

# $K$ -theory, LQEL manifolds and Severi varieties

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We use topological  $K$ -theory to study nonsingular varieties with quadratic entry locus. We thus obtain a new proof of Russo's divisibility property for locally quadratic entry locus manifolds. In particular we obtain a  $K$ -theoretic proof of Zak's theorem that the dimension of a Severi variety must be 2, 4, 8 or 16 and so answer a question of Atiyah and Berndt. We also show how the same methods applied to dual varieties recover the Landman parity theorem.

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## 1 Introduction

Zak's celebrated classification of Severi varieties [21] establishes that there are only four such varieties and that they correspond to projective planes over the four division algebras. Taking into account the classical results relating  $K$ -theory, division algebras and projective planes, Atiyah and Berndt [2] asked whether there might be a  $K$ -theoretic proof that the dimension of a Severi variety was necessarily 2, 4, 8 or 16.

By taking up an old approach of Fujita and Roberts [6] and Tango [19] but replacing characteristic classes with  $K$ -theory, we are able to provide a  $K$ -theoretic proof of the Severi variety dimension restriction. In fact our results sit naturally in the domain of Russo's LQEL manifolds [16] and we provide a new  $K$ -theoretic proof of his divisibility property for LQEL manifolds.

The method we employ is to consider the  $K$ -theoretic consequences of the existence of the generalized Euler sequence associated to a vector bundle. The generalized Euler sequence of a vector bundle  $V$  over a base  $B$  is the natural exact sequence on the total space of the projectivization  $\mathbb{P}(V)$ ,

$$(1) \quad 0 \longrightarrow \mathcal{O} \longrightarrow p^*V(1) \longrightarrow T\mathbb{P}(V) \longrightarrow p^*TB \longrightarrow 0,$$

where  $p: \mathbb{P}(V) \rightarrow B$  is the bundle map,  $p^*V(1) = p^*V \otimes \mathcal{O}(1)$  and  $\mathcal{O}(1)$  is the dual of the tautological line bundle on  $\mathbb{P}(V)$ . In the special case  $B$  is a point this is the familiar Euler sequence on projective space (see eg Hartshorne [8, II.8.13]) and in the general case as above, it essentially reduces to this since  $\mathbb{P}(V) \rightarrow B$  is locally trivial.

We obtain our results by taking  $V$  to be the (extended) tangent bundle of a projective variety and noting that  $\mathbb{P}(V)$  also fibres over the secant variety. In the case that the variety is an LQEL manifold, the irreducible components of a general fibre of the map to the secant variety are nonsingular quadrics. As a result, the topological  $K$ -theory of such a quadric carries a special relation in  $K$ -theory which turns out to be very restrictive.

The problem of classifying Severi varieties was first posed by Hartshorne in his influential paper [7] and is closely related to his complete intersection conjecture. Since Hartshorne's motivation for this conjecture was partly topological (specifically, the Barth–Larsen theorems) it is tempting to wonder, in view of the results here and of Ionescu and Russo's recent proof [12] of the complete intersection conjecture for quadratic manifolds, what relevance topological  $K$ -theory may have for the complete intersection conjecture.

## 2 LQEL manifolds

We recall the basic definitions for the reader's convenience and to fix notation and terminology. For examples, further details and proofs of the assertions below we recommend Russo [15; 16], Fujita and Roberts [6] and of course Zak's excellent foundational monograph [21]. Our definitions are slightly simpler because we stick to nonsingular varieties. We work over  $\mathbb{C}$  throughout as we will obtain our results by using topological  $K$ -theory.

**Definition 2.1** Let  $Y \subseteq \mathbb{P}^N$  be a nonsingular irreducible projective variety with secant variety  $\text{Sec}(Y) \subseteq \mathbb{P}^N$  and  $z \in \text{Sec}(Y) - Y$ . The entry locus of  $Y$  with respect to  $z$  is defined to be

$$\Sigma_z(Y) = \{y \in Y \mid \text{the line } yz \text{ is a tangent or secant of } Y\}.$$

The general entry locus is a projective variety with pure dimension equal to the secant deficiency, ie, for general  $z$ ,

$$\dim \Sigma_z(Y) = \delta = 2n + 1 - \dim \text{Sec}(Y).$$

**Definition 2.2** Let  $Y \subseteq \mathbb{P}^N$  be a nonsingular irreducible projective variety. Following Russo [16] we say  $Y$  is a locally quadratic entry locus (LQEL) manifold of type  $\delta$  if each irreducible component of a general entry locus is a nonsingular,  $\delta$ -dimensional quadric.

**Definition 2.3** Let  $Y \subseteq \mathbb{P}^N$  be a nonsingular irreducible projective variety with tangent variety  $\text{Tan}(Y) \subseteq \mathbb{P}^N$  and  $z \in \text{Tan}(Y) - Y$ . The tangent locus of  $Y$  with respect to  $z$  is defined to be

$$\tau_z(Y) = \{y \in Y \mid z \in \mathbb{T}_y Y\},$$

where  $\mathbb{T}_y Y \subseteq \mathbb{P}^N$  is the embedded tangent space of  $Y$  at  $y$ .

The general tangent locus is a projective variety with pure dimension equal to the tangent deficiency, ie, for general  $z$ ,

$$\dim \tau_z(Y) = \delta_\tau = 2n - \dim \text{Tan}(Y).$$

We recall Zak’s theorem that  $\delta > 0$  if and only if  $\text{Tan}(Y) = \text{Sec}(Y)$  so that in this case we have  $\delta_\tau = \delta - 1$ .

**Lemma 2.4** Let  $Y \subseteq \mathbb{P}^N$  be an LQEL manifold of type  $\delta > 0$  and  $z \in \text{Sec}(Y) - Y$  a general point. For each irreducible component  $Q$  of the entry locus  $\Sigma_z(Y)$ , the polar of  $z$  with respect to  $Q$  determines a nonsingular hyperplane section  $F$  of  $Q$ . These nonsingular  $(\delta - 1)$ -dimensional quadrics  $F$  are the irreducible components of the tangent locus  $\tau_z(Y)$ .

**Proof** Let  $F$  be an irreducible component of  $\tau_z(Y)$ . Since  $\tau_z(Y) \subset \Sigma_z(Y)$  we must have  $F \subset Q$  for some irreducible component  $Q$  of  $\Sigma_z(Y)$ . Since any tangent line of  $Y$  passing through  $z$  can be obtained as a limit of secants of passing through  $z$  we have

$$F = \{y \in Q \mid z \in \mathbb{T}_y Y\} = \{y \in Q \mid z \in \mathbb{T}_y Q\} = Q \cap H_z,$$

where  $H_z = \{x \in M \mid q(x, z) = 0\}$  is the polar of  $z$  with respect to  $Q$ ,  $M \subseteq \mathbb{P}^N$  is the  $(\delta + 1)$ -dimensional linear span of  $Q$  and  $q$  is the quadratic form on  $M$  cutting out  $Q$ . □

Our key observation is that an irreducible component of a general tangent locus supports some rather special topology as a result of the ambient LQEL geometry.

**Proposition 2.5** Let  $Y \subseteq \mathbb{P}^N$  be an  $n$ -dimensional LQEL manifold of type  $\delta > 0$  and let  $F \subseteq Y$  be an irreducible component of a general tangent locus. Then

$$1 + \mathcal{O}(1) \mid 2(n - \delta)$$

in  $K(F)$ , where  $K(F)$  is the topological (complex)  $K$ -theory of  $F$  (with its analytic topology) and  $\mathcal{O}(1)$  is the class in  $K(F)$  represented by the restriction of the hyperplane section bundle to  $F$ .

**Proof** We take up the ideas of Fujita and Roberts [6] and Tango [19] except that instead of computing Chern classes, we derive a relation in  $K$ -theory. Thus let

$$\Theta = \{(y, z) \in Y \times \text{Sec}(Y) \mid z \in \mathbb{T}_y Y\}.$$

Recall that the embedded tangent space  $\mathbb{T}_y Y \subset \mathbb{P}^N$  used above is related to the intrinsic tangent space  $TY$  by the exact sequence of bundles

$$(2) \quad 0 \longrightarrow \mathcal{O} \longrightarrow \widehat{\mathbb{T}}Y(1) \longrightarrow TY \longrightarrow 0,$$

where we have that  $\widehat{\mathbb{T}}_y Y \subset \mathbb{C}^{N+1}$  is the vector subspace lying over  $\mathbb{T}_y Y \subset \mathbb{P}^N$  and  $\widehat{\mathbb{T}}Y(1) = \widehat{\mathbb{T}}Y \otimes \mathcal{O}(1)$ . We thus see that<sup>1</sup>  $\Theta = \mathbb{P}(\widehat{\mathbb{T}}Y)$ .

Note that we have natural maps

$$(3) \quad \begin{array}{ccc} & \Theta & \\ f \swarrow & & \searrow g \\ Y & & \text{Sec}(Y) \end{array}$$

and that the fibre of  $g$  above a point  $z \in \text{Sec}(Y) - Y$  is naturally identified by  $f$  with the corresponding tangent locus in  $Y$ .

With this setup in place, the proof is mostly formal. The result is a consequence of the relation that exists in  $K(F)$  as a result of the generalized Euler sequence (1) with  $V = \widehat{\mathbb{T}}Y$  restricted to  $F$  together with the fact that  $F$  is a quadric. We thus consider the following exact sequence on  $\Theta$ :

$$(4) \quad 0 \longrightarrow \mathcal{O} \longrightarrow f^* \widehat{\mathbb{T}}Y(1) \longrightarrow T\Theta \longrightarrow f^* TY \longrightarrow 0$$

Furthermore there is a natural isomorphism  $\mathcal{O}_\Theta(1) \simeq g^* \mathcal{O}(1)$  and so when we restrict (4) to an irreducible component  $F$  of a fibre of the map  $g$  we have

$$(5) \quad f^* \widehat{\mathbb{T}}Y(1)|_F \simeq \widehat{\mathbb{T}}Y|_F.$$

Now we simply collect up all the natural exact sequences to hand and interpret them as relations in  $K(F)$  (forgetting the holomorphic structures). At the risk of being overly explicit, we list all the exact sequences we need below. We use the notation  $\mathbb{P}^\delta$  to

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<sup>1</sup>Those comparing with [6] should note that the authors realize  $\Theta$  as  $\mathbb{P}(E^*)$ , where  $E = \widehat{\mathbb{T}}^*Y(-1)$  (though they use Grothendieck's convention for projectivization so the dual on  $E$  does not appear). It is slightly simpler to realize  $\Theta$  as we do since then the tautological bundle  $\mathcal{O}_\Theta(-1)$  (which appears later) is not twisted.

denote the linear subspace of  $\mathbb{P}^N$  that is the span of the quadric  $F$ :

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathbb{P}^\delta} \longrightarrow \mathcal{O}_{\mathbb{P}^\delta}(1)^{\delta+1} \longrightarrow T\mathbb{P}^\delta \longrightarrow 0 \\ 0 &\longrightarrow TF \longrightarrow T\mathbb{P}^\delta|_F \longrightarrow \mathcal{O}_F(2) \longrightarrow 0 \\ 0 &\longrightarrow TF \longrightarrow T\Theta|_F \longrightarrow \mathcal{O}_F^{2n+1-\delta} \longrightarrow 0 \end{aligned}$$

Regarding these three sequences together with (2) and (4) as five equations in  $K(F)$  in five unknowns we can solve for the class of  $\widehat{\mathbb{T}}Y$ . Bearing in mind (5) we get the following equation in  $K(F)$ :

$$\begin{aligned} \widehat{\mathbb{T}}Y(1 + \mathcal{O}(1)) &= 2n + 2 - \delta + (\delta + 1)\mathcal{O}(1) - \mathcal{O}(2) \\ &= 2(n - \delta) + (2 + \delta - \mathcal{O}(1))(1 + \mathcal{O}(1)) \end{aligned}$$

Thus, letting  $W = \widehat{\mathbb{T}}Y - 2 - \delta + \mathcal{O}(1)$  we have

$$(6) \quad (1 + \mathcal{O}(1))W = 2(n - \delta),$$

which proves the result. □

We can already extract useful information from this proposition using characteristic classes. Taking the first Chern class of the identity (6) we get

$$2c_1(W) = -(n - \delta)c_1(\mathcal{O}(1)).$$

Thus if  $\dim F \geq 3$  since  $c_1(\mathcal{O}(1)) \in H^2(F, \mathbb{Z}) \simeq \mathbb{Z}$  is a generator we must have  $2 \mid n - \delta$  as integers.<sup>2</sup> However as we shall see a much stronger relationship holds.

Fujita and Roberts [6] and Tango [19] essentially pursued this characteristic class approach (for Severi varieties) but only obtained partial results. To bring this approach to fruition it would be necessary to fully characterize the image of  $K(F)$  under the Chern character, as a maximal-rank lattice in  $H^*(F, \mathbb{Q})$ . In fact it is easier to dispense with ordinary cohomology entirely and stay in *K*-theory.

Thus to take full advantage of the result of Proposition 2.5 we need to know the ring structure of  $K(F)$  explicitly. We have relegated a discussion of this purely topological result to Proposition A.1 in the Appendix. With this in hand we can state the following.

**Corollary 2.6** *Let  $Y \subseteq \mathbb{P}^N$  be an  $n$ -dimensional LQEL manifold of type  $\delta \geq 3$  then*

$$(7) \quad 2^{[(\delta-1)/2]} \mid n - \delta$$

*in  $\mathbb{Z}$ .*

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<sup>2</sup>In fact although  $H^2(F, \mathbb{Z})$  is not cyclic for  $\dim_{\mathbb{C}} F = 2$  we can still deduce that  $2 \mid n - \delta$  in this case since  $c_1(\mathcal{O}(1))$  is not even and thus the relation holds as long as  $\delta \geq 3$ .

**Proof** This corollary is an immediate consequence of [Proposition 2.5](#) together with [Corollary A.2](#). □

In other words, we have a new proof of Russo’s divisibility property for LQEL manifolds (see [\[16, Theorem 2.8 \(2\)\]](#)) showing that it holds for topological reasons.

**Remark 2.7** In fact [Proposition 2.5](#) can be refined slightly: the class in  $K(F)$  denoted  $W$  in [\(6\)](#) can be represented by the normal bundle of the entry locus (restricted to the tangent locus). Indeed if  $F \subset Q \subset Y$  is the inclusion of a (general) tangent locus in an entry locus of  $Y$  then we have the following natural exact sequences involving normal bundles:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \widehat{\mathbb{T}}Y & \longrightarrow & \mathcal{O}^{N+1} & \longrightarrow & N_{Y|\mathbb{P}^N}(-1) \longrightarrow 0 \\
 0 & \longrightarrow & N_{F|Y} & \longrightarrow & N_{F|\mathbb{P}^N} & \longrightarrow & N_{Y|\mathbb{P}^N} \longrightarrow 0 \\
 0 & \longrightarrow & N_{F|\mathbb{P}^\delta} & \longrightarrow & N_{F|\mathbb{P}^N} & \longrightarrow & N_{\mathbb{P}^\delta|\mathbb{P}^N} \longrightarrow 0 \\
 0 & \longrightarrow & N_{F|Q} & \longrightarrow & N_{F|Y} & \longrightarrow & N_{Q|Y} \longrightarrow 0
 \end{array}$$

Since  $N_{F|Q} \simeq \mathcal{O}(1)$ ,  $N_{F|\mathbb{P}^\delta} \simeq \mathcal{O}(2)$ ,  $N_{\mathbb{P}^\delta|\mathbb{P}^N} \simeq \mathcal{O}(1)^{N-\delta}$  we get

$$W = N_{Q|Y}(-1)$$

in  $K(F)$ . In other words, we can refine [Proposition 2.5](#) to

$N_{Q|Y} \oplus N_{Q|Y}(-1)$  is topologically stably trivial restricted to  $F$ .

Also, there is presumably a holomorphic counterpart of this statement, just as there is for the analogous statement [\(11\)](#) discussed in the next section (though it is certainly not that the above holds as holomorphic bundles).

Finally we wish to comment on Severi varieties. We thus recall the following.

**Definition 2.8** A Severi variety is a nondegenerate nonsingular irreducible variety  $Y \subseteq \mathbb{P}^N$  of dimension  $n$  such that  $3n = 2(N - 2)$  and  $\text{Sec}(Y) \neq \mathbb{P}^N$ .

As we have noted, Zak [\[21\]](#) provided a beautiful classification of Severi varieties showing that there are just four and that they correspond to projective planes over the four division algebras. The hard part of the classification is proving that  $n \in \{2, 4, 8, 16\}$ .

The first step toward understanding Severi varieties is the following result of Zak.

**Proposition 2.9** *A Severi variety is an LQEL<sup>3</sup> manifold of type  $\delta = n/2$ .*

**Proof** See Zak [21, Proposition 2.1] or Russo [15, Proposition 3.2.3]. □

Our motivation for this work was the remark of Atiyah and Berndt [2, pages 25, 26], concerning a possible *K*-theoretic proof of the dimension restriction for Severi varieties:

“There is a striking resemblance between Zak’s theorem in complex algebraic geometry and the classical results about division algebras and projective planes [...] One is therefore tempted to expect a *K*-theory proof of Zak’s theorem.”

For emphasis we thus explicitly state the following.

**Corollary 2.10** *If  $Y \subset \mathbb{P}^N$  is an  $n$ -dimensional Severi variety, then  $n \in \{2, 4, 8, 16\}$ .*

**Proof** By definition  $n$  is even and if  $n > 4$  then by (7) with  $\delta = n/2$  we immediately find  $4 \mid n$  and thence  $2^{n/4} \mid n$  from which the result follows. □

We thus answer Atiyah and Berndt’s implied question affirmatively. Moreover, granting the purely topological result Proposition A.1 describing the ring structure of the *K*-theory of the quadric, our methods provide an extremely short (and easy) proof that the dimension of a Severi variety must be as above.

For the sake of completeness we provide the chronology of proofs of this result. It has been proved by

- Zak (1982) [21] (see also Lazarsfeld and Van de Ven [14]) who used a detailed algebro-geometric study of the entry loci and their mutual intersection properties,
- Landsberg (1996) [13] who studied the local differential geometry via the second fundamental form and appealed to classification of Clifford modules,
- Chaput (2002) [4] who showed how to see a priori that a Severi variety is projectively homogeneous,
- Russo (2009) [16] who established Corollary 2.6 by inductively studying the variety of lines through a point in an LQEL manifold,
- Schillewaert, Van Maldegham (2013) [18] who show how to obtain the classification over arbitrary fields using only the axioms of what they call a Mazzocca–Melone set.

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<sup>3</sup>In fact Zak’s result is slightly stronger: a Severi variety is a QEL manifold (in the terminology of [16]) ie, the entry loci are irreducible.

### 3 Dual varieties

Proposition 2.5 is really just an examination of the consequences that exist in  $K$ -theory as a result of the relation obtained from the generalized Euler sequence on the bundle of embedded tangent spaces.

However there is another bundle of embedded linear spaces associated to any nonsingular variety, the (twisted) conormal bundle, ie, if  $N_{Y|\mathbb{P}^N}$  is the normal bundle of a nonsingular variety  $Y \subseteq \mathbb{P}^N$  and  $y \in Y$  then there is a natural embedding of the fibre

$$\mathbb{P}(N_{Y|\mathbb{P}^N}^*(1))_y \subseteq \mathbb{P}^{N^*}.$$

It is thus natural to examine what consequences the generalized Euler sequence for the projectivized conormal bundle has in  $K$ -theory.

Unsurprisingly, we will end up recovering known results (the Landman parity theorem and a weak version of a result due to Ein) but it is instructive to see the parallels with Section 2 and to obtain these results with such ease.

We thus define  $\Phi = \mathbb{P}(N_{Y|\mathbb{P}^N}^*(1))$  and note that naturally  $\Phi \subseteq Y \times Y^* \subseteq \mathbb{P}^N \times \mathbb{P}^{N^*}$ , where  $Y^*$  is the dual variety of  $Y$ . The analogue of the diagram (3) in this case is then:

$$(8) \quad \begin{array}{ccc} & \Phi & \\ f \swarrow & & \searrow g \\ Y & & Y^* \end{array}$$

This time the fibre of  $g$  above a general point  $H \in Y^*$  is the contact locus  $C_H(Y)$ . Identifying this fibre with its image under  $f$  we have

$$C_H(Y) = \{y \in Y \mid \mathbb{T}_y Y \subseteq H\}.$$

The contact locus is well known to be a linear space of dimension  $k = N - 1 - \dim Y^*$ , the dual deficiency of  $Y$ . Since we will obtain a relation in  $K(C_H(Y))$  we must assume  $k > 0$  in order to have nontrivial content.

**Proposition 3.1** *Let  $Y \subset \mathbb{P}^N$  be an irreducible nonsingular variety of dual deficiency  $k > 0$ , let  $H \in Y^*$  be a general point and let  $N_{C|Y}$  be the normal bundle of the contact locus  $C_H(Y)$  in  $Y$  then we have*

$$N_{C|Y} = N_{C|Y}^*(1)$$

in  $K(C_H(Y))$ .



**Proof** Referring to (8), we have the generalized Euler sequence for  $\Phi$ ,

$$(9) \quad 0 \longrightarrow \mathcal{O} \longrightarrow f^* N_{Y|\mathbb{P}^N}^*(1) \otimes g^* \mathcal{O}(1) \longrightarrow T\Phi \longrightarrow f^* TY \longrightarrow 0,$$

where we have used  $\mathcal{O}_{\mathbb{P}(N_{Y|\mathbb{P}^N}^*(1))}(1) \simeq g^* \mathcal{O}(1)$  naturally.

Restricting to the fibre  $C_H(Y)$  of  $g$  as in the proof of Proposition 2.5 and bearing in mind that  $C_H(Y)$  is a linear space we thus have the following natural exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{k+1} \longrightarrow TC_H(Y) \longrightarrow 0 \\ 0 &\longrightarrow TY \longrightarrow T\mathbb{P}^N|_Y \longrightarrow N_{Y|\mathbb{P}^N} \longrightarrow 0 \\ 0 &\longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{N+1} \longrightarrow T\mathbb{P}^N \longrightarrow 0 \\ 0 &\longrightarrow TC_H(Y) \longrightarrow T\Phi|_{C_H(Y)} \longrightarrow \mathcal{O}^{N-1-k} \longrightarrow 0 \end{aligned}$$

Regarding these four exact sequences together with (9) as relations in  $K(C_H(Y))$  we thus obtain

$$(10) \quad N_{Y|\mathbb{P}^N}^*(1) - N_{Y|\mathbb{P}^N} = (k - N)(\mathcal{O}(1) - 1)$$

in  $K(C_H(Y))$ . Since we are restricting to  $C_H(Y) \subset Y$  we can instead express this in terms of the normal bundle  $N_{C|Y}$  of  $C_H(Y)$  in  $Y$  instead of  $N_{Y|\mathbb{P}^N}$ . These are related by the natural exact sequence of bundles on  $C_H(Y)$

$$0 \longrightarrow N_{C|Y} \longrightarrow N_{C|\mathbb{P}^N} \longrightarrow N_{Y|\mathbb{P}^N}|_{C_H(Y)} \longrightarrow 0,$$

and since  $C_H(Y) \subset \mathbb{P}^N$  is linearly embedded  $N_{C|\mathbb{P}^N} \simeq \mathcal{O}(1)^{N-k}$ . Thus eliminating  $N_{Y|\mathbb{P}^N}$  the identity (10) becomes

$$(11) \quad N_{C|Y} = N_{C|Y}^*(1)$$

in  $K(C_H(Y))$ . This completes the proof. □

**Corollary 3.2** *Let  $Y \subset \mathbb{P}^N$  be an  $n$ -dimensional nonsingular irreducible projective variety with dual deficiency  $k > 0$  then*

$$2 \mid n - k.$$

**Proof** Take first Chern classes of each side in (11). Since  $\text{rank } N_{C|Y} = n - k$ , we get

$$2c_1(N_{C|Y}) = (n - k)c_1(\mathcal{O}(1)).$$

The result then follows since  $c_1(\mathcal{O}(1)) \in H^2(C_H(Y), \mathbb{Z}) \simeq \mathbb{Z}$  is a generator. □

The above corollary is known as the Landman parity theorem and was first proved by Landman using Picard–Lefschetz theory (though not published). Subsequently Ein [5] (using a result of Kleiman) provided a proof in which he established that (11) in fact holds as holomorphic bundles rather than just as stable topological bundles as we have shown (see also Tevelev [20, Theorem 7.1] and Ionescu and Russo [11, Proposition 3.1]).

We note that in contrast to Proposition 2.5, the fact that there exists a bundle satisfying (11) in  $K(C_H(Y))$  does not contain more information than we have obtained by noting that  $c_1(N_{C|Y})$  is integral. For example the bundle  $V = (1 \oplus \mathcal{O}(1)) \otimes \mathcal{O}^{(n-k)/2}$  has rank  $n - k$  and satisfies  $V \simeq V^*(1)$  for any  $n, k$  as long as  $2 \mid n - k$ . There is thus no analogue of the stronger Corollary 2.6 in this context.

On the other hand, the fact that it is not just any bundle but  $N_{C|Y}$  that appears in (11) does of course contain more data. For example if  $Y$  is a nonsingular scroll of fibre dimension  $l$  and base dimension  $m < l$  we can use it to calculate  $k$ .

Indeed since the contact locus for a scroll is necessarily contained in a fibre, ie,  $C_H(Y) \subseteq L \subseteq Y$  for a fibre  $L$ , we have the natural exact sequence of normal bundles

$$0 \longrightarrow N_{C|L} \longrightarrow N_{C|Y} \longrightarrow N_{L|Y}|_{C_H(Y)} \longrightarrow 0,$$

but of course  $N_{C|L} \simeq \mathcal{O}(1)^{l-k}$  and  $N_{L|Y} \simeq \mathcal{O}^m$  and so in  $K(C_H(Y))$  we have

$$N_{C|Y} = m + (l - k)\mathcal{O}(1)$$

in  $K(C_H(Y))$ . The only way this is compatible with (11) is if  $k = l - m$ .

## Appendix: $K$ -theory of the quadric

To take full advantage of Proposition 2.5 we need to know the ring structure of the  $K$ -theory of a nonsingular quadric. Surprisingly, this does not seem<sup>4</sup> to appear in the literature so we provide the necessary results here.

The calculation falls into two cases depending on whether the dimension of the quadric is odd or even. As a CW-complex, the quadric has a cell decomposition with no odd-dimensional cells and one cell in each even dimension except for the middle dimension

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<sup>4</sup>We should qualify this remark by saying that since the  $n$ -dimensional complex quadric is diffeomorphic to the oriented real Grassmannian  $\tilde{G}(2, n + 2)$ , it might be possible to extract the result we need from Sankaran and Zvengrowski [17]. However as  $\tilde{G}(2, n + 2)$  is an edge case for the calculations in [17], it was difficult to be certain if it was really covered. Furthermore the polynomial ring representation of the  $K$ -theory given in [17] is not perfectly suited to our needs. For these reasons and because we needed to be sure of the correctness of this crucial result, we decided to work from first principles.

in the case of the even-dimensional quadric where there are two cells. Thus<sup>5</sup> if  $F$  is our quadric then  $K^1(F)$  vanishes and  $K^0(F) = K(F)$  is free abelian with rank equal to the number of cells, ie

$$(1) \quad \text{rank } K(F) = \begin{cases} 1 + \dim F & \text{for } \dim F \text{ odd,} \\ 2 + \dim F & \text{for } \dim F \text{ even.} \end{cases}$$

To determine the ring structure of  $K(F)$ , we need to use more sophisticated techniques. We shall represent  $F$  as a homogeneous space so that we can use the methods of Atiyah and Hirzebruch [3] and Hodgkin [9]. Thus let  $\dim F = m - 1$  and recall that there is a natural diffeomorphism

$$F \simeq \frac{\text{SO}(m + 1)}{\text{SO}(2) \times \text{SO}(m - 1)},$$

where of course by  $\text{SO}(k)$  we mean the real Lie group. In fact we need  $F$  to be a homogeneous space of a simply connected group. Thus we lift to the double-cover and so regard<sup>6</sup>

$$(2) \quad F \simeq \frac{\text{Spin}(m + 1)}{\text{Spin}^c(m - 1)}.$$

(We need to be a little careful with the above for  $m = 2, 3$  but there is no real problem.)

In view of (2) we see that representations of  $\text{Spin}^c(m - 1)$  give vector bundles on  $F$ . We wish to highlight the bundles corresponding to certain special representations.

Thus consider the double cover  $\text{Spin}(2) \times \text{Spin}(m - 1)$  of  $\text{Spin}^c(m - 1)$  and suppose for now that  $m$  is even. If we let  $\mathbb{Z}[t, t^{-1}]$  be the representation ring of  $\text{SO}(2)$ , then  $R \text{Spin}(2) = \mathbb{Z}[t^{1/2}, t^{-1/2}]$ . In addition there is the unique irreducible spin representation  $\delta$  of  $\text{Spin}(m - 1)$  since  $m - 1$  is odd. Neither  $t^{1/2}$  nor  $\delta$  descends to  $\text{Spin}^c(m - 1)$  but their product does. We thus let

$$X = \text{bundle on } F \text{ obtained from representation } t^{-1/2}\delta \text{ of } \text{Spin}^c(m - 1).$$

Similarly for  $m$  odd we define the bundles  $X^+, X^-$  by

$$X^\pm = \text{bundle on } F \text{ obtained from representation } t^{-1/2}\delta^\pm \text{ of } \text{Spin}^c(m - 1),$$

where  $\delta^\pm$  are the irreducible components of the spin representation (since  $m - 1$  is even). Note that  $\text{rank } X = 2^{m/2-1}$  and  $\text{rank } X^\pm = 2^{(m-1)/2}$ .

<sup>5</sup>See eg Atiyah [1, Proposition 2.5.2].

<sup>6</sup>We often define  $\text{Spin}^c(k)$  as  $S^1 \times_{\pm 1} \text{Spin}(k)$ . Fixing  $S^1 \simeq \text{Spin}(2)$ , it is clear that  $\text{Spin}^c(k)$  is naturally the double cover of  $\text{SO}(2) \times \text{SO}(k)$  as we have defined it.

**Proposition A.1** *Let  $F \subset \mathbb{P}^m$  be an  $(m-1)$ -dimensional nonsingular quadric,  $m \geq 3$ . Let  $L = \mathcal{O}(1) - 1 \in K(F)$ . Suppose  $m$  is even and let  $X$  be the bundle defined above, then*

- $1, L, L^2, \dots, L^{m-2}, X$  are a  $\mathbb{Z}$ -basis for the torsion-free ring  $K(F)$ ,
- $L^m = 0$  (obviously, for dimensional reasons),
- $LX = 2^{m/2} - 2X$ ,
- $2^{m/2}X = 2^{m-1} - 2^{m-2}L + \dots + 2L^{m-2} - L^{m-1}$  (this is equivalent to previous bullet but shows why we need  $X$  instead of  $L^{m-1}$ ).

Similarly if  $m$  is odd and  $X^\pm$  are the bundles defined above, then

- $1, L, L^2, \dots, L^{m-2}, X^+, X^-$  are a  $\mathbb{Z}$ -basis for the torsion-free ring  $K(F)$ ,
- $L^m = 0$ ,
- $LX^\pm = 2^{(m-1)/2} - X^\pm - X^\mp$ ,
- $2^{(m-1)/2}(X^+ + X^-) = 2^{m-1} - 2^{m-2}L + \dots - 2L^{m-2} + L^{m-1}$ .

**Proof** For brevity, let  $G = \text{Spin}(m+1)$  and  $H = \text{Spin}^c(m-1)$ . We will use the methods of Atiyah and Hirzebruch [3, Section 5] as well as Hodgkin [9] to compute  $K(G/H)$ . Indeed as pointed out by Atiyah and Hirzebruch, there is a natural map

$$RH \longrightarrow K(G/H).$$

(This is simply the map induced by associating a vector bundle to a representation of  $H$  and then extending to the full representation ring.) Now  $H$  is a maximal-rank subgroup of  $G$  and so  $RG \subset RH$ . The restriction to  $RG$  gives only trivial bundles so if we let  $RG$  act on  $\mathbb{Z}$  by dimension then we have a natural map

$$RH \otimes_{RG} \mathbb{Z} \longrightarrow K(G/H).$$

Hodgkin [9, page 71] proves this map is an isomorphism since  $\pi_1(G) = 1$  and  $H$  has maximal rank. Furthermore there is a natural exact sequence of  $RH$ -modules

$$0 \longrightarrow RH \cdot I \longrightarrow RH \longrightarrow RH \otimes_{RG} \mathbb{Z} \longrightarrow 0,$$

where  $I \subset RG \subset RH$  is the augmentation ideal of  $RG$  (ie, the kernel of the dimension map  $RG \rightarrow \mathbb{Z}$ ). In other words for general reasons we have a natural ring isomorphism

$$(3) \quad K(F) \simeq RH/RH \cdot I.$$

To put this to use we need an explicit realization of three things:

- $RG$  and the dimension map  $RG \rightarrow \mathbb{Z}$  with kernel  $I$
- $RH$
- the inclusion  $RG \hookrightarrow RH$

We must now separately consider the two cases  $m$  even and  $m$  odd. We consider first the slightly simpler case  $m$  even.

We shall follow the notation of Husemoller [10]; by his Theorem 10.3 we have that  $RG$  is a polynomial ring:

$$RG \simeq \mathbb{Z}[\Lambda_1, \Lambda_2, \dots, \Lambda_{m/2-1}, \Delta],$$

and  $\Delta^2 = 1 + \Lambda_1^2 + \dots + \Lambda_{m/2-1}^2 + \Lambda_{m/2}^2$ .

Now  $H = (\text{Spin}(2) \times \text{Spin}(m-1))/\{\pm 1\}$  and so we have

$$RH \simeq (R \text{Spin}(2) \otimes R \text{Spin}(m-1))^{\mathbb{Z}/(2)}.$$

If we let<sup>7</sup>

$$R \text{Spin}(2) = \mathbb{Z}[t^{1/2}, t^{-1/2}],$$

$$R \text{Spin}(m-1) = \mathbb{Z}[\lambda_1, \dots, \lambda_{m/2-2}, \delta],$$

then as above  $\delta^2 = 1 + \lambda_1^2 + \dots + \lambda_{m/2-1}^2$ . The  $\mathbb{Z}/(2)$  action fixes the  $\lambda_i$  and changes the sign of  $\delta$  as well as the half-integral powers of  $t$ . We thus obtain

$$(4) \quad RH \simeq \mathbb{Z}[t, t^{-1}, \lambda_1, \dots, \lambda_{m/2-1}, X],$$

where  $X = t^{-1/2}\delta$ . Note that the above ring is not quite a polynomial ring, it is a quotient by the ideal generated by the relation

$$(5) \quad X^2 = t^{-1}(1 + \lambda_1^2 + \dots + \lambda_{m/2-1}^2).$$

Finally the map  $RG \hookrightarrow RH$  is described by

$$\Delta = (t^{1/2} + t^{-1/2})\delta = (1 + t)X,$$

$$\Lambda_i = \lambda_i + (t + t^{-1})\lambda_{i-1} + \lambda_{i-2},$$

---

<sup>7</sup>We need to be a little careful for the case  $m = 4$  below but there is no real problem. However the statement clearly does not hold for  $m = 2$ ; hence the assumption  $m \geq 3$  in the proposition statement.

for  $1 \leq i \leq m/2 - 1$  provided  $\lambda_0 = 1$  and  $\lambda_{-1} = 0$ . (These equations follow from the expressions given by Husemoller in [10, Chapter 14, Sections 9.4 and 9.2].) By (3) we thus have the following relations between the images of elements of  $RH$  in  $K(G/H)$ :

$$(6) \quad (1 + t)X = \dim \Delta = 2^{m/2},$$

$$(7) \quad \lambda_i + (t + t^{-1})\lambda_{i-1} + \lambda_{i-2} = \dim \Lambda_i.$$

Using (7) inductively we remove the  $\lambda_i$  from any polynomial expression in  $RH$  given by (4) and have only expressions involving  $t, t^{-1}$  instead. In other words we thus have a surjection from  $\mathbb{Z}[t, t^{-1}, X]$  to  $K(F)$ .

Now it is easier to work with nilpotent elements so let  $L = t - 1$ . Note that  $t$  corresponds to  $\mathcal{O}(1)$  so this is indeed the  $L$  in the proposition statement. Then  $L^m = 0$  for dimensional reasons (its image under the Chern character would lie in cohomology of degree at least  $2m$  and  $\dim_{\mathbb{R}} F = 2m - 2$ ) and so we have

$$t^{-1} = 1 - L + L^2 - \dots - L^{m-1}.$$

We thus have a surjection  $\mathbb{Z}[L, X]$  to  $K(F)$ . Combining this with the relation (5) we see that  $K(F)$  is spanned over  $\mathbb{Z}$  by the classes represented by  $L^i, XL^i$  for  $0 \leq i \leq m - 1$ . From here using (6) we see that  $K(F)$  is spanned by  $L^i, X$  for  $0 \leq i \leq m - 1$  and then finally elementary computation reveals

$$2^{m/2}X = 2^{m-1} - 2^{m-2}L + \dots + 2L^{m-2} - L^{m-1}.$$

Thus we can omit  $L^{m-1}$  and still have a spanning set. Since there are  $m$  elements in this set and we know by (1) that the rank of  $K(F)$  is  $m$ , this must be a  $\mathbb{Z}$ -basis as required. This deals with the case  $m$  even.

The argument for the case  $m$  odd is extremely similar. For the methods below we need to assume  $m \geq 5$  but the result for the case  $m = 3$  is easily verified since in this case  $F \simeq S^2 \times S^2$ .

This time we have

$$\begin{aligned} RG &\simeq \mathbb{Z}[\Lambda_1, \dots, \Lambda_{(m-3)/2}, \Delta^+, \Delta^-], \\ R \text{Spin}(m-1) &\simeq \mathbb{Z}[\lambda_1, \dots, \lambda_{(m-5)/2}, \delta^+, \delta^-], \\ (\Delta^\pm)^2 &= \Lambda_\pm + \Lambda_{(m-3)/2} + \Lambda_{(m-7)/2} + \dots, \\ \Delta^+ \Delta^- &= \Lambda_{(m-1)/2} + \Lambda_{(m-5)/2} + \dots, \end{aligned}$$

where  $\Lambda_{(m+1)/2} = \Lambda_+ + \Lambda_-$  and the series end in 1 or  $\Lambda_1$  according to parity (and similarly for  $\delta^\pm$  and  $\lambda_\pm$ ). Then similarly to the case  $m$  even we have

$$RH \simeq \mathbb{Z}[t, t^{-1}, \lambda_1, \dots, \lambda_{(m-5)/2}, \lambda_+, \lambda_-, X^+, X^-],$$

where  $X^\pm = t^{-1/2}\delta^\pm$  and the map  $RG \hookrightarrow RH$  is given by the same relation between the  $\lambda_i$  and  $\Lambda_i$  as for  $m$  even but

$$\begin{aligned}\Delta^+ &= t^{1/2}\delta^+ + t^{-1/2}\delta^-, \\ \Delta^- &= t^{1/2}\delta^- + t^{-1/2}\delta^+.\end{aligned}$$

Using these formulae, the same argument goes through just as for  $m$  even to yield the stated results. □

**Corollary A.2** *Let  $F \subset \mathbb{P}^m$  be a nonsingular quadric hypersurface,  $m \geq 3$ , and suppose  $1 + \mathcal{O}(1)$  divides  $l$  in  $K(F)$  for some  $l \in \mathbb{Z}$  then*

$$2^{\lfloor (m+1)/2 \rfloor} \mid l$$

in  $\mathbb{Z}$ . (The brackets in the power denote the integer part.)

**Proof** Set  $L = \mathcal{O}(1) - 1$  as in the proposition. If  $m$  is even, let  $X$  be as in the proposition and if  $m$  is odd, let  $X = X^+ + X^-$ . Note that in either case we then have

$$(1 + \mathcal{O}(1))X = (2 + L)X = 2^{\lfloor (m+1)/2 \rfloor}.$$

Since  $2 + L$  is not a zero divisor and  $X$  is part of a  $\mathbb{Z}$ -basis the result follows. □

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