}}(TA)$ and $\omega \in \Gamma_{\mathrm{Bor}}(T^*A)$, in the same way as in the previous subsection, define $X^* \in \Gamma_{\mathrm{Bor}}(T^*A)$, $\omega^* \in \Gamma_{\mathrm{Bor}}(TA)$ and $\nabla^g f \in \Gamma_{\mathrm{Bor}}(TA)$.

Remark 3.22 We *cannot* discuss twice differentiability for a vector field (or $p(\geq 1)$–form) on X in the same way as above.

Remark 3.23 We give a relationship between Ambrosio and Kirchheim's metric currents theory [1] and the weakly second-order differential structure. Let A be a Borel subset of X and let $\mathcal{D}^k(A)$ be the set of $(k+1)$–tuples $\omega = (f, \pi_1, \dots, \pi_k)$ of Lipschitz functions on A such that f is bounded. In [1], they defined the notion of currents in A by using $\mathcal{D}^k(A)$. See [1, Definition 3.1] for the precise definition. Let ι be the map from $\mathcal{D}^k(A)$ to $\Gamma_{\mathrm{Bor}}(\bigwedge^k T^*A)$ defined by

$$\iota(\omega) := f\, d\pi_1 \wedge \cdots \wedge d\pi_k.$$

Note that if each π_i is weakly twice differentiable on A, then $\iota(\omega)$ is weakly differentiable on A. Then by Proposition 3.18, if each π_i is weakly twice differentiable on A, then we have the following compatibility for the exterior derivative:

$$\iota(d\omega) = d\iota(\omega),$$

where d in the left-hand side is the exterior differential defined in [1]. Similarly, we can check the compatibility for the pullback.

3.4 Weakly Lipschitz Riemannian metric on weakly rectifiable metric measure spaces

In this subsection, we will study Riemannian metrics on weakly rectifiable metric measure spaces. Let (X, υ) be a weakly rectifiable metric measure space with respect to $\{(C_i^l, \phi_i^l)\}_{l,i}$, and $g = \{g_i^l\}_{l,i} \in \Gamma_{\mathrm{Bor}}(T^*X \otimes T^*X)$ a Borel Riemannian metric on X.

Definition 3.24 (Weakly Lipschitz Riemannian metric on weakly rectifiable metric measure spaces) We say that:

(i) g is a *weakly α–Hölder Riemannian metric on X with respect to $\{(C_i^l, \phi_i^l)\}_{l,i}$* if $g_i^l \in \mathrm{Riem}_\alpha(\phi_i^l(C_i^l))$.

(ii) g is a *weakly Lipschitz Riemannian metric on X with respect to $\{(C_i^l, \phi_i^l)\}_{l,i}$* if $g_i^l \in \mathrm{Riem}_1(\phi_i^l(C_i^l))$.

Proposition 3.25 *Let $0 < \alpha \leq 1$. Assume that g is a weakly α–Hölder Riemannian metric on X with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$. Then we see that (X, υ) has a weakly $C^{1,\alpha}$–structure with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$ and that $g \in \Gamma_\alpha(T^*X \otimes T^*X, \{(C^l_i, \phi^l_i)\}_{l,i})$.*

Proof Since g^l_i is weakly α–Hölder continuous, by simple calculation, we see that every map $J(\Phi^l_{ij})$ is weakly α–Hölder continuous on $\phi^l_i(C^l_i \cap C^l_j)$. Therefore we have the proposition. □

Assume that g is a weakly Lipschitz Riemannian metric on X with respect to

$$\{(C^l_i, \phi^l_i)\}_{l,i}.$$

The following is a direct consequence of Proposition 3.11 and Corollary 3.13:

Theorem 3.26 (Levi-Civita connection) *There exists a Levi-Civita connection ∇^g on X, defined uniquely in the following sense:*

(i) ∇^g *is a map from* $\Gamma_{\mathrm{Bor}}(TX) \times \Gamma_1(TX)$ *to* $\Gamma_{\mathrm{Bor}}(TX)$.

(ii) $\nabla^g_U(V + W) = \nabla^g_U V + \nabla^g_U W$ *for every* $U \in \Gamma_{\mathrm{Bor}}(TX)$ *and every* $V, W \in \Gamma_1(TX)$.

(iii) $\nabla^g_{fU+hV} W = f\nabla^g_U W + h\nabla^g_V W$ *for every* $U, V \in \Gamma_{\mathrm{Bor}}(TX)$, *every* $W \in \Gamma_1(TX)$ *and every pair of Borel functions* f, h *on* X.

(iv) $\nabla^g_U(fV) = U(f)V + f\nabla^g_U V$ *for every* $U \in \Gamma_{\mathrm{Bor}}(TX)$, *every* $V \in \Gamma_1(TX)$ *and every weakly Lipschitz function* f *on* X.

(v) $\nabla^g_U V - \nabla^g_V U = [U, V]$ *for every* $U, V \in \Gamma_1(TX)$.

(vi) $Ug(V, W) = g(\nabla^g_U V, W) + g(V, \nabla^g_U W)$ *for every* $U \in \Gamma_{\mathrm{Bor}}(TX)$ *and every* $V, W \in \Gamma_1(TX)$.

The following is a direct consequence of Corollary 3.15:

Proposition 3.27 *Let A be a Borel subset of X, f a weakly twice differentiable function on A, $\omega \in \Gamma_1(T^*A)$ and $Y \in \Gamma_1(TA)$. Then there are unique elements:*

(i) $\nabla^g \omega \in \Gamma_{\mathrm{Bor}}(T^*A \otimes T^*A)$ *satisfying* $\nabla^g \omega|_{C^l_i \cap A} = \nabla^{g^l_i}(\omega|_{C^l_i \cap A})$.

(ii) *A Hessian* $\mathrm{Hess}^g_f \in \Gamma_{\mathrm{Bor}}(T^*A \otimes T^*A)$ *satisfying* $\mathrm{Hess}^g_f|_{C^l_i \cap A} = \mathrm{Hess}^{g^l_i}_{f \circ \phi^l_i}$.

(iii) *A Borel function* $\mathrm{div}^g\, Y$ *(called the divergence of Y) on A satisfying*

$$\mathrm{div}^g\, Y(x) = \mathrm{div}^{g^l_i}\,(Y|_{C^l_i})(\phi^l_i(x))$$

for almost every $x \in \phi^l_i(A \cap C^l_i)$.

(iv) *A Borel function* $\Delta^g f$ *on* A *satisfying* $\Delta^g f(x) = \Delta^{g^l_i}(f \circ (\phi^l_i)^{-1})(\phi^l_i(x))$ *for almost every* $x \in \phi^l_i(A \cap C^l_i)$.

Moreover we have the following:

(a) $\mathrm{Hess}^g_f(x)$ *is symmetric for almost every* $x \in A$.

(b) $\mathrm{div}^g\,(h(\nabla^g f)) = -h\Delta^g f + g(\nabla^g f, \nabla^g h)$ *for every weakly twice differentiable function* h *on* A.

(c) $\Delta^g(fh) = h\Delta^g f - 2g(\nabla^g f, \nabla^g h) + f\Delta^g h$ *for every weakly twice differentiable function* h *on* A.

Finally, we end this subsection by giving the definition of the canonical Riemannian metric on a rectifiable metric measure space:

Definition 3.28 Let $(\widehat{X}, \widehat{\upsilon})$ be a rectifiable metric measure space with respect to $\{(\widehat{C}^l_i, \widehat{\phi}^l_i)\}_{l,i}$, and $\{\langle \cdot, \cdot \rangle_w\}_w$ the canonical family of inner products $\langle \cdot, \cdot \rangle_w$ on $T^*_w \widehat{X}$ as in Theorem 2.2. Define

$$\widehat{g} = \{\widehat{g}^l_i\}_{l,i} \in \Gamma_{\mathrm{Bor}}(T^*X \otimes T^*X), \quad (\widehat{g}^l_i)^{st} = \langle d\widehat{\phi}^l_{i,s}, d\widehat{\phi}^l_{i,t} \rangle,$$

where $\widehat{\phi}^l_i = (\widehat{\phi}^l_{i,1}, \dots, \widehat{\phi}^l_{i,k})$, and call $\widehat{g} = \{\widehat{g}^l_i\}_{l,i} \in \Gamma_{\mathrm{Bor}}(T^*\widehat{X} \otimes T^*\widehat{X})$ the *Riemannian metric of* $(\widehat{X}, \widehat{\upsilon})$ *with respect to* $\{(\widehat{C}^l_i, \widehat{\phi}^l_i)\}_{l,i}$.

Remark 3.29 In [3; 8], Cheeger and Colding refer to the family $\{\langle \cdot, \cdot \rangle_w\}_w$ as the Riemannian metric of $(\widehat{X}, \widehat{\upsilon})$.

4 Ricci limit spaces

In this section, we will give the proofs of Theorems 1.2 and 1.5. Let (M_∞, m_∞) be a Ricci limit space. Cheeger and Colding showed that any two limit measures υ_1, υ_2 on M_∞ are mutually absolutely continuous. See [8, Theorem 4.17]. Therefore, for instance, note that the notion of weak Hölder continuity for functions on M_∞ does not depend on the choice of the limit measures.

4.1 Angles, their weak Hölder continuity and bi-Lipschitz embedding

In this subsection, we will give a proof of Theorem 1.2 and discuss the weak Hölder continuity of angles. The proof of the following proposition is based on the proof of Cheeger and Colding's splitting theorem [5, Theorem 6.64]:

Proposition 4.1 *For every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n) > 0$ such that the following property holds: Let M be an n–dimensional complete Riemannian manifold with $\mathrm{Ric}_M \ge -\epsilon^2(n-1)$, and $m, p_1, p_2, q_1, q_2 \in M$. Assume that $\overline{p_i, m} \ge \epsilon^{-1}$, $\overline{q_i, m} \ge \epsilon^{-1}$ and $\overline{m, p_i} + \overline{m, q_i} - \overline{p_i, q_i} \le \delta$ for $i = 1, 2$. Then we have*

$$\frac{1}{\mathrm{vol}\, B_1(m)} \int_{B_1(m)} \left| \langle dr_{p_1}, dr_{p_2} \rangle - \frac{1}{\mathrm{vol}\, B_1(m)} \int_{B_1(m)} \langle dr_{p_1}, dr_{p_2} \rangle \, d\mathrm{vol} \right| d\mathrm{vol} \le C(n)\epsilon^{\alpha(n)},$$

where $0 < \alpha(n) < 1$ and $C(n) \ge 1$ are constants depending only on n.

Proof Cheeger and Colding's proof of [5, Lemmas 6.15, 6.25 and Proposition 6.60] yields that there exists $\delta = \delta(\epsilon, n) > 0$ with the following properties: Let M, m, p_1, p_2, q_1, q_2 be as above. Then for every $i = 1, 2$, there exists a harmonic function $\boldsymbol{b}_i$ on $B_1(m)$ such that $|r_{p_i} - \boldsymbol{b}_i|_{L^\infty(B_1(m))} \le C_1(n)\epsilon^{\alpha_1(n)}$ and

$$\frac{1}{\mathrm{vol}\, B_1(m)} \int_{B_1(m)} \left(|dr_{p_i} - d\boldsymbol{b}_i|^2 + |\operatorname{Hess}_{\boldsymbol{b}_i}|^2 \right) d\mathrm{vol} \le C_2(n)\epsilon^{\alpha_2(n)}.$$

Therefore, by the Poincaré inequality of type $(1, 2)$ on M, we have

$$\frac{1}{\mathrm{vol}\, B_1(m)} \int_{B_1(m)} \left| \langle d\boldsymbol{b}_1, d\boldsymbol{b}_2 \rangle - \frac{1}{\mathrm{vol}\, B_1(m)} \int_{B_1(m)} \langle d\boldsymbol{b}_1 . d\boldsymbol{b}_2 \rangle \, d\mathrm{vol} \right| d\mathrm{vol}$$
$$\le C(n) \sqrt{\frac{1}{\mathrm{vol}\, B_1(m)} \int_{B_1(m)} \left(|\operatorname{Hess}_{\boldsymbol{b}_1}|^2 + |\operatorname{Hess}_{\boldsymbol{b}_2}|^2 \right) d\mathrm{vol}} \le C_3(n)\epsilon^{\alpha_3(n)}.$$

This completes the proof. □

Remark 4.2 These Hölder estimates as above essentially follows from Laplacian comparison theorem and the Abresch–Gromoll excess estimate

$$e(x) \le C(n)\epsilon^{\alpha(n)}$$

for M, m, p_1, p_2, q_1, q_2 as in Proposition 4.1 and for every $x \in B_{100}(m)$, where $e(x) = \overline{p_1, x} + \overline{q_1, x} - \overline{p_1, q_2}$. See [5, Proposition 6.2] by Cheeger and Colding (or [4, Theorem 9.1] by Cheeger) for the details. We can also see a valuable survey about this excess estimate in [11, Section 1.7] by Colding and Naber and a new very useful excess estimate in [11, Theorem 2.6].

For every $p \in M_\infty$ and every $\tau > 0$, put

$$\mathcal{D}_p^\tau = \{z \in M_\infty : \text{there exists } w \in M_\infty \text{ with } \overline{w, z} \ge \tau \text{ and } \overline{p, z} + \overline{z, w} = \overline{p, w}\}.$$

Note that $C_p = M_\infty \setminus \bigcup_{\tau > 0} \mathcal{D}_p^\tau$.

Corollary 4.3 *Let υ be a limit measure on M_∞, $\beta > 0$, $\tau > 0$, $p, q \in M_\infty$ and $x \in \mathcal{D}_p^\tau \cap \mathcal{D}_q^\tau \setminus (B_\beta(p) \cup B_\beta(q))$. Then we have*

$$\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left|\langle dr_p, dr_q\rangle - \frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dr_p, dr_q\rangle\, d\upsilon\right| d\upsilon \le C(n)\max\left\{r, \frac{r}{\beta}, \frac{r}{\tau}\right\}^{\alpha(n)}$$

for every $0 < r < \min\{\beta, \tau\}$.

Proof Let $\{(M_i, m_i)\}_i$ be a sequence of pointed n–dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$ such that

$$(M_i, m_i, \mathrm{vol}/\mathrm{vol}\, B_1(m_i)) \to (M_\infty, m_\infty, \upsilon).$$

Fix sequences $p_i, q_i, x_i \in M_i$ with $p_i \to p, q_i \to q$ and $x_i \to x$. By considering the rescaled metric $r^{-1} d_{M_i}$, it follows from Proposition 4.1 that

$$\frac{1}{\mathrm{vol}\, B_r(x_i)}\int_{B_r(x_i)}\left|\langle dr_{p_i}, dr_{q_i}\rangle - \frac{1}{\mathrm{vol}\, B_r(x_i)}\int_{B_r(x_i)}\langle dr_{p_i}, dr_{q_i}\rangle\, d\mathrm{vol}\right| d\mathrm{vol} \le C(n)\max\left\{r, \frac{r}{\beta}, \frac{r}{\tau}\right\}^{\alpha(n)}$$

for every sufficiently large i. By the property (i) at the end of Section 2.2, since $dr_{p_i} \to dr_p$, $dr_{q_i} \to dr_q$ and $dr_{x_i} \to dr_x$ on M_∞, by letting $i \to \infty$ and using the property (iii) at the end of Section 2.2, we have the corollary. □

Let υ be a limit measure on M_∞ and $p, q, x \in M_\infty$ with $x \in M_\infty \setminus (C_p \cup C_q \cup \{p\} \cup \{q\})$. It follows directly from Corollary 4.3 that the limit

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dr_p, dr_q\rangle\, d\upsilon$$

exists. Define the angle $\angle^\upsilon pxq$ of pxq with respect to υ by

$$\angle^\upsilon pxq = \arccos\left(\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dr_p, dr_q\rangle\, d\upsilon\right).$$

Theorem 4.4 *We have*

$$\cos\angle^\upsilon pxq = \lim_{t\to 0}\frac{2t^2 - \overline{\gamma_p(t), \gamma_q(t)}^2}{2t^2}$$

for any minimal geodesics γ_p from x to p, and γ_q from x to q. In particular, we have Theorem 1.2.

Proof Recall that a pointed proper geodesic space (Y, y) is said to be a *tangent cone of M_∞ at $x \in M_\infty$* if there exists a sequence of positive numbers $\{r_i\}_i$ such that $r_i \to 0$ and $(M_\infty, r_i^{-1} d_{M_\infty}, x) \to (Y, y)$. Fix

(i) a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$,

(ii) a tangent cone (Y, y) of M_∞ at x, and a Radon measure υ_Y on Y such that $(M_\infty, r_i^{-1} d_{M_\infty}, x, \upsilon_i) \to (Y, y, \upsilon_Y)$, where $\upsilon_i = \upsilon/\upsilon(B_{r_i}(x))$,

(iii) two geodesics γ_p, γ_q on M_∞ beginning at p, q, respectively, such that x is an interior point of both γ_p and γ_q.

Then it is easy to check that there exist lines l_p, l_q of Y such that $y \in \mathrm{Image}(l_p) \cap \mathrm{Image}(l_q)$ and that $(\gamma_p, r_i^{-1} d_Y) \to l_p$ and $(\gamma_q, r_i^{-1} d_Y) \to l_q$ with respect to the Gromov–Hausdorff topology (recall that a map $l\colon \mathbb{R} \to Y$ is said to be a *line of Y* if l is an isometric embedding).

Claim 4.5 *Let $b_p^i = r_i^{-1} d_{M_\infty}(p, \cdot) - r_i^{-1} d_{M_\infty}(p, x)$. Then $(b_p^i, db_p^i) \to (b_{l_p}, db_{l_p})$ on Y with respect to the convergence $(M_\infty, r_i^{-1} d_{M_\infty}, x, \upsilon_i) \to (Y, y, \upsilon_Y)$, where b_{l_p} is the Busemann function of l_p.*

The proof is as follows. It is easy to check $b_p^i \to b_{l_p}$ on Y. Let $R > 0$, $x_j, p_j \in M_j$ with $x_j \to x$, $p_j \to p$ and let $b^{i,j} = r_i^{-1} d_{M_j}(p_j, \cdot) - r_i^{-1} d_{M_j}(p_j, x_j)$. [5, Lemmas 6.15 and 6.25] by Cheeger and Colding yield that there exists a sequence $\{\boldsymbol{b}^{i,j}\}_{i<\infty, j<\infty}$ of $C(n)$–Lipschitz harmonic functions $\boldsymbol{b}^{i,j}$ on $B_R^{r_i^{-1} d_{M_j}}(x_j)$ such that for every i there exists i_0 such that

$$\left\|\boldsymbol{b}^{i,j} - b^{i,j}\right\|_{L^\infty\left(B_R^{r_i^{-1} d_{M_j}}(x_j)\right)} + \left\|d\boldsymbol{b}^{i,j} - db^{i,j}\right\|_{L^2\left(B_R^{r_i^{-1} d_{M_j}}(x_j)\right)} \le \Psi(r_i^{-1}; n, R)$$

for every $j \ge i_0$. Without loss of generality we can assume that for every i there exists a $C(n)$–Lipschitz function $\boldsymbol{b}^{i,\infty}$ on $B_R^{r_i^{-1} d_{M_\infty}}(x)$ such that $\boldsymbol{b}^{i,j} \to \boldsymbol{b}^{i,\infty}$ on $B_R^{r_i^{-1} d_{M_\infty}}(x)$. Note that the property (ii) at the end of Section 2.2 yields $d\boldsymbol{b}^{i,j} \to d\boldsymbol{b}^{i,\infty}$ on $B_R^{r_i^{-1} d_{M_\infty}}(x)$. In particular we have

$$\left\|\boldsymbol{b}^{i,\infty} - b_p^i\right\|_{L^\infty\left(B_R^{r_i^{-1} d_{M_\infty}}(x)\right)} + \left\|d\boldsymbol{b}^{i,\infty} - db_p^i\right\|_{L^2\left(B_R^{r_i^{-1} d_{M_\infty}}(x)\right)} \le \Psi(r_i^{-1}; n, R).$$

Thus $\boldsymbol{b}^{i,\infty} \to b_{l_p}$ on $B_R(y)$. Since there exists a subsequence $\{j(i)\}_i$ such that $\boldsymbol{b}^{i,j(i)} \to b_{l_p}$ on $B_R(y)$ with respect to the convergence $(M_{j(i)}, x_{j(i)}, r_i^{-1} d_{M_{j(i)}}) \to (Y, y)$, applying the property (ii) at the end of Section 2.2 again yields $d\boldsymbol{b}^{i,\infty} \to db_{l_p}$ on $B_R(y)$. Thus we have $db_p^i \to db_{l_p}$ on $B_R(y)$. Since R is arbitrary, we have Claim 4.5.

Therefore we have

$$\begin{aligned}\cos \angle^{\upsilon} pxq &= \lim_{i\to\infty} \frac{1}{\upsilon(B_{r_i}(x))} \int_{B_{r_i}(x)} \langle dr_p, dr_q\rangle \, d\upsilon \\ &= \lim_{i\to\infty} \frac{1}{\upsilon_i(B_1^{r_i^{-1}d_{M_\infty}}(x))} \int_{B_1^{r_i^{-1}d_{M_\infty}}(x)} \langle db_p^i, db_q^i\rangle \, d\upsilon_i \\ &= \frac{1}{\upsilon_Y(B_1(y))} \int_{B_1(y)} \langle db_{l_p}, db_{l_q}\rangle \, d\upsilon_Y \\ &= \cos(\text{the angle between } l_p \text{ and } l_q) = \lim_{i\to\infty} \frac{2r_i^2 - \overline{\gamma_p(r_i), \gamma_q(r_i)}^2}{2r_i^2}.\end{aligned}$$

Thus Gromov's compactness theorem yields the theorem. □

Since $\angle^{\upsilon} pxq$ is independent of υ, we set $\angle pxq = \angle^{\upsilon} pxq$ and call it *the angle of* pxq.

Remark 4.6 By the proof of Theorem 4.4 and the property (iii) at the end of Section 2.2, we have

$$\lim_{\epsilon\to 0} \sup_{s,t\in[\tau_1,\tau_2]} \left| \cos \angle pxq - \frac{(s\epsilon)^2 + (t\epsilon)^2 - \overline{\gamma_p(s\epsilon), \gamma_q(t\epsilon)}^2}{2st\epsilon^2} \right| = 0$$

for every $0 < \tau_1 < \tau_2 < \infty$ and

$$\lim_{r\to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} |\cos \angle pzq - \cos \angle pxq| \, d\upsilon(z) = 0.$$

Remark 4.7 Let $p, q \in M_\infty$. Then [17, Theorem 3.3] yields that there exists $A \subset M_\infty$ such that $\upsilon(M_\infty \setminus A) = 0$ and that

$$\langle dr_p, dr_q\rangle(x) = \lim_{\delta\to 0} \frac{\overline{q, \gamma_p(\overline{x,p}+\delta)} - \overline{q,x}}{\delta}$$

for every $x \in A$ and every γ_p. Thus Theorem 4.4 and Lebesgue's differentiation theorem yield the following *almost first variation formula for the distance function* (in some weak sense):

$$\overline{q, \gamma_p(\overline{x,p}+\delta)} = \overline{q,x} + \delta \cos \angle pxq + o(\delta)$$

for almost every $x \in A$ and every γ_p, γ_q.

The next corollary is a direct consequence of Corollary 4.3 and [16, Theorem 3.2]:

Corollary 4.8 (Weak Hölder continuity of angles) *Let $\tau > 0$, $R > 1$, $p, q, x \in M_\infty$ with $p, q \in B_R(x) \setminus B_{R^{-1}}(x)$ and $x \in \mathcal{D}^\tau_p \cap \mathcal{D}^\tau_q$. Then there exists $r = r(n, \tau, R) > 0$ such that a function $\Phi(z) = \cos \angle pzq$ is $\alpha(n)$–Hölder continuous on $B_r(x) \cap \mathcal{D}^\tau_p \cap \mathcal{D}^\tau_q$. In particular, Φ is weakly $\alpha(n)$–Hölder continuous on M_∞ with respect to any limit measure.*

The next corollary is a direct consequence of Proposition 3.25 and Corollary 4.8:

Corollary 4.9 *Let $\{(C^l_i, \phi^l_i)\}_{l,i}$ be a rectifiable coordinate system constructed by distance functions on M_∞ as in Theorem 2.4. Then M_∞ has a weakly $C^{1,\alpha(n)}$–structure with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$. Moreover for every $p \in M_\infty$, dr_p is a weakly $\alpha(n)$–Hölder continuous 1–form on M_∞ with respect to $\{(C^l_i, \phi^l_i)\}_{l,i}$.*

Remark 4.10 Let (Y, p) be as in Colding and Naber Theorem 1.1. According to [10, Theorem 1.2] by Colding and Naber, we see that Y does *not* have a $C^{1,\beta}$–structure in the ordinary sense for *any* $0 < \beta \le 1$.

Next, we will discuss the continuity of angles with respect to the Gromov–Hausdorff topology. Recall that a map ϕ from a metric space X_1 to a metric space X_2 is said to be an *ϵ–Gromov–Hausdorff approximation* if $X_2 \subset B_\epsilon(\mathrm{Image}(\phi))$ and $|\overline{x, y} - \overline{\phi(x), \phi(y)}| < \epsilon$ for every $x, y \in X_1$.

Proposition 4.11 *Let (Y, y) be a Ricci limit space, $R > 1$, $0 < \tau < 1$, $0 < \beta < 1$ and $p, q \in B_R(y)$ with $y \in M_\infty \setminus (C_p \cup C_q \cup \{p, q\})$. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that the following property holds: Let $(\hat{Y}, \hat{y})$ be a Ricci limit space and $\hat{p}, \hat{q} \in B_R(\hat{y})$ with $\hat{y} \in \mathcal{D}^\tau_{\hat{p}} \cap \mathcal{D}^\tau_{\hat{q}} \setminus (B_\beta(\hat{p}) \cup B_\beta(\hat{q}))$. Assume that there exists a δ–Gromov–Hausdorff approximation ϕ from $(B_R(\hat{y}), \hat{y})$ to $(B_R(y), y)$ such that $\overline{\phi(\hat{p}), p} < \delta$, $\overline{\phi(\hat{q}), q} < \delta$. Then we have $|\angle pyq - \angle \hat{p}\hat{y}\hat{q}| < \epsilon$.*

Proof The proof is done by contradiction. Suppose that the assertion is false. Then there exist $\epsilon_0 > 0$, $R > 1$, $\tau > 0$, $\beta > 0$, sequences of Ricci limit spaces $\{(Y_i, y_i)\}_{i<\infty}$ and of points $p_i, q_i \in B_R(y_i)$ such that $(B_R(y_i), y_i) \to (B_R(y), y)$, $p_i \to p, q_i \to q$, $y_i \in \mathcal{D}^\tau_{p_i} \cap \mathcal{D}^\tau_{q_i} \setminus (B_\beta(p_i) \cup B_\beta(q_i))$ for every i and that $|\cos \angle p_i y_i q_i - \cos \angle pyq| \ge \epsilon_0$ for every i. Moreover, by Gromov's compactness theorem, without loss of generality, we can assume that there exist a limit measure υ on Y and a sequence $\{\upsilon_i\}_i$ of limit measures υ_i on Y_i such that υ is the limit measure of $\{\upsilon_i\}_i$. By Corollary 4.3, there exists $r > 0$ such that

$$\left|\cos \angle p_i y_i q_i - \frac{1}{\upsilon_i(B_r(y_i))} \int_{B_r(y_i)} \langle dr_{p_i}, dr_{q_i} \rangle \, d\upsilon_i\right| + \left|\cos \angle pyq - \frac{1}{\upsilon(B_r(y))} \int_{B_r(y)} \langle dr_p, dr_q \rangle \, d\upsilon\right| < \frac{\epsilon_0}{3}$$

for every i. On the other hand, since $dr_{p_i} \to dr_p$ and $dr_{q_i} \to dr_q$ on Y, we have

$$\left| \frac{1}{\upsilon_i(B_r(y_i))} \int_{B_r(y_i)} \langle dr_{p_i}, dr_{q_i} \rangle \, d\upsilon_i - \frac{1}{\upsilon(B_r(y))} \int_{B_r(y)} \langle dr_p, dr_q \rangle \, d\upsilon \right| < \frac{\epsilon_0}{3}$$

for every sufficiently large i. Thus we have $|\cos \angle p_i y_i q_i - \cos \angle pyq| < \epsilon_0$ for every sufficiently large i. This is a contradiction. □

The following theorem is about the continuity of angles with respect to the Gromov–Hausdorff topology:

Theorem 4.12 (GH-continuity of angles) *Let $R > 1$, $\beta > 0$ and $0 < \tau < 1$. Then for every $\epsilon > 0$, there exists $\delta = \delta(n, R, \tau, \beta, \epsilon) > 0$ such that the following property holds: Let (Y_1, y_1) and (Y_2, y_2) be Ricci limit spaces, and $a_i, b_i \in B_R(y_i)$ with $y_i \in \mathcal{D}^\tau_{a_i} \cap \mathcal{D}^\tau_{b_i} \setminus (B_\beta(a_i) \cup B_\beta(b_i))$ for every $i = 1, 2$. Assume that there exists a δ–Gromov–Hausdorff approximation ϕ from $(B_R(y_1), y_1)$ to $(B_R(y_2), y_2)$ such that $\overline{\phi(a_1), a_2} < \delta$ and $\overline{\phi(b_1), b_2} < \delta$. Then we have $|\angle a_1 y_1 b_1 - \angle a_2 y_2 b_2| < \epsilon$.*

Proof The proof is done by contradiction. Suppose that the assertion is false. Then by Gromov's compactness theorem, there exists

(i) $R > 1$, $\beta > 0$, $0 < \tau < 1$, $\epsilon_0 > 0$,

(ii) a Ricci limit space (Z, z), points $a, b \in Z$,

(iii) a sequence of Ricci limit spaces $\{(Z^j_i, z^j_i)\}_{1 \le i < \infty, j=1,2}$,

(iv) a sequence of positive numbers $\{\delta_i\}_i$ with $\delta_i \to 0$,

(v) sequences of points $a^j_i, b^j_i \in Z^j_i$ with

$$z^j_i \in \mathcal{D}^\tau_{a^j_i} \cap \mathcal{D}^\tau_{b^j_i} \cap \big(B_R(a^j_i) \setminus B_\beta(a^j_i)\big) \cap \big(B_R(b^j_i) \setminus B_\beta(b^j_i)\big),$$

(vi) a sequence of δ_i–Gromov–Hausdorff approximations ϕ_i from $(B_R(z^1_i), z^1_i)$ to $(B_R(z^2_i), z^2_i)$ with $\overline{\phi_i(a^1_i), a^2_i} < \delta_i$ and $\overline{\phi_i(b^1_i), b^2_i} < \delta_i$,

such that

$$(B_R(z^j_i), z^j_i) \to (B_R(z), z), \quad a^j_i \to a, \quad b^j_i \to b \quad \text{as } i \to \infty$$

for every $j = 1, 2$ and that $|\angle a^1_i z^1_i b^1_i - \angle a^2_i z^2_i b^2_i| \ge \epsilon_0$. But by Proposition 4.11, we have $\lim_{i\to\infty} \angle a^j_i z^j_i b^j_i = \angle azb$. This is a contradiction. □

We end this subsection by giving an application of the weak Hölder continuity of angles to a bi-Lipschitz embedding from a subset of M_∞ to a Euclidean space. Let $(\mathcal{R}_k)_{\delta,r} = \{x \in M_\infty : d_{GH}((\overline{B}_t(x), x), (\overline{B}_t(0_k), 0_k)) < \delta t$ for every $0 < t < r\}$, where $0_k \in \mathbb{R}^k$, and d_{GH} is the Gromov–Hausdorff distance between pointed metric spaces. See also Cheeger and Colding [6; 8], Colding and Naber [10] and Gromov [14].

Proposition 4.13 *Let $R > 1$, $r > 0$, $\delta > 0$, $\tau > 0$ and $x \in (\mathcal{R}_k)_{\delta,r}$. Assume that there exists $\{p_i\}_{1 \le i \le k} \subset M_\infty$ such that $x \in \bigcap_i \big((B_R(p_i) \setminus B_{R^{-1}}(p_i)) \cap \mathcal{D}^\tau_{p_i}\big)$ and $\det(\cos \angle p_i x p_j)_{ij} \neq 0$. Then the map*

$$\phi_t = (r_{p_1}, \dots, r_{p_k}) \sqrt{(\cos \angle p_i x p_j)_{ij}}^{\,-1}$$

from $B_t(x) \cap (\mathcal{R}_k)_{\delta,r} \cap \bigcap_i \mathcal{D}^\tau_{p_i}$ to $\mathbb{R}^k$ is an $(1 \pm \Psi(\delta, t; R, \beta, \tau, r))$–bi-Lipschitz embedding for every $0 < t < r$, where $\Psi(a, b; c, d, e, f)$ is a positive definite function on $\mathbb{R}^6$ satisfying $\lim_{a \to 0, b \to 0} \Psi(a, b; c, d, e, f) = 0$ for every fixed c, d, e, f.

Proof Let υ be a limit measure on M_∞. An argument similar to that of the proof of Theorem 4.4 yields

$$\lim_{r \to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} \det(\langle dr_{p_i}, dr_{p_j} \rangle)_{ij} \, d\upsilon = \det(\cos \angle p_i x p_j)_{ij}.$$

Then the proposition follows from an argument similar to that of the proof of [17, Lemma 3.14]. □

Remark 4.14 Assume (M_∞, m_∞) is a *noncollapsing limit*, ie, there exists a sequence of pointed n–dimensional complete Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \ge -(n-1)$ such that $\lim_{i \to \infty} \mathrm{vol}\, B_1(m_i) > 0$ and $(M_i, m_i) \to (M_\infty, m_\infty)$. Then in [5, Theorem 5.11], Cheeger and Colding showed that for every $x \in (\mathcal{R}_n)_{\delta,r}$, we have $B_{r/32}(x) \subset (\mathcal{R}_n)_{\Psi(\delta,r;n),r/32}$. See also [6, Remark 5.15 and Theorem B.2] by Cheeger and Colding, and [10, Theorem 1.1] by Colding and Naber, for related results.

4.2 Weak Lipschitz continuity of the Riemannian metric on a Ricci limit space

In this subsection, we will show that the Riemannian metric of a Ricci limit space is weakly Lipschitz.

Assume that $(M_i, m_i, \mathrm{vol} / \mathrm{vol}\, B_1(m_i)) \to (M_\infty, m_\infty, \upsilon)$. The following proposition is an essential result to get Theorem 1.5. See [18; 19; 20; 21] by Kasue and Kumura for related important interesting results.

Proposition 4.15 *Let $R > 0$ and let $\{f_i\}_{1 \le i \le \infty}$ be a sequence of Lipschitz functions f_i on $B_R(m_i)$ with $\sup_i \mathbf{Lip} f_i < \infty$. Assume that the following hold:*

(i) $(f_i, df_i) \to (f_\infty, df_\infty)$ *on* $B_R(m_\infty)$.

(ii) *There exists $r > 0$ with $r < R$ such that* $\mathrm{supp}(f_i) \subset B_r(m_i)$ *for every i.*

(iii) $|df_i|^2 \in H_{1,2}(B_R(m_i))$ *for every $i < \infty$, and*

$$\sup_{i<\infty} \frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} \left| d|df_i|^2 \right|^2 d\mathrm{vol} < \infty. \tag{1}$$

Then we have $|df_\infty|^2 \in H_{1,2}(B_R(m_\infty))$ and

$$\frac{1}{\upsilon(B_R(m_\infty))} \int_{B_R(m_\infty)} \left| d|df_\infty|^2 \right|^2 d\upsilon \le \liminf_{i\to\infty} \frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} \left| d|df_i|^2 \right|^2 d\mathrm{vol}.$$

Proof [23, Lemma 5.8] by Kuwae and Shioya (or [12, Lemma 5.17] by Ding) yields that there exists an orthonormal basis $\{\phi_i^j\}_j$ on $L^2(B_R(m_i))$ consisting of eigenfunctions ϕ_i^j associated with the j^{th} eigenvalue λ_i^j with respect to the Dirichlet problem on $B_R(m_i)$ such that $\lambda_i^j \to \lambda_\infty^j$ and that $\phi_i^j \to \phi_\infty^j$ with respect to the L^2–topology (see [23, Definition 2.3] by Kuwae and Shioya, or [15], for the definition of an L^2–topology with respect to the measured Gromov–Hausdorff topology). Put $|df_i|^2 = \sum_{j=0}^\infty a_i^j \phi_i^j$ in $L^2(B_R(m_i))$ for every $i \le \infty$. Let $L \ge 1$ with

$$\frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} |d\, |df_i|^2|^2 \, d\mathrm{vol} = \sum_{j=0}^\infty \lambda_i^j (a_i^j)^2 \le L$$

for every $i < \infty$. [12, Lemma 5.11] by Ding yields

$$\sum_{j=N+1}^\infty (a_i^j)^2 \le \frac{1}{(\lambda_i^{N+1})^{1/2}} \| f_i \|_{L^2(B_R(m_i))} \left\| d|df_i|^2 \right\|_{L^2(B_R(m_i))} \le \frac{C(n,R,L)}{N^{1/n}}$$

for every $i < \infty$ and every N. Fix $\epsilon > 0$. Then there exists N_0 such that

$$\sum_{j=N_0+1}^\infty (a_i^j)^2 < \epsilon$$

for every $i < \infty$. Since $|df_i|^2 \to |df_\infty|^2$ on $B_R(m_\infty)$ with respect to the L^2–topology, we have

$$a_i^j = \frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} |df_i|^2 \phi_i^j \, d\mathrm{vol} \xrightarrow{i\to\infty} \frac{1}{\upsilon(B_R(m_\infty))} \int_{B_R(m_\infty)} |df_\infty|^2 \phi_\infty^j \, d\upsilon = a_\infty^j.$$

Thus we have

$$\left\| |df_\infty|^2 - \sum_{j=0}^{N} a_\infty^j \phi_\infty^j \right\|_{L^2(B_R(m_\infty))} = \lim_{i\to\infty} \left\| |df_i|^2 - \sum_{j=0}^{N} a_i^j \phi_f i^j \right\|_{L^2(B_R(m_i))} \le \epsilon,$$

for every $N \ge N_0$, ie, $\sum_{j=0}^N a_\infty^j \phi_\infty^j \to |df_\infty|^2$ in $L^2(B_R(m_\infty))$ as $N \to \infty$. Since

$$\left\| d\left(\sum_{j=0}^{N} a_i^j \phi_i^j\right) \right\|_{L^2(B_R(m_i))} \to \left\| d\left(\sum_{j=0}^{N} a_\infty^j \phi_\infty^j\right) \right\|_{L^2(B_R(m_\infty))}$$

as $i \to \infty$ for every N, this completes the proof. □

Corollary 4.16 *Let $R > 0$, $L \ge 1$ and let $\{f_i\}_i$ be a sequence of Lipschitz functions f_i on $B_R(m_i)$. Assume that the following properties hold:*

(i) *f_i is a C^2–function for every $i < \infty$.*

(ii) $\sup_{i<\infty}\left(\|f_i\|_{L^\infty} + \mathbf{Lip} f_i + \frac{1}{\mathrm{vol}\, B_R(m_i)} \int_{B_R(m_i)} (\Delta f_i)^2 d\mathrm{vol} \right) \le L.$

(iii) *$f_i \to f_\infty$ on $B_R(m_\infty)$.*

Then we have $|df_\infty|^2 \in H_{1,2}(B_r(m_\infty))$ for every $r < R$, and

$$\frac{1}{\upsilon(B_r(m_\infty))} \int_{B_r(m_\infty)} \left| d|df_\infty|^2 \right|^2 d\upsilon \le C(n, L, r, R).$$

In particular, we see that $|df_\infty|^2$ is weakly Lipschitz on $B_R(m_\infty)$.

Proof The existence of a good cutoff function [5, Theorem 6.33] by Cheeger and Colding yields that there exists a sequence $\{\phi_i\}_{i<\infty}$ of smooth functions ϕ_i on $B_R(m_i)$ such that $\|\nabla\phi_i\|_{L^\infty} \le C(n, r, R)$, $\|\Delta\phi_i\|_{L^\infty} \le C(n, r, R)$, $0 \le \phi \le 1$, $\phi_i|_{B_r(m_i)} \equiv 1$ and $\mathrm{supp}(\phi_i) \subset B_{(r+R)/2}(m_i)$. By applying Proposition 4.15 for $\phi_i f_i$, the property of (ii) at the end of Section 2.2 and [17, Remark 4.2], it follows that

$$\frac{1}{\upsilon(B_r(m_\infty))} \int_{B_r(m_\infty)} \left| d|df_\infty|^2 \right|^2 d\upsilon \le C(n, L, r, R).$$

On the other hand, Cheeger and Colding proved in [8, Theorem 2.15] that the Poincaré inequality of type $(1, 2)$ on M_∞ holds. Thus [3, Theorem 4.14] by Cheeger yields that any Sobolev function is weakly Lipschitz. Therefore we have the corollary. □

The following is a direct consequence of Corollary 4.16:

Theorem 4.17 (Weak twice differentiability of Ricci limit spaces) *Let* $\{(C_i^l, \phi_i^l)\}_{l,i}$ *be a rectifiable coordinate system of* (M_∞, υ). *Assume that for every* i, l, *there exist* $r > 0$, *a sequence* $\{x_j\}_j$ *of points* $x_j \in M_j$ *with* $C_i^l \subset B_r(x_\infty)$ *and* $x_j \to x_\infty$, *a sequence* $\{f_{j,s}\}_{j<\infty, 1 \le s \le l}$ *of* C^2*–functions* $f_{j,s}$ *on* $B_r(x_j)$ *such that*

$$\sup_{j,s} \mathbf{Lip} f_{j,s} < \infty, \quad f_{j,s} \to \phi_{i,s}^l$$

on C_i^l *as* $j \to \infty$ *for every* s *and that*

$$\sup_{j,s} \frac{1}{\operatorname{vol} B_r(x_j)} \int_{B_r(x_j)} (\Delta f_{j,s})^2 d\operatorname{vol} < \infty, \qquad \text{where } \phi_i^l = (\phi_{i,1}^l, \dots \phi_{i,l}^l).$$

Then the Riemannian metric g *of* M_∞ *is weakly Lipschitz with respect to* $\{(C_i^l, \phi_i^l)\}_{l,i}$. *In particular,* M_∞ *has a weakly second-order differential structure with respect to* $\{(C_i^l, \phi_i^l)\}_{l,i}$.

We now are in a position to prove Theorem 1.5:

Proof of Theorem 1.5 It follows directly from Theorems 2.3 and 4.17. □

Definition 4.18 We say that a rectifiable coordinate system $\{(C_i^l, \phi_i^l)\}_{l,i}$ of (M_∞, υ) as in Theorem 4.17 is a *weakly second-order differential structure associated with* $\{(M_j, m_j, \operatorname{vol} / \operatorname{vol} B_1(m_j))\}_j$.

Assume that $\{(C_i^l, \phi_i^l)\}_{l,i}$ is a weakly second-order differential structure associated with $\{(M_j, m_j, \operatorname{vol} / \operatorname{vol} B_1(m_j))\}_j$.

Proposition 4.19 *Let* $R > 0$ *and let* f_∞ *be a Lipschitz function on* $B_R(m_\infty)$. *Assume that there exists a sequence* $\{f_j\}_{j<\infty}$ *of* C^2*–functions* f_j *on* $B_R(m_j)$ *such that* $\sup_j \mathbf{Lip} f_j < \infty$, $f_j \to f_\infty$ *on* $B_R(m_\infty)$ *and*

$$\sup_{j<\infty} \frac{1}{\operatorname{vol} B_R(m_j)} \int_{B_R(m_j)} (\Delta f_j)^2 d\operatorname{vol} < \infty.$$

Then f_∞ *is weakly twice differentiable on* $B_R(m_\infty)$ *with respect to* $\{(C_i^l, \phi_i^l)\}_{l,i}$.

Proof The proposition follows from Corollary 4.16. □

Finally, we end this section by giving the following corollary:

Corollary 4.20 (Weak twice differentiability of eigenfunctions) *Let f_∞ be an eigenfunction associated with the eigenvalue λ_∞ with respect to the Dirichlet problem on $B_R(m_\infty)$. Then f_∞ is weakly twice differentiable on $B_R(m_\infty)$ with respect to $\{(C_i^l, \phi_i^l)\}_{l,i}$.*

Proof [23, Lemma 5.8] by Kuwae and Shioya (or [12, Lemma 5.17] by Ding) yields that there exists a sequence $\{f_i\}_i$ of eigenfunctions f_i associated with the eigenvalue λ_i with respect to the Dirichlet problem on $B_R(x_i)$ such that $\lambda_i \to \lambda_\infty$ and that $f_i \to f_\infty$ with respect to the L^2–topology. Note that it follows from Cheng and Yau's gradient estimate [9] that $\sup_i \mathbf{Lip}(f_i|_{B_r(x_i)}) < \infty$ for every $r < R$. Thus the corollary follows directly from Proposition 4.19. $\square$

Remark 4.21 See [15, Theorem 1.3] for a generalization of Corollary 4.16 and Proposition 4.19. Moreover, in [15], we will prove that for f_∞ as in Corollary 4.20, if M_∞ is *noncollapsing*, then

$$\Delta^{g_{M_\infty}} f_\infty = \lambda_\infty f_\infty = \Delta^{\upsilon} f_\infty,$$

where Δ^{υ} is the Dirichlet Laplacian defined by Cheeger and Colding in [8]. In particular, in noncollapsing setting, the Laplacian defined as in Proposition 3.27 coincides with the Dirichlet Laplacian on a dense subspace in L^2. However, in collapsing setting, the equality above does *not* hold in general. See [15, Theorem 1.4, Remark 4.33] for the details.

References

[1] **L Ambrosio**, **B Kirchheim**, *Currents in metric spaces*, Acta Math. 185 (2000) 1–80 MR1794185

[2] **Y Burago**, **M Gromov**, **G Perelman**, *A D Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk 47 (1992) 3–51 MR1185284 In Russian; translated in Russian Math. Surveys 47 (1992) 1–58

[3] **J Cheeger**, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. 9 (1999) 428–517 MR1708448

[4] **J Cheeger**, *Degeneration of Riemannian metrics under Ricci curvature bounds*, Lezioni Fermiane, Scuola Normale Superiore, Pisa (2001) MR2006642

[5] **J Cheeger**, **T H Colding**, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. 144 (1996) 189–237 MR1405949

[6] **J Cheeger**, **T H Colding**, *On the structure of spaces with Ricci curvature bounded below, I*, J. Differential Geom. 46 (1997) 406–480 MR1484888

[7] **J Cheeger**, **T H Colding**, *On the structure of spaces with Ricci curvature bounded below, II*, J. Differential Geom. 54 (2000) 13–35 MR1815410

[8] **J Cheeger**, **T H Colding**, *On the structure of spaces with Ricci curvature bounded below, III*, J. Differential Geom. 54 (2000) 37–74 MR1815411

[9] **S Y Cheng**, **S T Yau**, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975) 333–354 MR0385749

[10] **T H Colding**, **A Naber**, *Lower Ricci curvature, branching and bi-Lipschitz structure of uniform Reifenberg spaces* `arXiv:1111.2184`

[11] **T H Colding**, **A Naber**, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Ann. of Math. 176 (2012) 1173–1229 MR2950772

[12] **Y Ding**, *Heat kernels and Green's functions on limit spaces*, Comm. Anal. Geom. 10 (2002) 475–514 MR1912256

[13] **K Fukaya**, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. 87 (1987) 517–547 MR874035

[14] **M Gromov**, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, Boston, MA (2007) MR2307192

[15] **S Honda**, *Ricci curvature and L^p–convergence*, to appear in J. Reine Angew. Math. `arXiv:1212.2052`

[16] **S Honda**, *Bishop–Gromov type inequality on Ricci limit spaces*, J. Math. Soc. Japan 63 (2011) 419–442 MR2793106

[17] **S Honda**, *Ricci curvature and convergence of Lipschitz functions*, Comm. Anal. Geom. 19 (2011) 79–158 MR2818407

[18] **A Kasue**, *Convergence of Riemannian manifolds and Laplace operators, I*, Ann. Inst. Fourier (Grenoble) 52 (2002) 1219–1257 MR1927079

[19] **A Kasue**, *Convergence of Riemannian manifolds and Laplace operators, II*, Potential Anal. 24 (2006) 137–194 MR2217418

[20] **A Kasue**, **H Kumura**, *Spectral convergence of Riemannian manifolds*, Tohoku Math. J. 46 (1994) 147–179 MR1272877

[21] **A Kasue**, **H Kumura**, *Spectral convergence of Riemannian manifolds, II*, Tohoku Math. J. 48 (1996) 71–120 MR1373175

[22] **S Keith**, *A differentiable structure for metric measure spaces*, Adv. Math. 183 (2004) 271–315 MR2041901

[23] **K Kuwae**, **T Shioya**, *Convergence of spectral structures: A functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. 11 (2003) 599–673 MR2015170

[24] **T J Laakso**, *Ahlfors Q–regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality*, Geom. Funct. Anal. 10 (2000) 111–123 MR1748917

[25] **J Lott**, **C Villani**, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. 169 (2009) 903–991 MR2480619

[26] **S-i Ohta**, *On the measure contraction property of metric measure spaces*, Comment. Math. Helv. 82 (2007) 805–828 MR2341840

[27] **Y Otsu**, *Almost everywhere existence of second differentiable structure of Alexandrov spaces*, preprint

[28] **Y Otsu**, **T Shioya**, *The Riemannian structure of Alexandrov spaces*, J. Differential Geom. 39 (1994) 629–658 MR1274133

[29] **P Pansu**, *Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. 129 (1989) 1–60 MR979599

[30] **G Perelman**, *A D Alexandrov spaces with curvature bounded from below, II*, preprint

[31] **G Perelman**, *DC–structure on Alexandrov space* Available at `www.math.psu.edu/petrunin/papers/alexandrov/Cstructure.pdf`

[32] **H Rademacher**, *Über partielle und totale differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale*, Math. Ann. 79 (1919) 340–359 MR1511935

[33] **L Simon**, *Lectures on geometric measure theory*, Proc. Cent. Math. Anal. Austr. Nat. Uni. 3, Australian National University Centre for Mathematical Analysis, Canberra (1983) MR756417

[34] **K-T Sturm**, *On the geometry of metric measure spaces, I*, Acta Math. 196 (2006) 65–131 MR2237206

[35] **K-T Sturm**, *On the geometry of metric measure spaces, II*, Acta Math. 196 (2006) 133–177 MR2237207

[36] **C Villani**, *Optimal transport: Old and new*, Grundl. Math. Wissen. 338, Springer, Berlin (2009) MR2459454

Faculty of Mathmatics, Kyushu University
744, Motooka, Nishi-Ku, Fukuoka 819-0395, Japan

`honda@math.kyushu-u.ac.jp`

Proposed: Dmitri Burago
Seconded: John Lott, Tobias H Colding

Received: 5 March 2013
Accepted: 28 September 2013