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Large scale geometry of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$

XIANGDONG XIE

We classify all negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$ up to quasi-isometry. We show that all quasi-isometries between such manifolds (except when they are bilipschitz to the real hyperbolic spaces) are almost similarities. We prove these results by studying the quasisymmetric maps on the ideal boundary of these manifolds.

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1 Introduction

In this paper we study quasi-isometries between negatively curved solvable Lie groups of the form $\mathbb{R}^n \rtimes \mathbb{R}$ and quasisymmetric maps between their ideal boundaries.

Given an $n \times n$ matrix A, we let G_A be the semidirect product $\mathbb{R}^n \rtimes_A \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^n$ by $x \mapsto e^{tA}x$ for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. Then G_A is a solvable Lie group.

Let G_A be equipped with any left-invariant Riemannian metric such that the $\mathbb{R}$ direction is perpendicular to the $\mathbb{R}^n$ factor. When $A = I_n$, G_A is isometric to $\mathbb{H}^{n+1}$. More generally, if the eigenvalues of A all have positive real parts, then it follows from Heintze's results [12] that G_A is Gromov hyperbolic. Hence G_A has a well-defined ideal boundary ∂G_A. The ideal boundary ∂G_A can be naturally identified with (the one-point compactification of) $\mathbb{R}^n$. On $\mathbb{R}^n$ (identified with the ideal boundary with one point removed), there is a parabolic visual (quasi)metric D_A, which is invariant under Euclidean translations and admits a family of dilations $\{\lambda_t = e^{tA}\}$. See Section 3 for more details.

Given an $n \times n$ matrix A, the *real part Jordan form* of A is obtained from the Jordan form of A by replacing each diagonal entry with its real part and reordering to make it canonical. Notice that the real part Jordan form is different from the real Jordan form and the absolute Jordan form. It is related to the absolute Jordan form through matrix exponential.

Here are the main results of the paper. See Theorem 5.12 for a more precise statement of Theorem 1.2. Also see Section 2 for basic definitions.

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Theorem 1.1 *Let A and B be $n \times n$ matrices whose eigenvalues all have positive real parts. Then $(\mathbb{R}^n, D_A)$ and $(\mathbb{R}^n, D_B)$ are quasisymmetric if and only if there is some $s > 0$ such that A and sB have the same real part Jordan form.*

Theorem 1.2 *Let A and B be $n \times n$ matrices whose eigenvalues all have positive real parts. Denote by λ_1 and μ_1 the smallest real parts of the eigenvalues of A and B respectively, and set $\epsilon = \lambda_1/\mu_1$. If the real part Jordan form of A is not a multiple of the identity matrix I_n, then for every quasisymmetric map $F\colon (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$, the map $F\colon \big(\mathbb{R}^n, D_A^{\epsilon}\big) \to (\mathbb{R}^n, D_B)$ is bilipschitz.*

When $A = I_n$, the manifold G_A is isometric to the real hyperbolic space $\mathbb{H}^{n+1}$. In this case, the ideal boundary is $\mathbb{R}^n$ with the Euclidean metric, and hence the claim in Theorem 1.2 fails: there are nonbilipschitz quasiconformal maps in the Euclidean space $\mathbb{R}^n$. More generally, if the real part Jordan form of A is a multiple of I_n, then it follows from the result of Farb and Mosher (see Section 3) that $(\mathbb{R}^n, D_A)$ is bilipschitz to $(\mathbb{R}^n, |\cdot|^{\epsilon})$, where $|\cdot|$ denotes the Euclidean metric and $\epsilon > 0$ is some constant. Hence the claim in Theorem 1.2 also fails.

There are several consequences of the main results.

Recall that two geodesic Gromov hyperbolic spaces admitting cocompact isometric group actions are quasi-isometric if and only if their ideal boundaries are quasisymmetric with respect to the visual metrics; see Paulin [17] or Bonk and Schramm [2]. Hence Theorem 1.1 yields the quasi-isometric classification of all negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$.

Corollary 1.3 *Let A and B be $n \times n$ matrices whose eigenvalues all have positive real parts. Then G_A and G_B are quasi-isometric if and only if there is some $s > 0$ such that A and sB have the same real part Jordan form.*

The next three results are consequences of Theorem 1.2.

A map $f\colon X \to Y$ between two metric spaces is called an *almost similarity* if there are constants $L > 0$ and $C \geq 0$ such that $Ld(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + C$ for all $x_1, x_2 \in X$ and $d(y, f(X)) \leq C$ for all $y \in Y$.

Corollary 1.4 *Let A and B be $n \times n$ matrices whose eigenvalues all have positive real parts. Suppose the real part Jordan form of A is not a multiple of the identity matrix I_n. Then every quasi-isometry $f\colon G_A \to G_B$ is an almost similarity.*

We view the canonical projection $h_A\colon G_A = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as the height function for G_A. Let A and B be two $n \times n$ matrices. A quasi-isometry $f\colon G_A \to G_B$ is *height-respecting* if it maps the fibers of h_A to within uniformly bounded Hausdorff distance from the fibers of h_B.

Corollary 1.5 *Let A and B be $n \times n$ matrices whose eigenvalues all have positive real parts. Suppose the real part Jordan form of A is not a multiple of the identity matrix I_n. Then every quasi-isometry $f\colon G_A \to G_B$ is height-respecting.*

Corollary 1.6 *Let A be a square matrix whose eigenvalues all have positive real parts. If the real part Jordan form of A is not a multiple of the identity matrix, then G_A is not quasi-isometric to any finitely generated group.*

A group G of bijections $g\colon X \to X$ of a quasimetric space is a *uniform quasimöbius group* if there is some homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that every element g of G is η–quasimöbius. The following result follows from Theorem 1.2 and a theorem of Dymarz and Peng [6].

Corollary 1.7 *Let A be a square matrix whose eigenvalues all have positive real parts. Suppose that the real part Jordan form of A is not a multiple of the identity matrix. Let G be a uniform quasimöbius group of ∂G_A (equipped with a visual metric). If the induced action of G on the space of distinct triples of ∂G_A is cocompact, then G can be conjugated by a bilipschitz map of $(\mathbb{R}^n, D_A)$ into the group of almost homotheties of $(\mathbb{R}^n, D_A)$.*

When A is a Jordan block, we describe all the quasisymmetric maps on $(\mathbb{R}^n, D_A)$. Consequently, we are able to prove a Liouville-type theorem. See Sections 7 and 8.

Theorem 1.2 was established in the diagonal case by Shanmugalingam and the author [20] and in the 2×2 Jordan block case by the author [22]. We believe that Theorem 1.2 holds true for most homogeneous manifolds with negative curvature (HMNs), with only a few exceptions. Recall that HMNs were characterized by Heintze in [12]: each such manifold is isometric to a solvable Lie group S with a left-invariant Riemannian metric and the group S has the form $S = N \rtimes \mathbb{R}$, where N is a simply connected nilpotent Lie group, and the action of $\mathbb{R}$ on N is generated by a derivation whose eigenvalues all have positive real parts. The only exceptions known to the author are (those HMNs that are bilipschitz to) the real and complex hyperbolic spaces: there are quasisymmetric maps in the Euclidean spaces (Gehring and Väisälä [10]) and the Heisenberg groups (Balogh [1]) that change Hausdorff dimensions (of certain subsets), so they can not be bilipschitz.

Our results concern the quasi-isometric rigidity and quasi-isometric classification of negatively curved solvable Lie groups. The first result in this area is Pansu's rigidity theorem [16] for the quarternionic hyperbolic spaces and Cayley plane. Later, by using L^p cohomology, Pansu [14, Corollaries 55, 93] established the quasi-isometric classification theorem for those G_A where A is diagonal. This same result also follows from a theorem of Tyson [21, Theorem 15.3]. All these results belong to the larger project of quasi-isometric rigidity and quasi-isometric classification of focal hyperbolic groups; see Conrulier [3]. Dymarz [5] recently extended Theorem 1.2 to the case of mixed type focal hyperbolic groups.

Recently, Eskin, Fisher and Whyte [7; 8] and Peng [18; 19] proved quasi-isometric rigidity and classification theorems for certain solvable Lie groups that admit lattices but do not admit negative curvature. By contrast, the solvable Lie groups we study in this paper have negative curvature but do not admit lattices (except for rank one symmetric spaces). The approaches taken are also very different. Eskin, Fisher and Whyte and Peng work directly on the solvable Lie groups by using coarse differentiation, while we do analysis on the ideal boundary.

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2 Some basic definitions

In this section we recall some basic definitions.

A *quasimetric* ρ on a set X is a function $\rho\colon X \times X \to \mathbb{R}$ satisfying the following three conditions:

(1) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$.

(2) $\rho(x,y) \geq 0$ for all $x, y \in X$, and $\rho(x,y) = 0$ if and only if $x = y$.

(3) There is some $M \geq 1$ such that $\rho(x,z) \leq M(\rho(x,y) + \rho(y,z))$ for all $x, y, z \in X$.

For each $M \geq 1$, there is a constant $\epsilon_0 > 0$ such that ρ^ϵ is bilipschitz equivalent to a metric for all quasimetric ρ with constant M and all $0 < \epsilon \leq \epsilon_0$; see [11, Proposition 14.5].

For any quadruple $Q = (x,y,z,w)$ of distinct points in a quasimetric space X, the *cross ratio* $\operatorname{cr}(Q)$ of Q is

$$\operatorname{cr}(Q) = \frac{\rho(x,w)\rho(y,z)}{\rho(x,z)\rho(y,w)}.$$

Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A bijection $F\colon X \to Y$ between two quasimetric spaces is *η–quasimöbius* if $\operatorname{cr}(F(Q)) \leq \eta(\operatorname{cr}(Q))$ for all quadruples

$Q = (x, y, z, w)$ of distinct points in X, where $F(Q) = (F(x), F(y), F(z), F(w))$. A bijection $F\colon X \to Y$ between two quasimetric spaces is *η–quasisymmetric* if for all distinct triples $x, y, z \in X$, we have

$$\frac{\rho(F(x), F(y))}{\rho(F(x), F(z))} \le \eta\left(\frac{\rho(x, y)}{\rho(x, z)}\right).$$

A map $F\colon X \to Y$ is quasisymmetric if it is η–quasisymmetric for some η.

Let $K \ge 1$ and $C > 0$. A bijection $F\colon X \to Y$ between two quasimetric spaces is called a *K–quasisimilarity* (with constant C) if

$$\frac{C}{K}\rho(x, y) \le \rho(F(x), F(y)) \le CK\rho(x, y)$$

for all $x, y \in X$. When $K = 1$, we say F is a *similarity*. It is clear that a map is a quasisimilarity if and only if it is a bilipschitz map. The point of using the notion of quasisimilarity is that sometimes there is control on K but not on C.

3 Negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$

In this section we first review some basics about negatively curved $\mathbb{R}^n \times \mathbb{R}$, then define the parabolic visual (quasi)metric on their ideal boundary and study its properties. We also recall a result of Farb and Mosher and the main results of [22] and [20].

3.1 Ideal boundary and parabolic visual quasimetric

Let A be an $n \times n$ matrix. Let $\mathbb{R}$ act on $\mathbb{R}^n$ by

$$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$(t, x) \mapsto e^{tA}x.$$

We denote the corresponding semidirect product by $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$. Then G_A is a solvable Lie group. Recall that the group operation in G_A is given by

$$(x_1, t_1) \cdot (x_2, t_2) = (x_1 + e^{t_1 A}x_2, t_1 + t_2).$$

We will always assume that the eigenvalues of A have positive real parts. An *admissible metric* on G_A is a left-invariant Riemannian metric such that the $\mathbb{R}$ direction is perpendicular to the $\mathbb{R}^n$ factor. The *standard metric* on G_A is the left-invariant Riemannian metric determined by the standard inner product on the tangent space of the identity element $(0, 0) \in \mathbb{R}^n \times \mathbb{R} = G_A$. We remark that G_A with the standard metric does not always have negative sectional curvature. However, Heintze's result [12] says that G_A has an admissible metric with negative sectional curvature. Since any two left-invariant Riemannian distances on a Lie group are bilipschitz equivalent, G_A with any left-invariant Riemannian metric is Gromov hyperbolic. From now on, unless indicated otherwise, G_A will always be equipped with the standard metric.

At a point $(x,t) \in \mathbb{R}^n \times \mathbb{R} \approx G_A$, the tangent space is identified with $\mathbb{R}^n \times \mathbb{R}$, and the standard metric is given by the symmetric matrix

$$\begin{pmatrix} Q_A(t) & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $Q_A(t) = e^{-tA^T} e^{-tA}$. Here T denotes matrix transpose.

For each $x \in \mathbb{R}^n$, the map $\gamma_x\colon \mathbb{R} \to G_A$, $\gamma_x(t) = (x,t)$ is a geodesic. We call such a geodesic a vertical geodesic. It can be checked that all vertical geodesics are asymptotic as $t \to +\infty$. Hence they define a point ξ_0 in the ideal boundary ∂G_A. The sets $\mathbb{R}^n \times \{t\}$ $(t \in \mathbb{R})$ are horospheres centered at ξ_0. For each $t \in \mathbb{R}$, the induced metric on the horosphere $\mathbb{R}^n \times \{t\} \subset G_A$ is determined by the quadratic form $Q_A(t)$. This metric has the distance formula $d_{A,t}((x,t),(y,t)) = |e^{-tA}(x-y)|$. Here $|\cdot|$ denotes the Euclidean norm.

Each geodesic ray in G_A is asymptotic to either an upward-oriented vertical geodesic or a downward-oriented vertical geodesic. The upward-oriented vertical geodesics are asymptotic to ξ_0 and the downward-oriented vertical geodesics are in one-to-one correspondence with $\mathbb{R}^n$. Hence $\partial G_A \backslash \{\xi_0\}$ can be naturally identified with $\mathbb{R}^n$.

We next define a parabolic visual quasimetric on $\partial G_A \backslash \{\xi_0\} = \mathbb{R}^n$. Given $x, y \in \mathbb{R}^n = \partial G_A \backslash \{\xi_0\}$, the parabolic visual quasimetric is defined as $D_A(x,y) = e^t$, where t is the smallest real number such that at height t the two vertical geodesics γ_x and γ_y are at distance one apart in the horosphere; that is,

$$d_{A,t}((x,t),(y,t)) = |e^{-tA}(x-y)| = 1.$$

For each $g = (x,t) \in G_A$, the Lie group left translation L_g is an isometry of G_A and fixes the point ξ_0. It shifts all the horospheres centered at ξ_0 in the vertical direction by the same amount. It follows that the boundary map of L_g is a similarity of $(\mathbb{R}^n, D_A)$. When $g = (z,0)$, L_g leaves invariant all the horospheres centered at ξ_0, and the boundary map is the Euclidean translation by z. Hence Euclidean translations are isometries with respect to D_A,

$$D_A(x+z, y+z) = D_A(x,y) \quad \text{for all } x, y, z \in \mathbb{R}^n.$$

When $g = (0,t)$, L_g shifts all the horospheres centered at ξ_0 by t, and the boundary map is the linear transformation e^{tA}. Hence e^{tA} is a similarity with similarity constant e^t,

$$D_A(e^{tA}x, e^{tA}y) = e^t D_A(x,y) \quad \text{for all } x,y \in \mathbb{R}^n \text{ and all } t \in \mathbb{R}.$$

We remark that D_A in general is not a metric, but merely a quasimetric. See the remark after the proof of Corollary 3.2.

For any integer $n \geq 2$, let

$$J_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

be the $n \times n$ Jordan matrix with eigenvalue 1. We write $J_n = I_n + N$. Here we omit the subscript n for N to simplify the notation. Notice that $e^{-tJ_n} = e^{-tI_n}e^{-tN} = e^{-t}e^{-tN}$. Hence $D_{J_n}(x,y) = e^t$ if and only if t is the smallest real number satisfying $e^t = |e^{-tN}(y-x)|$. For later use, we notice the following:

$$e^{tN} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n-4}}{(n-4)!} & \frac{t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3-1}$$

3.2 Reduction to the real part Jordan form case

Here we state a corollary of a result of Farb and Mosher [9], which implies that the main results in this paper can be reduced to the case when the matrices are already in real part Jordan form.

Let P be a nonsingular $n \times n$ matrix. Define a map $f\colon G_A \to G_{PAP^{-1}}$ by $f(x,t) = (Px,t)$. Then it is easy to check that f is a Lie group isomorphism. Hence f is an isometry if $G_{PAP^{-1}}$ is equipped with the standard metric and G_A has the admissible metric in which $P^{-1}e_1, \dots, P^{-1}e_n, e_{n+1}$ is orthonormal at the identity element of G_A. Here $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$, and e_{n+1} is the standard basis for $\mathbb{R}$. Hence, G_A with any admissible metric is isometric to $G_{PAP^{-1}}$ with the standard metric for some nonsingular matrix P. By Heintze's result [12], there is a nonsingular matrix P such that $G_{PAP^{-1}}$ with the standard metric has negative sectional curvature.

Now we suppose both G_A and $G_{PAP^{-1}}$ are equipped with the standard metric. Then it is easy to check that for each $t\in\mathbb{R}$, the restricted map

$$f|_{\mathbb{R}^n\times\{t\}}\colon (\mathbb{R}^n\times\{t\}, d_{A,t})\to(\mathbb{R}^n\times\{t\}, d_{PAP^{-1},t})$$

is K–bilipschitz, where $K:=\max\{\|P\|,\|P^{-1}\|\}$. Here $\|M\|=\sup\{|Mx|\mid x\in\mathbb{R}^n, |x|=1\}$ denotes the operator norm of an $n\times n$ matrix M. We next recall a more general result by Farb and Mosher [9].

Proposition 3.1 [9, Proposition 4.1] *Let A and B be two $n\times n$ matrices. Suppose there are constants $r,s>0$ such that rA and sB have the same real part Jordan form. Then there is a height-respecting quasi-isometry $f\colon G_A\to G_B$. To be more precise, there exist an $n\times n$ matrix M and $K\geq 1$ such that for each $t\in\mathbb{R}$, the map $v\to Mv$ is a K–bilipschitz homeomorphism from $(\mathbb{R}^n, d_{A,t})$ to $(\mathbb{R}^n, d_{B,(s/r)t})$; it follows that the map $f\colon G_A\to G_B$ given by*

$$(x,t)\longmapsto\left(Mx,\tfrac{s}{r}\cdot t\right)$$

is bilipschitz with bilipschitz constant $\sup\{K,\frac{s}{r},\frac{r}{s}\}$.

Corollary 3.2 *Suppose we are in the setting of Proposition 3.1. Assume further that $r=1$ and G_A has negative sectional curvature. Then:*

(1) *The boundary map $\partial f\colon(\mathbb{R}^n, D_A^s)\to(\mathbb{R}^n, D_B)$ is bilipschitz.*

(2) *f is an almost similarity.*

Proof (1) We observe that the boundary map is given by $\partial f(x)=Mx$. Let $x,y\in\mathbb{R}^n$ and assume $D_A^s(x,y)=e^t$. Then $D_A(x,y)=e^{t/s}$. By the definition of D_A, we have $d_{A,t/s}((x,t/s),(y,t/s))=1$. Since G_A has pinched negative sectional curvature, there is a constant a depending only on the curvature bounds of G_A, such that $d_{A,t'}((x,t'),(y,t'))<1/K$ for $t'>t/s+a$ and $d_{A,t'}((x,t'),(y,t'))>K$ for $t'<t/s-a$. It now follows from Proposition 3.1 that $d_{B,t''}((Mx,t''),(My,t''))<1$ for $t''>t+sa$ and $d_{B,t''}((Mx,t''),(My,t''))>1$ for $t''<t-sa$. By the definition of D_B we have $e^{-sa}e^t\leq D_B(Mx,My)\leq e^{sa}e^t$. Hence $\partial f\colon(\mathbb{R}^n, D_A^s)\to(\mathbb{R}^n, D_B)$ is bilipschitz with bilipschitz constant e^{sa}.

(2) Let $p=(x_1,t_1)$, $q=(x_2,t_2)\in G_A$ be arbitrary. We may assume $t_1\leq t_2$. If $x_1=x_2$, then it is clear from the definition of f that $d(f(p),f(q))=s\cdot d(p,q)$. So we assume $x_1\neq x_2$ and that $d_{A,t_0}((x_1,t_0),(x_2,t_0))=1$ for some t_0. First assume $t_0\leq t_2$. Then $d((x_1,t_2),q)<d_{A,t_2}((x_1,t_2),q)\leq 1$ as G_A has negative sectional curvature. By the triangle inequality, we have $|d(p,q)-(t_2-t_1)|\leq 1$. By Proposition 3.1,

$d((Mx_1, st_2), f(q)) \le d_{B,st_2}((Mx_1, st_2), f(q)) \le K$. By the triangle inequality again we have $|d(f(p), f(q)) - (st_2 - st_1)| \le K$. Hence $|d(f(p), f(q)) - s\cdot d(p,q)| \le s + K$.

Next assume $t_0 > t_2$. By [20, Lemma 6.3 (1)] we have $|d(p,q) - (t_0 - t_1) - (t_0 - t_2)| \le C_1$ for some constant C_1 depending only on the curvature bounds of G_A. By [20, Lemma 6.2], the point (x_1, t_0) is a C_2–quasicenter of $x_1, x_2, \xi_0 \in \partial G_A$ for some constant C_2 depending only on the curvature bounds of G_A. Since f is a quasi-isometry, $f(x_1, t_0) = (Mx_1, st_0)$ is a C_3–quasicenter of $Mx_1, Mx_2, \eta_0 \in \partial G_B$ (here η_0 denotes the point in ∂G_B corresponding to upward-oriented vertical geodesics), where C_3 depends only on C_2, the quasi-isometry constants of f and the Gromov hyperbolicity constant of G_B. Similarly, the point (Mx_2, st_0) is also a C_3–quasicenter of $Mx_1, Mx_2, \eta_0 \in \partial G_B$. Now consider the geodesic triangle consisting of $\{Mx_1\} \times \mathbb{R}$, $\{Mx_2\} \times \mathbb{R}$ and a geodesic joining Mx_1, Mx_2. Notice that $f(p) \in \{Mx_1\} \times \mathbb{R}$ lies between Mx_1 and (Mx_1, st_0) and $f(q) \in \{Mx_2\} \times \mathbb{R}$ lies between Mx_2 and (Mx_2, st_0). It follows that

$$\begin{aligned}
&|d(f(p), f(q)) - (st_0 - st_1) - (st_0 - st_2)| \\
&\qquad = |d(f(p), f(q)) - d(f(p), (Mx_1, st_0)) - d(f(q), (Mx_2, st_0))| \le C_4
\end{aligned}$$

for some constant C_4 depending only on C_3 and the Gromov hyperbolicity constant of G_B. This combined with $|d(p,q) - (t_0 - t_1) - (t_0 - t_2)| \le C_1$ implies $|d(f(p), f(q)) - s\cdot d(p,q)| \le C_4 + sC_1$. □

We notice that Corollary 3.2 (1) implies that D_A is indeed a quasimetric: by Heintze's result, there is some nonsingular P such that $G_{PAP^{-1}}$ has pinched negative sectional curvature and hence $D_{PAP^{-1}}$ is a quasimetric (this can be proved by the arguments of Coornaert, Delzant and Papadopoulos [4, page 124] or by using the relation between parabolic visual quasimetric and visual quasimetric [20, Section 5]); since $(\mathbb{R}^n, D_A)$ and $(\mathbb{R}^n, D_{PAP^{-1}})$ are bilipschitz, D_A is also a quasimetric.

3.3 Distance between certain subsets

In this subsection we show that certain subsets of $(\mathbb{R}^n, D_A)$ are "parallel." These results will be used in Section 5.

Let A be an $n \times n$ matrix in real part Jordan form with positive eigenvalues

$$\lambda_1 < \cdots < \lambda_{k_A}.$$

Let $V_i \subset \mathbb{R}^n$ be the generalized eigenspace of λ_i, and let $d_i = \dim V_i$.

If $k := k_A \ge 2$, we write A in the block diagonal form $A = [A_1, \dots, A_k]$, where A_i is the block corresponding to the eigenvalue λ_i; we also denote $A' = [A_1, \dots, A_{k-1}]$.

Correspondingly, $\mathbb{R}^n$ admits the decomposition $\mathbb{R}^n = V_1 \times \cdots \times V_k$. Hence each point $x \in \mathbb{R}^n$ can be written $x = (x_1, \dots, x_k)$, where $x_i \in V_i$. Observe that, for each $x_k \in V_k$, if we identify $V_1 \times \cdots \times V_{k-1} \times \{x_k\}$ with $V_1 \times \cdots \times V_{k-1}$, then the restriction of D_A to $V_1 \times \cdots \times V_{k-1} \times \{x_k\}$ agrees with $D_{A'}$. It is not hard to check that for all $x_k, y_k \in V_k$, the following holds for the distance with respect to the quasimetric D_A:

$$D_A(V_1 \times \cdots \times V_{k-1} \times \{x_k\}, V_1 \times \cdots \times V_{k-1} \times \{y_k\}) = D_{A_k}(x_k, y_k) \tag{3-2}$$

Also, for any $x = (x_1, \dots, x_k) \in \mathbb{R}^n$ and any $y_k \in V_k$,

$$D_A(x, V_1 \times \cdots \times V_{k-1} \times \{y_k\}) = D_{A_k}(x_k, y_k). \tag{3-3}$$

When $k = 1$, that is, when A has only one eigenvalue $\lambda := \lambda_1 > 0$, the matrix A also has a block diagonal form $A = [\lambda I_{n_0}, \lambda I_{n_1} + N, \dots, \lambda I_{n_r} + N]$, where $n_0 \geq 0$ and $\lambda I_{n_i} + N$ is a Jordan block. We allow the case $A = \lambda I_n$. We write a point $p \in \mathbb{R}^n$ as $p = (z, (x_1, y_1), \dots, (x_r, y_r))^T$, where T denotes matrix transpose, $z \in \mathbb{R}^{n_0}$ corresponds to λI_{n_0} and $(x_i, y_i)^T \in \mathbb{R}^{n_i}$ ($x_i \in \mathbb{R}^{n_i - 1}$, $y_i \in \mathbb{R}$) corresponds to $\lambda I_{n_i} + N$. Let $\pi_A \colon \mathbb{R}^n \to \mathbb{R}^{n_0 + r}$ be the projection given by

$$\pi_A(p) = (z, y_1, \dots, y_r)^T \quad \text{for } p = (z, (x_1, y_1), \dots, (x_r, y_r))^T \in \mathbb{R}^n.$$

Set

$$A(1) = [\lambda I_{n_1 - 1} + N, \dots, \lambda I_{n_r - 1} + N],$$

where $\lambda I_1 + N$ is understood to be λI_1.

Lemma 3.3 *The restriction of D_A to the fibers of π_A agrees with $D_{A(1)}$. To be more precise, for all $p = (z, (x_1, y_1), \dots, (x_r, y_r))^T$, $p' = (z, (x_1', y_1), \dots, (x_r', y_r))^T$ we have*

$$D_A(p, p') = D_{A(1)}(x, x'),$$

where $x = (x_1, \dots, x_r)^T$ and $x' = (x_1', \dots, x_r')^T$.

Proof Assume $D_A(p, p') = e^t$ and $D_{A(1)}(x, x') = e^s$. By the definition, s is the smallest real number such that $|e^{-sA(1)}(x' - x)| = 1$. We calculate

$$e^{-sA(1)}(x' - x) = e^{-\lambda s}(e^{-sN_{n_1 - 1}}(x_1' - x_1), \dots, e^{-sN_{n_r - 1}}(x_r' - x_r))^T.$$

Similarly, t is the smallest real number such that $|e^{-tA}(p' - p)| = 1$. We calculate

$$e^{-tA}(p' - p) = e^{-\lambda t}\big(\mathbf{0}, (e^{-tN_{n_1 - 1}}(x_1' - x_1), 0), \dots, (e^{-tN_{n_r - 1}}(x_r' - x_r), 0)\big)^T.$$

It follows that the two equations $|e^{-sA(1)}(x' - x)| = 1$ and $|e^{-tA}(p' - p)| = 1$ are the same. Hence $s = t$. □

Lemma 3.4 *The following hold for all $y, y' \in \mathbb{R}^{n_0+r}$:*

(1) $D_A(\pi_A^{-1}(y), \pi_A^{-1}(y')) = |y-y'|^{1/\lambda}$

(2) *For any $p \in \pi_A^{-1}(y)$, we have $D_A(p, \pi_A^{-1}(y')) = |y-y'|^{1/\lambda}$*

Proof Let

$$p = (z, (x_1, y_1), \dots, (x_r, y_r))^T \in \pi_A^{-1}(y),$$
$$p' = (z', (x_1', y_1'), \dots, (x_r', y_r'))^T \in \pi_A^{-1}(y'),$$

where y and y' are written $y = (z, y_1, \dots, y_r)$, $y' = (z', y_1', \dots, y_r')$. Assume $D_A(p, p') = e^t$. Then t is the smallest real number such that

$$\left|\left(z'-z, e^{-tN_{n_1}}(x_1'-x_1, y_1'-y_1)^T, \dots, e^{-tN_{n_r}}(x_r'-x_r, y_r'-y_r)^T\right)\right| = e^{\lambda t}.$$

Notice that the last entry of $e^{-tN_{n_i}}(x_i'-x_i, y_i'-y_i)^T$ is $y_i'-y_i$, which is independent of t. It follows that $e^{\lambda t} \geq |(z'-z, y_1'-y_1, \dots, y_r'-y_r)| = |y'-y|$, and hence $D_A(p, p') = e^t \geq |y'-y|^{1/\lambda}$.

Set $t_0 = \ln|y'-y|/\lambda$. Then $e^{\lambda t_0} = |y'-y|$. Now let $p = (z, (x_1, y_1), \dots, (x_r, y_r))^T \in \pi_A^{-1}(y)$ be arbitrary. Since the matrix $e^{-t_0 N_{n_i}}$ is nonsingular, the equation

$$e^{-t_0 N_{n_i}}(u_i, v_i)^T = (0, \dots, 0, y_i'-y_i)^T$$

has a unique solution $(u_i, v_i)^T$, where $u_i \in \mathbb{R}^{n_i-1}$ and $v_i \in \mathbb{R}$. Notice that $v_i = y_i'-y_i$. Set $x_i' = u_i + x_i$ and $p' = (z', (x_1', y_1'), \dots, (x_r', y_r'))^T$. Then $p' \in \pi_A^{-1}(y')$ and

$$e^{-t_0 A}(p'-p) = e^{-t_0\lambda}(z'-z, (\mathbf{0}, y_1'-y_1), \dots, (\mathbf{0}, y_r'-y_r))^T.$$

It follows that t_0 is a solution of $|e^{-tA}(p'-p)| = 1$ and so $D_A(p, p') \leq e^{t_0} = |y-y'|^{1/\lambda}$. This together with the first paragraph implies $D_A(p, p') = |y-y'|^{1/\lambda}$. So each point $p \in \pi_A^{-1}(y)$ is within $|y-y'|^{1/\lambda}$ of $\pi_A^{-1}(y')$. Similarly, every point $p' \in \pi_A^{-1}(y')$ is also within $|y-y'|^{1/\lambda}$ of $\pi_A^{-1}(y)$. Therefore, (1) holds.

Part (2) also follows from the above two paragraphs. □

3.4 Previous results

The following two results will be used in the proof of Theorems 1.1 and 1.2. They are the basic steps in the induction.

Theorem 3.5 [20, Theorem 4.1] *Suppose A is diagonal with positive eigenvalues $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ ($r \geq 2$). Then every η–quasisymmetry $F\colon (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_A)$ is a K–quasisimilarity, where K depends only on η and r.*

Theorem 3.6 [22, Theorems 4.1, 5.1] *Every η–quasisymmetric map $F\colon (\mathbb{R}^2, D_{J_2}) \to (\mathbb{R}^2, D_{J_2})$ is a K–quasisimilarity, where K depends only on η. Furthermore, a bijection $F\colon (\mathbb{R}^2, D_{J_2}) \to (\mathbb{R}^2, D_{J_2})$ is a quasisymmetric map if and only of it has the following form: $F(x,y) = (ax + c(y), ay + b)$ for all $(x,y) \in \mathbb{R}^2$, where $a \neq 0$, b are constants and $c\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz map.*

4 Q–variation on the ideal boundary

In this section we introduce the main tool in the proof of the main results: Q–variation for maps between quasimetric spaces. It is a discrete version of the notion of capacity. The advantage of this notion is that it makes sense for quasimetric spaces and does not require the existence of rectifiable curves. We remark that, while dealing with ideal boundary of negatively curved spaces, very often either one has to work with quasimetric spaces in which the triangle inequality is not available, or one needs to work with metric spaces that contain no rectifiable curves. Both scenarios are unpleasant from the point of view of classical quasiconformal analysis.

The notion of Q–variation is due to Bruce Kleiner [13].

Let (X, ρ) be a quasimetric space and $L \geq 1$. A subset $A \subset X$ is called an *L–quasiball* if there is some $x \in X$ and some $r > 0$ such that $B(x,r) \subset A \subset B(x, Lr)$. Here $B(x,r) := \{y \in X \mid \rho(y,x) < r\}$.

For any ball $B := B(x,r)$ and any $\kappa > 0$, we sometimes denote $B(x, \kappa r)$ by κB.

For a subset E of a quasimetric space (Y, ρ), the ρ–diameter of E is

$$\mathrm{diam}_\rho(E) := \sup\{\rho(e_1, e_2) \mid e_1, e_2 \in E\}.$$

Let $u\colon (X, \rho_1) \to (Y, \rho_2)$ be a map between two quasimetric spaces. For any subset $A \subset X$, the oscillation of u over A is

$$\mathrm{osc}(u|_A) = \mathrm{diam}_{\rho_2}(u(A)).$$

Let $Q \geq 1$. For a collection of disjoint subsets $\mathcal{A} = \{A_i\}$ of X, the *Q–variation of u over $\mathcal{A}$*, denoted by $V_Q(u, \mathcal{A})$, is the quantity

$$\sum_i [\mathrm{osc}(u|_{A_i})]^Q.$$

For $\delta > 0$ and $K \geq 1$, set

$$V^\delta_{Q,K}(u) = \sup\{V_Q(u, \mathcal{A})\},$$

where $\mathcal{A}$ ranges over all disjoint collections of K–quasiballs in (X, ρ_1) with ρ_1–diameter at most δ. Finally, the *(Q, K)–variation $V_{Q,K}(u)$ of u* is

$$V_{Q,K}(u) = \lim_{\delta \to 0} V^{\delta}_{Q,K}(u).$$

We notice that $V_{Q,K}(u|_{E_1}) \le V_{Q,K}(u|_{E_2})$ whenever $E_1 \subset E_2 \subset X$.

There are useful variants of this definition, for instance one can look at the infimum over all coverings. Or one can take the infimum over all coverings followed by the sup as the mesh size tends to zero. As a tool, Q–variation could be compared with Pansu's modulus [15], but seems slightly easier to work with in our context.

Since quasisymmetric maps send quasiballs to quasiballs quantitatively, it is easy to see that Q–variation is a quasisymmetric invariant. To be more precise, we recall the following lemma.

Lemma 4.1 [22, Lemma 3.1] *Let X be a bounded quasimetric space and $F\colon X \to Z$ an η–quasisymmetric map. Then for every map $u\colon X \to Y$ we have $V_{Q,K}(u) \le V_{Q,\eta(K)}(u \circ F^{-1})$.*

We next calculate the Q–variation of certain functions defined on the ideal boundary of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$. These calculations will be used in the next section to show that certain foliations on the ideal boundary are preserved by quasisymmetric maps.

For later use we recall that, for any $Q > 1$, any integer $k \ge 1$ and any nonnegative numbers $a_1, \dots, a_k$, Jensen's inequality states

$$\frac{\sum_{i=1}^k a_i^Q}{k} \ge \left(\frac{\sum_{i=1}^k a_i}{k} \right)^Q,$$

and equality holds if and only if all the a_i are equal. In our applications, the a_i will be the oscillations of a function u along a "stack" of quasiballs.

Let A be an $n \times n$ matrix in real part Jordan form with positive eigenvalues

$$\lambda_1 < \lambda_2 < \dots < \lambda_k,$$

let $V_i \subset \mathbb{R}^n$ be the generalized eigenspace of λ_i, and let $d_i = \dim V_i$. Then $\mathbb{R}^n$ admits the decomposition: $\mathbb{R}^n = V_1 \times \cdots \times V_k$. Since e^{tA} is a linear transformation with $\det(e^{tA}) = e^{t(\sum_i d_i \lambda_i)}$, for any subset $U \subset \mathbb{R}^n$, we have $\mathrm{Vol}(e^{tA}(U)) = e^{t(\sum_i d_i \lambda_i)} \mathrm{Vol}(U)$.

There are constants C_1, C_2, C_3 depending only on the dimension n with the following properties. If $B := B(o, 1) \subset \mathbb{R}^n$ is the unit ball (in the Euclidean metric), and $t \le -1$, then

$$B(o, C_1 e^{t\lambda_k}|t|^{-n+1}) \subset e^{tA}B \subset B(o, C_2 e^{t\lambda_1}|t|^{n-1}), \tag{4-1}$$

while

$$\mathrm{Vol}(e^{tA}B) = C_3 e^{t(\sum_i d_i\lambda_i)}. \tag{4-2}$$

Let $S = \prod_{i=1}^n [0,1] \subset \mathbb{R}^n$ be the unit cube. We notice that both S and B are K_0–quasiballs with respect to D_A for some K_0 depending only on A. Hence there is some $r > 0$ such that $B_A(o,r) \subset B \subset B_A(o, K_0 r)$. Here the subscript A refers to D_A. Also recall that D_A is a quasimetric: there is a constant $M \geq 1$ such that $D_A(x,z) \leq M(D_A(x,y) + D_A(y,z))$ for all $x, y, z \in \mathbb{R}^n$.

In the following, when we say a subset $E \subset \mathbb{R}^n$ is convex, we mean it is convex with respect to the Euclidean metric. The continuity of a function $u\colon E \to \mathbb{R}$ is with respect to the topology induced from the usual topology on $\mathbb{R}^n$.

Lemma 4.2 *Let $E \subset \mathbb{R}^n$ be a convex open subset. If $u\colon (E, D_A) \to \mathbb{R}$ is a nonconstant continuous function, then $V_{Q,K}(u) = \infty$ for all $Q < \left(\sum_i d_i\lambda_i\right)/\lambda_k$ and all $K \geq K_0$.*

Proof Let $p, q \in E$ with $u(p) \neq u(q)$. Let $C \subset E$ be a fixed cylinder with axis $\overline{pq}$, such that the minimum of u on one cap of C is strictly greater than its maximum on the other cap. We pack C with translates of $e^{tA}B$, for $t \ll 0$, as follows. First pick a maximal set of lines $\mathcal{L} = \{L_j\}$ in $\mathbb{R}^n$ satisfying the following conditions:

(1) Each line is parallel to $\overline{pq}$.

(2) Each line intersects C.

(3) The Hausdorff distance (with respect to D_A) between any two of the lines is at least $2MK_0re^t$.

The maximality implies that for each $x \in C$, we have $D_A(x, L_j) \leq 2MK_0re^t$ for some j. For each j, consider a translate B_j of $e^{tA}B$ centered at some point on L_j. Then we move B_j along L_j (in both directions) by translations until the translates just touch B_j. Repeat this and we obtain a "stack" of K_0–quasiballs centered on L_j. Do this for each j and we obtain a packing $\mathcal{P} = \{P\}$ of C by translates of $e^{tA}B$, after removing those that are disjoint from C.

We claim that the collection $\mathcal{P}$ covers a fixed fraction of the volume of C. To see this, first notice that the D_A–distance between the centers x_1, x_2 of two consecutive K_0–quasiballs along L_j is at most $M(K_0re^t + K_0re^t) = 2MK_0re^t$, due to the generalized triangle inequality for D_A. Assume $D_A(x_1, x_2) = e^s$. Then

$$e^{(\ln r - s)A}(x_2 - x_1) \in e^{(\ln r - s)A}\,\overline{B}_A(o, e^s) = \overline{B}_A(o, r) \subset \overline{B} \subset \overline{B}_A(o, K_0 r).$$

Since $\bar{B}$ is convex, the line segment joining o and $e^{(\ln r - s)A}(x_2 - x_1)$ is contained in $\bar{B} \subset \bar{B}_A(o, K_0 r)$. It follows that the segment joining o and $x_2 - x_1$ lies in $e^{(s - \ln r)A}\bar{B}_A(o, K_0 r) = \bar{B}_A(o, K_0 e^s)$. Hence $\overline{x_1 x_2} \subset \bar{B}_A(x_1, K_0 e^s)$. This shows that every point on $L_j \cap C$ is within $K_0 e^s \le K_1 := 2MK_0^2 r e^t$ of the center of some $P \in \mathcal{P}$. Now the choice of the lines $\{L_j\}$ and the generalized triangle inequality for D_A imply that C is covered by D_A–balls with radius $K_2 := M(2MK_0 r e^t + K_1)$ and centers at the centers of $\{P\}$. Since the volumes of $e^{tA}B$ and $B_A(o, K_2)$ are comparable, the claim follows.

The number of K_0–quasiballs in $\mathcal{P}$ along each line L_j is $\lesssim e^{-t\lambda_k}|t|^{n-1}$ in view of the estimate (4-1). By Jensen's inequality, the Q–variation of u for the K_0–quasiballs along L_j is at least as large as the Q–variation when the oscillations of u on these quasiballs are equal. This common oscillation is $\gtrsim e^{t\lambda_k}|t|^{-n+1}$. Since $\mathcal{P}$ covers a fixed fraction of C, the cardinality of $\mathcal{P}$ is $\gtrsim e^{-t(\sum_i d_i\lambda_i)}$. Hence the Q–variation of u on $\mathcal{P}$ is

$$\gtrsim e^{-t(\sum_i d_i\lambda_i)}(e^{t\lambda_k}|t|^{-n+1})^Q = e^{t(Q\lambda_k - \sum_i d_i\lambda_i)}|t|^{(-n+1)Q},$$

which tends to ∞ as $t \to -\infty$ for $Q < \left(\sum_i d_i\lambda_i\right)/\lambda_k$. Hence $V_{Q,K}(u) = \infty$. □

Notice that $\left(\sum_i d_i\lambda_i\right)/\lambda_k < n$ if $k \ge 2$ and $\left(\sum_i d_i\lambda_i\right)/\lambda_k = n$ if $k = 1$. Hence we have the following corollary.

Corollary 4.3 *Suppose $k=1$. Let $E \subset \mathbb{R}^n$ be a convex open subset. If $u\colon (E, D_A) \to \mathbb{R}$ is a nonconstant continuous function, then $V_{Q,K}(u) = \infty$ for all $Q < n$ and all $K \ge K_0$.*

Lemma 4.4 *Let $E \subset \mathbb{R}^n$ be a convex open subset. Let $u\colon (E, D_A) \to \mathbb{R}$ be a continuous function. Suppose there is an affine subspace W parallel to the subspace $\prod_{i \le l} V_i$ such that $u|_{W \cap E}$ is not constant. Then $V_{Q,K}(u) = \infty$ for all $Q < \left(\sum_i d_i\lambda_i\right)/\lambda_l$ and all $K \ge K_0$.*

Proof Note that in the proof of Lemma 4.2, if $\overline{pq}$ is parallel to the subspace $\prod_{i \le l} V_i$, then the number of quasiballs in $\mathcal{P}$ along a line L_j is $\lesssim e^{-t\lambda_l}|t|^{n-1}$, so the lower bound on Q–variation becomes

$$Ce^{t(Q\lambda_l - \sum_i d_i\lambda_i)}|t|^{(-n+1)Q},$$

which tends to ∞ as $t \to -\infty$ if $Q < \left(\sum_i d_i\lambda_i\right)/\lambda_l$. □

Let $\pi\colon \mathbb{R}^n = V_1 \times \cdots \times V_k \to V_k$ be the natural projection.

Lemma 4.5 *Let $\pi'\colon V_k \to \mathbb{R}$ be a coordinate function on V_k, and set $u = \pi' \circ \pi$. Then $V_{Q,K}(u|_E) = 0$ for all $Q > \big(\sum_i d_i\lambda_i\big)/\lambda_k$, all $K \geq K_0$ and all bounded subsets $E \subset \mathbb{R}^n$.*

Proof Let E be a bounded open subset. Let $0 < \delta \ll 1$ and $\{B_j\}_{j\in I}$ be a packing of E by K–quasiballs with size less than δ. Then for each j there is some $x_j \in \mathbb{R}^n$ and some t_j such that

$$B_A(x_j, e^{t_j}) \subset B_j \subset B_A(x_j, Ke^{t_j}).$$

Since $B_A(o,r) \subset B \subset B_A(o, K_0 r)$, we have $e^{t'_j A}B \subset B_A(o, e^{t_j})$ and $B_A(o, Ke^{t_j}) \subset e^{t''_j A}B$, where $t'_j = t_j - \ln r - \ln K_0$ and where $t''_j = t_j - \ln r + \ln K$. Set $B'_j = x_j + e^{t'_j A}B$ and $B''_j = x_j + e^{t''_j A}B$. Then $B'_j \subset B_j \subset B''_j$. It follows that

$$\operatorname{osc}(u|_{B_j}) \leq \operatorname{osc}(u|_{B''_j}) \lesssim e^{t''_j\lambda_k}|t''_j|^{d_k-1},$$
$$(\operatorname{osc}(u|_{B_j}))^Q \lesssim e^{t''_j(Q\lambda_k)}|t''_j|^{Q(d_k-1)} \lesssim e^{t'_j(Q\lambda_k)}|t'_j|^{Q(d_k-1)}.$$

If we have $Q > \big(\sum_i d_i\lambda_i\big)/\lambda_k$, then this will be $\lesssim (\operatorname{Vol}(B'_j))^s \leq (\operatorname{Vol}(B_j))^s$ for $s = \big(Q\lambda_k + \sum_i d_i\lambda_i\big)/\big(2\sum_i d_i\lambda_i\big) > 1$, which implies that the Q–variation is zero. □

For the rest of this section, we will assume $k = 1$ and use the notation introduced before Lemma 3.3.

Lemma 4.6 *Suppose A has only one eigenvalue $\lambda > 0$. Let $\pi'\colon \mathbb{R}^{n_0+r} \to \mathbb{R}$ be a coordinate function and $u = \pi' \circ \pi_A$. Then for any bounded open subset E:*

(1) *$V_{Q,K}(u|_E) = 0$ for all $Q > n$ and all $K \geq K_0$*

(2) *$0 < V_{n,K}(u|_E) < \infty$ for all $K \geq K_0$*

Proof Let P be a K–quasiball. Then there is a D_A–ball U with $U \subset P \subset KU$. Let $t_0 = \ln(KK_0)$. For some $t \in \mathbb{R}$ there is a translate $S(t)$ of $e^{tA}S$ and a translate $S(t+t_0)$ of $e^{(t+t_0)A}S$ such that $(1/K_0)U \subset S(t) \subset U$ and $KU \subset S(t+t_0) \subset KK_0U$. Observe that for any translate S' of $e^{tA}S$, we have $\operatorname{osc}(u|_{S'}) = e^{\lambda t}$. It follows that

$$\operatorname{osc}(u|_P) \geq \operatorname{osc}(u|_{S(t)}) = \frac{1}{(KK_0)^\lambda}\operatorname{osc}(u|_{S(t+t_0)}) \geq \frac{1}{(KK_0)^\lambda}\operatorname{osc}(u|_P).$$

Also notice that $\operatorname{osc}(u|_{S(t)}) = (\operatorname{Vol}(S(t)))^{1/n} \leq (\operatorname{Vol}(P))^{1/n}$.

Now let E be a bounded open subset and $\{P_i\}$ a packing of E by a disjoint collection of K–quasiballs with size less than δ. For each P_i, let U_i be a D_A–ball with $U_i \subset$

$P_i \subset KU_i$ and let S_i be a translate of some $e^{t_i A}S$ with $(1/K_0)U_i \subset S_i \subset U_i$. Then the preceding paragraph implies

$$\sum_i \mathrm{osc}(u|_{P_i})^Q \le (KK_0)^{Q\lambda} \sum_i \mathrm{osc}(u|_{S_i})^Q \le (KK_0)^{Q\lambda} \sum_i \mathrm{Vol}(P_i)^{Q/n}.$$

From this it is clear that $V_{Q,K}(u|_E) = 0$ if $Q > n$ and $V_{n,K}(u|_E) < \infty$ since $\{P_i\}$ is a disjoint collection in E.

Now consider a particular packing $\{P_i\}$ of E by the images of the integral unit cubes in $\mathbb{R}^n$ under e^{tA}. Then $\mathrm{osc}(u|_{P_i}) = e^{\lambda t}$. The cardinality of $\{P_i\}$ is approximately $\mathrm{Vol}(E)/e^{n\lambda t}$. Hence $V^{\delta}_{n,K}(u|_E) \ge \sum_i \mathrm{osc}(u|_{P_i})^n \approx \mathrm{Vol}(E)$. Hence we have $0 < V_{n,K}(u|_E) < \infty$. □

Lemma 4.7 *Suppose A has only one eigenvalue $\lambda > 0$. Let $E \subset \mathbb{R}^n$ be a rectangular box whose edges are parallel to the coordinate axes. Let $u\colon (E, D_A) \to \mathbb{R}$ be a continuous function. Suppose there is some fiber H of $\pi_A\colon \mathbb{R}^n \to \mathbb{R}^{n_0+r}$ such that $u|_{H\cap E}$ is not constant. Then $V_{Q,K}(u) = \infty$ for all $Q \le n$ and all $K \ge K_0$.*

Proof Suppose there is some fiber H of π_A such that $u|_{H\cap E}$ is not constant. Then there is some Jordan block J in A with the following property: if we denote by $x = (x_1, \dots, x_m)$ the coordinates corresponding to J, then there is some index k, $1 \le k \le m-1$, such that u is constant along every line parallel to the x_j–axis for $j \le k-1$, but is not constant along some line L parallel to the x_k–axis. We write $\mathbb{R}^n = \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k}$, where the $\mathbb{R}$ corresponds to the x_k–axis and the $\mathbb{R}^{k-1}$ is spanned by the x_j–axes ($j \le k-1$). After composing u with an affine function, we may assume that for some rectangular box $C = \prod_{i=1}^n [a_i, b_i] \subset E$, we have $u \le 0$ on the codimension-1 face $F_0 := \{x \in C \mid x_k = a_k\}$ of C and $u \ge 1$ on the codimension-1 face $F_1 := \{x \in C \mid x_k = b_k\}$ of C. We will induct on k.

Recall that for a Jordan block $J = \lambda I_m + N$, we have $e^{tJ} = e^{\lambda t} e^{tN}$. See (3-1) for an expression of e^{tN}.

We first assume $k = 1$. For $t \ll 0$, consider the images of the integral unit cubes under e^{tA}. Let $\{B_i\}$ be the collection of all those images that intersect the box C. Notice that a vertical stack (ie parallel to the x_m–axis) of integral cubes is mapped by e^{tA} to a sequences of K_0–quasiballs which is almost parallel to the x_1–axis. We divide $\{B_i\}$ into such sequences which join F_0 and F_1. Note that the projection of each B_i to the x_1–axis has length comparable to $e^{\lambda t}|t|^{m-1}$. Hence the cardinality of each sequence is comparable to $e^{-\lambda t}|t|^{1-m}$. The Q–variation of u along each sequence is at least the Q–variation of u when oscillations of u on the members of the sequence are equal. Since $u \le 0$ on F_0 and $u \ge 1$ on F_1, this common oscillation is at least comparable to $e^{\lambda t}|t|^{m-1}$. Since each B_i has volume $e^{n\lambda t}$, the cardinality of $\{B_i\}$ is comparable to $e^{-n\lambda t}$. It follows that the Q–variation of $u|_C$ is at least comparable to

$$e^{-n\lambda t}\cdot(e^{\lambda t}|t|^{m-1})^Q=e^{\lambda t(Q-n)}|t|^{Q(m-1)},$$

which tends to ∞ as $t\to-\infty$ if $Q\le n$. Hence $V_{Q,K}(u|_C)=\infty$ for $Q\le n$.

Now we assume $m-1\ge k\ge 2$. Then u is constant along affine subspaces parallel to $\mathbb{R}^{k-1}\times\{0\}\times\{0\}\subset\mathbb{R}^n$. Let

$$U=\{x\in F_0\mid (3a_i+b_i)/4\le x_i\le (a_i+3b_i)/4 \text{ for all } i\ne k\}\subset F_0.$$

For $t\ll 0$, denote by

$$v(t)=(-1)^{m-k}e^{-\lambda t}e^{tA}\vec{e}_m,$$

where $\vec{e}_m$ is the m^{th} vector in the standard basis for $\mathbb{R}^n$. Notice that the components of $v(t)$ corresponding to the Jordan block J is

$$(-1)^{m-k}\left(\frac{t^{m-1}}{(m-1)!},\frac{t^{m-2}}{(m-2)!},\dots,t,1\right)$$

and all other components are 0. Hence for $t\ll 0$, lines parallel to $v(t)$ travel much faster in the x_i $(1\le i\le m-1)$ direction than in the x_{i+1} direction. Let $Z\subset\mathbb{R}^n$ be the subset given by

$$Z=\left\{f+sv(t)\;\middle|\; f\in U,\ 0\le s\le\frac{(m-k)!}{|t|^{m-k}}(b_k-a_k)\right\}.$$

Note that for each fixed f, the segment $\{f+sv(t)\mid 0\le s\le((m-k)!/|t|^{m-k})(b_k-a_k)\}$ joins the two hyperplanes $x_k=a_k$ and $x_k=b_k$. Also notice that these segments are parallel to the images of vertical stacks (ie, parallel to the x_m–axis) of integral cubes under e^{tA}. Hence Z has a packing $\mathcal{P}$ that can be divided into sequences such that each sequence joins $x_k=a_k$ and $x_k=b_k$ and is the image (under e^{tA}) of a vertical stack of integral cubes.

For $p=(x,y,z)$, $q=(x',y',z')\in\mathbb{R}^{k-1}\times\mathbb{R}\times\mathbb{R}^{n-k}$, define $p\sim q$ if $y'=y$, $z'=z$ and x'_i-x_i is an integral multiple of b_i-a_i for $1\le i\le k-1$. Set $Y=\mathbb{R}^n/\sim$ and let $\pi\colon\mathbb{R}^n\to Y$ be the natural projection. Also let $\pi_C\colon C\to Y$ be the composition of the inclusion $C\subset\mathbb{R}^n$ and π. It is clear that π_C is injective on the interior of C. It is also easy to check that $\pi|_Z$ is injective. Now the packing $\mathcal{P}$ of Z projects onto a packing of Y, which can then be pulled back through π_C to obtain a packing $\mathcal{P}'$ of C (since $\pi(Z)\subset\pi_C(C)$). A sequence in $\mathcal{P}$ gives rise to a broken sequence in $\mathcal{P}'$: the broken sequence will first hit the boundary of C at a point of $\partial\big(\prod_{i=1}^{k-1}[a_i,b_i]\big)\times\prod_k^n[a_i,b_i]\subset\partial C$, it continues after a translation by an element of the form $\left(\sum_{i=1}^{k-1} m_i(b_i - a_i), 0, 0\right) \in \mathbb{R}^n$, where $m_i \in \mathbb{Z}$; this can be repeated until the sequence hits $x_k = b_k$. Note that we can apply Jensen's inequality to each broken sequence while considering Q–variations of u since by assumption u is constant along affine spaces parallel to $\mathbb{R}^{k-1} \times \{0\} \times \{0\}$.

Each broken sequence joins F_0 to F_1. Since the projection of $e^{tA}S$ to the x_k–axis has length comparable to $e^{\lambda t}|t|^{m-k}$, the cardinality of each sequence is comparable to $e^{-\lambda t}|t|^{k-m}$. The Q–variation of u along the sequence is at least the Q–variation when the oscillations of u are the same on all members of the sequence. The common oscillation is at least comparable to $e^{\lambda t}|t|^{m-k}$. Hence the Q–variation of u is at least comparable to

$$\frac{1}{e^{n\lambda t}} \cdot (e^{\lambda t}|t|^{m-k})^Q = |t|^{Q(m-k)} e^{(Q-n)\lambda t},$$

which tends to ∞ when $t \to -\infty$ if $Q \le n$. Hence $V_{Q,K}(u|_C) = \infty$ for $Q \le n$. □

5 Proof of the main theorems

In this section we prove the main results of the paper. The main tools are the notion of Q–variation (Section 4) and the arguments from [22, Section 4] and [20]. The main results of [20] and [22] are the basic steps in the induction.

We first fix the notation. Let A be an $n \times n$ matrix in real part Jordan form with positive eigenvalues

$$\lambda_1 < \cdots < \lambda_{k_A}.$$

Let $V_i \subset \mathbb{R}^n$ be the generalized eigenspace of λ_i, and set $d_i = \dim V_i$. If $k_A \ge 2$, we write A in the block diagonal form $A = [A_1, \ldots, A_{k_A}]$, where A_i is the block corresponding to the eigenvalue λ_i; we also denote $A' = [A_1, \ldots, A_{k_A - 1}]$. If $k_A = 1$, that is, if A has only one eigenvalue $\lambda = \lambda_1$, we also write $A = [\lambda I_{n_0}, \lambda I_{n_1} + N, \ldots, \lambda I_{n_r} + N]$ in the block diagonal form, and we let $\pi_A \colon \mathbb{R}^n \to \mathbb{R}^{n_0 + r}$ be the projection defined before Lemma 3.3. If $k_A = 1$ and $r \ge 1$, we set $l_A = \max\{n_1, \ldots, n_r\}$.

Similarly, let B be an $n \times n$ matrix in real part Jordan form with positive eigenvalues

$$\mu_1 < \cdots < \mu_{k_B}.$$

Let $W_j \subset \mathbb{R}^n$ be the generalized eigenspace of μ_j, and set $e_j = \dim W_j$. If $k_B \ge 2$, we write B in the block diagonal form $B = [B_1, \ldots, B_{k_B}]$, where B_j is the block corresponding to the eigenvalue μ_j; we also denote $B' = [B_1, \ldots, B_{k_B - 1}]$. If $k_B = 1$, that is, if B has only one eigenvalue $\mu = \mu_1$, we write $B = [\mu I_{m_0}, \mu I_{m_1} + N, \ldots, \mu I_{m_s} + N]$

in the block diagonal form, and we let $\pi_B\colon \mathbb{R}^n \to \mathbb{R}^{m_0+s}$ be the projection defined before Lemma 3.3. If $k_B = 1$ and $s \geq 1$, we set $l_B = \max\{m_1, \dots, m_s\}$.

Suppose there is an η–quasisymmetric map $F\colon (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$.

Lemma 5.1 *$k_A = 1$ if and only if $k_B = 1$.*

Proof Suppose $k_A = 1$ and $k_B \geq 2$. Fix any Q with $\big(\sum_j \mu_j e_j\big)/\mu_{k_B} < Q < n$. Let $\pi\colon (\mathbb{R}^n, D_B) \to W_{k_B}$ be the projection onto W_{k_B}, and $\pi'\colon W_{k_B} \to \mathbb{R}$ a coordinate function on W_{k_B}. Set $u = \pi' \circ \pi$. Then Lemma 4.5 implies $V_{Q,\eta(K)}(u|_{F(E)}) = 0$ for all sufficiently large K and all bounded subsets $E \subset (\mathbb{R}^n, D_A)$. By Lemma 4.1 $V_{Q,K}(u \circ F|_E) = 0$. But this contradicts Corollary 4.3. □

Lemma 5.2 *Suppose $k_A = 1$. Then $A = \lambda_1 I_n$ if and only if $B = \mu_1 I_n$.*

Proof Suppose $B = \mu_1 I_n$. Let π_i $(i = 1, 2, \dots, n)$ be the coordinate functions on $(\mathbb{R}^n, D_B)$. Then by Lemma 4.6 we have $V_{n,\eta(K)}(\pi_i|_{F(E)}) < \infty$ for all i, all sufficiently large K and all rectangular boxes $E \subset (\mathbb{R}^n, D_A)$. Hence $V_{n,K}(\pi_i \circ F|_E) < \infty$ by Lemma 4.1. Now Lemma 4.7 implies that $\pi_i \circ F$ is constant on the fibers of π_A. Since this is true for all $1 \leq i \leq n$, the fibers of π_A must have dimension 0. Hence A must also be a multiple of I_n. □

Lemma 5.3 *Suppose $k_A = 1$ and $r \geq 1$. Then F maps each fiber of π_A onto some fiber of π_B.*

Proof Lemmas 5.1 and 5.2 imply that $k_B = 1$ and $s \geq 1$. Notice that it suffices to show that each fiber of π_A is mapped by F into some fiber of π_B: by symmetry each fiber of π_B is mapped by F^{-1} into some fiber of π_A and hence the lemma follows. We shall prove this by contradiction and so assume that there is some fiber H of π_A such that $F(H)$ is not contained in any fiber of π_B. Then there is some coordinate function $\pi'\colon \mathbb{R}^{m_0+s} \to \mathbb{R}$ such that $u \circ F$ is not constant on H, where $u := \pi' \circ \pi_B$. Now Lemma 4.6 implies that $V_{n,\eta(K)}(u|_{F(E)}) < \infty$ for all sufficiently large K and all rectangular boxes $E \subset (\mathbb{R}^n, D_A)$. By Lemma 4.1 we have $V_{n,K}(u \circ F|_E) < \infty$. This contradicts Lemma 4.7 since we can choose E such that $u \circ F$ is not constant on $H \cap E$. □

It follows from Lemma 5.3 that F induces a map $G\colon \mathbb{R}^{n_0+r} \to \mathbb{R}^{m_0+s}$ such that $F(\pi_A^{-1}(y)) = \pi_B^{-1}(G(y))$ for all $y \in \mathbb{R}^{n_0+r}$. Define

$$\tau_A\colon \mathbb{R}^n = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_r} \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-n_0-r} \times \mathbb{R}^{n_0+r}$$

by

$$\tau_A(z, (x_1, y_1), \dots, (x_r, y_r)) = ((x_1, \dots, x_r), (z, y_1, \dots, y_r)),$$

where $(x_i, y_i) \in \mathbb{R}^{n_i} = \mathbb{R}^{n_i - 1} \times \mathbb{R}$. Similarly, there is an identification

$$\tau_B\colon \mathbb{R}^n = \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s} \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-m_0-s} \times \mathbb{R}^{m_0+s}.$$

With the identifications τ_A and τ_B, we have $\pi_A^{-1}(y) = \mathbb{R}^{n-n_0-r} \times \{y\}$, $\pi_B^{-1}(G(y)) = \mathbb{R}^{n-m_0-s} \times \{G(y)\}$, and $F(\mathbb{R}^{n-n_0-r} \times \{y\}) = \mathbb{R}^{n-m_0-s} \times \{G(y)\}$. Hence for each $y \in \mathbb{R}^{n_0+r}$, there is a map

$$H(\,\cdot\,, y)\colon \mathbb{R}^{n-n_0-r} \to \mathbb{R}^{n-m_0-s}$$

such that $F(x, y) = (H(x, y), G(y))$ for all $x \in \mathbb{R}^{n-n_0-r}$.

In the following $|\cdot|$ denotes the Euclidean norm.

Lemma 5.4 *Suppose $k_A = 1$ and $r \geq 1$. Then:*

1. *The map $G\colon (\mathbb{R}^{n_0+r}, |\cdot|^{1/\lambda}) \to (\mathbb{R}^{m_0+s}, |\cdot|^{1/\mu})$ is η–quasisymmetric.*
2. *For each $y \in \mathbb{R}^{n_0+r}$, the map $H(\,\cdot\,, y)\colon (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$ is η–quasisymmetric.*

Proof Statement (1) follows from Lemma 3.4 and the arguments in [22, page 10]. Statement (2) follows from Lemma 3.3. □

Suppose $k_A = 1$. Set $\epsilon = \lambda/\mu$ and $\eta_1(t) = \eta(t^{1/\epsilon})$. We notice that all the following maps are η_1–quasisymmetric:

1. $F\colon \big(\mathbb{R}^n, D_A^\epsilon\big) \to (\mathbb{R}^n, D_B)$
2. $G\colon (\mathbb{R}^{n_0+r}, |\cdot|^{1/\mu}) \to (\mathbb{R}^{m_0+s}, |\cdot|^{1/\mu})$
3. $H(\,\cdot\,, y)\colon \big(\mathbb{R}^{n-n_0-r}, D_{A(1)}^\epsilon\big) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$, for each $y \in \mathbb{R}^{n_0+r}$

Let $g\colon (X_1, \rho_1) \to (X_2, \rho_2)$ be a bijection between two quasimetric spaces. Suppose g satisfies the following condition: for any fixed $x \in X_1$, $\rho_1(y, x) \to 0$ if and only if $\rho_2(g(y), g(x)) \to 0$. We define for every $x \in X_1$ and $r > 0$,

$$L_g(x, r) = \sup\{\rho_2(g(x), g(x')) \mid \rho_1(x, x') \leq r\},$$
$$l_g(x, r) = \inf\{\rho_2(g(x), g(x')) \mid \rho_1(x, x') \geq r\},$$

and set

$$L_g(x) = \limsup_{r \to 0} \frac{L_g(x, r)}{r}, \quad l_g(x) = \liminf_{r \to 0} \frac{l_g(x, r)}{r}.$$

Lemma 5.5 *Consider the map* $G\colon (\mathbb{R}^{n_0+r}, |\cdot|^{1/\mu}) \to (\mathbb{R}^{m_0+s}, |\cdot|^{1/\mu})$ *and the map* $H(\cdot, y)\colon \big(\mathbb{R}^{n-n_0-r}, D^{\epsilon}_{A(1)}\big) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$. *The following hold for all* $y \in \mathbb{R}^{n_0+r}$, $x \in \mathbb{R}^{n-n_0-r}$:

(1) $L_G(y, r) \le \eta_1(1) l_{H(\cdot,y)}(x, r)$ *for any* $r > 0$

(2) $\eta_1^{-1}(1) l_{H(\cdot,y)}(x) \le l_G(y) \le \eta_1(1) l_{H(\cdot,y)}(x)$

(3) $\eta_1^{-1}(1) L_{H(\cdot,y)}(x) \le L_G(y) \le \eta_1(1) L_{H(\cdot,y)}(x)$

Proof The proof is very similar to that of [22, Lemma 4.3]. Let $y \in \mathbb{R}^{n_0+r}$, $x \in \mathbb{R}^{n-n_0-r}$ and $r > 0$. Let $y' \in \mathbb{R}^{n_0+r}$ with $|y-y'|^{1/\mu} \le r$ and $x' \in \mathbb{R}^{n-n_0-r}$ with $D^{\epsilon}_{A(1)}(x, x') \ge r$. Set $t_0 = \ln|y'-y|/\lambda$. Let (u_i, v_i) $(u_i \in \mathbb{R}^{n_i-1}, v_i \in \mathbb{R}, 1 \le i \le r)$ be the unique solution of $e^{-t_0 N_{n_i}}(u_i, v_i)^T = (0, \dots, 0, y'_i - y_i)^T$. Let $x''_i = u_i + x_i$ and $x'' = (x''_1, \dots, x''_r)$. Then $D^{\epsilon}_A((x, y), (x'', y')) = |y-y'|^{1/\mu} \le r \le D^{\epsilon}_A((x, y), (x', y))$. Since $F\colon \big(\mathbb{R}^n, D^{\epsilon}_A\big) \to (\mathbb{R}^n, D_B)$ is η_1–quasisymmetric, we have

$$\begin{aligned}|G(y) - G(y')|^{1/\mu} \le D_B(F(x'', y'), F(x, y)) &\le \eta_1(1) D_B(F(x, y), F(x', y)) \\ &= \eta_1(1) D_{B(1)}(H(x, y), H(x', y)).\end{aligned}$$

Since y' and x' are chosen arbitrarily, (1) follows.

The proofs of (2) and (3) are exactly the same as those in [22, Lemma 4.3]. □

Recall that, when A has only one eigenvalue $\lambda = \lambda_1$ and is written in the block diagonal form $A = [\lambda I_{n_0}, \lambda I_{n_1} + N, \dots, \lambda I_{n_r} + N]$ with $r \ge 1$, we denote $l_A = \max\{n_1, \dots, n_r\}$.

Lemma 5.6 *Suppose* $k_A = 1$ *and* $l_A = 2$. *Then* $l_B = 2$ *and for* $\epsilon = \lambda/\mu$:

(1) A *and* ϵB *have the same real part Jordan form.*

(2) *The map* $F\colon (\mathbb{R}^n, D^{\epsilon}_A) \to (\mathbb{R}^n, D_B)$ *is a* K*–quasisimilarity, where* K *depends only on* A, B *and* η.

Proof (1) By Lemma 5.4 (2), $H(\cdot, y)\colon (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$ is η–quasisymmetric for each $y \in \mathbb{R}^{n_0+r}$. Since $l_A = 2$, all Jordan blocks of A have size 2 and $A(1) = \lambda I_r$. Now Lemma 5.2 applied to $H(\cdot, y)$ implies that $B(1) = \mu I_r$. It follows that all Jordan blocks of B also have size 2, and hence $l_B = 2$ and $B(1) = \mu I_s$. So we have $r = s$. That is, A and B have the same number of 2×2 Jordan blocks. Now (1) follows.

(2) The proof of (2) is very similar to the arguments in [20, Section 4] and [22]. We will only indicate the differences here. First we notice that $G\colon (\mathbb{R}^{n_0+r}, |\cdot|) \to (\mathbb{R}^{m_0+s}, |\cdot|)$ is also quasisymmetric, and hence is differentiable ae. Since $F\colon \big(\mathbb{R}^n, D^{\epsilon}_A\big) \to (\mathbb{R}^n, D_B)$

is η_1–quasisymmetric, the arguments in [20, Section 4] and [22] imply that there is a constant K_1 depending only on η_1, such that for every $y \in \mathbb{R}^{n_0+r}$ where $G\colon (\mathbb{R}^{n_0+r}, |\cdot|) \to (\mathbb{R}^{m_0+s}, |\cdot|)$ is differentiable, we have $0 < l_G(y) < \infty$ and the map

$$H(\cdot, y)\colon \big(\mathbb{R}^{n-n_0-r}, D^{\epsilon}_{A(1)}\big) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$$

is a K_1–quasisimilarity with constant $l_G(y)$.

Now let $y, y' \in \mathbb{R}^{n_0+r}$ be two points where G is differentiable. We will show that $l_G(y)$ and $l_G(y')$ are comparable. Let $x \in \mathbb{R}^{n-n_0-r}$ and choose $x' \in \mathbb{R}^{n-n_0-r}$ so that $D_{A(1)}(x, x') \gg |y'-y|^{1/\lambda}$. Let (u_i, v_i) be as in the proof of Lemma 5.5. Let $x''_i = x_i + u_i$, $x'''_i = x'_i + u_i$ $(1 \le i \le r)$, and set $x'' = (x''_1, \dots, x''_r)$, $x''' = (x'''_1, \dots, x'''_r)$. Then

$$D_A((x, y), (x'', y')) = D_A((x', y), (x''', y')) = |y'-y|^{1/\lambda}.$$

Now the generalized triangle inequality implies

$$\begin{aligned} D_A((x'', y'), (x', y)) &\le M\big\{D_A((x'', y'), (x, y)) + D_A((x, y), (x', y))\big\} \\ &\le 2M D_A((x, y), (x', y)). \end{aligned}$$

By the quasisymmetry condition we have

$$D_B(F(x'', y'), F(x', y)) \le \eta(2M) D_B(F(x, y), F(x', y)).$$

Similarly, $D_B(F(x'', y'), F(x''', y')) \le \eta(2M) D_B(F(x'', y'), F(x', y))$. So we have

$$D_B(F(x'', y'), F(x''', y')) \le (\eta(2M))^2 D_B(F(x, y), F(x', y)).$$

This together with the quasisimilarity properties of $H(\cdot, y)$ and $H(\cdot, y')$ mentioned above implies that

$$l_G(y') D^{\epsilon}_{A(1)}(x'', x''') \le K_1^2 (\eta(2M))^2 l_G(y) D^{\epsilon}_{A(1)}(x, x').$$

Since $D_{A(1)}(x'', x''') = D_{A(1)}(x, x')$, we have $l_G(y') \le K_1^2(\eta(2M))^2 l_G(y)$. By symmetry, we also have $l_G(y) \le K_1^2(\eta(2M))^2 l_G(y')$. Now fix y and set $C = l_G(y)$. Then at every y' where G is differentiable, $H(\cdot, y')$ is a K_2–quasisimilarity with constant C, where $K_2 = K_1^3(\eta(2M))^2$. Now a limiting argument shows that this is true for every $y' \in \mathbb{R}^{n_0+r}$. The arguments in [22, Section 4] (using Lemma 5.5 from above instead in [22, Lemma 4.3]) then show that there is a constant $K_3 = K_3(K_2, \eta_1)$ such that $G\colon (\mathbb{R}^{n_0+r}, |\cdot|^{1/\mu}) \to (\mathbb{R}^{m_0+s}, |\cdot|^{1/\mu})$ and all $H(\cdot, y)$ are K_3–quasisimilarities with constant C.

The final difference is in finding a lower bound for $D_B(F(x, y), F(x', y'))$. If

$$D^{\epsilon}_A((x, y), (x', y')) \le (2M)^{\epsilon} |y'-y|^{1/\mu},$$

then

$$\begin{aligned} D_B(F(x,y),F(x',y')) \geq |G(y')-G(y)|^{1/\mu} &\geq \frac{C}{K_3}|y'-y|^{1/\mu} \\ &\geq \frac{C}{(2M)^{\epsilon}K_3} D_A^{\epsilon}((x,y),(x',y')). \end{aligned}$$

Now assume $D_A^{\epsilon}((x,y),(x',y')) \geq (2M)^{\epsilon}|y'-y|^{1/\mu}$. Let (u_i,v_i) be as in the above paragraph. Let $x_i''=x_i'-u_i$ and set $x''=(x_1'',\dots,x_r'')$. Then $D_A^{\epsilon}((x'',y),(x',y'))=|y'-y|^{1/\mu}$. The generalized triangle inequality implies

$$\frac{1}{2M} \leq \frac{D_A((x,y),(x'',y))}{D_A((x,y),(x',y'))} \leq 2M.$$

Now the quasisymmetric condition implies

$$\begin{aligned} D_B(F(x,y),F(x',y')) &\geq \frac{1}{\eta(2M)} D_B(F(x,y),F(x'',y)) \\ &\geq \frac{C}{K_3\eta(2M)} D_A^{\epsilon}((x,y),(x'',y)) \\ &\geq \frac{C}{(2M)^{\epsilon}K_3\eta(2M)} D_A^{\epsilon}((x,y),(x',y')). \end{aligned}$$

So we have found a lower bound for $D_B(F(x,y),F(x',y'))$. The rest of the proof is the same as in [22, Section 4]. We notice that the constant M depends only on A, and ϵ depends only on A and B. Hence F is a K–quasisimilarity with K depending only on A, B and η. □

Lemma 5.7 *Suppose $k_A=1$ and $l_A \geq 2$. Then for $\epsilon=\lambda/\mu$:*

(1) *A and ϵB have the same real part Jordan form.*

(2) *The map $F\colon (\mathbb{R}^n, D_A^{\epsilon}) \to (\mathbb{R}^n, D_B)$ is a K–quasisimilarity, where K depends only on A, B and η.*

Proof We induct on l_A. The basic step $l_A=2$ is Lemma 5.6. Now assume $l_A=l\geq 3$ and that the lemma holds for $l_A=l-1$. For any $y\in\mathbb{R}^{n_0+r}$, the induction hypothesis applied to the η–quasisymmetric map $H(\cdot,y)\colon (\mathbb{R}^{n-n_0-r}, D_{A(1)}) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$ implies that for $\epsilon=\lambda/\mu$:

(a) $A(1)$ and $\epsilon B(1)$ have the same real part Jordan form.

(b) $H(\cdot,y)\colon (\mathbb{R}^{n-n_0-r}, D_{A(1)}^{\epsilon}) \to (\mathbb{R}^{n-m_0-s}, D_{B(1)})$ is a K–quasisimilarity with K depending only on $A(1)$, $B(1)$ and η.

Now (1) follows from (a), and (2) follows from (b), Lemma 5.5 and the arguments in [22, Section 4]; see the proof of Lemma 5.6 (2). □

Lemma 5.8 *Suppose $k_A \geq 2$. Then $k_B \geq 2$ and $\big(\sum_i d_i\lambda_i\big)/\lambda_{k_A} = \big(\sum_j e_j\mu_j\big)/\mu_{k_B}$.*

Proof Lemma 5.1 implies $k_B \geq 2$. Suppose $\big(\sum_i d_i\lambda_i\big)/\lambda_{k_A} > \big(\sum_j e_j\mu_j\big)/\mu_{k_B}$. Pick any Q with $\big(\sum_i d_i\lambda_i\big)/\lambda_{k_A} > Q > \big(\sum_j e_j\mu_j\big)/\mu_{k_B}$. Let $\pi\colon (\mathbb{R}^n, D_B) \to W_{k_B}$ be the projection onto W_{k_B}, and $\pi'\colon W_{k_B} \to \mathbb{R}$ a coordinate function on W_{k_B}. Set $u = \pi' \circ \pi$. By Lemma 4.5 we have $V_{Q,\eta(K)}(u|_{F(E)}) = 0$ for all sufficiently large K and all Euclidean balls $E \subset (\mathbb{R}^n, D_A)$. Lemma 4.1 implies $V_{Q,K}(u \circ F|_E) = 0$. This contradicts Lemma 4.2 since $Q < \big(\sum_i d_i\lambda_i\big)/\lambda_{k_A}$ and the function $u \circ F$ is nonconstant. Similarly there is a contradiction if $\big(\sum_i d_i\lambda_i\big)/\lambda_{k_A} < \big(\sum_j e_j\mu_j\big)/\mu_{k_B}$. The lemma follows. □

Recall that (see Section 3), if $k_A \geq 2$, then the restriction of D_A to each affine subspace H parallel to $\prod_{i<k_A} V_i$ agrees with $D_{A'}$, where $A' = [A_1, \dots, A_{k_A-1}]$.

Lemma 5.9 *Denote $k = k_A$ and $k' = k_B$. Suppose $k \geq 2$. Then each affine subspace H of $\mathbb{R}^n$ parallel to $\prod_{i<k} V_i$ is mapped by F onto an affine subspace parallel to $\prod_{j<k'} W_j$. Furthermore, $F|_H\colon (H, D_{A'}) \to (F(H), D_{B'})$ is η–quasisymmetric, and F induces an η–quasisymmetric map $G\colon (V_k, D_{A_k}) \to (W_{k'}, D_{B_{k'}})$ such that $F\big(\big(\prod_{i<k} V_i\big) \times \{y\}\big) = \big(\prod_{j<k'} W_j\big) \times \{G(y)\}$.*

Proof As in the proof of Lemma 5.3, to establish the first claim it suffices to show that each affine subspace parallel to $\prod_{i<k} V_i$ is mapped into an affine subspace parallel to $\prod_{j<k'} W_j$. By Lemma 5.8 we have $\big(\sum_i d_i\lambda_i\big)/\lambda_k = \big(\sum_j e_j\mu_j\big)/\mu_{k'}$. Pick any Q with

$$\frac{\sum_i d_i\lambda_i}{\lambda_k} < Q < \min\left\{\frac{\sum_i d_i\lambda_i}{\lambda_{k-1}}, \frac{\sum_j e_j\mu_j}{\mu_{k'-1}}\right\}.$$

Suppose there is an affine subspace H parallel to $\prod_{i<k} V_i$ such that $F(H)$ is not contained in any affine subspace parallel to $\prod_{j<k'} W_j$. Let $\pi\colon \prod_j W_j \to W_{k'}$ be the canonical projection. Then there is some coordinate function $\pi'\colon W_{k'} \to \mathbb{R}$ such that $u \circ F$ is not constant on H, where $u = \pi' \circ \pi$. As $Q > \big(\sum_j e_j\mu_j\big)/\mu_{k'}$, Lemma 4.5 implies $V_{Q,\eta(K)}(u|_{F(E)}) = 0$ for all sufficiently large K and all rectangular boxes $E \subset (\mathbb{R}^n, D_A)$. By Lemma 4.1 $V_{Q,K}(u \circ F|_E) = 0$. This contradicts Lemma 4.4 since $Q < \big(\sum_i d_i\lambda_i\big)/\lambda_{k-1}$ and we can choose a rectangular box E such that $u \circ F$ is not constant on $H \cap E$.

Since by assumption F is η–quasisymmetric, it follows from the remark preceding the lemma that $F|_H\colon (H, D_{A'}) \to (F(H), D_{B'})$ is η–quasisymmetric.

The first claim implies there is a map $G\colon V_k \to W_{k'}$ such that $F\big(\big(\prod_{i<k} V_i\big) \times \{y\}\big) = \big(\prod_{j<k'} W_j\big) \times \{G(y)\}$ for any $y \in V_k$. That $G\colon (V_k, D_{A_k}) \to (W_{k'}, D_{B_{k'}})$ is η–quasisymmetric follows from (3-2), (3-3) and the arguments of [22, page 10]. □

Lemma 5.10 *Suppose $k_A = 2$. Then $k_B = 2$ and for $\epsilon = \lambda_1/\mu_1$:*

(1) *A and ϵB have the same real part Jordan form.*

(2) *The map $F\colon \big(\mathbb{R}^n, D_A^{\epsilon}\big) \to (\mathbb{R}^n, D_B)$ is a K–quasisimilarity, where K depends only on A, B and η.*

Proof Let H be an affine subspace of $\mathbb{R}^n$ parallel to $\prod_{i<k_A} V_i$. By Lemma 5.9 $F(H)$ is an affine subspace parallel to $\prod_{j<k_B} W_j$, and $F|_H\colon (H, D_{A'}) \to (F(H), D_{B'})$ is η–quasisymmetric. Since $k_A = 2$, we have $k_{A'} = 1$. Now Lemma 5.1 applied to $F|_H$ implies $k_{B'} = 1$, so $k_B = k_{B'} + 1 = 2$. Now the η–quasisymmetric map $F|_H\colon (H, D_{A'}) \to (F(H), D_{B'})$ becomes $(V_1, D_{A_1}) \to (W_1, D_{B_1})$, and Lemmas 5.7 and 5.2 imply that A_1 and $\epsilon_1 B_1$ have the same real part Jordan form, where $\epsilon_1 = \lambda_1/\mu_1$. By Lemma 5.9 F induces an η–quasisymmetric map $G\colon (V_2, D_{A_2}) \to (W_2, D_{B_2})$, and hence Lemmas 5.7 and 5.2 again imply that A_2 and $\epsilon_2 B_2$ have the same real part Jordan form, where $\epsilon_2 = \lambda_2/\mu_2$. Lemma 5.9 also implies $d_1 = e_1$ and $d_2 = e_2$. Now Lemma 5.8 implies $\lambda_1/\mu_1 = \lambda_2/\mu_2$. Hence (1) holds.

To prove (2), we consider two cases. First assume that $A_1 = \lambda_1 I$ and $A_2 = \lambda_2 I$. In this case, (2) follows from (1) and Theorem 3.5. Next we assume that either $A_1 \neq \lambda_1 I$ or $A_2 \neq \lambda_2 I$ holds. Then Lemma 5.7 implies that either $F|_H\colon \big(H, D_{A_1}^{\epsilon}\big) \to (F(H), D_{B_1})$ is a K_1–quasisimilarity with K_1 depending only on A_1, B_1 and η, or $G\colon \big(V_2, D_{A_2}^{\epsilon}\big) \to (W_2, D_{B_2})$ is a K_2–quasisimilarity with K_2 depending only on A_2, B_2 and η. Then the arguments similar to those in the proof of Lemma 5.6 (2) (also compare with [20, Section 4]) show that $F\colon \big(\mathbb{R}^n, D_A^{\epsilon}\big) \to (\mathbb{R}^n, D_B)$ is a K–quasisimilarity with K depending only on A, B and η. □

Lemma 5.11 *Suppose $k_A \geq 2$. Then for $\epsilon = \lambda_1/\mu_1$:*

(1) *A and ϵB have the same real part Jordan form.*

(2) *The map $F\colon \big(\mathbb{R}^n, D_A^{\epsilon}\big) \to (\mathbb{R}^n, D_B)$ is a K–quasisimilarity, where K depends only on A, B and η.*

Proof We induct on k_A. The basic step $k_A = 2$ is Lemma 5.10. Now we assume $k_A = k \geq 3$ and that the lemma holds for $k_A = k - 1$. For each affine subspace H of $\mathbb{R}^n$ parallel to $\prod_{i<k_A} V_i$, the induction hypothesis applied to $F|_H\colon (H, D_{A'}) \to (F(H), D_{B'})$ implies that for $\epsilon = \lambda_1/\mu_1$:

(a) A' and $\epsilon B'$ have the same real part Jordan form.

(b) The map $F|_H \colon \big(H, D^{\epsilon}_{A'}\big) \to (F(H), D_{B'})$ is a K–quasisimilarity, where K depends only on A', B' and η.

The statement (a) implies in particular $k_A - 1 = k_B - 1$ (hence $k_A = k_B$), $\lambda_i = \epsilon \mu_i$ and $e_i = d_i$ for $i < k_A$. Now it follows from Lemma 5.8 that $\lambda_{k_A} = \epsilon \mu_{k_A}$. If A_{k_A} is a multiple of I, then (1) follows from Lemmas 5.9 and 5.2. If A_{k_A} is not a multiple of I, then Lemmas 5.9 and 5.7 (1) imply that A_{k_A} and ϵB_{k_B} have the same real part Jordan form. Hence (1) holds as well in this case.

If A_{k_A} is a multiple of I, then (2) follows from the statement (b) above and the arguments in the proof of Lemma 5.6 (2). If A_{k_A} is not a multiple of I, then Lemma 5.7 (2) implies that $G \colon \big(V_k, D^{\epsilon}_{A_k}\big) \to (W_{k'}, D_{B_{k'}})$ is a K_1–quasisimilarity with K_1 depending only on A_{k_A}, B_{k_B} and η. In this case, (2) follows from this, (b) and the arguments in the proof of Lemma 5.6 (2). $\square$

Next we will finish the proofs of the main theorems. So let A, B be two arbitrary $n \times n$ matrices whose eigenvalues have positive real parts. Let G_A, G_B be equipped with arbitrary admissible metrics. Then there are nonsingular matrices P, Q such that G_A is isometric to $G_{PAP^{-1}}$ (equipped with the standard metric) and G_B is isometric to $G_{QBQ^{-1}}$ (equipped with the standard metric). Hence below in the proofs we will replace $(\mathbb{R}^n, D_A)$ and $(\mathbb{R}^n, D_B)$ with $(\mathbb{R}^n, D_{PAP^{-1}})$ and $(\mathbb{R}^n, D_{QBQ^{-1}})$ respectively. There also exist nonsingular matrices P_0, Q_0 such that $G_{P_0AP_0^{-1}}$ and $G_{Q_0BQ_0^{-1}}$ have pinched negative sectional curvature. We may choose the same $P_0AP_0^{-1}$ for all conjugate matrices A. Denote by J and J' the real part Jordan forms of A and B respectively. By Proposition 3.1, there are bilipschitz maps $f_J \colon G_{P_0AP_0^{-1}} \to G_J$ and $f_P \colon G_{P_0AP_0^{-1}} \to G_{PAP^{-1}}$. Then Corollary 3.2 implies their boundary maps $\partial f_J \colon (\mathbb{R}^n, D_{P_0AP_0^{-1}}) \to (\mathbb{R}^n, D_J)$ and $\partial f_P \colon (\mathbb{R}^n, D_{P_0AP_0^{-1}}) \to (\mathbb{R}^n, D_{PAP^{-1}})$ are also bilipschitz. Similarly, there are bilipschitz maps $f_{J'} \colon G_{Q_0BQ_0^{-1}} \to G_{J'}$ and $f_Q \colon G_{Q_0BQ_0^{-1}} \to G_{QBQ^{-1}}$, whose boundary maps $\partial f_{J'} \colon (\mathbb{R}^n, D_{Q_0BQ_0^{-1}}) \to (\mathbb{R}^n, D_{J'})$ and $\partial f_Q \colon (\mathbb{R}^n, D_{Q_0BQ_0^{-1}}) \to (\mathbb{R}^n, D_{QBQ^{-1}})$ are also bilipschitz.

Completing the proof of Theorem 1.1 The "if" part follows from Proposition 3.1 since the boundary map of a quasi-isometry between Gromov hyperbolic spaces is quasisymmetric. We will prove the "only if" part. So we suppose $(\mathbb{R}^n, D_{PAP^{-1}})$ and $(\mathbb{R}^n, D_{QBQ^{-1}})$ are quasisymmetric. Since the four maps ∂f_P, ∂f_J, ∂f_Q and $\partial f_{J'}$ are bilipschitz, we see that $(\mathbb{R}^n, D_J)$ and $(\mathbb{R}^n, D_{J'})$ are quasisymmetric. Now it follows from Lemmas 5.2, 5.7 (1) and 5.11 (1) that J and $\epsilon J'$ have the same real part Jordan form, where $\epsilon = \lambda_1/\mu_1$. Hence A and ϵB also have the same real part Jordan form. $\square$

Theorem 5.12 *Let A and B be $n \times n$ matrices whose eigenvalues all have positive real parts, and let G_A and G_B be equipped with arbitrary admissible metrics. Denote by λ_1 and μ_1 the smallest real parts of the eigenvalues of A and B respectively, and set $\epsilon = \lambda_1/\mu_1$. If the real part Jordan form of A is not a multiple of the identity matrix I_n, then for every η–quasisymmetric map $F\colon (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$, the map $F\colon \left(\mathbb{R}^n, D_A^{\epsilon}\right) \to (\mathbb{R}^n, D_B)$ is a K–quasisimilarity, where K depends only on η, A, B and the metrics on G_A, G_B.*

Completing the proof of Theorem 5.12 Let $F\colon (\mathbb{R}^n, D_{PAP^{-1}}) \to (\mathbb{R}^n, D_{qbq^{-1}})$ be an η–quasisymmetric map. Notice that the bilipschitz constant of the map ∂f_J depends only on A (actually the conjugacy class of A) as the same $P_0 A P_0^{-1}$ is chosen for all matrices A in the same conjugate class. However, the bilipschitz constant of ∂f_P depends on P and hence on the admissible metric on G_A. Hence $\partial f_J \circ \partial f_P^{-1}$ is L_1–bilipschitz for some constant L_1 depending only on A and the admissible metric on G_A. Similarly, $\partial f_{J'} \circ \partial f_Q^{-1}$ is L_2–bilipschitz for some constant L_2 depending only on B and the admissible metric on G_B. It follows that

$$G := (\partial f_{J'} \circ \partial f_Q^{-1}) \circ F \circ (\partial f_J \circ \partial f_P^{-1})^{-1}\colon (\mathbb{R}^n, D_J) \to (\mathbb{R}^n, D_{J'})$$

is η_1–quasisymmetric, where η_1 depends only on L_1, L_2 and η. Now Lemmas 5.7 (2) and 5.11 (2) imply that G is a K–quasisimilarity, where K depends only on J, J' and η_1. Consequently, F is a KL_1L_2–quasisimilarity. □

6 Proof of the corollaries

In this section we prove the corollaries from the introduction and also derive a local version of Theorem 1.1.

Let M be a Hadamard manifold with pinched negative sectional curvature, $\xi_0 \in \partial M$, and $x_0 \in M$ a base point. Let γ be the geodesic with $\gamma(0) = x_0$ and $\gamma(\infty) = \xi_0$. Let $h_M = -B_\gamma\colon M \to \mathbb{R}$, where B_γ is the Busemann function associated with γ. Set $H_t = h_M^{-1}(t)$. A parabolic visual quasimetric D_{ξ_0} on $\partial M\backslash\{\xi_0\}$ is defined as follows. For $\xi, \eta \in \partial M\backslash\{\xi_0\}$, $D_{\xi_0}(\xi, \eta) = e^t$ if and only if $\xi\xi_0 \cap H_t$ and $\eta\xi_0 \cap H_t$ have distance 1 in the horosphere H_t.

Let N be another Hadamard manifold with pinched negative sectional curvature, and $f\colon M \to N$ a quasi-isometry. For any $\xi \in \partial M$ and $x \in M$, we set $\xi' = \partial f(\xi)$ and $x' = f(x)$. Let γ' be the geodesic with $\gamma'(0) = x_0'$ and $\gamma'(\infty) = \xi_0'$. Set $h_N = -B_{\gamma'}$, where $B_{\gamma'}$ is the Busemann function associated with γ'. Denote $H_t' = h_N^{-1}(t)$. Let $D_{\xi_0'}$ be the parabolic visual quasimetric on $\partial N\backslash\{\xi_0'\}$ with respect to the base point x_0'.

Lemma 6.1 *Let $s > 0$. Then the following three conditions are equivalent.*

(1) *There is a constant $C \geq 0$ such that the Hausdorff distance $HD(f(H_t), H'_{st}) \leq C$ for all t.*

(2) *The boundary map $\partial f\colon (\partial M \backslash \{\xi_0\}, D^s_{\xi_0}) \to (\partial N \backslash \{\xi'_0\}, D_{\xi'_0})$ is bilipschitz.*

(3) *There exists a constant $C \geq 0$ such that $s \cdot d(x,y) - C \leq d(f(x), f(y)) \leq s \cdot d(x,y) + C$ for all $x, y \in M$.*

Proof The arguments in the proof of [20, Lemma 6.4] shows (2)$\Rightarrow$(1), while the arguments at the end of [20, proof of Corollary 1.2] yield (1)$\Rightarrow$(3). We shall prove (3)$\Rightarrow$(1) and (1)$\Rightarrow$(2).

(1)$\Rightarrow$(2): Suppose (1) holds. Let $\xi \neq \eta \in \partial M \backslash \{\xi_0\}$. Assume $D_{\xi_0}(\xi, \eta) = e^t$ and $D_{\xi'_0}(\xi', \eta') = e^{t'}$. Let γ_ξ be the geodesic joining ξ and ξ_0 with $\gamma_\xi(0) \in H_0$ and $\gamma_\xi(\infty) = \xi_0$. By [20, Lemma 6.2], $\gamma_\xi(t)$ is a C_1–quasicenter of ξ, η, ξ_0, and $\gamma_{\xi'}(t')$ is a C_1–quasicenter of ξ', η', ξ'_0, where C_1 depends only on the curvature bounds of M and N. Since f is a quasi-isometry, $f(\gamma_\xi(t))$ is a C_2–quasicenter of ξ', η', ξ'_0, where C_2 depends only on C_1, the quasi-isometry constants of f and the curvature bounds of N. It follows that $d(f(\gamma_\xi(t)), \gamma_{\xi'}(t')) \leq C_3$, where C_3 depends only on C_1, C_2 and the curvature bounds of N. By condition (1), the point $f(\gamma_\xi(t))$ is within C of H'_{st}. It follows that $\gamma_{\xi'}(t') \in H'_{t'}$ is within $C + C_3$ of H'_{st} so $|t' - st| \leq C + C_3$. Therefore, $e^{-(C+C_3)} e^{st} \leq D_{\xi'_0}(\xi', \eta') = e^{t'} \leq e^{C+C_3} e^{st}$.

(3)$\Rightarrow$(1): Suppose (3) holds. Let $\omega\colon \mathbb{R} \to M$ be any geodesic with $\omega(0) \in H_0$ and $\omega(\infty) = \xi_0$. Then $f \circ \omega$ is a (L_1, C_1)–quasigeodesic in N, where L_1 and C_1 depend only on s and C. By the stability of quasigeodesics in a Gromov hyperbolic space, there is a constant C_2 depending only on L_1, C_1 and the Gromov hyperbolicity constant of N, and a complete geodesic ω' in N with one endpoint ξ'_0 such that the Hausdorff distance between $\omega'(\mathbb{R})$ and $f \circ \omega(\mathbb{R})$ is at most C_2. Let $t_1 < t_2$. Then it follows from condition (3) and the triangle inequality that

$$|h_N(f(\omega(t_2))) - h_N(f(\omega(t_1))) - s(t_2 - t_1)| \leq C_3,$$

where C_3 depends only on C, C_2 and the Gromov hyperbolicity constant of N. In particular, this applied to $\omega = \gamma$, $t_2 = t$ and $t_1 = 0$ (or $t_2 = 0$ and $t_1 = t$ if $t < 0$) implies $|h_N(f(\gamma(t))) - st| \leq C_3$.

Let $x \in H_t$ be arbitrary. Let ω_1 be the geodesic with $\omega_1(t) = x$ and $\omega_1(\infty) = \xi_0$. Pick any $t_2 \geq t$ with $d(\gamma(t_2), \omega_1(t_2)) \leq 1$. By condition (3),

$$|h_N(f(\gamma(t_2))) - h_N(f(\omega_1(t_2)))| \leq d(f(\gamma(t_2)), f(\omega_1(t_2))) \leq s + C.$$

The discussion from the preceding paragraph implies

$$|h_N(f(\omega_1(t_2))) - h_N(f(\omega_1(t))) - s(t_2 - t)| \le C_3,$$
$$|h_N(f(\gamma(t_2))) - h_N(f(\gamma(t))) - s(t_2 - t)| \le C_3.$$

These inequalities together with the one at the end of last paragraph imply

$$|h_N(f(\omega_1(t))) - st| \le C_4 := 3C_3 + s + C.$$

Hence $f(x) = f(\omega_1(t))$ is within C_4 of H'_{st}. This shows $f(H_t)$ lies in the C_4–neighborhood of H'_{st}. By considering a quasi-inverse of f, we see that the Hausdorff distance $HD(f(H_t), H'_{st}) \le C_5$, where C_5 depends only on s, C and the Gromov hyperbolicity constants of M and N. □

A local version of Theorem 1.1 also holds.

Theorem 6.2 *Let A, B be $n \times n$ matrices whose eigenvalues have positive real parts, and let G_A and G_B be equipped with arbitrary admissible metrics. Let $U \subset (\mathbb{R}^n, D_A)$, $V \subset (\mathbb{R}^n, D_B)$ be open subsets, and $F\colon (U, D_A) \to (V, D_B)$ an η–quasisymmetric map. Then A and sB have the same real part Jordan form for some $s > 0$.*

Proof By Corollary 3.2 and the discussion before the proof of Theorem 1.1 we may assume A and B are in real part Jordan form. Fix a base point $x \in U$. We may assume both x and $F(x)$ are the origin o. Then there is some constant $a > 1$ and a sequence of distinct triples (x_k, y_k, z_k) from U satisfying $x_k = o$, $D_A(x_k, y_k) \to 0$ and

$$\frac{D_A(x_k, y_k)}{D_A(x_k, z_k)}, \frac{D_A(y_k, x_k)}{D_A(y_k, z_k)}, \frac{D_A(z_k, x_k)}{D_A(z_k, y_k)} \in (1/a, a).$$

Such a triple can be chosen from the eigenspace of λ_1 (the smallest eigenvalue of A) so that $x_k = o$ is the middle point of the segment $y_k z_k$. Since F is η–quasisymmetric, there is a constant $b > 0$ depending only on a and η such that

$$\frac{D_B(F(x_k), F(y_k))}{D_B(F(x_k), F(z_k))}, \frac{D_B(F(y_k), F(x_k))}{D_B(F(y_k), F(z_k))}, \frac{D_B(F(z_k), F(x_k))}{D_B(F(z_k), F(y_k))} \in (1/b, b).$$

Assume $D_A(x_k, y_k) = e^{-t_k}$ and $D_B(F(x_k), F(y_k)) = e^{-t'_k}$. Then we have that $e^{t_k A}\colon (U, e^{t_k} D_A) \to (e^{t_k A} U, D_A)$ is an isometry. Hence the sequence of pointed metric spaces $(U, e^{t_k} D_A, o)$ converges (as $k \to \infty$) in the pointed Gromov–Hausdorff topology towards $(\mathbb{R}^n, D_A)$. Similarly, the sequence of pointed metric spaces $(V, e^{t'_k} D_B, o)$ converges (as $k \to \infty$) in the pointed Gromov–Hausdorff topology towards $(\mathbb{R}^n, D_B)$. On the other hand, the sequence of maps $F_k = F\colon (U, e^{t_k} D_A) \to (V, e^{t'_k} D_B)$ are all η–quasisymmetric, and the triples $(x_k, y_k, z_k) \in (U, e^{t_k} D_A)$, $(F(x_k), F(y_k), F(z_k)) \in$

$(V, e^{t'_k} D_B)$ are uniformly separated and uniformly bounded. Now the compactness property of quasisymmetric maps implies that a subsequence of $\{F_k\}$ converges in the pointed Gromov–Hausdorff topology towards an η–quasisymmetric map $F'\colon (\mathbb{R}^n, D_A) \to (\mathbb{R}^n, D_B)$. Now the theorem follows from Theorem 1.1. $\square$

Lemma 6.3 *Let $F\colon \partial G_A \to \partial G_B$ be a quasisymmetric map, where ∂G_A and ∂G_B are equipped with visual metrics. Let $\xi_0 \in \partial G_A, \xi'_0 \in \partial G_B$ be the points corresponding to upward-oriented vertical geodesic rays. If the real part Jordan form of A is not a multiple of the identity matrix, then $F(\xi_0) = \xi'_0$.*

Proof The proof is similar to that of [22, Proposition 3.5]. Suppose $F(\xi_0) \neq \xi'_0$. By the relation between visual metrics and parabolic visual metrics [20, Section 5], the map

$$F\colon (\mathbb{R}^n \backslash \{F^{-1}(\xi'_0)\}, D_A) \to (\mathbb{R}^n \backslash \{F(\xi_0), D_B)$$

is locally quasisymmetric. By Theorem 6.2, A and sB have the same real part Jordan form for some $s > 0$. In particular, we have $k_B = k_A$; the fibers of π_A and π_B have the same dimension if $k_A = 1$, and the subspaces $\prod_{i<k_A} V_i$ and $\prod_{j<k_B} W_j$ have the same dimension if $k_A \geq 2$. If $k_A = 1$, let H be a fiber of π_A not containing $F^{-1}(\xi'_0)$; if $k_A \geq 2$, then let H be an affine subspace parallel to $\prod_{i<k_A} V_i$ and not containing $F^{-1}(\xi'_0)$. Let m be the topological dimension of H. Then $H \cup \{\xi_0\} \subset \partial G_A$ is an m–dimensional topological sphere. Since $F(\xi_0) \neq \xi'_0$ and $F^{-1}(\xi'_0) \notin H$, the image $F(H \cup \{\xi_0\})$ is a m–dimensional topological sphere in $\mathbb{R}^n = \partial G_B \backslash \{\xi'_0\}$. In particular, $F(H \cup \{\xi_0\})$ (and hence $F(H)$) is not contained in any fiber of π_B (if $k_A = 1$) or any affine subspace parallel to $\prod_{j<k_B} W_j$ (if $k_A \geq 2$). Now the arguments of Lemmas 5.3 and 5.9 yield a contradiction. Hence $F(\xi_0) = \xi'_0$. $\square$

Now Corollary 1.3 follows from Proposition 3.1, Lemma 6.3, Theorem 1.1 and the fact that a quasi-isometry between Gromov hyperbolic spaces induces a quasisymmetric map between the ideal boundaries.

Proofs of Corollaries 1.4 and 1.5 We use the notation introduced before the proof of Theorem 1.1. Let $f\colon G_{PAP^{-1}} \to G_{QBQ^{-1}}$ be a quasi-isometry. By Lemma 6.3, f induces a boundary map $\partial f\colon (\mathbb{R}^n, D_{PAP^{-1}}) \to (\mathbb{R}^n, D_{QBQ^{-1}})$, which is quasisymmetric. By Theorem 1.2, there is some $s > 0$ such that $\partial f\colon (\mathbb{R}^n, D^s_{PAP^{-1}}) \to (\mathbb{R}^n, D_{QBQ^{-1}})$ is bilipschitz. Since ∂f_P and ∂f_Q are also bilipschitz, we have that the boundary map $\partial(f_Q^{-1} \circ f \circ f_P)\colon (\mathbb{R}^n, D^s_{P_0AP_0^{-1}}) \to (\mathbb{R}^n, D_{Q_0BQ_0^{-1}})$ of $f_Q^{-1} \circ f \circ f_P\colon G_{P_0AP_0^{-1}} \to G_{Q_0BQ_0^{-1}}$ is bilipschitz. Since $G_{P_0AP_0^{-1}}$ and $G_{Q_0BQ_0^{-1}}$ have pinched negative sectional curvature, Lemma 6.1 implies the map $f_Q^{-1} \circ f \circ f_P$ is height-respecting and is an almost similarity. By Proposition 3.1 and Corollary 3.2 the two maps f_P and f_Q

are height-respecting and are almost similarities. Hence f is height-respecting and is an almost similarity. □

The proof of Corollary 1.6 is the same as in [20, Corollary 1.3].

Next we give a proof of Corollary 1.7. Recall that a group G of quasisimilarity maps of $(\mathbb{R}^n, D_A)$ is a uniform group if there is some $K \geq 1$ such that every element of G is a K–quasisimilarity. Dymarz and Peng have established the following theorem; see [6] for the definition of almost homotheties.

Theorem 6.4 [6] *Let A be a square matrix whose eigenvalues all have positive real parts, and G be a uniform group of quasisimilarity maps of $(\mathbb{R}^n, D_A)$. If the induced action of G on the space of distinct couples of $\mathbb{R}^n$ is cocompact, then G can be conjugated by a bilipschitz map into the group of almost homotheties.*

Proof of Corollary 1.7 Let G be a group of quasimöbius maps of $(\partial G_A, d)$ such that every element of G is η–quasimöbius, where d is a fixed visual metric on ∂G_A. Let $\xi_0 \in \partial G_A$ be the point corresponding to vertical geodesic rays. Since the real part Jordan form of A is not a multiple of the identity matrix, Lemma 6.3 implies that the point ξ_0 is fixed by all quasisymmetric maps $\partial G_A \to \partial G_A$. Hence G restricts to a group of quasisymmetric maps of $(\mathbb{R}^n, D_A)$. For any three distinct points $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^n = \partial G_A \backslash \{\xi_0\}$, the quasimöbius condition applied to the quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_0)$ implies that every element of G is an η–quasisymmetric map of $(\mathbb{R}^n, D_A)$. Now Theorem 1.2 implies that there is some $K \geq 1$ such that every element of G is a K–quasisimilarity. In other words, G is a uniform group of quasisimilarities of $(\mathbb{R}^n, D_A)$.

Since the induced action of G on the space of distinct triples of $(\partial G_A, d)$ is cocompact, there is some $\delta > 0$ such that for any distinct triple (ξ_1, ξ_2, ξ_3), there is some $g \in G$ such that $d(g(\xi_i), g(\xi_j)) \geq \delta$ for all $1 \leq i \neq j \leq 3$. Now let $\xi \neq \xi_2 \in \mathbb{R}^n = \partial G_A \backslash \{\xi_0\}$ be any distinct couple. Then there is an element $g \in G$ as above corresponding to the triple (ξ_0, ξ_1, ξ_2). Since $g(\xi_0) = \xi_0$, there are two constants $a, b > 0$ depending only on δ such that $D_A(g(\xi_1), o) \leq b$, $D_A(g(\xi_2), o) \leq b$ and $D_A(g(\xi_1), g(\xi_2)) \geq a$. This shows that G acts cocompactly on the space of distinct couples of $(\mathbb{R}^n, D_A)$.

Now the corollary follows from the theorem of Dymarz and Peng. □

7 QS maps in the Jordan block case

In this section we describe all the quasisymmetric maps on the ideal boundary in the case when A is a Jordan block.

Theorem 7.1 *Let $J_n = I_n + N$ be the $n \times n$ $(n \geq 2)$ Jordan block with eigenvalue 1. Then a bijection $F\colon (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ is a quasisymmetric map if and only of there are constants $a_0 \neq 0, a_1, \dots, a_{n-2} \in \mathbb{R}$, a vector $v \in \mathbb{R}^n$ and a Lipschitz map $C\colon \mathbb{R} \to \mathbb{R}$ such that*

$$F(x) = (a_0 I_n + a_1 N + \cdots + a_{n-2} N^{n-2})x + v + \widetilde{C}(x)$$

for all $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, where $\widetilde{C}(x) = (C(x_n), 0, \dots, 0)^T$. Here T indicates matrix transpose.

We first prove that every map of the indicated form is actually bilipschitz. Notice that the map F described in the theorem decomposes as $F = F_1 \circ F_2 \circ F_3$, with $F_1(x) = x + v$, $F_2(x) = x + \widetilde{C}_1(x)$ and $F_3(x) = (a_0 I_n + a_1 N + \cdots + a_{n-2} N^{n-2})x$, where $C_1\colon \mathbb{R} \to \mathbb{R}$ is defined by $C_1(t) = C(t/a_0)$. Since D_{J_n} is invariant under Euclidean translations, F_1 is an isometry. We shall prove that F_2 and F_3 are bilipschitz in the next two lemmas.

For an $n \times n$ matrix $M = (m_{ij})$, set $Q(M) = \sum_{i,j} m_{ij}^2$. We will use the fact $\|M\| \leq Q(M)^{1/2}$, where $\|M\|$ denotes the operator norm of M.

Lemma 7.2 *Suppose $C\colon \mathbb{R} \to \mathbb{R}$ is L–Lipschitz for some $L > 0$. Then we have $F_2\colon (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$, $F_2(x) = x + \widetilde{C}(x)$ is L'–bilipschitz, where L' depends only on L and the dimension n.*

Proof Let $x = (x_1, \dots, x_n)^T$ and $x' = (x'_1, \dots, x'_n)^T$ be two arbitrary points in $\mathbb{R}^n$. Then $F_2(x) = (x_1 + C(x_n), x_2, \dots, x_n)^T$ and $F_2(x') = (x'_1 + C(x'_n), x'_2, \dots, x'_n)^T$. Assume $D_{J_n}(x, x') = e^t$ and $D_{J_n}(F_2(x), F_2(x')) = e^s$. We need to show that there is some constant a depending only on L and n such that $|t - s| \leq a$.

Since $D_{J_n}(x, x') = e^t$, we have $e^t = |e^{-tN}(x' - x)|$; see Section 3. Similarly, $D_{J_n}(F_2(x), F_2(x')) = e^s$ gives $e^s = |e^{-sN}(F_2(x') - F_2(x))|$. Note $F_2(x') - F_2(x) = (x' - x) + w$, where $w = (C(x'_n) - C(x_n), 0, \dots, 0)^T$. The only nonzero entry in $e^{-tN} w$ is $C(x'_n) - C(x_n)$. So we have

$$|e^{-tN} w| = |C(x'_n) - C(x_n)| \leq L|x'_n - x_n|.$$

On the other hand, the last entry in $e^{-tN}(x' - x)$ is $(x'_n - x_n)$, hence

$$|e^{-tN} w| \leq L|x'_n - x_n| \leq L|e^{-tN}(x' - x)| = Le^t.$$

We write

$$e^{-sN}(F_2(x') - F_2(x)) = e^{(t-s)N} e^{-tN}[(x' - x) + w] = e^{(t-s)N}[e^{-tN}(x' - x) + e^{-tN} w].$$

Now

$$\begin{aligned}e^s &= |e^{-sN}(F_2(x')-F_2(x))| = |e^{(t-s)N}[e^{-tN}(x'-x)+e^{-tN}w]| \\ &\le \|e^{(t-s)N}\|\cdot|e^{-tN}(x'-x)+e^{-tN}w| \le \|e^{(t-s)N}\|\cdot\{|e^{-tN}(x'-x)|+|e^{-tN}w|\} \\ &\le \|e^{(t-s)N}\|\cdot\{e^t+Le^t\} \le e^t(1+L)\sqrt{Q(e^{(t-s)N})}.\end{aligned}$$

From this we derive $e^{s-t} \le (1+L)\sqrt{Q(e^{(t-s)N})}$. Notice that $Q(e^{(t-s)N})$ is a polynomial of degree $2(n-1)$ in $t-s$ that depends only on n. It follows that there is a constant a depending only on n and L such that $s-t\le a$. Since the inverse of F_2 is $F_2^{-1}(x) = x + (-C(x_n), 0, \dots, 0)^T$, the above argument applied to F_2^{-1} yields $t-s\le a$. Hence $|s-t|\le a$, and we are done. □

Lemma 7.3 *Let $F_3\colon (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ be given by*

$$F_3(x) = (a_0 I_n + a_1 N + \cdots + a_{n-1}N^{n-1})x,$$

where $a_0 \ne 0, a_1, \dots, a_{n-1} \in \mathbb{R}$ are constants. Then F_3 is L–bilipschitz for some L depending only on n and $a_0, a_1, \dots, a_{n-1}$.

Proof The proof is similar to that of Lemma 7.2. Let $x, x' \in \mathbb{R}^n$ be arbitrary. Assume $D_{J_n}(x,x') = e^t$ and $D_{J_n}(F_3(x), F_3(x')) = e^s$. Then we have $e^t = |e^{-tN}(x'-x)|$ and $e^s = |e^{-sN}(F_3(x')-F_3(x))|$. We need to find a constant a that depends only on n and the numbers $a_0, \cdots, a_{n-1}$ such that $|s-t|\le a$.

Set $B_1 = e^{(t-s)N}$ and $B_2 = a_0 I_n + a_1 N + \cdots + a_{n-1}N^{n-1}$. Notice that B_2 commutes with N. We have

$$\begin{aligned}e^s &= |e^{-sN}(F_3(x')-F_3(x))| = |e^{(t-s)N}e^{-tN}B_2(x'-x)| \\ &= |B_1B_2e^{-tN}(x'-x)| \le \|B_1\|\cdot\|B_2\|\cdot|e^{-tN}(x'-x)| \\ &\le \sqrt{Q(B_1)}\sqrt{Q(B_2)}\,e^t.\end{aligned}$$

Hence $e^{s-t} \le \sqrt{Q(B_1)Q(B_2)}$. Since $Q(B_1)Q(B_2)$ is a polynomial in $t-s$ that depends only on n and the numbers $a_0, \dots, a_{n-1}$, there is some constant $a>0$ depending only on n and $a_0, \dots, a_{n-1}$ such that $s-t\le a$.

Notice that $F_3^{-1}(x) = B_2^{-1}x$. Set

$$\beta = -\left(\frac{a_1}{a_0}N + \cdots + \frac{a_{n-1}}{a_0}N^{n-1}\right).$$

Then $\beta^n = 0$. We have $B_2 = a_0(I-\beta)$ and $B_2^{-1} = a_0^{-1}(I+\beta+\beta^2+\cdots+\beta^{n-1})$. It follows that B_2^{-1} has the expression $B_2^{-1} = a_0^{-1}I + b_1N + \cdots + b_{n-2}N^{n-2} +$

$b_{n-1}N^{n-1}$, where $b_1, \dots, b_{n-1}$ are constants depending only on $a_0, \dots, a_{n-1}$. Now the preceding paragraph implies that $t - s \le a'$ for some constant a' depending only on n and $a_0^{-1}, b_1, \dots, b_{n-1}$, hence only on n and $a_0, \dots, a_{n-1}$. Therefore $|s-t| \le \max\{a, a'\}$, and the proof of Lemma 7.3 is complete. □

To prove that every quasisymmetric map has the described type, we induct on n. The basic step $n = 2$ is given by Theorem 3.6. Now we assume $n \ge 3$ and that Theorem 7.1 holds for J_{n-1}.

Let $F\colon (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ be a quasisymmetric map. Let $\mathcal{F}_i$ $(i = 1, \dots, n-1)$ be the foliation of $\mathbb{R}^n$ consisting of affine subspaces parallel to the linear subspace

$$H_i := \{x = (x_1, \dots, x_n)^T \in \mathbb{R}^n \mid x_{i+1} = \dots = x_n = 0\}.$$

Then the proof of Theorem 1.2 shows that the foliation $\mathcal{F}_i$ is preserved by F. To be more precise, if H is an affine subspace parallel to H_i, then $F(H)$ is also an affine subspace parallel to H_i. In particular, F maps every line parallel to the x_1–axis (that is, parallel to H_1) to a line parallel to the x_1–axis, and maps every horizontal hyperplane (that is, parallel to H_{n-1}) to a horizontal hyperplane. It follows that there is a map $G\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that for any $y \in \mathbb{R}^{n-1}$, $F(\mathbb{R} \times \{y\}) = \mathbb{R} \times \{G(y)\}$. For each $y \in \mathbb{R}^{n-1}$, there is a map $H(\cdot, y)\colon \mathbb{R} \to \mathbb{R}$ such that $F(x_1, y) = (H(x_1, y), G(y))$.

Arguments similar to the proofs of Lemmas 3.3 and 3.4 show the following:

(1) For each $y \in \mathbb{R}^{n-1}$, the restriction of D_{J_n} to $\mathbb{R} \times \{y\}$ agrees with the Euclidean distance on $\mathbb{R}$.

(2) For any two $y_1, y_2 \in \mathbb{R}^{n-1}$, the Hausdorff distance with respect to D_{J_n} satisifes $HD(\mathbb{R} \times \{y_1\}, \mathbb{R} \times \{y_2\}) = D_{J_{n-1}}(y_1, y_2)$.

(3) For any $p = (x_1, y_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $y_2 \in \mathbb{R}^{n-1}$, we have $D_{J_n}(p, \mathbb{R} \times \{y_2\}) = D_{J_{n-1}}(y_1, y_2)$.

Hence each $H(\cdot, y)\colon (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is quasisymmetric, and the arguments of [22, page 11] shows that $G\colon (\mathbb{R}^{n-1}, D_{J_{n-1}}) \to (\mathbb{R}^{n-1}, D_{J_{n-1}})$ is also quasisymmetric.

The induction hypothesis applied to G establishes constants $a_0 \ne 0$, $a_1, \dots, a_{n-3}$, b_i $(2 \le i \le n)$ and a Lipschitz map $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$G\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_0x_2 + a_1x_3 + \dots + a_{n-3}x_{n-1} + b_2 + g(x_n) \\ a_0x_3 + a_1x_4 + \dots + a_{n-3}x_n + b_3 \\ \vdots \\ a_0x_{n-1} + a_1x_n + b_{n-1} \\ a_0x_n + b_n \end{pmatrix}.$$

Notice that the horizontal hyperplane $\mathbb{R}^{n-1} \times \{x_n\}$ at height x_n is mapped by F to the horizontal hyperplane $\mathbb{R}^{n-1} \times \{a_0 x_n + b_n\}$ at height $a_0 x_n + b_n$. Since the restriction of D_{J_n} to a horizontal hyperplane agrees with $D_{J_{n-1}}$ (Lemma 3.3), the map

$$F\colon (\mathbb{R}^{n-1} \times \{x_n\}, D_{J_{n-1}}) \to (\mathbb{R}^{n-1} \times \{a_0 x_n + b_n\}, D_{J_{n-1}})$$

is quasisymmetric. Now the induction hypothesis, the fact $F(x_1, y) = (H(x_1, y)G(y))$ and the expression of G imply that

$$H(x_1, y) = a_0 x_1 + a_1 x_2 + \cdots + a_{n-3} x_{n-2} + c_1(x_n) + c_2(x_{n-1}, x_n),$$

where $c_1\colon \mathbb{R} \to \mathbb{R}$ and $c_2\colon \mathbb{R}^2 \to \mathbb{R}$ are two maps and for each fixed v, $c_2(u, v)$ is Lipschitz in u. Since F is a homeomorphism, c_1 and c_2 are continuous. Define $c_3\colon \mathbb{R}^2 \to \mathbb{R}$ by $c_3(u, v) = c_1(v) + c_2(u, v)$. After composing F with a map of the described type, we may assume F has the following form

$$F(x_1, x_2, \dots, x_n) = (x_1 + c_3(x_{n-1}, x_n), x_2 + g(x_n), x_3, \dots, x_n).$$

We need to show that there are constants a_{n-2}, d_2 and a Lipschitz map $C\colon \mathbb{R} \to \mathbb{R}$ such that $g(x_n) = a_{n-2} x_n + d_2$ and $c_3(x_{n-1}, x_n) = a_{n-2} x_{n-1} + C(x_n)$.

Lemma 7.4 *There is a constant L such that the following holds for all $u, v, v' \in \mathbb{R}$:*

$$\Big|\big\{c_3(u + (v' - v)\ln|v' - v|, v') - c_3(u, v)\big\} - \ln|v' - v|\{g(v') - g(v)\}\Big| \le L|v' - v|$$

Proof Let $u, v, v' \in \mathbb{R}$. Let $x \in \mathbb{R}^n$ with $x_{n-1} = u$, $x_n = v$. Set $t = \ln|v' - v|$ and let $y = (y_1, \dots, y_n)^T$ be the unique solution of $e^{-tN} y = (0, \dots, 0, v' - v)^T$. Let $x' = x + y$. Notice $y_n = v' - v$, $y_{n-1} = (v' - v)\ln|v' - v|$, $x'_n = v'$ and

$$x'_{n-1} = x_{n-1} + y_{n-1} = u + (v' - v)\ln|v' - v|.$$

Notice also that t is the smallest solution for $e^t = |e^{-tN}(x' - x)|$ and so $D_{J_n}(x, x') = e^t$. Suppose $D_{J_n}(F(x), F(x')) = e^s$. Then $e^s = |e^{-sN}(F(x') - F(x))|$. By Theorem 1.2, F is L_1–bilipschitz for some $L_1 \ge 1$. Hence $e^t/L_1 \le e^s \le L_1 e^t$. It follows that $|t - s| \le \ln L_1$. Now we write the following:

$$e^{-sN}(F(x') - F(x))$$

$$= e^{-sN}(x' - x) + e^{-sN} \begin{pmatrix} c_3(x'_{n-1}, x'_n) - c_3(x_{n-1}, x_n) \\ g(x'_n) - g(x_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= e^{(t-s)N} e^{-tN} (x'-x) + e^{(t-s)N} e^{-tN} \begin{pmatrix} c_3(x'_{n-1}, x'_n) - c_3(x_{n-1}, x_n) \\ g(x'_n) - g(x_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= e^{(t-s)N} \begin{pmatrix} \{c_3(x'_{n-1}, x'_n) - c_3(x_{n-1}, x_n)\} - t\{g(x'_n) - g(x_n)\} \\ g(x'_n) - g(x_n) \\ 0 \\ \vdots \\ 0 \\ x'_n - x_n \end{pmatrix}$$

Set

$$\tau = \{c_3(x'_{n-1}, x'_n) - c_3(x_{n-1}, x_n)\} - t\{g(x'_n) - g(x_n)\}.$$

The first entry of $e^{-sN}(F(x') - F(x))$ is

$$q := \tau + (t-s)\{g(x'_n) - g(x_n)\} + \frac{(t-s)^{n-1}}{(n-1)!}(x'_n - x_n).$$

We have

$$|q| \le |e^{-sN}(F(x') - F(x))| = e^s \le L_1 e^t = L_1 |v' - v|.$$

Recall that g is L_2–Lipschitz for some $L_2 \ge 0$. Hence,

$$|g(x'_n) - g(x_n)| \le L_2 |x'_n - x_n| = L_2 |v' - v|.$$

Now it follows from $|t-s| \le \ln L_1$ and the triangle inequality that

$$|\tau| \le \left(L_1 + L_2 \ln L_1 + \frac{(\ln L_1)^{n-1}}{(n-1)!} \right) |v' - v|. \qquad \square$$

Recall that the map g is Lipschitz and for each fixed v, $c_3(u, v)$ is Lipschitz in u. Hence g is differentiable ae, and for each fixed v, the partial derivative $\partial c_3 / \partial u$ exists for ae u.

Lemma 7.5 *Let v be any point such that $g'(v)$ exists. Then $c_3(u, v) = c_3(0, v) + g'(v)u$ for all u.*

Proof Fix an arbitrary $u \in \mathbb{R}$. Let $a > 0$. For any positive integer n, define $(y_0, z_0) = (u, v)$ and $(y_i, z_i) = (u + i(a/n)\ln(a/n), v + i(a/n))$ $(1 \le i \le n)$. Applying Lemma 7.4

to y_{i-1}, z_{i-1}, z_i we obtain

$$\left|\{c_3(y_i, z_i) - c_3(y_{i-1}, z_{i-1})\} - \ln\left(\frac{a}{n}\right)\{g(z_i) - g(z_{i-1})\}\right| \le L\frac{a}{n}.$$

Now let $k = k(n)$ be the integer part of $n/\ln(n/a)$. Then $(k/n)\ln(a/n) \to -1$ as $n \to \infty$. Combining the above inequalities for $1 \le i \le k$ and using the triangle inequality, we obtain

$$\left|\{c_3(y_k, z_k) - c_3(u, v)\} - \ln\left(\frac{a}{n}\right)\{g(z_k) - g(v)\}\right| \le L\frac{ak}{n}.$$

Now divide both sides by $(ak/n)\ln(n/a)$ (which converges to a as $n \to \infty$), we get

$$\left|\frac{\{c_3(y_k, z_k) - c_3(u, v)\}}{(ak/n)\ln(n/a)} + \frac{\{g(z_k) - g(v)\}}{(ak/n)}\right| \le \frac{L}{\ln(n/a)}.$$

As $n \to \infty$, we have $z_k = v + (ak/n) \to v$, $y_k \to u - a$. Also, since $g'(v)$ exists, we have

$$\frac{\{g(z_k) - g(v)\}}{(ak/n)} \to g'(v).$$

Consequently,

$$\frac{c_3(u-a, v) - c_3(u, v)}{a} + g'(v) = 0.$$

Hence $c_3(u-a, v) - c_3(u, v) = -ag'(v)$ for all $u \in \mathbb{R}$ and all $a > 0$. It follows that $c_3(u, v) = c_3(0, v) + g'(v)u$ for all u. □

Lemma 7.6 *Suppose g is differentiable at v_1 and v_2. Then $g'(v_1) = g'(v_2)$.*

Proof By Lemma 7.5, we have $c_3(u, v_1) = c_3(0, v_1) + ug'(v_1)$ and

$$c_3(u + [v_2 - v_1]\ln|v_2 - v_1|, v_2) = c_3(0, v_2) + (u + [v_2 - v_1]\ln|v_2 - v_1|)g'(v_2)$$

for all u. Now Lemma 7.4 applied to u, v_1, v_2 implies that $\left|u(g'(v_2) - g'(v_1))\right| \le C$ holds for all u, where C is a quantity independent of u. Thus $g'(v_2) - g'(v_1) = 0$. □

Completing the proof of Theorem 7.1 Lemma 7.6 implies that g is an affine function and hence there are constants a, b such that $g(v) = av + b$. It now follows from Lemma 7.5 that for any v we have $c_3(u, v) = c_3(0, v) + au$. To finish the proof of Theorem 7.1, it remains to show that $c_3(0, v)$ is Lipschitz in v. This follows immediately from Lemma 7.4 after plugging in the formulas for g and c_3.

Now the proof of Theorem 7.1 is complete. □

8 A Liouville-type theorem

In this section we prove a Liouville-type theorem for G_A in the case when A is a Jordan block: every conformal map of the ideal boundary of G_A extends to an isometry of G_A.

Let X and Y be quasimetric spaces with finite Hausdorff dimension. Denote by H_X and H_Y their Hausdorff dimensions and by $\mathcal{H}_X$ and $\mathcal{H}_Y$ their Hausdorff measures. We say a quasisymmetric map $f\colon X \to Y$ is conformal if

(1) $L_f(x) = l_f(x) \in (0, \infty)$ for $\mathcal{H}_X$–almost every $x \in X$,

(2) $L_{f^{-1}}(y) = l_{f^{-1}}(y) \in (0, \infty)$ for $\mathcal{H}_Y$–almost every $y \in Y$.

We now describe some isometries of G_A. For any $g = (x, t) \in G_A = \mathbb{R}^n \rtimes \mathbb{R}$, the Lie group left translation L_g is an isometry. If $g = (x, 0)$, then the boundary map $\partial L_g\colon \mathbb{R}^n \to \mathbb{R}^n$ of L_g is translation by x. If $g = (0, t)$, then the boundary map of L_g is the similarity e^{tA}. Let $\tau'\colon G_A \to G_A$ be given by $\tau'(x, t) = (-x, t)$. Then τ' is an isometry, and its boundary map is $\tau\colon \mathbb{R}^n \to \mathbb{R}^n$, $\tau(x) = -x$.

Theorem 8.1 *Let J_n be the $n \times n$ $(n \geq 2)$ Jordan matrix with eigenvalue 1. Then every conformal map $F\colon (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ is the boundary map of an isometry $G_{J_n} \to G_{J_n}$.*

We first prove the case $n = 2$.

Lemma 8.2 *Every conformal map $F\colon (\mathbb{R}^2, D_{J_2}) \to (\mathbb{R}^2, D_{J_2})$ is the boundary map of an isometry $G_{J_2} \to G_{J_2}$.*

Proof Since F is conformal, it is quasisymmetric in particular. By Theorem 7.1, F has the following form: $F(x, y) = (ax + c(y), ay + b)$, where $a \neq 0$, b are constants and $c\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz map. By composing F with the boundary maps of the isometries described before Theorem 8.1, we may assume $a = 1$ and $b = 0$; that is, F has the form $F(x, y) = (x + c(y), y)$. We shall prove that $c(y)$ is a constant function.

Since $c\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, it is differentiable ae. We shall show that $c'(y) = 0$ for ae $y \in \mathbb{R}$. By the definition of a conformal map, $L_F(x, y) = l_F(x, y)$ for ae $(x, y) \in \mathbb{R}^2$ with respect to the Lebesgue measure in $\mathbb{R}^2$. It follows from Fubini's Theorem that for ae $y \in \mathbb{R}$, the derivative $c'(y)$ exists and $L_F(x, y) = l_F(x, y)$ for ae $x \in \mathbb{R}$. Let y_0 be an arbitrary such point and $x_0 \in \mathbb{R}$ be such that $L_F(x_0, y_0) = l_F(x_0, y_0)$. We will show $c'(y_0) = 0$.

By precomposing and postcomposing with Euclidean translations if necessary, we may assume that $(x_0, y_0) = (0, 0)$ and $c(y_0) = 0$. We need to show $c'(0) = 0$. We will suppose $c'(0) \neq 0$ and get a contradiction. Notice that $F(x, 0) = (x, 0)$ for all $x \in \mathbb{R}$. It follows that $L_F(0, 0) \geq 1$ and $l_F(0, 0) \leq 1$. Combining this with the assumption $L_F(0, 0) = l_F(0, 0)$, we obtain $L_F(0, 0) = l_F(0, 0) = 1$. First suppose $c'(0) > 0$. Then $c(y) > 0$ for sufficiently small $y > 0$. Let $p = (0, 0)$ and $q = (r + r \ln r, r)$ with $r > 0$. Then $F(p) = p$ and $F(q) = (r + r \ln r + c(r), r)$. One calculates $D(p, q) = r$ and $D(F(p), F(q)) = r + c(r)$. It follows that $L_F(p, r) \geq r + c(r)$ and hence $L_F(p) \geq 1 + c'(0) > 1$, contradicting $L_F(0, 0) = 1$. If $c'(0) < 0$, then letting $q = (-r + r \ln r, r)$ one similarly obtains a contradiction. $\square$

Proof of Theorem 8.1 Let $F\colon (\mathbb{R}^n, D_{J_n}) \to (\mathbb{R}^n, D_{J_n})$ be a quasisymmetric map. After composing with the boundary maps of isometries described before Theorem 8.1, we may assume F has the following form

$$F(x) = (I + a_1 N + \cdots + a_{n-2} N^{n-2})x + (C(x_n), 0, \ldots, 0)^T,$$

where $C\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz. We will prove the following statement by inducting on n:

If F as above is conformal, then $a_1 = \cdots = a_{n-2} = 0$ and C is constant.

The basic step $n = 2$ is Lemma 8.2. Now we assume $n \geq 3$ and that the statement holds for Jordan matrices with sizes less than or equal to $n - 1$. Notice that F maps every horizontal hyperplane $H(x_n) := \mathbb{R}^{n-1} \times \{x_n\}$ to itself. By Lemma 3.3 the restriction of D_{J_n} on $H(x_n)$ agrees with the metric $D_{J_{n-1}}$. It now follows from Fubini's Theorem that for ae $x_n \in \mathbb{R}$, the restricted map

$$F|_{H(x_n)}\colon (H(x_n), D_{J_{n-1}}) \to (H(x_n), D_{J_{n-1}})$$

is also conformal. Now the induction hypothesis applied to $F|_{H(x_n)}$ implies that $a_i = 0$ for $1 \leq i \leq n - 2$. It remains to show C is constant.

Suppose C is not constant. Then there is some $u \in \mathbb{R}$ such that $C'(u) \neq 0$ and $L_F(p) = l_F(p)$ for some $p \in H(u)$. After precomposing and postcomposing with Euclidean translations, we may assume $u = 0$, $C(0) = 0$ and p is the origin o. Notice that the restriction of F to the x_1–axis is the identity, so $L_F(o) = l_F(o) = 1$. Now for any $x_n > 0$, choose $x_1, \ldots, x_{n-1}$ such that $x = (x_1, \ldots, x_n)^T$ satisfies $e^{-tN} x = (0, \ldots, 0, x_n)^T$, where $t = \ln x_n$. It follows that $D_{J_n}(o, x) = e^t = x_n$. Suppose $D_{J_n}(F(o), F(x)) = e^s$. Then $e^s = |e^{-sN} F(x)|$. We calculate as before that

$$e^{-sN} F(x) = \left(C(x_n) + \frac{(t-s)^{n-1}}{(n-1)!} x_n, \frac{(t-s)^{n-2}}{(n-2)!} x_n, \ldots, (t-s)x_n, x_n \right)^T.$$

Since $L_F(o) = l_F(o) = 1$, we must have $e^s/e^t = D_{J_n}(F(x), F(o))/D_{J_n}(x, o) \to 1$ as $x_n \to 0$ and hence $t - s \to 0$. Now

$$e^s = \left|e^{-sN} F(x)\right| = x_n \left|\left(\frac{C(x_n)}{x_n} + \frac{(t-s)^{n-1}}{(n-1)!}, \frac{(t-s)^{n-2}}{(n-2)!}, \dots, (t-s), 1\right)^T\right|.$$

Since $x_n = e^t$, we have

$$e^{s-t} = \left|\left(\frac{C(x_n)}{x_n} + \frac{(t-s)^{n-1}}{(n-1)!}, \frac{(t-s)^{n-2}}{(n-2)!}, \dots, (t-s), 1\right)^T\right|.$$

Now as $x_n \to 0$, the right hand side converges to

$$\left|(C'(0), 0, \dots, 0, 1)^T\right| = \sqrt{1 + (C'(0))^2},$$

which is greater than 1 since $C'(0) \neq 0$. However, the left-hand side converges to 1. The contradiction shows C must be a constant function. □

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Department of Mathematics and Statistics, Bowling Green State University
Bowling Green, OH 43403, USA

`xiex@bgsu.edu`

`http://personal.bgsu.edu/~xiex/`

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