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Rational curves and special metrics on twistor spaces

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A Hermitian metric ω on a complex manifold is called *SKT* or *pluriclosed* if $dd^c\omega = 0$. Let M be a twistor space of a compact, anti-selfdual Riemannian manifold, admitting a pluriclosed Hermitian metric. We prove that in this case M is Kähler, hence isomorphic to $\mathbb{C}P^3$ or a flag space. This result is obtained from rational connectedness of the twistor space, due to F Campana. As an aside, we prove that the moduli space of rational curves on the twistor space of a $K3$ surface is Stein.

53C28; 32Q15, 53C26

To Professor H Blaine Lawson on his 70^{th} birthday

1 Introduction

1.1 Special Hermitian metrics on complex manifolds

The world of non-Kähler complex geometry is infinitely bigger than that inhabited by Kähler manifolds. For instance, as shown by Taubes [29] (see also Panov and Petrunin [24]), any finitely-generated group can be realized as a fundamental group of a compact complex manifold. On the contrary, the Kähler condition puts big restrictions on the fundamental group.

However, there are not many constructions which lead to explicit non-Kähler complex manifolds. There are many homogeneous and locally homogeneous manifolds (such as complex nilmanifolds), which are known to be non-Kähler. The locally conformally Kähler manifolds are non-Kähler by a theorem of Vaisman [30]. Kodaira class VII surfaces (forming a vast and still not completely understood class of complex surfaces) are never Kähler. Finally, the twistor spaces, as shown by Hitchin, are never Kähler, except two examples: $\mathbb{C}P^3$, being a twistor space of S^4, and the flag space, being a twistor space of $\mathbb{C}P^2$; see Hitchin [17].

There are many ways to weaken the Kähler condition $d\omega = 0$. Given a Hermitian form ω on a complex n–manifold, one may consider an equation $d(\omega^k) = 0$. For $1 < k < n-1$, this equation is equivalent to $d\omega = 0$, but the equation $d(\omega^{n-1})$ is quite nontrivial. Such metrics are called *balanced*. All twistor spaces are balanced (see

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Mourougane [22]); also, all Moishezon manifolds are balanced (see Alessandrini and Bassanelli [3]). Another way to weaken the Kähler condition is to consider the equation $dd^c(\omega^k)=0$, where $d^c=-IdI$; this equation is nontrivial for all $0<k<n$. When $k=1$, a metric satisfying $dd^c\omega=0$ is called *pluriclosed*, or *strong Kähler torsion (SKT) metric*; such metrics are quite important in physics and in generalized complex geometry. A Hermitian metric satisfying $dd^c(\omega^{n-1})=0$ is called *Gauduchon*. As shown by P Gauduchon [14], every Hermitian metric is conformally equivalent to a Gauduchon metric, which is unique in its conformal class up to a constant multiplier.

Since a twistor space has complex dimension 3, and is balanced, the only nontrivial metric condition (among those mentioned above) for the twistor space is $dd^c(\omega)=0$.

The main result of this paper is the following theorem, which can be considered as a generalization of Hitchin's theorem on non-Kählerianity of twistor spaces.

Theorem 1.1 *Let M be a twistor space of a compact 4–dimensional anti-selfdual Riemannian manifold. Assume that M admits a pluriclosed Hermitian form ω: $dd^c(\omega)=0$. Then M is Kähler.*

Proof See Corollary 3.4. □

1.2 Strongly Gauduchon and symplectic Hermitian metrics

The Gauduchon, pluriclosed and all the rest of the $dd^c(\omega^k)=0$ Hermitian metrics have an interesting variation of a cohomological nature.

Definition 1.2 Let (M,I) be a complex manifold, and ω a Hermitian form. We say that ω^k is *strongly pluriclosed* if either of the following equivalent conditions are satisfied:

(i) $d(\omega^k)$ is dd^c–exact.

(ii) ω^k is the (k,k)–part of a closed $2k$–form.

Notice that either of these conditions easily implies $dd^c(\omega^k)=0$, but these conditions are significantly stronger.

For $k=1$ and $n-1$ this condition is especially interesting. When a pluriclosed Hermitian form ω is the $(1,1)$–part of a closed (and hence symplectic) form $\widetilde{\omega}$, ω is called *taming* or *Hermitian symplectic*, and when $(\omega)^{n-1}$ is the $(n-1,n-1)$–part of a closed form, ω is called *strongly Gauduchon*; see Popovici [26].

In [28] Streets and Tian have constructed a parabolic flow of pluriclosed metrics, analogous to the Kähler–Ricci flow. If the initial condition is Hermitian symplectic, it is easy to see that the solution also remains Hermitian symplectic. Streets and Tian asked whether there exists a compact complex Hermitian symplectic manifold not admitting a Kähler structure. This question was considered by Enrietti, Fino and Grantcharov in [11] and by Enrietti, Fino and Vezzoni in [12] for complex nilmanifolds. In [12] it was shown that complex nilmanifolds cannot admit Hermitian symplectic metrics. However, the pluriclosed metrics exist on many complex nilmanifolds.

The present paper grew as an attempt to answer the Streets–Tian question for twistor spaces. However, it was found that the twistor spaces are not only never Hermitian symplectic, they never admit a pluriclosed metric unless they are Kähler.

1.3 Rational curves and pluriclosed metrics

The results of the present paper are based on the study of the moduli of rational curves. Unlike many complex nonalgebraic manifolds, the twistor spaces are very rich in curves: there exists a smooth rational curve passing through any finite subset of a twistor space; see Claim 2.8.

For an almost complex structure I equipped with a taming symplectic form, all components of the space of complex curves are compact, by Gromov's compactness theorem [15; 4]. I will show that the same is true for pluriclosed metrics if I is integrable; see Corollary 2.19. This is used to prove that a twistor space admitting a pluriclosed metric is actually Moishezon; see Theorem 2.23. However, Moishezon varieties satisfy the dd^c–lemma. This is used to show that any pluriclosed metric is in fact Hermitian symplectic. Finally, by using the Peternell's theorem from [25], we prove that no Moishezon manifold can be Hermitian symplectic; see Corollary 3.4.

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2 Twistor spaces for 4–dimensional Riemannian manifolds and the space of rational curves

2.1 Twistor spaces for 4–dimensional Riemannian manifolds: Definition and basic results

Definition 2.1 Let M be a Riemannian 4–manifold. Consider the action of the Hodge $*$–operator, $*\colon \Lambda^2 M \to \Lambda^2 M$. Since $*^2 = 1$, the eigenvalues are ± 1, and one has a decomposition $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ onto *selfdual* ($*\eta = \eta$) and *anti-selfdual* ($*\eta = -\eta$) forms.

Remark 2.2 If one changes the orientation of M, leaving metric the same, $\Lambda^+ M$ and $\Lambda^- M$ are exchanged. Therefore, $\dim \Lambda^2 M = 6$ implies $\dim \Lambda^\pm(M) = 3$.

Remark 2.3 Using the isomorphism $\Lambda^2 M = \mathfrak{so}(TM)$, we interpret $\eta \in \Lambda^2_m M$ as endomorphisms of $T_m M$. Then the unit vectors $\eta \in \Lambda^+_m M$ correspond to oriented, orthogonal complex structures on $T_m M$.

Definition 2.4 Let $\mathrm{Tw}(M) := S\Lambda^+ M$ be the set of unit vectors in $\Lambda^+ M$. At each point $(m,s) \in \mathrm{Tw}(M)$, consider the decomposition $T_{m,s}\,\mathrm{Tw}(M) = T_m M \oplus T_s S\Lambda^+_m M$ induced by the Levi-Civita connection. Let I_s be the complex structure on $T_m M$ induced by s, $I_{S\Lambda^+_m M}$ the complex structure on $S\Lambda^+_m M = S^2$ induced by the metrics and orientation, and $\mathcal{I}\colon T_{m,s}\,\mathrm{Tw}(M) \to T_{m,s}\,\mathrm{Tw}(M)$ be equal to $\mathcal{I}_s \oplus I_{S\Lambda^+_m M}$. The almost complex manifold $(\mathrm{Tw}(M), \mathcal{I})$ is called *the twistor space* of M.

The following results about twistor spaces are well known (see eg Besse [5]).

Theorem 2.5 *The almost complex structure on* $(\mathrm{Tw}(M), \mathcal{I})$ *is a conformal invariant of M. Moreover, one can reconstruct the conformal structure on M from the almost complex structure on* $\mathrm{Tw}(M)$ *and its anticomplex involution $(m,s) \to (m,-s)$.*

Theorem 2.6 *The almost complex manifold* $(\mathrm{Tw}(M), \mathcal{I})$ *is a complex manifold if and only if $W^+ = 0$, where W^+ ("self-dual conformal curvature") is an autodual component of the curvature tensor. Such manifolds are called* conformally half-flat *or* ASD (anti-selfdual).

2.2 Rational curves on $\mathrm{Tw}(M)$

Definition 2.7 *An ample rational curve* on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$ with $i_k > 0$. It is called *a quasiline* if all $i_k = 1$.

Claim 2.8 *Let M be a compact complex manifold containing a an ample rational line. Then any N points $z_1, \dots, z_N$ can be connected by an ample rational curve.*

Proof This fact is well known in algebraic geometry; see Kollár [21]. However, its proof is valid for all complex manifolds. □

Claim 2.9 *Let M be a Riemannian 4–manifold, $\mathrm{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point and $S_m := \sigma^{-1}(m) = S\Lambda^+_m(M)$ the corresponding S^2 in $\mathrm{Tw}(M)$. Then S_m is a quasiline.*

Proof Since the claim is essentially infinitesimal, it suffices to check it when M is flat. Then $\mathrm{Tw}(M) = \mathrm{Tot}(\mathcal{O}(1)^{\oplus 2}) \cong \mathbb{C}P^3 \backslash \mathbb{C}P^1$ and S_m is a section of $\mathcal{O}(1)^{\oplus 2}$. □

Corollary 2.10 *Any N points $z_1, \dots, z_N$ on a twistor space can be connected by an smooth, ample rational curve* □

2.3 Rational curves and plurinegative metrics

For other applications of Gromov's compactness theorem on manifolds with pluriclosed metrics, please see Ivashkovich [18].

Definition 2.11 Let S be a complex curve on a Hermitian manifold (M, I, g, ω). Define *the Riemannian volume* as $\mathrm{Vol}(S) := \int_S \omega$.

Definition 2.12 A Hermitian form ω is called *plurinegative* (*pluripositive*) if the (2,2)–form $dd^c\omega$ is negative (positive).

Example 2.13 As shown by Kaledin and the author in [20, (8.12)], a standard Hermitian form on a twistor space of a hyperkähler manifold is pluripositive.

Remark 2.14 It is interesting to ask for a meaningful differential-geometric condition on a Riemannian manifold M which can insure pluripositivity or plurinegativity of a form $\partial\bar{\partial}\omega$ on its twistor space $\mathrm{Tw}(M)$, where ω is the standard Hermitian form on twistors. Some computations in this direction were done by Mourougane in [23]. However, the only example they have is the one obtained in [20, (8.12)] (rediscovered independently). As follows from Corollary 2.25, on a compact twistor manifold any plurinegative metric is pluriclosed.

Remark 2.15 The notion of "positive (k,k)–form" comes in two flavours: *weakly positive* and *strongly positive*; see Demailly [9]. When $k=1$ or $k=\dim_{\mathbb{C}} M-1$, these two notions coincide. Since in the present paper we are interested mostly in 3–dimensional complex manifolds, this distinction becomes irrelevant. For the sake of a definition, we shall consider, in the present paper, "positive" as a synonym to "strongly positive".

Of course, pluriclosed Hermitian metrics are both pluripositive and plurinegative.

Claim 2.16 *Let X be a component of the moduli of complex curves on a given complex manifold, $\tilde{X}$ the set of pairs $\{S\in X, z\in S\subset M\}$, ("the universal family"), and $\pi_M\colon \tilde{X}\to M$, $\pi_X\colon \tilde{X}\to X$ the forgetful maps. Then the volume function* $\operatorname{Vol}\colon X\to\mathbb{R}^{>0}$ *can be expressed as* $\operatorname{Vol}=(\pi_X)_*\pi_M^*\omega$. □

Remark 2.17 Since the pullback and pushforward of differential forms commute with d, d^c, this gives $dd^c\operatorname{Vol}=(\pi_X)_*\pi_M^*(dd^c\omega)$ (see eg [20, (8.12)], the author [31, Theorem 2.10] or Ivashkovich [19, Proposition 1.9]). Therefore, $-\operatorname{Vol}$ is plurisubharmonic on X whenever ω is plurinegative.

Theorem 2.18 (Gromov) *Let M be a compact Hermitian almost complex manifold, $\mathfrak{X}$ the space of all complex curves on M and* $\operatorname{Vol}\colon\mathfrak{X}\to\mathbb{R}^{>0}$ *the volume function. Then* Vol *is proper (that is, preimage of a compact set is compact).*

Proof See [15; 4]. □

Corollary 2.19 *Let M be a complex manifold, equipped with a plurinegative Hermitian form ω and X a component of the moduli of complex curves. Then the function* $\operatorname{Vol}\colon X\to\mathbb{R}^{>0}$ *is constant, and X is compact.*

Proof Since $\operatorname{Vol}\geqslant 0$, the set $\operatorname{Vol}^{-1}(]-\infty,C])$ is compact for all $C\in\mathbb{R}$, hence $-\operatorname{Vol}$ has a maximum somewhere in X. However, a plurisubharmonic function which has a maximum is necessarily constant by E Hopf's strong maximum principle. Therefore, Vol is constant: $\operatorname{Vol}=A$. Now compactness of $X=\operatorname{Vol}^{-1}(A)$ follows from Gromov's theorem. □

2.4 Quasilines and Moishezon manifolds

Let M be a compact complex manifold, and $S\subset M$ an ample rational curve. Assume that the space of deformations of S in M is compact. From Campana [6, Theorem 3] it follows that M is Moishezon; see Campana [7, Remark 3.2 and Theorem 4.5]. For the convenience of the reader, I will give an independent proof of this result here.

Recall that *a quasiline* is a smooth rational curve $S \subset M$ such that its normal bundle is isomorphic to $\mathcal{O}(1)^n$. A neighbourhood of a quasiline shares many properties with a neighbourhood of a line in $\mathbb{C}P^n$. Heuristically, this can be stated as follows.

An imprecise statement Let $S \subset M$ be a quasiline. Then, for an appropriate tubular neighbourhood $U \subset M$ of S, "for every two points $x, y \in U$ close to S and far from each other, there is a unique deformation of S containing x and y."

More precisely, we have the following.

Claim 2.20 *Let $S \subset M$ be a quasiline. Then, for any sufficiently small tubular neighbourhood $U \subset M$ of S, there exists a smaller tubular neighbourhood $W \subset U$, satisfying the following condition. Let Δ_S be the image of the diagonal embedding $\Delta_S\colon S \to W \times W$. Then there exists an open neighbourhood V of Δ_S, properly contained in $W \times W$, such that for any pair $(x, y) \in W \times W \backslash V$, there exists a unique deformation $S' \subset U$ of S containing x and y.* □

Claim 2.21 *A small deformation $S' \subset U$ of S passing through $z \in S$ is uniquely determined by a 1–jet of S' at z.* □

Both of these claims follow from a general results of deformation theory: the first cohomology of the normal bundle NS vanishes, hence there are no obstructions to a deformation, and the deformation space is locally modeled on the space of sections of NS. However, since $NS = \mathcal{O}(1)^n$, any section is uniquely determined by its values in two different points, or by its 1–jet at any given point.

Further on, we shall need the following simple lemma.

Lemma 2.22 *Let $X \to Y$ be a dominant map of complex varieties, which is finite at a general point. Assume that X is Moishezon. Then Y is also Moishezon.*

Proof Replacing X by its ramified covering, we may assume that outside of its singularities, the map $X \to Y$ is a Galois covering, with the Galois group G. Then $X/G \to Y$ is bimeromorphic. Replacing X by its resolution, we can also assume that X is projective. Then X/G is also projective, by Noether's theorem on invariant rings. □

Now we can prove the main result of this subsection (see also [6, Theorem 3]).

Theorem 2.23 *Let M be a complex manifold, $S \subset M$ a quasiline and W its deformation space. Assume that W is compact. Then M is Moishezon.*

Proof **Step 1** Let $z \in M$ be a point, containing a quasiline $S \in W$, W_z the set of all curves $S_1 \in W$ containing z, and $\widetilde{W}_z$, the set of all pairs $\{x \in S_1, S_1 \in W_z\}$. From Claim 2.20, it follows that the map $\widetilde{W}_z \to M$, $(S, x) \to x$ is surjective and finite at a generic point.

Step 2 By Lemma 2.22, it would suffice to prove that $\widetilde{W}_z$ is Moishezon.

Step 3 After an appropriate bimeromorphic transform, we may assume that $\widetilde{W}_z \to W_z$ is a smooth, proper map with rational, 1–dimensional fibers. Then $\widetilde{W}_z$ is Moishezon if and only if W_z is Moishezon. Indeed, the space of cycles in a Moishezon manifold is Moishezon.

Step 4 By Claim 2.21, the map from W_z to $\mathbb{P}T_zM$ mapping a quasiline to its 1–jet is also generically finite to its image. Therefore, W_z is Moishezon. □

Corollary 2.24 *Let M be a twistor space admitting a pluriclosed (or plurinegative) Hermitian metric. Then M is Moishezon.* □

Corollary 2.25 *Let M be a twistor space, and ω a plurinegative metric on M. Then ω is pluriclosed.*

Proof By Corollary 2.24, M is Moishezon. This implies that M admits a positive Kähler current Θ (see eg Demailly and Paun [10]), that is, a positive, closed (1,1)–current satisfying $\Theta \geqslant \varepsilon\omega$ for some $\varepsilon > 0$. Then $\int_M dd^c\omega \wedge \Theta = 0$, because $dd^c\omega$ is exact and Θ closed. However, this integral can be nonzero only if $dd^c\omega = 0$. □

3 Pluriclosed and Hermitian symplectic metrics on twistor spaces

Recall the following classical theorem of Harvey and Lawson [16].

Theorem 3.1 *Let M be a compact, complex n–manifold. Then the following conditions are equivalent:*

(i) *M does not admit a Kähler metric.*

(ii) *M has a nonzero, positive $(n-1, n-1)$–current Θ which is the $(n-1, n-1)$–part of an exact current.*

The same argument, applied to pluriclosed or taming metrics, gives the following (see also Alessandrini and Bassanelli [2]).

Theorem 3.2 *Let M be a compact, complex n–manifold. Then:*

(a) *M admits no Hermitian symplectic metrics if and only if M admits a positive, exact, nonzero $(n-1,n-1)$–current.*

(b) *M admits no pluriclosed metrics if and only if M admits a positive, nonzero, dd^c–exact $(n-1,n-1)$–current.*

Proof (a) Let $A\subset\Lambda^2(M)$ be the cone of real 2–forms η such that the $(1,1)$–part $\eta^{1,1}$ is strictly positive. Then $A\cap\ker d=0$ is equivalent, by the Hahn–Banach theorem, to existence of a $(2n-2)$–current vanishing on $\ker d$ (hence, exact) and positive on A, hence of type $(n-1,n-1)$ and positive.

(b) Let A be the same as above. Then $A\cap\ker dd^c=0$ is equivalent, by Hahn–Banach, to existence of a $(2n-2)$–current Θ positive on A (hence, of type $(n-1,n-1)$ and positive) and vanishing on $\ker dd^c$. A positive dd^c–exact $(n-1,n-1)$–current clearly vanishes on $\ker dd^c$. It remains to show, conversely, that Θ is dd^c–exact whenever Θ vanishes on $\ker dd^c$.

Since $\ker dd^c$ contains the space $\ker d$, the current Θ is exact. Let

$$H^{n-1,n-1}_{BC}(M,\mathbb{R}):=\frac{\ker d|_{\Lambda^{n-1,n-1}(M)}}{dd^c(\Lambda^{n-2,n-2}(M))}$$

be the *Bott–Chern cohomology group*, and

$$H^{1,1}_{AE}(M,\mathbb{R}):=\frac{\ker dd^c|_{\Lambda^{1,1}(M)}}{\operatorname{im} d\cap\Lambda^{1,1}(M)}$$

be the Aeppli cohomology; see Aeppli [1] and Schweitzer [27]. The exterior multiplication induces a pairing between these two groups, and it is not hard to see that they are dual. Since Θ vanishes on $\ker dd^c$, the pairing with its Bott–Chern cohomology class $\langle[\Theta],\cdot\rangle\colon H^{1,1}_{AE}(M,\mathbb{R})\to\mathbb{R}$ vanishes. Therefore, the class of Θ in $H^{n-1,n-1}_{BC}(M,\mathbb{R})$ is equal zero, hence $\eta\in\operatorname{im} dd^c$. Theorem 3.2 is proved. □

This leads to the following useful proposition.

Proposition 3.3 *Any Moishezon space which admits a pluriclosed metric also admits a Hermitian symplectic structure. In particular, any twistor space M which admits a pluriclosed metric also admits a Hermitian symplectic structure.*

Proof By Corollary 2.24, M is Moishezon. Then Deligne, Griffiths, Morgan and Sullivan [8] imply that M satisfies dd^c–lemma. Therefore, any exact (2,2)-current is dd^c–exact. Applying Theorem 3.2, we obtain that M is Hermitian symplectic. □

Corollary 3.4 *Let M be a twistor space admitting a pluriclosed (or Hermitian symplectic) metric. Then M is Kähler.*

Proof Peternell [25, Corollary 2.3] has shown that any compact non-Kähler Moishezon n–manifold admits an exact, positive $(n-1,n-1)$–current. Therefore, it is never Hermitian symplectic Theorem 3.2. By Proposition 3.3, M cannot be pluriclosed. □

Appendix: Rational lines on the twistor space of a $K3$ surface

For a complex manifold Z equipped with a pluripositive Hermitian form, the same argument as used in Corollary 2.19 implies that any component of the moduli of curves on Z is pseudoconvex. In particular, this is true on twistor spaces of hyperkähler manifolds [20; 23]. For a twistor space of $K3$, a stronger result can be achieved.

Theorem A.1 *Let M be a $K3$ surface equipped with a hyperkähler metric and $\mathrm{Tw}(M)$ its twistor space. Denote by X a connected component of the moduli of rational curves on $\mathrm{Tw}(M)$. Then X is Stein.*

Proof The proof is based on the following useful theorem of Fornæss–Narasimhan.

Definition A.2 Let X be a complex variety (possibly singular), and $\varphi\colon X \to [-\infty,\infty[$ an upper semicontinuous function. We say that φ is *plurisubharmonic (in the weak sense)* if for any holomorphic map $f\colon D \to X$ from a disc in $\mathbb{C}$, the composition $f\circ\varphi\colon D \to \mathbb{R}$ is plurisubharmonic (or identically $-\infty$). This function is called *strongly plurisubharmonic* if any perturbation of φ which is small in C^2–topology remains plurisubharmonic.[1]

Theorem A.3 *Let X be a complex variety admitting an exhaustion function which is strictly plurisubharmonic. Then X is Stein.*

Proof See Fornæss and Narasimhan [13, Theorem 6.1]. □

[1] To define precisely what it means "small in C^2–topology", we embed an open subset $U \subset X$ in $\mathbb{C}^n$. Suppose that there exists $\varepsilon > 0$ such that for any function f on $\mathbb{C}^n$ with $|f|_{C^2} < \varepsilon$, the sum $\varphi + f|_U$ is plurisubharmonic. Then φ is called *strongly plurisubharmonic in U*. If X admits a covering by such U, then φ is called *strongly plurisubharmonic on X*.

Now we can prove Theorem A.1. By Gromov's compactness theorem (Theorem 2.18), Vol: $X \to \mathbb{R}$ is exhaustion, and by [20] it is plurisubharmonic. It remains to show that this function is strictly plurisubharmonic.

Let ω be the standard Hermitian form on $\mathrm{Tw}(M) = M \times \mathbb{C}P^1$. Denote by $\omega_{\mathbb{C}P^1}$ the Fubini–Study 2–form on $\mathbb{C}P^1$ and let $\pi^*\omega_{\mathbb{C}P^1}$ be its lifting to $\mathrm{Tw}(M)$, where $\pi\colon \mathrm{Tw}(M) \to \mathbb{C}P^1$ is the twistor projection. Then $dd^c\omega = \omega \wedge \pi^*\omega_{\mathbb{C}P^1}$ [20, (8.12)].

Let $v \in Z_x X$ be a vector from the Zariski tangent cone of X. The strict plurisubharmonicity of Vol would follow if the second derivative $\sqrt{-1}\ \mathrm{Lie}_v \mathrm{Lie}_{\overline{v}} \mathrm{Vol}$ were positive for all $v \neq 0$.

Now, let $x = [S] \in X$ be a point represented by a curve S. Then $Z_x X$ is a subspace of $H^0(NS)$, where NS is the normal sheaf of S. A priori, S can have several irreducible components, some of them sitting in the fibers of the twistor projection $\pi\colon \mathrm{Tw}(M) \to \mathbb{C}P^1$, others transversal to these fibers. However, all components sitting in the fibers of π are fixed, because the rational curves on $K3$ are fixed. Therefore, v in nontrivial along the fibers of π. Now,

$$\sqrt{-1}\ \mathrm{Lie}_v \mathrm{Lie}_{\overline{v}} \mathrm{Vol} = dd^c \mathrm{Vol}(v, \overline{v}) = \int_S (dd^c\omega)(v, \overline{v}) \geqslant \int_S \pi^*\omega_{\mathbb{C}P^1} \cdot \omega(v, \overline{v}).$$

The last integral is positive, because v vanishes on those components of S which belong to the fibers of π, hence $v \neq 0$ on a component S_1 which is transversal to π. Then,

$$\int_S \pi^*\omega_{\mathbb{C}P^1} \cdot \omega(v, \overline{v}) \geqslant \int_{S_1} \pi^*\omega_{\mathbb{C}P^1} \cdot \omega(v, \overline{v}),$$

but this integral is positive, because $\pi^*\omega_{\mathbb{C}P^1}$ is positive on each transversal component of S. This proves that $\sqrt{-1}\ \mathrm{Lie}_v \mathrm{Lie}_{\overline{v}} \mathrm{Vol} > 0$, implying strict plurisubharmonicity of Vol. □

Remark A.4 The variety X, which is shown to be Stein in Theorem A.1, could be singular (a complex variety is *Stein* if it admits a closed holomorphic embedding to $\mathbb{C}^n$). However, there is not a single known example of a singular point in any component of the space $\mathcal{S}(M)$ of rational curves on $\mathrm{Tw}(M)$, when M is a $K3$. It is not hard to see that $\mathcal{S}(M)$ is smooth when M is a compact torus. It is not entirely impossible that it is also smooth for a $K3$.

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