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Grothendieck ring of semialgebraic formulas and motivic real Milnor fibers

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We define a Grothendieck ring for basic real semialgebraic formulas, that is, for systems of real algebraic equations and inequalities. In this ring the class of a formula takes into consideration the algebraic nature of the set of points satisfying this formula and this ring contains as a subring the usual Grothendieck ring of real algebraic formulas. We give a realization of our ring that allows us to express a class as a $\mathbb{Z}[\frac{1}{2}]$–linear combination of classes of real algebraic formulas, so this realization gives rise to a notion of virtual Poincaré polynomial for basic semialgebraic formulas. We then define zeta functions with coefficients in our ring, built on semialgebraic formulas in arc spaces. We show that they are rational and relate them to the topology of real Milnor fibers.

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Introduction

Let us consider the category $SA(\mathbb{R})$ of real semialgebraic sets, the morphisms being the semialgebraic maps. We denote by $(K_0(SA(\mathbb{R})), +, \cdot)$, or simply $K_0(SA(\mathbb{R}))$, the Grothendieck ring of $SA(\mathbb{R})$, that is to say the free ring generated by all semialgebraic sets A, denoted by $[A]$ as viewed as element of $K_0(SA(\mathbb{R}))$, in such a way that for all objects A, B of $SA(\mathbb{R})$ one has $[A \times B] = [A] \cdot [B]$ and for all closed semialgebraic sets F in A one has $[A \setminus F] + [F] = [A]$ (this implies that for all semialgebraic sets A, B one has $[A \cup B] = [A] + [B] - [A \cap B]$).

When an equivalence relation for semialgebraic sets is also considered when defining $K_0(SA(\mathbb{R}))$, one has to be aware that the induced quotient ring, still denoted by $K_0(SA(\mathbb{R}))$ for simplicity, may dramatically collapse. For instance, let us consider the equivalence relation $A \sim B$ if and only if there exists a semialgebraic bijection from A to B. In this case we simply say that A and B are isomorphic. Then for the definition of $K_0(SA(\mathbb{R}))$, starting from classes of isomorphic sets instead of simply

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sets, one obtains a quite trivial Grothendieck ring, namely $K_0(SA(\mathbb{R})) = \mathbb{Z}$. Indeed, denoting $[\mathbb{R}]$ by $\mathbb{L}$ and $[\{*\}]$ by $\mathbb{P}$, from the fact that $\{*\} \times \{*\} \sim \{*\}$, one gets

$$\mathbb{P}^k = \mathbb{P}, \quad \forall k \in \mathbb{N}^*$$

and from the fact that $\mathbb{R} =]-\infty, 0[\cup \{0\} \cup]0, +\infty[$ and that intervals of the same type are isomorphic, one gets

$$\mathbb{L} = -\mathbb{P}.$$

On the other hand, by the semialgebraic cell decomposition theorem, we obtain that a real semialgebraic set is a finite union of disjoint open cells, each of which is isomorphic to $\mathbb{R}^k$ with $k \in \mathbb{N}$ (with the convention that $\mathbb{R}^0 = \{*\}$). It follows that $K_0(SA(\mathbb{R})) = \langle \mathbb{P} \rangle$, the ring generated by $\mathbb{P}$. At this point, the ring $\langle \mathbb{P} \rangle$ could be trivial. But one knows that the Euler–Poincaré characteristic with compact supports $\chi_c : SA(\mathbb{R}) \to \mathbb{Z}$ is surjective. Let us recall that the Euler–Poincaré characteristic with compact supports is a topological invariant defined on locally compact semialgebraic sets and uniquely extended to an additive invariant on all semialgebraic sets (see for instance Coste [4, Theorem 1.22]). Since χ_c is additive, multiplicative and invariant under isomorphisms it factors through $K_0(SA(\mathbb{R}))$, giving a surjective morphism of rings and finally an isomorphism of rings, still denoted by χ_c for simplicity (cf also Quarez [17]):

$$\begin{tikzcd} SA(\mathbb{R}) \arrow[r, "\chi_c"] \arrow[d] & \mathbb{Z} \\ \langle \mathbb{P} \rangle = K_0(SA(\mathbb{R})) \arrow[ur, "\chi_c"'] & \end{tikzcd}$$

The characteristic $\chi_c(A)$ of a semialgebraic set A is in fact defined in the same way, so we obtain the equality $K_0(SA(\mathbb{R})) = \langle \mathbb{P} \rangle$, that is from a specific cell decomposition of A, where $\langle \mathbb{P} \rangle$ is replaced by $\chi_c(\{*\}) = 1$. The difficulty in the definition of χ_c is then to show that χ_c is independent of the choice of the cell decomposition of A (it technically consists in showing that the definition of $\chi_c(A)$ does not depend on the isomorphism class of A; see van den Dries [9] for instance).

When one starts from the category of real algebraic varieties $\mathrm{Var}_{\mathbb{R}}$ or from the category of real algebraic sets $\mathbb{R}\mathrm{Var}$, as we do not have algebraic cell decompositions, we could expect that the induced Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{R}})$ is no longer trivial. This is indeed the case, since for instance the virtual Poincaré polynomial morphism factors through $K_0(\mathrm{Var}_{\mathbb{R}})$ and has image $\mathbb{Z}[u]$ (see McCrory and Parusiński [15]).

The first part of this article is devoted to the construction of nontrivial Grothendieck ring $K_0(BSA_{\mathbb{R}})$ associated to $SA(\mathbb{R})$, with a canonical inclusion

$$K_0(\mathrm{Var}_{\mathbb{R}}) \hookrightarrow K_0(BSA_{\mathbb{R}})$$

that gives rise to a notion of virtual Poincaré polynomial for basic real semialgebraic formulas extending the virtual Poincaré polynomial of real algebraic sets and that allows factorization of the Euler–Poincaré characteristic of real semialgebraic sets of points satisfying the formulas.

To be more precise, we first construct $K_0(BSA_{\mathbb{R}})$, the Grothendieck ring of basic real semialgebraic formulas (which are quantifier free real semialgebraic formulas or simply systems of real algebraic equations and inequalities) where the class of basic formulas without inequality is considered up to algebraic isomorphism of the underlying real algebraic varieties. In general a class in $K_0(BSA_{\mathbb{R}})$ of a basic real semialgebraic formula depends strongly on the formula itself rather than only on the geometry of the real semialgebraic set of points satisfying this formula. This construction is achieved in Section 2.

In order to make some computations more convenient we present a realization, denoted by χ, of the ring $K_0(BSA_{\mathbb{R}})$ in the somewhat more simple ring $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$, that is a morphism of rings $\chi : K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ that restricts to the identity map on $K_0(\mathrm{Var}_{\mathbb{R}}) \hookrightarrow K_0(BSA_{\mathbb{R}})$. The morphism χ provides an explicit computation (see Proposition 2.2), presenting a class of $K_0(BSA_{\mathbb{R}})$ as a $\mathbb{Z}[\frac{1}{2}]$–linear combination of classes of $K_0(\mathrm{Var}_{\mathbb{R}})$. When one wants to further simplify the computation of a class of a basic real semialgebraic formula, one can shrink the original ring $K_0(BSA_{\mathbb{R}})$ a little bit more from $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ to $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$, where for instance algebraic formulas with empty set of real points have trivial class. However, as noted in point (2) of Remark 2.5, the class of a basic real semialgebraic formula with empty set of real points may be not trivial in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$. The ring $K_0(BSA_{\mathbb{R}})$ is not defined with an a priori notion of isomorphism relation, contrary to the ring $K_0(\mathrm{Var}_{\mathbb{R}})$ where algebraic isomorphism classes of varieties are generators. Nevertheless we indicate a notion of isomorphism for basic semialgebraic formulas that factors through $K_0(BSA_{\mathbb{R}})$ (see Proposition 2.8). This is done in Section 2.

The realization $\chi : K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ naturally allows us to define in Section 4 a notion of virtual Poincaré polynomial for basic real semialgebraic formulas: For a class $[F]$ in $K_0(BSA_{\mathbb{R}})$ that is written as a $\mathbb{Z}[\frac{1}{2}]$–linear combination $\sum_{i=1}^{q} a_i[A_i]$ of classes $[A_i] \in K_0(\mathrm{Var}_{\mathbb{R}})$ of real algebraic varieties A_i, we simply define the virtual Poincaré polynomial of F as the corresponding $\mathbb{Z}[\frac{1}{2}]$–linear combination $\sum_{i=1}^{q} a_i \beta(A_i)$ of virtual Poincaré polynomials $\beta(A_i)$ of the varieties A_i. The virtual

Poincaré polynomial of F is thus a polynomial $\beta(F)$ in $\mathbb{Z}[\frac{1}{2}][u]$. It is then shown that the evaluation at -1 of $\beta(F)$ is the Euler–Poincaré characteristic of the real semialgebraic set of points satisfying the basic formula F (Proposition 3.4).

These constructions are summed up in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Var}_{\mathbb{R}} \arrow[rrrr, hook] \arrow[d] \arrow[drr] & & & & BSA_{\mathbb{R}} \arrow[dll] \arrow[ddd, "\chi_c"] \\
K_0(\mathrm{Var}_{\mathbb{R}}) \arrow[rr, hook] \arrow[dd, "\beta"'] \arrow[drr, hook] & & K_0(BSA_{\mathbb{R}}) \arrow[d, "\chi"] \arrow[dr, "\chi"] & & \\
 & & K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}] \arrow[r] \arrow[d, "\beta"] & K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}] \arrow[dl, "\beta"] & \\
\mathbb{Z}[u] \arrow[rr, hook] & & \mathbb{Z}[\frac{1}{2}][u] \arrow[rr, "u=-1"] & & \mathbb{Z}
\end{tikzcd}$$

The second and last part of this article concerns the real Milnor fibers of a given polynomial function $f \in \mathbb{R}[x_1, \dots, x_d]$. As geometrical objects, we consider real semialgebraic Milnor fibers of the following types: $f^{-1}(\pm c) \cap \bar{B}(0,\alpha)$, $f^{-1}(]0, \pm c[) \cap \bar{B}(0,\alpha)$, $f^{-1}(]0, \pm\infty[) \cap S(0,\alpha)$, for $0 < |c| \ll \alpha \ll 1$, $\bar{B}(0,\alpha)$ the closed ball of $\mathbb{R}^d$ of center 0 and radius α and $S(0,\alpha)$ the sphere of center 0 and radius α. The topological types of these fibers are easily comparable, and in order to present a motivic version of these real semialgebraic Milnor fibers we define appropriate zeta functions with coefficients in $(K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}]$ (the localization of the ring $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ with respect to the multiplicative set generated by $\mathbb{L}$). As in the complex context (see Denef and Loeser [5; 6]), we prove that these zeta functions are rational functions expressed in terms of an embedded resolution of f (see Theorem 4.2). For a complex hypersurface f, the rationality of the corresponding zeta function allows the definition of the motivic Milnor fiber S_f as the negative of the limit at infinity of the rational expression of the zeta function. In the real semialgebraic case, the same definition makes sense but we obtain a class S_f in $K_0(\mathrm{Var}_{\mathbb{R}})) \otimes \mathbb{Z}[\frac{1}{2}]$ having a realization under the Euler–Poincaré characteristic of greater combinatorial complexity in terms of the data of the resolution of f than in the complex case. Indeed, all the strata of the natural stratification of the exceptional divisor of the resolution of f appear in the expression of $\chi_c(S_f)$ in the real case. Nevertheless we show that the motivic real semialgebraic Milnor fibers have for value under the Euler–Poincaré characteristic morphism the Euler–Poincaré characteristic of the corresponding set-theoretic real semialgebraic Milnor fibers (Theorem 4.12).

In what follows we sometimes simply say *measure* for the class of an object in a given Grothendieck ring. The term *inequation* refers to the symbol $\neq$, and the term *inequality* refers to the symbol $>$.

1 The Grothendieck ring of basic semialgebraic formulas

1.1 Affine real algebraic varieties

By an affine algebraic variety over $\mathbb{R}$ we mean an affine reduced and separated scheme of finite type over $\mathbb{R}$. The category of affine algebraic varieties over $\mathbb{R}$ is denoted by $\mathrm{Var}_{\mathbb{R}}$. An affine real algebraic variety X is then defined by a subset of $\mathbb{A}^n$ together with a finite number of polynomial equations. Namely, there exist $P_i \in \mathbb{R}[X_1, \dots, X_n]$ for $i = 1, \dots, r$, such that the real points $X(\mathbb{R})$ of X are given by

$$X(\mathbb{R}) = \{x \in \mathbb{A}^n \mid P_i(x) = 0,\ i = 1, \dots, r\}.$$

A Zariski-constructible subvariety Z of $\mathbb{A}^n$ is similarly defined by real polynomial equations and inequations. Namely there exist $P_i, Q_j \in \mathbb{R}[X_1, \dots, X_n]$ for $i = 1, \dots, p$ and $j = 1, \dots, q$, such that the real points $Z(\mathbb{R})$ of Z are given by

$$Z(\mathbb{R}) = \{x \in \mathbb{A}^n \mid P_i(x) = 0,\ Q_j(x) \neq 0,\ i = 1, \dots, p,\ j = 1, \dots, q\}.$$

As an abelian group, the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{R}})$ of affine real algebraic varieties is formally generated by isomorphism classes $[X]$ of Zariski-constructible real algebraic varieties, subject to the additivity relation

$$[X] = [Y] + [X \setminus Y]$$

in case $Y \subset X$ is a closed subvariety of X. Here $X \setminus Y$ is the Zariski-constructible variety defined by combining the equations and inequations that define X together with the equations and inequations obtained by reversing the equations and inequations that define Y. The product of constructible sets induces a ring structure on $K_0(\mathrm{Var}_{\mathbb{R}})$. We denote by $\mathbb{L}$ the class of $\mathbb{A}^1$ in $K_0(\mathrm{Var}_{\mathbb{R}})$.

1.2 Real algebraic sets

The real points $X(\mathbb{R})$ of an affine algebraic variety X over $\mathbb{R}$ form a real algebraic set (in the sense of [3]). The Grothendieck ring $K_0(\mathbb{R}\mathrm{Var})$ of affine real algebraic sets [15] is defined in a similar way to that of real algebraic varieties over $\mathbb{R}$. Taking the real points of an affine real algebraic variety over $\mathbb{R}$ gives a ring morphism from $K_0(\mathrm{Var}_{\mathbb{R}})$ to $K_0(\mathbb{R}\mathrm{Var})$. A great advantage of $K_0(\mathbb{R}\mathrm{Var})$ from a geometrical point of view is that the additivity property implies that the measure of an algebraic set without real points is zero in $K_0(\mathbb{R}\mathrm{Var})$.

We already know some realizations of $K_0(\mathbb{R}\mathrm{Var})$ in simpler rings, such as the Euler characteristics with compact supports in $\mathbb{Z}$ or the virtual Poincaré polynomial in $\mathbb{Z}[u]$ (cf [15]). We obtain therefore similar realizations for $K_0(\mathrm{Var}_{\mathbb{R}})$ by composition with the realizations of $K_0(\mathrm{Var}_{\mathbb{R}})$ in $K_0(\mathbb{R}\mathrm{Var})$.

1.3 Basic semialgebraic formulas

Let us now specify the definition of the Grothendieck ring $K_0(BSA_{\mathbb{R}})$ of basic semialgebraic formulas. This definition is inspired by [7]. The ring $K_0(BSA_{\mathbb{R}})$ will contain $K_0(\mathrm{Var}_{\mathbb{R}})$ as a subring (Proposition 1.3) and will be projected on the ring $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ (Theorem 2.3) by an explicit computational process.

A basic semialgebraic formula A in n variables is defined as a finite number of equations, inequations and inequalities, namely there exist $P_i, Q_j, R_k \in \mathbb{R}[X_1, \dots, X_n]$ for $i = 1, \dots, p$, $j = 1, \dots, q$ and $k = 1, \dots, r$ such that $A(\mathbb{R})$ is equal to the set of points $x \in \mathbb{A}^n$ such that

$$\begin{aligned} P_i(x) &= 0, \quad i = 1, \dots p, \\ Q_j(x) &\neq 0, \quad j = 1, \dots, q, \\ R_k(x) &> 0, \quad k = 1, \dots, r. \end{aligned}$$

The relations $Q_j(x) \neq 0$ are called inequations and the relations $R_k(x) > 0$ are called inequalities. We will simply denote a basic semialgebraic formula by

$$A = \{P_i = 0,\ Q_j \neq 0,\ R_k > 0 \mid i = 1, \dots, p,\ j = 1, \dots, q,\ k = 1, \dots, r\}.$$

In particular A is not characterized by its real points $A(\mathbb{R})$, that is by the real solutions of these equations, inequations and inequalities, but by the equations, inequations and inequalities themselves.

We will consider basic semialgebraic formulas up to algebraic isomorphisms, when the basic semialgebraic formulas are defined without inequality.

Remark 1.1 In the sequel, we will allow ourselves to use the notation $\{P < 0\}$ for the basic semialgebraic formula $\{-P > 0\}$ and similarly $\{P > 1\}$ instead of $\{P - 1 > 0\}$, where P denotes a polynomial with real coefficients. Furthermore given two basic semialgebraic formulas A and B, the notation $\{A, B\}$ will denote the basic formula with equations, inequations and inequalities coming from both A and B.

We define the Grothendieck ring $K_0(BSA_{\mathbb{R}})$ of basic semialgebraic formulas as the free abelian ring generated by basic semialgebraic formulas $[A]$, up to algebraic isomorphism when the formula A has no inequality, and subject to the three following relations:

(1) (*Algebraic additivity*)

$$[A]=[A, S=0]+[A,\{S\neq 0\}],$$

where A is a basic semialgebraic formula in n variables and $S\in\mathbb{R}[X_1,\dots,X_n]$.

(2) (*Semialgebraic additivity*)

$$[A, R\neq 0]=[A, R>0]+[A,-R>0],$$

where A is a basic semialgebraic formula in n variables and $R\in\mathbb{R}[X_1,\dots,X_n]$.

(3) (*Product*) The product of basic semialgebraic formulas, defined by taking the conjunction of the formulas with disjoint sets of free variables, induces the ring product on $K_0(BSA_{\mathbb{R}})$. In other words we consider the relation

$$[A, B]=[A]\cdot[B]$$

for basic real semialgebraic formulas A and B with disjoint sets of variables.

Remark 1.2 (1) Contrary to the Grothendieck ring of algebraic varieties or algebraic sets, we do not consider isomorphism classes of basic real semialgebraic formulas in the definition of $K_0(BSA_{\mathbb{R}})$. As a consequence the realization we are interested in does depend in a crucial way on the description of the basic semialgebraic set as a basic semialgebraic formula. For instance $\{X-1>0\}$ and $\{X>0, X-1>0\}$ will have different measures.

(2) One may decide to enlarge the basic semialgebraic formulas with nonstrict inequalities by imposing, by convention, that the measure of $\{A, R\geq 0\}$ for a basic semialgebraic formula A in n variables and $R\in\mathbb{R}[X_1,\dots,X_n]$ is the sum of the measures of $\{A, R>0\}$ and of $\{A, R=0\}$.

Proposition 1.3 *The natural map i from $K_0(\mathrm{Var}_{\mathbb{R}})$ that associates to an affine real algebraic variety its value in the Grothendieck ring $K_0(BSA_{\mathbb{R}})$ of basic real semialgebraic formulas is an injective morphism*

$$i\colon K_0(\mathrm{Var}_{\mathbb{R}})\longrightarrow K_0(BSA_{\mathbb{R}}).$$

We therefore identify $K_0(\mathrm{Var}_{\mathbb{R}})$ with a subring of $K_0(BSA_{\mathbb{R}})$.

Proof We construct a left inverse j of i as follows. Let $a\in K_0(BSA_{\mathbb{R}})$ be a sum of products of measures of basic semialgebraic formulas. If there exist Zariski constructible real algebraic sets $Z_1,\dots,Z_m$ such that $[Z_1]+\cdots+[Z_m]$ is equal to a in $K_0(BSA_{\mathbb{R}})$, then we define the image of a by j to be

$$j(a) = [Z_1] + \cdots + [Z_m] \in K_0(\mathrm{Var}_{\mathbb{R}}).$$

Otherwise, the image of a by j is defined to be zero in $K_0(\mathrm{Var}_{\mathbb{R}})$. The map j is well defined. Indeed, if $Y_1, \dots, Y_l$ are other Zariski constructible sets such that $[Y_1] + \cdots + [Y_l]$ is equal to a in $K_0(BSA_{\mathbb{R}})$, then

$$[Y_1] + \cdots + [Y_l] = [Z_1] + \cdots + [Z_m]$$

in $K_0(BSA_{\mathbb{R}})$. This equality still holds in $K_0(\mathrm{Var}_{\mathbb{R}})$ by definition of the structure ring of $K_0(\mathrm{Var}_{\mathbb{R}})$ and the fact that j defines a left inverse of i is immediate. □

Remark 1.4 Note however that the map j constructed in the proof of Proposition 1.3 is not a group morphism. For instance $j([X > 0]) = j([X < 0]) = 0$ while $j([X \neq 0]) = \mathbb{L} - 1$.

2 A realization of $K_0(BSA_{\mathbb{R}})$

An example of a ring morphism from $K_0(BSA_{\mathbb{R}})$ to $\mathbb{Z}$ is the Euler characteristic with compact supports χ_c. We construct in this section a realization for elements in $K_0(BSA_{\mathbb{R}})$ with values in the ring of polynomials with coefficient in $\mathbb{Z}[\frac{1}{2}]$. This realization specializes to the Euler characteristic with compact supports. To this aim, we construct a ring morphism from $K_0(BSA_{\mathbb{R}})$ to the tensor product of $K_0(\mathrm{Var}_{\mathbb{R}})$ with $\mathbb{Z}[\frac{1}{2}]$.

2.1 The realization

We define a morphism χ from the ring $K_0(BSA_{\mathbb{R}})$ to the ring $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ as follows. Let A be a basic semialgebraic formula without inequalities. We assign to A its value $\chi(A) = [A]$ in $K_0(\mathrm{Var}_{\mathbb{R}})$ as a constructible set. We proceed now by induction on the number of inequalities in the description of the basic semialgebraic formulas. Assuming that we have defined χ for basic semialgebraic formulas with at most k inequalities, $k \in \mathbb{N}$, let A be a basic real semialgebraic formula with n variables and at most k inequalities and let us consider $R \in \mathbb{R}[X_1, \dots, X_n]$. Define $\chi([A, R > 0])$ by

$$\chi([A, R > 0]) := \tfrac{1}{4}\big(\chi([A, Y^2 = R]) - \chi([A, Y^2 = -R])\big) + \tfrac{1}{2}\chi([A, R \neq 0]),$$

where $\{A, Y^2 = \pm R\}$ is a basic real semialgebraic formula with $n+1$ variables and at most k inequalities and $\{A, R \neq 0\}$ is a basic semialgebraic formula with n variables and at most k inequalities.

Remark 2.1 The way of defining χ may be seen as an average of two different natural ways of understanding a basic semialgebraic formula as a quotient of algebraic varieties. Namely, for a basic semialgebraic formula in n variables of the form $\{R>0\}$, we may see its set of real points as the projection, with two-point fibers, of $\{Y^2=R\}$ minus the zero set of R, or as the complement of the projection of $Y^2=-R$. The algebraic average of these two possible points of view is

$$\tfrac{1}{2}\big((\tfrac{1}{2}[Y^2=R]-[R=0])+(\mathbb{L}^n-\tfrac{1}{2}[Y^2=-R])\big),$$

which, considering that $\mathbb{L}^n-[R=0]=[R\neq 0]$, gives for $\chi(R>0)$ the expression just defined above.

We give below the general formula that computes the measure of a basic semialgebraic formula in terms of the measure of real algebraic varieties.

Proposition 2.2 *Let Z be a constructible set in $\mathbb{R}^n$ and take $R_k\in\mathbb{R}[X_1,\dots,X_n]$, with $k=1,\dots,r$. For a subset $I\subset\{1,\dots,r\}$ of cardinal $\sharp I=i$ and $\varepsilon\in\{\pm 1\}^i$, we denote by $R_{I,\varepsilon}$ the real constructible set defined by*

$$R_{I,\varepsilon}=\{Y_j^2=\varepsilon_j R_j(X),\ R_k(X)\neq 0,\ j\in I,\ k\notin I\}.$$

Then $\chi([Z,R_k>0,\ k=1,\dots,r])$ is equal to

$$\sum_{i=0}^{r}\frac{1}{2^{r+i}}\sum_{\substack{I\subset\{1,\dots,r\},\\ \sharp I=i}}\ \sum_{\varepsilon\in\{\pm 1\}^i}\Big(\prod_{j\in I}\varepsilon_j\Big)[Z,R_{I,\varepsilon}].$$

Proof If $r=1$, this follows from the definition of χ. We prove the general result by induction on $r\in\mathbb{N}$. Assume $Z=\mathbb{R}^n$ to simplify notation. Take $R_k\in\mathbb{R}[X_1,\dots,X_n]$, with $k=1,\dots,r+1$. Denote by A the formula $R_1>0,\dots,R_r>0$. By definition of χ we obtain

$$\begin{aligned}&\chi([A,R_{r+1}>0])\\ &\qquad=\tfrac{1}{4}\big(\chi([A,Y^2=R_{r+1}])-\chi([A,Y^2=-R_{r+1}])\big)+\tfrac{1}{2}\chi([A,R_{r+1}\neq 0]).\end{aligned}$$

Now we can use the induction assumption to express the terms in the right-hand side of the formula upstairs as

$$\begin{aligned}\sum_{i=0}^{r}\frac{1}{2^{r+i}}\sum_{\substack{I\subset\{1,\dots,r\},\\ \sharp I=i}}\ \sum_{\varepsilon\in\{\pm 1\}^i}\Big(\prod_{j\in I}\varepsilon_j\Big)\big(\tfrac{1}{4}([R_{I,\varepsilon},Y^2=R_{r+1}]-[R_{I,\varepsilon},Y^2=-R_{r+1}])&\\ +\tfrac{1}{2}[R_{I,\varepsilon},R_{r+1}\neq 0]\big).&\end{aligned}$$

Choose a subset $I \subset \{1, \dots, r\}$ of cardinal $\sharp I = i$ and $\varepsilon \in \{\pm 1\}^i$. Then we obtain from the definition of χ that

$$\tfrac{1}{4}\big([R_{I,\varepsilon}, Y^2 = R_{r+1}] - [R_{I,\varepsilon}, Y^2 = -R_{r+1}]\big) + \tfrac{1}{2}[R_{I,\varepsilon}, R_{r+1} \neq 0] \\ = \tfrac{1}{4}\big([R_{I\cup\{r+1\},\varepsilon^+}] - [R_{I\cup\{r+1\},\varepsilon^-}]\big) + \tfrac{1}{2}[R_{\widetilde{I},\varepsilon}],$$

where $\varepsilon^+ = (\varepsilon_1, \dots, \varepsilon_r, 1)$, $\varepsilon^- = (\varepsilon_1, \dots, \varepsilon_r, -1)$ and $\widetilde{I}$ denotes I as a subset of $\{1, \dots, r+1\}$. Therefore

$$\tfrac{1}{2^{r+i}}\Big(\prod_{j\in I} \varepsilon_j\Big)[R_{r+1} > 0, R_{I,\varepsilon}] \\ = \tfrac{1}{2^{(r+1)+(i+1)}}\Big(\prod_{j\in I} \varepsilon_j\Big)\big([R_{I\cup\{r+1\},\varepsilon^+}] - [R_{I\cup\{r+1\},\varepsilon^-}]\big) + \tfrac{1}{2^{(r+1)+i}}\Big(\prod_{j\in I} \varepsilon_j\Big)[R_{\widetilde{I},\varepsilon}],$$

which gives the result. □

The morphism χ is then defined on $K_0(BSA_{\mathbb{R}})$.

Theorem 2.3 *The map*

$$\chi\colon K_0(BSA_{\mathbb{R}}) \longrightarrow K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

is a ring morphism that is the identity on $K_0(\mathrm{Var}_{\mathbb{R}}) \subset K_0(BSA_{\mathbb{R}})$.

Proof We must prove that the given definition of χ is compatible with the algebraic and semialgebraic additivities. However the semialgebraic additivity follows directly from the definition of χ. Indeed, if A is a basic semialgebraic formula and R a real polynomial, then the sum of $\chi([A, R > 0])$ and $\chi([A, -R > 0])$ is equal to

$$\tfrac{1}{4}\big(\chi([A, Y^2 = R]) - \chi([A, Y^2 = -R])\big) + \tfrac{1}{2}\chi([A, R \neq 0]) \\ + \tfrac{1}{4}\big(\chi([A, Y^2 = -R]) - \chi([A, Y^2 = R])\big) + \tfrac{1}{2}\chi([A, -R \neq 0]) \\ = \chi([A, -R \neq 0]).$$

The algebraic additivity as well as the multiplicativity follow from Proposition 2.2, which enables us to express the measure of a basic semialgebraic formula in terms of algebraic varieties for which additivity and multiplicativity hold. We conclude by noting that we may construct a left inverse to χ restricted to $K_0(\mathrm{Var}_{\mathbb{R}})$ in the same way as in the proof of Proposition 1.3. □

Example 2.4 (1) A half-line defined by $X > 0$ has measure in $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\tfrac{1}{2}]$ half of the value of the line minus one point, as expected, since by definition

$$\chi([X > 0]) = \tfrac{1}{4}(\mathbb{L} - \mathbb{L}) + \tfrac{1}{2}(\mathbb{L} - 1) = \tfrac{1}{2}(\mathbb{L} - 1).$$

However, if we add one more inequality, like $\{X > 0, X > -1\}$, then the measure has more complexity. We will see in Section 3.1 that, evaluated in the polynomial ring $\mathbb{Z}[\frac{1}{2}][u]$, we obtain in that case

$$\beta([X > 0, X > -1]) = \frac{5u - 11}{16}.$$

(2) Using the multiplicativity, we find the measure of the half-plane and the measure of the quarter plane, as expected, to be

$$\chi([X_1 > 0]) = \tfrac{1}{2}(\mathbb{L}^2 - \mathbb{L}) \quad \text{and} \quad \chi([X_1 > 0, X_2 > 0]) = \tfrac{1}{4}(\mathbb{L} - 1)^2.$$

Remark 2.5 (1) Let $R \in \mathbb{R}[X_1, \dots, X_n]$ be odd. Then

$$\chi([R > 0]) = \chi([R < 0]) = \tfrac{1}{2}[R \neq 0].$$

Indeed, the varieties $Y^2 = R(X)$ and $Y^2 = -R(X)$ are isomorphic via $X \mapsto -X$, and the result follows from the definition of χ.

(2) The ring morphism from $K_0(\mathrm{Var}_{\mathbb{R}})$ to $K_0(\mathbb{R}\mathrm{Var})$ gives a realization from the ring $K_0(BSA_{\mathbb{R}})$ to the ring $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$ for which the measure of a real algebraic variety without real point is zero. This is why it is often convenient to push the computations to the ring $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$ rather than staying at the higher level of $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$. However we have to notice that the measure of a basic real semialgebraic formula without real points is not necessarily zero in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$. For instance, let us compute the measure of $X^2 + 1 > 0$ in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$. By definition of χ we obtain that $\chi([X^2 + 1 > 0])$ is equal to

$$\tfrac{1}{4}\big(\chi([Y^2 = X^2 + 1]) - \chi([Y^2 = -X^2 - 1])\big) + \tfrac{1}{2}\chi([X^2 + 1 \neq 0]) = \tfrac{1}{4}(\mathbb{L} - 1) + \tfrac{1}{2}\mathbb{L} = \tfrac{1}{4}(3\mathbb{L} - 1).$$

By additivity we have

$$\begin{aligned}\chi([X^2 + 1 < 0]) &= \chi([X^2 + 1 \neq 0]) - \chi([X^2 + 1 > 0]) \\ &= \mathbb{L} - \chi([X^2 + 1 = 0]) - \chi([X^2 + 1 > 0]).\end{aligned}$$

But since $\chi([X^2 + 1 = 0]) = 0$ in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$, we obtain that the measure of $\{X^2 + 1 < 0\}$ in $K_0(\mathbb{R}\mathrm{Var}) \otimes \mathbb{Z}[\frac{1}{2}]$, whose set of real points is empty, is

$$\chi([X^2 + 1 < 0]) = \tfrac{1}{4}(\mathbb{L} + 1).$$

(3) In a similar way, the basic semialgebraic formula $\{P > 0, -P > 0\}$ with $P(X) = 1 + X^2$, whose set of real points is empty, has measure

$$\chi([P > 0, -P > 0]) = \tfrac{1}{8}(\mathbb{L} + 1).$$

2.2 Isomorphism between basic semialgebraic formulas

In this section we give a condition for two basic semialgebraic formulas to have the same realization by χ. It deals with the complexification of the algebraic liftings of the basic semialgebraic formulas.

Let X be a real algebraic subvariety of $\mathbb{R}^n$, defined by $P_i \in \mathbb{R}[X_1, \dots, X_n]$ for $i = 1, \dots, r$. The complexification $X_{\mathbb{C}}$ of X is defined to be the complex algebraic subvariety of $\mathbb{C}^n$ defined by the same polynomials $P_1, \dots, P_r$. We define similarly the complexification of a real algebraic map.

Let $Y \subset \mathbb{R}^n$ be a Zariski constructible subset of $\mathbb{R}^n$ and take $R_1, \dots, R_r \in \mathbb{R}[X_1, \dots, X_n]$. Let A denote the basic semialgebraic formula of $\mathbb{R}^n$ defined by Y together with the inequalities $R_1 > 0, \dots, R_r > 0$ and V denote the Zariski constructible subset of $\mathbb{R}^{n+r}$ defined by

$$V = \{Y,\ Y_1^2 = R_1,\ \dots,\ Y_r^2 = R_r\}.$$

Note that V is endowed with an action of $\{\pm 1\}^r$, defined by multiplication by -1 on the indeterminates $Y_1, \dots, Y_r$.

Let $Z \subset \mathbb{R}^n$ be a Zariski constructible subset of $\mathbb{R}^n$ and take similarly $S_1, \dots, S_r \in \mathbb{R}[X_1, \dots, X_n]$. Let B denote the basic semialgebraic formula of $\mathbb{R}^n$ defined by Z together with the inequalities $S_1 > 0, \dots, S_r > 0$ and W denote the Zariski constructible subset of $\mathbb{R}^{n+r}$ defined by

$$W = \{Z,\ Y_1^2 = S_1,\ \dots,\ Y_r^2 = S_r\}.$$

Definition 2.6 We say that the basic semialgebraic formulas A and B are isomorphic if there exists a real algebraic isomorphism $\phi\colon V \to W$ between V and W that is equivariant with respect to the action of $\{\pm 1\}^r$ on V and W, and whose complexification $\phi_{\mathbb{C}}$ induces a complex algebraic isomorphism between the complexifications $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ of V and W.

Remark 2.7 Let us consider first the particular case $Y = \mathbb{R}^n$, $Z = \mathbb{R}^n$ and $r = 1$. Change moreover the notation as follows. Put $V^+ = V$ and $W^+ = W$, and define $V^- = \{y^2 = -R(x)\}$ and $W^- = \{y^2 = -S(x)\}$.

Then the complex points $V^+_{\mathbb{C}}$ and $V^-_{\mathbb{C}}$ of V^+ and V^- are isomorphic via the complex (and not real) isomorphism $(x, y) \mapsto (x, iy)$. Now, suppose that the basic semialgebraic formula $\{R > 0\}$ is isomorphic to $\{S > 0\}$. Let $\phi = (f, g)\colon (x, y) \mapsto (f(x, y), g(x, y))$ be the real isomorphism involved in the definition (that is f and g are defined by real equations, and moreover $f(x,-y)=f(x,y)$ and $g(x,-y)=-g(x,y)$). Then the diagram

$$\begin{tikzcd} V_{\mathbb{C}}^{+} \arrow[r, "(f,g)"] \arrow[d, "(x,y)\mapsto(x,iy)"'] & W_{\mathbb{C}}^{+} \arrow[d, "(x,y)\mapsto(x,iy)"] \\ V_{\mathbb{C}}^{-} & W_{\mathbb{C}}^{-} \end{tikzcd}$$

induces a complex isomorphism (F,G) between $V_{\mathbb{C}}^{-}$ and $W_{\mathbb{C}}^{-}$ given by

$$(x,y)\mapsto (f(x,-iy), ig(x,-iy)).$$

In fact, this isomorphism is defined over $\mathbb{R}$ since

$$\overline{F(x,y)}=\overline{f(x,-iy)}=f(\overline{x},\overline{-iy})=f(\overline{x},i\overline{y})=f(\overline{x},-i\overline{y})=F(\overline{x},\overline{y}),$$
$$\overline{G(x,y)}=\overline{ig(x,-iy)}=-ig(\overline{x},\overline{-iy})=-ig(\overline{x},i\overline{y})=ig(\overline{x},-i\overline{y})=G(\overline{x},\overline{y}),$$

where the bar denotes complex conjugation. Therefore it induces a real algebraic isomorphism between V^- and W^-.

Moreover $g(x,0)=-g(x,0)$ so $g(x,0)=0$ and then the real algebraic sets $\{R=0\}$ and $\{S=0\}$ are also isomorphic.

Proposition 2.8 *If the basic semialgebraic formulas A and B are isomorphic, then $\chi([A])=\chi([B])$.*

Proof Thanks to Proposition 2.2, we only need to prove that the real algebraic varieties $R_{I,\varepsilon}$ corresponding to A and B are isomorphic two by two, which is a direct generalization of Remark 2.7. □

3 Virtual Poincaré polynomial

3.1 Polynomial realization

The best realization known (with respect to the highest algebraic complexity of the realization ring) of the Grothendieck ring of real algebraic varieties is given by the virtual Poincaré polynomial [15]. This polynomial, whose coefficients coincide with the Betti numbers with coefficients in $\mathbb{Z}/2\mathbb{Z}$ when sets are compact and nonsingular, has coefficients in $\mathbb{Z}$. As a corollary of Theorem 2.3 we obtain the following realization of $K_0(BSA_{\mathbb{R}})$ in $\mathbb{Z}[\frac{1}{2}][u]$.

Proposition 3.1 *There exists a ring morphism*

$$\beta\colon K_0(BSA_{\mathbb{R}}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}][u]$$

whose restriction to $K_0(\mathrm{Var}_{\mathbb{R}}) \subset K_0(BSA_{\mathbb{R}})$ coincides with the virtual Poincaré polynomial.

The interest of such a realization is that it enables us to make concrete computations.

Example 3.2 (1) The virtual Poincaré polynomial of the open disc $X_1^2+X_2^2<1$ is equal to

$$\begin{aligned}\tfrac{1}{4}\big(\beta([Y^2=1-(X_1^2+X_2^2)])-\beta([Y^2=X_1^2+X_2^2-1])\big)+\tfrac{1}{2}\beta([X_1^2+X_2^2\neq 1])\\ =\tfrac{1}{4}(u^2+1-u(u+1))+\tfrac{1}{2}(u^2-u-1)=\tfrac{1}{4}(2u^2-3u-1).\end{aligned}$$

(2) Let us compute the measure of the formula $X>a, X>b$ with $a\neq b\in\mathbb{R}$. By Proposition 2.2, we are lead to compute the virtual Poincaré polynomial of the real algebraic subsets of $\mathbb{R}^3$ defined by $\{y^2=\pm(x-a),\ z^2=\pm(x-b)\}$. These sets are isomorphic to $\{y^2\pm z^2=\pm(a-b)\}$, and we recognize either a circle, a hyperbola or the empty set.

In particular, using the formula in Proposition 2.2, we obtain

$$\begin{aligned}&\beta([X>a, X>b])\\ &\quad=\tfrac{1}{16}(2(u-1)-(u+1))+\tfrac{1}{8}(2u-2u)+\tfrac{1}{8}(2-2)+\tfrac{1}{4}(u-2)=\frac{5u-11}{16}.\end{aligned}$$

Remark 3.3 In case the set of real points of a basic semialgebraic formula is a real algebraic set (or even an arc symmetric set [13; 10]), its virtual Poincaré polynomial does not coincide in general with the virtual Poincaré polynomial of the real algebraic set. For instance, the basic semialgebraic formula $X^2+1>0$, considered in Remark 2.5, has virtual Poincaré polynomial equal to $\frac{1}{4}(3u-1)$, whereas its set of points is a real line whose virtual Poincaré polynomial equals u as a real algebraic set.

Evaluating u at an integer gives another realization, with coefficients in $\mathbb{Z}[\frac{1}{2}]$. The virtual Poincaré polynomial of a real algebraic variety, evaluated at $u=-1$, coincides with its Euler characteristic with compact supports [15]. Indeed, evaluating the virtual Poincaré polynomial of a basic semialgebraic formula gives also the Euler characteristic with compact supports of its set of real points, and therefore has its values in $\mathbb{Z}$.

Proposition 3.4 *The virtual Poincaré polynomial $\beta(A)$ of a basic semialgebraic formula A is equal to the Euler characteristic with compact supports of its set of real points $A(\mathbb{R})$ when evaluated at $u=-1$. In other words,*

$$\beta(A)(-1)=\chi_c(A(\mathbb{R})).$$

Proof We recall that in Proposition 2.2 we explain how to express the class of A as a linear combination of classes of real algebraic varieties for which the virtual Poincaré polynomial evaluated at $u=-1$ coincides with the Euler characteristic with compact supports. At each step of our inductive process to obtain such a linear combination, we introduce a new variable and a double covering of the set of points satisfying one less inequality. The inductive formula

$$\chi([B, R>0]) := \tfrac{1}{4}\big(\chi([B, Y^2=R]) - \chi([B, Y^2=-R])\big) + \tfrac{1}{2}\chi([B, R\neq 0])$$

used at this step to eliminate one inequality by replacing the system $\{B, R>0\}$ by other systems $\{B, Y^2=R\}, \{B, Y^2=-R\}, \{B, R\neq 0\}$ is compatible with the Euler characteristic of the underlying sets of points, that is to say that our induction formula is true for $\chi=\chi_c$. The geometric reason for this fact is explained in Remark 2.1, and is the intuitive motivation for defining the realization χ by induction precisely as it is defined. □

3.2 Homogeneous case

We propose some computations of the virtual Poincaré polynomial of basic real semi-algebraic formulas of the form $\{R>0\}$, where R is homogeneous. Looking at its Euler characteristic with compact supports, it is equal to the product of the Euler characteristics with compact supports of $\{X>0\}$ with $\{R=1\}$. We investigate the case of virtual Poincaré polynomial. A key point in the proofs will be the invariance of the virtual Poincaré polynomial of constructible sets under regular homeomorphisms (see [16, Proposition 4.3]).

Proposition 3.5 *Let $R\in\mathbb{R}[X_1,\dots,X_n]$ be a homogeneous polynomial of degree d. Assume d is odd. Then*

$$\beta([R>0]) = \beta([X>0])\beta([R=1]).$$

Proof The algebraic varieties defined by $Y^2=R(X)$ and $Y^2=-R(X)$ are isomorphic since $R(-X)=-R(X)$, therefore

$$\beta([R>0]) = \tfrac{1}{2}\beta([R\neq 0]).$$

The map $(\lambda, x)\mapsto \lambda x$ from $\mathbb{R}^*\times\{R=1\}$ to $R\neq 0$ is a regular homeomorphism with inverse $y\mapsto (R(y)^{1/d}, y/R(y)^{1/d})$, therefore

$$\beta([R\neq 0]) = \beta(\mathbb{R}^*)\beta([R=1])$$

so that

$$\beta([R>0]) = \tfrac{1}{2}\beta(\mathbb{R}^*)\beta([R=1]) = \beta([X>0])\beta([R=1]).$$ □

The result is no longer true when the degree is even. However, in the particular case of the square of a homogeneous polynomial of odd degree, the relation of Proposition 3.5 remains valid.

Proposition 3.6 *Let $P \in \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial of degree k. Assume k is odd, and define $R \in \mathbb{R}[X_1, \dots, X_n]$ by $R = P^2$. Then*

$$\beta([R > 0]) = \beta([X > 0])\beta([R = 1]).$$

Proof Note first that $\{Y^2 - R\}$ can be factorized as $(Y - P)(Y + P)$ therefore the virtual Poincaré polynomial of $Y^2 - R$ is equal to

$$\beta(Y - P = 0) + \beta(Y + P = 0) - \beta(P = 0).$$

However the algebraic varieties $Y - P = 0$ and $Y + P = 0$ are isomorphic to a $n-$dimensional affine space, whereas $Y^2 + R = 0$ is isomorphic to $P = 0$ since $R = P^2$ is positive, so that the virtual Poincaré polynomial of $R > 0$ is equal to

$$\tfrac{1}{4}(2\beta(\mathbb{R}^n) - 2\beta([P = 0])) + \tfrac{1}{2}\beta([P \neq 0]) = \beta([P \neq 0]).$$

To compute $\beta([P \neq 0]$, note that the map $(\lambda, x) \mapsto \lambda x$ from $\mathbb{R}^* \times \{P = 1\}$ to $\{P \neq 0\}$ is a regular homeomorphism with inverse $y \mapsto (R(y)^{1/k}, y/R(y)^{1/k})$, therefore

$$\beta([P \neq 0]) = \beta(\mathbb{R}^*)\beta([P = 1]).$$

We achieve the proof by noticing that $R - 1 = (P - 1)(P + 1)$ so that $\beta([P = 1]) = \frac{1}{2}\beta([R = 1])$, because the degree of the homogeneous polynomial P is odd. Finally

$$\beta([R > 0]) = \tfrac{1}{2}\beta(\mathbb{R}^*)\beta([R = 1])$$

and the proof is achieved. □

More generally, for a homogeneous polynomial R of degree twice a odd number, we can express the virtual Poincaré polynomial of $[R > 0]$ in terms of that of $[R = 1]$, $[R = -1]$ and $[R \neq 0]$ as follows.

Proposition 3.7 *Let $k \in \mathbb{N}$ be odd and put $d = 2k$. Let $R \in \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial of degree d. Then*

$$\beta([R > 0]) = \tfrac{1}{4}\beta(\mathbb{R}^*)(\beta([R = 1]) - \beta([R = -1])) + \tfrac{1}{2}\beta([R \neq 0]).$$

Example 3.8 We cannot do better in general as illustrated by the following examples. For $R_1 = X_1^2 + X_2^2$ we obtain

$$\beta([R_1 > 0]) = \tfrac{3}{2}\beta([X > 0])\beta([R_1 = 1])$$

whereas for $R_2 = X_1^2 - X_2^2$ we have

$$\beta([R_2 > 0]) = \beta([X > 0])\beta([R_2 = 1]).$$

The proof of Proposition 3.7 is a direct consequence of the next lemma.

Lemma 3.9 *Let $k \in \mathbb{N}$ be odd and put $d = 2k$. Let $R \in \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial of degree d. Then*

$$\beta([Y^2 = R]) = \beta([R = 0]) + \beta(\mathbb{R}^*)\beta([R = 1]).$$

Proof Note first that the algebraic varieties $Y^2 = R$ and $Y^d = R$ have the same virtual Poincaré polynomial. Indeed the map $(x, y) \mapsto (x, y^k)$ realizes a regular homeomorphism between $Y^2 = R$ and $Y^d = R$, whose inverse is given by $(x, y) \mapsto (x, y^{1/k})$. However the polynomial $Y^d - R$ being homogeneous, we obtain a regular homeomorphism

$$\mathbb{R}^* \times (\{R = 1\} \cap \{Y^d = R\}) \longrightarrow \{R \neq 0\} \cap \{Y^d = R\}$$

defined by $(\lambda, x, y) \mapsto (\lambda x, \lambda y)$. As a consequence

$$\beta([Y^d - R = 0]) = \beta([R = 0]) + \beta(\mathbb{R}^*)\beta([R = 1]). \qquad \square$$

4 Zeta functions and motivic real Milnor fibers

We apply in this section the construction of $\chi\colon K_0(BSA_{\mathbb{R}}) \to K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$ to define, for a given polynomial $f \in \mathbb{R}[X_1, \cdots, X_d]$, zeta functions whose coefficients are classes in $(K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}]$ of real semialgebraic formulas. We then show that these zeta functions are deeply related to the topology of some corresponding set-theoretic real semialgebraic Milnor fibers of f.

4.1 Semialgebraic zeta functions and real Denef–Loeser formulas

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial function with coefficients in $\mathbb{R}$ sending 0 to 0. We denote by $\mathcal{L}$ or $\mathcal{L}(\mathbb{R}^d, 0)$ the space of formal arcs $\gamma(t) = (\gamma_1(t), \cdots, \gamma_d(t))$ in $\mathbb{R}^d$, with $\gamma_j(0) = 0$ for all $j \in \{1, \cdots, d\}$, by $\mathcal{L}_n$ or $\mathcal{L}_n(\mathbb{R}^d, 0)$ the space of truncated arcs $\mathcal{L}/(t^{n+1})$ and by $\pi_n\colon \mathcal{L} \to \mathcal{L}_n$ the truncation map. More generally, for M a variety and W a closed subset of M, $\mathcal{L}(M, W)$ (resp. $\mathcal{L}_n(M, W)$) will denote the space of arcs in M (resp. the n^{th} jet-space on M) with endpoints in W.

Let ϵ be one of the symbols in the set $\{naive, -1, 1, >, <\}$. For such a symbol ϵ, via the realization of $K_0(BSA_{\mathbb{R}})$ in $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$, we define a zeta function $Z_f^\epsilon(T) \in (K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][\![T]\!]$ by

$$Z_f^\epsilon(T) := \sum_{n\geq 1} [X_{n,f}^\epsilon]\mathbb{L}^{-nd}T^n,$$

where $X_{n,f}^\epsilon$ is defined in the following way:

- $X_{n,f}^{naive} = \{\gamma \in \mathcal{L}_n \mid f(\gamma(t)) = at^n + \cdots,\ a \neq 0\}$
- $X_{n,f}^{-1} = \{\gamma \in \mathcal{L}_n \mid f(\gamma(t)) = at^n + \cdots,\ a = -1\}$
- $X_{n,f}^{1} = \{\gamma \in \mathcal{L}_n \mid f(\gamma(t)) = at^n + \cdots,\ a = 1\}$
- $X_{n,f}^{>} = \{\gamma \in \mathcal{L}_n \mid f(\gamma(t)) = at^n + \cdots,\ a > 0\}$
- $X_{n,f}^{<} = \{\gamma \in \mathcal{L}_n \mid f(\gamma(t)) = at^n + \cdots,\ a < 0\}$

Note that $X_{n,f}^\epsilon$ is a real algebraic variety for $\epsilon = -1$ or 1, a real algebraic constructible set for $\epsilon = naive$ and a semialgebraic set, given by an explicit description involving one inequality, for ϵ being the symbol $>$ or the symbol $<$. Consequently, $Z_f^\epsilon(T) \in K_0(\mathrm{Var}_{\mathbb{R}})[\mathbb{L}^{-1}][\![T]\!]$ for $\epsilon \in \{naive, -1, 1\}$ and $Z_f^\epsilon(T) \in (K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][\![T]\!]$ for $\epsilon \in \{>, <\}$.

We show in this section that $Z_f^\epsilon(T)$ is a rational function expressed in terms of the combinatorial data of a resolution of f. To define those data let us consider a proper birational map $\sigma\colon (M, \sigma^{-1}(0)) \to (\mathbb{R}^d, 0)$ which is an isomorphism over the complement of $\{f = 0\}$ in $(\mathbb{R}^d, 0)$, such that $f \circ \sigma$ and the jacobian determinant $\mathrm{jac}\,\sigma$ are normal crossings and $\sigma^{-1}(0)$ is a union of components of the exceptional divisor. We denote by E_j, for $j \in \mathcal{J}$, the irreducible components of $(f \circ \sigma)^{-1}(0)$ and assume that E_k are the irreducible components of $\sigma^{-1}(0)$ for $k \in \mathcal{K} \subset \mathcal{J}$. For $j \in \mathcal{J}$ we denote by N_j the multiplicity $\mathrm{mult}_{E_j} f \circ \sigma$ of $f \circ \sigma$ along E_j and for $k \in \mathcal{K}$ by ν_k the number $\nu_k = 1 + \mathrm{mult}_{E_k} \mathrm{jac}\ \sigma$. For any $I \subset \mathcal{J}$, we put $E_I^0 = (\bigcap_{i\in I} E_i) \setminus (\bigcup_{j\in\mathcal{J}\setminus I} E_j)$. These sets E_I^0 are constructible sets and the collection $(E_I^0)_{I\subset\mathcal{J}}$ gives a canonical stratification of the divisor $f \circ \sigma = 0$, compatible with $\sigma = 0$, such that in some affine open subvariety U in M we have $f \circ \sigma(x) = u(x) \prod_{i\in I} x_i^{N_i}$, where u is a unit, that is to say a rational function which does not vanish on U, and $x = (x', (x_i)_{i\in I})$ are local coordinates.

Finally for $\epsilon \in \{-1, 1, >, <\}$ and $I \subset \mathcal{J}$, we define $\widetilde{E}_I^{0,\epsilon}$ as the gluing along E_I^0 of the sets

$$R_U^\epsilon = \{(x, t) \in (E_I^0 \cap U) \times \mathbb{R} \mid t^m \cdot u(x)\ ?_\epsilon\},$$

where $?_\epsilon$ is $= -1$, $= 1$, > 0 or < 0 in case ϵ is $-1, 1, >$ or $<$ and $m = \gcd_{i\in I}(N_i)$.

Remark 4.1 Up to isomorphism, the definition of the R_U^ϵ is independent of the choice of the coordinates as well as the gluing of the R_U^ϵ: In another coordinate system $z = z(x) = (z', (z_i)_{i\in I})$ in a Zariski neighborhood of E_I^0 we have $f \circ \sigma(z) = v(z)\prod_{i\in I} z^{N_i}$, and there exist nonvanishing functions α_i so that $z_i = \alpha_i(z)\cdot x_i$. We thus obtain $v(z)\prod_{i\in I}\alpha_i^{N_i}(z) = u(x)$, and the transformation

$$\{(x,t)\in(E_I^0\cap U)\times\mathbb{R} \mid t^m\cdot u(x)\ ?_\epsilon\}\to\{(z,s)\in(E_I^0\cap U)\times\mathbb{R}\mid s^m\cdot v(z)\ ?_\epsilon\},$$
$$(x,t)\mapsto\Bigl(z, s = t\prod_{i\in I}\alpha_i(z)^{N_i/m}\Bigr)$$

is an isomorphism in case $?_\epsilon$ is $=1$ or $=-1$, and induces an isomorphism between the associate double covers $\mathcal{R}_U^\epsilon = \{(x,t,y)\in(E_I^0\cap U)\times\mathbb{R}\times\mathbb{R}\mid t^m\cdot u(x)\cdot y^2 = \eta(\epsilon)\}$ and $\mathcal{R}_U'^\epsilon = \{(z,s,w)\in(E_I^0\cap U)\times\mathbb{R}\times\mathbb{R}\mid s^m\cdot v(z)\cdot w^2 = \eta(\epsilon)\}$, with $\eta(\epsilon)=1$ when ϵ is the symbol $>$ and $\eta(\epsilon)=-1$ when ϵ is the symbol $<$; the induced isomorphism is simply

$$\mathcal{R}_U^\epsilon\to\mathcal{R}_U'^\epsilon,\quad (x,t,y)\mapsto(z,s,w=y).$$

Also notice that $\widetilde{E}_I^{0,\epsilon}$ is a constructible set when ϵ is -1 or 1 and a semialgebraic set with explicit description over the constructible set E_I^0 when ϵ is $<$ or $>$.

We can thus define the class $[\widetilde{E}_I^{0,\epsilon}]\in\chi(K_0(BSA_{\mathbb{R}}))$ as follows. Choosing a finite covering $(U_l)_{l\in L}$ of M by affine open subvarieties U_l, for $l\in L$, we set

$$[\widetilde{E}_I^{0,\epsilon}] = \sum_{S\subset L}(-1)^{|S|+1}\bigl[R^\epsilon_{\bigcap_{s\in S}U_s}\bigr].$$

The class $[\widetilde{E}_I^{0,\epsilon}]$ does not depend on the choice of the covering thanks to Remark 4.1 and the algebraic additivity in $K_0(BSA_{\mathbb{R}})$.

With this notation one can give an expression of $Z_f^\epsilon(T)$ in terms of $[\widetilde{E}_I^{0,\epsilon}]$, as for instance in Denef and Loeser [5; 6; 8] or Looijenga [14], essentially using the Kontsevitch change of variables formula in motivic integration (see [6; 12] for instance).

Theorem 4.2 *With the notation above, one has*

$$Z_f^\epsilon(T) = \sum_{I\cap\mathcal{K}\neq\varnothing}(\mathbb{L}-1)^{|I|-1}[\widetilde{E}_I^{0,\epsilon}]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{N_i}}$$

for ϵ being $-1, 1, >$ or $<$.

Remark 4.3 Classically, the right-hand side of equality of Theorem 4.2 does not depend, as a formal series in $(K_0(\mathrm{Var}_{\mathbb{R}})\otimes\mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][\![T]\!]$, on the choice of the resolution σ as the definition of $Z_f^\epsilon(T)$ does not depend itself on any choice of resolution.

To prove this theorem, we first start with a lemma that needs the following notation. We denote by

$$\sigma_*\colon \mathcal{L}(M,\sigma^{-1}(0)) \to \mathcal{L}(\mathbb{R}^d,0),$$
$$\sigma_{n,*}\colon \mathcal{L}_n(M,\sigma^{-1}(0)) \to \mathcal{L}_n(\mathbb{R}^d,0), \quad n\in\mathbb{N}$$

the natural mappings induced by $\sigma\colon (M,\sigma^{-1}(0)) \to (\mathbb{R}^d,0)$. Let

$$Y^\epsilon_{n,f} = \pi_n^{-1}(X^\epsilon_{n,f}).$$

Then $Y^\epsilon_{n,f\circ\sigma} = \{\gamma\in\mathcal{L}(M,\sigma^{-1}(0)) \mid f(\sigma(\pi_n(\gamma)))(t) = at^n+\cdots,\ a\,?_\epsilon\}$, where $?_\epsilon$ is $=-1$, $=1$, >0 or <0 in case ϵ is $-1,1,>$ or $<$, and $Y^\epsilon_{n,f\circ\sigma} = \sigma_*^{-1}(Y^\epsilon_{n,f})$. Finally for $e\geq 1$, let

$$\Delta_e = \{\gamma\in\mathcal{L}(M,\sigma^{-1}(0)) \mid \operatorname{mult}_t\ (\operatorname{jac}\ \sigma)(\gamma(t)) = e\},$$
$$Y^\epsilon_{e,n,f\circ\sigma} = Y^\epsilon_{n,f\circ\sigma}\cap\Delta_e.$$

Lemma 4.4 *With the notation above, there exists $c\in\mathbb{N}$ such that*

$$Z^\epsilon_f(T) = \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\varnothing} \mathbb{L}^{-(n+1)d}[\mathcal{L}_n(M,E^0_I\cap\sigma^{-1}(0))\cap\pi_n(\Delta_e)\cap X^\epsilon_{n,f\circ\sigma}].$$

Proof As usual in motivic integration, the class of the cylinder $Y^\epsilon_{n,f} = \pi_n^{-1}(X^\epsilon_{n,f})$, $n\geq 1$, is an element of $(K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac12])[\mathbb{L}^{-1}]$, the localization of the ring $K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac12]$ with respect to the multiplicative set generated by $\mathbb{L}$, and defined by $[Y^\epsilon_{n,f}] := \mathbb{L}^{-(n+1)d}[X^\epsilon_{n,f}]$, since the truncation morphisms $\pi_{k+1,k}\colon \mathcal{L}_{k+1}(\mathbb{R}^d,0)\to\mathcal{L}_k(\mathbb{R}^d,0)$, $k\geq 1$, are locally trivial fibrations with fiber $\mathbb{R}^d$. Hence $Z^\epsilon_f(T) = \mathbb{L}^d\sum_{n\geq 1}[Y^\epsilon_{n,f}]T^n$.

Take now $\gamma\in\sigma_*^{-1}(Y^\epsilon_{n,f})$ and let $I\subset\mathcal{J}$ such that $\gamma(0)\in E^0_I$. In some neighborhood of E^0_I, one has coordinates such that $f\circ\sigma(x) = u(x)\prod_{i\in I}x_i^{N_i}$ and $\operatorname{jac}(\sigma)(x) = v(x)\prod_{i\in I}x_i^{\nu_i-1}$, with u and v units. If one denotes $\gamma = (\gamma_1,\cdots,\gamma_d)$ in these coordinates, with k_i the multiplicity of γ_i at 0 for $i\in I$, then we have $\operatorname{mult}_t(f\circ\sigma\circ\gamma(t)) = \sum_{i\in I}k_iN_i = n$. Now

$$\operatorname{mult}_t(\operatorname{jac}\sigma)(\gamma(t)) = \sum_{i\in I}k_i(\nu_i-1) \leq \max_{i\in I}\Big(\frac{\nu_i-1}{N_i}\Big)\sum_{i\in I}N_ik_i = \max_{i\in I}\Big(\frac{\nu_i-1}{N_i}\Big)n.$$

Therefore if one sets $c = \max_{i\in I}(\frac{\nu_i-1}{N_i})$, one has

$$Y^\epsilon_{n,f\circ\sigma} = \bigcup_{e\geq 1}Y^\epsilon_{e,n,f\circ\sigma} = \bigcup_{1\leq e\leq cn}Y^\epsilon_{e,n,f\circ\sigma}$$

as disjoint unions. Now we can apply the change of variables theorem (see [6; 12]) to compute $[Y^{\epsilon}_{n,f}]$ in terms of $[Y^{\epsilon}_{e,n,f\circ\sigma}]$:

$$[Y^{\epsilon}_{n,f}] = \sum_{e\leq cn} \mathbb{L}^{-e}[Y^{\epsilon}_{e,n,f\circ\sigma}],$$

and summing over the subsets I of $\mathcal{J}$, as $Y^{\epsilon}_{e,n,f\circ\sigma}$ is the disjoint union

$$\bigcup_{I\neq\varnothing} Y^{\epsilon}_{e,n,f\circ\sigma} \cap \pi_0^{-1}(E_I^0\cap\sigma^{-1}(0)),$$

we obtain

$$\begin{aligned}
&Z^{\epsilon}_f(T)\\
&\ = \mathbb{L}^d \sum_{n\geq 1}[Y^{\epsilon}_{n,f}]T^n\\
&\ = \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\varnothing}[Y^{\epsilon}_{e,n,f\circ\sigma} \cap \pi_0^{-1}(E_I^0\cap\sigma^{-1}(0))]\\
&\ = \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\varnothing} \mathbb{L}^{-(n+1)d}[\pi_n(Y^{\epsilon}_{e,n,f\circ\sigma} \cap \pi_0^{-1}(E_I^0\cap\sigma^{-1}(0)))]\\
&\ = \mathbb{L}^d \sum_{n\geq 1} T^n \sum_{e\leq cn} \mathbb{L}^{-e} \sum_{I\neq\varnothing} \mathbb{L}^{-(n+1)d}[\mathcal{L}_n(M, E_I^0\cap\sigma^{-1}(0))\cap\pi_n(\Delta_e)\cap X^{\epsilon}_{n,f\circ\sigma}]. \ \square
\end{aligned}$$

Proof of Theorem 4.2 Considering the expression of $Z^{\epsilon}_f(T)$ given by Lemma 4.4, we have to compute the class of $[\mathcal{L}_n(M, E_I^0\cap\sigma^{-1}(0))\cap\pi_n(\Delta_e)\cap X^{\epsilon}_{n,f\circ\sigma}]$. For this we notice that on some neighborhood U of the end point $\gamma(0)\in E_I^0\cap\sigma^{-1}(0)$, one has coordinates such that

$$f\circ\sigma(x) = u(x)\prod_{i\in I} x_i^{N_i} \quad\text{and}\quad \operatorname{jac}(\sigma)(x) = v(x)\prod_{i\in I} x_i^{\nu_i-1},$$

with u and v units. As a consequence $\mathcal{L}_n(M, E_I^0\cap U\cap\sigma^{-1}(0))\cap\pi_n(\Delta_e)\cap X^{\epsilon}_{n,f\circ\sigma}$ is isomorphic to

$$\Big\{\gamma\in\mathcal{L}_n(M,\sigma^{-1}(0)) \;\Big|\; \gamma(0)\in E_I^0\cap U\cap\sigma^{-1}(0),\ \sum_{i\in I} N_i k_i = n,\\
\sum_{i\in I} k_i(\nu_i-1) = e,\ f\circ\sigma(\gamma(t)) = at^n+\cdots,\ a\,?_{\epsilon}\Big\},$$

where $?_\epsilon$ is $=-1$, $=1$, >0 or <0 in case ϵ is $-1, 1, >$ or $<$ and k_i is the multiplicity of γ_i for $i \in I$. Now denoting by $A(I,n,e)$ the set

$$A(I,n,e) := \left\{ k=(k_1,\cdots,k_d)\in\mathbb{N}^d \;\middle|\; \sum_{i\in I} N_i k_i = n,\ \sum_{i\in I} k_i(\nu_i-1)=e \right\}$$

and identifying for simplicity x and $((x_i)_{i\notin I},(x_i)_{i\in I})$, the set

$$\mathcal{L}_n\big(M, E_I^0\cap U\cap\sigma^{-1}(0)\big)\cap\pi_n(\Delta_e)\cap X^{\epsilon}_{n,f\circ\sigma}$$

is isomorphic to the product

$$\begin{aligned}(\mathbb{R}^n)^{d-|I|}&\\ \times\bigcup_{k\in A(I,n,e)}&\left\{x\in\big(E_I^0\cap U\cap\sigma^{-1}(0)\big)\times(\mathbb{R}^*)^{|I|}\;\middle|\; u((x_i)_{i\notin I},0)\prod_{i\in I}x_i^{N_i}\ ?_\epsilon\right\}\\ &\qquad\qquad\qquad\qquad\times\prod_{i\in I}(\mathbb{R}^{n-k_i}).\end{aligned}$$

Indeed, denoting an arc $\gamma=(\gamma_1,\ldots,\gamma_d)$ of $\mathcal{L}_n(M, E_I^0\cap U\cap\sigma^{-1}(0))$ by $\gamma_i(t)=a_{i,0}+\cdots+a_{i,n}t^n$ for $i\notin I$ and $\gamma_i(t)=a_{i,k_i}t^{k_i}+\cdots+a_{i,n}t^n$ for $i\in I$, the first factor of the product comes from the free choice of the coefficients $a_{i,j}$, $i\notin I$, $j=1,\ldots,n$, the last factor of the product comes from the free choice of the coefficients $a_{i,j}$, $i\in I$, $j=k_i+1,\ldots,n$ and the middle factor of the product comes from the choice of the coefficients $a_{i,0}\in E_I^0\cap U\cap\sigma^{-1}(0)$, $i\notin I$, and from the choice of the coefficients a_{i,k_i}, $i\in I$, subject to the condition

$$\begin{aligned}f\circ\sigma(\gamma(t)) &= u(\gamma(t))\prod_{i\in I}\gamma_i^{N_i}(t)\\ &= u((a_{i,0})_{i\notin I},0)\Big(\prod_{i\in I}a_{i,k_i}^{N_i}\Big)t^n+\cdots = at^n+\cdots,\quad a\ ?_\epsilon.\end{aligned}$$

We now choose $n_i\in\mathbb{Z}$ such that $\sum_{i\in I} n_iN_i = m = \gcd_{i\in I}(N_i)$ and consider the two semialgebraic sets

$$\begin{aligned}W_U^\epsilon &= \left\{x\in\big(E_I^0\cap U\cap\sigma^{-1}(0)\big)\times(\mathbb{R}^*)^{|I|}\;\middle|\; u((x_i)_{i\notin I},0)\prod_{i\in I}x_i^{N_i}\ ?_\epsilon\right\},\\ W_U'^\epsilon &= \left\{(x',t)\in\big(E_I^0\cap U\cap\sigma^{-1}(0)\big)\times(\mathbb{R}^*)^{|I|}\times\mathbb{R}^*\right.\\ &\qquad\qquad\qquad\left.\middle|\; u((x_i')_{i\notin I},0)t^m\ ?_\epsilon,\ \prod_{i\in I}x_i'^{N_i/m}=1\right\},\end{aligned}$$

where $?_\epsilon$ is $=-1$, $=1$, >0 or <0 in case ϵ is $-1, 1, >$ or $<$. In case $?_\epsilon=1$ or $?_\epsilon=-1$, the mapping

$$\begin{aligned} W_U^{\prime\epsilon} &\to W_U^\epsilon, \\ (x',t) &\mapsto x=\big((x_i')_{i\notin I},(t^{n_i}x_i')_{i\in I}\big)\end{aligned}$$

is an isomorphism with inverse

$$\begin{aligned} &W_U^\epsilon \to W_U^{\prime\epsilon}, \\ &x\mapsto \left(x'=\left((x_i)_{i\notin I},\left(\left(\prod_{\ell\in I}x_\ell^{N_\ell/m}\right)^{-n_i}x_i\right)_{i\in I}\right), t=\prod_{\ell\in I}x_\ell^{N_\ell/m}\right).\end{aligned}$$

In the semialgebraic case, this isomorphism induces a natural isomorphism on the double covers $\mathcal{W}_U^\epsilon$ and $\mathcal{W}_U^{\prime\epsilon}$ associated to W_U^ϵ and $W_U^{\prime\epsilon}$ and defined by

$$\mathcal{W}_U^\epsilon=\left\{(x,y)\in\big(E_I^0\cap U\cap\sigma^{-1}(0)\big)\times(\mathbb{R}^*)^{|I|}\times\mathbb{R}\;\middle|\; y^2u((x_i')_{i\notin I},0)\prod_{i\in I}x_i^{N_i}=\eta(\epsilon)\right\},$$

$$\begin{aligned}\mathcal{W}_U^{\prime\epsilon}=\Big\{(x,t,w)\in\big(E_I^0\cap U\cap\sigma^{-1}(0)\big)\times(\mathbb{R}^*)^{|I|}\times\mathbb{R}^*\times\mathbb{R}& \\ \Big|\; w^2u((x_i')_{i\notin I},0)t^m=\eta(\epsilon),\ \prod_{i\in I}x_i^{\prime N_i/m}=1\Big\},&\end{aligned}$$

where $\eta(\epsilon)=1$ when ϵ is the symbol $>$ and $\eta(\epsilon)=-1$ when ϵ is the symbol $<$. In consequence, $[W_U^\epsilon]=[W_U^{\prime\epsilon}]$ in the algebraic case ($\epsilon=-1$ or 1) as well as in the semialgebraic case ($\epsilon=<$ or $>$) considering our realization formula for basic semialgebraic formulas in $K_0(\mathrm{Var}_\mathbb{R})\otimes\mathbb{Z}[\frac{1}{2}]$. Now we observe in the case where ϵ is -1 or 1 that $W_U^{\prime\epsilon}$ is isomorphic to $R_U^\epsilon\times(\mathbb{R}^*)^{|I|-1}$ (see [8, Lemma 2.5]) whereas in the case where ϵ is $<$ or $>$, we obtain that the class of $W_U^{\prime\epsilon}$ is equal to the class of $R_U^\epsilon\times(\mathbb{R}^*)^{|I|-1}$, considering again the double coverings associated to the basic semialgebraic formulas defining these two sets.

We finally obtain

$$\begin{aligned}&\big[\mathcal{L}_n\big(M,E_I^0\cap\sigma^{-1}(0)\big)\cap\pi_n(\Delta_e)\cap X_{n,f\circ\sigma}^\epsilon\big] \\ &\qquad=\sum_{k\in A(I,n,e)}\mathbb{L}^{nd-\sum_{i\in I}k_i}[W_U^{\prime\epsilon}] \\ &\qquad=\sum_{k\in A(I,n,e)}\mathbb{L}^{nd-\sum_{i\in I}k_i}\times[R_U^\epsilon]\times(\mathbb{L}-1)^{|I|-1}.\end{aligned}$$

Summing over the charts U, the expression of $Z_f^\epsilon(T)$ given by Lemma 4.4 is now:

$$\begin{aligned}
&Z_f^\epsilon(T)\\
&\quad=\sum_{I\cap\mathcal{K}\neq\varnothing}\mathbb{L}^d\sum_{n\geq 1}T^n\sum_{e\leq cn}\mathbb{L}^{-e}(\mathbb{L}-1)^{|I|-1}\mathbb{L}^{-(n+1)d}\big[\widetilde{E}_I^{0,\epsilon}\big]\sum_{k\in A(I,n,e)}\mathbb{L}^{nd-\sum_{i\in I}k_i}\\
&\quad=\sum_{I\cap\mathcal{K}\neq\varnothing}(\mathbb{L}-1)^{|I|-1}\big[\widetilde{E}_I^{0,\epsilon}\big]\sum_{n\geq 1}T^n\sum_{e\leq cn}\sum_{k\in A(I,n,e)}\mathbb{L}^{-e-\sum_{i\in I}k_i}
\end{aligned}$$

Noticing that the $(k_i)_{i\in I}$ such that $k=((k_i)_{i\notin I},(k_i)_{i\in I})\in\bigcup_{e\leq cn,\,n\geq 1}A(I,n,e)$ are in bijection with $\mathbb{N}^{*|I|}$, we have:

$$\begin{aligned}
Z_f^\epsilon(T)&=\sum_{I\cap\mathcal{K}\neq\varnothing}(\mathbb{L}-1)^{|I|-1}\big[\widetilde{E}_I^{0,\epsilon}\big]\sum_{(k_i)_{i\in I}\in\mathbb{N}^{|I|}}\prod_{i\in I}\big(\mathbb{L}^{-\nu_i}T^{N_i}\big)^{k_i}\\
&=\sum_{I\cap\mathcal{K}\neq\varnothing}(\mathbb{L}-1)^{|I|-1}\big[\widetilde{E}_I^{0,\epsilon}\big]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{N_i}}
\end{aligned}$$ □

4.2 Motivic real Milnor fibers and their realizations

We can now define a motivic real Milnor fiber by taking the constant term of the rational function $Z_f^\epsilon(T)$ viewed as a power series in T^{-1}. This process formally consists in letting T going to ∞ in the rational expression of $Z_f^\epsilon(T)$ given by Theorem 4.2 and using the usual computation rules as in the convergent case (see for instance [5; 8]).

Definition 4.5 Let $f\colon\mathbb{R}^d\to\mathbb{R}$ be a polynomial function and ϵ be one of the symbols *naive*, 1, -1, $>$ or $<$. Consider a resolution of f as above and let us adopt the same notation $(E_I^0)_I$ for the stratification of the exceptional divisor of this resolution, leading to the notation $\widetilde{E}_I^{0,\epsilon}$. The real motivic Milnor ϵ–fiber S_f^ϵ of f is defined as (see [8] for the complex case)

$$S_f^\epsilon:=-\lim_{T\to\infty}Z_f^\epsilon(T):=-\sum_{I\cap\mathcal{K}\neq\varnothing}(-1)^{|I|}\big[\widetilde{E}_I^{0,\epsilon}\big](\mathbb{L}-1)^{|I|-1}\in K_0(\mathrm{Var}_{\mathbb{R}})\otimes\mathbb{Z}[\tfrac{1}{2}].$$

It does not depend on the choice of the resolution σ.

For ϵ being the symbol 1 for instance, we have $S_f^1\in K_0(\mathrm{Var}_{\mathbb{R}})$. We can consider, first in the complex case, the realization of S_f^1 via the Euler–Poincaré characteristic ring morphism $\chi_c\colon K_0(\mathrm{Var}_{\mathbb{C}})\to\mathbb{Z}$. Note that in the complex case, the Euler characteristics with and without compact supports are equal. For $f\colon\mathbb{C}^d\to\mathbb{C}$, since $\chi_c(\mathbb{L}-1)=0$, we obtain

$$\chi_c(S_f^1) = \sum_{|I|=1, I\subset\mathcal{K}} \chi_c\big(\widetilde{E}_I^{0,1}\big) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c\big(E_I^0 \cap \sigma^{-1}(0)\big).$$

Now denote by F the set-theoretic Milnor fiber of the fibration

$$f_{|B(0,\alpha)\cap f^{-1}(D_\eta^\times)} \colon B(0,\alpha) \cap f^{-1}(D_\eta^\times) \longrightarrow D_\eta^\times,$$

with $B(0,\alpha)$ the open ball in $\mathbb{C}^d$ of radius α centered at 0, D_η the disc in $\mathbb{C}$ of radius η centered at 0 and $D_\eta^\times = D_\eta \setminus \{0\}$, with $0 < \eta \ll \alpha \ll 1$. Compare the above expression $\chi_c(S_f^1) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c(E_I^0)$ with the following A'Campo formula of [1] for the first Lefschetz number of the iterates of the monodromy $M\colon H^*(F,\mathbb{C}) \to H^*(F,\mathbb{C})$ of f, that is for the Euler–Poincaré characteristic of the fiber F:

$$\chi_c(F) = \sum_{|I|=1, I\subset\mathcal{K}} N_I \cdot \chi_c\big(E_I^0 \cap \sigma^{-1}(0)\big)$$

We simply observe that

$$\chi_c(S_f^1) = \chi_c(F).$$

Let $\bar{F}$ be the closure $f^{-1}(c) \cap \bar{B}(0,\alpha)$, $0 < |c| \ll \alpha \ll 1$, of the Milnor fiber F and note that the boundary of $\bar{F}$ is the odd-dimensional compact manifold $f^{-1}(c) \cap S(0,\alpha)$. Then $\chi_c(f^{-1}(c) \cap S(0,\alpha)) = 0$ and we finally have

$$\chi_c(S_f^1) = \chi_c(F) = \chi_c(\bar{F}).$$

Remark 4.6 There is a priori no hint in the definition of $Z_f^\epsilon(T)$ that the opposite of the constant term S_f^1 of the power series in T^{-1} induced by the rationality of $Z_f^\epsilon(T)$ could be the motivic version of the Milnor fiber of f (as well as, for instance, there is no evident hint that the expression of Z_f^ϵ in Theorem 4.2 does not depend on the resolution σ). As mentioned above, in the complex case, we just observe that the expression of $\chi_c(S_f^1)$ is the expression of $\chi_c(F)$ provided by the A'Campo formula. Exactly in the same way there is no a priori reason for $\chi_c(S_f^\epsilon)$, regarding the definition of Z_f^ϵ, to be so accurately related to the topology of $f^{-1}(\epsilon|c|) \cap B(0,\alpha)$. Nevertheless we prove that it is actually the case (Theorem 4.12).

In order to establish this result we start with a geometric proof of the formula in the complex case (compare with [1] where only $\Lambda(M^0)$ is considered, M^k being the k^{th} iterate of the monodromy $M\colon H^*(F,\mathbb{C}) \to H^*(F,\mathbb{C})$ of f). We will then extend to the reals this computational proof in the proof of Theorem 4.12, allowing us interpret the complex proof as the first complexity level of its real extension.

Remark 4.7 Note that in the complex case a proof of the fact that $\Lambda(M^k)=\chi_c(X^1_{k,f})$ for $k\geq 1$ is given in [11] without the help of resolution of singularities, that is to say without help of A'Campo's formulas (see [11, Theorem 1.1.1]). As a direct corollary it is thus proved that $\chi_c(S^1_f)=\chi_c(F)$ in the complex case, without using A'Campo formulas.

Realization of the complex motivic Milnor fiber under χ_c The fiber

$$F=\{f=c\}\cap B(0,\alpha)$$

is homeomorphic to the fiber $\mathcal{F}=\{f\circ\sigma=c\}\cap\sigma^{-1}(B(0,\alpha))$, with $\sigma^{-1}(S(0,\alpha))$ viewed as the boundary of a tubular neighborhood of $\sigma^{-1}(0)=\bigcup_{E^0_J\subset\sigma^{-1}(0)}E^0_J$, keeping the same notation $(E^0_J)_J$ as before for the natural stratification of the strict transform $\sigma^{-1}(\{f=0\})$ of $f=0$. Now the formula may be established for $\mathcal{F}$ in some chart of $M\cap\sigma^{-1}(B(0,\alpha))$, by additivity. In such a chart, where $f\circ\sigma$ is normal crossing, we consider:

- The set $E_J=\bigcap_{i\in J}E_i\subset\sigma^{-1}(0)$, given by $x_i=0$, $i\in J$.
- A closed small enough tubular neighborhood V_J in M of $\bigcup_{J\subset K,K\neq J}E^0_K$, that is a tubular neighborhood of all the E^0_K bounding E^0_J, such that $E^0_J\setminus V_J$ is homeomorphic to E^0_J.
- π_J the projection onto E_J along the x_j coordinates, for $j\in J$.
- An open neighborhood $\mathcal{E}_J$ of $E^0_J\setminus V_J$ in $\sigma^{-1}(B(0,\alpha))$ given by $\pi_J^{-1}(E^0_J\setminus V_J)$, $|x_j|\leq\eta_J$, $j\in J$, with $\eta_J>0$ small enough.

Remark 4.8 For $I=\{i\}$, we remark that $\mathcal{F}\cap\mathcal{E}_I$ is homeomorphic to N_i copies of $E^0_I\cap\mathcal{E}_I$, and thus to N_i copies of E^0_I. Indeed, assuming $f\circ\sigma=u(x)x_i^{N_i}$ in $\mathcal{E}_I$, we observe that the family $(f_t)_{t\in[0,1]}$ with $f_t=u((x_j)_{j\notin I},t\cdot x_i)x_i^{N_i}-c$ has homeomorphic fibers $\{f_t=0\}\cap\mathcal{E}_J$, $t\in[0,1]$ by Thom's isotopy lemma, since

$$\frac{\partial f_t}{\partial x_i}(x)=t\frac{\partial u}{\partial x_i}(x)x_i^{N_i}+u(x)x_i^{N_i-1}=0$$

would imply

$$t\frac{\partial u}{\partial x_i}(x)x_i+u(x)=0.$$

But the first term in this sum goes to 0 as x_i goes to 0, since the derivatives of u are bounded on the compact $\mathrm{cl}(\mathcal{E}_I)$ by a nonzero constant, since u is a unit. Finally, as $\{f_1=0\}\cap\mathcal{E}_I$ is homeomorphic to $\{f_0=0\}\cap\mathcal{E}_I$ and $\{f_0=0\}\cap\mathcal{E}_I$ is a N_i–graph over $E^0_I\cap\mathcal{E}_I$, $\mathcal{F}\cap\mathcal{E}_I$ is homeomorphic to N_i copies of E^0_I.

By this remark, $\mathcal{F}$ covers maximal dimensional stratum E_I^0, $|I| = 1$, $I \subset \mathcal{K}$, with N_i copies of a leaf $\mathcal{F}_I$ of $\mathcal{F}$. To be more accurate, with the notation introduced above, $\mathcal{F}_I$ covers the neighborhood $E_I^0 \cap \mathcal{E}_I$ of $E_I^0 \setminus V_I$. Moreover the $\mathcal{F}_I$ overlap in $\mathcal{F}$ over the open set $E_J^0 \cap \mathcal{E}_J$ of the strata E_J^0 that bound the E_I^0 for $|I| = 1$, $|J| = 2$ and $I \subset J$ in bundles over the $E_J^0 \cap \mathcal{E}_J$ of fiber $\mathbb{C}^*$. Those subleaves $\mathcal{F}_J$ of $\mathcal{F}$ overlap in turn over the open $E_Q^0 \cap \mathcal{E}_Q$ of the strata E_Q^0, $|Q| = 3$, $J \subset Q$, that bound the E_J^0, in bundles over the $E_Q^0 \cap \mathcal{E}_Q$ of fibers $(\mathbb{C}^*)^2$ and so forth… For instance when

$$f \circ \sigma = u(x) \prod_{i \in I} x_i^{N_i} \quad \text{in } \mathcal{E}_I,\ I = \{i\},$$

$$f \circ \sigma = v(x) x_i^{N_i} x_j^{N_j} \quad \text{in } \mathcal{E}_J,\ J = \{i, j\},$$

the N_i leaves $\mathcal{F}_I$, homeomorphic to the N_i copies $x_i^{N_i} = c/u(x)$ of E_I^0, overlap over $E_J^0 \cap \mathcal{E}_J$ in subleaves $\mathcal{F}_J$ of $\mathcal{F}_I$, given by $v(x)\, x_i^{N_i} x_j^{N_j} = c$, fibering over E_J^0 with fiber $\gcd(\{N_i, N_j\})$ copies of $(\mathbb{C}^*)^{|J|-1}$ and so forth (see Figure 1).

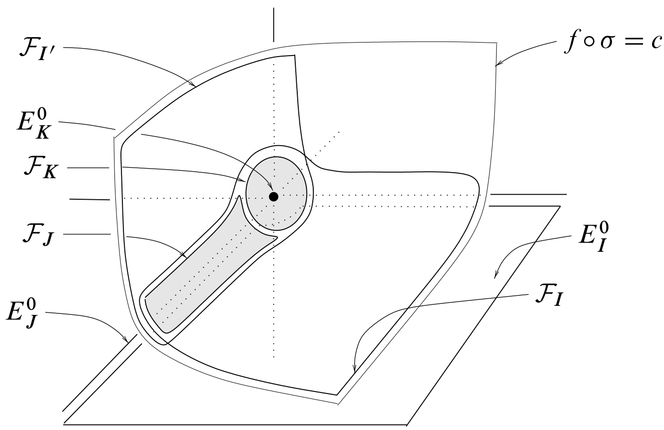

Figure 1

Remark 4.9 Note that the topology of $\mathcal{F} = \{f \circ \sigma = c\} \cap \sigma^{-1}(B(0,\alpha))$ is the same as the topology of $\bigcup_{J \cap \mathcal{K} \neq 0} \mathcal{F}_J$ (that is the topology of $\mathcal{F}$ above the strata E_J^0 of $\sigma^{-1}(0)$) since the retraction of $\mathcal{F}$ onto $\bigcup_{J \cap \mathcal{K} \neq \varnothing} \mathcal{F}_J$, as α goes to 0, induces a homeomorphism from $\mathcal{F}$ to $\bigcup_{J \cap \mathcal{K} \neq \varnothing} \mathcal{F}_J$.

From Remark 4.9, by additivity, it follows that the Euler–Poincaré characteristic of $\mathcal{F}$ (in our chart) is the sum

$$(*) \qquad \sum_{|I|=1, I \subset \mathcal{K}} N_I \cdot \chi_c\big(E_I^0 \cap \sigma^{-1}(0)\big) + L,$$

where L is some $\mathbb{Z}$–linear combination of Euler–Poincaré characteristics of bundles over the open sets $E_J \cap \mathcal{E}_J^0$, $|J| > 1$, of fiber a power of tori $\mathbb{C}^*$. Now the A'Campo formula

$$\chi_c(F) = \sum_{|I|=1, I \subset \mathcal{K}} N_I \cdot \chi_c\big(E_I^0 \cap \sigma^{-1}(0)\big)$$

for the Milnor number follows from the fact that $\chi_c(\mathbb{C}^*) = 0$ implies $L = 0$.

Realization of the real motivic Milnor fibers under χ_c The partial covering of $\mathcal{F}$ by the pieces $\mathcal{F}_J$, for $J \cap \mathcal{K} \neq \varnothing$, over the strata of the stratification $(E_J^0)_{J \cap \mathcal{K} \neq \varnothing}$ of $\sigma^{-1}(0)$ allows us to compute the Euler–Poincaré characteristic of the Milnor fiber $\mathcal{F}$ in terms of the Euler–Poincaré characteristic of the strata E_J^0, in the complex as well as in the real case. In the complex case, as noted above, for J with $|J| > 1$, one has $\chi_c(\mathcal{F}_J) = 0$. This cancellation provides a quite simple formula for $\chi_c(F)$: Only the strata of the maximal dimension of the divisor $\sigma^{-1}(0)$ appear in this formula, as expected from the A'Campo formula.

In the real case one does not have such cancellations: On one hand the expression of $\chi_c(F)$ in terms of $\chi_c(\widetilde{E}_J^{0,\epsilon})$ is no more trivial (the remaining term L of Equation $(*)$ is not zero and consequently terms $\chi_c(\widetilde{E}_J^{0,\epsilon})$ for $|J| > 1$ and $E_j \cap \sigma^{-1}(0) \neq \varnothing$ appear) and on the other hand the expression of $\chi_c(S_f^\epsilon)$ given by the real Denef–Loeser formula in Definition 4.5 has terms $2^{|J|-1}\chi_c(\widetilde{E}_J^{0,\epsilon})$ for $|J| > 1$ and $J \cap \mathcal{K} \neq \varnothing$ (since $\chi_c(\mathbb{L} - 1) = -2$ in the real case).

Nevertheless, in the real case we show that $\chi_c(S_f^\epsilon)$ is again $\chi_c(\overline{F})$, justifying the terminology of motivic real semialgebraic Milnor fiber of f at 0 for S_f^ϵ. The formula stated in Theorem 4.12 below is the real analogue of the A'Campo–Denef–Loeser formula for complex hypersurface singularities and thus appears as the extension to the reals of this complex formula, or, in other words, the complex formula is the notably first level of complexity of the more general real formula.

Notation 4.10 Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial function such that $f(0) = 0$ and with isolated singularity at 0, that is $\operatorname{grad} f(x) = 0$ only for $x = 0$ in some open neighborhood of 0. Let $0 < \eta \ll \alpha$ be such that the topological type of $f^{-1}(c) \cap B(0,\alpha)$ does not depend on c and α for $0 < c < \eta$ or for $-\eta < c < 0$.

- Let us denote, for $\epsilon \in \{-1, 1\}$ and $\epsilon \cdot c > 0$, this topological type by F_ϵ, by $\overline{F}_\epsilon$ the topological type of the closure of the Milnor fiber F_ϵ and by $Lk(f)$ the link $f^{-1}(0) \cap S(0,\alpha)$ of f at the origin. We recall that the topology of $Lk(f)$ is the same as the topology of the boundary $f^{-1}(c) \cap S(0,\alpha)$ of the Milnor fiber $\overline{F}_\epsilon$ when f has an isolated singularity at 0.

- Let us denote, for $\epsilon \in \{<, >\}$, the topological type of $f^{-1}(]0, c_\epsilon[) \cap B(0, \alpha)$ by F_ϵ, and the topological type of $f^{-1}(]0, c_\epsilon[) \cap \bar{B}(0, \alpha)$ by $\bar{F}_\epsilon$, where $c_< \in]-\eta, 0[$ and $c_> \in]0, \eta[$.

- Let us denote, for $\epsilon \in \{<, >\}$, the topological type of $\{f \mathrel{\bar{\epsilon}} 0\} \cap S(0, \alpha)$ by G_ϵ, where $\bar{\epsilon}$ is $\le$ when ϵ is $<$ and $\bar{\epsilon}$ is $\ge$ when ϵ is $>$.

Remark 4.11 When d is odd, $Lk(f)$ is a smooth odd-dimensional submanifold of $\mathbb{R}^d$ and consequently $\chi_c(Lk(f)) = 0$. For $\epsilon \in \{-1, 1, <, >\}$, we thus have in $\chi_c(F_\epsilon) = \chi_c(\bar{F}_\epsilon)$ this situation. This is the situation in the complex setting. When d is even and for $\epsilon \in \{-1, 1\}$, since $\bar{F}_\epsilon$ is a compact manifold with boundary $Lk(f)$, one knows that

$$\chi_c(\bar{F}_\epsilon) = -\chi_c(F_\epsilon) = \tfrac{1}{2}\chi_c(Lk(f)).$$

For general $d \in \mathbb{N}$ and for $\epsilon \in \{-1, 1, <, >\}$, we thus have

$$\chi_c(\bar{F}_\epsilon) = (-1)^{d+1}\chi_c(F_\epsilon).$$

On the other hand we recall that for $\epsilon \in \{<, >\}$

$$\chi_c(G_\epsilon) = \chi_c(\bar{F}_{\delta_\epsilon}),$$

where $\delta_>$ is 1 and $\delta_<$ is -1 (see [2; 18]).

Theorem 4.12 *With Notation 4.10 we have, for $\epsilon \in \{-1, 1, <, >\}$,*

$$\chi_c(S_f^\epsilon) = \chi_c(\bar{F}_\epsilon) = (-1)^{d+1}\chi_c(F_\epsilon)$$

and for $\epsilon \in \{<, >\}$,

$$\chi_c(S_f^\epsilon) = -\chi_c(G_\epsilon).$$

Proof Assume first that $\epsilon \in \{-1, 1\}$. We denote by $\mathcal{F}$ the fiber $\sigma^{-1}(F_\epsilon)$ and recall that $\mathcal{F}$ and F_ϵ have the same topological type. Let us denote $\bar{\mathcal{K}}$ the set of multi-indices $J \subset \mathcal{I}$ such that $\bar{E}_J \cap \sigma^{-1}(0) \neq \varnothing$, with $\bar{E}_J$ the closure of $E_J = \bigcap_{i \in J} E_i$. In what follows only $J \in \bar{\mathcal{K}}$ are concerned, since we study the local Milnor fiber at 0. The proof consists in the computation of the Euler–Poincaré characteristic of $\mathcal{F}$ using the decomposition of $\mathcal{F}$ by the overlapping components $\mathcal{F}_I$ introduced just before Figure 1 and illustrated there. We simply count the number of these overlapping components in the decomposition of $\mathcal{F}$ they provide. Note that a connected component of E_J^0 (still denoted E_J^0 for simplicity in the sequel) for $J \subset \mathcal{J}$ is covered by $n_J := M_J \cdot 2^{|J|-1}$ connected components $\mathcal{G}$ of $\mathcal{F}$, where M_J is 0, 1 or 2 depending on the fact that the multiplicity $m_J = \gcd_{j \in J}(N_j)$ defining $\widetilde{E}_J^{0,\epsilon}$ is odd or even, and on sign condition on c (remember from Figure 1 how E_J^0 is covered by $\mathcal{F}_J$; here the term covered simply means that one can naturally project the component $\mathcal{F}_J$ onto E_J^0). Note furthermore that M_J is the degree of the covering $\widetilde{E}_J^{0,\epsilon}$ of E_J^0. Now expressing a connected component $\mathcal{G}$ of $\mathcal{F}$ as the union

$$\bigcup_{|I|=1,\mathcal{F}_I\subset\mathcal{G}} \mathcal{F}_I,$$

where the (connected) leaves $\mathcal{F}_I$ cover (the open subset $E_I^0\cap\mathcal{E}_I^0$ of E_I^0 homeomorphic to) E_I^0, and using the additivity of χ_c, one has that $\chi_c(\mathcal{G})$ is expressed as a sum of characteristics of the overlapping connected subleaves $\mathcal{F}_J$ of the $\mathcal{F}_I$, each of them with sign coefficient $s_J := (-1)^{|J|-1}$. Note that (a connected component of) E_J^0 is covered by n_J copies of such a $\mathcal{F}_J$, coming from the n_J connected components of $\mathcal{F}$ above $E_J^0\cap\mathcal{E}_J^0$, and that a connected subleaf $\mathcal{F}_J$ has the topology of $(E_J^0\cap\mathcal{E}_J^0)\times\mathbb{R}^{|J|-1}$. We denote by t_J the characteristic $t_J := \chi_c(\mathbb{R}^{|J|-1}) = (-1)^{|J|-1}$.

With this notation, summing over all the connected components $\mathcal{G}$ of $\mathcal{F}$, one gets

$$\begin{aligned}
\chi_c(\mathcal{F}) &= \sum_{J\in\bar{\mathcal{K}}} s_J\times n_J\times\chi_c(E_J^0)\times t_J\\
&= \sum_{J\in\bar{\mathcal{K}}} (-1)^{|J|-1}\times 2^{|J|-1}M_J\times\chi_c(E_J^0)\times(-1)^{|J|-1}\\
&= \sum_{J\in\bar{\mathcal{K}}} 2^{|J|-1}\chi_c\big(\widetilde{E}_J^{0,\epsilon}\big) = \sum_{J\cap\mathcal{K}\neq\varnothing} 2^{|J|-1}\chi_c\big(\widetilde{E}_J^{0,\epsilon}\big) + \sum_{\substack{J\cap\mathcal{K}=\varnothing,\\ J\in\bar{\mathcal{K}}}} 2^{|J|-1}\chi_c\big(\widetilde{E}_J^{0,\epsilon}\big)\\
&= \chi_c(S_f^\epsilon) + \sum_{J\cap\mathcal{K}=\varnothing,J\in\bar{\mathcal{K}}} 2^{|J|-1}\chi_c\big(\widetilde{E}_J^{0,\epsilon}\big) = \chi_c(S_f^\epsilon)+\chi_c\Big(\bigcup_{J\cap\mathcal{K}=\varnothing,J\in\bar{\mathcal{K}}}\mathcal{F}_J\Big).
\end{aligned}$$

Note that the subleaves $\mathcal{F}_J$ for $J\cap\mathcal{K}=\varnothing$ and $J\in\bar{\mathcal{K}}$ cover the set $\{f\circ\sigma=c\}\cap\widehat{S}(0,\alpha)$ for $\epsilon\cdot c>0$, where $\widehat{S}(0,\alpha)$ is a neighborhood $\sigma^{-1}(S(0,\alpha)\times]0,\beta[)$ of $\sigma^{-1}(S(0,\alpha))$ with $0<\beta\ll\alpha$. It follows that

$$\begin{aligned}
\chi_c\Big(\bigcup_{J\cap\mathcal{K}=\varnothing,J\in\bar{\mathcal{K}}}\mathcal{F}_J\Big) &= \chi_c\big(F_\epsilon\cap\big(S(0,\alpha)\times]0,\beta[\big)\big)\\
&= \chi_c\big(Lk(f)\times]0,\beta[\big) = -\chi_c(Lk(f)).
\end{aligned}$$

We finally obtain

$$\begin{aligned}
\chi_c(F_\epsilon) &= \chi_c(S_f^\epsilon)-\chi_c(Lk(f)),\\
\chi_c(\bar{F}_\epsilon) &= \chi_c(F_\epsilon)+\chi_c(Lk(f)) = \chi_c(S_f^\epsilon).
\end{aligned}$$

This proves the first equality of our statement, the equality $\chi_c(\bar{F}_\epsilon)=(-1)^{d+1}\chi_c(F_\epsilon)$ being proved in Remark 4.11.

Assume now that $\epsilon \in \{<,>\}$ and denote $\delta_< := -1$ and $\delta_> := 1$, like in Remark 4.11. With this notation $\bar{F}_\epsilon = \bar{F}_{\delta_\epsilon} \times]0,1[$, and by the formula proved above in the case $\epsilon \in \{-1,1\}$, we obtain

$$\chi_c(\bar{F}_\epsilon) = \chi_c(\bar{F}_{\delta_\epsilon})\chi_c(]0,1[) = -\chi_c(\bar{F}_{\delta_\epsilon}) = -\chi_c(S_f^{\delta_\epsilon}) = -\sum_{J\cap\mathcal{K}\neq\varnothing} 2^{|J|-1}\chi(\widetilde{E}_J^{0,\delta_\epsilon}).$$

But since $\widetilde{E}_J^{0,\epsilon} = \widetilde{E}_J^{0,\delta_\epsilon} \times \mathbb{R}_+$, it follows that

$$\chi_c(\bar{F}_\epsilon) = \sum_{J\cap\mathcal{K}\neq\varnothing} 2^{|J|-1}\chi(\widetilde{E}_J^{0,\delta_\epsilon})\chi_c(\mathbb{R}_+) = \sum_{J\cap\mathcal{K}\neq\varnothing} 2^{|J|-1}\chi(\widetilde{E}_J^{0,\epsilon}) = \chi_c(S_f^\epsilon).$$

This proves the first equality of our statement. The equality $\chi_c(\bar{F}_\epsilon) = (-1)^{d+1}\chi_c(F_\epsilon)$ is the consequence of $\chi_c(\bar{F}_\epsilon) = \chi_c(\bar{F}_{\delta_\epsilon})\chi_c(]0,1[)$, $\chi_c(F_\epsilon) = \chi_c(F_{\delta_\epsilon})\chi_c(]0,1[)$ and $\chi_c(\bar{F}_{\delta_\epsilon}) = (-1)^{d+1}\chi_c(F_{\delta_\epsilon})$.

To finish, equality $\chi_c(S_f^\epsilon) = -\chi_c(G_\epsilon)$ comes from the equality $\chi_c(G_\epsilon) = \chi_c(\bar{F}_{\delta_\epsilon})$ recalled in Remark 4.11 and from $\chi_c(\bar{F}_\epsilon) = -\chi_c(\bar{F}_{\delta_\epsilon})$, $\chi_c(S_f^\epsilon) = \chi_c(\bar{F}_\epsilon)$. □

Remark 4.13 As stated in Theorem 4.12, the realization via χ_c of the motivic Milnor fiber S_f^ϵ for $\epsilon \in \{-1,1,<,>\}$ gives the Euler–Poincaré characteristic of the corresponding set theoretic semialgebraic closed Milnor fiber $\bar{F}_\epsilon$. Nevertheless it is worth noting that this equality is in general not true at the higher level of $\chi(K_0[BSA_\mathbb{R}])$. Even computed in $K_0(\mathrm{Var}_\mathbb{R}) \otimes \mathbb{Z}[\frac{1}{2}]$, we may have $S_f^\epsilon \neq [A_{f,\epsilon}]$, for a given semialgebraic formula $A_{f,\epsilon}$ with real points $\bar{F}_\epsilon$. Let us illustrate this remark by the following quite trivial example.

Example 4.14 Let us consider the simple case where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x,y) = xy$. After one blowing-up $\sigma\colon M \to \mathbb{R}^2$ of the origin of $\mathbb{R}^2$, the situation is as required by Theorem 4.2. We denote by E_1 the exceptional divisor $\sigma^{-1}(0)$ (which is isomorphic to $\mathbb{P}_1$) and by E_2, E_3 the irreducible components of the strict transform $\sigma^{-1}(\{f=0\})$. The induced stratification of E_1 is given by $E_{1,2}^0 = E_1 \cap E_2$, $E_{1,3}^0 = E_1 \cap E_3$ and the two connected components $E_1'^0, E_1''^0$ of $E_1 \setminus (E_2 \cup E_3)$. We consider a chart (X,Y) of M such that $\sigma(X,Y) = (x=Y, y=XY)$. In this chart, $(f\circ\sigma)(X,Y) = XY^2$. The multiplicity of $f\circ\sigma$ along E_1 is $N_1 = 2$ and the multiplicity of $\mathrm{jac}\,\sigma$ along E_1 is 1, thus $\nu_1 = 2$. Assuming that $E_1'^0$ corresponds to $X > 0$ and $E_1''^0$ corresponds to $X < 0$, it follows that

$$\widetilde{E}_1'^{0,\epsilon} = \{(X,t) \mid X \in E_1'^0,\ t \in \mathbb{R},\ Xt^2 ?_\epsilon\},$$
$$\widetilde{E}_1''^{0,\epsilon} = \{(X,t) \mid X \in E_1''^0,\ t \in \mathbb{R},\ Xt^2 ?_\epsilon\},$$

where $?_\epsilon$ is $=1$, $=-1$, $>$ or <0 in case ϵ is 1, -1, $>$ or $<$. In case $\epsilon=1$ we obtain

$$\big[\widetilde{E}_1^{\prime 0,1}\big]=\mathbb{L}-1 \quad \text{and} \quad \big[\widetilde{E}_1^{\prime\prime 0,1}\big]=0$$

since $\widetilde{E}_1^{\prime 0,1}$ has a one-to-one projection onto $\{(X,Y) \mid X=0,\ Y\neq 0\}$ and $\widetilde{E}_1^{\prime\prime 0,1}$ is empty. Now in a neighborhood of $E^0_{1,2}$, $f\circ\sigma(X,Y)=XY^2$, giving $N_1=1$, $N_2=2$ and $m=\gcd(N_1,N_2)=1$. We also have $\nu_1=2$ and $\nu_2=1$. It follows that

$$\widetilde{E}^{0,1}_{1,2}=\{(0,t) \mid t\in\mathbb{R}, t=1\} \quad \text{thus} \quad \big[\widetilde{E}^{0,1}_{1,2}\big]=1.$$

In the same way, using another chart, one finds

$$\big[\widetilde{E}^{0,1}_{1,3}\big]=1.$$

By Theorem 4.2 we then have

$$\begin{aligned} Z_f^1(T) &= (\mathbb{L}-1)^{1-1}(\mathbb{L}-1)\left(\frac{\mathbb{L}^{-2}T^2}{1-\mathbb{L}^{-2}T^2}\right) \\ &\qquad + 2(\mathbb{L}-1)^{2-1}\left(\frac{\mathbb{L}^{-2}T^2}{1-\mathbb{L}^{-2}T^2}\right)\left(\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T}\right), \\ Z_f^1(T) &= \frac{\mathbb{L}-1}{(\mathbb{L}T^{-1}-1)^2}, \\ S_f^1 &= -(\mathbb{L}-1). \end{aligned}$$

Of course we find that $\chi_c(S_f)=\chi_c(\{f=c\}\cap\overline{B}(0,1))=2$, $0<c\ll 1$.

Now let us for instance choose $\{xy=c, 1-x^2-y^2>0\}$ for $0<c\ll 1$ as a basic semialgebraic formula to represent the open Milnor fiber of $f=0$ and let us compute $\beta([xy=c,1-x^2-y^2>0])$ (rather than $[xy=c,1-x^2-y^2>0]$ itself, since we use regular homeomorphisms in our computations). By definition of the realization $\beta\colon K_0(BSA_{\mathbb{R}})\to\mathbb{Z}[\frac{1}{2}][u]$, we have

$$\begin{aligned} \beta\big([xy=c,1-x^2-y^2>0]\big) &= \tfrac{1}{4}\beta\big([xy=c,z^2=1-x^2-y^2]\big) \\ &\qquad -\tfrac{1}{4}\beta\big([xy=c,z^2=x^2+y^2-1]\big) \\ &\qquad +\tfrac{1}{2}\beta\big([xy=c,1-x^2-y^2\neq 0]\big). \end{aligned}$$

Projecting the algebraic set $\{xy=c, z^2=1-x^2-y^2\}$ orthogonally to the plane $x=-y$ with coordinates $(X=\frac{1}{\sqrt{2}}(x-y),z)$ one finds twice the quadric $z^2+2X^2=1-2c$, that is, up to regular homeomorphism, two circles. A circle having class $u+1$, we have

$$\beta\big([xy=c,z^2=1-x^2-y^2]\big)=2(u+1).$$

Projecting the algebraic set $\{xy = c, z^2 = x^2 + y^2 - 1\}$ to the plane $x = -y$ with coordinates $(X = \frac{1}{\sqrt{2}}(x - y), z)$ one finds twice the hyperbola $2X^2 - z^2 = 1 - 2c$. Projecting orthogonally again the hyperbola onto one of its asymptotic axes we see that this hyperbola has class $u - 1$. It gives

$$\beta\big([xy = c, z^2 = x^2 + y^2 - 1]\big) = 2(u - 1).$$

Finally the constructible set $\{xy = c, 1 - x^2 - y^2 \neq 0\}$ being the hyperbola without 4 points, we have

$$\beta\big([xy = c, 1 - x^2 - y^2 > 0]\big) = \tfrac{1}{2}(u + 1) - \tfrac{1}{2}(u - 1) + \tfrac{1}{2}(u - 1) - 2 = \frac{u - 3}{2}.$$

Of course, $\chi_c(\chi([xy = c, 1 - x^2 - y^2 > 0])) = \chi_c(\{f = c\} \cap B(0, 1)) = -2$.

The simple semialgebraic formula representing the set theoretic closed Milnor fiber is $\{xy = c, 1 - x^2 - y^2 \geq 0\}$, it has class $\beta([xy = c, 1 - x^2 - y^2 > 0]) + 4\beta([\{*\}]) = \frac{1}{2}(u + 5)$ in $\mathbb{Z}[\frac{1}{2}][u]$. But although

$$\chi_c\big(\chi\big([xy = c, 1 - x^2 - y^2 \geq 0]\big)\big) = \chi_c(S^1_f) = \chi_c\big(\{f = c\} \cap \bar{B}(0, 1)\big) = 2$$

as expected from Theorem 4.12, we observe that

$$\frac{u + 5}{2} = \beta\big([xy = c, 1 - x^2 - y^2 \geq 0]\big) \neq \beta(S^1_f) = -(u - 1).$$

As a final consequence, we certainly cannot have this equality between

$$\chi\big([xy = c, 1 - x^2 - y^2 \geq 0]\big) \quad \text{and} \quad S^1_f$$

at the level of $K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$.

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