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The genus 0 Gromov–Witten invariants of projective complete intersections

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We describe the structure of mirror formulas for genus 0 Gromov–Witten invariants of projective complete intersections with any number of marked points and provide an explicit algorithm for obtaining the relevant structure coefficients. As an application, we give explicit closed formulas for the genus 0 Gromov–Witten invariants of Calabi–Yau complete intersections with 3 and 4 constraints. The structural description alone suffices for some qualitative applications, such as vanishing results and the bounds on the growth of these invariants predicted by R Pandharipande. The resulting theorems suggest intriguing conjectures relating GW–invariants to the energy of pseudoholomorphic maps and the expected dimensions of their moduli spaces.

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1 Introduction

Gromov–Witten invariants of a smooth projective variety X are certain counts of curves in X. In many cases, these invariants are known or conjectured to possess rather amazing structure which is often completely unexpected from the classical point of view. For example, the genus 0 GW–invariants of a quintic threefold, ie a degree 5 hypersurface in $\mathbb{P}^4$, are related by a so-called *mirror formula* to hypergeometric series. This relation was explicitly predicted by Candelas, de la Ossa, Green and Parkes in [7] and mathematically confirmed by Bertram [5], Gathmann [11], Givental [13], Lee [20] and Lian, Liu and Yau [22]. In fact, the prediction of [7] has been shown to be a special case of closed formulas for 1–pointed genus 0 GW–invariants (counts of curves passing through one constraint) of complete intersections of sufficiently small total multidegree (see Givental [12] and [22]). It is shown by Bertram and Kley in [6] and the author in [30] that closed formulas for 2–pointed genus 0 GW–invariants of hypersurfaces are explicit transforms of the 1–pointed formulas; this is extended to projective complete intersections by Cherveny in [8] and Popa and the author [27].

The classical localization theorem of Atiyah and Bott [3] reduces the computation of genus 0 GW–invariants of projective complete intersections to a sum over decorated graphs. In this paper, we use the method of the author [31] for breaking such graphs at

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special nodes to show that closed formulas for N–pointed genus 0 GW–invariants of projective complete intersections are explicit transforms of the 1–pointed formulas, with the key link provided by the transform for the 2–pointed invariants obtained in [27]. We show that closed formulas for N–pointed genus 0 GW–invariants of projective complete intersections, with $N \geq 3$, are linear combinations of N–fold products of derivatives of 1–pointed formulas with coefficients that are polynomials of total degree at most $N-3$. While we describe two explicit ways of computing the coefficients of these polynomials, the final formulas become rather complicated as N increases. Nevertheless, our qualitative description of generating functions for N–pointed GW–invariants as linear combinations of N–fold products of derivatives leads to some simple-to-state qualitative results concerning these invariants; see Theorems 1 and 2 below.

Throughout the paper $N \geq 3$, $n \geq 2$ and $l \geq 0$ will be fixed integers and

$$\boldsymbol{a} \equiv (a_k)_{k=1,2,\dots,l} \equiv (a_1, \dots, a_l)$$

a tuple of positive integers, with N and $\boldsymbol{a}$ denoting the number of marked points and the multidegree of a fixed complete intersection $X_{\boldsymbol{a}} \subset \mathbb{P}^{n-1}$, respectively. Let

$$[N] = \{1, 2, \dots, N\}, \quad |\boldsymbol{a}| = \sum_{k=1}^{k=l} a_k, \quad \nu_{\boldsymbol{a}} = n - |\boldsymbol{a}|,$$

$$\|\boldsymbol{a}\| = \sum_{k=1}^{k=l} k a_k, \quad \langle \boldsymbol{a} \rangle = \prod_{k=1}^{k=l} a_k, \quad \boldsymbol{a}^{\boldsymbol{a}} = \prod_{k=1}^{k=l} a_k^{a_k}, \quad \boldsymbol{a}! = \prod_{k=1}^{k=l} a_k!$$

For any nonnegative integer d, we denote by $\overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}}, d)$ the moduli space of genus 0 degree d N–marked stable maps to $X_{\boldsymbol{a}}$. For each $s = 1, \dots, N$, let

$$\mathrm{ev}_s \colon \overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}}, d) \to X_{\boldsymbol{a}}, \quad \psi_s \equiv c_1(L_s^*) \in H^2(\overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}}, d)),$$

be the evaluation map and the first Chern class of the universal tangent line bundle at the s^{th} marked point. Denote by $H \in H^2(\mathbb{P}^{n-1})$ the hyperplane class.

The main theorem of this paper, Theorem A in Section 2.3, provides a closed formula for the N–pointed version of the standard (one-pointed) Givental's J–function. This is a generating function for genus 0 GW–invariants,

$$\langle \tau_{b_1} H^{c_1}, \dots, \tau_{b_N} H^{c_N} \rangle_{0,d}^{X_{\boldsymbol{a}}} \equiv \int_{[\overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}},d)]^{\mathrm{vir}}} \prod_{s=1}^{s=N} (\psi_s^{b_s} \, \mathrm{ev}_s^* H^{c_s}), \tag{1-1}$$

of a complete intersection $X_{\boldsymbol{a}} \subset \mathbb{P}^{n-1}$ of multidegree $\boldsymbol{a}$ with $|\boldsymbol{a}| \leq n$. In particular, it encodes the famous big J–function (which allows powers of only one ψ–class). The precise statement of this formula is quite involved and is thus deferred until Section 2. We instead begin by describing some qualitative corollaries of Theorem A, Theorems 1 and 2 and special cases, Theorems 3 and 4.

Theorem 1 *If $n \in \mathbb{Z}^+$, $\boldsymbol{a} \in (\mathbb{Z}^+)^l$ and $X_{\boldsymbol{a}} \subset \mathbb{P}^{n-1}$ is a complete intersection of multidegree $\boldsymbol{a}$, there exists $C_{\boldsymbol{a}} \in \mathbb{R}^+$ such that*

$$\left|\frac{\langle b_1!\tau_{b_1}H^{c_1},\dots,b_N!\tau_{b_N}H^{c_N}\rangle_{0,d}^{X_{\boldsymbol{a}}}}{N!}\right| \le C_{\boldsymbol{a}}^{N+d}$$

for all $N \in \mathbb{Z}^+$, $d,b_1,\dots,b_N,c_1,\dots,c_N \in \mathbb{Z}$.

This bound holds for $d=0$, since

$$\begin{aligned}\langle\tau_{b_1}H^{c_1},\dots,\tau_{b_N}H^{c_N}\rangle_{0,0}^{X_{\boldsymbol{a}}} &= \langle\boldsymbol{a}\rangle\bigg(\int_{\mathbb{P}^{n-1}} H^{c_1+\dots+c_N+l}\bigg)\bigg(\int_{\overline{\mathcal{M}}_{0,N}}\psi_1^{b_1}\cdots\psi_N^{b_N}\bigg)\\ &= \langle\boldsymbol{a}\rangle\delta_{c_1+\dots+c_N,n-1-l}\binom{N-3}{b_1,\dots,b_N}\end{aligned}$$

for all $c_1,\dots,c_N \ge 0$, where $\overline{\mathcal{M}}_{0,N}$ is the Deligne–Mumford moduli space of genus 0 curves with N marked points. Theorem 1 implies that for every Calabi–Yau complete intersection threefold $X_{\boldsymbol{a}} \subset \mathbb{P}^{n-1}$ ($|\boldsymbol{a}|=n$, $l=n-4$) there exists $C \in \mathbb{R}^+$ such that

$$\big|\langle\cdot\rangle_{0,d}^{X_{\boldsymbol{a}}}\big| \le \frac{N!C^N}{d^N}\cdot C^d \quad \text{for all } d,N \in \mathbb{Z}^+;$$

for $N \le 2$, this bound also follows from the one-point mirror formulas. According to Maulik and Pandharipande [23], the $X_{\boldsymbol{a}} = \mathbb{P}^3$ case of Theorem 1 (Pandharipande's conjecture) and Givental [14, Theorem 1] should imply such bounds in all genera via Maulik and Pandharipande [24]. In turn, the latter imply that generating functions for GW–invariants of any genus have positive radii of convergence, as expected from physical considerations. If n_d is the number of degree d rational curves passing through $3d-1$ general points in $\mathbb{P}^2$, by Theorem 1

$$\frac{n_d}{(3d-1)!} \le C^d$$

for some $C>0$. This recovers the bound established in the proof of [9, Proposition 3] by Di Francesco and Itzykson using Kontsevich's formula (see Ruan, Yongbin and Tian [28, Theorem 10.4]).[1]

[1] This bound for n_d is implied by the statement of [9, Proposition 3], but the argument in [9] does not establish the proposition itself. It only establishes a positive lower bound on lim inf and an upper bound on lim sup for the sequence $\sqrt[d]{n_d/(3d-1)!}$ and not even that it converges.

The rather direct approach of [9] can be used to obtain a bound as in Theorem 1 on primary GW–invariants ($b_s = 0$ for all s in Theorem 1) of $\mathbb{P}^3$ and perhaps of $\mathbb{P}^n$. While the recursions of Lee and Pandharipande [21, (1), (2)] reduce descendant GW–invariants ($b_s \neq 0$) to primary GW–invariants, they involve a significant number of cancellations and do not appear to lead to the bound of Theorem 1, even for $\mathbb{P}^n$. We instead deduce the nontrivial cases ($|\boldsymbol{a}| \leq n$, $N \geq 3$) of this theorem from Theorem A; see Section 5.

Theorem 2 *Suppose $n, N \in \mathbb{Z}^+$ with $N \geq 3$, $\boldsymbol{a} \in (\mathbb{Z}^+)^l$, $X_{\boldsymbol{a}} \subset \mathbb{P}^{n-1}$ is a complete intersection of multidegree $\boldsymbol{a}$ and $(b_s)_{s\in[N]}$ and $(c_s)_{s\in[N]}$ are N–tuples of nonnegative integers. If there exists $S \subset [N]$ such that $b_s + c_s < \nu_{\boldsymbol{a}}$ for every $s \in S$ and $\sum_{s\in S} b_s > N-3$, then*

$$\langle \tau_{b_1} H^{c_1}, \dots, \tau_{b_N} H^{c_N} \rangle_{0,d}^{X_{\boldsymbol{a}}} = 0.$$

This theorem is an immediate consequence of Theorem A; see Remark 5.1. Because of the condition on b_s, the assumptions of this theorem are never satisfied if $\nu_{\boldsymbol{a}} = 0, 1$ (Calabi–Yau and borderline Fano cases). For the same reason, it is most useful if $|\boldsymbol{a}| = 0$ (projective case). For example,

$$\langle \underbrace{\tau_b H^{n-b}, \dots, \tau_b H^{n-b}}_{N-2}, \cdot, \cdot \rangle_{0,d}^{\mathbb{P}^n} = 0 \quad \text{for all } N \geq 3,\ b = 1, 2, \dots, n. \tag{1-2}$$

The $\mathbb{P}^1$–case of (1-2) follows from the dilaton relation; see Hori, Katz, Klemm, Pandharipande, Thomas, Vafa, Vakil and Zaslow [17, page 527]. For $n \geq 2$, $\tau_b H^{n-b}$ is not a divisor on $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^n, d)$ and there appears to be no direct geometric reason for the vanishing (1-2).

Theorems 1 and 2 are potential indications of fundamental properties of GW–invariants that are out of reach of the current methods. Their statements have natural intrinsic extensions to more general symplectic manifolds, formulated in the two conjectures below. The failure of these conjectures would suggest that GW–invariants detect whether a symplectic manifold is projective or even of some more restricted class (such as a toric complete intersection); this would perhaps be even more astounding than if the conjectures were true. Note that in Conjecture 1 the exponent $\langle \omega, \beta \rangle$ is the energy of the J–holomorphic maps of class β, while $N + \langle \omega, \beta \rangle$ is essentially the energy of the induced "graph map". Theorem 1 establishes the first conjecture for projective complete intersections X, H_i being in the image of the cohomology pullback for the inclusion map $X \to \mathbb{P}^n$, and $g = 0$. The approach of [23] should remove the genus restriction and establish the dependence of $C_{X,g}$ on g and even on X. Theorem 2 establishes the second conjecture for projective complete intersections X and $H_s = H^{c_s}$.

Conjecture 1 *If (X,ω) is a compact symplectic manifold, $g\in\mathbb{Z}$ and $H_1,\dots,H_k\in H^*(X)$, then there exists $C_{X,g}\in\mathbb{R}^+$ such that*

$$\left|\frac{\langle b_1!\tau_{b_1}H_{c_1},\dots,b_N!\tau_{b_N}H_{c_N}\rangle_{g,\beta}^X}{N!}\right|\le C_{X,g}^{N+\langle\omega,\beta\rangle}$$

for all $\beta\in H_2(X)$, $N,b_s\ge 0$, $c_s\in[k]$.

Conjecture 2 *Let (X,ω) be a compact monotone symplectic manifold with minimal Chern number ν.*[2] *If $N\ge 3$, $(b_s)_{s\in[N]}$ and $(c_s)_{s\in[N]}$ are N–tuples of nonnegative integers, and $H_s\in H^{2c_s}(X)$ for every $s\in[N]$, then*

$$\langle\tau_{b_1}H_1,\dots,\tau_{b_N}H_N\rangle_{0,\beta}^X=0$$

if there exists $S\subset[N]$ such that $b_s+c_s<\nu$ for every $s\in S$ and $\sum_{s\in S}b_s>N-3$.

The genus 0 GW–invariants of a complete intersection $X_{\boldsymbol{a}}\subset\mathbb{P}^{n-1}$ are related to certain twisted GW–invariants of $\mathbb{P}^{n-1}$. Let

$$\begin{array}{ccc}\mathfrak{U} & \xrightarrow{\ \mathrm{ev}\ } & \mathbb{P}^{n-1}\\ \big\downarrow{\scriptstyle\pi} & & \\ \overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d) & & \end{array}$$

be the universal curve over $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)$. The GW–invariants of (1-1) then satisfy

$$\text{(1-3)}\quad \langle\tau_{b_1}H^{c_1},\dots,\tau_{b_N}H^{c_N}\rangle_{0,d}^{X_{\boldsymbol{a}}} = \int_{\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)}\prod_{k=1}^{k=l}e(\pi_*\,\mathrm{ev}^*\,\mathcal{O}_{\mathbb{P}^{n-1}}(a_k))\prod_{s=1}^{s=N}(\psi_s^{b_s}\,\mathrm{ev}_s^*\,H^{c_s}).$$

Since the moduli space $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)$ is a smooth stack (orbifold) and

$$\bigoplus_{k=1}^{k=l}\pi_*\,\mathrm{ev}^*\,\mathcal{O}_{\mathbb{P}^{n-1}}(a_k)\longrightarrow\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)$$

is a locally free sheaf, that is, the sheaf of sections of a vector orbibundle $\mathcal{V}_d$ over $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)$, the right-hand side of (1-3) is well-defined; its computation will be the main focus of this paper. In (2-1), we combine all GW–invariants (1-3) with

[2]Thus, $c_1(X)=\lambda[\omega]\in H^2(X;\mathbb{R})$ for some $\lambda\in\mathbb{R}^+$ and ν is the minimal value of $c_1(X)$ on the homology classes representable by nonconstant J–holomorphic maps $S^2\to X$ for every ω–compatible almost complex structure on X.

fixed N into a generating function. We show that for $N \ge 3$ this generating function is a certain transform of the $N=1$ generating function.

The main splitting principle of this paper described in Section 4.1 is valid for all $\boldsymbol{a}$, but the explicit expressions for the transforms apply only for $\nu_{\boldsymbol{a}} \ge 0$. This means that the main equivariant statement of this paper, ie Theorem B, holds for any $\boldsymbol{a}$ for *some* structure coefficients $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}$; the main nonequivariant statement, ie Theorem A along with (2-34), holds for any $\boldsymbol{a}$ for *some* structure coefficients $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ if $\Delta_{\boldsymbol{p}}$ is replaced by its geometric analogue or equivalently by the nonequivariant analogue of (3-5). In the $\nu_{\boldsymbol{a}} \ge 0$ cases, we specify the structure coefficients $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ and $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}$ completely based on the hypergeometric series

$$F(w,q) \equiv \sum_{d=0}^{\infty} q^d \frac{w^{\nu_{\boldsymbol{a}} d} \prod\limits_{k=1}^{k=l} \prod\limits_{r=1}^{a_k d} (a_k w + r)}{\prod\limits_{r=1}^{r=d} \big((w+r)^n - w^n\big)}. \tag{1-4}$$

This series also describes the one- and two-pointed GW–invariants of $X_{n;\boldsymbol{a}}$ if $\nu_{\boldsymbol{a}} \ge 0$.[3] In the remainder of this paper, we assume that $\nu_{\boldsymbol{a}} \ge 0$ for the purposes of all statements directly related to explicit hypergeometric series.

The power series (1-4) in q is an element of $1+q\mathbb{Q}(w)[\![q]\!]$ such that the coefficient of each power of q is holomorphic at $w=0$. The subgroup

$$\mathcal{P} \subset 1+q\mathbb{Q}(w)[\![q]\!]$$

of such power series is preserved by the operator

$$\boldsymbol{M}\colon 1+q\mathbb{Q}(w)[\![q]\!] \to 1+q\mathbb{Q}(w)[\![q]\!], \quad \{\boldsymbol{M}H\}(w,q) = \left\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\right\}\left(\frac{H(w,q)}{H(0,q)}\right).$$

We define $I_c \in 1+q\mathbb{Q}[\![q]\!]$ for $c=0,1,\dots$ and $J \in q\mathbb{Q}[\![q]\!]$ by

$$I_c(q) = \begin{cases} 1 & \text{if } |\boldsymbol{a}|<n, \\ \{\boldsymbol{M}^c F\}(0,q) & \text{if } |\boldsymbol{a}|=n, \end{cases}$$
$$J(q) = \begin{cases} 0 & \text{if } |\boldsymbol{a}| \le n-2, \\ \boldsymbol{a}!q & \text{if } |\boldsymbol{a}|=n-1, \\ \frac{1}{I_0(q)} \sum\limits_{d=1}^{\infty} q^d \left(\frac{\prod_{k=1}^{k=l}(a_k d)!}{(d!)^n} \left(\sum\limits_{k=1}^{k=l} \sum\limits_{r=d+1}^{a_k d} \frac{a_k}{r} \right)\right) & \text{if } |\boldsymbol{a}|=n. \end{cases} \tag{1-5}$$

[3] For the purposes of Theorems 3 and 4, the term w^n can be dropped from the definition of F.

The power series $J(q)$ is the coefficient of w in the power series expansion of $F(w,q)/I_0(q)$ at $w=0$; thus, $I_1(q)=1+q\frac{\mathrm{d}}{\mathrm{d}q}J(q)$ if $|\boldsymbol{a}|\neq n-1$. Similarly to the 1– and 2–pointed cases, the explicit expressions of Theorem A for generating functions for $N\geq 3$ involve the power series $I_0, I_1, \dots, I_{n-l}$ and J; see Section 1.1 for some examples.

1.1 The Calabi–Yau case

If $|\boldsymbol{a}|=n$, $X_{\boldsymbol{a}}$ is a Calabi–Yau $(n-1-l)$–fold. The virtual dimension of $\overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}},d)$ and of the space of N–marked rational curves in $X_{\boldsymbol{a}}$,

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}},d)=n-4-l+N,$$

is independent of d in this case. If $c_1,\dots,c_N$ are nonnegative integers such that

$$c_1+\cdots+c_N=n-4-l+N,$$

the corresponding genus 0 degree d GW–invariant of $X_{\boldsymbol{a}}$,

$$N_d^{X_{\boldsymbol{a}}}(c_1,\dots,c_N)\equiv\int_{[\overline{\mathfrak{M}}_{0,N}(X_{\boldsymbol{a}},d)]^{\mathrm{vir}}}(\mathrm{ev}_1^*H^{c_1})\cdots(\mathrm{ev}_N^*H^{c_N}),\tag{1-6}$$

is a rational number. These numbers define BPS states of $X_{\boldsymbol{a}}$ via Klemm and Pandharipande [19, (2)], that are intended to be virtual counts of curves (rather than maps) and are conjectured to be integers (see also Footnote 6). For a sufficiently small value of the degree d, the corresponding BPS number is known to be the number of rational degree d curves in a general complete intersection of multidegree $\boldsymbol{a}$ that pass through general linear subspaces of codimensions $c_1,\dots,c_N$.

Theorem A yields fairly simple closed formulas for the numbers (1-6) with $N=3,4$. Theorem 3 below follows immediately from (1-3), (2-1), (2-36), (2-20), (2-18), (2-41), (2-43), (2-30), (2-37), (2-42), (2-3), (2-23) and (2-25).[4]

Theorem 3 *Suppose $n\in\mathbb{Z}^+$, $X_{\boldsymbol{a}}\subset\mathbb{P}^{n-1}$ is a nonsingular Calabi–Yau complete intersection of multidegree $\boldsymbol{a}$, I_c and J are given by* (1-5) *and $Q=q\cdot e^{J(q)}\in q\mathbb{Q}[\![q]\!]$. If c_1,c_2,c_3 are nonnegative integers such that $c_1+c_2+c_3=n-1-l$, then*

$$\sum_{d=0}^{\infty}Q^dN_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3)=\frac{\langle\boldsymbol{a}\rangle}{(1-\boldsymbol{a}^{\boldsymbol{a}}q)I_0(q)^2\prod_{s=1}^{s=3}\prod_{c=1}^{c=c_s}I_c(q)}.\tag{1-7}$$

[4]Equation (2-41) is needed for (1-7) only; (2-43), (2-30), (2-37), (2-42) and (2-25) are needed for (1-8) only.

If c_1, c_2, c_3, c_4 are nonnegative integers such that $c_1+c_2+c_3+c_4=n-l$, then

$$\sum_{d=0}^{\infty} Q^d N_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3,c_4) \tag{1-8}$$
$$= \frac{\langle \boldsymbol{a}\rangle}{(1-\boldsymbol{a}^{\boldsymbol{a}}q) I_0^2(q) \prod\limits_{s=1}^{s=4}\prod\limits_{c=1}^{c=c_s} I_c(q)} \Bigg\{ \frac{n-l-2c_4}{2}\bigg(\frac{\boldsymbol{a}^{\boldsymbol{a}}q}{1-\boldsymbol{a}^{\boldsymbol{a}}q} - 2\frac{I_0'(q)}{I_0(q)}\bigg)$$
$$+\sum_{s=1}^{s=4}\frac{\mathcal{S}_{c_s}'(q)}{\mathcal{S}_{c_s}(q)} - \frac{\mathcal{S}_{c_1+c_2}'(q)}{\mathcal{S}_{c_1+c_2}(q)} - \frac{\mathcal{S}_{c_1+c_3}'(q)}{\mathcal{S}_{c_1+c_3}(q)} - \frac{\mathcal{S}_{c_2+c_3}'(q)}{\mathcal{S}_{c_2+c_3}(q)}\Bigg\},$$

where $'$ denotes the operator $q\frac{\mathrm{d}}{\mathrm{d}q}$ and $\mathcal{S}_c = I_1^{c-1} I_2^{c-2}\cdots I_c^0$.

Since $J(q)\in q\mathbb{Q}[\![q]\!]$, there exists $\tilde{J}(Q)\in Q\mathbb{Q}[\![Q]\!]$ such that $q=Qe^{\tilde{J}(Q)}$. Thus, (1-7) and (1-8) determine the numbers $N_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3)$ and $N_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3,c_4)$, respectively. Since

$$\frac{\mathcal{S}_c'(q)}{\mathcal{S}_c(q)} = \frac{\mathcal{S}_{n-l-c}(q)}{\mathcal{S}_{n-l-c}(q)} - \frac{n-l-2c}{2}\bigg(\frac{\boldsymbol{a}^{\boldsymbol{a}}q}{1-\boldsymbol{a}^{\boldsymbol{a}}q} - 2\frac{I_0'(q)}{I_0(q)}\bigg) \tag{1-9}$$

for all $c=0,1,\dots,n-l$ by (2-23)–(2-25) and (2-3), (1-8) is equivalent to

$$\sum_{d=0}^{\infty} Q^d N_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3,c_4) \tag{1-10}$$
$$= \frac{\langle \boldsymbol{a}\rangle}{(1-\boldsymbol{a}^{\boldsymbol{a}}q) I_0^2(q) \prod\limits_{s=1}^{s=4}\prod\limits_{c=1}^{c=c_s} I_c(q)} \Bigg\{ c_1\bigg(\frac{\boldsymbol{a}^{\boldsymbol{a}}q}{1-\boldsymbol{a}^{\boldsymbol{a}}q} - 2\frac{I_0'(q)}{I_0(q)}\bigg)$$
$$+\sum_{s=1}^{s=4}\frac{\mathcal{S}_{c_s}'(q)}{\mathcal{S}_{c_s}(q)} - \frac{\mathcal{S}_{c_1+c_2}'(q)}{\mathcal{S}_{c_1+c_2}(q)} - \frac{\mathcal{S}_{c_1+c_3}'(q)}{\mathcal{S}_{c_1+c_3}(q)} - \frac{\mathcal{S}_{c_1+c_4}'(q)}{\mathcal{S}_{c_1+c_4}(q)}\Bigg\}.$$

By (1-9), the right-hand side of (1-8) is symmetric in c_1,c_2,c_3,c_4, as expected. By (1-10),

$$N_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3,c_4)=0 \quad \text{if } 0\in\{c_1,c_2,c_3,c_4\},$$

as expected. By (1-7), (2-23), (2-24) and (2-3),

$$N_d^{X_{\boldsymbol{a}}}(c_1,c_2,c_3) = \begin{cases} \langle \boldsymbol{a}\rangle & \text{if } d=0, \\ 0 & \text{if } d>0, \end{cases} \qquad \text{if } 0\in\{c_1,c_2,c_3\}.$$

Since $I_1(q)=1+q\frac{\mathrm{d}}{\mathrm{d}q}J(q)$, (1-7) and (1-10) immediately give

$$dN_d(c_2,c_3,c_4)=N_d^{X_{\boldsymbol{a}}}(1,c_2,c_3,c_4),$$

as expected from the divisor relation [17, page 527]. By the divisor relation and (1-7),

$$\langle \boldsymbol{a}\rangle+\left\{Q\frac{\mathrm{d}}{\mathrm{d}Q}\right\}^3\sum_{d=0}^{\infty}Q^dN_d^{X_{\boldsymbol{a}}}(\cdot)=\frac{\langle \boldsymbol{a}\rangle}{(1-\boldsymbol{a}^{\boldsymbol{a}}q)I_0(q)^2}\left(\frac{Q}{q}\frac{\mathrm{d}q}{\mathrm{d}Q}\right)^3,$$

whenever $X_{\boldsymbol{a}}$ is a Calabi–Yau threefold, which recovers the famous mirror symmetry formula [7, (5.13)]; see [31, Appendix B] for a comparison of notation. By the divisor relation, (1-7), (2-23), (2-24) and (2-3),

$$\langle \boldsymbol{a}\rangle+\sum_{d=1}^{\infty}Q^d dN_d^{X_{\boldsymbol{a}}}(c_1,c_2)=\langle \boldsymbol{a}\rangle\frac{I_{c_1+1}(q)}{I_1(q)}\qquad \text{if } c_1+c_2=n-2-l,$$

$$\langle \boldsymbol{a}\rangle+\sum_{d=1}^{\infty}Q^d d^2N_d^{X_{\boldsymbol{a}}}(n-3-l)=\langle \boldsymbol{a}\rangle\frac{I_2(q)}{I_1(q)};$$

these identities are [27, Equations (1.5), (1.6)].

The first true cases of (1-7) and (1-8) occur for Calabi–Yau 6–folds and 7–folds:

$$(n,\boldsymbol{a},c_1,c_2,c_3)=(8,(8),2,2,2),\quad (n,\boldsymbol{a},c_1,c_2,c_3,c_4)=(9,(9),2,2,2,2).$$

Tables 1–4 show some low-degree BPS counts obtained from (1-7) and (1-8) via [19, (2)] for all complete intersections $X_{\boldsymbol{a}}\subset\mathbb{P}^{n-1}$, with $n\le 10$, of suitable dimensions, with H^{c_i} indicating that one of the constraints is a general linear subspace of $\mathbb{P}^{n-1}$ of codimension c_i. All degree 1 and 2 numbers agree with the corresponding lines and conics counts obtained via classical Schubert calculus computations (the 3–pointed numbers for hypersurfaces can be found in Katz [18], which also describes the classical methods). The degree 3 numbers for the hypersurfaces X_8 and X_9 agree with Ellingsrud and Strømme [10]; the remaining degree 3 numbers can be confirmed by similar computations. Most noteworthy is the agreement of the 4–pointed numbers, since these do not naturally arise in the physics view of mirror symmetry as originally presented by Greene, Morrison and Plesser in [15].[5] There are currently no direct methods of counting curves of degree 4 or higher on projective complete intersections; so the numbers in these degrees obtained from (1-7) and (1-8) cannot be compared to anything. Finally, these BPS counts for $d\le 100$ and all compete intersections $X_{\boldsymbol{a}}\subset\mathbb{P}^{n-1}$ with $n\le 10$ are integers, as expected.[6]

d	1	2	3	4
X_8	59021312	821654025830400	12197109744970010814464	186083410628492378226388631552
X_{27}	19133912	52069545843672	150771900962422866056	448721851648931529402358688
X_{36}	9303984	9656915909184	10669913703022812624	12119013327306237518117376
X_{45}	6536800	4306289363200	3019921285456823200	2177140100777199737600000
X_{226}	7036416	4323279882240	2819049510852887040	1889305224389886741405696
X_{235}	3936600	1091194853400	321105896368043400	97128823290992207460000
X_{244}	3252224	699998060544	159942140236292096	37565431180080918822912
X_{334}	2589408	396151430400	64359976334347296	10748812573405031454720

Table 1: Low-degree genus 0 BPS numbers (H^2, H^2, H^2) for some Calabi–Yau 6–folds

d	1	2	3
(H^2, H^3, H^3)	51415320000	444475303469701680000	4089048226644406809222184680000
(H^2, H^2, H^4)	38922224000	295035175517918176000	2467449594491156931046837776000
(H^2, H^2, H^2, H^3)	75062592000	1394799570099498816000	20109980886063766606715932224000

Table 2: Low-degree genus 0 BPS numbers for X_{10} in $\mathbb{P}^9$

d	1	2	3
X_9	1579510449	506855012110118424	174633921378662035929052320
X_{28}	466477056	25865899481481216	1538349758855955308748800
X_{37}	200848599	3684692607275358	72513809257771729565550
X_{46}	122812416	1209608310822912	12780622639872867502080
X_{55}	104480625	841277146035000	7266883194629367785000

Table 3: Low-degree genus 0 BPS numbers (H^2, H^2, H^3) for some Calabi–Yau 7–folds

d	1	2	3
X_9	2395066806	1718927099008463268	957208127608222375829677128
X_{28}	702562304	86939314932416512	8348345278919524413816832
X_{37}	302321376	12364886269091538	392695531026064094763648
X_{46}	184771584	4056318495977472	69156291871338627290112
X_{55}	157178750	2820556380767500	39310596116635041745000

Table 4: Low-degree genus 0 BPS numbers (H^2, H^2, H^2, H^2) for some Calabi–Yau 7–folds

1.2 The projective case

Throughout the paper, we denote by $\overline{\mathbb{Z}}^+$ the set of nonnegative integers. If $N, d, n \in \overline{\mathbb{Z}}^+$, let

$$\mathcal{P}_N(d) = \left\{ \boldsymbol{d} \equiv (d_1, d_2, \dots, d_N) \in (\overline{\mathbb{Z}}^+)^N \;\middle|\; \sum_{s=1}^{s=N} d_s = d \right\}, \tag{1-11}$$
$$\mathcal{P}_N^n(d) = \{ \boldsymbol{d} \equiv (d_1, d_2, \dots, d_N) \in \mathcal{P}_N(d) \mid d_s < n \text{ for all } s \in [N] \}.$$

For any $\boldsymbol{p} \in \mathcal{P}_N^n(d)$, let

$$(\!|\boldsymbol{p}|\!) = \min\{p_s + 1, n-1-p_s \mid s \in [N]\}.$$

If $(c_s)_{s \in [N]} \in (\overline{\mathbb{Z}}^+)^N$, let

$$\left\langle \prod_{s=1}^{s=N} \frac{H^{c_s}}{\hbar_s - \psi} \right\rangle_{0,d}^{\mathbb{P}^{n-1}} = \sum_{b_1, b_2, \dots, b_N \ge 0} \left(\prod_{s=1}^{s=N} \hbar_s^{-1-b_s} \right) \langle \tau_{b_1} H^{c_1}, \dots, \tau_{b_N} H^{c_N} \rangle_{0,d}^{\mathbb{P}^{n-1}}.$$

[5] This viewpoint is extended to an arbitrary number of marked points in Barannikov [4].

[6] The genus 0 GW–invariants of CY's with at least 3 marked points are integers; see McDuff and Salamon [25, Section 7.3] and [28]. Since the GW–BPS transform of [19, (2)] is always lower-triangular with 1's on the diagonal and integers everywhere else if the number of marked points is at least 3, it follows that the BPS numbers are integers as well in this case.

Theorem A yields fairly simple closed formulas for the genus 0 GW–invariants of projective spaces with 3 and 4 insertions. Theorem 4 below follows immediately from (2-1), (2-36), (2-34), (2-41), (2-45), (2-20), (2-18), (2-14) and (1-4).[7]

Theorem 4 *The* 3*– and* 4*–pointed Gromov–Witten invariants of* $\mathbb{P}^{n-1}$ *are given by*

$$\sum_{p_1,p_2,p_3\ge 0}\left\langle\prod_{s=1}^{s=3}\frac{H^{n-1-p_s}}{\hbar_s-\psi}\right\rangle_{0,d}^{\mathbb{P}^{n-1}} H_1^{p_1}H_2^{p_2}H_3^{p_3} = \sum_{d'=0}^{d'=1}\sum_{\substack{\boldsymbol{d}\in\mathcal{P}_3(d-d')\\ \boldsymbol{p}\in\mathcal{P}_3^n((2-d')n-2)}}\prod_{s=1}^{s=3}\frac{(H_s+d_s\hbar_s)^{p_s}}{\hbar_s\prod_{r=1}^{d_s}(H_s+r\hbar_s)^n},$$

and

$$\sum_{p_1,p_2,p_3,p_4\ge 0}\left\langle\prod_{s=1}^{s=4}\frac{H^{n-1-p_s}}{\hbar_s-\psi}\right\rangle_{0,d}^{\mathbb{P}^{n-1}} H_1^{p_1}H_2^{p_2}H_3^{p_3}H_4^{p_4}$$
$$=\left\{\sum_{\substack{\boldsymbol{d}\in\mathcal{P}_3(d-1)\\ \boldsymbol{p}\in\mathcal{P}_4^n(2n-4)}}(\!|\boldsymbol{p}|\!)+\left(\sum_{s=1}^{s=4}\hbar_s^{-1}\right)\sum_{d'=0}^{d'=2}\sum_{\substack{\boldsymbol{d}\in\mathcal{P}_4(d-d')\\ \boldsymbol{p}\in\mathcal{P}_4^n((3-d')n-3)}}\right\}\prod_{s=1}^{s=4}\frac{(H_s+d_s\hbar_s)^{p_s}}{\hbar_s\prod_{r=1}^{d_s}(H_s+r\hbar_s)^n}.$$

Both identities hold modulo H_s^n *and as power series in* $\hbar_s^{-1}$.

Since the $d=1$ Gromov–Witten invariant counts lines in $\mathbb{P}^{n-1}$, the $d=1$ case of the 4–pointed formula in Theorem 4 gives

$$\langle\sigma_{c_1}\sigma_{c_2}\sigma_{c_3}\sigma_{c_4},\mathbb{G}(2,n)\rangle=\min\{c_s+1,n-1-c_s\mid s=1,2,3,4\}$$

if $c_s\in\bar{\mathbb{Z}}^+$, $\sum_{s=1}^{s=4}c_s=2n-4$, where σ_c is the usual codimension c Schubert cycle on $\mathbb{G}(2,n)$. As pointed out to the author by A Buch, this identity can be confirmed by applying Pieri's rule (see Griffiths and Harris [16, page 203]) to $\sigma_{c_1}\sigma_{c_2}$ and $\sigma_{c_3}\sigma_{c_4}$ and counting pairs of dual cycles in its outputs. The $d=2$ case of the 4–pointed formula gives

$$\langle H^{c_1},H^{c_2},H^{c_3},H^{c_4}\rangle_{0,2}^{\mathbb{P}^{n-1}}=0.$$

This is indeed as expected, since every conic lies in a $\mathbb{P}^2$ [16, page 177] and no $\mathbb{P}^2$ meets general linear subspaces of $\mathbb{P}^{n-1}$ of total codimension $3n$ (the space of planes meeting the constraints is the intersection of Schubert cycles in $\mathbb{G}(3,n)$ of total codimension $3n-8$ and is thus empty).

[7] In this case, $\tilde{\mathfrak{c}}_{p,s}^{(d)}=\delta_{0,d}\delta_{p,s}$ in (2-18) and (2-41); (2-45) is needed for the second identity in Theorem 4 only.

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2 Main Theorem

In addition to the notation introduced at the beginning of Section 1.2, for any $m, l \in \bar{\mathbb{Z}}^+$ we define

$$\llbracket m \rrbracket = \{s \in \bar{\mathbb{Z}}^+ \mid s < m\}, \quad \llbracket m \rrbracket_l = \{s \in \llbracket m \rrbracket \mid s \ge l\}.$$

We denote by $\mathcal{P}_m([N])$ the set of unordered partitions $\boldsymbol{S} \equiv \{S_i\}_{i \in [m]}$ of $[N]$ into nonempty subsets S_i such that one of them is $\{N\}$.[8] If $\boldsymbol{p}$ is an N–tuple of integers, $S \subset [N]$ and $p' \in \mathbb{Z}$, let $\boldsymbol{p}|_S$ and $\boldsymbol{p}p'$ denote the S–tuple consisting of the elements of $\boldsymbol{p}$ indexed by S and the $(N+1)$–tuple obtained by adjoining p' to $\boldsymbol{p}$ at the end, respectively, and set

$$|\boldsymbol{p}|_S \equiv |\boldsymbol{p}|_S| \equiv \sum_{s \in S} p_s.$$

If R is a ring and $\underline{x} = (x_1, \dots, x_N)$ is a tuple of variables, let

$$R[\underline{x}] = R[x_1, \dots, x_N]$$

be the ring of polynomials in $x_1, \dots, x_N$. If $\Phi \in R\llbracket q \rrbracket$ and $d \in \mathbb{Z}$, let $\llbracket \Phi \rrbracket_{q;d} \in R$ denote the coefficient of q^d ($\llbracket \Phi \rrbracket_{q;d} \equiv 0$ if $d < 0$).

Let $\mathbb{P}^{n-1}_N = (\mathbb{P}^{n-1})^N$. For each $s = 1, \dots, N$, we set

$$H_s = \pi_s^* H \in H^2(\mathbb{P}^{n-1}_N),$$

where $\pi_s \colon \mathbb{P}^{n-1}_N \to \mathbb{P}^{n-1}$ is the projection onto the s^{th} coordinate. Since $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1}, d)$ is smooth, there is a well-defined cohomology pushforward

$$\mathrm{ev}_* \equiv \{\mathrm{ev}_1 \times \dots \times \mathrm{ev}_N\}_* \colon H^*(\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1}, d)) \to H^*(\mathbb{P}^{n-1}_N).$$

With $\underline{\hbar} = (\hbar_1, \dots, \hbar_N)$, $\underline{\hbar}^{-1} = (\hbar_1^{-1}, \dots, \hbar_N^{-1})$ and $\underline{H} = (H_1, \dots, H_N)$, let

$$Z(\underline{\hbar}, \underline{H}, Q) = \sum_{d=0}^{\infty} Q^d \, \mathrm{ev}_* \left\{ \frac{e(\mathcal{V}_d)}{\prod_{s=1}^{s=N} (\hbar_s - \psi_s)} \right\} \in H^*(\mathbb{P}^{n-1}_N)[\underline{\hbar}^{-1}]\llbracket Q \rrbracket. \tag{2-1}$$

[8] More precisely, $\mathcal{P}_m([N])$ consists of unordered partitions with a choice of some ordering for each of the partitions.

By (1-3), this power series encodes all genus 0 GW–invariants of $X_{\boldsymbol{a}}$ with constrains that arise from $\mathbb{P}^{n-1}$. If $\boldsymbol{b} \equiv (b_1, \dots, b_N) \in \mathbb{Z}^N$, let

$$\underline{\hbar}^{-\boldsymbol{b}} = \prod_{s=1}^{s=N} (\hbar_s^{-1})^{b_s}.$$

2.1 An asymptotic expansion

The power series F defined by (1-4) admits an asymptotic expansion $w \to \infty$ which plays a central role in this paper and which we now describe.

Define

$$L(q) \in 1 + q\mathbb{Q}[\![q]\!] \quad \text{by} \quad L(q)^n - \boldsymbol{a}^{\boldsymbol{a}} q L(q)^{|\boldsymbol{a}|} = 1 \in \mathbb{Q}[\![q]\!],$$

$$\chi_0, \chi_1, \dots, \chi_{|\boldsymbol{a}|} \in \mathbb{Q} \quad \text{by} \quad \prod_{k=1}^{k=l} \prod_{r=1}^{r=a_k} (a_k D + r) \equiv \boldsymbol{a}^{\boldsymbol{a}} \sum_{i=0}^{i=|\boldsymbol{a}|} \chi_{|\boldsymbol{a}|-i} D^i \in \mathbb{Z}[D]. \tag{2-2}$$

In the two extremal cases, (2-2) gives

$$L(q) = \begin{cases} (1+q)^{1/n} & \text{if } |\boldsymbol{a}| = 0, \\ (1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-1/n} & \text{if } |\boldsymbol{a}| = n. \end{cases} \tag{2-3}$$

Setting $\chi_i \equiv 0$ if $i < 0$ or $i > |\boldsymbol{a}|$, we find that

$$\chi_0 = 1, \quad \chi_1 = \frac{|\boldsymbol{a}| + l}{2}. \tag{2-4}$$

For $m, j \in \mathbb{Z}$, we define $\mathcal{H}_{m,j} \in \mathbb{Q}(u)$ recursively by

$$\mathcal{H}_{m,j} \equiv 0 \quad \text{unless } 0 \le j \le m, \quad \mathcal{H}_{0,0} \equiv 1, \tag{2-5}$$
$$\mathcal{H}_{m,j}(u) \equiv \mathcal{H}_{m-1,j}(u) + \frac{u-1}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} u} \left(nu \frac{\mathrm{d}}{\mathrm{d}u} + m - j \right) \mathcal{H}_{m-1,j-1}(u),$$

if $m \ge 1$, $0 \le j \le m$. In particular, for $m \ge 0$,

$$\mathcal{H}_{m,0}(u) = 1, \quad \mathcal{H}_{m,1}(u) = \binom{m}{2} \frac{u-1}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} u}. \tag{2-6}$$

Finally, we define differential operators $\mathfrak{L}_1, \dots, \mathfrak{L}_n$ on $\mathbb{Q}[\![q]\!]$ by

$$\mathfrak{L}_k = \sum_{i=0}^{k} \bigg[\binom{n}{i} L^n \mathcal{H}_{n-i,k-i}(L^n) - (L^n - 1) \sum_{r=0}^{k-i} \binom{|\boldsymbol{a}| - r}{i} \chi_r \mathcal{H}_{|\boldsymbol{a}|-i-r,k-i-r}(L^n) \bigg] D^i, \tag{2-7}$$

where $D = q\frac{\mathrm{d}}{\mathrm{d}q}$. By (2-6), (2-4) and (2-2), the first of these operators is

$$\mathfrak{L}_1 = (|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n)\left\{D + \frac{L^n - 1}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n}\left(\frac{\nu_{\boldsymbol{a}} n L^n}{2(|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n)} - \frac{l+1}{2}\right)\right\} \tag{2-8}$$
$$= (|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n)\left\{\left(\left(\frac{n}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n}\right)^{1/2} L^{(l+1)/2}\right) \times D\left(\left(\frac{n}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n}\right)^{1/2} L^{(l+1)/2}\right)^{-1}\right\}.$$

Proposition 2.1 *The power series F of* (1-4) *admits an asymptotic expansion*

$$F(w,q) \sim e^{\xi(q)w}\sum_{b=0}^{\infty}\Phi_b(q)w^{-b} \quad \text{as } w \to \infty, \tag{2-9}$$

with $\xi, \Phi_1, \ldots \in q\mathbb{Q}[\![q]\!]$ and $\Phi_0 \in 1 + q\mathbb{Q}[\![q]\!]$ determined by the first-order ODE's

$$1 + \xi'(q) = L(q), \quad \mathfrak{L}_1\Phi_b + \frac{1}{L}\mathfrak{L}_2\Phi_{b-1} + \cdots + \frac{1}{L^{n-1}}\mathfrak{L}_n\Phi_{b+1-n} = 0, \tag{2-10}$$

where $\Phi_b \equiv 0$ for $b < 0$.

From (2-8) and (2-10), we immediately find that

$$\Phi_0(q) = \left(\frac{n}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L(q)^n}\right)^{1/2} L(q)^{(l+1)/2}. \tag{2-11}$$

In the extremal cases, this reduces to

$$\Phi_0(q) = \begin{cases} L(q)^{-(n-1)/2} = (1+q)^{-(n-1)/2n} & \text{if } |\boldsymbol{a}| = 0, \\ L(q)^{(l+1)/2} = (1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-(l+1)/2n} & \text{if } |\boldsymbol{a}| = n. \end{cases} \tag{2-12}$$

Proposition 2.1 in the $|\boldsymbol{a}| = n$ case is proved by Popa in [26, Section 4], building up on the $\boldsymbol{a} = (n)$ case contained in [29, Lemma 1.3 and Theorems 1.1, 1.2 and 1.4] by Zagier and the author. The remaining cases are addressed in Appendix A.

2.2 One- and two-pointed formulas

By the dilaton relation [17, page 527] and [12, Theorems 9.5,10.7,11.8], the generating function (2-1) with $N = 1$ and the degree 0 term defined to be $\langle \boldsymbol{a} \rangle H_1^l \hbar_1$ is given by

$$Z(\hbar_1, H_1, Q) = \langle \boldsymbol{a} \rangle H_1^l e^{-J(q_1)w_1}\hbar_1 \frac{F(w_1, q_1)}{I_0(q_1)}, \tag{2-13}$$

where $w_1 = H_1/\hbar_1$, $q_1 e^{\delta_{0\nu_{\boldsymbol{a}}} J(q_1)} = Q/H_1^{\nu_{\boldsymbol{a}}}$. The generating function (2-1) for $N = 2$ is given in [27, Section 2] in terms of certain transforms of F, which we describe next.

Define

$$\boldsymbol{D}\colon \mathbb{Q}(w)[\![q]\!] \to \mathbb{Q}(w)[\![q]\!], \quad \boldsymbol{D}H(w,q) \equiv \{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\}H(w,q), \tag{2-14}$$

$$F_0(w,q) \equiv \sum_{d=0}^{\infty} q^d w^{\nu_{\boldsymbol{a}} d} \frac{\prod_{k=1}^{k=l}\prod_{r=0}^{a_k d-1}(a_k w+r)}{\prod_{r=1}^{d}((w+r)^n - w^n)} \in \mathcal{P}, \tag{2-15}$$

$$F_p \equiv \boldsymbol{D}^p F_0 = \boldsymbol{M}^p F_0 \in \mathcal{P} \quad \text{for all } p=1,2,\dots,l. \tag{2-16}$$

In particular, $F_l = F$. For $\nu_{\boldsymbol{a}} > 0$, we also define $\mathrm{c}_{p,s}^{(d)}, \tilde{\mathrm{c}}_{l+p,l+s}^{(d)} \in \mathbb{Q}$ with $p,s,d \geq 0$ by

$$\begin{aligned} \sum_{d=0}^{\infty}\sum_{s=0}^{\infty} \mathrm{c}_{p,s}^{(d)} w^s q^d &= \sum_{d=0}^{\infty} q^d \frac{(w+d)^p \prod_{k=1}^{l}\prod_{r=1}^{a_k d}(a_k w+r)}{\prod_{r=1}^{d}(w+r)^n} \\ &= w^p \boldsymbol{D}^p F(w, q/w^{\nu_{\boldsymbol{a}}}), \end{aligned} \tag{2-17}$$

$$\sum_{\substack{d_1+d_2=d \\ d_1,d_2\geq 0}} \sum_{r=0}^{p-\nu_{\boldsymbol{a}} d_1} \tilde{\mathrm{c}}_{l+p,l+r}^{(d_1)} \mathrm{c}_{r,s}^{(d_2)} = \delta_{0,d}\delta_{p,s} \quad \text{for all } d,s \in \overline{\mathbb{Z}}^+, s \leq p-\nu_{\boldsymbol{a}} d.$$

The second equation in (2-17) expresses the numbers $\tilde{\mathrm{c}}_{l+p,l+s}^{(d)}$ with $s \leq p-\nu_{\boldsymbol{a}} d$ in terms of $\tilde{\mathrm{c}}_{l+p,l+r}^{(d_1)}$ with $d_1 < d$, since $\mathrm{c}_{p,s}^{(0)} = \delta_{p,s}$; the numbers $\tilde{\mathrm{c}}_{l+p,l+s}^{(d)}$ with $s > p-\nu_{\boldsymbol{a}} d$ will not be needed. In particular, $\tilde{\mathrm{c}}_{p,s}^{(0)} = \delta_{p,s}$ for all $p,s \geq l$. For $p > l$, set

$$F_p(w,q) = \begin{cases} \boldsymbol{M}^p F(w,q) & \text{if } \nu_{\boldsymbol{a}} = 0, \\ \sum_{d=0}^{\infty}\sum_{s=0}^{p-l-\nu_{\boldsymbol{a}} d} \frac{\tilde{\mathrm{c}}_{p,l+s}^{(d)} q^d}{w^{p-l-\nu_{\boldsymbol{a}} d-s}} \boldsymbol{D}^s F(w,q) & \text{if } \nu_{\boldsymbol{a}} > 0. \end{cases} \tag{2-18}$$

Thus, $F_p \in \mathcal{P}$ for all $p \in \mathbb{Z}^+$ by (2-17) and $F_p = \boldsymbol{D}^{p-l}F$ unless $p \geq l+\nu_{\boldsymbol{a}}$. By [27, Theorem 3], the generating function (2-1) with $N=2$ and the degree 0 term defined to be the image of $(\langle \boldsymbol{a}\rangle H_1^l H_2^l/\hbar_1+\hbar_2)((H_1^{n-l}-H_2^{n-l})/(H_1-H_2))$ is given by

$$\begin{aligned} &Z(\hbar_1,\hbar_2,H_1,H_2,Q) \\ &\qquad = \frac{\langle \boldsymbol{a}\rangle}{\hbar_1+\hbar_2} e^{-J(q_1)w_1 - J(q_2)w_2} \sum_{\substack{p_1+p_2=n-1+l \\ p_1,p_2\geq l}} \prod_{s=1}^{s=2} H_s^{p_s} \frac{F_{p_s}(w_s,q_s)}{I_{p_s-l}(q)}, \end{aligned} \tag{2-19}$$

where $w_s=H_s/\hbar_s$, $q_se^{\delta_{0\nu_{\boldsymbol{a}}}J(q_s)}=Q/H_s^{\nu_{\boldsymbol{a}}}$.

Remark 2.2 The mismatch in the indexing of I_* and F_* is unfortunate for the purposes of this section. However, the choice of the indexing for the former is intended to simplify the explicit formulas for the Calabi–Yau complete intersections in Section 1.1, while the choice of the indexing for the latter is intended to simplify some of the formulas in the proof of Theorem A in the rest of the paper.

2.3 Multipointed formulas

Similarly to (2-19), the generating function (2-1) for $N\ge 3$ is a linear combination of the N–fold products

$$\Delta_{\boldsymbol{p}}(\underline{\hbar},\underline{H},Q)\equiv\prod_{s=1}^{s=N}\frac{H_s^{p_s}}{\hbar_s}\frac{F_{p_s}(w_s,q_s)}{\prod\limits_{r=p_s-l}^{n-l-1}I_r(q_s)},\tag{2-20}$$

where $w_s=H_s/\hbar_s$, $q_se^{\delta_{0\nu_{\boldsymbol{a}}}J(q_s)}=Q/H_s^{\nu_{\boldsymbol{a}}}$, with $\boldsymbol{p}=(p_1,p_2,\dots,p_N)\in\llbracket n\rrbracket_l^N$ and with coefficients that are polynomials in $\hbar_1^{-1},\dots,\hbar_N^{-1}$ of total degree at most $N-3$. These coefficients are described below inductively using the coefficients $\tilde{\mathrm{c}}_{p,s}^{(d)}$ defined above and the asymptotic expansion of $F(w,q)$ provided by Proposition 2.1.

For $r<0$, we set $I_r(q)=1$. By Proposition 2.1, (2-14)–(2-16) and (2-18), there are asymptotic expansions

$$\frac{F_p(w,p)}{\prod\limits_{r=p-l}^{n-l-1}I_r(q)}\sim e^{\xi(q)w}\frac{I_0(q)}{L(q)^{\delta_{0\nu_{\boldsymbol{a}}}n}}\sum_{b=0}^{\infty}\Phi_{p;b}(q)w^{-b}\quad\text{as } w\to\infty,\tag{2-21}$$

with $\Phi_{p;0}\in 1+q\mathbb{Q}\llbracket q\rrbracket$ and $\Phi_{p;1},\Phi_{p;2},\ldots\in q\mathbb{Q}\llbracket q\rrbracket$ given by

$$\begin{gathered}\widehat{\Phi}_{p+1;b}=L\widehat{\Phi}_{p;b}+\widehat{\Phi}'_{p;b-1}-\Bigl(\sum_{r=0}^{r=p}\frac{I'_r}{I_r}\Bigr)\widehat{\Phi}_{p;b-1}\qquad\text{for all } p\in\mathbb{Z},\ \widehat{\Phi}_{0;b}=\Phi_b,\\ \Phi_{p;b}(q)=\begin{cases}\sum\limits_{d=0}^{\infty}\sum\limits_{s=0}^{p-\nu_{\boldsymbol{a}}d}\tilde{\mathrm{c}}_{p,s}^{(d)}q^d\widehat{\Phi}_{s-l;b-(p-\nu_{\boldsymbol{a}}d-s)}(q) & \text{if } \nu_{\boldsymbol{a}}>0,\\ \widehat{\Phi}_{p-l;b}(q) & \text{if } \nu_{\boldsymbol{a}}=0,\end{cases}\end{gathered}\tag{2-22}$$

where $\widehat{\Phi}_{p;b}\equiv 0$ if $b<0$, $\tilde{\mathrm{c}}_{p,s}^{(d)}\equiv\delta_{0,d}\delta_{p,s}$ unless $p,s\ge l$, and $'$ denotes $q\frac{\mathrm{d}}{\mathrm{d}q}$ as before. In the Calabi–Yau case, $\nu_{\boldsymbol{a}}=0$, the recursion (2-22) for the coefficients

$\Phi_{p;b} = \widehat{\Phi}_{p-l;b}$ in the asymptotic expansion (2-21) is obtained using the first two identities in the following lemma.[9]

Lemma 2.3 [26, Proposition 4.4] *If* $|\boldsymbol{a}| = n$*, the power series* I_p *defined by* (1-4) *and* (1-5) *satisfy*

$$I_{n-l-p} = I_p \quad \text{for all } p = 0, 1, \dots, n-l, \tag{2-23}$$

$$I_0 I_1 \cdots I_{n-l} = L^n, \tag{2-24}$$

$$I_0^{n-l} I_1^{n-l-1} \cdots I_{n-l}^0 = L^{(n(n-l))/2}. \tag{2-25}$$

For example, by (2-22),

$$\widehat{\Phi}_{p;0} = L^p \Phi_0, \quad \widehat{\Phi}_{p;1} = L^p (\Phi_1 + \mathbb{A}_p^{(1)} \Phi_0), \tag{2-26}$$

$$\frac{\Phi_{p;0}(q)}{\Phi_0(q)} = L(q)^{p-l} \begin{cases} \sum\limits_{d=0}^{\infty} \frac{\tilde{\mathrm{c}}_{p,p-\nu_{\boldsymbol{a}} d}^{(d)} q^d}{L(q)^{\nu_{\boldsymbol{a}} d}} & \text{if } \nu_{\boldsymbol{a}} > 0, \\ 1 & \text{if } \nu_{\boldsymbol{a}} = 0, \end{cases} \tag{2-27}$$

and

$$\frac{\Phi_{p;1}(q)}{\Phi_0(q)} = L(q)^{p-l} \begin{cases} \sum_{d=0}^{\infty} \frac{\tilde{\mathrm{c}}_{p,p-\nu_{\boldsymbol{a}} d}^{(d)} q^d}{L(q)^{\nu_{\boldsymbol{a}} d}} \Big(\frac{\Phi_1(q)}{\Phi_0(q)} \\ \qquad + \mathbb{A}_{p-l-\nu_{\boldsymbol{a}} d}^{(1)}(q) \Big) \\ \qquad\qquad + \sum_{d=0}^{\infty} \frac{\tilde{\mathrm{c}}_{p,p-\nu_{\boldsymbol{a}} d-1}^{(d)} q^d}{L(q)^{\nu_{\boldsymbol{a}} d+1}} & \text{if } \nu_{\boldsymbol{a}} > 0, \\ \frac{\Phi_1(q)}{\Phi_0(q)} + \mathbb{A}_{p-l}^{(1)}(q) & \text{if } \nu_{\boldsymbol{a}} = 0, \end{cases} \tag{2-28}$$

where

$$\tilde{\mathrm{c}}_{p,s}^{(d)} \equiv 0 \quad \text{if } s + \nu_{\boldsymbol{a}} d > p,$$
$$\mathbb{A}_p^{(1)} = L^{-1} \bigg(p \frac{\Phi_0'}{\Phi_0} + \frac{p(p-1)}{2} \frac{L'}{L} - \sum_{r=0}^{r=p} (p-r) \frac{I_r'}{I_r} \bigg). \tag{2-29}$$

In the two extremal cases, (2-12) gives

$$\mathbb{A}_p^{(1)} = L^{-1} \begin{cases} -\frac{(n-p)p}{2} \frac{L'}{L} & \text{if } |\boldsymbol{a}| = 0, \\ \frac{(p+l)p}{2} \frac{L'}{L} - \sum\limits_{r=0}^{r=p} (p-r) \frac{I_r'}{I_r} & \text{if } |\boldsymbol{a}| = n. \end{cases} \tag{2-30}$$

[9]The last identity in Lemma 2.3 follows from the first two; it was used in Section 1.1.

If $m \in \bar{\mathbb{Z}}^+$, $d, t \in \mathbb{Z}$ and $\boldsymbol{c} \equiv (c_r)_{r \in \mathbb{Z}^+} \in (\bar{\mathbb{Z}}^+)^\infty$, let

$$\mathcal{S}_m(d,t) = \{(\boldsymbol{p}, \boldsymbol{b}) \in [\![n]\!]^m \times \mathbb{Z}^m \mid |\boldsymbol{p}| - |\boldsymbol{b}| = n - 2 + (m-1)(l+2) + \nu_{\boldsymbol{a}} d + nt\},$$

(2-31)
$$\Phi_{m,\boldsymbol{c}} = \frac{\Phi_0^2}{I_0^2}(-1)^{m+|\boldsymbol{c}|}(m+|\boldsymbol{c}|)! \prod_{r=1}^{\infty} \frac{1}{c_r!}\left(\frac{\Phi_r}{(r+1)!\Phi_0}\right)^{c_r}.$$

For any $p, p' \in [\![n]\!]$ and $b, b', d, t \in \mathbb{Z}$, let

$$\text{(2-32)} \quad \mathrm{c}^{(d,t)}_{(p,p'),(b,b')} = \begin{cases} (-1)^b \left[\!\!\left[\frac{L(q)^{\delta_{0\nu_{\boldsymbol{a}}}(1+t)n}}{I_0(q)^2} \right]\!\!\right]_{q;d} & \text{if } b \geq 0,\ b+b' = -1,\ p+p'+nt = n-1+l, \\ 0 & \text{otherwise.} \end{cases}$$

For any N–tuples $\boldsymbol{p} \in [\![n]\!]^N$, $\boldsymbol{b} \in (\bar{\mathbb{Z}}^+)^N$ with $N \geq 3$ and $d, t \in \bar{\mathbb{Z}}^+$, we inductively define

$$\text{(2-33)} \quad \mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}} = \sum_{\substack{m,d',t' \in \mathbb{Z} \\ m \geq 3}} \sum_{\substack{\boldsymbol{S} \in \mathcal{P}_m([N]) \\ \boldsymbol{d} \in \mathcal{P}_m(d-d') \\ \boldsymbol{t} \in \mathcal{P}_m(t-t') \\ (\boldsymbol{p}',\boldsymbol{b}') \in \mathcal{S}_m(d',t')}} \sum_{\substack{\boldsymbol{b}'' \in (\bar{\mathbb{Z}}^+)^m \\ \boldsymbol{c} \in (\bar{\mathbb{Z}}^+)^\infty \\ |\boldsymbol{b}''| + \|\boldsymbol{c}\| = m-3}} \left(\left(\prod_{i=1}^{i=m} \mathrm{c}^{(d_i,t_i)}_{\boldsymbol{p}|_{S_i} p'_i, \boldsymbol{b}|_{S_i} b'_i} \right) \times \left[\!\!\left[\Phi_{m-3,\boldsymbol{c}}(q) \prod_{i=1}^{i=m} \frac{I_0(q)^2 \Phi_{p'_i; b'_i+1+b''_i}(q)}{b''_i! L(q)^{\delta_{0\nu_{\boldsymbol{a}}} n} \Phi_0(q)} \right]\!\!\right]_{q;d'} \right),$$

where $\Phi_{p;b} \equiv 0$ if $b < 0$ and $\mathrm{c}^{(d_i,t_i)}_{\boldsymbol{p}|_{S_i} p'_i, \boldsymbol{b}|_{S_i} b'_i} \equiv 0$ if $b'_i < 0$ and $|S_i| \geq 2$. By induction,

$$\text{(2-34)} \quad \mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}} \neq 0 \Rightarrow |\boldsymbol{b}| \leq N-3, \quad |\boldsymbol{p}| - |\boldsymbol{b}| + \nu_{\boldsymbol{a}} d + nt = (N-1)(n-2) + 2 + l.$$

Since $\Phi_{m-3,\boldsymbol{c}}, \Phi_{p'_i; b'_i+1+b''_i} \in q\mathbb{Q}[\![q]\!]$ unless $\boldsymbol{c} = \boldsymbol{0}$ and $b'_i + 1 + b''_i = 0$,

$$\text{(2-35)} \quad \mathrm{c}^{(0,t)}_{\boldsymbol{p},\boldsymbol{b}} = \delta_{|\boldsymbol{p}|+nt,(N-1)(n-1)+l} \binom{N-3}{\boldsymbol{b}}.$$

Theorem A *Suppose $n, N \in \mathbb{Z}^+$, with $N \geq 3$, and $\boldsymbol{a} \in (\mathbb{Z}^+)^l$ is such that $\|\boldsymbol{a}\| \leq n$. The generating function (2-1) for N–pointed genus 0 GW–invariants of a complete intersection $X_{\boldsymbol{a}} \subset \mathbb{P}^{n-1}$ is given by*

$$(2\text{-}36)\quad Z(\underline{\hbar},\underline{H},Q)=\langle\boldsymbol{a}\rangle\exp\Big(-\sum_{s=1}^{s=N}J(q_s)w_s\Big)\times\sum_{\boldsymbol{p}\in\llbracket n\rrbracket_l^N}\sum_{\boldsymbol{b}\in(\overline{\mathbb{Z}}^+)^N}\sum_{d=0}^{\infty}\mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,0)}q^d\underline{\hbar}^{-\boldsymbol{b}}\Delta_{\boldsymbol{p}}(\underline{\hbar},\underline{H},Q),$$

where $w_s=H_s/\hbar_s$, $q_s e^{\delta_{0\nu_{\boldsymbol{a}}}J(q_s)}=Q/H_s^{\nu_{\boldsymbol{a}}}$ *and* $q e^{\delta_{0\nu_{\boldsymbol{a}}}J(q)}=Q$.

We show in Section 3 that this theorem follows from Theorem B.

By (1-3), (2-1), (2-36), (2-35) and (2-20),

$$\langle\tau_{b_1}(H^{c_1}),\dots,\tau_{b_N}(H^{c_N})\rangle_{0,0}^{X_{\boldsymbol{a}}}=\delta_{|\boldsymbol{c}|,n-1-l}\langle\boldsymbol{a}\rangle\binom{N-3}{\boldsymbol{b}}$$

whenever $b_i,c_i\ge 0$, as expected.

By (2-32), for each $p\in\llbracket n\rrbracket$, there exists a unique pair $(\widehat{p},t_p)\in\llbracket n\rrbracket\times\mathbb{Z}$ such that $\mathrm{c}_{(p,\widehat{p}),(b,b')}^{(d,t_p)}\neq 0$ at least for some $b,b',d\in\mathbb{Z}$:

$$(2\text{-}37)\qquad (\widehat{p},t_p)=\begin{cases}(n-1+l-p,0) & \text{if } p\ge l,\\ (l-1-p,1) & \text{if } p<l.\end{cases}$$

For any $\boldsymbol{p}\in\llbracket n\rrbracket^N$, let

$$(2\text{-}38)\qquad t_{\boldsymbol{p}}=\sum_{s=1}^{s=N}t_{p_s}=|\{s\in[N]\mid p_s<l\}|.$$

We note that

$$(2\text{-}39)\qquad \widetilde{\mathrm{c}}_{\widehat{p},\widehat{p}-\nu_{\boldsymbol{a}}d}^{(d)}\neq 0\Rightarrow p+\nu_{\boldsymbol{a}}d+(n-l)t_p\le n-1.$$

If $N\ge 3$, $\boldsymbol{p}\in\llbracket n\rrbracket^N$, $\boldsymbol{b}\in(\overline{\mathbb{Z}}^+)^N$, $d\in\overline{\mathbb{Z}}^+$ and $t\in\mathbb{Z}$ satisfy the last property in (2-34) and $|\boldsymbol{b}|=N-3$, the only nonzero terms in (2-33) arise from $(m,\boldsymbol{c})=(N,\boldsymbol{0})$, $p_i'=\widehat{p}_i$, $b_i'=-1-b_i$ and $b_i''=b_i$. If in addition $\nu_{\boldsymbol{a}}\neq 0$, by (2-27), (2-11) and Lemma B.4,

$$(2\text{-}40)\quad \mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,t)}=\binom{N-3}{\boldsymbol{b}}\sum_{d'=0}^{d'=d}\widetilde{\mathrm{c}}_{\widehat{\boldsymbol{p}}}^{(d-d')}\left\llbracket\frac{nL(q)^{\nu_{\boldsymbol{a}}d'+n(1+t-t_{\boldsymbol{p}})}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\right\rrbracket_{q;d'}=\binom{N-3}{\boldsymbol{b}}\sum_{d'=0}^{d'=d}(\boldsymbol{a}^{\boldsymbol{a}})^{d'}\binom{d'+t-t_{\boldsymbol{p}}}{d'}\widetilde{\mathrm{c}}_{\widehat{\boldsymbol{p}}}^{(d-d')},$$

with the binomial coefficients defined as in (B-5) and

$$\tilde{c}^{(d)}_{\hat{\boldsymbol{p}}} \equiv \sum_{\boldsymbol{d}\in\mathcal{P}_N(d)} \tilde{c}^{(\boldsymbol{d})}_{\hat{\boldsymbol{p}}}, \quad \tilde{c}^{(\boldsymbol{d})}_{\hat{\boldsymbol{p}}} \equiv \prod_{s=1}^{s=N} \tilde{c}^{(d_s)}_{\hat{p}_s,\hat{p}_s-\nu_{\boldsymbol{a}}d_s}.$$

If $\nu_{\boldsymbol{a}}=0$, the last property in (2-34) imposes no restriction on d. In this case, we find that

$$\sum_{d=0}^{\infty} c^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}} q^d = \binom{N-3}{\boldsymbol{b}} \frac{L(q)^{n(1+t)}}{I_0(q)^2}. \tag{2-41}$$

In the $\nu_{\boldsymbol{a}}=0$ case, the last property in (2-34) forces $t\ge 0$ and $t_{\boldsymbol{p}}=0$ if $t=0$, whenever $|\boldsymbol{b}|=N-3$. The proof of Theorem A implies that the right-hand side of (2-40) also vanishes if either $t<0$ or $t=0$ and $t_{\boldsymbol{p}}>0$. By (2-39) and the last property in (2-34),

$$(n-l)(d'+t+1-t_{\boldsymbol{p}})-(|\boldsymbol{a}|-l)d'+lt-1\ge 0$$

whenever the d'–summand in (2-40) is nonzero; this implies that

$$1\le t_{\boldsymbol{p}}-t\le d'$$

whenever the triple product in (2-40) is nonzero and either $t<0$ or $t=0$ and $t_{\boldsymbol{p}}>0$. The explicit expression on the right-hand side of (2-40) thus provides a direct reason for the vanishing of $c^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}$ in these cases.

If $N=3$, the only possibly nonzero coefficients in (2-36) are $c^{(d,0)}_{\boldsymbol{p},\boldsymbol{0}}$; these are given by (2-40) and (2-41). If $N=4$, the only possibly nonzero coefficients in (2-36) are $c^{(d,0)}_{\boldsymbol{p},\boldsymbol{0}}$ and

$$c^{(d,0)}_{\boldsymbol{p},1000}=c^{(d,0)}_{\boldsymbol{p},0100}=c^{(d,0)}_{\boldsymbol{p},0010}=c^{(d,0)}_{\boldsymbol{p},0001},$$

with $\boldsymbol{p}\in[\![n]\!]^4$; the latter set of coefficients is given by (2-40) and (2-41) whenever $\boldsymbol{p}$ satisfies the last property in (2-34) with $N=4$, $|\boldsymbol{b}|=1$ and $t=0$. We next give a formula for the former set of coefficients. For $p,d\in\mathbb{Z}$, define

$$\begin{aligned}&[\![p]\!]_d,[\![\hat{p}]\!]_d,\tau_d(p),t_d(p)\in\mathbb{Z} \quad \text{by} \quad 0\le[\![p]\!]_d,[\![\hat{p}]\!]_d\le n-1,\\ &[\![p]\!]_d+\nu_{\boldsymbol{a}}d+n\tau_d(p)=p, \quad [\![p]\!]_d+[\![\hat{p}]\!]_d+nt_d(p)=n-1+l.\end{aligned} \tag{2-42}$$

If $\boldsymbol{p},\boldsymbol{d}\in\mathbb{Z}^4$, let

$$\Sigma_2(\boldsymbol{p},\boldsymbol{d})=\{p_1+p_2+\nu_{\boldsymbol{a}}(d_1+d_2),\,p_1+p_3+\nu_{\boldsymbol{a}}(d_1+d_3),\,p_2+p_3+\nu_{\boldsymbol{a}}(d_2+d_3)\}.$$

If $\nu_{\boldsymbol{a}}=0$, $[\![p]\!]_d$, $[\![\hat{p}]\!]_d$ and $\Sigma_2(\boldsymbol{p},\boldsymbol{d})$ do not depend on d or $\boldsymbol{d}$, and so we omit $[\![\cdot]\!]_d$ and $\boldsymbol{d}$ from the notation in this case. In the $\nu_{\boldsymbol{a}}=0$ case, a direct computation from (2-33), (2-41), (2-32), (2-26), (2-27) and (2-28) gives

$$\sum_{d=0}^{\infty} \mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{0}} q^d = \frac{L(q)^{n+1}}{I_0(q)^2}\bigg\{\sum_{p'-1\in\Sigma_2(\boldsymbol{p})} \mathbb{A}^{(1)}_{\hat{p}'-l}(q) - \sum_{s=1}^{s=4} \mathbb{A}^{(1)}_{\hat{p}_s-l}(q)\bigg\}, \tag{2-43}$$

whenever $\boldsymbol{p}$ satisfies the last property in (2-34) with $N=4$, $|\boldsymbol{b}|=0$ and $t=0$.

If $\nu_{\boldsymbol{a}}\neq 0$, $d,d',p\in\bar{\mathbb{Z}}^+$ and $t=0,1$, let

$$\begin{aligned}
\tilde{\mathrm{c}}^{(d,t)}_{p,d'} &\equiv \left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol{a}}d'+n(1-t)}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\big(\tilde{\mathrm{c}}^{(d)}_{p,p-\nu_{\boldsymbol{a}}d}L(q)\mathbb{A}^{(1)}_{p-l-\nu_{\boldsymbol{a}}d}(q)+\tilde{\mathrm{c}}^{(d)}_{p,p-\nu_{\boldsymbol{a}}d-1}\big)\right]\!\!\right]_{q;d'}\\
&= \tilde{\mathrm{c}}^{(d)}_{p,p-\nu_{\boldsymbol{a}}d}\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol{a}}d'+n(1-t)+1}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\mathbb{A}^{(1)}_{p-l-\nu_{\boldsymbol{a}}d}(q)\right]\!\!\right]_{q;d'}\\
&\qquad\qquad + \binom{d'-t}{d'}(\boldsymbol{a}^{\boldsymbol{a}})^{d'}\tilde{\mathrm{c}}^{(d)}_{p,p-\nu_{\boldsymbol{a}}d-1};
\end{aligned}$$

the equality above holds by Lemma B.4. On the other hand, by the second equation in (2-29), (2-11), (2-2) and Corollary B.5,

$$\begin{aligned}
&\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol{a}}d'+n(1-t)+1}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\mathbb{A}^{(1)}_{p-l}(q)\right]\!\!\right]_{q;d'}\\
&\qquad = \frac{p-l}{2}\left(\frac{\boldsymbol{a}^{\boldsymbol{a}}}{n}\right)^{d'}\Big(d'|\boldsymbol{a}|^{d'}-(n-p)\sum_{\substack{d_1+d_2=d'-1\\ d_1,d_2\geq 0}}|\boldsymbol{a}|^{d_1}(n-\nu_{\boldsymbol{a}}t)^{d_2}\\
&\qquad\qquad\qquad -(d'-1+\delta_{0,d'})tp|\boldsymbol{a}|^{d'-1}\Big)
\end{aligned}$$

whenever $t=0,1$. In particular, we have that $\tilde{\mathrm{c}}^{(0,t)}_{p,0}=0$. For $d,p\in\bar{\mathbb{Z}}^+$ such that $p\leq 2n-1$, let

$$\tilde{\mathrm{C}}^{(d)}_p = \sum_{\boldsymbol{d}\in\mathcal{P}_4(d)}(\boldsymbol{a}^{\boldsymbol{a}})^{d_3}\binom{d_3+\tau_{d_2+d_3}(p)-t_{d_2+d_3}(p)}{d_3}\times\tilde{\mathrm{c}}^{(d_2)}_{[\![\hat{p}]\!]_{d_2+d_3},[\![\hat{p}]\!]_{d_2+d_3}-\nu_{\boldsymbol{a}}d_2}\tilde{\mathrm{c}}^{(d_1,\tau_{d_2+d_3}(p))}_{[\![p]\!]_{d_2+d_3},d_4}.$$

Since $0\leq p\leq 2n-1$,

$$\binom{d_3+\tau_{d_2+d_3}(p)-t_{d_2+d_3}(p)}{d_3}\tilde{\mathrm{c}}^{(d_2)}_{[\![\hat{p}]\!]_{d_2+d_3},[\![\hat{p}]\!]_{d_2+d_3}-\nu_{\boldsymbol{a}}d_2}\neq 0\Rightarrow \tau_{d_2+d_3}(p)\in\{0,1\}$$

by (2-39), and so $\widetilde{\mathrm{C}}_p^{(d)}$ is well-defined. For example, $\widetilde{\mathrm{C}}_p^{(0)} = 0$. If $\nu_{\boldsymbol{a}} \neq 0$, $t_{\boldsymbol{p}} = 0$ and $\boldsymbol{p}$ satisfies the last property in (2-34) with $N = 4$, $|\boldsymbol{b}| = 0$ and $t = 0$, then

$$\mathrm{c}_{\boldsymbol{p},\boldsymbol{0}}^{(d,0)} = \sum_{d'=0}^{d'=d} \sum_{\boldsymbol{d}\in\mathcal{P}_4(d-d')} \Bigg(\sum_{2n-2+l-p'\in\Sigma_2(\boldsymbol{p},\boldsymbol{d})} \widetilde{\mathrm{C}}_{p'}^{(d')} \widetilde{\mathrm{c}}_{\widehat{\boldsymbol{p}}}^{(\boldsymbol{d})} - \sum_{r=1}^{r=4} \Bigg(\prod_{s\in[4]-r} \widetilde{\mathrm{c}}_{\widehat{p}_s,\widehat{p}_s-\nu_{\boldsymbol{a}} d_s}^{(d_s)} \Bigg) \widetilde{\mathrm{c}}_{\widehat{p}_r,d'}^{(d_r,0)} \Bigg). \tag{2-44}$$

This is obtained by a direct computation from (2-33), (2-40), (2-32), (2-26), (2-27) and (2-28) except the vanishing of the coefficient of $\Phi_1(q)$ follows from Corollary B.8. If $\widetilde{\mathrm{c}}_{\widehat{\boldsymbol{p}}}^{(\boldsymbol{d})} \neq 0$ in (2-44), then

$$l \le p_s + \nu_{\boldsymbol{a}} d_s \le n-1 \quad \text{for all } s \in [4]$$

by the assumption that $t_{\boldsymbol{p}} = 0$ and (2-39), and so

$$l \le p' \le 2n-2-l \quad \text{if } 2n-2+l-p' \in \Sigma_2(\boldsymbol{p},\boldsymbol{d});$$

thus, the right-hand side of (2-44) is well-defined. In the case of a projective space, $\boldsymbol{a} = \varnothing$, the above formulas give

$$\widetilde{\mathrm{c}}_{p,d'}^{(d,t)} = \begin{cases} -\frac{p(n-p)}{2n} & \text{if } d=0, d'>0, t=0, \\ -\frac{p(n-p)}{2n} & \text{if } (d,d',t)=(0,1,1), \\ 0 & \text{otherwise;} \end{cases} \qquad \widetilde{\mathrm{C}}_p^{(d)} = \begin{cases} -\frac{[\![p]\!]_0(n-[\![p]\!]_0)}{2n} & \text{if } d>0, \\ 0 & \text{if } d=0; \end{cases}$$

$$\mathrm{c}_{\boldsymbol{p},\boldsymbol{0}}^{(d,0)} = \begin{cases} 0 & \text{if } d=0,2, \\ \min\{p_s+1, n-1-p_s\} & \text{if } d=1; \end{cases} \tag{2-45}$$

the last statement holds under the assumption that $|\boldsymbol{p}| + nd = 3n-4$.

The N–pointed formula of Theorem A takes the simplest form in the two extremal cases, $\nu_{\boldsymbol{a}} = 0$ (Calabi–Yau) and $\nu_{\boldsymbol{a}} = n$ (projective space), as $\widetilde{\mathrm{c}}_{p,s}^{(d)} = \delta_{0,d}\delta_{p,s}$ in these two cases. However, it is also straightforward to compute all the relevant coefficients in the intermediate cases. For example, for a cubic threefold $X_3 \subset \mathbb{P}^4$, the only nontrivial coefficients $\widetilde{\mathrm{c}}_{p,s}^{(d)}$ are

$$\widetilde{\mathrm{c}}_{3,1}^{(1)} = \widetilde{\mathrm{c}}_{4,1}^{(1)} = -6, \quad \widetilde{\mathrm{c}}_{4,2}^{(1)} = -21,$$

as computed in [27, Section 2].[10] From this, (2-40) and (2-44), we find that the only nonzero coefficients in the $N = 3, 4$ cases of (2-36) with $d \in \mathbb{Z}^+$ and $\boldsymbol{b} = \boldsymbol{0}$ are

[10] In this paper, the subscripts on $\widetilde{\mathrm{c}}$ are shifted up by l from [27].

$$\begin{aligned}
&\mathrm{c}^{(1,0)}_{133,\mathbf{0}}=6, \quad \mathrm{c}^{(1,0)}_{223,\mathbf{0}}=15, \quad \mathrm{c}^{(2,0)}_{113,\mathbf{0}}=36, \quad \mathrm{c}^{(2,0)}_{122,\mathbf{0}}=126, \quad \mathrm{c}^{(3,0)}_{111,\mathbf{0}}=216,\\
&\mathrm{c}^{(1,0)}_{1333,\mathbf{0}}=6, \quad \mathrm{c}^{(1,0)}_{2233,\mathbf{0}}=15, \quad \mathrm{c}^{(2,0)}_{1133,\mathbf{0}}=72, \quad \mathrm{c}^{(2,0)}_{1223,\mathbf{0}}=252, \quad \mathrm{c}^{(3,0)}_{1113,\mathbf{0}}=648,\\
&\mathrm{c}^{(2,0)}_{2222,\mathbf{0}}=729, \quad \mathrm{c}^{(3,0)}_{1122,\mathbf{0}}=2484, \quad \mathrm{c}^{(4,0)}_{1111,\mathbf{0}}=5184,
\end{aligned}$$

where 133 denotes any of the tuples $(1,3,3)$, $(3,1,3)$ and $(3,3,1)$ and similarly with the other subscripts. From (2-36), we then find that

$$\begin{aligned}
\langle H^3,H,H\rangle^{X_3}_{0,1}&=\langle H^3,H,H,H\rangle^{X_3}_{0,1}=18,\\
\langle H^2,H^2,H\rangle^{X_3}_{0,1}&=\langle H^2,H^2,H,H\rangle^{X_3}_{0,1}=45,\\
\langle H^3,H^3,H\rangle^{X_3}_{0,2}&=\tfrac{1}{2}\langle H^3,H^3,H,H\rangle^{X_3}_{0,2}=108,\\
\langle H^2,H^2,H^2,H^2\rangle^{X_3}_{0,2}&=2187,\\
\langle H^3,H^2,H^2\rangle^{X_3}_{0,2}&=\tfrac{1}{2}\langle H^3,H^2,H^2,H\rangle^{X_3}_{0,2}=378,\\
\langle H^3,H^3,H^2,H^2\rangle^{X_3}_{0,3}&=7452,\\
\langle H^3,H^3,H^3\rangle^{X_3}_{0,3}&=\tfrac{1}{3}\langle H^3,H^3,H^3,H\rangle^{X_3}_{0,3}=648,\\
\langle H^3,H^3,H^3,H^3\rangle^{X_3}_{0,4}&=15552.
\end{aligned}$$

These conclusions are consistent with the divisor relation. The above invariants are enumerative at least for $d=1,2,3$. The degree 1 and 2 numbers agree with the classical Schubert calculus computations on $G(2,5)$ and $G(3,5)$, respectively. The approach of [10] can be used to test the two degree 3 numbers.

Based on (2-33), the coefficient $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ in (2-36) with $\boldsymbol{p}\in[\![n]\!]^N$ involves the power series Φ_r of Proposition 2.1 with $r=0,1,\dots,N-3-|\boldsymbol{b}|$ only. By (2-43) and (2-44), only the power series Φ_0 enters in the $N=4$ case. For $N=5$, the power series Φ_1 and Φ_2 do enter in the final expression for $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\mathbf{0}}$. However, at least for $\boldsymbol{a}=(n)$, ie when $X_{\boldsymbol{a}}$ is a Calabi–Yau hypersurface, Φ_2 cancels with Φ_1^2/Φ_0 (these two power series are equal in this case).

2.4 Alternative description of the structure constants

We now describe the constants $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ defined above as sums over N–marked trivalent trees.[11] It is fairly straightforward to see that the two descriptions are equivalent; this also follows from the two variations of the main localization computation in Section 4.

[11] The constants $\mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}$ with $t>0$ can be described in the same way as well, but they are not needed in this approach.

A *graph* consists of a set Ver of *vertices* and a collection Edg of *edges*, ie of two-element subsets of Ver. In Figure 1, the vertices are represented by dots, while each edge $\{v_1, v_2\}$ is shown as the line segment between v_1 and v_2. For such a graph Γ and $v \in \mathrm{Ver}$, let

$$\mathrm{E}_v(\Gamma) = \{e \in \mathrm{Edg} \mid v \in e\}$$

be the set of edges leaving v. A graph $(\mathrm{Ver}, \mathrm{Edg})$ is a *tree* if it is connected and contains no loops, ie for all $v, v' \in \mathrm{Ver}$ with $v \neq v'$ there exists a unique ordered collection

$$v_1 \equiv v, v_2, \dots, v_{m-1}, v_m \equiv v' \in \mathrm{Ver},$$

with $m \geq 2$, such that

$$\{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{m-1}, v_m\}\} \subset \mathrm{Edg}\,.$$

An N*–marked graph* is a tuple $\Gamma = (\mathrm{Ver}, \mathrm{Edg}; \eta)$, where $(\mathrm{Ver}, \mathrm{Edg})$ is a graph and $\eta\colon [N] \to \mathrm{Ver}$ is a map. In Figure 1, which shows examples of 4-marked graphs, the elements of the set $[N] = [4]$ are shown in bold face and are linked by line segments to their images under η. An N–marked graph $\Gamma = (\mathrm{Ver}, \mathrm{Edg}; \eta)$ is called *trivalent* if

$$m_v \equiv \mathrm{val}_\Gamma(v) - 3 \equiv |\mathrm{E}_v(\Gamma)| + |\eta^{-1}(v)| - 3 \geq 0$$

for every vertex $v \in \mathrm{Ver}$. There is a unique trivalent 3-marked tree; the four trivalent 4-marked trees are shown in Figure 1. For any N–marked tree,

$$\sum_{v \in \mathrm{Ver}} m_v + |\,\mathrm{Edg}\,| = N - 3. \tag{2-46}$$

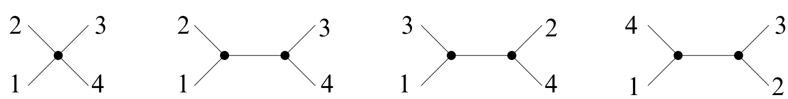

Figure 1: The trivalent 4–marked trees

We will call a partial ordering $\prec$ on a set Ver *linear* if for any pair of distinct incomparable elements $v_1, v_2 \in \mathrm{Ver}$ there exists a third element $v \in \mathrm{Ver}$ such that $v \prec v_1, v_2$. A finite linearly ordered set Ver has a unique minimal element $v_0 \in \mathrm{Ver}$. For each trivalent N–marked tree $\Gamma = (\mathrm{Ver}, \mathrm{Edg}; \eta)$, we fix a linear partial ordering $\prec$ on Ver so that if $v \prec v'$, then there exist

$$v_1, \dots, v_m \in \mathrm{Ver}\,\text{such that}\; v_{i-1} \prec v_i, \{v_{i-1}, v_i\} \in \mathrm{Edg}\,\text{for all}\;\; i \in [m+1],$$

where $v_0 \equiv v,\; v_{m+1} \equiv v'$.[12]

[12]Such a partial ordering is determined by the minimal vertex v_0, which could be taken to be $\eta(N)$, for example.

For every edge $e \in \mathrm{Edg}$, let $v_e^-, v_e^+ \in \mathrm{Ver}$ be the elements of $e \subset \mathrm{Ver}$ with $v_e^- \prec v_e^+$. For each $v \in \mathrm{Ver}$, let

$$\mathrm{E}_v^-(\Gamma) = \{e \in \mathrm{Edg} \mid v_e^- = v\}$$

be the set of edges descending to v. If $v \neq v_0$, let $e_v \in \mathrm{Edg}$ be the unique edge descending from v.

Let $(\boldsymbol{p}, \boldsymbol{b}, d) \in \llbracket n \rrbracket_l^N \times (\bar{\mathbb{Z}}^+)^N \times \bar{\mathbb{Z}}^+$ be a tuple satisfying the two properties on the right-hand side of (2-34) with $t = 0$, $\Gamma = (\mathrm{Ver}, \mathrm{Edg}; \eta)$ be a trivalent N–marked tree, and

$$\boldsymbol{d} \equiv (d_v)_{v \in \mathrm{Ver}} \in \mathcal{P}_\Gamma(d) \equiv \mathcal{P}_{\mathrm{Ver}}(d)$$

be a partition of d into nonnegative integers. We denote by

$$\mathcal{S}_\Gamma(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{d}) \subset \llbracket n \rrbracket^{\mathrm{Edg}} \times (\bar{\mathbb{Z}}^+)^{\mathrm{Edg}} \times \mathbb{Z}^{\mathrm{Ver}}$$

the subset of triples $(\boldsymbol{p}', \boldsymbol{b}', \boldsymbol{t})$ such that

$$\text{(2-47)} \quad \sum_{s \in \eta^{-1}(v)} (\hat{p}_s + b_s) + \sum_{e \in \mathrm{E}_v^-(\Gamma)} (\hat{p}_e' - 1 - b_e') + (p_{e_v}' + b_{e_v}') \\ = n - 3 + (m_v + 2)(l + 1) + \nu_{\boldsymbol{a}} d_v + n t_v$$

for all $v \in \mathrm{Ver}$, where $\hat{p}$ is as in (2-37) and we set $p_{e_v}' + b_{e_v} \equiv 0$ if $v = v_0$. Each choice of $\boldsymbol{b}'$ determines $\boldsymbol{p}'$ and $\boldsymbol{t}$ uniquely by solving (2-47) for p_v and t_v starting with maximal elements of Ver and moving down; the equation for $v = v_0$ will then be automatically solvable for t_v because of the last property in (2-34). Furthermore, for every $(\boldsymbol{p}', \boldsymbol{b}', \boldsymbol{t}) \in \mathcal{S}_\Gamma(\boldsymbol{p}, \boldsymbol{b}, \boldsymbol{d})$,

$$t_{\boldsymbol{p}'} + \sum_{v \in \mathrm{Ver}} t_v = 0,$$

with $t_{\boldsymbol{p}'}$ as in (2-38).

If $(\boldsymbol{p}, \boldsymbol{b}) \in \llbracket n \rrbracket_l^N \times (\bar{\mathbb{Z}}^+)^N$, $d \in \bar{\mathbb{Z}}^+$ satisfy the last property in (2-34) with $t = 0$, set

$$\text{(2-48)} \quad \mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,0)} = \sum_{\Gamma} \sum_{\substack{\boldsymbol{d} \in \mathcal{P}_\Gamma(d) \\ (\boldsymbol{p}',\boldsymbol{b}',\boldsymbol{t}) \in \mathcal{S}_\Gamma(\boldsymbol{p},\boldsymbol{b},\boldsymbol{d})}} (-1)^{|\boldsymbol{b}|+|\boldsymbol{b}'|} \sum_{\substack{\boldsymbol{b}'' \in (\bar{\mathbb{Z}}^+)^N; \boldsymbol{b}^-, \boldsymbol{b}^+ \in (\bar{\mathbb{Z}}^+)^{\mathrm{Edg}} \\ (\boldsymbol{c}_v)_{v \in \mathrm{Ver}} \in ((\bar{\mathbb{Z}}^+)^\infty)^{\mathrm{Ver}} \\ |\boldsymbol{b}''|_{\eta^{-1}(v)} + |\boldsymbol{b}^-|_{\mathrm{E}_v^-(\Gamma)} + b_{e_v}^+ + \|\boldsymbol{c}_v\| = m_v}} \prod_{v \in \mathrm{Ver}} \Bigg[\!\!\Bigg[\Phi_{m_v, \boldsymbol{c}_v}(q) \\ \times \prod_{s \in \eta^{-1}(v)} \frac{\Phi_{\hat{p}_s; b_s'' - b_s}(q)}{b_s''! \Phi_0(q)} \times \prod_{e \in \mathrm{E}_v^-(\Gamma)} \frac{L(q)^{\delta_{0 \nu_{\boldsymbol{a}}} n t_{p_e'}} \Phi_{\hat{p}_e'; b_e^- + 1 + b_e'}(q)}{b_e^-! \Phi_0(q)} \\ \times \frac{I_0(q)^2 \Phi_{p_{e_v}'; b_{e_v}^+ - b_{e_v}'}(q)}{b_{e_v}^+! L(q)^{\delta_{0 \nu_{\boldsymbol{a}}} n} \Phi_0(q)} \Bigg]\!\!\Bigg]_{q; d_v},$$

where $b^+_{e_{v_0}}\equiv 0$, the last fraction is defined to be 1 for $v=v_0$, and the outer sum is taken over all trivalent N–marked trees $\Gamma=(\mathrm{Ver},\mathrm{Edg};\eta)$. For example, the contribution of the one-vertex N–marked tree is

$$(-1)^{|\boldsymbol{b}|}\sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^N,\boldsymbol{c}\in(\bar{\mathbb{Z}}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=N-3}}\left[\!\!\left[\Phi_{N-3,\boldsymbol{c}}(q)\prod_{s=1}^{s=N}\frac{\Phi_{\hat{p}_s;b''_s-b_s}(q)}{b''_s!\Phi_0(q)}\right]\!\!\right]_{q;d}.$$

If $|\boldsymbol{b}|=N-3$, this gives (2-40) and (2-41) with $t,t_{\boldsymbol{p}}=0$.

For a nonzero summand in (2-48),

$$b_s\le b''_s \text{ for all } s\in[N],\quad |\boldsymbol{b}''|\le N-3-|\mathrm{Edg}|;$$

the latter inequality follows from (2-46). This implies the bound on $\boldsymbol{b}$ in (2-34). If $d\in\bar{\mathbb{Z}}^+$ and $(\boldsymbol{p},\boldsymbol{b})\in[\![n]\!]_l^N\times(\bar{\mathbb{Z}}^+)^N$ do not satisfy the last condition in (2-34) with $t=0$, set $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}=0$.

In the Calabi–Yau case, $\nu_{\boldsymbol{a}}=0$, the collection $\mathcal{S}_\Gamma(\boldsymbol{p},\boldsymbol{b})\equiv\mathcal{S}_\Gamma(\boldsymbol{p},\boldsymbol{b},\boldsymbol{d})$ does not depend on $\boldsymbol{d}$. In the projective case, $\nu_{\boldsymbol{a}}=n$, the collection of pairs $(\boldsymbol{p}',\boldsymbol{b}')$ does not depend on $\boldsymbol{d}$. As t_v in (2-47) is determined by $\boldsymbol{b}'$, we abbreviate the elements of $\mathcal{S}_\Gamma(\boldsymbol{p},\boldsymbol{b})$ as $(\boldsymbol{p}',\boldsymbol{b}')$ in either case. In these extremal cases, (2-31) and (2-12) reduce (2-48) to

$$\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}=\sum_\Gamma\sum_{(\boldsymbol{p}',\boldsymbol{b}')\in\mathcal{S}_\Gamma(\boldsymbol{p},\boldsymbol{b})}(-1)^{|\boldsymbol{b}|+|\boldsymbol{b}'|}\sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^N;\boldsymbol{b}^-,\boldsymbol{b}^+\in(\bar{\mathbb{Z}}^+)^{\mathrm{Edg}}\\ (\boldsymbol{c}_v)_{v\in\mathrm{Ver}}\in((\bar{\mathbb{Z}}^+)^\infty)^{\mathrm{Ver}}\\ |\boldsymbol{b}''|_{\eta^{-1}(v)}+|\boldsymbol{b}^-|_{e^-_\Gamma(v)}+b^+_{e_v}+\|\boldsymbol{c}_v\|=m_v}}\left[\!\!\left[L(q)^{|\boldsymbol{a}|t_{\boldsymbol{p}'}}\Phi_{\Gamma,(\boldsymbol{c}_v)_{v\in\mathrm{Ver}}}(q)\right.\right.$$
$$\left.\left.\times\prod_{s=1}^{s=N}\frac{\Phi_{\hat{p}_s;b''_s-b_s}(q)}{b''_s!\Phi_0(q)}\times\prod_{e\in\mathrm{Edg}}\frac{\Phi_{\hat{p}'_e;b^-_e+1+b'_e}(q)\Phi_{p'_e;b^+_e-b'_e}(q)}{b^-_e!b^+_e!\Phi_0(q)^2}\right]\!\!\right]_{q;d},$$

where

$$\Phi_{\Gamma,(\boldsymbol{c}_v)_{v\in\mathrm{Ver}}}=\frac{L^{|\boldsymbol{a}|-(n-1-l)|\mathrm{Ver}|}}{I_0^2}\prod_{v\in\mathrm{Ver}}\left((-1)^{m_v+|\boldsymbol{c}_v|}(m_v+|\boldsymbol{c}_v|)!\prod_{r=1}^\infty\frac{1}{c_{v;r}!}\left(\frac{\Phi_r}{(r+1)!\Phi_0}\right)^{c_{v;r}}\right).$$

The coefficients $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ must be invariant under the permutations of $[N]$ (same permutations in the components of $\boldsymbol{p}$ and $\boldsymbol{b}$). For $N\ge 4$, this is not apparent from either of the above two descriptions of these coefficients, even in the extremal cases; thus, this is a consequence of the proof of Theorem A below. In the case of (1-8), this invariance can be seen directly using Lemma 2.3, as indicated in Section 1.1.

3 Equivariant GW–invariants

In this section we first review the relevant aspects of equivariant cohomology; a more detailed discussion can be found in [31, Section 1.1]. We then state an equivariant version of Theorem A and use it to obtain Theorem A.

We denote by $\mathbb{T}$ the n–torus $(\mathbb{C}^*)^n$. Its group cohomology is the polynomial algebra on n generators:

$$H^*_{\mathbb{T}} \equiv H^*(B\mathbb{T};\mathbb{Q}) = \mathbb{Q}[\alpha] \equiv \mathbb{Q}[\alpha_1,\dots,\alpha_n],$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\alpha_i=\pi_i^*c_1(\gamma^*)$ if

$$\pi_i\colon B\mathbb{T}\to B\mathbb{C}^*=\mathbb{P}^\infty \quad \text{and} \quad \gamma\to\mathbb{P}^\infty$$

are the projection onto the i^{th} component and the tautological line bundle, respectively. Let

$$\mathcal{H}^*_{\mathbb{T}}=\mathbb{Q}_\alpha\equiv\mathbb{Q}(\alpha_1,\dots,\alpha_n) \quad \text{and} \quad \mathcal{I}\subset\mathbb{Q}[\alpha_1,\dots,\alpha_n]\subset\mathcal{H}^*_{\mathbb{T}}$$

be the field of fractions of $H^*_{\mathbb{T}}$ and the ideal in $\mathbb{Q}[\underline{\alpha}]$ generated by the elementary symmetric polynomials $\sigma_1,\sigma_2,\dots,\sigma_{n-1}$ in $\alpha_1,\alpha_2,\dots,\alpha_n$, respectively. Let

$$\widehat{\sigma}_r=(-1)^{r-1}\sigma_r\in\mathbb{Q}_\alpha, \quad r=0,1,2,\dots, \qquad D_\alpha=\prod_{j\neq k}(\alpha_j-\alpha_k),$$

where $\sigma_0\equiv 1$.

If $\mathbb{T}$ is acting on a topological space M, let

$$H^*_{\mathbb{T}}(M)\equiv H^*(BM;\mathbb{Q}), \quad \text{where } BM=E\mathbb{T}\times_{\mathbb{T}}M,$$

be the *equivariant cohomology* of M. The projection map $BM\to B\mathbb{T}$ induces an action of $H^*_{\mathbb{T}}$ on $H^*_{\mathbb{T}}(M)$. We define

$$\mathcal{H}^*_{\mathbb{T}}(M)=H^*_{\mathbb{T}}(M)\otimes_{H^*_{\mathbb{T}}}\mathcal{H}^*_{\mathbb{T}}.$$

If the $\mathbb{T}$–action on M lifts to an action on a (complex) vector bundle $V\to M$, let

$$\boldsymbol{e}(V)\equiv e(BV)\in H^*_{\mathbb{T}}(M)\subset\mathcal{H}^*_{\mathbb{T}}(M)$$

denote the *equivariant euler class of* V.

Throughout the paper we work with the standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$:

$$(e^{i\theta_1},\dots,e^{i\theta_n})\cdot[z_1,\dots,z_n]=[e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n];$$

it has n fixed points

$$P_1=[1,0,\dots,0],\, P_2=[0,1,0,\dots,0],\,\dots,\, P_n=[0,\dots,0,1].$$

The $\mathbb{T}$–equivariant cohomology of $\mathbb{P}^{n-1}_N$ with respect to the induced diagonal $\mathbb{T}$–action on $\mathbb{P}^{n-1}_N$ is given by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}_N) = \mathbb{Q}[\alpha, \underline{\boldsymbol{x}}]/\{(\boldsymbol{x}_s - \alpha_1)\cdots(\boldsymbol{x}_s - \alpha_n) \mid s = 1, \dots, N\}, \tag{3-1}$$

where $\underline{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ and $\boldsymbol{x}_s = \pi_s^* \boldsymbol{x}$ if $\pi_s\colon \mathbb{P}^{n-1}_N \to \mathbb{P}^{n-1}$ is the projection onto the s^{th} component and $\boldsymbol{x} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$ is the equivariant hyperplane class. For each $\boldsymbol{p} \in \llbracket n \rrbracket^N$, let

$$\underline{\boldsymbol{x}}^{\boldsymbol{p}} = \prod_{i=1}^{i=N} \boldsymbol{x}_s^{p_s} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}_N);$$

these elements form a basis for $H^*_{\mathbb{T}}(\mathbb{P}^{n-1}_N)$ as a module over $H^*_{\mathbb{T}} = \mathbb{Q}[\alpha]$.

The action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ naturally lifts to the tautological line bundle γ, the vector bundle

$$\mathcal{L} \equiv \bigoplus_{k=1}^{k=l} \gamma^{*\otimes a_k} = \bigoplus_{k=1}^{k=l} \mathcal{O}_{\mathbb{P}^{n-1}}(a_k) \to \mathbb{P}^{n-1},$$

and the tangent bundle $T\mathbb{P}^{n-1}$ so that

$$\boldsymbol{e}(\mathcal{L})|_{P_i} = \langle \boldsymbol{a} \rangle \alpha_i^l, \quad \boldsymbol{e}(T\mathbb{P}^{n-1})|_{P_i} = \prod_{\substack{1 \le k \le n \\ k \ne i}} (\alpha_i - \alpha_k) \quad \text{for all } i = 1, 2, \dots, n. \tag{3-2}$$

Via composition of maps, the action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ and $\mathcal{L}$ induces actions on the moduli spaces $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1}, d)$ and

$$\mathcal{V}_d = \overline{\mathfrak{M}}_{0,N}(\mathcal{L}, d) \to \overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1}, d)$$

so that the evaluation maps

$$\mathrm{ev} \equiv \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_N\colon \overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1}, d) \to \mathbb{P}^{n-1}_N, \quad \widetilde{\mathrm{ev}}_s\colon \overline{\mathfrak{M}}_{0,N}(\mathcal{L}, d) \to \mathrm{ev}_s^* \mathcal{L},$$

are $\mathbb{T}$–equivariant. In particular, $\mathcal{V}_d$ has a well-defined equivariant euler class

$$\boldsymbol{e}(\mathcal{V}_d) \in H^*_{\mathbb{T}}(\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1}, d)).$$

Since the bundle homomorphisms $\widetilde{\mathrm{ev}}_s$ are surjective, their kernels are again equivariant vector bundles. Let

$$\mathcal{V}''_d = \ker \widetilde{\mathrm{ev}}_2 \to \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d).$$

With $\underline{\hbar}$ and $\underline{\hbar}^{-1}$ as in (2-1) and $\underline{\boldsymbol{x}}$ as in (3-1), let

$$\mathcal{Z}(\underline{\hbar}, \underline{\boldsymbol{x}}, Q) = \sum_{d=0}^{\infty} Q^d\, \mathrm{ev}_* \left\{ \frac{\boldsymbol{e}(\mathcal{V}_d)}{\prod_{s=1}^{s=N} (\hbar_s - \psi_s)} \right\} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}_N)\llbracket \underline{\hbar}^{-1}, Q \rrbracket, \tag{3-3}$$

where ev: $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}_N$; for $N=1,2$, we define the coefficient of Q^0 to be

$$\langle\boldsymbol{a}\rangle\boldsymbol{x}_1^l \quad\text{and}\quad -\frac{\langle\boldsymbol{a}\rangle\boldsymbol{x}_1^l}{\hbar_1+\hbar_2}\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge 0}}\hat{\sigma}_r\boldsymbol{x}_1^{p_1}\boldsymbol{x}_2^{p_2},$$

respectively. For each $p\in\llfloor n\rrfloor$, let

$$\mathcal{Z}_p(\hbar,\boldsymbol{x},Q)=\boldsymbol{x}^p+\sum_{d=1}^{\infty}Q^d\,\mathrm{ev}_{1*}\left\{\frac{e(\mathcal{V}_d'')\,\mathrm{ev}_2^*\boldsymbol{x}^p}{\hbar-\psi}\right\}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},Q]\!], \tag{3-4}$$

where $\mathrm{ev}_1,\mathrm{ev}_2\colon\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}$. Similarly to (2-20), let

$$\mathcal{Z}_{\boldsymbol{p}}(\underline{\hbar},\underline{\boldsymbol{x}},Q)\equiv\prod_{s=1}^{s=N}\frac{1}{\hbar_s}\frac{\mathcal{Z}_{p_s}(\hbar,\boldsymbol{x}_s,Q)}{\prod\limits_{r=p_s-l+1}^{n-l-1}I_r(q_s)}. \tag{3-5}$$

Theorem B *Suppose $n,N\in\mathbb{Z}^+$, with $N\ge 3$, and $\boldsymbol{a}\in(\mathbb{Z}^+)^l$ is such that $\|\boldsymbol{a}\|\le n$. The generating function (3-3) for equivariant N–pointed genus 0 GW–invariants of a complete intersection $X_{\boldsymbol{a}}\subset\mathbb{P}^{n-1}$ is given by*

$$\mathcal{Z}(\underline{\hbar},\underline{\boldsymbol{x}},Q)=\langle\boldsymbol{a}\rangle\sum_{\boldsymbol{p}\in\llfloor n\rrfloor^N}\sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^N}\sum_{d=0}^{\infty}\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}q^d\underline{\hbar}^{-\boldsymbol{b}}\mathcal{Z}_{\boldsymbol{p}}(\underline{\hbar},\underline{\boldsymbol{x}},Q) \tag{3-6}$$

for some $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}\in\mathbb{Q}[\alpha]$ such that

$$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}-\sum_{t=0}^{\infty}\mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}\hat{\sigma}_n^t\in\mathcal{I}, \tag{3-7}$$

where $\mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}\in\mathbb{Q}$ are the numbers defined above Theorem A.

Setting $\alpha=0$ in Theorem B and using [27, Theorem 3], we obtain

$$\begin{aligned}Z(\underline{\hbar},\underline{H},Q)=\langle\boldsymbol{a}\rangle\exp\Big(-\sum_{s=1}^{s=N}J(q_s)w_s\Big)\\ \times\sum_{\boldsymbol{p}\in\llfloor n\rrfloor^N}\sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^N}\sum_{d=0}^{\infty}\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}q^d\underline{\hbar}^{-\boldsymbol{b}}\Delta_{\boldsymbol{p}}(\underline{\hbar},\underline{H},Q).\end{aligned} \tag{3-8}$$

This implies Theorem A provided $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}=0$ if $p_s<l$ for some $s\in[N]$; this is shown in the next paragraph.

Suppose instead $\mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,0)} \neq 0$ for some triple $(\boldsymbol{p},\boldsymbol{b},d)$ with $p_1 < l$. Choose $(\boldsymbol{p},\boldsymbol{b},d)$ minimizing p_1, as well as minimizing d for the smallest possible p_1. We show that

$$\langle \tau_{b_1} H^{n-1-p_1}, \dots, \tau_{b_N} H^{n-1-p_N} \rangle_{0,d}^{X_{\boldsymbol{a}}} = \langle \boldsymbol{a} \rangle \mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,0)}. \tag{3-9}$$

By (1-3) and (2-1), this GW–invariant is the coefficient of

$$Q^d \prod_{s=1}^{s=N} \hbar_s^{-(b_s+1)} H_s^{p_s}$$

of the right-hand side of (3-8). Suppose a triple $(\boldsymbol{p}',\boldsymbol{b}',d')$, with $\mathrm{c}_{\boldsymbol{p}',\boldsymbol{b}'}^{(d',0)} \neq 0$, contributes to this coefficient. Since the lowest power of H in the coefficient of a product of powers of q and $\hbar^{-1}$ in $H^p F_p(w,q)$ is $\min(p,l)$, $p_1' = p_1$ by the minimality of p_1 and thus $d' = d$ by the minimality of d. Since the coefficient of q^0 in $H^p F_p(w,q)$ is H^p, $p_s' = p_s$ for all $s \in [N]$ and thus $b_s' = b_s$ for all $s \in [N]$; this gives (3-9). Since $H^{n-1-p_1}|_{X_{\boldsymbol{a}}} = 0$ for $p_1 < l$, we conclude that $\mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,0)} = 0$.

The proof of Theorem B below provides an algorithm for computing the structure coefficients $\mathcal{C}_{\boldsymbol{p},\boldsymbol{b}}^{(d)}$ completely. On the other hand, they may be irrelevant in many applications. For example, the one- and two-point equivariant generating functions (3-3) play a key in the localization computation of the genus 1 GW–invariants of Calabi–Yau complete intersections in [31; 26], but the structure coefficients lying in $\mathcal{I}$ are ignored. Similarly, the equivariant generating functions with $N \leq g$ with the structure coefficients lying in $\mathcal{I}$ ignored should play a key role in computing genus $g \geq 2$ GW–invariants of complete intersections.

4 Proof of Theorem A

4.1 Localization setup

If $\mathbb{T}$ acts smoothly on a smooth compact oriented manifold M, there is a well-defined integration-along-the-fiber homomorphism

$$\int_M : H_{\mathbb{T}}^*(M) \to H_{\mathbb{T}}^*$$

for the fiber bundle $BM \to B\mathbb{T}$. The classical localization theorem of [3] relates it to integration along the fixed locus of the $\mathbb{T}$–action. The latter is a union of smooth compact orientable manifolds F and $\mathbb{T}$ acts on the normal bundle $\mathcal{N}F$ of each F. Once an orientation of F is chosen, there is a well-defined integration-along-the-fiber homomorphism

$$\int_F : H_{\mathbb{T}}^*(F) \to H_{\mathbb{T}}^*.$$

The localization theorem states that

$$\int_M \psi = \sum_F \int_F \frac{\psi|_F}{e(\mathcal{N}F)} \in \mathcal{H}_{\mathbb{T}}^* \quad \text{for all } \psi \in H_{\mathbb{T}}^*(M), \tag{4-1}$$

where the sum is taken over all components F of the fixed locus of $\mathbb{T}$. Part of the statement of (4-1) is that $e(\mathcal{N}F)$ is invertible in $\mathcal{H}_{\mathbb{T}}^*(F)$.

The standard $\mathbb{T}$–action on $\mathbb{P}_N^{n-1}$ has n^N fixed points

$$P_{i_1\cdots i_N} \equiv P_{i_1} \times \cdots \times P_{i_N}.$$

The restriction maps on the equivariant cohomology induced by $P_{i_1\cdots i_N} \to \mathbb{P}_N^{n-1}$ are the homomorphisms

$$H_{\mathbb{T}}^*(\mathbb{P}_N^{n-1}) \to \mathbb{Q}[\alpha_1,\dots,\alpha_n], \quad \boldsymbol{x}_s \to \alpha_{i_s}, \quad s=1,\dots,N. \tag{4-2}$$

By (3-1) and (4-2),

$$\eta = 0 \in H_{\mathbb{T}}^*(\mathbb{P}_N^{n-1}) \Leftrightarrow \eta|_{P_{i_1\cdots i_N}} = 0 \in H_{\mathbb{T}}^* \quad \text{for all } i_s = 1,2,\dots,n,\ s=1,\dots,N,$$

ie an element of $H_{\mathbb{T}}^*(\mathbb{P}_N^{n-1})$ is determined by its restrictions to the n^N $\mathbb{T}$–fixed points. For each $i=1,2,\dots,n$, the equivariant Poincaré dual of P_i in $\mathbb{P}^{n-1}$ is given by

$$\phi_i = \prod_{k\neq i}(\boldsymbol{x}-\alpha_k) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1}). \tag{4-3}$$ [13]

Thus, by the defining property of the cohomology pushforward [31, (1.11)], the power series $\mathcal{Z}(\underline{\hbar},\underline{\boldsymbol{x}},Q)$ in (3-3) is completely determined by the n^N power series

$$\mathcal{Z}(\underline{\hbar},\alpha_{i_1,\dots,i_N},Q) = \sum_{d=0}^{\infty} Q^d \int_{\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_d) \prod_{s=1}^{s=N}\left(\frac{\mathrm{ev}_s^*\,\phi_{i_s}}{\hbar_s-\psi_s}\right), \tag{4-4}$$

where $\alpha_{i_1\cdots i_N} \equiv (\alpha_{i_1},\dots,\alpha_{i_N})$.

As described in detail in [17, Section 27.3], the fixed loci $\mathcal{Z}_\Gamma$ of the $\mathbb{T}$–action on $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)$ are indexed by *N–marked decorated trees* Γ. An *N–marked decorated tree* is a tuple

$$\Gamma = (\mathrm{Ver},\mathrm{Edg};\mu,\mathfrak{d},\eta), \tag{4-5}$$

where $(\mathrm{Ver},\mathrm{Edg})$ is a tree and

$$\mu\colon \mathrm{Ver}\to[n]\equiv\{1,\dots,n\}, \quad \mathfrak{d}:\mathrm{Edg}\to\mathbb{Z}^+, \quad \eta\colon [N]\to\mathrm{Ver},$$

[13] In other words, if $\eta\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})$, then $\eta|_{P_i} \equiv \int_{P_i}\eta|_{P_i} = \int_{\mathbb{P}^{n-1}}\eta\phi_i$.

are maps such that

$$\mu(v_1) \neq \mu(v_2) \quad \text{if } \{v_1, v_2\} \in \mathrm{Edg}\,. \tag{4-6}$$

In the first diagram of Figure 2, the value of the map μ on each vertex is indicated by the number next to the vertex. Similarly, the value of the map $\mathfrak{d}$ on each edge is indicated by the number next to the edge. By (4-6), no two consecutive vertex labels are the same. Let

$$|\Gamma| = \sum_{e \in \mathrm{Edg}} \mathfrak{d}(e).$$

For each $e = \{v, v'\} \in \mathrm{E}_v(\Gamma)$, let $\mu_v(e) = \mu(v') \in [n]$.

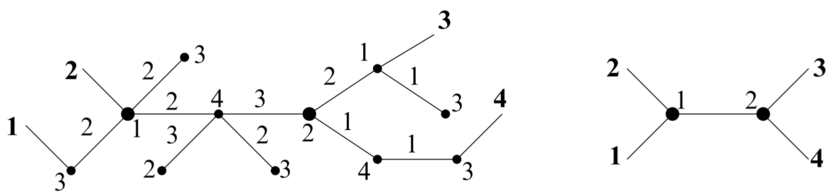

Figure 2: A decorated tree, with special vertices indicated by larger dots, and its decorated core

If Γ is a decorated tree as in (4-5) and $v \in \mathrm{Ver}$, let

$$\mathrm{val}_\Gamma(v) = |\mathrm{E}_v(\Gamma)| + |\eta^{-1}(v)|$$

be the *valence* of v in Γ. If in addition we have that $N \geq 3$, *the core of* Γ is the tuple $\overline{\Gamma} \equiv (\overline{\mathrm{Ver}}, \overline{\mathrm{Edg}}; \overline{\mu}, \overline{\eta})$ such that

(R1) $(\overline{\mathrm{Ver}}, \overline{\mathrm{Edg}})$ is a tree, $\overline{\mathrm{Ver}} = \{v \in \mathrm{Ver} \mid \mathrm{val}_\Gamma(v) \geq 3\}$ and $\overline{\mu} = \mu|_{\overline{\mathrm{Ver}}}$;

(R2) $\{v, v'\} \in \overline{\mathrm{Edg}}$ if and only if $v, v' \in \overline{\mathrm{Ver}}$, $v \neq v'$ and for some $m \geq 0$ there exist distinct

$$v_1, \dots, v_m \in \mathrm{Ver} - \overline{\mathrm{Ver}} \quad \text{such that } \{v_{i-1}, v_i\} \in \mathrm{Edg} \text{ for all } i \in [m+1],$$

where $v_0 \equiv v$ and $v_{m+1} \equiv v'$;

(R3) if $s \in \eta^{-1}(\overline{\mathrm{Ver}}) \subset [N]$, $\overline{\eta}(s) = \eta(s)$; if $s \in \eta^{-1}(\mathrm{Ver} - \overline{\mathrm{Ver}})$, there exist distinct elements

$$v_1, \dots, v_m \in \mathrm{Ver} - \overline{\mathrm{Ver}} \quad \text{such that } \{v_{i-1}, v_i\} \in \mathrm{Edg} \text{ for all } i \in [m+1],$$

where $v_0 \equiv \overline{\eta}(s)$ and $v_{m+1} = \eta(s)$.

The core of a graph with $N \geq 3$ is obtained by repeatedly collapsing all vertices with valence less than 3 onto their neighbors, until no such vertices are left; see Figure 2. We will call the vertices $\overline{\mathrm{Ver}}$ of the core $\overline{\Gamma}$ the *special vertices of* Γ.

The localization formula (4-1) reduces the restriction of (3-3) to each fixed point $P_{i_1\cdots i_N} \in \mathbb{P}^{n-1}_N$ to a sum over decorated trees. This sum can be computed by breaking each such tree Γ at its special vertices into *strands*, with each of the strands keeping a copy of the special vertex, with its label, which will have a new marked point attached; see Figure 3. There are three types of strands:

(S1) one-marked strands

(S2) strands with two new marked points

(S3) strands with one new marked points and one of the original N marked points

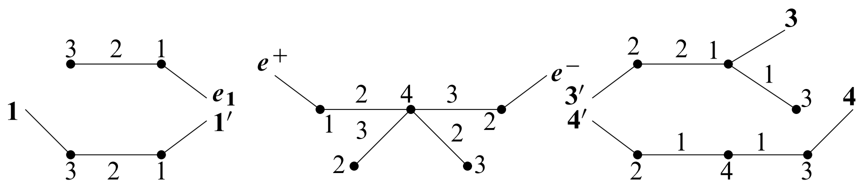

Figure 3: The strands of the graph in the first diagram in Figure 2.

By (4-1), each one-pointed strand at a special vertex $v \in \overline{\mathrm{Ver}} \subset \mathrm{Ver}$ contributes to

$$\mathcal{Z}'^*(\hbar, \alpha_j, Q) \equiv \sum_{d=1}^{\infty} Q^d \int_{\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1},d)} e(\mathcal{V}'_d) \frac{\mathrm{ev}_1^* \phi_j}{\hbar - \psi_1}, \tag{4-7}$$

where $j = \mu(v) \in [n]$ is the label of the vertex v of Γ and

$$\mathcal{V}'_d \to \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1}, d)$$

is the kernel of the surjective vector bundle homomorphism $\widetilde{\mathrm{ev}}_1\colon \mathcal{V}_d \to \mathrm{ev}_1^* \mathcal{L}$. By the dilaton relation [17, page 527],

$$\widetilde{\mathcal{Z}}^*(\hbar, \alpha_j, Q) \equiv \sum_{d=1}^{\infty} Q^d \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} e(\mathcal{V}'_d) \left(\frac{\mathrm{ev}_1^* \phi_j}{\hbar - \psi_1} \right) = \hbar^{-1} \mathcal{Z}'^*(\hbar, \alpha_j, Q).$$

Each of the two-pointed strands contributes to

$$\mathcal{Z}^*(\hbar_1, \hbar_2, \alpha_{j_1}, \alpha_{j_2}, Q) \equiv \sum_{d=1}^{\infty} Q^d \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_d) \frac{\mathrm{ev}_1^* \phi_{j_1}}{\hbar_1 - \psi_1} \frac{\mathrm{ev}_2^* \phi_{j_2}}{\hbar_2 - \psi_2},$$

where $j_1, j_2 \in [n]$ are the labels of the vertices to which the marked points are attached. Thus, the power series $\mathcal{Z}(\underline{\hbar}, \underline{\boldsymbol{x}}, Q)$ in (3-3) is determined by the previously computed power series for one- and two-pointed GW–invariants.

While the number of one-marked strands at each node can be arbitrary large, as indicated in [31, Sections 2.1,2.2] it is possible to sum over all possibilities for these strands at each special vertex; see Corollary 4.3 below. On the other hand, the number of special vertices, the number of two-pointed strands of type (S2), and the number of two-pointed strands of type (S3), are bounded (by $N-2$, $N-3$ and N, respectively). Using the Residue Theorem for S^2, one can then sum up over all possibilities of the markings for each of the distinguished nodes. Thus, the approach of breaking trees at special vertices reduces (3-3) to a finite sum, with one summand for each trivalent N–marked tree.

The description of the structure constants $\mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}$ in Section 2.4 is obtained by breaking the trees at all special vertices. On the other hand, the description in Section 2.3 is obtained by breaking at the special vertex $\overline{\eta}(N)$ only. In addition to the strands (S1), we would then obtain strands with marked points indexed by the sets $S_i \sqcup \{0\}$, for a partition $\{S_i\}_{i\in[m]}$ of $[N]$ so that one of the sets S_i is $\{N\}$. With either approach, the main step is summing over all possibilities for the strands (S1), as done in Corollary 4.3.

4.2 Notation and preliminaries

If $f = f(\hbar)$ is a rational function in $\hbar$ and $\hbar_0 \in S^2$, let

$$\mathop{\mathfrak{R}}_{\hbar=\hbar_0}\{f(\hbar)\} = \frac{1}{2\pi\mathrm{i}}\oint f(\hbar)\mathrm{d}\hbar,$$

where the integral is taken over a positively oriented loop around $\hbar = \hbar_0$ containing no other singular points of f, denote the residue of $f(\hbar)\mathrm{d}\hbar$ at $\hbar = \hbar_0$. With this definition,

$$\mathop{\mathfrak{R}}_{\hbar=\infty}\{f(\hbar)\} = -\mathop{\mathfrak{R}}_{w=0}\{w^{-2}f(w^{-1})\}.$$

If f involves variables other than $\hbar$, $\mathfrak{R}_{\hbar=\hbar_0}\{f(\hbar)\}$ will be a function of such variables. If f is a power series in q with coefficients that are rational functions in $\hbar$ and possibly other variables, denote by $\mathfrak{R}_{\hbar=\hbar_0}\{f(\hbar)\}$ the power series in q obtained by replacing each of the coefficients by its residue at $\hbar = \hbar_0$. If $\hbar_1, \dots, \hbar_k$ is a collection of distinct points in S^2, let

$$\mathop{\mathfrak{R}}_{\hbar=\hbar_1,\dots,\hbar_k}\{f(\hbar)\} = \sum_{i=1}^{i=k}\mathop{\mathfrak{R}}_{\hbar=\hbar_i}\{f(\hbar)\}$$

be the sum of the residues at the specified values of $\hbar$.

We denote by

$$\mathbb{Q}'_\alpha \equiv \mathbb{Q}[\alpha, \sigma_n^{-1}, D_\alpha^{-1}] \subset \mathbb{Q}_\alpha$$

the subring of rational functions in $\alpha_1, \dots, \alpha_n$ with denominators that are products of σ_n and D_α. Let

$$\mathbb{Q}'_{\alpha;\hbar,x} \equiv \mathbb{Q}'_\alpha[\hbar, x^{\pm 1}]_{\langle (x+r\hbar)^n - x^n, \prod_{k=1}^{k=n}(x-\alpha_k+r\hbar) - \prod_{k=1}^{k=n}(x-\alpha_k) | r\in\mathbb{Z}^+\rangle} \subset \mathbb{Q}_\alpha(\hbar, x)$$

be the subring of rational functions in $\alpha_1, \dots, \alpha_n$, $\hbar$ and x with numerators that are polynomials in $\alpha_1, \dots, \alpha_n$, $\hbar$ and x and with denominators that are products of

$$\sigma_n, \quad D_\alpha, \quad x, \quad (x+r\hbar)^n - x^n, \quad \prod_{k=1}^{k=n}(x-\alpha_k+r\hbar) - \prod_{k=1}^{k=n}(x-\alpha_k), \quad \text{with } r\in\mathbb{Z}^+.$$

If R is one of the rings $\mathbb{Q}'_\alpha$, $\mathbb{Q}'_\alpha[x^{\pm 1}]$, or $\mathbb{Q}'_{\alpha;\hbar,x}$ and f_1 and f_2 are elements of R or $R[\![Q]\!]$, we will write $f_1 \sim f_2$ if $f_1 - f_2$ lies in $\mathcal{I}\cdot R$ or $\mathcal{I}\cdot R[\![Q]\!]$, respectively. By the next lemma, certain operations on these rings respect these equivalence relations.

Lemma 4.1 (1) *If $f \in \mathbb{Q}'_{\alpha;\hbar,x}$, there exists $g \in \mathbb{Q}'_\alpha[x^{\pm 1}]$ such that*

$$\mathfrak{R}_{\hbar=0}\{f(\hbar, x=\alpha_j)\} = g(x=\alpha_j) \quad \textit{for all } j\in[n].$$

(2) *If $g \in \mathbb{Q}'_\alpha[x^{\pm 1}]$,*

$$\mathfrak{R}_{x=0,\infty}\left\{\frac{g(x)}{\prod_{k=1}^{k=n}(x-\alpha_k)}\right\} \in \mathbb{Q}'_\alpha.$$

(3) *For every $p\in\mathbb{Z}$,*

$$-\mathfrak{R}_{x=0,\infty}\left\{\frac{x^p}{\prod_{k=1}^{k=n}(x-\alpha_k)}\right\} \sim \begin{cases} \hat\sigma_n^t & \textit{if } p = n-1+nt \textit{ with } t\in\mathbb{Z}, \\ 0 & \textit{if } p+1\notin n\mathbb{Z}. \end{cases}$$

Proof If $f\in\mathbb{Q}'_{\alpha;\hbar,x}$, then

$$\text{(4-8)} \qquad \mathfrak{R}_{\hbar=0}\{f(\hbar, x=\alpha_j)\} = \big(\mathfrak{R}_{\hbar=0}\{f(\hbar, x)\}\big)\big|_{x=\alpha_j},$$

$$\mathfrak{R}_{\hbar=0}\{f(\hbar,x)\} \in \mathbb{Q}'_\alpha[x^{\pm 1}, \sigma_{n-1}(x)^{-1}],$$

where $\sigma_{n-1}(x) = \sum_{i=1}^{i=n}\prod_{k\neq i}(x-\alpha_k)$. The first claim of this lemma thus follows from the observation that

$$\frac{1}{\sigma_{n-1}(x)}\bigg|_{x=\alpha_j} = \frac{1}{D_\alpha^2}\bigg(\sum_{i=1}^{i=n}\Big(\prod_{\substack{i'\neq i\\ k\neq i'}}(\alpha_{i'}-\alpha_k)^2\Big)\Big(\prod_{k\neq i}(x-\alpha_k)\Big)\bigg)\bigg|_{x=\alpha_j} \qquad \text{for all } j\in[n].$$

The second claim is immediate from the third. The third claim of this lemma follows from the power series expansions

$$-\frac{1}{\boldsymbol{x}^n-\widehat{\sigma}_n}=\sum_{r=0}^{\infty}\widehat{\sigma}_n^{-r-1}\boldsymbol{x}^{nr}, \qquad \frac{1}{1-\widehat{\sigma}_n w^n}=\sum_{r=0}^{\infty}\widehat{\sigma}_n^{r}w^{nr}$$

around $\boldsymbol{x}=0$ and $w=0$, respectively. □

We will also use the residue theorem on S^2:

$$\sum_{\boldsymbol{x}_0\in S^2}\mathop{\mathfrak{R}}_{\boldsymbol{x}=\boldsymbol{x}_0}\{f(\boldsymbol{x})\}=0$$

for every rational function $f=f(\boldsymbol{x})$ on $S^2\supset\mathbb{C}$.

4.3 Equivariant one- and two-pointed formulas

The most fundamental generating function for GW–invariants in the mirror symmetry computations following [12] is

$$\text{(4-9)}\quad \widetilde{\mathcal{Z}}(\hbar,\boldsymbol{x},Q)\equiv 1+\widetilde{\mathcal{Z}}^*(\hbar,\boldsymbol{x},Q) \equiv 1+\sum_{d=1}^{\infty}Q^d\,\mathrm{ev}_{1*}\left\{\frac{e(\mathcal{V}_d')}{\hbar-\psi_1}\right\}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},Q]\!],$$

where $\mathrm{ev}_1\colon \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}$ and $\mathcal{V}_d'\to\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$ is the kernel of the surjective vector bundle homomorphism $\widetilde{\mathrm{ev}}_1\colon\mathcal{V}_d\to\mathrm{ev}_1^*\mathcal{L}$. By [12],

$$\widetilde{\mathcal{Z}}(\hbar,\alpha_j,Q)\in\mathbb{Q}_\alpha(\hbar)\quad\text{for all } j\in[n].$$

Thus, we can define $\zeta(\alpha_j,Q)\in Q\cdot\mathbb{Q}_\alpha[\![Q]\!]$ and $\widetilde{\mathcal{Z}}_{m,B}(\alpha_j,Q)\in\mathbb{Q}_\alpha[\![Q]\!]$ by

$$\zeta(\alpha_j,Q)=\mathop{\mathfrak{R}}_{\hbar=0}\{\ln(1+\widetilde{\mathcal{Z}}^*(\hbar,\alpha_j,Q))\},$$

$$\widetilde{\mathcal{Z}}_{m,B}(\alpha_j,Q)=\sum_{m'=0}^{\infty}\frac{(m'+m)!}{m'!} \times\sum_{\boldsymbol{b}\in\mathcal{P}_{m'}(m-B+m')}\left(\prod_{k=1}^{k=m'}\frac{(-1)^{b_k}}{b_k!}\mathop{\mathfrak{R}}_{\hbar=0}\{\hbar^{-b_k}\widetilde{\mathcal{Z}}^*(\hbar,\alpha_j,Q)\}\right),$$

for $m,B\in\overline{\mathbb{Z}}^+$. Since the power series $\widetilde{\mathcal{Z}}^*(\hbar,\boldsymbol{x},Q)$ has no Q–constant term, the above sum is finite in each Q–degree. It is shown Section 4.4 that the power series $\widetilde{\mathcal{Z}}_{m,B}(\boldsymbol{x},Q)$ describe the contributions of the strands (S1) at a vertex v of the core of a tree with $m_v=m$ (with m_v computed with respect to the core). Define

$$\mathcal{Z}(\hbar_1,\hbar_2,\boldsymbol{x}_1,\boldsymbol{x}_2,Q)\in-\frac{\langle\boldsymbol{a}\rangle\boldsymbol{x}_1^l}{\hbar_1+\hbar_2}+H^*_{\mathbb{T}}(\mathbb{P}_2^{n-1})[\![\hbar_1^{-1},\hbar_2^{-1},Q]\!]$$

by

$$\mathcal{Z}^*(\hbar_1,\hbar_2,\boldsymbol{x}_1,\boldsymbol{x}_2,Q)=\sum_{d=1}^{\infty}Q^d\,\mathrm{ev}_*\left\{\frac{\boldsymbol{e}(\mathcal{V})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\right\},$$

$$\mathcal{Z}(\hbar_1,\hbar_2,\boldsymbol{x}_1,\boldsymbol{x}_2,Q)=-\frac{\langle\boldsymbol{a}\rangle\boldsymbol{x}_1^l}{\hbar_1+\hbar_2}\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge 0}}\hat{\sigma}_r\boldsymbol{x}_1^{p_1}\boldsymbol{x}_2^{p_2}+\mathcal{Z}^*(\hbar_1,\hbar_2,\boldsymbol{x}_1,\boldsymbol{x}_2,Q), \tag{4-10}$$

where ev: $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}_2$ is the total evaluation map.

Proposition 4.2 *The power series* (4-9) *and* (3-4) *admit expansions*

$$\tilde{\mathcal{Z}}(\hbar,\alpha_j,Q)=e^{\zeta(\alpha_j,Q)/\hbar}\sum_{b=0}^{\infty}\Psi_b(\alpha_j,Q)\hbar^b, \tag{4-11}$$

$$\frac{\mathcal{Z}_p(\hbar,\alpha_j,Q)}{\prod_{r=p-l+1}^{n-l-1}I_r(q)}=e^{\zeta(\alpha_j,Q)/\hbar}\sum_{b=0}^{\infty}\Psi_{p;b}(\alpha_j,Q)\hbar^b, \tag{4-12}$$

for some $\zeta,\Psi_b,\Psi_{p;b}\in\mathbb{Q}'_\alpha[\boldsymbol{x}^{\pm 1}][\![Q]\!]$ *such that*

$$\Psi_b(\boldsymbol{x},Q)\sim\frac{\Phi_b(\boldsymbol{q})}{I_0(\boldsymbol{q})}\boldsymbol{x}^{-b},\quad \Psi_{p;b}(\boldsymbol{x},Q)\sim\frac{I_0(\boldsymbol{q})\Phi_{p;b}(\boldsymbol{q})}{L(\boldsymbol{q})^{\delta_{0\nu_{\boldsymbol{a}}}n}}\boldsymbol{x}^{p-b}, \tag{4-13}$$

where $\boldsymbol{q}e^{\delta_{0\nu_{\boldsymbol{a}}}J(\boldsymbol{q})}=Q/\boldsymbol{x}^{\nu_{\boldsymbol{a}}}$.

Proof The existence of the expansion (4-11) follows from [31, Lemmas 2.2 and 2.3], but a direct argument is provided below and in Appendix A. Let

$$\mathcal{Y}(\hbar,\boldsymbol{x},q)=\sum_{d=0}^{\infty}q^d\frac{\prod_{k=1}^{k=l}\prod_{r=1}^{r=a_kd}(a_k\boldsymbol{x}+r\hbar)}{\prod_{r=1}^{r=d}\left(\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k+r\hbar)-\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k)\right)}\in(\mathbb{Q}'_{\alpha;\hbar,\boldsymbol{x}}\cap\mathbb{Q}_\alpha[\boldsymbol{x}][\![\hbar^{-1}]\!])[\![Q]\!].$$

By [17, Section 29.1],

$$\tilde{\mathcal{Z}}(\hbar,\boldsymbol{x},Q)=\exp\!\left(-J(q)\frac{\boldsymbol{x}^{\delta_{0\nu_{\boldsymbol{a}}}}}{\hbar}+f(q)\frac{\sigma_1}{\hbar}\right)\frac{\mathcal{Y}(\hbar,\boldsymbol{x},q)}{I_0(q)} \tag{4-14}$$

for some $f \in q\mathbb{Q}[\![q]\!]$ (which is 0 unless $\nu_{\boldsymbol{a}} = 0$), where $qe^{\delta_{0\nu_{\boldsymbol{a}}} J(q)} = Q$. Since

$$\mathcal{Y}(\hbar, \boldsymbol{x}, q) = \left\{1 + \frac{\hbar}{\boldsymbol{x}} q \frac{\mathrm{d}}{\mathrm{d}q}\right\}^l \mathcal{Y}_0(\hbar, \boldsymbol{x}, q),$$

with $\mathcal{Y}_0(\hbar, \boldsymbol{x}, q)$ given by (A-1), Lemma A.1 implies that $\mathcal{Y}(\hbar, \boldsymbol{x}, q)$ admits an expansion of the form

$$\mathcal{Y}(\hbar, \boldsymbol{x}, q) = e^{\xi(\boldsymbol{x}, q)/\hbar} \sum_{b=0}^{\infty} \Phi_b(\boldsymbol{x}, q) \hbar^b \tag{4-15}$$

with $\xi(\boldsymbol{x}, q), \Phi_0(\boldsymbol{x}, q), \Phi_1(\boldsymbol{x}, q), \ldots \in \mathbb{Q}_\alpha(\boldsymbol{x})[\![q]\!]$. Since

$$\begin{gathered}
\xi(\boldsymbol{x}, q) = \mathfrak{R}_{\hbar=0} \{\ln \mathcal{Y}(\hbar, \boldsymbol{x}, q)\}, \\
\Phi_b(\boldsymbol{x}, q) = \mathfrak{R}_{\hbar=0} \{\hbar^{-b-1} e^{-\xi(\boldsymbol{x}, q)/\hbar} \mathcal{Y}(\hbar, \boldsymbol{x}, q)\}, \\
\mathcal{Y}(\hbar, \boldsymbol{x}, q) - F(w, \boldsymbol{q}) \in q \cdot \mathcal{I}\mathbb{Q}'_{\alpha;\hbar,\boldsymbol{x}},
\end{gathered}$$

where $w = \boldsymbol{x}/\hbar$, Proposition 2.1 and the first statement of Lemma 4.1 imply that there exist

$$\tilde{\xi}(\boldsymbol{x}, q), \tilde{\Phi}_0(\boldsymbol{x}, q), \tilde{\Phi}_1(\boldsymbol{x}, q), \ldots \in \mathbb{Q}'_\alpha[\boldsymbol{x}^{\pm 1}][\![q]\!]$$

such that

$$\begin{gathered}
\mathcal{Y}(\hbar, \alpha_j, q) = e^{\tilde{\xi}(\alpha_j, q)/\hbar} \sum_{b=0}^{\infty} \tilde{\Phi}_b(\alpha_j, q) \hbar^b \quad \text{for all } j \in [n], \\
\tilde{\xi}(\boldsymbol{x}, q) \sim \xi(\boldsymbol{q})\boldsymbol{x}, \quad \tilde{\Phi}_b(\boldsymbol{x}, q) \sim \Phi_b(\boldsymbol{q})\boldsymbol{x}^{-b} \quad \text{for all } b \in \mathbb{Z}^+.
\end{gathered} \tag{4-16}$$

By (4-14) and (4-16), (4-11) and the first statement in (4-13) hold with

$$\zeta(\boldsymbol{x}, Q) = \tilde{\xi}(\boldsymbol{x}, q) - J(\boldsymbol{q})\boldsymbol{x} + f(q)\sigma_1, \quad \Psi_b(\boldsymbol{x}, Q) = \frac{\tilde{\Phi}_b(\boldsymbol{x}, \boldsymbol{q})}{I_0(q)} = \frac{\tilde{\Phi}_b(\boldsymbol{x}, \boldsymbol{q})}{I_0(\boldsymbol{q})}.$$

The existence of the expansion (4-12) follows from the existence of the expansion (4-11) and the description of $\mathcal{Z}_p(\hbar, \boldsymbol{x}, Q)$ as a linear combination of the derivatives of $\tilde{\mathcal{Z}}(\hbar, \boldsymbol{x}, Q)$ in [27, Theorem 4]. By [27, Theorem 4],

$$\mathcal{Z}_p(\hbar, \boldsymbol{x}, Q) \sim e^{-J(\boldsymbol{q})w} \boldsymbol{x}^p \frac{F_p(w, \boldsymbol{q})}{I_{p-l}(\boldsymbol{q})}.$$

Along with the first statement in Lemma 4.1 and (2-21), this gives the second claim in (4-13). □

Corollary 4.3 *For all $m \in \overline{\mathbb{Z}}^+$ and $\boldsymbol{c} \in (\overline{\mathbb{Z}}^+)^\infty$, there exists $\Psi_{m,\boldsymbol{c}} \in \mathbb{Q}'_\alpha[\boldsymbol{x}^{\pm 1}][\![Q]\!]$ such that*

$$\widetilde{\mathcal{Z}}_{m,B}(\alpha_j, Q) = \sum_{\boldsymbol{c}\in(\overline{\mathbb{Z}}^+)^\infty} \left((-1)^{m-\|\boldsymbol{c}\|}\binom{B}{m-\|\boldsymbol{c}\|}\zeta(\alpha_j, Q)^{B-(m-\|\boldsymbol{c}\|)}\Psi_{m,\boldsymbol{c}}(\alpha_j, Q)\right) \tag{4-17}$$

for all $B \in \overline{\mathbb{Z}}^+$ and $j \in [n]$ and

$$\Psi_{m,\boldsymbol{c}}(\boldsymbol{x}, Q) \sim \left(\frac{I_0(\boldsymbol{q})}{\Phi_0(\boldsymbol{q})}\right)^{m+3} \Phi_{m,\boldsymbol{c}}(\boldsymbol{q})\boldsymbol{x}^{-\|\boldsymbol{c}\|}, \tag{4-18}$$

where $\boldsymbol{q}e^{\delta_{0\nu_{\boldsymbol{a}}}J(\boldsymbol{q})} = Q/\boldsymbol{x}^{\nu_{\boldsymbol{a}}}$.

Proof By Lemma B.2 and (4-11), (4-17) holds with

$$\Psi_{m,\boldsymbol{c}}(\boldsymbol{x}, Q) = (-1)^{m+|\boldsymbol{c}|}(m+|\boldsymbol{c}|)!\frac{1}{\Psi_0(\boldsymbol{x}, Q)^{m+1}}\prod_{r=1}^{\infty}\frac{1}{c_r!}\left(\frac{1}{(r+1)!}\frac{\Psi_r(\boldsymbol{x}, Q)}{\Psi_0(\boldsymbol{x}, Q)}\right)^{c_r}. \tag{4-19}$$

Along with the first statement in (4-13) and (2-31), this implies (4-18). □

Lemma 4.4 *There exists a collection $\{\mathcal{C}_{p_-p_+}\}_{p_\pm\in\llbracket n\rrbracket} \subset \mathbb{Q}[\alpha][\![Q]\!]$ such that*

$$\begin{aligned}\frac{1}{\langle\boldsymbol{a}\rangle}\mathop{\mathfrak{R}}_{\hbar_+=0}&\left\{\frac{1}{\hbar_+^{1+b_+}}\exp\left(-\frac{\zeta(\alpha_{j_+}, Q)}{\hbar_+}\right)\mathcal{Z}(\hbar_-,\hbar_+,\alpha_{j_-},\alpha_{j_+}, Q)\right\}\\ &= \sum_{b_-=0}^{b_-=b_+}\left(\frac{(-1)^{b_-}}{\hbar_-^{b_-}}\sum_{p_+,p_-\in\llbracket n\rrbracket}\mathcal{C}_{p_-p_+}(Q)\Psi_{p_+;b_+-b_-}(\alpha_{j_+}, Q) \times\frac{\mathcal{Z}_{p_-}(\hbar_-,\alpha_{j_-}, Q)}{\hbar_-\prod_{r=p_--l+1}^{n-l-1}I_r(q)}\right)\end{aligned} \tag{4-20}$$

for all $b_+ \in \overline{\mathbb{Z}}^+$ and $j_-, j_+ \in [n]$ and

$$\mathcal{C}_{p_-p_+}(Q) \sim \begin{cases}\frac{L(q)^{\delta_{0\nu_{\boldsymbol{a}}}(1+t)n}}{I_0(q)^2}\widehat{\sigma}_n^t & \text{if } p_-+p_++nt = n-1+l, t=0,1,\\ 0 & \text{otherwise,}\end{cases} \tag{4-21}$$

where $qe^{\delta_{0\nu_{\boldsymbol{a}}}J(q)} = Q$.

Proof By [27, Theorem 4],

$$\frac{\hbar_-+\hbar_+}{\langle \boldsymbol{a}\rangle}\mathcal{Z}(\hbar_-,\hbar_+,\boldsymbol{x}_-,\boldsymbol{x}_+,Q) = \Bigg\{-\sum_{\substack{p_-+p_++r=n-1+l\\ p_-,p_+\in\llbracket n\rrbracket, r\in\bar{\mathbb{Z}}^+\\ p_-,p_+\ge l}} + \sum_{\substack{p_-+p_++r=n-1+l\\ p_-,p_+\in\llbracket n\rrbracket, r\in\bar{\mathbb{Z}}^+\\ p_-,p_+< l}}\Bigg\} \times \hat{\sigma}_r\mathcal{Z}_{p_-}(\hbar_-,\boldsymbol{x}_-,Q)\mathcal{Z}_{p_+}(\hbar_+,\boldsymbol{x}_+,Q). \tag{4-22}$$

Combining this identity with (4-12), we find that (4-20) holds with

$$\mathcal{C}_{p_-p_+}(Q) = \Bigg(\prod_{r=p_+-l+1}^{n-l-1} I_r(q)\Bigg)\Bigg(\prod_{r=p_--l+1}^{n-l-1} I_r(q)\Bigg)\hat{\sigma}_{n-1+l-p_--p_+} \times\begin{cases}1 & \text{if } p_-,p_+<l,\\ -1 & \text{if } p_-,p_+\ge l,\\ 0 & \text{otherwise.}\end{cases} \tag{4-23}$$

Along with the first two statements in Lemma 2.3, this implies (4-21). □

4.4 Main localization computation

We now prove Theorem B, with each of the two definitions of the structure constants $\mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,t)}$, by summing up the contributions of the $\mathbb{T}$–fixed loci $\mathcal{Z}_\Gamma$ of

$$\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d),$$

with $d\in\bar{\mathbb{Z}}^+$. As outlined in Section 4.1, this will be done by breaking each Γ (and correspondingly each fixed locus $\mathcal{Z}_\Gamma$) at either one special vertex, $v=\bar{\mu}(N)$, or at every special vertex of Γ.

Let Γ be a decorated tree with N marked points as in (4-5). Let

$$\bar{\Gamma}\equiv(\overline{\mathrm{Ver}},\overline{\mathrm{Edg}};\bar{\mu},\bar{\eta})$$

be the core of Γ as in Section 4.1 and $v=\bar{\eta}(N)$. Similarly to Figure 3, we break Γ at the vertex $v\in\overline{\mathrm{Ver}}\subset\mathrm{Ver}$ into strands Γ_e indexed by the set $\mathrm{E}_v(\Gamma)$ of the edges with vertex v in Γ; each strand Γ_e keeps a copy of the vertex v and gains an extra marked point, which will be labeled e, attached at v. For each $e\in\mathrm{E}_v(\Gamma)$, denote by $S_e\subset[N]$

the subset of the original marked points carried by the strand Γ_e. Let

$$\begin{aligned}
\mathrm{E}^*_v(\Gamma) &= \{e \in \mathrm{E}_v(\Gamma) \mid S_e \neq \varnothing\} \sqcup \eta^{-1}(v),\\
\mathrm{E}'_v(\Gamma) &= \{e \in \mathrm{E}_v(\Gamma) \mid S_e = \varnothing\},\\
\bar{\mathrm{E}}_v(\Gamma) &= \mathrm{E}^*_v(\Gamma) \cup \mathrm{E}'_v(\Gamma) \subset \mathrm{E}_v(\Gamma) \sqcup [N].
\end{aligned}$$

Thus, $|\mathrm{E}'_v(\Gamma)| \geq 0$, $|\mathrm{E}^*_v(\Gamma)| \geq 3$ (because $\bar{\Gamma}$ is a trivalent tree) and $\{S_e\}_{e \in \mathrm{E}^*_v(\Gamma)} \in \mathcal{P}_{\mathrm{E}^*_v(\Gamma)}([N])$, where $S_e \equiv \{e\}$ if $e \in \eta^{-1}(v)$.

The fixed locus $\mathcal{Z}_\Gamma$ corresponding to Γ, the restriction of $\boldsymbol{e}(\mathcal{V})$ to $\mathcal{Z}_\Gamma$, and the euler class of the normal bundle of $\mathcal{Z}_\Gamma$ are given by

$$\begin{aligned}
\mathcal{Z}_\Gamma &= \bar{\mathcal{M}}_{0,\bar{\mathrm{E}}_v(\Gamma)} \times \prod_{e \in \mathrm{E}_v(\Gamma)} \mathcal{Z}_{\Gamma_e},\\
\frac{\boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{L}_{\mu(v)})} &= \prod_{e \in \mathrm{E}_v(\Gamma)} \frac{\pi_e^* \boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{L}_{\mu(v)})}, \qquad \text{(4-24)}\\
\frac{\boldsymbol{e}(T_{\mu(v)}\mathbb{P}^{n-1})}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_\Gamma)} &= \prod_{e \in \mathrm{E}_v(\Gamma)} \frac{\boldsymbol{e}(T_{\mu(v)}\mathbb{P}^{n-1})}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})(\hbar'_e - \pi_e^*\psi_e)},
\end{aligned}$$

where $\bar{\mathcal{M}}_{0,\bar{\mathrm{E}}_v(\Gamma)} \approx \bar{\mathcal{M}}_{0,|\mathrm{E}_v(\Gamma)|+|\eta^{-1}(v)|}$ is the moduli space of stable rational $\bar{\mathrm{E}}_v(\Gamma)$–marked curves,

$$\hbar'_e \equiv c_1(L'_e) \in H^*(\bar{\mathcal{M}}_{0,\bar{\mathrm{E}}_v(\Gamma)})$$

is the first Chern class of the universal tangent line bundle for the marked point corresponding to the edge e, and

$$\pi_e \colon \mathcal{Z}_\Gamma \to \mathcal{Z}_{\Gamma_e} \subset \bigcup_{d_e=1}^{\infty} \bar{\mathfrak{M}}_{0,S_e \sqcup \{e\}}(\mathbb{P}^{n-1}, d_e)$$

is the projection map. By [17, Section 27.2],

$$\psi_e|_{\mathcal{Z}_{\Gamma_e}} = \frac{\alpha_{\mu_v(e)} - \alpha_{\mu(v)}}{\mathfrak{d}(e)}.$$

Thus, by [17, Exercise 25.2.8],

$$\text{(4-25)} \qquad \int_{\bar{\mathcal{M}}_{0,\bar{\mathrm{E}}_v(\Gamma)}} \left\{ \left(\prod_{e \in \mathrm{E}_v(\Gamma)} \frac{1}{\hbar'_e - \pi_e^*\psi_e} \right) \left(\prod_{e \in \eta^{-1}(v)} \frac{1}{\hbar_e - \psi_e} \right) \right\}$$

$$= (-1)^{|\mathrm{E}_v(\Gamma)|} \sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^{\bar{\mathrm{E}}_v(\Gamma)}} \int_{\bar{\mathcal{M}}_{0,\bar{\mathrm{E}}_v(\Gamma)}} \bigg\{ \bigg(\prod_{e\in \mathrm{E}_v(\Gamma)} \psi_e^{-b_e-1} \hbar_e'^{b_e} \bigg) \times \bigg(\prod_{e\in\eta^{-1}(v)} \hbar_e^{-b_e-1} \psi_e^{b_e} \bigg) \bigg\}$$

$$= \sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^{\bar{\mathrm{E}}_v(\Gamma)}} \bigg\{ \binom{|\bar{\mathrm{E}}_v(\Gamma)|-3}{\boldsymbol{b}} \bigg(\prod_{e\in \mathrm{E}_v(\Gamma)} \Big(\frac{\alpha_{\mu(v)}-\alpha_{\mu_v(e)}}{\mathfrak{d}(e)} \Big)^{-b_e-1} \bigg) \times \bigg(\prod_{e\in\eta^{-1}(v)} \hbar_e^{-b_e-1} \bigg) \bigg\}.$$

Combining this with (4-24), (3-2) and (4-3), we obtain

$$\text{(4-26)}\quad \frac{\prod_{k\neq\mu(v)}(\alpha_{\mu(v)}-\alpha_k)}{\langle\boldsymbol{a}\rangle\alpha_{\mu(v)}^l} \int_{\mathcal{Z}_\Gamma} \frac{\boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_\Gamma)} \prod_{s=1}^{s=N} \Big(\frac{\mathrm{ev}_s^*\,\phi_{i_s}}{\hbar_s-\psi_s} \Big)$$
$$= \sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^{\bar{\mathrm{E}}_v(\Gamma)}} \bigg\{ \binom{|\bar{\mathrm{E}}_v(\Gamma)|-3}{\boldsymbol{b}} \prod_{s\in\eta^{-1}(v)} \Big(\hbar_s^{-b_s-1} \prod_{k\neq i_s} (\alpha_{\mu(v)}-\alpha_k) \Big) \times \prod_{e\in\mathrm{E}_v(\Gamma)} \bigg(\Big(\frac{\alpha_{\mu(v)}-\alpha_{\mu_v(e)}}{\mathfrak{d}(e)} \Big)^{-b_e-1} \times \int_{\mathcal{Z}_{\Gamma_e}} \frac{\boldsymbol{e}(\mathcal{V})\,\mathrm{ev}_e^*\,\phi_{\mu(v)}}{\langle\boldsymbol{a}\rangle\alpha_{\mu(v)}^l \boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})} \prod_{s\in S_e} \Big(\frac{\mathrm{ev}_s^*\,\phi_{i_s}}{\hbar_s-\psi_s} \Big) \bigg) \bigg\}.$$

The equality holds after dividing the right-hand side by the order of the appropriate group of symmetries; see [17, Section 27.3]. This group is taken into account in the next paragraph.

We now sum up (4-26) over all possibilities for Γ. If $e\in\mathrm{E}_v'(\Gamma)$,

$$\frac{\boldsymbol{e}(\mathcal{V})}{\langle\boldsymbol{a}\rangle\boldsymbol{x}^l} = \boldsymbol{e}(\mathcal{V}'),$$

with $\mathcal{V}'=\mathcal{V}'_{|\Gamma_e|}$ as in (4-7). Thus, in this case, by [31, Section 2.2]

$$\text{(4-27)}\quad \sum_{\Gamma_e} Q^{|\Gamma_e|} \Big(\frac{\alpha_{\mu_v(e)}-\alpha_{\mu(v)}}{\mathfrak{d}(e)} \Big)^{-b_e-1} \int_{\mathcal{Z}_{\Gamma_e}} \frac{\boldsymbol{e}(\mathcal{V})\,\mathrm{ev}_e^*\,\phi_{\mu(v)}}{\langle\boldsymbol{a}\rangle\alpha_{\mu(v)}^l \boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})} \prod_{s\in S_e} \Big(\frac{\mathrm{ev}_s^*\,\phi_{i_s}}{\hbar_s-\psi_s} \Big)$$
$$= \sum_{\Gamma_e} Q^{|\Gamma_e|} \Big(\frac{\alpha_{\mu_v(e)}-\alpha_{\mu(v)}}{\mathfrak{d}(e)} \Big)^{-b_e-1} \int_{\mathcal{Z}_{\Gamma_e}} \frac{\boldsymbol{e}(\mathcal{V}')\,\mathrm{ev}_e^*\,\phi_{\mu(v)}}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}$$
$$= -\mathop{\mathfrak{R}}_{\hbar_e=0} \{ \hbar_e^{-b_e} \tilde{\mathcal{Z}}^*(\hbar_e, \alpha_{\mu(v)}, Q) \},$$

where the sum is taken over all possibilities for the strand Γ_e, leaving the vertex v, with $\mu(v)$ fixed. By a similar reasoning, if $e \in \mathrm{E}_v^*(\Gamma)$,

$$\text{(4-28)}\quad \sum_{\Gamma_e} Q^{|\Gamma_e|}\left(\frac{\alpha_{\mu_v(e)}-\alpha_{\mu(v)}}{\mathfrak{d}(e)}\right)^{-b_e-1}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\boldsymbol{e}(\mathcal{V})\,\mathrm{ev}_e^*\,\phi_{\mu(v)}}{\langle\boldsymbol{a}\rangle\alpha_{\mu(v)}^l\boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\prod_{s\in S_e}\left(\frac{\mathrm{ev}_s^*\,\phi_{i_s}}{\hbar_s-\psi_s}\right)$$
$$=-\frac{1}{\langle\boldsymbol{a}\rangle\alpha_{\mu(v)}^l}\mathop{\mathfrak{R}}_{\hbar_e=0}\{\hbar_e^{-b_e-1}$$
$$\times\mathcal{Z}^*((\hbar_s)_{s\in S_e},\hbar_e,(\boldsymbol{x}_s=\alpha_{i_s})_{s\in S_e},\boldsymbol{x}_e=\alpha_{\mu(v)},Q)\},$$

where the sum is taken over all possibilities for the strand Γ_e, leaving the vertex v, with $\mu(v)$ fixed, $|\Gamma_e|>0$, and carrying the marked points $S_e\subset[N]$, and $\mathcal{Z}^*$ is the positive-degree part of the power series (4-4) with $[N]$ replaced by $S_e\sqcup\{e\}$ if $|S_e|\ge 2$ (for $|S_e|=1$, $\mathcal{Z}^*$ is defined in (4-10)).[14] Finally, if $s\in\eta^{-1}(v)$,

$$\text{(4-29)}\quad \hbar_s^{-b_s-1}\prod_{k\neq i_s}(\alpha_{\mu(v)}-\alpha_k)$$
$$=\frac{(-1)^{b_s}}{\langle\boldsymbol{a}\rangle\alpha_{\mu(v)}^l}\mathop{\mathfrak{R}}_{\hbar_e=0}\{\hbar_e^{-b_s-1}[\![\mathcal{Z}(\hbar_s,\hbar_e,\alpha_{i_s},\alpha_{\mu(v)},Q)]\!]_{Q;0}\}.$$

This corresponds to the strand Γ_e in (4-28) with $|\Gamma_e|=0$ whenever $S_e=\{s\}$ is a single element set. On the other hand, if $|S_e|\ge 2$,

$$\mathop{\mathfrak{R}}_{\hbar_e=0}\{\hbar_e^{-b_e-1}[\![\mathcal{Z}((\hbar_s)_{s\in S_e},\hbar_e,(\boldsymbol{x}_s=\alpha_{i_s})_{s\in S_e},\boldsymbol{x}_e=\alpha_{\mu(v)},Q)]\!]_{Q;0}\}=0.$$

Putting this all together, taking into account the group of symmetries (permutations of the one-marked strands), and summing over all possibilities for $m'\equiv|\mathrm{E}_v'(\Gamma)|$, while keeping

$$m\equiv|\mathrm{E}_v^*(\Gamma)|\ge 3,$$
$$\{S_i\}_{i\in[m]}\equiv\{S_e\}_{e\in\mathrm{E}_v^*(\Gamma)}\in\mathcal{P}_m([N]),$$
$$j\equiv\eta(v)\in[n],$$

[14] By the proof of [17, Chapter 30, (3.21)], the left-hand side of (4-27) summed over Γ_e with $\mathfrak{d}(e)=d$ and $\mu_v(e)=i$ fixed is the residue of $\hbar^{-b}\widetilde{\mathcal{Z}}^*(\hbar,\alpha_{\mu(v)},Q)$ at $\hbar=(\alpha_i-\alpha_{\mu(v)})/d$; see also [31, Section 2.2]. Since $\widetilde{\mathcal{Z}}^*(\hbar,\alpha_{\mu(v)},Q)$ vanishes to second order at $\hbar=\infty$, $\hbar^{-b}\widetilde{\mathcal{Z}}^*(\hbar,\alpha_{\mu(v)},Q)\mathrm{d}\hbar$ has no residue at $\hbar=\infty$ for all $b\in\mathbb{Z}^+$. Since $\widetilde{\mathcal{Z}}^*(\hbar,\alpha_{\mu(v)},Q)\mathrm{d}\hbar$ has poles only at $\hbar=(\alpha_i-\alpha_{\mu(v)})/d$ with $i\in[n]-\mu(v)$ and $d\in\mathbb{Z}^+$, and at $\hbar=0$, (4-27) follows from the residue theorem on S^2. By (4-22), the same reasoning applies to $\hbar^{-1}\mathcal{Z}^*(\hbar_s,\hbar,\alpha_{i_s},\alpha_{\mu(v)},Q)$, giving the $|S_e|=1$ case of (4-28). Since $|\mathrm{E}_v^*(\Gamma)|\ge 3$, $|S_e\sqcup\{e\}|<N$; by Theorem B and induction on N, the same reasoning is applicable to (4-28) for $|S_e|\ge 2$ as well.

fixed, we find that

$$\text{(4-30)}\quad \frac{\prod_{k\neq j}(\alpha_j-\alpha_k)}{\langle\boldsymbol{a}\rangle\alpha_j^l}\sum_{\Gamma}Q^{|\Gamma|}\int_{\mathcal{Z}_\Gamma}\frac{\boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_\Gamma)}\prod_{s=1}^{s=N}\left(\frac{\mathrm{ev}_s^*\phi_{i_s}}{\hbar_s-\psi_s}\right)$$
$$=\sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^m}\left\{\tilde{\mathcal{Z}}_{m-3,\|\boldsymbol{b}\|}(\alpha_j,Q)\prod_{i=1}^{i=m}\left(\frac{1}{\langle\boldsymbol{a}\rangle\alpha_j^l}\frac{(-1)^{b_i}}{b_i!}\right.\right.$$
$$\left.\left.\times\mathfrak{R}_{\hbar_i'=0}\left\{\hbar_i'^{-b_i-1}\mathcal{Z}((\hbar_s)_{s\in S_i},\hbar_i',(\alpha_{i_s})_{s\in S_i},\alpha_j,Q)\right\}\right)\right\}.$$

By (4-17) and the first statement of Lemma B.1, the right-hand side of this expression reduces to

$$\sum_{\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^m}\ \sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{c}\in(\bar{\mathbb{Z}}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=m-3}}\left\{\Psi_{m-3,\boldsymbol{c}}(\alpha_j,Q)\prod_{i=1}^{i=m}\left(\frac{1}{\langle\boldsymbol{a}\rangle\alpha_j^l}\frac{1}{b_i!}\binom{b_i}{b_i''}\right.\right.$$
$$\left.\left.\times\mathfrak{R}_{\hbar_i'=0}\left\{\hbar_i'^{-b_i''-1}\left(-\frac{\zeta(\alpha_j,Q)}{\hbar_i'}\right)^{b_i-b_i''}\mathcal{Z}((\hbar_s)_{s\in S_i},\hbar_i',(\alpha_{i_s})_{s\in S_i},\alpha_j,Q)\right\}\right)\right\}$$

which is equal to

$$\sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{c}\in(\bar{\mathbb{Z}}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=m-3}}\left\{\Psi_{m-3,\boldsymbol{c}}(\alpha_j,Q)\prod_{i=1}^{i=m}\left(\frac{1}{\langle\boldsymbol{a}\rangle\alpha_j^l}\frac{1}{b_i''!}\right.\right.$$
$$\left.\left.\times\mathfrak{R}_{\hbar=0}\left\{\frac{e^{-(\zeta(\alpha_j,Q)/\hbar)}}{\hbar^{b_i''+1}}\mathcal{Z}((\hbar_s)_{s\in S_i},\hbar,(\alpha_{i_s})_{s\in S_i},\alpha_j,Q)\right\}\right)\right\}.$$

Since $m\geq 3$, $|S_i|\leq N-2$ for every $i\in[m]$. Thus, each of the power series $\mathcal{Z}$ appearing in the last expression above is given either by (4-22) or Theorem B with N replaced by $|S_i|+1<N$ (which we can assume to hold by induction). By the last expression for the left-hand side of (4-30), Lemma 4.4, (3-6) with N replaced by $|S_i|+1<N$ whenever $|S_i|\geq 2$, and (4-12), the sum on the left-hand side of (4-30) equals

$$\langle\boldsymbol{a}\rangle\sum_{\substack{\boldsymbol{p}\in\llbracket n\rrbracket^N\\ \boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^N}}\left\{\frac{\underline{\hbar}^{-\boldsymbol{b}}\mathcal{Z}_{\boldsymbol{p}}(\underline{\hbar},\alpha_{i_1\cdots i_N},Q)}{\alpha_j^{l(m-1)}\prod_{k\neq j}(\alpha_j-\alpha_k)}\sum_{\substack{\boldsymbol{d}\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{p}'\in\llbracket n\rrbracket^m\\ \boldsymbol{b}'\in\mathbb{Z}^m}}\ \sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{c}\in(\bar{\mathbb{Z}}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=m-3}}q^{|\boldsymbol{d}|}\Psi_{m-3,\boldsymbol{c}}(\alpha_j,Q)\right.$$
$$\left.\times\prod_{i=1}^{i=m}\frac{\mathcal{C}^{(d_i)}_{\boldsymbol{p}|_{S_i}p_i',\boldsymbol{b}|_{S_i}b_i'}\Psi_{p_i';b_i'+1+b_i''}(\alpha_j,Q)}{b_i''!}\right\}$$

with $\Psi_{p;b}\equiv 0$ if $b<0$. In the two-pointed case (for $|S_i|=1$), the above structure constants are given by

$$\sum_{d=0}^{\infty} q^d \mathcal{C}^{(d)}_{pp',bb'} = \delta_{b+b',-1}(-1)^b \mathcal{C}_{pp'}(Q), \tag{4-31}$$

with $\mathcal{C}_{pp'}$ as in (4-23). Summing over all

$$j\in[n], \quad \boldsymbol{S}\equiv\{S_i\}_{\in[m]}\in\mathcal{P}_m([N]), \quad m\ge 3,$$

and using the residue theorem on S^2, we obtain a recursion for the coefficients $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}$ in Theorem B:

$$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} = -\sum_{\substack{m,d'\in\bar{\mathbb{Z}}^+\\ m\ge 3}} \sum_{\substack{\boldsymbol{S}\in\mathcal{P}_m([N])\\ \boldsymbol{d}\in\mathcal{P}_m(d-d')\\ (\boldsymbol{p}',\boldsymbol{b}')\in\llbracket n\rrbracket^m\times\mathbb{Z}^m}} \sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{c}\in(\bar{\mathbb{Z}}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=m-3}} \mathfrak{R}_{\boldsymbol{x}=0,\infty} \left[\!\!\left[\frac{\Psi_{m-3,\boldsymbol{c}}(\boldsymbol{x},Q)}{\boldsymbol{x}^{l(m-1)}\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k)} \prod_{i=1}^{i=m} \frac{\mathcal{C}^{(d_i)}_{\boldsymbol{p}|_{S_i}p'_i,\boldsymbol{b}|_{S_i}b'_i}\Psi_{p'_i;b'_i+1+b''_i}(\boldsymbol{x},Q)}{b''_i!} \right]\!\!\right]_{q;d'} \tag{4-32}$$

By (3-6) and (3-4), if $\boldsymbol{b}\in(\bar{\mathbb{Z}}^+)^N$ and $d\in\bar{\mathbb{Z}}^+$, the coefficient of

$$q^d\prod_{s=1}^{s=N}((\hbar_s^{-1})^{b_s+1})$$

in the power series $\mathcal{Z}(\underline{\hbar},\underline{\boldsymbol{x}},Q)$ is

$$\begin{aligned}\llbracket\mathcal{Z}(\underline{\hbar},\underline{\boldsymbol{x}},Q)\rrbracket_{\underline{\hbar}^{-1},q;\boldsymbol{b}+\boldsymbol{1},d} &= \sum_{\boldsymbol{p}\in\llbracket n\rrbracket^N}\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}\underline{\boldsymbol{x}}^{\boldsymbol{p}}_s + \sum_{\substack{d'\in\llbracket d\rrbracket\\ \boldsymbol{d}\in\mathcal{P}_N(d-d')}}\sum_{\boldsymbol{p}\in\llbracket n\rrbracket^N}\sum_{\substack{\boldsymbol{b}'\in(\bar{\mathbb{Z}}^+)^N\\ b'_s\le b_s}}\mathcal{C}^{(d')}_{\boldsymbol{p},\boldsymbol{b}'}\\ &\qquad\times\prod_{s=1}^{s=N}\llbracket\mathcal{Z}_{(p_s)}(\hbar_s,\boldsymbol{x}_s,Q)\rrbracket_{\hbar_s^{-1},q;b_s-b'_s,d_s},\end{aligned} \tag{4-33}$$

where $\llbracket\mathcal{Z}_{(p)}(\hbar,\boldsymbol{x},Q)\rrbracket_{\hbar^{-1},q;b,d'}$ is the coefficient of $q^{d'}(\hbar^{-1})^b$ in

$$\mathcal{Z}_{(p)}(\hbar,\boldsymbol{x},Q)\equiv\frac{\mathcal{Z}_p(\hbar,\boldsymbol{x},Q)}{\prod_{r=p_s-l+1}^{n-l-1}I_r(q_s)}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\llbracket\hbar^{-1},Q\rrbracket = H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\llbracket\hbar^{-1},Q\rrbracket.$$

Since $H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$ and $H^*_{\mathbb{T}}(\mathbb{P}^{n-1}_N)$ are free modules over $\mathbb{Q}[\alpha]$ with bases $\{x^p\}_{p\in\llbracket n\rrbracket}$ and $\{\underline{x}^{\boldsymbol{p}}\}_{\boldsymbol{p}\in\llbracket n\rrbracket^N}$, respectively, and

$$\llbracket \mathcal{Z}_{(p)}(\hbar,x,Q)\rrbracket_{\hbar^{-1},q;b,d'} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}), \quad \llbracket \mathcal{Z}(\underline{\hbar},\underline{x},Q)\rrbracket_{\underline{\hbar}^{-1},q;\boldsymbol{b}+\boldsymbol{1},d} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}_N),$$

by (3-4) and (3-3), (4-33) and induction on d imply that $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} \in \mathbb{Q}[\alpha]$ as claimed in Theorem B.

We now confirm (3-7) by induction on N. For $N=2$, (3-7) holds by (4-31), (4-21) and (2-32). On the other hand, by (4-32), (4-18), the second statement in (4-13), the inductive assumption (3-7) and the last two statements in Lemma 4.1, we have:

$$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} \sim -\sum_{\substack{m,d'\in\bar{\mathbb{Z}}^+\\ m\ge 3}} \sum_{\substack{\boldsymbol{S}\in\mathcal{P}_m([N])\\ \boldsymbol{d}\in\mathcal{P}_m(d-d')\\ \boldsymbol{t}\in(\bar{\mathbb{Z}}^+)^m\\ (\boldsymbol{p}',\boldsymbol{b}')\in\llbracket n\rrbracket^m\times\mathbb{Z}^m}} \sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{c}\in(\mathbb{Z}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=m-3}} \hat{\sigma}_n^{|\boldsymbol{t}|} \mathop{\mathfrak{R}}_{x=0,\infty} \left\llbracket \frac{x^{|\boldsymbol{p}'|-|\boldsymbol{b}'|-(l+2)(m-1)+1}}{\prod_{k=1}^{k=n}(x-\alpha_k)} \right.$$
$$\left. \times\, \Phi_{m-3,\boldsymbol{c}}(\boldsymbol{q}) \prod_{i=1}^{i=m} \left(\mathrm{c}^{(d_i,t_i)}_{\boldsymbol{p}|_{S_i}p'_i,\boldsymbol{b}|_{S_i}b'_i} \frac{I_0(\boldsymbol{q})^2\Phi_{p'_i;b'_i+1+b''_i}(\boldsymbol{q})}{b''_i!L(\boldsymbol{q})^{\delta_{0\nu_{\boldsymbol{a}}}n}\Phi_0(\boldsymbol{q})} \right) \right\rrbracket_{q;d'}$$

Since $\boldsymbol{q}=q/x^{\nu_{\boldsymbol{a}}}$, by the last statement of Lemma 4.1 the negative of the expression on the last line is equivalent to

$$\left\llbracket \Phi_{m-3,\boldsymbol{c}}(q) \prod_{i=1}^{i=m} \left(\mathrm{c}^{(d_i,t_i)}_{\boldsymbol{p}|_{S_i}p'_i,\boldsymbol{b}|_{S_i}b'_i} \frac{I_0(q)^2\Phi_{p'_i;b'_i+1+b''_i}(q)}{b''_i!L(q)^{\delta_{0\nu_{\boldsymbol{a}}}n}\Phi_0(q)} \right) \right\rrbracket_{q;d'} \hat{\sigma}_n^{t'}$$

with $t'\in\mathbb{Z}$ defined by

$$|\boldsymbol{p}'|-|\boldsymbol{b}'|-(l+2)(m-1)+1-\nu_{\boldsymbol{a}}d' = n-1+nt' \Leftrightarrow (\boldsymbol{p}',\boldsymbol{b}')\in\mathcal{S}_m(d',t');$$

if such an integer t' does not exist, the above residue is equivalent to 0. Since $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} \in \mathbb{Q}[\alpha]$ by the previous paragraph, we conclude that

$$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} \sim \sum_{t=0}^{\infty} \hat{\sigma}_n^t \sum_{\substack{m,d',t'\in\mathbb{Z}\\ m\ge 3}} \sum_{\substack{\boldsymbol{S}\in\mathcal{P}_m([N])\\ \boldsymbol{d}\in\mathcal{P}_m(d-d')\\ \boldsymbol{t}\in\mathcal{P}_m(t-t')\\ (\boldsymbol{p}',\boldsymbol{b}')\in\mathcal{S}_m(d',t')}} \sum_{\substack{\boldsymbol{b}''\in(\bar{\mathbb{Z}}^+)^m\\ \boldsymbol{c}\in(\mathbb{Z}^+)^\infty\\ |\boldsymbol{b}''|+\|\boldsymbol{c}\|=m-3}} \left(\left(\prod_{i=1}^{i=m} \mathrm{c}^{(d_i,t_i)}_{\boldsymbol{p}|_{S_i}p'_i,\boldsymbol{b}|_{S_i}b'_i} \right) \right.$$
$$\left. \times \left\llbracket \Phi_{m-3,\boldsymbol{c}}(q) \prod_{i=1}^{i=m} \frac{I_0(q)^2\Phi_{p'_i;b'_i+1+b''_i}(q)}{b''_i!L(q)^{\delta_{0\nu_{\boldsymbol{a}}}n}\Phi_0(q)} \right\rrbracket_{q;d'} \right).$$

Comparing this expression with (2-33), we conclude that (3-7) holds.[15]

We next show that (3-7) holds with the coefficients $\mathrm{c}_{\boldsymbol{p},\boldsymbol{b}}^{(d,t)}$ as defined in (2-48). Let Γ be an N–marked decorated tree and $\overline{\Gamma}$ its core as before, with a partial ordering $\prec$ as in Section 2.4. This time, we break Γ and $\overline{\mathcal{Z}}_\Gamma$ at all vertices $\overline{\mathrm{Ver}} \subset \mathrm{Ver}$ of $\overline{\Gamma}$, adding a marked point to each of the strands; see Figure 3. There are now three types of strands, (S1)–(S3), described in Section 4.1. Each strand of type (S3) carries one of the original marked points $s \in [N]$ and an added marked point s', which we associate with the element of $\mathrm{E}_v(\Gamma)$ that leaves v in the direction of $\eta(s)$. These strands are thus naturally indexed by the complement of the subset $\eta^{-1}(\overline{\mathrm{Ver}}) \subset [N]$ of the marked points attached to a vertex of the core in Γ. Each strand of type (S2) runs between vertices in $\overline{\mathrm{Ver}} \subset \mathrm{Ver}$ in Γ that are joined by an edge $e = \{v_e^-, v_e^+\}$ in $\overline{\Gamma}$, with $v_e^- \prec v_e^+$. It carries two added marked points, which we label e^- and e^+, attached to the vertices v_e^- and v_e^+, respectively, in the strand Γ_e. We associate the marked point e^- (resp. e^+) with the element of $\mathrm{E}_{v_e^-}(\Gamma)$ (resp. $\mathrm{E}_{v_e^+}(\Gamma)$) that leaves v_e^- (resp. v_e^+) in the directions of v_e^+ (resp. v_e^-). Similarly to the first approach, for each $v \in \overline{\mathrm{Ver}}$, denote by $\mathrm{E}'_v(\Gamma) \subset \mathrm{E}_v(\Gamma)$ the set of one-marked edges at v and set

$$\begin{aligned}\overline{\mathrm{E}}_v(\Gamma) &= \mathrm{E}'_v(\Gamma) \cup \eta^{-1}(v) \cup \mathrm{E}_v(\overline{\Gamma}) \subset \mathrm{E}_v(\Gamma) \sqcup [N],\\ \overline{\mathrm{E}}(\Gamma) &= \bigsqcup_{v \in \overline{\mathrm{Ver}}} \overline{\mathrm{E}}_v(\Gamma).\end{aligned}$$

As before, this set indexes the marked points on the contracted component.

The analogues of the decompositions (4-24) in this case are

$$\mathcal{Z}_\Gamma = \prod_{v \in \overline{\mathrm{Ver}}} \Big(\overline{\mathcal{M}}_{0,\overline{\mathrm{E}}_v(\Gamma)} \times \prod_{e \in \mathrm{E}'_v(\Gamma)} \mathcal{Z}_{\Gamma_e} \Big) \times \prod_{e \in \overline{\mathrm{Edg}}} \mathcal{Z}_{\Gamma_e},$$

$$\frac{\boldsymbol{e}(\mathcal{V})}{\prod_{v \in \overline{\mathrm{Ver}}} \boldsymbol{e}(\mathcal{L}_{\overline{\mu}(v)})} = \prod_{v \in \overline{\mathrm{Ver}}} \prod_{e \in \mathrm{E}'_v(\Gamma)} \frac{\pi_e^* \boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{L}_{\overline{\mu}(v)})} \times \prod_{e \in \overline{\mathrm{Edg}}} \frac{\pi_e^* \boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{L}_{\overline{\mu}(v_e^-)}) \boldsymbol{e}(\mathcal{L}_{\overline{\mu}(v_e^+)})},$$

$$\begin{aligned}\frac{\prod_{v \in \overline{\mathrm{Ver}}} \boldsymbol{e}(T_{\overline{\mu}(v)} \mathbb{P}^{n-1})}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_\Gamma)} &= \prod_{v \in \overline{\mathrm{Ver}}} \Big(\prod_{e \in \mathrm{E}_v(\Gamma)} \frac{\boldsymbol{e}(T_{\overline{\mu}(v)} \mathbb{P}^{n-1})}{\hbar'_e - \pi_e^* \psi_e} \times \prod_{e \in \mathrm{E}'_v(\Gamma)} \frac{1}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})} \Big)\\ &\qquad \times \prod_{e \in \overline{\mathrm{Edg}}} \frac{1}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}.\end{aligned}$$

[15] As can be seen by induction on n, $\mathcal{I}\mathbb{Q}'_\alpha \cap \mathbb{Q}[\alpha] = \mathcal{I}$. Since $\mathcal{C}_{\boldsymbol{p},\boldsymbol{b}}^{(d)}$ is a symmetric function in $\alpha_1, \dots, \alpha_n$, it is even sufficient to check that the symmetric polynomials in $\mathcal{I}\mathbb{Q}'_\alpha \cap \mathbb{Q}[\alpha]$ are contained in $\mathcal{I}$; this is immediate from the algebraic independence of the elementary symmetric functions.

For each $v \in \overline{\mathrm{Ver}}$, (4-25) still applies. The analogue of (4-26), but weighted by the automorphism group, is then

$$\left(\prod_{v\in\overline{\mathrm{Ver}}}\frac{\prod_{k\neq\overline{\mu}(v)}(\alpha_{\overline{\mu}(v)}-\alpha_k)}{\langle\boldsymbol{a}\rangle\alpha_{\overline{\mu}(v)}^{l}}\right)\int_{\mathcal{Z}_\Gamma}\frac{e(\mathcal{V})}{e(\mathcal{N}\mathcal{Z}_\Gamma)}\prod_{s=1}^{s=N}\left(\frac{\mathrm{ev}_s^*\,\phi_{i_s}}{\hbar_s-\psi_s}\right) \tag{4-34}$$
$$= \sum_{\substack{\boldsymbol{b}\in(\overline{\mathbb{Z}}^+)^{\overline{\mathrm{E}}(\Gamma)}\\ |\boldsymbol{b}|_{\overline{\mathrm{E}}_v(\Gamma)}=|\overline{\mathrm{E}}_v(\Gamma)|-3}}\left\{\prod_{v\in\overline{\mathrm{Ver}}}\left(\frac{(|\overline{\mathrm{E}}_v(\Gamma)|-3)!}{|\mathrm{E}'_v(\Gamma)|!}\right.\right.$$
$$\times\prod_{e\in\mathrm{E}'_v(\Gamma)}\left(\frac{1}{b_e!}\left(\frac{\alpha_{\overline{\mu}(v)}-\alpha_{\mu_v(e)}}{\mathfrak{d}(e)}\right)^{-b_e-1}\int_{\mathcal{Z}_{\Gamma_e}}\frac{e(\mathcal{V})\,\mathrm{ev}_e^*\,\phi_{\overline{\mu}(v)}}{\langle\boldsymbol{a}\rangle\alpha_{\overline{\mu}(v)}^{l}e(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\right)$$
$$\times\prod_{s\in\overline{\eta}^{-1}(v)}\left(\frac{1}{b_{s'}!}\left(\frac{\alpha_{\overline{\mu}(v)}-\alpha_{\mu_v(s')}}{\mathfrak{d}(s')}\right)^{-b_{s'}-1}\right.$$
$$\left.\left.\times\int_{\mathcal{Z}_{\Gamma_{s'}}}\frac{e(\mathcal{V})\,\mathrm{ev}_{s'}^*\,\phi_{\overline{\mu}(v)}\,\mathrm{ev}_s^*\,\phi_{i_s}}{\langle\boldsymbol{a}\rangle\alpha_{\overline{\mu}(v)}^{l}e(\mathcal{N}\mathcal{Z}_{\Gamma_{s'}})(\hbar_s-\psi_s)}\right)\right)$$
$$\times\prod_{e\in\overline{\mathrm{Edg}}}\left(\frac{1}{b_{e^-}!b_{e^+}!}\prod_{*=-,+}\left(\frac{\alpha_{\overline{\mu}(v_e^*)}-\alpha_{\mu_{v_e^*}(e^*)}}{\mathfrak{d}(e^*)}\right)^{-b_{e^*}-1}\right.$$
$$\left.\left.\times\int_{\mathcal{Z}_{\Gamma_e}}\frac{e(\mathcal{V})\,\mathrm{ev}_{e^-}^*\,\phi_{\overline{\mu}(v_e^-)}\,\mathrm{ev}_{e^+}^*\,\phi_{\overline{\mu}(v_e^+)}}{\langle\boldsymbol{a}\rangle^2\alpha_{\overline{\mu}(v_e^-)}^{l}\alpha_{\overline{\mu}(v_e^+)}^{l}e(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\right)\right\},$$

where

$$\frac{1}{b_{s'}!}\left(\frac{\alpha_{\overline{\mu}(v)}-\alpha_{\mu_v(s')}}{\mathfrak{d}(s')}\right)^{-b_{s'}-1}\int_{\mathcal{Z}_{\Gamma_{s'}}}\frac{e(\mathcal{V})\,\mathrm{ev}_{s'}^*\,\phi_{\overline{\mu}(v)}\,\mathrm{ev}_s^*\,\phi_{i_s}}{\langle\boldsymbol{a}\rangle\alpha_{\overline{\mu}(v)}^{l}e(\mathcal{N}\mathcal{Z}_{\Gamma_{s'}})(\hbar_s-\psi_s)}$$
$$\equiv\frac{1}{b_s!}(\hbar_s^{-b_s-1}\prod_{k\neq i_s}(\alpha_{\overline{\mu}(v)}-\alpha_k))$$

if $s\in\eta^{-1}(v)$.

For each $v\in\overline{\mathrm{Ver}}$, (4-27) still reduces the summation of the factor on the second line in (4-34) over all possibilities for Γ_e with $e\in\mathrm{E}'_v(\Gamma)$ and for $m'_v\equiv|\mathrm{E}'_v(\Gamma)|$ to $\widetilde{\mathcal{Z}}_{m_v,\|\boldsymbol{b}_v\|}(\alpha_{j_v},Q)$, where

$$m_v\equiv m_v(\overline{\Gamma})=|\overline{\eta}^{-1}(v)|+|\mathrm{E}_v(\overline{\Gamma})|-3,\quad \boldsymbol{b}_v=\boldsymbol{b}|_{\overline{\eta}^{-1}(v)\cup\mathrm{E}_v(\overline{\Gamma})},\quad j_v=\overline{\mu}(v).$$

For each $s\in\overline{\eta}^{-1}(v)$, (4-28) and (4-29) with $v=\overline{\eta}(s)$ and $S_e=\{s\}$ still compute the sum of the factors on the third line in (4-34) over all possibilities for $\Gamma_{s'}$ of positive and zero degree, respectively. By a similar reasoning (see Footnote 14), for each $e\in\overline{\mathrm{Edg}}$

$$\sum_{\Gamma_e}\Biggl(\biggl(\frac{\alpha_{\mu_{v_e^-}(e^-)}-\alpha_{j_{v_e^-}}}{\mathfrak{d}(e^-)}\biggr)^{-b_e^--1}\biggl(\frac{\alpha_{\mu_{v_e^+}(e^+)}-\alpha_{j_{v_e^+}}}{\mathfrak{d}(e^+)}\biggr)^{-b_e^+-1}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\boldsymbol{e}(\mathcal{V})\,\mathrm{ev}_{e^-}^*\phi_{j_{v_e^-}}\,\mathrm{ev}_{e^+}^*\phi_{j_{v_e^+}}}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\Biggr)$$

is equal to

$$\mathop{\mathfrak{R}}_{\hbar_-=0}\Bigl\{\mathop{\mathfrak{R}}_{\hbar_+=0}\{\hbar_-^{-b_e^--1}\hbar_+^{-b_e^+-1}\mathcal{Z}^*(\hbar_-,\hbar_+,\alpha_{j_{v_e^-}},\alpha_{j_{v_e^+}},Q)\}\Bigr\},$$

where the sum is taken over all possibilities for the strand Γ_e between the vertices v_{e^-} and v_{e^+} in Γ with $\mu(v_e^-)=j_{v_e^-}$ and $\mu(v_e^+)=j_{v_e^+}$ fixed. Since

$$\mathop{\mathfrak{R}}_{\hbar_-=0}\left\{\mathop{\mathfrak{R}}_{\hbar_+=0}\left\{\hbar_-^{-b_e^--1}\hbar_+^{-b_e^+-1}\frac{\langle\boldsymbol{a}\rangle\alpha_{j_{e^-}}^l}{\hbar_-+\hbar_+}\sum_{\substack{p_-+p_++r=n-1\\ p_-,p_+,r\ge0}}\hat{\sigma}_r\alpha_{j_{v_e^-}}^{p_-}\alpha_{j_{v_e^+}}^{p_+}\right\}\right\}=0$$

for all $b_e^-,b_e^+\in\overline{\mathbb{Z}}^+$, we can replace $\mathcal{Z}^*$ in the previous expression by $\mathcal{Z}$.

Putting this all together, we obtain a replacement for (4-30), involving products over $v\in\overline{\mathrm{Ver}}$ and $e\in\overline{\mathrm{Edg}}$, which (4-17) and the first statement of Lemma B.1, reduce to

$$\begin{aligned}
&\Biggl(\prod_{v\in\overline{\mathrm{Ver}}}\frac{\prod_{k\ne j_v}(\alpha_{j_v}-\alpha_k)}{\langle\boldsymbol{a}\rangle\alpha_{j_v}^l}\Biggr)\sum_{\Gamma}Q^{|\Gamma|}\int_{\mathcal{Z}_\Gamma}\frac{\boldsymbol{e}(\mathcal{V})}{\boldsymbol{e}(\mathcal{N}\mathcal{Z}_\Gamma)}\prod_{s=1}^{s=N}\biggl(\frac{\mathrm{ev}_s^*\phi_{i_s}}{\hbar_s-\psi_s}\biggr)\\
&=\sum_{\substack{\boldsymbol{b}''\in(\overline{\mathbb{Z}}^+)^N\\ \boldsymbol{b}^-,\boldsymbol{b}^+\in(\overline{\mathbb{Z}}^+)^{\overline{\mathrm{Edg}}}\\ (\boldsymbol{c}_v)_{v\in\overline{\mathrm{Ver}}}\in((\overline{\mathbb{Z}}^+)^\infty)^{\overline{\mathrm{Ver}}}\\ |\boldsymbol{b}''|_{\eta^{-1}(v)}+|\boldsymbol{b}^-|_{\mathrm{E}_v^-(\overline{\Gamma})}+b_{ev}^++\|\boldsymbol{c}_v\|=m_v}}\Biggl\{\prod_{v\in\overline{\mathrm{Ver}}}\Psi_{m_v,\boldsymbol{c}_v}(\alpha_{j_v},Q)\times\prod_{s=1}^{s=N}\Biggl(\frac{1}{\langle\boldsymbol{a}\rangle\alpha_{j_s}^l}\frac{1}{b_s''!}\\
&\qquad\times\mathop{\mathfrak{R}}_{\hbar=0}\biggl\{\frac{e^{-(\zeta(\alpha_{j_s},Q)/\hbar)}}{\hbar^{b_s''+1}}\mathcal{Z}(\hbar_s,\hbar,\alpha_{i_s},\alpha_{j_s},Q)\biggr\}\Biggr)\times\prod_{e\in\overline{\mathrm{Edg}}}\Biggl(\frac{1}{\langle\boldsymbol{a}\rangle^2\alpha_{j_{v_e^-}}^l\alpha_{j_{v_e^+}}^l}\\
&\qquad\times\frac{1}{b_e^-!b_e^+!}\mathop{\mathfrak{R}}_{\hbar_-=0}\Biggl\{\mathop{\mathfrak{R}}_{\hbar_+=0}\Biggl\{\frac{\exp\Bigl(-\frac{\zeta(\alpha_{j_{v_e^-}},Q)}{\hbar_-}-\frac{\zeta(\alpha_{j_{v_e^+}},Q)}{\hbar_+}\Bigr)}{\hbar_-^{b_e^-+1}\hbar_+^{b_e^++1}}\\
&\qquad\times\mathcal{Z}(\hbar_-,\hbar_+,\alpha_{j_{v_e^-}},\alpha_{j_{v_e^+}},Q)\Biggr\}\Biggr\}\Biggr)\Biggr\},
\end{aligned}$$

where $b_{e_{v_0}} \equiv 0$ for the minimal element $v_0 \in \overline{\mathrm{Ver}}$, $j_s = j_{\overline{\mu}(\overline{\eta}(s))}$, and the sum is taken over all possibilities for Γ with the core $\overline{\Gamma} = (\overline{\mathrm{Ver}}, \overline{\mathrm{Edg}}; \overline{\mu}, \overline{\eta})$ fixed. Using Lemma 4.4 and (4-12) to compute the residues, we find that the sum on the left-hand side of the above expression equals

$$\langle \boldsymbol{a}\rangle \sum_{\substack{\boldsymbol{p}\in\llbracket n\rrbracket^N\\ \boldsymbol{b}\in(\overline{\mathbb{Z}}^+)^N}} \Biggl\{ \underline{\hbar}^{-\boldsymbol{b}} \mathcal{Z}_{\boldsymbol{p}}(\underline{\hbar}, \alpha_{i_1\cdots i_N}, Q) \sum_{\substack{\widetilde{\boldsymbol{p}}\in\llbracket n\rrbracket^N\\ \boldsymbol{p}',\widetilde{\boldsymbol{p}}'\in\llbracket n\rrbracket^{\overline{\mathrm{Edg}}}\\ \boldsymbol{b}'\in(\overline{\mathbb{Z}}^+)^{\overline{\mathrm{Edg}}}}} (-1)^{|\boldsymbol{b}|+|\boldsymbol{b}'|} \sum_{\substack{\boldsymbol{b}''\in(\overline{\mathbb{Z}}^+)^N\\ \boldsymbol{b}^-,\boldsymbol{b}^+\in(\overline{\mathbb{Z}}^+)^{\overline{\mathrm{Edg}}}\\ (\boldsymbol{c}_v)_{v\in\overline{\mathrm{Ver}}}\in((\overline{\mathbb{Z}}^+)^\infty)^{\overline{\mathrm{Ver}}}\\ |\boldsymbol{b}''|_{\eta^{-1}(v)}+|\boldsymbol{b}^-|_{\mathrm{E}_v^-(\overline{\Gamma})}+b^+_{e_v}+\|\boldsymbol{c}_v\|=m_v}}$$

$$\prod_{v\in\overline{\mathrm{Ver}}} \frac{\Psi_{m_v,\boldsymbol{c}_v}(\alpha_{j_v}, Q)}{\alpha_{j_v}^{l(m_v+2)} \prod_{k\neq j_v}(\alpha_{j_v}-\alpha_k)} \times \prod_{s=1}^{s=N} \frac{\mathcal{C}_{p_s\widetilde{p}_s}(Q)\Psi_{\widetilde{p}_s;b''_s-b_s}(\alpha_{j_s}, Q)}{b''_s!}$$

$$\times \prod_{e\in\overline{\mathrm{Edg}}} \frac{\mathcal{C}_{p'_e\widetilde{p}'_e}(Q)\Psi_{p'_e;b^+_e-b'_e}(\alpha_{j_{v_e^+}}, Q)\Psi_{\widetilde{p}'_e;b^-_e+1+b'_e}(\alpha_{j_{v_e^-}}, Q)}{b^-_e!\,b^+_e!}\Biggr\}.$$

For each $v \in \overline{\mathrm{Ver}}$, we now sum up the product of the corresponding factors above over all possibilities for $j_v \in [n]$ (which also determines j_s and $j_{v_e^\pm}$ whenever $\eta(s) = v$ and $v_e^\pm = v$). Using the residue theorem on S^2, we now obtain an explicit formula for the coefficients $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}$ in Theorem B:

$$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} = \sum_{\Gamma} \sum_{\boldsymbol{d}\in\mathcal{P}_\Gamma(d)} \sum_{\substack{\widetilde{\boldsymbol{p}}\in\llbracket n\rrbracket^N\\ \boldsymbol{p}',\widetilde{\boldsymbol{p}}'\in\llbracket n\rrbracket^{\mathrm{Edg}}\\ \boldsymbol{b}'\in(\overline{\mathbb{Z}}^+)^{\mathrm{Edg}}}} (-1)^{|\boldsymbol{b}|+|\boldsymbol{b}'|} \sum_{\substack{\boldsymbol{b}''\in(\overline{\mathbb{Z}}^+)^N\\ \boldsymbol{b}^-,\boldsymbol{b}^+\in(\overline{\mathbb{Z}}^+)^{\mathrm{Edg}}\\ (\boldsymbol{c}_v)_{v\in\mathrm{Ver}}\in((\overline{\mathbb{Z}}^+)^\infty)^{\mathrm{Ver}}\\ |\boldsymbol{b}''|_{\eta^{-1}(v)}+|\boldsymbol{b}^-|_{\mathrm{E}_v^-(\Gamma)}+b^+_{e_v}+\|\boldsymbol{c}_v\|=m_v}}$$

$$\prod_{v\in\mathrm{Ver}} (-1) \mathop{\mathfrak{R}}_{\boldsymbol{x}=0,\infty} \Biggl[\!\!\Biggl[\frac{\Psi_{m_v,\boldsymbol{c}_v}(\boldsymbol{x}, Q)}{\boldsymbol{x}^{l(m_v+2)} \prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k)} \times \prod_{s\in\eta^{-1}(v)} \frac{\mathcal{C}_{p_s\widetilde{p}_s}(Q)\Psi_{\widetilde{p}_s;b''_s-b_s}(\boldsymbol{x}, Q)}{b''_s!}$$

$$\times \prod_{e\in\mathrm{E}_v^-(\Gamma)} \frac{\mathcal{C}_{p'_e\widetilde{p}'_e}(Q)\Psi_{\widetilde{p}'_e;b^-_e+1+b'_e}(\boldsymbol{x}, Q)}{b^-_e!} \times \frac{\Psi_{p'_{e_v};b^+_{e_v}-b'_{e_v}}(\boldsymbol{x}, Q)}{b^+_{e_v}!} \Biggr]\!\!\Biggr]_{q;d_v} \tag{4-35}$$

where the outer sum is taken over all N–marked trivalent trees $\Gamma \equiv (\mathrm{Ver}, \mathrm{Edg}; \eta)$ and

$$\frac{\Psi_{p'_{e_{v_0}}; b^+_{e_{v_0}} - b'_{e_{v_0}}}(\boldsymbol{x}, Q)}{b^+_{e_{v_0}}!} \equiv 1$$

for the minimal element $v_0 \in \mathrm{Ver}$. Using (4-18), (4-21), the second statement in (4-13) and the last two statements in Lemma 4.1 as before, we conclude that

$$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}} \sim \sum_{\Gamma} \sum_{\substack{\boldsymbol{d} \in \mathcal{P}_\Gamma(d) \\ \boldsymbol{p}' \in [\![n]\!]^{\mathrm{Edg}}, \boldsymbol{b}' \in (\bar{\mathbb{Z}}^+)^{\mathrm{Edg}}}} (-1)^{|\boldsymbol{b}|+|\boldsymbol{b}'|} \hat{\sigma}_n^{t_{\boldsymbol{p}} + t_{\boldsymbol{p}'} + |\boldsymbol{t}|} \sum_{\substack{\boldsymbol{b}'' \in (\bar{\mathbb{Z}}^+)^N, \boldsymbol{b}^-, \boldsymbol{b}^+ \in (\bar{\mathbb{Z}}^+)^{\mathrm{Edg}} \\ (\boldsymbol{c}_v)_{v \in \mathrm{Ver}} \in ((\bar{\mathbb{Z}}^+)^\infty)^{\mathrm{Ver}} \\ |\boldsymbol{b}''|_{\eta^{-1}(v)} + |\boldsymbol{b}^-|_{\mathrm{E}_v^-(\Gamma)} + b^+_{e_v} + \|\boldsymbol{c}_v\| = m_v}}$$

$$\prod_{v \in \mathrm{Ver}} \Bigg[\!\!\Bigg[\Phi_{m_v, \boldsymbol{c}_v}(q) \times \prod_{s \in \eta^{-1}(v)} \frac{L(q)^{\delta_{0v} \boldsymbol{a} n t_{p_s}} \Phi_{\hat{p}_s; b''_s - b_s}(q)}{b''_s! \Phi_0(q)}$$

$$\times \prod_{e \in \mathrm{E}_v^-(\Gamma)} \frac{L(q)^{\delta_{0v} \boldsymbol{a} n t_{p'_e}} \Phi_{\hat{p}'_e; b^-_e + 1 + b'_e}(q)}{b^-_e! \Phi_0(q)} \times \frac{I_0(q)^2 \Phi_{p'_{e_v}; b^+_{e_v} - b'_{e_v}}(q)}{b^+_{e_v}! L(q)^{\delta_{0v} \boldsymbol{a} n} \Phi_0(q)} \Bigg]\!\!\Bigg]_{q; d_v}$$

with the last fraction above set to 1 for $v = v_0$ and $\boldsymbol{t} \in (\bar{\mathbb{Z}}^+)^{\mathrm{Ver}}$ defined by (2-47); if an integer t_v satisfying (2-47) does not exist for some $v \in \mathrm{Ver}$, the corresponding summand above is defined to be 0. This confirms (3-7) with $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ as defined in Section 2.4 (and describes $\mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}$ with $t \in \mathbb{Z}^+$ as well).

Remark 4.5 The recursion (4-32) and separately the closed formula (4-35) compute the coefficients $\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}$ in (3-6) and thus provide a straightforward algorithm for computing the equivariant N–pointed generating function (3-3). Following the proof of the first statement in Lemma 4.1, the power series $\Psi_{m,\boldsymbol{c}}(\boldsymbol{x}, Q)$ and $\Psi_{p;b}(\boldsymbol{x}, Q)$ can be computed directly from the power series $\Phi_b(\boldsymbol{x}, q)$ appearing in (4-15). The latter can be computed similarly to the power series $\Phi_b(q)$ appearing in Proposition 2.1; see Appendix A. For example, we first find that the power series ξ appearing in (4-15) is given by

$$\xi \in \boldsymbol{x} q \cdot \mathbb{Q}[\alpha, \boldsymbol{x}, \sigma_{n-1}(\boldsymbol{x})^{-1}][\![q]\!], \quad \boldsymbol{x} + \xi'(\boldsymbol{x}, q) = L(\boldsymbol{x}, q),$$

where $'$ denotes $q \frac{\mathrm{d}}{\mathrm{d}q}$ as before; $L(\boldsymbol{x}, q)$ is defined by

$$L(\boldsymbol{x}, q) \in \boldsymbol{x} + \boldsymbol{x}^{|\boldsymbol{a}|} q \cdot \mathbb{Q}[\alpha, \boldsymbol{x}, \sigma_{n-1}(\boldsymbol{x})^{-1}][\![\boldsymbol{x}^{|\boldsymbol{a}|-1} q]\!], \qquad \sigma_n(L(\boldsymbol{x}, q)) - q \boldsymbol{a}^{\boldsymbol{a}} L(\boldsymbol{x}, q)^{|\boldsymbol{a}|} = \sigma_n(\boldsymbol{x}), \tag{4-36}$$

with $\sigma_n(\cdot)$ defined analogously to (4-8); setting $\alpha = 0$ and $\boldsymbol{x} = 1$ above gives (2-2).

We then find that

$$\Phi_0(\boldsymbol{x},q) = \left(\frac{\boldsymbol{x}\cdot\sigma_{n-1}(\boldsymbol{x})}{L(\boldsymbol{x},q)\sigma_{n-1}(L(\boldsymbol{x},q)) - |\boldsymbol{a}|(\sigma_n(L(\boldsymbol{x},q)) - \sigma_n(\boldsymbol{x}))}\right)^{1/2}\left(\frac{L(\boldsymbol{x},q)}{\boldsymbol{x}}\right)^{(l+1)/2};$$

setting $\alpha=0$ and $\boldsymbol{x}=1$ above gives (2-11). This suffices for the $N=3$ case of (3-6).

5 Proof of Theorem 1

In this section we prove the bound of Theorem 1 for $d\in\mathbb{Z}^+$ by considering four separate cases: $|\boldsymbol{a}|>n$ and $|\boldsymbol{a}|\le n$ with $N=1,2,3+$. The first case is fairly straightforward, since there are only finitely many nonzero GW–invariants modulo the string, dilaton and divisor relations [17, page 527]. In the $|\boldsymbol{a}|\le n$ cases, we use explicit mirror formulas. For $N=1,2$, (2-13) and (2-19) reduce Theorem 1 to extracting the coefficients of $w^b q^d$ from the power series $F(w,q)$ and $F_p(w,q)$ defined in (1-4) and (2-18); Corollary 5.3 below presents them in a convenient form. For $N\ge 3$, the coefficients $\boldsymbol{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ in Theorem A must also be suitably bounded. This is done by Proposition 5.4; its proof constitutes most of this section.

We begin by considering the $|\boldsymbol{a}|>n$ case. Let

$$d_{\max}=\max\{d\in\mathbb{Z}\mid(|\boldsymbol{a}|-n)d\le n-4-l\}.$$

If $d>d_{\max}$, the virtual dimension of $\overline{\mathfrak{M}}_{0,0}(X_{\boldsymbol{a}},d)$ is negative, and so all genus 0 degree d GW–invariants vanish. Thus, we can assume that $d_{\max}\in\bar{\mathbb{Z}}^+$. Let $C\in\mathbb{R}^+$ be such that

$$\left|\langle b_1!\tau_{b_1}H^{c_1},\dots,b_N!\tau_{b_N}H^{c_N}\rangle^{X_{\boldsymbol{a}}}_{0,d}\right|\le C$$

whenever $b_s+c_s\ge 2$ for all s or $N\le d_{\max}$; the number of nonzero invariants of this form is finite. Let $b_{\max}$ be the largest of the sums $b_1+\cdots+b_N$ for nonzero invariants of this form. It then follows by induction via the dilaton, string and divisor relations that

$$\begin{aligned}&\Big|\langle b_1!\tau_{b_1}H^{c_1},\dots,b_N!\tau_{b_N}H^{c_N},\underbrace{\tau_0H^1,\dots,\tau_0H^1}_{k_1},\underbrace{\tau_0H^0,\dots,\tau_0H^0}_{k_2},\\&\qquad\qquad\underbrace{\tau_1H^0,\dots,\tau_1H^0}_{k_3}\rangle^{X_{\boldsymbol{a}}}_{0,d}\Big|\\&\qquad\le C(b_{\max}+d_{\max})^{k_1}\cdot\frac{(b_{\max}+k_2)!}{b_{\max}!}\cdot\frac{(N+k_1+k_2+k_3)!}{(N+k_1+k_2)!}\\&\qquad\le C\cdot C'^{k_1}\cdot 2^{b_{\max}+k_2}\cdot(N+k_1+k_2+k_3)!.\end{aligned}$$

This implies the bound in Theorem 1.

In the remainder of this section, we treat the $|\boldsymbol{a}|<n$ cases.

5.1 Outline of proof

By (1-3) and (2-1), the GW–invariant in Theorem 1 is the coefficient of

$$Q^d \boldsymbol{H}^{\boldsymbol{p}} \underline{\hbar}^{-\boldsymbol{b}-\mathbf{1}} \equiv Q^d \prod_{s=1}^{s=N} H_s^{p_s} \hbar_s^{-b_s-1}, \quad \text{where } p_s = n-1-c_s,$$

of the right-hand side of the identity in (2-13) if $N=1$, in (2-19) if $N=2$ and in (2-36) if $N\ge 3$. In particular, we need to bound the growth of the coefficients of

$$e^{-J(q)H/\hbar} H^p \frac{F_p(H/\hbar,q)}{I_{p-l}(q)} \in \mathbb{Q}[H][\![\hbar^{-1},Q]\!], \quad \text{where } qe^{\delta_{0\nu_{\boldsymbol{a}}}J(q)} = Q/H^{\nu_{\boldsymbol{a}}}.$$

By (1-4), (2-14)–(2-18), for every $p\in\mathbb{Z}^+$ there exists $\widehat{F}_p\in\mathbb{Q}(w)[\![q]\!]$ such that

$$e^{-J(q)H/\hbar} H^p \frac{F_p(H/\hbar,q)}{I_{p-l}(q)} = \hbar^p \widehat{F}_p(H/\hbar, Q/\hbar^{\nu_{\boldsymbol{a}}}), \tag{5-1}$$

and the coefficient of each power of q is holomorphic at $w=0$.

If $b_1+c_1=\nu_{\boldsymbol{a}}d+n-3-l$, (1-3), (2-1), (2-13) and (5-1) give

$$\langle \tau_{b_1}H^{c_1}\rangle_{0,d}^{X_{\boldsymbol{a}}} = [\![[\![[\![Z(\hbar_1,H_1,Q)]\!]_{Q;d}]\!]_{\hbar_1^{-1};b_1+1}]\!]_{H_1;p_1} = \langle\boldsymbol{a}\rangle[\![[\![\widehat{F}_l(w,q)]\!]_{q;d}]\!]_{w;p_1},$$

where $p_1=n-1-c_1$ as before. Thus, by Corollary 5.3 below,

$$\begin{aligned} |\langle b_1!\tau_{b_1}H^{c_1}\rangle_{0,d}^{X_{\boldsymbol{a}}}| \le \langle\boldsymbol{a}\rangle C_{\boldsymbol{a}}^d \frac{b_1!}{(\nu_{\boldsymbol{a}}d)!} &\le \langle\boldsymbol{a}\rangle C_{\boldsymbol{a}}^d (n-3-l)! \binom{\nu_{\boldsymbol{a}}d+n-3-l}{\nu_{\boldsymbol{a}}d} \\ &\le (n-3-l)!\langle\boldsymbol{a}\rangle C_{\boldsymbol{a}}^d \cdot 2^{\nu_{\boldsymbol{a}}d+n-3-l}; \end{aligned}$$

this confirms the statement of Theorem 1 for $N=1$.

If $b_1+c_1+b_2+c_2=\nu_{\boldsymbol{a}}d+n-3-l$, (1-3), (2-1), (2-19) and (5-1) give

$$\begin{aligned} &\sum_{\substack{\delta_1+\delta_2=1\\ \delta_1,\delta_2\ge 0}} \langle \tau_{b_1+\delta_1}H^{c_1}, \tau_{b_2+\delta_2}H^{c_2}\rangle_{0,d}^{X_{\boldsymbol{a}}} \\ &\qquad = \sum_{\substack{\delta_1+\delta_2=1\\ \delta_1,\delta_2\ge 0}} [\![[\![[\![Z(\underline{\hbar},\boldsymbol{H},Q)]\!]_{Q;d}]\!]_{\underline{\hbar}^{-1};(b_1+1+\delta_1,b_2+1+\delta_2)}]\!]_{\boldsymbol{H};(p_1,p_2)} \\ &\qquad = \langle\boldsymbol{a}\rangle \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 0\\ \nu_{\boldsymbol{a}}d_s\ge l+1+b_s-p_s}} \prod_{s=1}^{s=2} [\![[\![\widehat{F}_{\nu_{\boldsymbol{a}}d_s+p_s-b_s-1}(w,q)]\!]_{q;d_s}]\!]_{w;p_s}, \end{aligned}$$

with $p_s = n-1-c_s$, $\underline{\hbar} = (\hbar_1, \hbar_2)$ and $\boldsymbol{H} = (H_1, H_2)$. This gives

$$\langle \tau_{b_1+1} H^{c_1}, \tau_{b_2} H^{c_2} \rangle_{0,d}^{X_{\boldsymbol{a}}} = \langle \boldsymbol{a} \rangle \sum_{\substack{b_1'+b_2'=b_1+b_2+2 \\ 0\le b_2'\le b_2}} (-1)^{b_2-b_2'} \sum_{\substack{d_1+d_2=d \\ d_1,d_2\ge 0 \\ \nu_{\boldsymbol{a}} d_s \ge l+b_s'-p_s}} \prod_{s=1}^{s=2} [\![[\![\widehat{F}_{\nu_{\boldsymbol{a}} d_s + p_s - b_s'}(w,q)]\!]_{q;d_s}]\!]_{w;p_s}.$$

Thus, by Corollary 5.3 below,

$$\begin{aligned} \big| \langle (b_1+1)! \tau_{b_1+1} H^{c_1}, b_2! \tau_{b_2} H^{c_2} \rangle_{0,d}^{X_{\boldsymbol{a}}} \big| &\le \langle \boldsymbol{a} \rangle (b_2+1) C_{\boldsymbol{a}}^d \frac{(b_1+1)! b_2!}{(\nu_{\boldsymbol{a}} d)!} \sum_{d_1=0}^{d_1=d} \binom{\nu_{\boldsymbol{a}} d}{\nu_{\boldsymbol{a}} d_1} \\ &\le \langle \boldsymbol{a} \rangle C_{\boldsymbol{a}}^d \cdot (n-1-l)! \binom{\nu_{\boldsymbol{a}} d + n-1-l}{\nu_{\boldsymbol{a}} d} \cdot 2^{\nu_{\boldsymbol{a}} d} \\ &\le (n-1-l)! \langle \boldsymbol{a} \rangle C_{\boldsymbol{a}}^d \cdot 2^{2\nu_{\boldsymbol{a}} d + n-1-l}; \end{aligned}$$

this confirms the statement of Theorem 1 for $N=2$.

Finally, we consider the $N \ge 3$ case. For each $p \in \llbracket n \rrbracket_l$, let

$$\widehat{F}_{(p)}(w,q) = \frac{\widehat{F}_p(w,q)}{\prod_{r=p-l+1}^{n-l-1} I_r(q)}. \tag{5-2}$$

It is sufficient to assume that the tuples $\boldsymbol{b} \equiv (b_s)_{s\in[N]}$ and $\boldsymbol{c} \equiv (c_s)_{s\in[N]}$ in the statement of Theorem 1 satisfy

$$|\boldsymbol{b}| + |\boldsymbol{c}| = \nu_{\boldsymbol{a}} d + n - 4 - l + N, \quad b_s, c_s \ge 0, \quad c_s \le n-1-l.$$

Let $p_s = n-1-c_s$. If $\boldsymbol{d}, \boldsymbol{b}' \in (\overline{\mathbb{Z}}^+)^N$, define

$$\boldsymbol{p}'(\boldsymbol{d}, \boldsymbol{b}') \in (\overline{\mathbb{Z}}^+)^N \quad \text{by} \quad p_s'(\boldsymbol{d}, \boldsymbol{b}') = \nu_{\boldsymbol{a}} d_s + p_s - b_s + b_s'.$$

By (1-3), (2-1), (2-36), (2-20) and (5-1),

$$\langle \tau_{b_1} H^{c_1}, \dots, \tau_{b_N} H^{c_N} \rangle_{0,d}^{X_{\boldsymbol{a}}} = \langle \boldsymbol{a} \rangle \sum_{\substack{0\le d'\le d \\ \boldsymbol{d} \in \mathcal{P}_N(d-d') \\ \boldsymbol{b}' \in (\overline{\mathbb{Z}}^+)^N}} \mathrm{c}_{\boldsymbol{p}'(\boldsymbol{d},\boldsymbol{b}'),\boldsymbol{b}'}^{(d',0)} \prod_{s=1}^{s=N} [\![[\![\widehat{F}_{(p_s'(\boldsymbol{d},\boldsymbol{b}'))}(w,q)]\!]_{q;d_s}]\!]_{w;p_s}; \tag{5-3}$$

the above summand vanishes unless $l \le p'_s(\boldsymbol{d},\boldsymbol{b}') \le n-1$ for all $s\in[N]$. Since $\mathrm{c}^{(d',0)}_{\boldsymbol{p}',\boldsymbol{b}'}=0$ unless $|\boldsymbol{b}'|\le N-3$, Corollary 5.3 and Proposition 5.4 thus give

$$\begin{aligned}
&|\langle b_1!\tau_{b_1}H^{c_1},\dots,b_N!\tau_{b_N}H^{c_N}\rangle^{X_{\boldsymbol{a}}}_{0,d}\\
&\qquad\le \langle\boldsymbol{a}\rangle N!C_{\boldsymbol{a}}^{N+d}\sum_{\substack{0\le d'\le d\\ \boldsymbol{d}\in\mathcal{P}_N(d-d')}}\ \sum_{\substack{\boldsymbol{b}'\in(\overline{\mathbb{Z}}^+)^N\\ |\boldsymbol{b}'|\le N-3\\ b'_s\ge b_s-\nu_{\boldsymbol{a}}d_s-p_s}}\ \prod_{s=1}^{s=N}\left(p_s!\frac{b_s!}{b'_s!(\nu_{\boldsymbol{a}}d_s)!p_s!}\right)\\
&\qquad\le \langle\boldsymbol{a}\rangle N!C_{\boldsymbol{a}}^{N+d}\sum_{\substack{0\le d'\le d\\ \boldsymbol{d}\in\mathcal{P}_N(d-d')}}\ \sum_{\substack{\boldsymbol{b}'\in(\overline{\mathbb{Z}}^+)^N\\ |\boldsymbol{b}'|\le N-3\\ b'_s\ge b_s-\nu_{\boldsymbol{a}}d_s-p_s}}\ \prod_{s=1}^{s=N}(n!3^{b_s})\\
&\qquad\le \langle\boldsymbol{a}\rangle N!C_{\boldsymbol{a}}^{N+d}\cdot(n!)^N3^{\nu_{\boldsymbol{a}}d+n+N}\cdot\binom{d+N}{N}\binom{N-3+N}{N}\\
&\qquad\le N!C_{\boldsymbol{a}}'^{N+d}\cdot 2^{d+N}\cdot 2^{2N-3}.
\end{aligned}$$

This confirms the statement of Theorem 1 for $N\ge 3$.

Remark 5.1 For any nonvanishing summand on the right-hand side of (5-3),

$$p'_s(\boldsymbol{d},\boldsymbol{b}')\le n-1 \Rightarrow b_s+c_s\ge \nu_{\boldsymbol{a}}d_s.$$

Thus, $d_s=0$ if $b_s+c_s<\nu_{\boldsymbol{a}}$. Since the coefficient of q^0 in $\widehat{F}_{(p)}(w,q)$ is w^p, it follows that $p'_s(\boldsymbol{d},\boldsymbol{b}')=p_s$ and $b'_s=b_s$ in such a case. Since $|\boldsymbol{b}'|\le N-3$, this implies Theorem 2.

5.2 Bounds on the coefficients of generating functions

In this section, we obtain the bounds on the coefficients of the series $F_p,\widehat{F}_p\in\mathbb{Q}(w)[\![q]\!]$ defined in (5-1) and (5-2) that are used in the proof of Theorem 1 above.

Lemma 5.2 *There exists $C_{\boldsymbol{a}}\in\mathbb{R}^+$ such that*

$$\big|[\![[\![F_{p'}(w,q)]\!]_{q;d}]\!]_{w;\nu_{\boldsymbol{a}}d-p'+p}\big|\le\frac{C_{\boldsymbol{a}}^d}{(\nu_{\boldsymbol{a}}d)!}$$

for all $p,p'=0,1,\dots,n-1$ and $d\in\overline{\mathbb{Z}}^+$.

Proof By (1-4), (2-14)–(2-16) and (2-18), it is sufficient to show that there exists $C\in\mathbb{R}^+$ such that

$$\big|[\![[\![F_0(w,q)]\!]_{q;d}]\!]_{w;\nu_{\boldsymbol{a}}d+p}\big|,\ \big|[\![[\![F(w,q)]\!]_{q;d}]\!]_{w;\nu_{\boldsymbol{a}}d-l+p}\big|\le\frac{C^d}{(\nu_{\boldsymbol{a}}d)!}$$

for all $p=0,1,\dots,n-1$ and $d\in\overline{\mathbb{Z}}^+$. Both numbers on the left-hand side vanish for $p<l$ (unless $d,p=0$ in the case of the first number). If $l\le p<n$,

$$\begin{aligned}\big|\llbracket\llbracket F(w,q)\rrbracket_{q;d}\rrbracket_{w;\nu_{\boldsymbol a}d-l+p}\big| &= \frac{\prod_{k=1}^{l}(a_kd)!}{(d!)^n}\left|\left\llbracket\frac{\prod_{k=1}^{l}\prod_{r=1}^{a_kd}(1+(a_k/r)w)}{\prod_{r=1}^{d}(1+w/r)^n}\right\rrbracket_{w;p-l}\right|\\ &\le n^{nd}\frac{(|\boldsymbol a|d)!}{(nd)!}\cdot\left\llbracket\frac{(1+|\boldsymbol a|w)^{(|\boldsymbol a|-l)d}}{(1-w)^{(n-l)d}}\right\rrbracket_{w;p-l}\\ &\le \frac{n^{nd}}{(\nu_{\boldsymbol a}d)!}\sum_{\substack{r+s=p-l\\ r,s\ge0}}\binom{(n-l)d+r-1}{r}\binom{(|\boldsymbol a|-l)d}{s}|\boldsymbol a|^s\\ &\le \frac{n^{nd}}{(\nu_{\boldsymbol a}d)!}2^{(n-l)d+p-l}(|\boldsymbol a|+1)^{(|\boldsymbol a|-l)d}.\end{aligned}$$

The first inequality follows from Stirling's formula (see Apostol [1, Section 15.22]),

$$1<\frac{e^d}{\sqrt{2\pi}\,d^{d+\frac12}}d!<e^{1/(8d)}\quad\text{for all } d\in\mathbb{Z}^+; \tag{5-4}$$

the following statement uses the binomial theorem. The desired bound for $F_0(w,q)$ is obtained similarly. □

Corollary 5.3 *There exists $C_{\boldsymbol a}\in\mathbb{R}^+$ such that*

$$\big|\llbracket\llbracket \widehat F_{p'}(w,q)\rrbracket_{q;d}\rrbracket_{w;p}\big|,\ \big|\llbracket\llbracket \widehat F_{(p')}(w,q)\rrbracket_{q;d}\rrbracket_{w;p}\big|\le\frac{C_{\boldsymbol a}^d}{(\nu_{\boldsymbol a}d)!}$$

for all $p,p'=0,1,\dots,n-1$ and $d\in\overline{\mathbb{Z}}^+$.

Proof If $\nu_{\boldsymbol a}\ge2$,

$$\llbracket\llbracket \widehat F_{p'}(w,q)\rrbracket_{q;d}\rrbracket_{w;p}=\llbracket\llbracket F_{p'}(w,q)\rrbracket_{q;d}\rrbracket_{w;\nu_{\boldsymbol a}d-p'+p},$$

and the claim follows immediately from Lemma 5.2. If $\nu_{\boldsymbol a}=1$, by (1-5)

$$\llbracket\llbracket \widehat F_{p'}(w,q)\rrbracket_{q;d}\rrbracket_{w;p}=\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}}\frac{(-\boldsymbol a!)^{d_1}}{d_1!}\llbracket\llbracket F_{p'}(w,q)\rrbracket_{q;d_2}\rrbracket_{w;d_2-p'+p}$$

implying

$$\big|\llbracket\llbracket \widehat F_{p'}(w,q)\rrbracket_{q;d}\rrbracket_{w;p}\big|\le\frac{(\boldsymbol a!+C_{\boldsymbol a})^d}{d!},$$

where $C_{\boldsymbol{a}}$ is as in Lemma 5.2. Finally, suppose $\nu_{\boldsymbol{a}}=0$. Define

$$\widetilde{J}\in Q\cdot\mathbb{Q}[\![Q]\!] \quad\text{by}\quad q=Qe^{\widetilde{J}(Q)}.$$

By Lemma 5.2,

$$\big|[\![I_0(q)]\!]_{q;d}\big|,\big|[\![I_1(q)]\!]_{q;d}\big|,\dots,\big|[\![I_{n-1}(q)]\!]_{q;d}\big|,\big|[\![J(q)]\!]_{q;d}\big|\le C^d$$

implying

$$|[\![\widetilde{J}(q)]\!]_{q;d}|\le C'^d;$$

the last implication follows from the inverse function theorem. Since

$$[\![\widehat{F}_{p'}(w,Q)]\!]_{w;p}=\sum_{\substack{p_1+p_2=p-p'\\ p_1,p_2\ge 0}}\frac{\widetilde{J}(Q)^{p_1}}{p_1!}\frac{[\![F_{p'}(w,q)]\!]_{w;p_2}}{\prod_{r=p-l}^{n-l-1}I_r(q)},$$

the claim again follows from Lemma 5.2. □

5.3 Bounds on the structure constants in Theorem A

In this section, we obtain an upper bound for the coefficients $\mathrm{c}^{(d,0)}_{(\boldsymbol{p},\boldsymbol{b})}$ in Theorem A. This is one of the two key ingredients in the proof of Theorem 1.

Proposition 5.4 *If $n,N\in\mathbb{Z}^+$ with $N\ge 3$ and $\boldsymbol{a}\in(\mathbb{Z}^+)^l$ with $|\boldsymbol{a}|\le n$, there exists $C_{\boldsymbol{a}}\in\mathbb{R}^+$ such that*

$$\big|\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}\big|\le\frac{N!}{\boldsymbol{b}!}C_{\boldsymbol{a}}^{N+d}\quad\text{for all } d\in\overline{\mathbb{Z}}^+,\ \boldsymbol{p}\in[\![n]\!]^N,\ \boldsymbol{b}\in(\overline{\mathbb{Z}}^+)^N.$$

Lemma 5.5 *If $n\in\mathbb{Z}^+$, $a\in(0,n)$ and $L\in 1+q\mathbb{Q}[\![q]\!]$ is defined by*

$$L(q)^n-qL(q)^a=1, \tag{5-5}$$

then there exists $C_a\in\mathbb{R}^+$ such that

$$\left|\left[\!\!\left[\frac{L(q)^{1-n+k}}{(1-q)^{\delta}(a+(n-a)L(q)^n)^{k'}}\right]\!\!\right]_{q;d}\right|\le C_a$$

for all $k,k'\in\overline{\mathbb{Z}}^+$, $k\le 2n^2$, $k'\le 2n+1$, $\delta=0,1$.

Proof Let $\nu=n-a$. We show that (5-5) defines a holomorphic map $q\to L(q)$ on a neighborhood of the closed unit disk $\overline{D}\subset\mathbb{C}$ such that

$$L(q),a+\nu L(q)\ne 0\quad\text{for all } q\in\overline{D}.$$

Thus, the radius of convergence of the Cauchy series around $q=0$ for the holomorphic function

$$q \to \frac{L(q)^k}{(a+\nu L(q)^n)^{k'}}$$

is greater than 1. Let

$$S=\{(q,z)\in\mathbb{C}^2 \mid z^n-qz^a=1\}.$$

Since the differential of the defining equation is surjective for $z\neq 0$, S is a smooth curve in $\mathbb{C}^2$. The projection map $\pi_1\colon S\to\mathbb{C}$ to the first coordinate is an n–fold cover branched at the points $(q,z)\in S$ such that

$$nz^{n-1}-qaz^{a-1}=0 \Rightarrow q=\tfrac{n}{a}z^{\nu} \Rightarrow z^n=-\tfrac{a}{\nu} \\ \Rightarrow |q|=\tfrac{n}{a}\cdot(\tfrac{a}{\nu})^{\nu/n}>(\tfrac{n}{a})^{a/n}>1.$$

Thus, π_1 is an unramified cover of an open neighborhood U of $\bar{D}$, and its restriction to the component of $\pi^{-1}(0)$ containing $(0,1)$ induces a holomorphic map

$$U\to\mathbb{C}, \quad q\to L(q),$$

solving (5-5). It is immediate from (5-5) that $L(q)\neq 0$ for all q, if $a>0$. On the other hand,

$$1+\tfrac{\nu}{n}qL(q)^a=0 \Rightarrow q=-\tfrac{n}{\nu}L(q)^{-a} \Rightarrow L(q)^n=-\tfrac{a}{\nu} \\ \Rightarrow |q|=\tfrac{n}{\nu}\cdot(\tfrac{\nu}{a})^{a/n}>(\tfrac{n}{\nu})^{\nu/n}>1,$$

as claimed. □

Lemma 5.6 *Let $\Phi_0,\Phi_1,\ldots\in\mathbb{Q}[\![q]\!]$ be as in Proposition 2.1. There exists $C_{\boldsymbol{a}}\in\mathbb{R}^+$ such that*

$$\left|\left[\!\!\left[\frac{\Phi_b(q)}{\Phi_0(q)}\right]\!\!\right]_{q;d}\right| \le b!C_{\boldsymbol{a}}^b[\![(1-C_{\boldsymbol{a}}q)^{-b}]\!]_{q;d} \quad \textit{for all } b,d\in\bar{\mathbb{Z}}^+.$$

Proof For $k=1,2,\ldots,n$, define

$$\widetilde{\mathfrak{L}}_k\colon \mathbb{Q}[\![q]\!]\to\mathbb{Q}[\![q]\!] \quad\text{by}\quad \widetilde{\mathfrak{L}}_k(\Phi)=\frac{1}{L(q)^{k-1}\Phi_0(q)(|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n)}\mathfrak{L}_k(\Phi_0\Phi),$$

with $\mathfrak{L}_k$ and Φ_0 given by (2-7) and (2-11), respectively. These differential operators are of the form

$$\widetilde{\mathfrak{L}}_k=\sum_{i=0}^{i=k}\tilde{h}_{k,k-i}(q)D^i \quad \text{with } \tilde{h}_{k,i}\in\mathbb{Q}[\![q]\!]. \tag{5-6}$$

Note that by (2-2),

$$\frac{L'}{L} = \frac{L^n - 1}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n} = \frac{\boldsymbol{a}^{\boldsymbol{a}} q L(q)^{|\boldsymbol{a}|}}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n}. \tag{5-7}$$

We now consider three separate cases.

(1) Suppose $0 < |\boldsymbol{a}| < n$. We show that there exists $C_{\boldsymbol{a}} \in \mathbb{R}^+$ such that

$$\left| \left[\!\!\left[\frac{\Phi_b(q)}{\Phi_0(q)} \right]\!\!\right]_{q;d} \right| \le b! C_{\boldsymbol{a}}^b [\![(1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-b}]\!]_{q;d} \quad \text{for all } b, d \in \overline{\mathbb{Z}}^+. \tag{5-8}$$

By (2-11) and (5-7), for each $j \in \mathbb{Z}^+$ there exists $p_j \in \mathbb{Q}[u]$ such that

$$\frac{D^j \Phi_0}{\Phi_0} = \frac{(L^n - 1) p_j(L^n)}{(|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n)^{2j}} = \frac{\boldsymbol{a}^{\boldsymbol{a}} q L(q)^{|\boldsymbol{a}|} p_j(L^n)}{(|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n)^{2j}}, \quad \deg p_j \le 2j - 1. \tag{5-9}$$

By (2-5), for each $j \in \mathbb{Z}^+$ there exist $p_{m,j} \in \mathbb{Q}[u]$ such that

$$\mathcal{H}_{m,j}(u) = \frac{(u-1) p_{m,j}(u)}{(|\boldsymbol{a}| + \nu_{\boldsymbol{a}} u)^{2j-1}}, \quad \deg p_{m,j} \le 2j - 2,$$

where $\mathcal{H}_{m,j} \in Q(u)$ is the function defined in Section 2.1. Thus, by (2-7) and (5-9), there exist $\widetilde{p}_{k,i} \in \mathbb{Q}[u]$ such that

$$\widetilde{h}_{k,i} = \frac{1}{L^{k-1}} \cdot \frac{(qL^{|\boldsymbol{a}|})^{\delta_{i,k}} \widetilde{p}_{k,i}(L^n)}{(|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L^n)^{2i+1}}, \quad \deg \widetilde{p}_{k,i} \le 2i + 1 - \delta_{i,k}.$$

Let $C \ge 1$ be the maximum of the absolute values of the coefficients of the polynomials $(2i+1)\widetilde{p}_{k,i}$, with $i = 0, 1, \dots, k$ and $k = 2, 3, \dots, n$. Thus,

$$\left| [\![(1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-b} \widetilde{h}_{k,i}(q)]\!]_{q;d} \right| \le C C_{|\boldsymbol{a}|} [\![q^{\delta_{i,k}} (1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-b}]\!]_{q;d} \tag{5-10}$$

for all $2 \le k \le n$, $b \in \mathbb{Z}^+$, where $C_{|\boldsymbol{a}|}$ is as in Lemma 5.5. We show that (5-8) holds with

$$C_{\boldsymbol{a}} = n^2 C C_{|\boldsymbol{a}|} \boldsymbol{a}^{\boldsymbol{a}}.$$

This is indeed the case for $b = 0$. Suppose $b^* \ge 1$ and the bound holds for all $b < b^*$. By (2-10), (5-6), (5-10) and the inductive assumption,

$$\begin{aligned} \left| \left[\!\!\left[D\Big(\frac{\Phi_{b^*}(q)}{\Phi_0(q)}\Big) \right]\!\!\right]_{q;d} \right| &\le \sum_{k=2}^{k=n} \left| \left[\!\!\left[\widetilde{\mathfrak{L}}_k\Big(\frac{\Phi_{b^*-k+1}(q)}{\Phi_0(q)}\Big) \right]\!\!\right]_{q;d} \right| \\ &\le n^2 C C_{|\boldsymbol{a}|} \cdot C_{\boldsymbol{a}}^{b^*-1} b^*! b^* (\boldsymbol{a}^{\boldsymbol{a}})^2 [\![q(1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-b^*-1}]\!]_{q;d}. \end{aligned}$$

Integrating this inequality, we find that (5-8) holds for $b = b^*$ as well.

(2) Suppose next that $|\boldsymbol{a}|=n$. We show that (5-8) still holds. Since $nDL/L=(L^n-1)$ in this case, for each $j\in\mathbb{Z}^+$ there exists $p_j\in\mathbb{Q}[u]$ such that

$$\frac{D^j\Phi_0}{\Phi_0}=(L^n-1)p_j(L^n)\;=\boldsymbol{a}^{\boldsymbol{a}}qL(q)^n p_j(L^n),\quad \deg p_j\le j-1. \tag{5-11}$$

On the other hand, by (2-5) for each $j\in\mathbb{Z}^+$ there exist $p_{m,j}\in\mathbb{Q}[u]$ such that

$$\mathcal{H}_{m,j}(u)=(u-1)p_{m,j}(u),\quad \deg p_{m,j}\le j-1.$$

It follows that there exists $\widetilde{p}_{k,i}\in\mathbb{Q}[u]$ such that

$$\widetilde{h}_{k,i}=\frac{1}{L^{k-1}}(qL(q)^n)^{\delta_{i,k}}\,\widetilde{p}_{k,i}(L^n),\quad \deg\widetilde{p}_{k,i}\le i-\delta_{i,k}. \tag{5-12}$$

Let $C\ge 1$ be the maximum of the absolute values of the coefficients of the polynomials $(i+1)\widetilde{p}_{k,i}$, with $i=0,1,\dots,k$ and $k=2,3,\dots,n$. Thus,

$$\big|[\![(1-\boldsymbol{a}^{\boldsymbol{a}}q)^{-b}\widetilde{h}_{k,i}(q)]\!]_{q;d}\big|\le C[\![q^{\delta_{i,k}}(1-\boldsymbol{a}^{\boldsymbol{a}}q)^{-b-k}]\!]_{q;d} \tag{5-13}$$

for all $2\le k\le n$, $b\in\mathbb{Z}^+$; see (2-3). The same inductive argument as at the end of (1) now shows that (5-8) holds with $C_{\boldsymbol{a}}=n^2C\boldsymbol{a}^{\boldsymbol{a}}$.

(3) Finally, suppose $|\boldsymbol{a}|=0$, ie $\boldsymbol{a}=(\cdot)$. We show that there exist $C_\varnothing,C_{b,r}\in\mathbb{Q}$ for $b,r\in\overline{\mathbb{Z}}^+$ such that

$$\frac{\Phi_b}{\Phi_0}=\sum_{r=0}^{(n+1)b}C_{b,r}L^{-r},\qquad \sum_{r=0}^{(n+1)b}|C_{b,r}|\le b!C_\varnothing^b\quad\text{for all } b\in\mathbb{Z}^+. \tag{5-14}$$

This implies the claim, since

$$\begin{aligned}|[\![L(q)^{-r}]\!]_{q;d}|\le|[\![L(q)^{-2nb}]\!]_{q;d}|=\left|\binom{-2b}{d}\right|&=\binom{2b+d-1}{d}\\ &\le 2^{2b+d}\le 2^{2b}[\![(1-2q)^b]\!]_{q;d}\end{aligned}$$

for all $r\le 2nb$ and $b\in\mathbb{Z}^+$.

Since $nDL/L=(1-L^{-n})$ in this case, there exist $C_{r;i}^{(j)}\in\mathbb{Q}$ such that

$$D^iL^{-r}=L^{-r}\frac{DL}{L}\sum_{j=0}^{i-1}(r+nj)C_{r;i}^{(j)}L^{-nj},\qquad \sum_{j=0}^{i-1}\big|C_{r;i}^{(j)}\big|\le 2^{i-1}\prod_{j=0}^{i-2}\frac{r+nj}{n}$$

for all $r\in\overline{\mathbb{R}}^+$ and $i\in\mathbb{Z}^+$. On the other hand, by (2-5) for each $j\in\mathbb{Z}^+$ there exist $p_{m,j}\in u\cdot\mathbb{Q}[u]$

$$\mathcal{H}_{m,j}(u)=(u-1)p_{m,j}(1/u),\quad \deg p_{m,j}\le j.$$

It follows that there exist $\widetilde{p}_{k,i} \in \mathbb{Q}[u]$ such that

$$\widetilde{h}_{k,i} = \frac{1}{L^{k-1}} \widetilde{p}_{k,i}(L^{-n}) \left(\frac{DL}{L} \right)^{\delta_{i,k}}, \quad \deg \widetilde{p}_{k,i} \le i - \delta_{i,k}, \quad \text{for all } i \in \bar{\mathbb{Z}}^+. \tag{5-15}$$

Thus, there exist $\widetilde{C}^{(j)}_{r;k} \in \mathbb{Q}$ such that

$$\begin{aligned} \widetilde{\mathfrak{L}}_k L^{-r} &= L^{-r-k} DL \sum_{j=0}^{k-1} (r+nj+1) \widetilde{C}^{(j)}_{r;k} L^{-nj}, \\ \sum_{j=0}^{k-1} \big|\widetilde{C}^{(j)}_{r;k}\big| &\le 2^k C \prod_{j=1}^{k-1} \frac{r+nj}{n}, \end{aligned} \tag{5-16}$$

for all $r \in \bar{\mathbb{R}}^+$ and $k \in \mathbb{Z}^+$, where $C \ge 1$ is the maximum of the absolute values of the coefficients of the polynomials $(k+1)\widetilde{p}_{k,i}$ with $i = 0, 1, \dots, k$ and $k = 1, 2, \dots, n$. We show that (5-14) holds with

$$C_\varnothing = 4(\tfrac{2n+2}{n})^n C.$$

This is indeed the case for $b = 0$. Suppose $b^* \ge 1$ and the claim holds for all $b < b^*$. By (2-10), the inductive assumption and (5-16), there exist $C'_{b^*,r} \in \mathbb{Q}$ such that

$$\begin{aligned} D\left(\frac{\Phi_{b^*}}{\Phi_0} \right) &= -\sum_{k=2}^{k=n} \widetilde{\mathfrak{L}}_k \left(\frac{\Phi_{b^*-k+1}}{\Phi_0} \right) = -\frac{DL}{L} \sum_{r=1}^{(n+1)b^*} r C_{b^*,r} L^{-r}, \\ \sum_{r=1}^{(n+1)b^*} \big|C_{b^*,r}\big| &\le C \sum_{k=2}^{n} \sum_{r=0}^{(n+1)(b^*-1+k)} \sum_{j=0}^{k-1} \big|\widetilde{C}^{(j)}_{r;k}\big| \big|C_{b^*-k+1,r}\big| \\ &\le C \sum_{k=2}^{n} \bigg(2^k \prod_{j=1}^{k-1} \left(\frac{(n+1)(b^*-k+1)+nj}{n} \right) \\ &\qquad \cdot (b^*-k+1)! C_\varnothing^{b^*-k+1} \bigg) \\ &\le 2C C_\varnothing^{b^*-1} \sum_{k=2}^{n} \bigg(\left(2\frac{(n+1)}{n} \right)^{k-1} \prod_{j=1}^{k-1} (b^*-k+1+j) \\ &\qquad \cdot (b^*-k+1)! \bigg) \\ &\le \frac{C_\varnothing^{b^*}}{2} b^*!. \end{aligned} \tag{5-17}$$

Thus, integrating (5-17) and using $\Phi_{b^*} \in q \cdot \mathbb{Q}[\![q]\!]$, we find that (5-14) holds for $b = b^*$ as well. □

Remark 5.7 In the above argument, we use that all coefficients of $(1-q)^{-\alpha}$ are nonnegative (actually positive) if $\alpha > 0$, nondecreasing with α, nondecreasing with d if $\alpha \geq 1$ and at least as large in the absolute values as the coefficients of $(1 \pm q)^{\alpha}$.

Corollary 5.8 *Let $\Phi_{p;b}, \Phi_{m,\boldsymbol{c}}(q) \in \mathbb{Q}[\![q]\!]$ be as in* (2-21) *and* (2-31). *There exists $C_{\boldsymbol{a}} \in \mathbb{R}^+$ such that*

$$\left| \left[\!\!\left[\frac{\Phi_{p;b}(q)}{\Phi_0(q)} \right]\!\!\right]_{q;d} \right| \leq b! C_{\boldsymbol{a}}^b [\![(1-C_{\boldsymbol{a}} q)^{-b-1}]\!]_{q;d},$$

$$|[\![\Phi_{m,\boldsymbol{c}}(q)]\!]_{q;d}| \leq \frac{(m+|\boldsymbol{c}|)!}{|\boldsymbol{c}|!} \binom{|\boldsymbol{c}|}{\boldsymbol{c}} \prod_{r=1}^{\infty} \left(\frac{1}{r+1} \right)^{c_r} C_{\boldsymbol{a}}^{\|\boldsymbol{c}\|} [\![(1-C_{\boldsymbol{a}} q)^{-\|\boldsymbol{c}\|-1}]\!]_{q;d}$$

for all $b, d \in \bar{\mathbb{Z}}^+$, $p \in \lfloor\!\lfloor n \rfloor\!\rfloor$ and $\boldsymbol{c} \in (\bar{\mathbb{Z}}^+)^{\infty}$.

Proof It is sufficient to obtain the first bound for the power series

$$\hat{\Phi}_{p;b} \in \mathbb{Q}[\![q]\!], \quad -l \leq p \leq n-1-l,$$

defined in (2-22). If $0 < |\boldsymbol{a}| < n$, it follows by induction on $b \in \bar{\mathbb{Z}}^+$ and p (from 0 up to $n-1-l$ and down to $--l$) from Lemma 5.6, the $j=1$ case of (5-9) and Lemma 5.5. For $|\boldsymbol{a}| = n$, Lemma 5.2 implies that there exists $C \in \mathbb{R}^+$ such that

$$\left| [\![I_0(q)^{k_0} I_1(q)^{k_1} \cdots I_{n-l}(q)^{k_{n-l}}]\!]_{q;d} \right| \leq C^d \tag{5-18}$$

for all $d \in \bar{\mathbb{Z}}^+$, $k_0, k_1, \dots, k_{n-l} \in \{0, \pm 1\}$. By induction on b and $|p|$ (with the base case being Lemma 5.6) along with (2-3) and the $j=1$ case of (5-11), this implies that

$$\left| \left[\!\!\left[\frac{\hat{\Phi}_{l+p;b}(q)}{\Phi_0(q)} \right]\!\!\right]_{q;d} \right| \leq C_{\boldsymbol{a};p}^b b! [\![(1-C_{\boldsymbol{a};p} q)^{-b-|p|/n}]\!]_{q;d}$$

for all $b, d \in \bar{\mathbb{Z}}^+$, for some $C_{\boldsymbol{a};p} \in \mathbb{R}^+$. The same estimate holds if $|\boldsymbol{a}| = 0$, by Lemma 5.6 and (2-3). The second bound follows directly from Lemma 5.6 and (2-11), along with Lemma 5.5 if $0 < |\boldsymbol{a}| < n$ and (5-18) if $|\boldsymbol{a}| = n$. □

Proof of Proposition 5.4 By Corollary 5.8, the absolute value of each nonzero factor $[\![\cdot]\!]$ in (2-48) is bounded above by

$$\frac{(m_v+|\boldsymbol{c}_v|)!}{|\boldsymbol{c}_v|!} \binom{|\boldsymbol{c}_v|}{\boldsymbol{c}_v} \prod_{r=1}^{\infty} \left(\frac{1}{r+1} \right)^{c_{v;r}} \prod_{s \in \eta^{-1}(v)} \frac{1}{b_s!}$$
$$\times \prod_{e \in \mathrm{E}_v^-(\Gamma)} \frac{(b_e^- + 1 + b_e')!}{b_e^-!} \cdot \frac{(b_{e_v}^+ - b_{e_v}')!}{b_{e_v}^+!} C_{\boldsymbol{a}}^{\Delta_v(\boldsymbol{b}')} [\![(1-C_{\boldsymbol{a}} q)^{-\Delta_v(\boldsymbol{b}')}]\!]_{q;d_v},$$

where

$$\Delta_v(\boldsymbol{b}')=4m_v+8-|\boldsymbol{b}|_{\eta^{-1}(v)}+|\boldsymbol{b}'|_{\mathrm{E}_v^-(\Gamma)}-b'_{e_v}.$$

Thus, by (2-46), the absolute value of each nonzero summand (product of factors over $v\in\mathrm{Ver}$) in (2-48) is bounded above by

$$\frac{C_{\boldsymbol{a}}^{8N}[\![(1-C_{\boldsymbol{a}}q)^{-8N}]\!]_{q;d}}{\boldsymbol{b}!}\prod_{v\in\mathrm{Ver}}\left(\frac{(m_v+|\boldsymbol{c}_v|)!}{|\boldsymbol{c}_v|!}\binom{|\boldsymbol{c}_v|}{\boldsymbol{c}_v}\prod_{r=1}^{\infty}\left(\frac{1}{r+1}\right)^{c_{v;r}}\right) \times\prod_{e\in\mathrm{Edg}}\frac{(b_e^-+1+b'_e)!(b_e^+-b'_e)!}{b_e^-!b_e^+!}.$$

Note that

$$\sum_{b_e^-+b_e^+=b_e^{\pm}}\frac{(b_e^-+1+b'_e)!(b_e^+-b'_e)!}{b_e^-!b_e^+!} \le\sum_{b_e^-+b_e^+=b_e^{\pm}}\frac{(b_e^-+1+b_e^+)!}{b_e^-!b_e^+!}=(b_e^{\pm}+1)\sum_{b_e^-+b_e^+=b_e^{\pm}}\binom{b_e^-+b_e^+}{b_e^-}=(b_e^{\pm}+1)2^{b_e^{\pm}}\le 4^{b_e^{\pm}}.$$

Since each tuple $\boldsymbol{b}''$ is a partition of $N-3-|\mathrm{Edg}|-|\boldsymbol{b}^-|-|\boldsymbol{b}^+|-\|\boldsymbol{c}\|$ into N ordered parts, where

$$\|\boldsymbol{c}\|=\sum_{v\in\mathrm{Ver}}\|\boldsymbol{c}_v\|,$$

the number of such tuples with $|\boldsymbol{b}^-|+|\boldsymbol{b}^+|$ and $\|\boldsymbol{c}\|$ fixed is at most

$$\binom{N-3-|\mathrm{Edg}|-|\boldsymbol{b}^-|-|\boldsymbol{b}^+|-\|\boldsymbol{c}\|+N-1}{N-1}\le 2^{2(N-2)-|\boldsymbol{b}^-|-|\boldsymbol{b}^+|-\|\boldsymbol{c}\|}.$$

Thus, the absolute value of the sum in (2-48) with Γ, $(\boldsymbol{p}',\boldsymbol{b}',\boldsymbol{t})$ and $\boldsymbol{c}$ fixed is bounded above by

$$\frac{C_{\boldsymbol{a}}'^{8N}[\![(1-C_{\boldsymbol{a}}q)^{-8N}]\!]_{q;d}}{\boldsymbol{b}!}2^{-\|\boldsymbol{c}\|}\prod_{v\in\mathrm{Ver}}\left(\frac{(m_v+|\boldsymbol{c}_v|)!}{|\boldsymbol{c}_v|!}\binom{|\boldsymbol{c}_v|}{\boldsymbol{c}_v}\prod_{r=1}^{\infty}\left(\frac{1}{r+1}\right)^{c_{v;r}}\right).$$

Since $|1-2\ln 2|<1$, by the binomial theorem

$$\begin{aligned}\sum_{(\boldsymbol{c})_{v\in\mathrm{Ver}}\in((\bar{\mathbb{Z}}^+)^{\infty})^{\mathrm{Ver}}}&2^{-\|\boldsymbol{c}\|}\prod_{v\in\mathrm{Ver}}\left(\frac{(m_v+|\boldsymbol{c}_v|)!}{m_v!}\binom{|\boldsymbol{c}_v|}{\boldsymbol{c}_v}\prod_{r=1}^{\infty}\left(\frac{1}{r+1}\right)^{c_{v;r}}\right)\\ &=\prod_{v\in\mathrm{Ver}}\left(1-\sum_{r=1}^{\infty}\frac{w^r}{r+1}\right)^{-m_v-1}\Big|_{w=1/2}=\left(2+\frac{\ln(1-w)}{w}\right)^{-(N-2)}\Big|_{w=1/2}\\ &=(2(1-\ln 2))^{N-2}.\end{aligned}$$

Since $b'_e \le b_e^+$ for $e \in \mathrm{Edg}$ and nonzero summands in (2-48), $|\boldsymbol{b}'| \le N-3-|\mathrm{Edg}|$. The number of such tuples is

$$\binom{N-3-|\mathrm{Edg}|+|\mathrm{Edg}|}{|\mathrm{Edg}|} \le 2^{N-3}.$$

Thus, the absolute value of the contribution of each trivalent N–marked tree Γ to $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ is bounded above by

$$\frac{\widetilde{C}^N_{\boldsymbol{a}}}{\boldsymbol{b}!}\binom{-8N}{d} C^d_{\boldsymbol{a}} \cdot \prod_{v\in\mathrm{Ver}} m_v! = \frac{\widetilde{C}^N_{\boldsymbol{a}}}{\boldsymbol{b}!}\binom{8N+d-1}{d} C^d_{\boldsymbol{a}} \cdot \prod_{v\in\mathrm{Ver}} m_v!$$
$$\le \frac{\widetilde{C}^N_{\boldsymbol{a}}}{\boldsymbol{b}!} 2^{8N+d} C^d_{\boldsymbol{a}} \cdot \prod_{v\in\mathrm{Ver}} m_v!.$$

Combining this with Lemma 5.10 below, we obtain the claimed bound for $\mathrm{c}^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$. □

Remark 5.9 In the $|\boldsymbol{a}|=0$ case (projective space), we can obtain a bound of the form $(N!/\boldsymbol{b}!)C^{N-3-|\boldsymbol{b}|}$ using the last description of $c^{(d,0)}_{\boldsymbol{p},\boldsymbol{b}}$ in Section 2.4 and (5-14).

Lemma 5.10 *There exist $C \in \mathbb{R}^+$ such that*

$$a_{N-1} \equiv \sum_{\Gamma} \prod_{v\in\mathrm{Ver}} m_v! \le C^N N! \quad \text{for all } N \ge 3,$$

where the sum is taken over all trivalent N–marked trees.

Proof Let $a_1 = 1$ and

$$f(q) = \sum_{N=1}^{\infty} \frac{a_N}{N!} q^N \in \mathbb{Q}[\![q]\!].$$

Considering the vertex of an $(N+1)$–marked tree Γ to which the last marked point is attached, we observe that

$$a_N = \sum_{k=2}^{k=N} \frac{1}{k!} \sum_{(N_1,\dots,N_k)\in\mathcal{P}_k(N)} \binom{N}{N_1,\dots,N_k} (k-2)! a_{N_1}\dots a_{N_k}$$
$$= N! \sum_{k=2}^{k=N} \left(\frac{1}{k-1} - \frac{1}{k}\right) \sum_{(N_1,\dots,N_k)\in\mathcal{P}_k(N)} \frac{a_{N_1}}{N_1!} \cdots \frac{a_{N_k}}{N_k!}.$$

This recursion is equivalent to the condition that

$$f(q) = q + f(q) + (f(q)-1)\sum_{k=1}^{\infty} \frac{f(q)^k}{k} \Leftrightarrow (1-f(q))\ln(1-f(q)) = -q. \tag{5-19}$$

By the inverse function theorem, $f(q)$ is an analytic function on a neighborhood of $q=0$ and so $a_N/N!\le C^N$ for some $C\in\mathbb{R}^+$. □

Remark 5.11 As noticed by the author for small values of N and confirmed in general by P Johnson on *Math Overflow*, $a_{N-1}=(N-2)^{N-2}$. By (5-19),

$$f(q)=1-e^{W(-q)}, \tag{5-20}$$

where $W\in\mathbb{Q}[\![q]\!]$ is the Lambert W function, ie the analytic function on a neighborhood of $0\in\mathbb{C}$ defined by

$$W(q)e^{W(q)}=q,\quad W(0)=0.$$

As can be seen from the Lagrange inversion formula,

$$e^{W(q)}=1+q-\sum_{N=2}^{\infty}\frac{(N-1)^{(N-1)}}{N!}(-q)^N. \tag{5-21}$$

Along with (5-20), this implies the claim.

Appendix A: Existence of asymptotic expansions

In this appendix, we show that power series

$$\mathcal{Y}_0(\hbar,\boldsymbol{x},q)\equiv\sum_{d=0}^{\infty}q^d\frac{\prod\limits_{k=1}^{k=l}\prod\limits_{r=0}^{a_kd-1}(a_k\boldsymbol{x}+r\hbar)}{\prod\limits_{r=1}^{r=d}\left(\prod\limits_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k+r\hbar)-\prod\limits_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k)\right)}\in\mathbb{Q}_\alpha(\boldsymbol{x},\hbar)[\![q]\!] \tag{A-1}$$

admits an expansion of the form (4-15) and then prove Proposition 2.1. The arguments here are motivated by [29, Section 2].

Lemma A.1 *The power series $\mathcal{Y}_0(\hbar,\boldsymbol{x},q)$ admits an expansion of the form*

$$\mathcal{Y}_0(\hbar,\boldsymbol{x},q)=e^{\xi(\boldsymbol{x},q)/\hbar}\sum_{b=0}^{\infty}\Phi_{0;b}(\boldsymbol{x},q)\hbar^b \tag{A-2}$$

with $\xi,\Phi_{0;1},\Phi_{0;2},\ldots\in q\mathbb{Q}_\alpha(\boldsymbol{x})[\![q]\!]$ and $\Phi_{0;0}\in 1+q\mathbb{Q}_\alpha(\boldsymbol{x})[\![q]\!]$.

Proof Since $\mathcal{Y}_0\in 1+q\mathbb{Q}_\alpha(\hbar,\boldsymbol{x})[\![q]\!]$, there is an expansion

$$\ln\mathcal{Y}_0(\hbar,\boldsymbol{x},q)=\sum_{d=1}^{\infty}\sum_{b=b_{\min}(d)}^{\infty}C_{d,b}(\boldsymbol{x})\hbar^bq^d \tag{A-3}$$

around $\hbar=0$, with $C_{d,b}(\boldsymbol{x})\in\mathbb{Q}_\alpha(\boldsymbol{x})$; we can assume that $C_{d,b_{\min}(d)}\neq 0$ if $b_{\min}(d)<0$. The claim of Lemma A.1 is equivalent to the statement $b_{\min}(d)\geq -1$ for all $d\in\mathbb{Z}^+$; in such a case

$$\xi(\boldsymbol{x},q)=\sum_{d=1}^{\infty}C_{d,-1}(\boldsymbol{x})q^d.$$

Suppose instead $b_{\min}(d)<-1$ for some $d\in\mathbb{Z}^+$. Let

$$d^*=\min\{d\in\mathbb{Z}^+\mid b_{\min}(d)<-1\}\geq 1,\quad b^*=b_{\min}(d^*)\leq -2. \tag{A-4}$$

The power series $\mathcal{Y}_0$ satisfies the differential equation

$$\Bigl\{\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k+\hbar D)-q\prod_{k=1}^{l=1}\prod_{r=0}^{a_k-1}(a_k\boldsymbol{x}+a_k\hbar D+r\hbar)\Bigr\}\mathcal{Y}_0(\hbar,\boldsymbol{x},q)=\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k)\cdot\mathcal{Y}_0(\hbar,\boldsymbol{x},q), \tag{A-5}$$

where $D=q\frac{\mathrm{d}}{\mathrm{d}q}$. By (A-3), (A-4) and induction on the number of derivatives taken,

$$\frac{\Bigl\{\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k+\hbar D)\Bigr\}\mathcal{Y}_0(\hbar,\boldsymbol{x},q)}{\prod_{k=1}^{k=n}(\boldsymbol{x}-\alpha_k)\cdot\mathcal{Y}_0(\hbar,\boldsymbol{x},q)}=1+\sum_{k=1}^{k=n}\frac{d^*C_{d^*,b^*}}{\boldsymbol{x}-\alpha_k}\hbar^{b^*+1}q^{d^*}+A(\hbar,\boldsymbol{x},q), \tag{A-6}$$

$$q\frac{\Bigl\{\prod_{k=1}^{l=1}\prod_{r=0}^{a_k-1}(a_k\boldsymbol{x}+a_k\hbar D+r\hbar)\Bigr\}\mathcal{Y}_0(\hbar,\boldsymbol{x},q)}{\prod_{k=1}^{l=1}\prod_{r=0}^{a_k-1}(a_k\boldsymbol{x}+r\hbar)\cdot\mathcal{Y}_0(\hbar,\boldsymbol{x},q)}=B(\hbar,\boldsymbol{x},q),$$

for some

$$A,B\in q\mathbb{Q}_\alpha(\hbar,\boldsymbol{x})_0[\![q]\!]+q^{d^*}\hbar^{b^*+2}\mathbb{Q}_\alpha(\hbar,\boldsymbol{x})_0[\![q]\!]+q^{d^*+1}\mathbb{Q}_\alpha(\hbar,\boldsymbol{x})[\![q]\!],$$

where $\mathbb{Q}_\alpha(\hbar,\boldsymbol{x})_0\subset\mathbb{Q}_\alpha(\hbar,\boldsymbol{x})$ is the subring of rational functions in α, $\hbar$ and $\boldsymbol{x}$ that are regular at $\hbar=0$. Combining (A-5) and (A-6), we conclude that $C_{d^*,b^*}=0$, contrary to the assumption. □

Corollary A.2 *The power series $F_0\in\mathbb{Q}(w)[\![q]\!]$ defined by* (2-15) *admits an asymptotic expansion of the form* (2-9).

Proof The existence of an asymptotic expansion (2-9) is equivalent to the existence of an expansion of the form (4-15) for

$$F_0(\hbar^{-1},q) \equiv \mathcal{Y}_0(\hbar,1,q)|_{\alpha=0}.$$

Thus, Corollary A.2 follows from Lemma A.1. □

Remark A.3 It is possible to give a somewhat different proof of Corollary A.2, without using Lemma A.1, which is more in line with [29]. By [29, Lemma 1.3], an element $H \in \mathcal{P}$ admits an asymptotic expansion (2-9) if $\boldsymbol{M}^k H = H$ for some $k \in \mathbb{Z}^+$. By [26, Lemma 4.1], $\boldsymbol{M}^n F = F$ if $|\boldsymbol{a}| = n$. In the $\nu_{\boldsymbol{a}} > 0$ case, the coefficients $\tilde{\mathrm{c}}_{p,s}^{(d)}$ in (2-18) with $d \geq 1$ and $\nu_{\boldsymbol{a}} d \leq p - s$ are determined by the requirement that the resulting function $F_p(w,q)$ is holomorphic at $w = 0$ with value $1 \in Q[\![q]\!]$; see (2-17). On the other hand, $F_n = F_0$ if these coefficients are given by

$$\sum_{s=0}^{|\boldsymbol{a}|-l} \tilde{\mathrm{c}}_{n,l+s}^{(1)} w^s = -\langle \boldsymbol{a} \rangle \prod_{k=1}^{k=l} \prod_{r=1}^{a_k-1} (a_k w + r), \quad \tilde{\mathrm{c}}_{n,s}^{(d)} = 0 \quad \text{for all } d \geq 2.$$

Since F_0 is holomorphic at $w=0$ with value $1 \in Q[\![q]\!]$, it follows that indeed $F_n = F_0$. The proof of [29, Lemma 1.3] can be adjusted to show that this in turn implies that F_0 admits an asymptotic expansion of the form (2-9).

In the remainder of this appendix, we prove Proposition 2.1. Since $F = \boldsymbol{D}^l F_0$ and F_0 admits an asymptotic expansion of the form (2-9), so does F. The function $F(w,q)$ defined by (1-4) satisfies the ODE

$$\left\{ D_w^n - w^n - q w^{\nu_{\boldsymbol{a}}} \prod_{k=1}^{k=l} \prod_{r=1}^{r=a_k} (a_k D_w + r) \right\} F = 0,$$

where $D_w = w + q\frac{\mathrm{d}}{\mathrm{d}q}$. Therefore, the power series $\xi, \Phi_0, \Phi_1, \ldots$ introduced in Proposition 2.1 satisfy

$$\text{(A-7)} \qquad \left\{ \widetilde{D}_w^n - w^n - q w^{\nu_{\boldsymbol{a}}} \prod_{k=1}^{k=l} \prod_{r=1}^{r=a_k} (a_k \widetilde{D}_w + r) \right\} \sum_{b=0}^{\infty} \Phi_b(q) w^{-b} = 0,$$

where $\widetilde{D}_w = (1 + \xi'(q))w + q\frac{\mathrm{d}}{\mathrm{d}q}$. The resulting equation for the coefficient of w^n gives

$$\text{(A-8)} \qquad \{(1+\xi'(q))^n - 1 - \boldsymbol{a}^{\boldsymbol{a}} q (1+\xi'(q))^{|\boldsymbol{a}|}\} \Phi_0(q) = 0.$$

Since $\Phi_0(0) = 1$, combining (A-8) with the condition $\xi'(0) = 0$ and comparing with (2-2), we obtain the first equation in (2-10).

By the above, $\widetilde{D}_w = L(q)w + q\frac{\mathrm{d}}{\mathrm{d}q}$. Proceeding as in [29, Section 2.4], but using (5-7), we find that

$$\widetilde{D}_w^s = \sum_{k=0}^{k=s}\sum_{i=0}^{i=k}\binom{s}{i}\mathcal{H}_{s-i,k-i}(L^n)(Lw)^{s-k}D^i,$$

where $\mathcal{H}_{m,j}$ are the rational functions defined by (2-5). Thus,

$$L(q)^n\left\{\widetilde{D}_w^n - w^n - qw^{\nu_a}\prod_{k=1}^{k=l}\prod_{r=1}^{r=a_k}(a_k\widetilde{D}_w + r)\right\} = \sum_{k=1}^{n}(Lw)^{n-k}\mathfrak{L}_k,$$

where $\mathfrak{L}_k$ is the differential operator of order k given by (2-7). It follows that the second equation in (2-10) is the coefficient of $(Lw)^{n-1-b}$ in (A-7) multiplied by $L(q)^n$.

Appendix B: Some combinatorics

Lemma B.1 *The following identities hold:*

$$\sum_{\boldsymbol{b}'\in\mathcal{P}_m(b')}\prod_{i=1}^{i=m}\binom{b_i}{b_i'} = \binom{b_1+\dots+b_m}{b'}$$

$$\sum_{b=0}^{\infty}(-1)^b\binom{p}{b}\prod_{t=B-s+1}^{t=B}(t+b) = (-1)^p s!\binom{B}{s-p}$$

$$\sum_{p=0}^{\infty}(-1)^p\binom{m+p}{p}\Psi^p = \frac{1}{(1+\Psi)^{m+1}}$$

for all $m\in\mathbb{Z}^+$, $b_1,\dots,b_m, b'\in\overline{\mathbb{Z}}^+$, $B,p,s\in\overline{\mathbb{Z}}^+$ *and* $m\in\overline{\mathbb{Z}}^+$*, respectively.*

The first two statements of this lemma are proved in [31, Appendix A]. The last statement is a special case of the binomial theorem; here is a direct argument:

$$\begin{aligned}\sum_{p=0}^{\infty}(-1)^p\binom{m+p}{p}\Psi^p &= \frac{1}{m!}\left\{\frac{\mathrm{d}}{\mathrm{d}\Psi}\right\}^m\sum_{p=0}^{\infty}(-1)^p\Psi^{m+p}\\ &= \frac{(-1)^m}{m!}\left\{\frac{\mathrm{d}}{\mathrm{d}\Psi}\right\}^m\sum_{p=0}^{\infty}(-1)^p\Psi^p\\ &= \frac{(-1)^m}{m!}\left\{\frac{\mathrm{d}}{\mathrm{d}\Psi}\right\}^m\frac{1}{1+\Psi} = \frac{1}{(1+\Psi)^{m+1}}.\end{aligned}$$

Lemma B.2 *If* $\zeta, \Psi_0, \Psi_1, \ldots \in Q\mathbb{Q}_\alpha(\hbar)[\![Q]\!]$ *and*

$$1+\mathcal{Z}^*(\hbar,Q)=e^{\zeta(Q)/\hbar}\bigg(1+\sum_{b=0}^{\infty}\Psi_b(Q)\hbar^b\bigg), \tag{B-1}$$

then

$$\sum_{m'=0}^{\infty}\frac{(m'+m)!}{m'!}\sum_{\boldsymbol{b}\in\mathcal{P}_{m'}(m-B+m')}\bigg(\prod_{k=1}^{k=m'}\frac{(-1)^{b_k}}{b_k!}\mathop{\mathfrak{R}}_{\hbar=0}\{\hbar^{-b_k}\mathcal{Z}^*(\hbar,Q)\}\bigg) \tag{B-2}$$
$$=\sum_{\boldsymbol{c}\in(\bar{\mathbb{Z}}^+)^\infty}\Bigg((-1)^{|\boldsymbol{c}|+\|\boldsymbol{c}\|}(m+|\boldsymbol{c}|)!\frac{\zeta(Q)^{B-m+\|\boldsymbol{c}\|}}{(1+\Psi_0(Q))^{m+1}}\binom{B}{m-\|\boldsymbol{c}\|}$$
$$\times\prod_{r=1}^{\infty}\frac{1}{c_r!}\bigg(\frac{\Psi_r(Q)}{(r+1)!(1+\Psi_0(Q))}\bigg)^{c_r}\Bigg)$$

for all $m, B \in \bar{\mathbb{Z}}^+$.

Proof If $\boldsymbol{c} \in (\bar{\mathbb{Z}}^+)^\infty$, let

$$\Psi^{\boldsymbol{c}}=\prod_{r=1}^{\infty}\Psi_r^{c_r}, \quad \omega(\boldsymbol{c})=\prod_{r=1}^{\infty}((r+1)!)^{c_r}.$$

We show that each $\Psi_0^{c_0}\Psi^{\boldsymbol{c}}$, with $c_0 \in \bar{\mathbb{Z}}^+$, enters with the same coefficient on the two sides of (B-2).

For $c_0 \in \bar{\mathbb{Z}}^+$ and $\boldsymbol{c} \in (\bar{\mathbb{Z}}^+)^\infty$, let

$$S(c_0,\boldsymbol{c})=\{(r,j)\in\bar{\mathbb{Z}}^+\times\mathbb{Z}^+ \mid (r,j)\in\{r\}\times[c_r] \text{ for all } r\in\bar{\mathbb{Z}}^+\}.$$

This is a finite set of cardinality $c_0+|\boldsymbol{c}|$. By (B-1), for all $b \in \bar{\mathbb{Z}}^+$

$$\mathop{\mathfrak{R}}_{\hbar=0}\{\hbar^{-b}\mathcal{Z}^*(\hbar,Q)\}=\sum_{r=\max(b-1,0)}^{\infty}\frac{\zeta(Q)^{r+1-b}}{(r+1-b)!}\Psi_r(Q)+\begin{cases}\zeta(Q) & \text{if } b=0,\\ 0 & \text{if } b>0.\end{cases}$$

Thus, for each $\boldsymbol{b} \in (\bar{\mathbb{Z}}^+)^{S(c_0,\boldsymbol{c})}$ and every choice of disjoint subsets $S_0, S_1, \ldots$ of $[m']$, where

$$m'=B-m+|\boldsymbol{b}|,$$

of cardinalities $c_0, c_1, \dots$, the term $\Psi_0^{c_0}\Psi^{\boldsymbol{c}}$ appears in the m'^{th} summand on the left-hand side of (B-2) with the coefficient

$$\text{(B-3)}\quad \frac{(m'+m)!}{m'!}\zeta^{m'-c_0-|\boldsymbol{c}|}\prod_{(r,j)\in S(c_0,\boldsymbol{c})}\Big(\frac{(-1)^{b_{r,j}}}{b_{r,j}!}\cdot\frac{\zeta^{r+1-b_{r,j}}}{(r+1-b_{r,j})!}\Big) = \frac{\zeta^{B-m+\|\boldsymbol{c}\|}}{\omega(\boldsymbol{c})}(-1)^{|\boldsymbol{b}|}\frac{(m'+m)!}{m'!}\prod_{(r,j)\in S(c_0,\boldsymbol{c})}\binom{r+1}{b_{r,j}}.$$ [16]

Since the number of above choices is

$$\binom{m'}{c_0, \boldsymbol{c}, m'-c_0-|\boldsymbol{c}|} \equiv \frac{m'!}{c_0!\boldsymbol{c}!(m'-c_0-|\boldsymbol{c}|)!},$$

it follows that the coefficient of $\Psi_0^{c_0}\Psi^{\boldsymbol{c}}$ on the left-hand side of (B-2) is

$$\text{(B-4)}\quad \frac{\zeta^{B-m+\|\boldsymbol{c}\|}}{\omega(\boldsymbol{c})c_0!\boldsymbol{c}!}\sum_{\boldsymbol{b}\in(\overline{\mathbb{Z}}^+)^{S(c_0,\boldsymbol{c})}}\Bigg((-1)^{|\boldsymbol{b}|}\prod_{t=B-m-c_0-|\boldsymbol{c}|+1}^{t=B}(t+|\boldsymbol{b}|)\prod_{(r,j)\in S(c_0,\boldsymbol{c})}\binom{r+1}{b_{r,j}}\Bigg).$$

If $(c_0,\boldsymbol{c})=(0,\boldsymbol{0})$ and thus $(\overline{\mathbb{Z}}^+)^{S(c_0,\boldsymbol{c})}\equiv\{\boldsymbol{0}\}$, this expression reduces to $m!\binom{B}{m}\zeta^{B-m}$. Otherwise, (B-4) becomes

$$\frac{\zeta^{B-m+\|\boldsymbol{c}\|}}{c_0!\boldsymbol{c}!\omega(\boldsymbol{c})}\sum_{b=0}^{\infty}\Bigg((-1)^b\binom{c_0+|\boldsymbol{c}|+\|\boldsymbol{c}\|}{b}\prod_{t=B-m-c_0-|\boldsymbol{c}|+1}^{t=B}(t+b)\Bigg) = \frac{\zeta^{B-m+\|\boldsymbol{c}\|}}{c_0!\boldsymbol{c}!\omega(\boldsymbol{c})}(-1)^{c_0+|\boldsymbol{c}|+\|\boldsymbol{c}\|}(m+c_0+|\boldsymbol{c}|)!\binom{B}{m-\|\boldsymbol{c}\|},$$

by the first two statements of Lemma B.1. Lemma B.2 now follows from the last statement of Lemma B.1. □

For any $d\in\overline{\mathbb{Z}}^+$ and $t\in\mathbb{Z}$, let

$$\text{(B-5)}\quad \binom{t}{d} = \frac{\prod_{r=0}^{d-1}(t-r)}{d!}.$$

[16]The factors in the m'–fold product in (B-2) that contribute Ψ_r are indexed by the elements of S_r; the j^{th} such factor arises from $\mathfrak{R}_{\hbar=0}\{\hbar^{-b_{r,j}}\mathcal{Z}^*(\hbar,Q)\}$ with $r\ge b_{r,j}-1$. This leaves $m'-c_0-|\boldsymbol{c}|$ factors that contribute $\zeta(Q)$ from $\mathfrak{R}_{\hbar=0}\{\mathcal{Z}^*(\hbar,Q)\}$. The first expression in (B-3) is defined to be 0 if $b_{r,j}>r+1$ for some $(r,j)\in S(c_0,\boldsymbol{c})$.

For $r \in \bar{\mathbb{Z}}^+$ and $\boldsymbol{p} \in (\bar{\mathbb{Z}}^+)^n$, define $w_r \in \mathbb{Q}[\underline{\alpha}]$ and $C_{r;\boldsymbol{p}} \in \mathbb{Q}$ by

$$w_r \equiv \sum_{i=1}^{i=n} \alpha_i^r \equiv \sum_{\boldsymbol{p} \in (\bar{\mathbb{Z}}^+)^n} C_{r;\boldsymbol{p}} \hat{\sigma}_1^{p_1} \hat{\sigma}_2^{p_2} \cdots \hat{\sigma}_n^{p_n}.$$

If $r_1, r_2 \in [n]$ with $r_1 \neq r_2$ and $b_1, b_2 \in \bar{\mathbb{Z}}^+$, let

$$\boldsymbol{p} = (p_1, \dots, p_n), \quad p_r = \begin{cases} b_i & \text{if } r = r_i, \\ 0 & \text{otherwise,} \end{cases} \quad C_{r_1,r_2}^{(b_1,b_2)} = C_{b_1 r_1 + b_2 r_2; \boldsymbol{p}}. \tag{B-6}$$

Thus, $C_{r_1,r_2}^{(b_1,b_2)}$ is the coefficient of $\hat{\sigma}_{r_1}^{b_1} \hat{\sigma}_{r_2}^{b_2}$ in the expansion of $w_{b_1 r_1 + b_2 r_2}$ in terms of products of the modified (by sign) elementary symmetric polynomials $\hat{\sigma}_r$. If $b_1 < 0$ or $b_2 < 0$, set $C_{r_1,r_2}^{(b_1,b_2)} = 0$.

Lemma B.3 *If $r_1, r_2 \in [n]$ with $r_1 \neq r_2$ and $b_1, b_2 \in \bar{\mathbb{Z}}^+$ with $b_1 + b_2 \neq 0$,*

$$C_{r_1,r_2}^{(b_1,b_2)} = \binom{b_1 + b_2 - 1}{b_2} r_1 + \binom{b_1 + b_2 - 1}{b_1} r_2.$$

Proof If $b_1 \in \mathbb{Z}^+$ and $\alpha_1, \dots, \alpha_n$ are the n roots of the polynomial $\alpha^n - \alpha^{n-r_1} = \alpha^{n-r_1}(\alpha^{r_1} - 1)$,

$$C_{r_1,r_2}^{(b_1,0)} = \sum_{i=1}^{i=n} \alpha_i^{b_1 r_1} = r_1 \cdot 1^{b_1} + (n - r_1) \cdot 0^{b_1 r_1} = r_1;$$

thus, the claim holds when either $b_1 = 0$ or $b_2 = 0$. If $b_1, b_2 \in \mathbb{Z}^+$,

$$w_{b_1 r_1 + b_2 r_2} = \sum_{r=1}^{b_1 r_1 + b_2 r_2 - 1} \hat{\sigma}_r w_{b_1 r_1 + b_2 r_2 - r} \;+\; (b_1 r_1 + b_2 r_2) \hat{\sigma}_{b_1 r_1 + b_2 r_2}$$

by Newton's identity; see Artin [2, page 577]. This gives

$$C_{r_1,r_2}^{(b_1,b_2)} = C_{r_1,r_2}^{(b_1-1,b_2)} + C_{r_1,r_2}^{(b_1,b_2-1)} \quad \text{for all } b_1, b_2 \in \mathbb{Z}^+.$$

Along with the $b_1 = 0$ or $b_2 = 0$ case, this implies the claim by induction. □

Lemma B.4 *The power series $L \in 1 + q\mathbb{Q}[\![q]\!]$ defined by (2-2) satisfies*

$$\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol{a}} d + nt}}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L(q)^n} \right]\!\!\right]_{q;d} = (\boldsymbol{a}^{\boldsymbol{a}})^d \binom{d + t - 1}{d} \tag{B-7}$$

for all $d \in \bar{\mathbb{Z}}^+$ and $t \in \mathbb{Z}$.

Proof In the two extremal cases, by (2-3)

$$\frac{nL(q)^{\nu_{\boldsymbol{a}} d + nt}}{|\boldsymbol{a}| + \nu_{\boldsymbol{a}} L(q)^n} = \begin{cases} (1+q)^{d+t-1} & \text{if } |\boldsymbol{a}| = 0, \\ (1 - \boldsymbol{a}^{\boldsymbol{a}} q)^{-t} & \text{if } |\boldsymbol{a}| = n. \end{cases}$$

Thus, the claim in these two cases follows from the binomial theorem; so, we can assume that $0<|\boldsymbol{a}|<n$. Replacing $\boldsymbol{a}^{\boldsymbol{a}}q$ by q in (2-2), we observe that it is sufficient to prove (B-7) with L defined by (2-2) with $\boldsymbol{a}^{\boldsymbol{a}}$ replaced by 1 and $|\boldsymbol{a}|$ by some $a\in\mathbb{Z}^+$ with $a<n$; thus, $\nu_{\boldsymbol{a}}=\nu\equiv n-a$.

With these reductions, for each n^{th} root of unity $\zeta\in\mathbb{C}$ let

$$L_\zeta(q)=\zeta L(\zeta^a q)\in\mathbb{Q}[\![q]\!].$$

Then,

$$\begin{aligned}L_\zeta(q)^n-qL_\zeta(q)^a=1 &\Rightarrow \frac{1}{a+\nu L_\zeta(q)^n}=\frac{L'_\zeta(q)}{qL_\zeta(q)^{a+1}},\\ \zeta^{\nu d+nt}\cdot(\zeta^a q)^d=q^d &\Rightarrow \left[\!\!\left[\frac{nL(q)^{\nu d+nt}}{a+\nu L(q)^n}\right]\!\!\right]_{q;d}=\sum_{\zeta^n=1}\left[\!\!\left[\frac{L_\zeta(q)^{\nu d+nt}}{a+\nu L_\zeta(q)^n}\right]\!\!\right]_{q;d},\end{aligned}$$

where $'$ denotes $q\frac{\mathrm{d}}{\mathrm{d}q}$ as before. Combining these two conclusions, we find that:

$$\left[\!\!\left[\frac{nL(q)^{\nu d+nt}}{a+\nu L(q)^n}\right]\!\!\right]_{q;d}=\sum_{\zeta^n=1}\left[\!\!\left[L_\zeta(q)^{\nu(d+1)+n(t-1)}\frac{L'_\zeta(q)}{L_\zeta(q)}\right]\!\!\right]_{q;d+1}\tag{B-8}$$

If $\nu(d+1)+n(t-1)=0$, this gives

$$\begin{aligned}\left[\!\!\left[\frac{nL(q)^{\nu d+nt}}{a+\nu L(q)^n}\right]\!\!\right]_{q;d}&=(d+1)\sum_{\zeta^n=1}[\![\ln L_\zeta(q)]\!]_{q;d+1}\\ &=(d+1)\left[\!\!\left[\ln\Bigl(\prod_{\zeta^n=1}L_\zeta(q)\Bigr)\right]\!\!\right]_{q;d+1}\\ &=(d+1)\left[\!\!\left[\ln(-1)^{n-1}\right]\!\!\right]_{q;d+1}=0,\end{aligned}$$

since $\{L_\zeta\}_{\zeta^n=1}$ is the set of the roots of $\ell^n-q\ell^a-1=0$. Since $\nu<n$, our assumption on (d,t) implies that $0\le d+t-1<d$, and so the right-hand side of (B-7) also vanishes. If $\nu(d+1)+n(t-1)>0$, (B-8) and Lemma B.3 give

$$\begin{aligned}\left[\!\!\left[\frac{nL(q)^{\nu d+nt}}{a+\nu L(q)^n}\right]\!\!\right]_{q;d}&=\frac{d+1}{\nu(d+1)+n(t-1)}\sum_{\zeta^n=1}[\![L_\zeta(q)^{\nu(d+1)+n(t-1)}]\!]_{q;d+1}\\ &=\frac{d+1}{\nu(d+1)+n(t-1)}C^{(d+1,t-1)}_{\nu,n}=\binom{d+t-1}{d},\end{aligned}$$

as claimed (the last equality holds even if $t\le 0$). If $\nu(d+1)+n(t-1)<0$, (B-8) and Lemma B.3 give

$$\left[\!\!\left[\frac{nL(q)^{\nu d+nt}}{a+\nu L(q)^n}\right]\!\!\right]_{q;d}=\frac{d+1}{\nu(d+1)+n(t-1)}\sum_{\zeta^n=1}\left[\!\!\left[\left(\frac{1}{L_\zeta(q)}\right)^{a(d+1)-n(d+t)}\right]\!\!\right]_{q;d+1}$$
$$=\frac{d+1}{\nu(d+1)+n(t-1)}C_{a,n}^{(d+1,-(d+t))}(-1)^{d+1}=(-1)^d\binom{-t}{d},$$

since $\{1/L_\zeta\}_{\zeta^n=1}$ is the set of the roots of $\ell^n+q\ell^\nu-1=0$; the last equality holds even if $d+t>0$. Since

$$(-1)^d\binom{-t}{d}=\binom{d+t-1}{d},$$

(B-7) holds in this last case as well. □

Corollary B.5 *The power series* $L\in 1+q\mathbb{Q}[\![q]\!]$ *defined by* (2-2) *satisfies*

$$\text{(B-9)}\qquad \left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}d+nt}}{(|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n)^k}\cdot\frac{L'(q)}{L(q)}\right]\!\!\right]_{q;d}=\frac{(\boldsymbol a^{\boldsymbol a})^d}{n^k}\sum_{r=0}^{d-1}\binom{k-1+r}{r}\binom{d-1+t}{d-1-r}\left(-\frac{\nu_{\boldsymbol a}}{n}\right)^r$$

for all $d\in\bar{\mathbb{Z}}^+$ *and* $k,t\in\mathbb{Z}$.

Proof For $d=0$, both sides of (B-9) vanish. By (5-7), the $k=0$ case of (B-9) reduces to Lemma B.4. For $k\neq 0$, by (5-7) and the binomial theorem

$$\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}d+nt}}{(|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n)^k}\cdot\frac{L'(q)}{L(q)}\right]\!\!\right]_{q;d}$$
$$=\boldsymbol a^{\boldsymbol a}\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}(d-1)+n(t+1)}}{|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n}\cdot\frac{1}{(n+\nu_{\boldsymbol a}\boldsymbol a^{\boldsymbol a}qL(q)^{|\boldsymbol a|})^k}\right]\!\!\right]_{q;d-1}$$
$$=\frac{\boldsymbol a^{\boldsymbol a}}{n^k}\sum_{r=0}^{d-1}\binom{-k}{r}\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}(d-1-r)+n(t+1+r)}}{|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n}\right]\!\!\right]_{q;d-1-r}\left(\frac{\nu_{\boldsymbol a}}{n}\boldsymbol a^{\boldsymbol a}\right)^r.$$

The claim now follows from Lemma B.4. □

For $p,d\in\mathbb{Z}$, let $[\![p]\!]_d,[\![\hat p]\!]_d,\tau_d(p),t_d(p)\in\mathbb{Z}$ be as in (2-42). In particular,

$$\text{(B-10)}\qquad (\tau_{d-1}(p)-\tau_d(p),t_d(p))\in\{(0,0),(1,0),(0,1)\},$$

$$\text{(B-11)}\qquad 1-t_1(p)-\tau_0(p)+\tau_1(p)=\begin{cases}1 & \text{if } t_1(p)=0 \text{ and } \tau_0(p)=\tau_1(p),\\ 0 & \text{otherwise.}\end{cases}$$

Let $A=\boldsymbol{a}^{\boldsymbol{a}}$ for the remainder of this section.

Lemma B.6 *For all $d\in\overline{\mathbb{Z}}^{+}$, $p\in\mathbb{Z}$ and $f\colon\mathbb{Z}^2\to\mathbb{R}$,*

$$\text{(B-12)}\qquad \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 0}} \tilde{\mathrm{c}}^{(d_1)}_{[\![p]\!]_{d_2},[\![p]\!]_{d_2}-\nu_{\boldsymbol{a}}d_1}\tilde{\mathrm{c}}^{(d_2)}_{[\![\hat{p}]\!]_{d_2},[\![\hat{p}]\!]_{d_2}-\nu_{\boldsymbol{a}}d_2} f(\tau_{d_2}(p),t_{d_2}(p))$$
$$=\begin{cases} f(\tau_0(p),t_0(p)) & \text{if } d=0,\\ -A(1-\tau_0(p)+\tau_1(p)-t_1(p))f(\tau_1(p),t_1(d)) & \text{if } d=1,\\ 0 & \text{if } d\ge 2.\end{cases}$$

Proof The $d=0$ case of (B-12) is immediate from $\tilde{\mathrm{c}}^{(0)}_{p,s}=\delta_{p,s}$. If $\tau_0(p)=\tau_d(p)$ and $t_d(p)=0$, (B-12) reduces to [27, (2.9)]. In general, let $d_1^*,\dots,d_k^*\in\mathbb{Z}^+$ be such that

$$\tau_0(p)=\tau_{d_1^*-1}(p)>\tau_{d_1^*}(p)=\tau_{d_2^*-1}(p)>\tau_{d_2^*}(p)=\tau_{d_3^*-1}(p)$$
$$>\cdots>\tau_{d_k^*}(p)=\tau_d(p);$$

if $\tau_0(p)=\tau_d(p)$, $k\equiv 0$. Let $d_0^*=0$ and $d_{k+1}^*=d+1$. If $1\le i\le k$, then $[\![p]\!]_{d_i^*-1}<\nu_{\boldsymbol{a}}$, $[\![\hat{p}]\!]_{d_i^*}<l+\nu_{\boldsymbol{a}}$, and so

$$d_{i-1}^*\le d_2<d_i^* \;\Rightarrow\; [\![p]\!]_{d_2}-\nu_{\boldsymbol{a}}(d-d_2)<0 \;\Rightarrow\; \tilde{\mathrm{c}}^{(d-d_2)}_{[\![p]\!]_{d_2},[\![p]\!]_{d_2}-\nu_{\boldsymbol{a}}(d-d_2)}=0,$$
$$d_i^*\le d_2<d_{i+1}^* \;\Rightarrow\; [\![\hat{p}]\!]_{d_2}-\nu_{\boldsymbol{a}}d_2<l \;\Rightarrow\; \tilde{\mathrm{c}}^{(d_2)}_{[\![\hat{p}]\!]_{d_2},[\![\hat{p}]\!]_{d_2}-\nu_{\boldsymbol{a}}d_2}=0.$$

Thus, all summands on the left-hand side of (B-12) vanish if $k\neq 0$. Finally, if $d>0$ and $k=0$, but $t_d(p)=1$, then $[\![p]\!]_d,[\![\hat{p}]\!]_d<l$, and so

$$\tilde{\mathrm{c}}^{(d)}_{[\![\hat{p}]\!]_d,[\![\hat{p}]\!]_d-\nu_{\boldsymbol{a}}d}=0,\qquad [\![p]\!]_{d_2}-\nu_{\boldsymbol{a}}(d-d_2)<l,$$

implying

$$\tilde{\mathrm{c}}^{(d-d_2)}_{[\![p]\!]_{d_2},[\![p]\!]_{d_2}-\nu_{\boldsymbol{a}}(d-d_2)}=0\quad\text{for all } d_2=0,1,\dots,d-1.$$

Thus, all summands on the left-hand side of (B-12) vanish in this case as well. In light of (B-11), this confirms (B-12). □

Lemma B.7 *For all $d\in\overline{\mathbb{Z}}^{+}$ and $p\in\mathbb{Z}$,*

$$\text{(B-13)}\qquad \sum_{\boldsymbol{d}\in\mathcal{P}_4(d)}\Bigg\{\tilde{\mathrm{c}}^{(d_1)}_{[\![p]\!]_{d_2+d_3},[\![p]\!]_{d_2+d_3}-\nu_{\boldsymbol{a}}d_1}\tilde{\mathrm{c}}^{(d_2)}_{[\![\hat{p}]\!]_{d_2+d_3},[\![\hat{p}]\!]_{d_2+d_3}-\nu_{\boldsymbol{a}}d_2}A^{d_3}$$
$$\times\binom{d_3+\tau_{d_2+d_3}(p)-t_{d_2+d_3}(p)}{d_3}\left[\!\!\left[\frac{nL(q)^{\nu_{\boldsymbol{a}}d_4-n\tau_{d_2+d_3}(p)}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\right]\!\!\right]_{q;d_4}\Bigg\}=\delta_{d,0}.$$

Proof The $d=0$ case is clear; so we assume $d>0$. Using Lemma B.6 to sum over $d_1+d_2=d'$ with d' fixed, we find that the left-hand side of B.7 equals

$$\left[\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}d-n\tau_0(p)}}{|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n}\right]\!\right]_{q;d}+\sum_{\substack{d_3+d_4=d\\ 1\le d_3\le d}}A^{d_3}\binom{d_3-1+\tau_{d_3}(p)-t_{d_3}(p)}{d_3-1}\\ \times\frac{d_3\tau_{d_3-1}(p)+(d_3-1)(t_{d_3}(p)-\tau_{d_3}(p))}{d_3}\left[\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}d_4-n\tau_{d_3}(p)}}{|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n}\right]\!\right]_{q;d_4}.$$

By (B-10),

$$\binom{d_3-1+\tau_{d_3}(p)-t_{d_3}(p)}{d_3-1}\frac{d_3\tau_{d_3-1}(p)+(d_3-1)(t_{d_3}(p)-\tau_{d_3}(p))}{d_3}\\ =\binom{d_3-1+\tau_{d_3-1}(p)}{d_3}.$$

It follows that the left-hand side of (B-13) equals

$$A^d\binom{d-1-\tau_0(p)}{d}+A^d\sum_{\substack{d_3+d_4=d\\ 1\le d_3\le d}}\binom{d_3-1+\tau_{d_3-1}(p)}{d_3}\binom{d_4-1-\tau_{d_3}(p)}{d_4};\tag{B-14}$$

see also Lemma B.4. By induction on $s=0,1,\dots,d-1$,

$$\sum_{\substack{d_3+d_4=d\\ d-s\le d_3\le d}}\binom{d_3-1+\tau_{d_3-1}(p)}{d_3}\binom{d_4-1-\tau_{d_3}(p)}{d_4}\\ =(-1)^s\binom{d-1}{s}\binom{d-1-s+\tau_{d-1-s}(p)}{d}.$$

Setting $s=d-1$ in the last identity, we conclude that the sum in (B-14) vanishes. □

Corollary B.8 *For all $d\in\overline{\mathbb Z}^+$, $p,t\in\mathbb Z$ and $f\in\mathbb R[\![q]\!]$,*

$$\sum_{\boldsymbol d\in\mathcal P_4(d)}\left\{\widetilde{\mathrm c}^{(d_1)}_{[\![p]\!]_{d_2+d_3},[\![p]\!]_{d_2+d_3}-\nu_{\boldsymbol a}d_1}\widetilde{\mathrm c}^{(d_2)}_{[\![\hat p]\!]_{d_2+d_3},[\![\hat p]\!]_{d_2+d_3}-\nu_{\boldsymbol a}d_2}(\boldsymbol a^{\boldsymbol a})^{d_3}\right.\\ \binom{d_3+\tau_{d_2+d_3}(p)-t_{d_2+d_3}(p)-t}{d_3}\\ \left.\times\left[\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}d_4+n(t-\tau_{d_2+d_3}(p))}}{|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n}f(q)\right]\!\right]_{q;d_4}\right\}\left[\!\left[\frac{nL(q)^{\nu_{\boldsymbol a}d}f(q)}{|\boldsymbol a|+\nu_{\boldsymbol a}L(q)^n}\right]\!\right]_{q;d}.$$

Proof Replacing p with $p-nt$, we can assume that $t=0$. The $d=0$ case is clear; so we assume that $d\ge 1$ and that the above identity holds with d replaced by any nonnegative integer $d'<d$. The left-hand side of this identity is given by

$$\mathrm{LHS}_d=\sum_{\substack{d'+d''=d\\ d',d''\ge 0}} C_{d',d''}[\![f(q)]\!]_{q;d''},$$

where

$$C_{d',d''}=\sum_{\boldsymbol{d}\in\mathcal{P}_4(d')}\bigg\{\widetilde{\mathrm{c}}^{(d_1)}_{[\![p]\!]_{d_2+d_3},[\![p]\!]_{d_2+d_3}-\nu_{\boldsymbol{a}}d_1}\widetilde{\mathrm{c}}^{(d_2)}_{[\![\hat{p}]\!]_{d_2+d_3},[\![\hat{p}]\!]_{d_2+d_3}-\nu_{\boldsymbol{a}}d_2}(\boldsymbol{a}^{\boldsymbol{a}})^{d_3} \\ \times\binom{d_3+\tau_{d_2+d_3}(p)-t_{d_2+d_3}(p)}{d_3}\bigg[\!\!\bigg[\frac{nL(q)^{\nu_{\boldsymbol{a}}(d_4+d'')-n\tau_{d_2+d_3}(p)}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\bigg]\!\!\bigg]_{q;d_4}\bigg\}.$$

So, it is sufficient to show that

$$C_{d',d''}=\bigg[\!\!\bigg[\frac{nL(q)^{\nu_{\boldsymbol{a}}d}}{|\boldsymbol{a}|+\nu_{\boldsymbol{a}}L(q)^n}\bigg]\!\!\bigg]_{q;d'}$$

for $d'=0,1,\dots,d$. For $d'<d$, this is the case by the inductive assumption applied with $f=L^{\nu_{\boldsymbol{a}}d''}$. For $d'=d$, this is the case by Lemmas B.7 and B.4. □

Appendix C: Summary of important notation

$\boldsymbol{a}$	$(a_1,\dots,a_l)$
$\lvert\boldsymbol{a}\rvert$, $\langle\boldsymbol{a}\rangle$, etc.	$a_1+\dots+a_l$, $a_1\cdots a_k$: page 1036
$\mathrm{c}^{(d,t)}_{\boldsymbol{p},\boldsymbol{b}}$	main nonequivariant structure coefficients: (2-34), (2-36), (2-48)
$\mathcal{C}^{(d)}_{\boldsymbol{p},\boldsymbol{b}}$	main equivariant structure coefficients: (3-6), (4-32), (4-35)
$\Delta_{\boldsymbol{p}}$	normalized products of F_p: (2-20)
F	hypergeometric series (1-4)
F_p	linear combinations of derivatives of F: (2-16), (2-18)
$I_c(q)$	$w=0$ reduction of derivatives of F: (1-5)
$J(q)$	mirror map power series: (1-5)
$[m]$	$\{1,2,\dots,m\}$
$[\![m]\!]$, $[\![m]\!]_l$	$\{0,1,\dots,m-1\}$, $\{l,l+1,\dots,m-1\}$
$L(q)$	power series in q related to F: (2-2)
$L(\boldsymbol{x},q)$	equivariant version of F: (4-36)
$\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^{n-1},d)$	moduli of stable N–marked genus 0 degree d morphisms to $\mathbb{CP}^{n-1}$

$\mathbb{P}^{n-1}_N$	$(\mathbb{P}^{n-1})^N$
$\mathcal{P}_N(d)$	set of ordered partitions of $d \in \bar{\mathbb{Z}}^+$ into nonnegative integers: (1-11)
$\mathcal{P}_m([N])$	set of partitions of $[N]$ into m nonempty subsets: page 1047
$\mathbb{P}^{n-1}$	$\mathbb{CP}^{n-1}$
$[\![\Phi]\!]_{q;d}$	coefficient of q^d in $\Phi \in R[\![q]\!]$
$\Phi_b(q)$	coefficients of expansion of $F(w,q)$around $w=\infty$: (2-9)
$\Phi_{p;b}(q)$	$F_p(w,q)$ around $w=\infty$: (2-21)
$\Phi_{m,\boldsymbol{c}}$	product of Φ_b's: (2-31)
$\Psi_b(q)$, $\Psi_{p;b}(q)$	equivariant versions of $\Phi_b(q)$, $\Phi_{p;b}(q)$: (4-11), (4-12)
$\Psi_{m,\boldsymbol{c}}$	equivariant version of $\Phi_{m,\boldsymbol{c}}$: (4-19)
$\mathbb{Z}^+$, $\bar{\mathbb{Z}}^+$	$\{1,2,\dots\}$, $\{0,1,2,\dots\}$
$Z(\cdot,\cdot,\cdot)$	generating functions for genus 0 invariants: (2-1)
$\mathcal{Z}(\cdot,\cdot,\cdot)$	equivariant version of $Z(\cdot,\cdot,\cdot)$: (3-3)
$\mathcal{Z}_p(\cdot,\cdot,\cdot)$	a coefficient of $N=2$ case of $\mathcal{Z}(\cdot,\cdot,\cdot)$: (3-4)
$\mathcal{Z}_{\boldsymbol{p}}$	equivariant geometric version of $\Delta_{\boldsymbol{p}}$: (3-5)

References

[1] **T M Apostol**, *Calculus, Vol. II: Multi-variable calculus and linear algebra, with applications to differential equations and probability*, 2nd edition, Blaisdell Publishing, Waltham, Mass. (1969) MR0248290

[2] **M Artin**, *Algebra*, Prentice Hall, Englewood Cliffs, NJ (1991) MR1129886

[3] **M F Atiyah**, **R Bott**, *The moment map and equivariant cohomology*, Topology 23 (1984) 1–28 MR721448

[4] **S Barannikov**, *Generalized periods and mirror symmetry in dimensions $n > 3$* `arXiv:math/9903124`

[5] **A Bertram**, *Another way to enumerate rational curves with torus actions*, Invent. Math. 142 (2000) 487–512 MR1804158

[6] **A Bertram**, **H P Kley**, *New recursions for genus-zero Gromov–Witten invariants*, Topology 44 (2005) 1–24 MR2103998

[7] **P Candelas**, **X C de la Ossa**, **P S Green**, **L Parkes**, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B 359 (1991) 21–74 MR1115626

[8] **L Cherveny**, *Genus-zero mirror principle with two marked points* `arXiv:1001.0242`

[9] **P Di Francesco**, **C Itzykson**, *Quantum intersection rings*, from: "RCP 25, Vol. 46", (R Dijkgraaf, C Faber, G van der Geer, editors), Prépubl. Inst. Rech. Math. Av. 1994/29, Univ. Louis Pasteur, Strasbourg (1994) 153–226 MR1331616

[10] **G Ellingsrud**, **S A Strømme**, *Bott's formula and enumerative geometry*, J. Amer. Math. Soc. 9 (1996) 175–193 MR1317230

[11] **A Gathmann**, *Absolute and relative Gromov–Witten invariants of very ample hypersurfaces*, Duke Math. J. 115 (2002) 171–203 MR1944571

[12] **A B Givental**, *Equivariant Gromov–Witten invariants*, Internat. Math. Res. Notices (1996) 613–663 MR1408320

[13] **A B Givental**, *The mirror formula for quintic threefolds*, from: "Northern California Symplectic Geometry Seminar", Amer. Math. Soc. Transl. Ser. 2 196, Amer. Math. Soc. (1999) 49–62 MR1736213

[14] **A B Givental**, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices (2001) 1265–1286 MR1866444

[15] **B R Greene**, **D R Morrison**, **M R Plesser**, *Mirror manifolds in higher dimension*, from: "Mirror symmetry, II", (B Greene, S-T Yau, editors), AMS/IP Stud. Adv. Math. 1, Amer. Math. Soc. (1997) 745–791 MR1416356

[16] **P Griffiths**, **J Harris**, *Principles of algebraic geometry*, Wiley Classics Library, Wiley, New York (1994) MR1288523

[17] **K Hori**, **S Katz**, **A Klemm**, **R Pandharipande**, **R Thomas**, **C Vafa**, **R Vakil**, **E Zaslow**, *Mirror symmetry*, Clay Mathematics Monographs 1, Amer. Math. Soc. (2003) MR2003030

[18] **S Katz**, *Rational curves on Calabi–Yau manifolds: Verifying predictions of mirror symmetry*, from: "Projective geometry with applications", (E Ballico, editor), Lecture Notes in Pure and Appl. Math. 166, Dekker, New York (1994) 231–239 MR1302954

[19] **A Klemm**, **R Pandharipande**, *Enumerative geometry of Calabi–Yau* 4*–folds*, Comm. Math. Phys. 281 (2008) 621–653 MR2415462

[20] **Y-P Lee**, *Quantum Lefschetz hyperplane theorem*, Invent. Math. 145 (2001) 121–149 MR1839288

[21] **Y-P Lee**, **R Pandharipande**, *A reconstruction theorem in quantum cohomology and quantum K–theory*, Amer. J. Math. 126 (2004) 1367–1379 MR2102400

[22] **B H Lian**, **K Liu**, **S-T Yau**, *Mirror principle, I*, Asian J. Math. 1 (1997) 729–763 MR1621573

[23] **D Maulik**, **R Pandharipande**, in progress

[24] **D Maulik**, **R Pandharipande**, *A topological view of Gromov–Witten theory*, Topology 45 (2006) 887–918 MR2248516

[25] **D McDuff**, **D Salamon**, *J–holomorphic curves and symplectic topology*, AMS Colloquium Publications 52, Amer. Math. Soc. (2004) MR2045629

[26] **A Popa**, *The genus one Gromov–Witten invariants of Calabi–Yau complete intersections*, Trans. Amer. Math. Soc. 365 (2013) 1149–1181 MR3003261

[27] **A Popa**, **A Zinger**, *Mirror symmetry for closed, open, and unoriented Gromov–Witten invariants* `arXiv:1010.1946`

[28] **Y Ruan**, **G Tian**, *A mathematical theory of quantum cohomology*, J. Differential Geom. 42 (1995) 259–367 MR1366548

[29] **D Zagier**, **A Zinger**, *Some properties of hypergeometric series associated with mirror symmetry*, from: "Modular forms and string duality", (N Yui, H Verrill, C F Doran, editors), Fields Inst. Commun. 54, Amer. Math. Soc., Providence, RI (2008) 163–177 MR2454324

[30] **A Zinger**, *Genus–zero two–point hyperplane integrals in the Gromov–Witten theory*, Comm. Anal. Geom. 17 (2009) 955–999 MR2643736

[31] **A Zinger**, *The reduced genus 1 Gromov–Witten invariants of Calabi–Yau hypersurfaces*, J. Amer. Math. Soc. 22 (2009) 691–737 MR2505298

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