Geometry & Topology 18 (2014) 1865–1910

Gromov–Witten invariants of $\mathbb{P}^1$ and Eynard–Orantin invariants

PAUL NORBURY
NICK SCOTT

We prove that genus-zero and genus-one stationary Gromov–Witten invariants of $\mathbb{P}^1$ arise as the Eynard–Orantin invariants of the spectral curve $x = z + 1/z$, $y = \ln z$. As an application we show that tautological intersection numbers on the moduli space of curves arise in the asymptotics of large-degree Gromov–Witten invariants of $\mathbb{P}^1$.

05A15; 14N35

1 Introduction

As a tool for studying enumerative problems in geometry, Eynard and Orantin [8] define invariants of any compact Torelli marked Riemann surface C, equipped with two meromorphic functions x and y with the property that the zeros of dx are simple and the map

$$\begin{aligned} C &\to \mathbb{C}^2, \\ p &\mapsto (x(p), y(p)) \end{aligned}$$

is an immersion. For every $(g,n) \in \mathbb{Z}^2$ with $g \geq 0$ and $n > 0$, the Eynard–Orantin invariant $\omega_n^g(p_1, \dots, p_n)$ for $p_i \in C$ is a multidifferential, ie a tensor product of meromorphic 1–forms on the product C^n. One can make sense of $F^g = \omega_0^g$ using a recursion between ω_{n+1}^g and ω_n^g known as the dilaton equation. See Section 2 for more details and the definition of the invariants.

Important examples of the Eynard–Orantin invariants, for different choices of (C, x, y), store intersection numbers over the moduli space of curves (Eynard [6]), simple Hurwitz numbers (Borot, Eynard, Mulase and Safnuk [1], Bouchard and Mariño [3], and Eynard, Mulase and Safnuk [7]), a count of lattice points in the moduli space of curves (Norbury [14]), and conjecturally local Gromov–Witten invariants of (non-compact) toric Calabi–Yau 3–folds (Bouchard, Klemm, Mariño and Pasquetti [2], and Mariño [13]), and Chern–Simons invariants of 3–manifolds (Dijkgraaf, Fuji and Manabe [4]).

Published: 2 October 2014 DOI: 10.2140/gt.2014.18.1865

The Gromov–Witten invariants of $\mathbb{P}^1$ have been studied and well understood over the last ten years (Getzler [12], and Okounkov and Pandharipande [16; 18; 17]). In this paper we show that the Gromov–Witten invariants of $\mathbb{P}^1$ arise as Eynard–Orantin invariants, and how this approach brings new insight to the Gromov–Witten invariants. We also hope to gain a better understanding of the Eynard–Orantin invariants. The example in this paper, together with the simple Hurwitz problem [7] and the count of lattice points in the moduli space of curves, which also corresponds to a Hurwitz problem [14], raises the question: is the relationship of Eynard–Orantin invariants to Hurwitz problems a more general phenomenon?

Assemble the connected stationary Gromov–Witten invariants

$$\left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_d^g = \int_{[\overline{\mathcal{M}}_n^g(\mathbb{P}^1,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{b_i} \, \mathrm{ev}_i^*(\omega), \tag{1-1}$$

where d is determined by $\sum_{i=1}^{n} b_i = 2g-2+2d$, into the generating function

$$\Omega_n^g(x_1,\dots,x_n) = \sum_{\mathbf{b}} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_d^g \cdot \prod_{i=1}^{n} (b_i+1)! \, x_i^{-b_i-2} \, dx_i$$

which is a multidifferential. See Section 3 for a more detailed definition of Gromov–Witten invariants.

The Eynard–Orantin invariants ω_n^g are defined for any genus-0 compact Riemann surface C equipped with two *meromorphic* functions x and y. Nevertheless, by taking sequences of meromorphic functions one can extend the definition to allow y to be any analytic function defined on a domain of C containing the zeros of dx. In particular, consider

$$C = \begin{cases} x = z + 1/z, \\ y = \ln z. \end{cases} \tag{1-2}$$

The Riemann surface C is defined via the meromorphic function $x(z)$. The function

$$y(z) = \ln z \sim \sum \frac{(1-z^2)^k}{-2k}$$

is to be understood as the sequence of partial sums

$$y_N = \sum_{1}^{N} \frac{(1-z^2)^k}{-2k}.$$

Each invariant requires only a finite y_N—for fixed (g,n) the sequence of invariants ω_n^g of (C, x, y_N) stabilises for $N \geq 6g-6+2n$.

Theorem 1.1 *For $g=0$ and 1 and $2g-2+n>0$, the Eynard–Orantin invariants of the curve C defined in (1-2) agree with the generating function for the Gromov–Witten invariants of $\mathbb{P}^1$:*

$$\omega_n^g \sim \Omega_n^g(x_1,\dots,x_n).$$

More precisely, $\Omega_n^g(x_1,\dots,x_n)$ gives an analytic expansion of ω_n^g around a branch of $\{x_i=\infty\}$.

In the two exceptional cases $(g,n)=(0,1)$ and $(0,2)$, the invariants ω_n^g are not analytic at $x_i=\infty$. We can again get analytic expansions around a branch of $\{x_i=\infty\}$ by removing their singularities at $x_i=\infty$ as

$$\omega_1^0+\ln x_1 dx_1 \sim \Omega_1^0(x_1), \quad \omega_2^0-\frac{dx_1\,dx_2}{(x_1-x_2)^2}\sim\Omega_2^0(x_1,x_2). \tag{1-3}$$

The Eynard–Orantin recursion expresses Gromov–Witten invariants in terms of simpler, ie smaller Euler characteristic, Gromov–Witten invariants, although of arbitrarily large degree. The need for arbitrarily large degree does not seem very geometric.

Theorem 1.1 gives an extremely efficient way to calculate the Gromov–Witten invariants of $\mathbb{P}^1$. It also produces a general form of the invariants that reduces to the calculation of a collection of polynomials.

Theorem 1.2 *For $g=0$ and 1, the stationary Gromov–Witten invariants of $\mathbb{P}^1$ are of the form*

$$\left\langle\prod_{i=1}^k\tau_{2u_i}(\omega)\prod_{i=k+1}^n\tau_{2u_i-1}(\omega)\right\rangle^g=\frac{u_{k+1}\cdots u_n}{\prod_{i=1}^n u_i!^2}\,p_{n,k}^g(u_1,\dots,u_n), \tag{1-4}$$

where $p_{n,k}^g(u_1,\dots,u_n)$ is a polynomial of degree $3g-3+n$ in the u_i, symmetric in the first k and the last $n-k$ variables, with top coefficients c_β of $u_1^{\beta_1}\cdots u_n^{\beta_n}$ given by

$$c_\beta=2^g\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\beta_1}\cdots\psi_n^{\beta_n} \tag{1-5}$$

for $|\beta|=3g-3+n$.

Again the exceptional cases are $(g,n)=(0,1)$ and $(0,2)$, where we interpret a degree $3g-3+n$ polynomial to mean a rational function given by the reciprocal of a degree 2, respectively degree 1, polynomial.

The asymptotic behaviour of Eynard–Orantin invariants near zeros of dx is governed by the local behaviour of the curve C there (Eynard and Orantin [9]). By assumption the local behaviour is described by $x=y^2$, which, as a global curve, has Eynard–Orantin invariants that store tautological intersection numbers over the compactified moduli space of curves $\overline{\mathcal{M}}_{g,n}$. This supplies the top coefficients (1-5) and enables one to relate the asymptotic behaviour of the Gromov–Witten invariants of $\mathbb{P}^1$ to tautological intersection numbers over the compactified moduli space of curves $\overline{\mathcal{M}}_{g,n}$.

Corollary 1.3 *For $g = 0$ and 1 and $2g-2+n>0$, the stationary Gromov–Witten invariants of $\mathbb{P}^1$ behave asymptotically as*

$$\left\langle \prod_{i=1}^{k} \tau_{2u_i}(\omega) \prod_{i=k+1}^{n} \tau_{2u_i-1}(\omega) \right\rangle^g \sim \frac{u_{k+1}\cdots u_n}{\prod_{i=1}^{n} u_i!^2} \sum_{|\beta|=3g-3+n} u_1^{\beta_1}\cdots u_n^{\beta_n}\cdot 2^g \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1}\cdots\psi_n^{\beta_n}. \tag{1-6}$$

In the exceptional cases $(g,n)=(0,1)$ and $(0,2)$, the asymptotic form is given by the exact formulae in Section 7.

Section 2 defines the Eynard–Orantin invariants and proves recursions for the Eynard–Orantin invariants of the curve (1-2) analogous to recursions satisfied by the Gromov–Witten invariants of $\mathbb{P}^1$. The definition of Gromov–Witten invariants is contained in Section 3. Section 4 begins by proving a weaker result than Theorem 1.2, which is essentially that Ω_n^g is analytic and extends to a meromorphic multidifferential on a compact Riemann surface, before proving the main results. Section 5 describes the relationship between the defining recursion relations for the Eynard–Orantin invariants and the Virasoro constraints satisfied by the Gromov–Witten invariants of $\mathbb{P}^1$. Numerical checks show that the genus constraint in Theorem 1.1 and hence also in Theorem 1.2 and Corollary 1.3 should be unnecessary. Section 6 gives a non-rigorous matrix model proof of Theorem 1.1 that holds for all genus. Section 7 contains explicit formulae for Eynard–Orantin invariants and Gromov–Witten invariants of $\mathbb{P}^1$.

Acknowledgements

The authors would like to thank Norman Do and the anonymous referee for useful comments.

2 Eynard–Orantin invariants

For every $(g,n)\in\mathbb{Z}^2$ with $g\geq 0$ and $n>0$, the Eynard–Orantin invariant of a Torelli marked Riemann surface with meromorphic functions (C,x,y) is a multidifferential $\omega_n^g(p_1,\dots,p_n)$, ie a tensor product of meromorphic 1–forms on the product C^n, where $p_i \in C$. Recall that a *Torelli marking* of C is a choice of symplectic basis $\{a_i, b_i\}_{i=1,..,g}$ of the first homology group $H_1(\overline{C})$ of the compact closure $\overline{C}$ of C. In particular, a genus-0 surface C requires no Torelli marking. When $2g-2+n>0$, $\omega_n^g(p_1,\dots,p_n)$ is defined recursively in terms of local information around the poles of $\omega_{n'}^{g'}(p_1,\dots,p_n)$ for $2g'+2-n'<2g-2+n$. Equivalently, the $\omega_{n'}^{g'}(p_1,\dots,p_n)$ are used as kernels on the Riemann surface. This is a familiar idea, the main example being the Cauchy kernel which gives the derivative of a function in terms of the bidifferential $dw\,dz/(w-z)^2$ as

$$f'(z)\,dz = \operatorname*{Res}_{w=z} \frac{f(w)\,dw\,dz}{(w-z)^2} = -\sum_\alpha \operatorname*{Res}_{w=\alpha} \frac{f(w)\,dw\,dz}{(w-z)^2},$$

where the sum is over all poles α of $f(w)$.

The Cauchy kernel generalises to a bidifferential $B(w,z)$ on any Riemann surface C that arises from the meromorphic differential $\eta_w(z)\,dz$, unique up to scale, that has a double pole at $w\in C$, no other poles, and all A–periods vanishing. The scale factor can be chosen so that $\eta_w(z)\,dz$ varies holomorphically in w and transforms as a 1–form in w, hence it is naturally expressed as the unique bidifferential on C:

$$B(w,z)=\eta_w(z)\,dw\,dz, \quad \oint_{A_i} B = 0, \quad B(w,z)\sim \frac{dw\,dz}{(w-z)^2} \quad \text{near } w=z.$$

It is symmetric in w and z. The bidifferential $B(w,z)$ is called the *Bergmann kernel* in [8], following Tjurin [19]. It is called the fundamental normalised differential of the second kind on C in Fay [10]. Recall that a meromorphic differential is *normalised* if its A–periods vanish and it is of the *second kind* if its residues vanish. The bidifferential $B(w,z)$ is used to express a normalised differential of the second kind in terms of local information around its poles.

Since each zero α of dx is simple, for any point $p\in C$ close to α there is a unique point $\hat{p}\neq p$ close to α such that $x(\hat{p})=x(p)$. The recursive definition of $\omega_n^g(p_1,\dots,p_n)$ uses only local information around zeros of dx and makes use of the well-defined map $p\mapsto\hat{p}$ there. The invariants are defined as

$$\omega_1^0=-y\,dx(z), \quad \omega_2^0=B(z_1,z_2).$$

For $2g-2+n>0$,

$$\text{(2-1)}\quad \omega_{n+1}^g(z_0,z_S) = \sum_\alpha \operatorname*{Res}_{z=\alpha} K(z_0,z)\Bigg[\omega_{n+2}^{g-1}(z,\hat{z},z_S)+\sideset{}{'}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}} \omega_{|I|+1}^{g_1}(z,z_I)\,\omega_{|J|+1}^{g_2}(\hat{z},z_J)\Bigg],$$

where the sum is over zeros α of dx, $S=\{1,\dots,n\}$, and the notation $\sum'$ means that no ω_1^0 terms appear, ie the sum excludes $(g_1,|I|)=(0,0)$ and $(g_2,|J|)=(0,0)$. The kernel defined by

$$K(z_0,z)=\frac{-\int_{\hat z}^{z}B(z_0,z')}{2(y(z)-y(\hat z))\,dx(z)}$$

is well-defined in the vicinity of each zero of dx. Note that the quotient of any two differentials, regarded as meromorphic sections of a line bundle, is a meromorphic function. Hence the role of the differential $dx(z)$ in the denominator of the kernel K is to send a differential to a meromorphic function, or as in this case send a bidifferential to a differential. The recursion (2-1) depends only on the meromorphic differential $y\,dx$ and the map $p\mapsto\hat p$ around zeros of dx. For $2g-2+n>0$, each ω_n^g is a symmetric multidifferential with poles only at the zeros of dx, of order $6g-4+2n$, and zero residues.

For $2g-2+n>0$, the invariants satisfy the identity

$$\sum_{x(z)=x}\omega_{n+1}^g(z,z_S)=0$$

and the string and dilaton equations [8]

$$\sum_\alpha \operatorname*{Res}_{z=\alpha} y(z)x(z)^m\,\omega_{n+1}^g(z,z_S)=-\sum_{i=1}^n dz_i\,\frac{\partial}{\partial z_i}\left(\frac{x(z_i)^m\omega_n^g(z_S)}{dx(z_i)}\right), \tag{2-2}$$

$$\sum_\alpha \operatorname*{Res}_{z=\alpha}\Phi(z)\,\omega_{n+1}^g(z,z_S)=(2g-2+n)\,\omega_n^g(z_S), \tag{2-3}$$

where $m=0,1$, the sum is over the zeros α of dx, $\Phi(z)=\int^z y\,dx(z')$ is an arbitrary antiderivative and $z_S=(z_1,\dots,z_n)$.

When y is not a meromorphic function on C and is merely analytic in a domain containing the zeros of dx, we approximate it by a sequence of meromorphic functions $y^{(N)}$ that agree with y at the zeros of dx up to the N^{th} derivatives. The sequence $y^{(N)}$ does not necessarily converge to y. For example, the partial sums $y^{(N)}$ of

$$y(z)=\ln z\sim\sum\frac{(1-z^2)^k}{-2k}$$

give a divergent asymptotic expansion for $\ln(z)$ at $z=0$ in the region $\mathrm{Re}(z^2)>0$.

The meromorphic functions $y^{(N)}$ can be used in the recursions defining ω_n^g in place of $y(z)$ since they contain the same local information around $z=\pm1$ up to order N. More precisely, to define ω_n^g for (C,x,y) it is sufficient to use $(C,x,y^{(N)})$ for any $N\ge 6g-6+2n$.

2.1 Polynomial behaviour

In this section we consider the family of curves

$$\widetilde{C} = \begin{cases} x = z + 1/z, \\ y = y(z), \end{cases} \tag{2-4}$$

for $y(z)$ any analytic function defined on a domain of $\mathbb{C}$ containing ± 1.

With respect to the local coordinate x on C each invariant ω_n^g has an analytic expansion around a branch of $x = \infty$. Define the coefficients of this expansion

$$\omega_n^g =: \sum_{b_1,\dots,b_n=1}^{\infty} \frac{M_{n,k}^g(b_1,\dots,b_n)}{x_1^{b_1+1}\cdots x_n^{b_n+1}} dx_1 \cdots dx_n$$

for k the number of odd b_i. We may abuse this notation by writing $M_n^g = M_{n,k}^g$ when k is clear.

In Norbury and Scott [15] it was shown that Eynard–Orantin invariants of such a curve can be expressed via polynomials:

Lemma 2.1 [15] *For the curve $x = z + 1/z$, $y = y(z)$ and $2g - 2 + n > 0$, $\omega_n^g(z_1,\dots,z_n)$ has an expansion around $\{z_i = 0\}$ given by*

$$\omega_n^g(z_1,\dots,z_n) = \frac{d}{dz_1}\cdots\frac{d}{dz_n}\sum_{b_i>0} N_n^g(b_1,\dots,b_n) z_1^{b_1}\cdots z_n^{b_n}\, dz_1\cdots dz_n, \tag{2-5}$$

where N_n^g is a symmetric quasi-polynomial in the b_i^2 of degree $3g-3+n$, dependent on the parity of the b_i.

Recall that a function on $\mathbb{Z}^n$ is quasi-polynomial if it is polynomial on each coset of a sublattice $\Gamma \subset \mathbb{Z}^n$ and it is symmetric if it is invariant under the permutation group S_n. In particular, each polynomial is invariant under permutations that preserve the corresponding coset. The function N_n^g is polynomial on each coset of $2\mathbb{Z}^n \subset \mathbb{Z}^n$. By symmetry, we can represent its 2^n polynomials by the n polynomials $N_{n,k}^g(b_1,\dots,b_n)$, for $k = 1,\dots,n$, symmetric in $b_1,\dots,b_k$ and $b_{k+1},\dots,b_n$, corresponding to the first k variables being odd.

Lemma 2.2 [15] *The coefficients of the top homogeneous degree terms in the polynomial $N_{n,k}^g(b_1,\dots,b_n)$, defined above, can be expressed in terms of intersection numbers of ψ classes on $\overline{\mathcal{M}}_{g,n}$. For $\sum_i \beta_i = 3g-3+n$, the coefficient v_β of $\prod b_i^{2\beta_i}$ is*

$$v_\beta = \frac{y'(1)^{2-2g-n} + (-1)^k y'(-1)^{2-2g-n}}{2^{5g-5+2n}\beta_1!\cdots\beta_n!}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1}\cdots\psi_n^{\beta_n}.$$

In particular, the proofs are constructive, showing how to calculate such polynomials from ω_n^g, and lead to explicit formulae for the M_n^g via the following lemma. It is important to point out that $z=0$ and $z=\infty$ correspond to the two branches at $x=\infty$. The expansion in z is around $z=0$ while the expansion in x is around the other branch $z=\infty$. This is essentially due to the need for both expansions to have positive coefficients.

Lemma 2.3 *For the curve $x=z+1/z$, $y=y(z)$,*

$$M_n^g(b_1,\dots,b_n)=\sum_{l_i>b_i/2}^{b_i} N_n^g(2l_1-b_1,\dots,2l_n-b_n)\prod_{i=1}^n(2l_i-b_i)\binom{b_i}{l_i}. \tag{2-6}$$

Proof Extract the coefficients of a local expansion of ω_n^g in x_i^{-1} by taking residues.

$$\begin{aligned}
M_n^g&(b_1,\dots,b_n)\\
&:=(-1)^n \operatorname*{Res}_{x_1=\infty}\cdots\operatorname*{Res}_{x_n=\infty} x_1^{b_1}\cdots x_n^{b_n}\cdot\omega_n^g(z_1,\dots,z_n)\\
&=(-1)^n \operatorname*{Res}_{z_1=\infty}\cdots\operatorname*{Res}_{z_n=\infty} x_1^{b_1}\cdots x_n^{b_n}\cdot\omega_n^g(z_1,\dots,z_n)\\
&=\operatorname*{Res}_{z_1=0}\cdots\operatorname*{Res}_{z_n=0} x_1^{b_1}\cdots x_n^{b_n}\cdot\omega_n^g(z_1,\dots,z_n)\\
&=\prod_{i=1}^n\operatorname*{Res}_{z_i=0}\Big(\frac{1}{z_i}+z_i\Big)^{b_i}\sum_{k_1,\dots,k_n=1}^{\infty}N_n^g(k_1,\dots,k_n)\prod_{i=1}^n k_i z_i^{k_i-1}dz_i\\
&=\prod_{i=1}^n\operatorname*{Res}_{z_i=0}\sum_{l_1,\dots,l_n=0}^{b_i}\sum_{k_1,\dots,k_n=1}^{\infty}N_n^g(k_1,\dots,k_n)\prod_{i=1}^n k_i\binom{b_i}{l_i}z_i^{b_i-2l_i+k_i-1}dz_i\\
&=\sum_{l_i>b_i/2}^{b_i}N_n^g(2l_1-b_1,\dots,2l_n-b_n)\prod_{i=1}^n(2l_i-b_i)\binom{b_i}{l_i},
\end{aligned}$$

where the step from line 3 to line 4 uses

$$\omega_n^g(1/z_1,\dots,1/z_n)=(-1)^n\omega_n^g(z_1,\dots,z_n). \qquad \square$$

Analogous to the notation $N_{n,k}^g(b_1,\dots,b_n)$, which is the polynomial expression for N_n^g corresponding to the first k variables being odd, since the sum (2-6) respects parity, we define $M_{n,k}^g(b_1,\dots,b_n)$ to be the expression for M_n^g with k odd variables, obtained by summing $N_{n,k}^g$ terms.

Lemma 2.4 *$M^g_{n,k}(b_1,\dots,b_n)$ can be obtained from $N^g_{n,k}(b_1,\dots,b_n)$ via the term-by-term transform on monomials*

$$b_1^{2\alpha_1}\cdots b_n^{2\alpha_n} \mapsto \prod_{i=1}^{k} b_i \binom{b_i-1}{\frac{b_i-1}{2}} q_{\alpha_i}\left(\frac{b_i-1}{2}\right) \prod_{i=k+1}^{n} \frac{b_i}{2}\binom{b_i}{\frac{b_i}{2}} p_{\alpha_i}\left(\frac{b_i}{2}\right), \tag{2-7}$$

where $q_\alpha(n)$ and $p_\alpha(n)$ are polynomials of degree α satisfying the recurrences

$$p_{\alpha+1}(n) = 4n^2(p_\alpha(n) - p_\alpha(n-1)) + 4np_\alpha(n-1), \qquad p_0(n) = 1, \tag{2-8}$$

$$q_{\alpha+1}(n) = (2n+1)^2 q_\alpha(n) - 4n^2 q_\alpha(n-1), \qquad q_0(n) = 1. \tag{2-9}$$

Proof As the sum (2-6) is over all combinations of l_i for each i, for monomial terms of several variables we can factorise

$$\sum_{l_i>b_i/2}^{b_i} \prod_{i=1}^{n} (2l_i-b_i)^{2\alpha_i+1}\binom{b_i}{l_i} = \prod_{i=1}^{n} \sum_{l_i>b_i/2}^{b_i} (2l_i-b_i)^{2\alpha_i+1}\binom{b_i}{l_i} \tag{2-10}$$

to reduce the problem to the one-variable case. For different parities $b=2n$ and $b=2n+1$, the sums become

$$\sum_{l>b/2}^{b} (2l-b)^{2\alpha+1}\binom{b}{l} = \begin{cases} \sum_{l=0}^{n} \binom{2n}{n-l}(2l)^{2\alpha+1} & b=2n, \\ \sum_{l=0}^{n} \binom{2n+1}{n-l}(2l+1)^{2\alpha+1} & b=2n+1. \end{cases} \tag{2-11}$$

after exchanging $l \mapsto n-l$. From Tuenter [20], the sum

$$\widetilde{p}_\alpha(n) := \sum_{l=0}^{n} \binom{2n}{n-l}(2l)^{2\alpha+1}$$

satisfies the three-term recurrence

$$\widetilde{p}_{\alpha+1}(n) = 4n^2\,\widetilde{p}_\alpha(n) - 8n(2n-1)\,\widetilde{p}_\alpha(n-1), \quad \widetilde{p}_0(n) = n\binom{2n}{n}.$$

Letting $\widetilde{p}_\alpha(n) = p_\alpha(n) n\binom{2n}{n}$ gives the required recursion (2-8) for p_α. The proof for the odd case proceeds in the same manner, this time starting from the three-term recursion

$$\begin{aligned} \widetilde{q}_{\alpha+1}(n) &= (2n+1)^2\widetilde{q}_\alpha(n) - 8n(2n+1)\widetilde{q}_\alpha(n-1), \\ \widetilde{q}_0(n) &= (2n+1)\binom{2n}{n}. \end{aligned}$$

□

The first few transformation polynomials (in the form useful for (2-7)) are:

$$p_0\Big(\frac{b}{2}\Big) = 1 \qquad q_0\Big(\frac{b-1}{2}\Big) = 1$$
$$p_1\Big(\frac{b}{2}\Big) = 2b \qquad q_1\Big(\frac{b-1}{2}\Big) = 2b-1$$
$$p_2\Big(\frac{b}{2}\Big) = 8b(b-1) \qquad q_2\Big(\frac{b-1}{2}\Big) = 8b^2-12b+5$$
$$p_3\Big(\frac{b}{2}\Big) = 16b(3b^2-8b+6) \qquad q_3\Big(\frac{b-1}{2}\Big) = 48b^3-152b^2+166b-61$$

The p_α and q_α are generalisations of the Gandhi polynomials, related to the Dumont–Foata polynomials. See [20] and the references therein for a survey and properties of these topics.

Proposition 2.5 *For $2g-2+n>0$, the coefficients $M^g_{n,k}$ in the expansion of the Eynard–Orantin invariants of* (2-4) *about $x=\infty$ can be expressed as*

$$M^g_{n,k} = \prod_{i=1}^n \Big\lfloor \frac{b_i+1}{2} \Big\rfloor \binom{b_i}{\lfloor \frac{1}{2} b_i \rfloor} m^g_{n,k}(b_1,\dots,b_n), \tag{2-12}$$

where $\lfloor r \rfloor$ is the integer part of r and $m^g_{n,k}(b_1,\dots,b_n)$ is a polynomial of degree $3g-3+n$, symmetric in variables of the same parity, with coefficient v_β of $b_1^{\beta_1}\cdots b_n^{\beta_n}$ given by

$$v_\beta = \frac{y'(1)^{2-2g-n}+(-1)^k y'(-1)^{2-2g-n}}{2^{2g-2+n}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1}\cdots\psi_n^{\beta_n} \tag{2-13}$$

for $|\beta| = 3g-3+n$.

Proof Expand the Eynard–Orantin invariants about $z=0$, and apply Lemmas 2.1, 2.2 and 2.4 to get expressions for M^g_n. To prove the proposition we need the polynomials $p_\alpha(b/2)$ and $q_\alpha((b-1)/2)$ used in the transformation (2-7) to have leading order coefficients $\alpha!\,2^\alpha$.

By induction, suppose $p_\alpha(n)$ has leading coefficient $\alpha!\,2^{2\alpha}$. Using the recursion for $p_{\alpha+1}(n)$, the leading part of $p_{\alpha+1}(n)$ is

$$\begin{aligned}\alpha!\,2^{2\alpha}(4n^2(n^\alpha-(n-1)^\alpha)) + 4\alpha!\,2^{2\alpha}n^{\alpha+1} + O(n^\alpha)&\\ &= \alpha!\,2^{2\alpha}(4n^2(\alpha n^{\alpha-1}) + 4n(n^\alpha)) + O(n^\alpha)\\ &= (\alpha+1)!\,2^{2\alpha+2}n^{\alpha+1} + O(n^\alpha).\end{aligned}$$

Similarly, the recursion for $q_{\alpha+1}(n)$ shows that the leading part of $q_{\alpha+1}(n)$ is

$$\alpha!2^{2\alpha}(4n^2(\alpha n^{\alpha-1}) + (4n+1)n^\alpha) + O(n^{\alpha-1}) = (\alpha+1)!2^{2\alpha+2}n^{\alpha+1} + O(n^\alpha)$$

so that all transformation polynomials have the required leading order coefficients. □

2.2 Divisor and string equations

For the remainder of the paper we specialise to the curve (1-2)

$$C = \begin{cases} x = z + 1/z, \\ y = \ln z. \end{cases}$$

The recursions (2-14) and (2-15) below use the terms *divisor* and *string* equations, which anticipate the corresponding recursions (3-4) and (3-5) satisfied by Gromov–Witten invariants.

Theorem 2.6 *The coefficients M_n^g in the expansion of the Eynard–Orantin invariants of* (1-2) *about $x = \infty$ satisfy the divisor and string equations. For $2d = 2 - 2g - n + \sum_{i=1}^n b_i$,*

$$M_{n+1}^g(1, b_1, \ldots, b_n) = d M_n^g(b_1, \ldots, b_n), \tag{2-14}$$

$$M_{n+1}^g(0, b_1, \ldots, b_n) = \sum_{i=1}^n b_i M_n^g(b_1, \ldots, b_i - 1, \ldots, b_n), \tag{2-15}$$

where

$$M_{n+1,k}^g(0, b_1, \ldots, b_n) := \prod_{i=1}^k b_i \binom{b_i - 1}{\frac{b_i-1}{2}} \prod_{i=k+1}^n \frac{b_i}{2} \binom{b_i}{\frac{b_i}{2}} m_{n+1,k}^g(0, b_1, \ldots, b_n)$$

is defined by substituting $b_0 = 0$ in the quasi-polynomial defined by Proposition 2.5. Equations (2-14) *and* (2-15) *uniquely determine all genus-zero terms and, together with any top-degree term known from Proposition 2.5, all genus-one terms, from the initial cases M_3^0 and M_1^1.*

Proof In the following we use $\int_0^z \omega_{n+1}^g(z', z_S)$, which is well-defined (independently of the choice of path) since the residues of ω_{n+1}^g are zero. The calculations below will not be sensitive to the constant term arising from the choice of initial point 0 in the integral. To prove Equation (2-14), we use the difference of the string and dilaton equations (2-2) minus (2-3):

$$\begin{aligned} \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha}\Big(yx - \int y\,dx\Big)\,\omega_{n+1}^g(z, z_S) &= \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha}\Big(z - \frac{1}{z} + c\Big)\,\omega_{n+1}^g(z, z_S) \\ &= -\sum_{\alpha=0,\infty} \operatorname*{Res}_{z=\alpha}\Big(z - \frac{1}{z}\Big)\,\omega_{n+1}^g(z, z_S) \\ &= -2 \operatorname*{Res}_{z=\infty} z\,\omega_{n+1}^g(z, z_S) \end{aligned}$$

$$= -2 \operatorname*{Res}_{z=\infty} x(z)\,\omega^g_{n+1}(z, z_S)$$
$$= 2 \sum_{b_1,\dots,b_n=1}^{\infty} \frac{M^g_{n+1}(1, b_1, \dots, b_n)}{x_1^{b_1+1} \cdots x_n^{b_n+1}}\, dx_1 \cdots dx_n,$$

where line 1 above has used

$$yx - \int y\,dx = \ln(z) + \frac{\ln(z)}{z} - \int_{z_0}^{z} \ln(t)\Big(1 - \frac{1}{t^2}\Big)\,dt = z - \frac{1}{z} + c$$

and we have used $\omega^g_{n+1}(1/z, z_S) = -\omega^g_{n+1}(z, z_S)$ to go from line 2 to line 3 and added a residue free term to go from line 3 to line 4.

The right-hand side of (2-2)–(2-3) gives

$$-\sum_{i=1}^{n} dz_i \frac{\partial}{\partial z_i}\bigg(\frac{x(z_i)\omega^g_n(z_S)}{dx(z_i)}\bigg) - (2g-2+n)\omega^g_n(z_S)$$
$$= -\sum_{i=1}^{n} dx_i\, \frac{\partial}{\partial x_i}\bigg(x_i \sum_{b_1,\dots,b_n=1}^{\infty} \frac{M^g_n(b_1, \dots, b_n)}{x_1^{b_1+1} \cdots x_n^{b_n+1}}\, dx_1 \cdots d\hat{x}_i \cdots dx_n\bigg)$$
$$- (2g-2+n) \sum_{b_1,\dots,b_n=1}^{\infty} \frac{M^g_n(b_1, \dots, b_n)}{x_1^{b_1+1} \cdots x_n^{b_n+1}}\, dx_1 \cdots dx_n$$
$$= \bigg(\sum_{i=1}^{n} b_i + 2 - 2g - n\bigg) \sum_{b_1,\dots,b_n=1}^{\infty} \frac{M^g_n(b_1, \dots, b_n)}{x_1^{b_1+1} \cdots x_n^{b_n+1}}\, dx_1 \cdots dx_n\,.$$

Equating coefficients and using $2d = 2 - 2g - n + \sum_{i=1}^{n} b_i$ gives (2-14) as required.

To prove (2-15) take $m = 0$ in (2-2). When expanded around $x_i = \infty$ the negative of the right-hand side gives

$$\sum_{i=1}^{n} dz_i\, \frac{\partial}{\partial z_i}\bigg(\frac{\omega^g_n(z_1, \dots, z_n)}{dx(z_i)}\bigg)$$
$$= \sum_{i=1}^{n} dx_i\, \frac{\partial}{\partial x_i} \sum_{b_1,\dots,b_n=1}^{\infty} \frac{M^g_n(b_1, \dots, b_n)}{x_1^{b_1+1} \cdots x_n^{b_n+1}}\, dx_1 \cdots d\hat{x}_i \cdots dx_n$$
$$= -\sum_{b_1,\dots,b_n=1}^{\infty} \frac{\sum_{i=1}^{n}(b_i+1) M^g_n(b_1, \dots, b_n)}{x_1^{b_1+1} \cdots x_i^{b_i+2} \cdots x_n^{b_n+1}}\, dx_1 \cdots dx_n$$
$$= -\sum_{b_1,\dots,b_n=1}^{\infty} \frac{\sum_{i=1}^{n} b_i M^g_n(b_1, \dots, b_i - 1, \dots, b_n)}{x_1^{b_1+1} \cdots x_n^{b_n+1}}\, dx_1 \cdots dx_n\,,$$

where $d\hat{x}_i$ means that dx_i is missing from the first term. For the left-hand side, we need the following lemma.

Lemma 2.7 *Let $F(z)=\sum_{n=1}^{\infty} p(n)z^n$ for a quasi-polynomial $p(n)$. Then $F(z)$ is a meromorphic function on $\mathbb{P}^1$, analytic at 0 and ∞, satisfying $F(\infty)-F(0)=-p(0)$.*

Proof Recall that $p(n)$ is quasi-polynomial in n if it is polynomial on each coset of a sublattice $m\mathbb{Z}\subset\mathbb{Z}$, ie it is represented by m polynomials $p_a(n)$, $a=1,\dots,m$, for $n\equiv a(m)$, and $p(0):=p_m(0)$.

Decompose $F(z)$ into

$$F(z)=\sum_{n=1}^{\infty} p(n)z^n=\sum_{a=1}^{m}\sum_{0<n\equiv a(m)} p_a(n)z^n$$

and further decompose $p_a(n)$ into linear combinations of monomials n^k. Then

$$\sum_{0<n\equiv a(m)} n^k z^n=\Big(z\frac{d}{dz}\Big)^k\sum_{0<n\equiv a(m)} z^n=\Big(z\frac{d}{dz}\Big)^k\frac{z^a}{1-z^m}$$

which vanishes at $z=\infty$, since the denominator has greater degree than the numerator, except when $k=0$ and $a=m$, where it evaluates to -1 at $z=\infty$. □

The left-hand side of (2-2) now becomes

$$\begin{aligned}
&\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\ln(z)\omega^g_{n+1}(z,z_S)\\
&\qquad=-\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\frac{dz}{z}\int_0^z\omega^g_{n+1}(z',z_S) &&\text{(Integrating by parts)}\\
&\qquad=\sum_{\alpha=0,\infty}\operatorname*{Res}_{z=\alpha}\frac{dz}{z}\int_0^z\omega^g_{n+1}(z',z_S)\\
&\qquad=\operatorname*{Res}_{z=\infty}\frac{dz}{z}\int_0^z\omega^g_{n+1}(z',z_S) &&\text{(Analytic at } z=0)\\
&\qquad=-\int_0^{\infty}\omega^g_{n+1}(z,z_S)\\
&\qquad=\sum_{k_1,\dots,k_n=1}^{\infty}k_1\cdots k_nN^g_{n+1}(0,k_S)z_S^{k_S-1}\,dz_S &&\text{(Lemmas 2.1, 2.7)}
\end{aligned}$$

$$= \sum_{b_1,\dots,b_n=1}^{\infty} \prod_{i=1}^{n} \left\lfloor \frac{b_i+1}{2} \right\rfloor \binom{b_i}{\lfloor \frac{1}{2} b_i \rfloor} m^g_{n+1,k}(0, b_S) \prod \frac{dx_i}{x_i^{b_i+1}}$$

$$= \sum_{b_1,\dots,b_n=1}^{\infty} M^g_{n+1}(0, b_1, \dots, b_n) \frac{dx_1 \cdots dx_n}{x_1^{b_1+1} \cdots x_n^{b_n+1}},$$

where in the final step we have changed the first n variables from expansions in z to x using the transform from Lemma 2.4. The last variable contains only a constant term, which remains unchanged under this transform. Comparing this expression with (2-16) implies the string equation (2-15).

To complete the proof of Theorem 2.6 we need to show that the divisor and string equations uniquely determine the genus-zero and genus-one Eynard–Orantin invariants. To do this, one needs to prove the following auxiliary lemma.

Lemma 2.8 *Let $f_k(t_1, \dots, t_n)$ be a polynomial symmetric in the variables $t_1, \dots, t_k$ and also symmetric in the variables $t_{k+1}, \dots, t_n$. Evaluation at, for any a and b, $f_k(a, t_2, \dots, t_n)$ and $f_k(t_1, \dots, t_{n-1}, b)$, determines any such f_k of degree less than n, and if the degree of f_k equals n, it determines f_k up to a constant. If $k = 0$ or n, ie f_k is symmetric in all of its variables, then we evaluate at only one variable a or b.*

Proof Suppose $g_k(t_1, \dots, t_n)$ were another polynomial of the same degree as f_k, symmetric in the corresponding variables and satisfying $g_k(a, t_2, \dots, t_n) = f_k(a, t_2, \dots, t_n)$ and $g_k(t_1, \dots, t_{n-1}, b) = f_k(t_1, \dots, t_{n-1}, b)$. Define $h_k(t_1, \dots, t_n) = g_k(t_1, \dots, t_n) - f_k(t_1, \dots, t_n)$. Then $h_k(a, t_2, \dots, t_n) = 0 = h_k(t_1, \dots, t_{n-1}, b)$. By symmetry,

$$h_k(t_1, \dots, t_n) = \prod_{i=1}^{k} (t_i - a) \prod_{i=k+1}^{n} (t_i - b) \widetilde{h}_k(t_1, \dots, t_n)$$

for some other polynomial $\widetilde{h}_k$. If $\deg f_k = \deg h_k < n$ then $\widetilde{h}_k \equiv 0$ and $g_k(t_1, \dots, t_n) = f_k(t_1, \dots, t_n)$. If $\deg f_k = \deg h_k$ then $\widetilde{h}_k \equiv \lambda$ is constant and

$$g_k(t_1, \dots, t_n) = f_k(t_1, \dots, t_n) + \lambda \prod_{i=1}^{k} (t_i - a) \prod_{i=k+1}^{n} (t_i - b).$$

If $k = 0$ or n, the argument requires evaluation at only $t_n = b$ or $t_1 = a$, respectively. □

For any $k = 1, \dots, n$, the divisor equation (2-14) and string equation (2-15) allow us to compute $m^g_{n+1,k}(0, b_1, \dots, b_n)$ and $m^g_{n+1,k}(1, b_1, \dots, b_n)$ from $m^g_{n,k}$. (If $k = 0$ or $n+1$, the string or divisor equation respectively are alone sufficient to determine $m^0_{n+1,k}$ using precisely the same argument.) For $g = 0$, and each k, $m^0_{n+1,k}(b_0, b_1, \dots, b_n)$

is a polynomial of degree $n-2$, symmetric in variables of the same parity. Hence Lemma 2.8 shows that $m^0_{n+1,k}(b_0,b_1,\dots,b_n)$ is uniquely determined from $m^0_{n,k}$.

For $g=1$ and each k, $m^1_{n+1,k}(b_0,b_1,\dots,b_n)$ is a polynomial of degree $n+1$, symmetric in variables of the same parity. Hence Lemma 2.8 shows that the string and dilaton equations determine $m^1_{n+1,k}(b_0,b_1,\dots,b_n)$ from $m^1_{n,k}$ up to $\lambda\cdot\prod_{i=0}^n b_i$. The constant λ can be determined from Proposition 2.5, which gives the coefficients of all top degree terms in terms of intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In particular, the coefficient of $b_0\cdots b_n$ in $m^1_{n+1,k}(b_0,\dots,b_n)$ is $2^{1-n}\langle\tau_1^n\rangle=2^{1-n}(n-1)!/24$. □

3 Gromov–Witten invariants

3.1 The moduli space of stable maps

Let X be a projective algebraic variety and consider $(C,x_1,\dots,x_n)$ a connected smooth curve of genus g with n distinct marked points. For $\beta\in H_2(X,\mathbb{Z})$, the moduli space of maps $\mathcal{M}^g_n(X,\beta)$ consists of morphisms

$$\pi\colon (C,x_1,\dots,x_n)\to X$$

satisfying $\pi_*[C]=\beta$ quotiented by isomorphisms of the domain C that fix each x_i. The moduli space has a compactification $\overline{\mathcal{M}}^g_n(X,\beta)$ given by the moduli space of stable maps: the domain C is a connected nodal curve, the distinct points $\{x_1,\dots,x_n\}$ avoid the nodes, any genus-zero irreducible component of C with fewer than three distinguished points (nodal or marked) must not be collapsed to a point and any genus-one irreducible component of C with no marked point must not be collapsed to a point. The moduli space of stable maps has irreducible components of different dimensions but its expected or virtual dimension is

$$\dim\overline{\mathcal{M}}^g_n(X,\beta)=\langle c_1(X),\beta\rangle+(\dim X-3)(1-g)+n.$$

3.1.1 Cohomology on $\overline{\mathcal{M}}^g_n(X,\beta)$ Let $\mathcal{L}_i$ be the cotangent bundle over the i^{th} marked point and $\psi_i\in H^2(\overline{\mathcal{M}}^g_n(X,\beta),\mathbb{Q})$ be the first Chern class of $\mathcal{L}_i$.

For $i=1,\dots,n$ there exist evaluation maps

(3-1) $$\mathrm{ev}_i\colon \overline{\mathcal{M}}^g_n(X,\beta)\to X,\quad \mathrm{ev}_i(\pi)=\pi(x_i)$$

and classes $\gamma\in H^*(X,\mathbb{Z})$ pull back to classes in $H^*(\overline{\mathcal{M}}^g_n(X,\beta),\mathbb{Q})$

(3-2) $$\mathrm{ev}_i^*\colon H^*(X,\mathbb{Z})\to H^*(\overline{\mathcal{M}}^g_n(X,\beta),\mathbb{Q}).$$

Gromov–Witten theory involves integrating cohomology classes, often called descendent classes, of the form

$$\tau_{b_i}(\gamma) = \psi_i^{b_i}\,\mathrm{ev}_i^*(\gamma).$$

These are integrated against the *virtual fundamental class*, $[\overline{\mathcal{M}}_n^g(X,\beta)]^{\mathrm{vir}}$, the existence and construction of which is highly nontrivial.

Gromov–Witten invariants quite generally satisfy divisor, string and dilaton equations (Witten [21]) and topological recursion relations arising from relations on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ (Getzler [11]). We will write these relations only in the special case when the target is $\mathbb{P}^1$.

3.2 Specialising to $\mathbb{P}^1$

We now only consider the specific case of Gromov–Witten invariants of $\mathbb{P}^1$. Let $\omega \in H^2(\mathbb{P}^1,\mathbb{Q})$ be the Poincaré dual class of a point and $1 \in H^0(\mathbb{P}^1,\mathbb{Q})$ Poincaré dual to the fundamental class. We consider the invariants

$$\left\langle \prod_{i=1}^{l} \tau_{b_i}(1) \prod_{i=l+1}^{n} \tau_{b_i}(\omega) \right\rangle_d^g = \int_{[\overline{\mathcal{M}}_n^g(\mathbb{P}^1,d)]^{\mathrm{vir}}} \prod_{i=1}^{l} \psi_i^{b_i} \prod_{i=l+1}^{n} \psi_i^{b_i}\,\mathrm{ev}_i^*(\omega), \tag{3-3}$$

where we consider only connected invariants and (3-3) is defined to be zero unless $\sum_{i=1}^n b_i = 2g-2+2d+l$. In our notation, often either g or d will be missing when clear, since the dimension restraints define one from the other. Our main interest is the case $l=0$, known as the (connected) stationary Gromov–Witten theory of $\mathbb{P}^1$ since the images of the marked points are fixed.

We collect here a few properties of Gromov–Witten invariants of $\mathbb{P}^1$ needed here. We recommend reading [16; 17; 18] for a thorough treatment of this case. We use the following divisor, string and dilaton equations [21] principally for stationary Gromov–Witten invariants. For $2d = 2-2g+\sum_{i=1}^n b_i - l$, and $\alpha_i \in \{1,\omega\}$ we have:

(3-4) Divisor equation

$$\langle \tau_0(\omega)\tau_{b_1}(\alpha_1)\cdots\tau_{b_n}(\alpha_n)\rangle_d = d\langle \tau_{b_1}(\alpha_1)\cdots\tau_{b_n}(\alpha_n)\rangle_d + \sum_{i=1}^n \langle \tau_{b_1}(\alpha_1)\cdots\tau_{b_i-1}(\alpha_i\cup\omega)\cdots\tau_{b_n}(\alpha_n)\rangle_d.$$

(3-5) String equation

$$\langle \tau_0(1)\tau_{b_1}(\alpha_1)\cdots\tau_{b_n}(\alpha_n)\rangle_d = \sum_{i=1}^n \langle \tau_{b_1}(\alpha_1)\cdots\tau_{b_i-1}(\alpha_i)\cdots\tau_{b_n}(\alpha_n)\rangle_d.$$

(3-6) Dilaton equation

$$\langle \tau_1(1)\tau_{b_1}(\alpha_1)\cdots\tau_{b_n}(\alpha_n)\rangle^g = (2g-2+n)\langle \tau_{b_1}(\alpha_1)\cdots\tau_{b_n}(\alpha_n)\rangle^g,$$

where we define $\tau_b(0)=0$. Consider the generating function for descendent classes,

$$F = \exp\sum_{b=0}^{\infty}(t_b\tau_b(\omega)+s_b\tau_b(1)).$$

For $\alpha_i \in \{1,\omega\}$ the genus-zero topological recursion [21] is

$$\begin{aligned}\text{(3-7)}\quad &\langle \tau_{b_1}(\alpha_1)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)F\rangle^0\\ &= \langle \tau_0(1)\tau_{b_1-1}(\alpha_1)F\rangle^0\langle \tau_0(\omega)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)F\rangle^0\\ &\quad + \langle \tau_0(\omega)\tau_{b_1-1}(\alpha_1)F\rangle^0\langle \tau_0(1)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)F\rangle^0\end{aligned}$$

and the genus-one topological recursion is

$$\begin{aligned}\text{(3-8)}\quad &\langle \tau_{b_1}(\alpha_1)F\rangle^1\\ &= \langle \tau_0(1)\tau_{b_1-1}(\alpha_1)F\rangle^0\langle \tau_0(\omega)F\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\alpha_1)F\rangle^0\langle \tau_0(1)F\rangle^1\\ &\quad + \tfrac{1}{12}\langle \tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\alpha_1)F\rangle^0.\end{aligned}$$

In [16], Okounkov and Pandharipande show that for Gromov–Witten invariants that allow disconnected domains (denoted by the superscript $^\bullet$) the following relation holds:

$$\text{(3-9)}\qquad \left\langle \prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle_d^\bullet = \sum_{|\lambda|=d}\left(\frac{\dim\lambda}{d!}\right)^2\prod_{i=1}^{n}\frac{\mathbf{p}_{b_i+1}(\lambda)}{(b_i+1)!},$$

where the sum is over all partitions of d and for a partition λ, $\mathbf{p}_k(\lambda)$ is the shifted symmetric power sum defined by

$$\mathbf{p}_k(\lambda) = \sum_{i=1}^{\infty}[(\lambda_i - i + \tfrac12)^k - (-i+\tfrac12)^k] + (1-2^{-k})\zeta(-k).$$

4 Proof of Theorem 1.1

The strategy of the proof of Theorem 1.1 will be to use recursions to uniquely determine both the Eynard–Orantin invariants and the Gromov–Witten invariants of $\mathbb{P}^1$ and compare. The obvious candidates for the genus-0 and 1 Eynard–Orantin invariant are the divisor and string equations, (2-14) and (2-15). The genus-0 and 1 Gromov–Witten invariants of $\mathbb{P}^1$ are determined by the topological recursion relations (3-7) and (3-8). However, the two sets of recursion relations are not compatible, so we first produce new recursion relations for the stationary Gromov–Witten invariants of $\mathbb{P}^1$, given in Section 4.2, which are interesting in their own right, and serve our purposes here.

4.1 Polynomial behaviour of Gromov–Witten invariants

We begin by proving the following weaker version of Theorem 1.2.

Proposition 4.1 *For $g=0$ and 1, the stationary Gromov–Witten invariants are of the form*

$$\Big\langle \prod_{i=1}^{k} \tau_{2u_i}(\omega) \prod_{i=k+1}^{n} \tau_{2u_i-1}(\omega)\Big\rangle^g = \frac{u_{k+1}\cdots u_n}{\prod_{i=1}^n u_i!^2}\, p^g_{n,k}(u_1,\dots,u_n), \tag{4-1}$$

where $p^g_{n,k}(u_1,\dots,u_n)$ is a polynomial of degree $3g-3+n$ in the u_i, symmetric in the first k and the last $n-k$ variables.

Proof We prove this by induction using the topological recursion relations for genus-zero and genus-one Gromov–Witten invariants.

Genus-zero case

4.1.1 Initial cases The recursion (3-7) can be used along with the string and divisor equations to explicitly find expressions for genus-zero 1, 2 and 3–point invariants. The topological recursion relation (3-7) shows that the genus-zero one point invariants satisfy $\langle\tau_{2d-2}(\omega)\rangle^0 = \frac{1}{d^2}\langle\tau_{2d-4}(\omega)\rangle^0$, hence

$$\langle\tau_{2u}(\omega)\rangle^0 = \frac{1}{(u+1)!^2} = \frac{1}{u!^2}\frac{1}{(u+1)^2}. \tag{4-2}$$

Set $\alpha_i=\omega$, $F=1$, $b_1=2u_1$, $b_2=2u_2$ and $b_3=0$ in (3-7) and apply the string and divisor equations to get

$$\begin{aligned}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_0(\omega)\rangle^0 &= (u_2+1)(u_2+1)\langle\tau_{2u_1-2}(\omega)\rangle^0\langle\tau_{2u_2}(\omega)\rangle^0+0\\ &= \frac{(u_2+1)^2}{u_1!^2\,(u_2+1)!^2} = \frac{1}{u_1!^2\,u_2!^2}\end{aligned}$$

and similarly, to calculate $\langle\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\tau_0(\omega)\rangle^0$, set $\alpha_i=\omega$, $F=1$, $b_1=2u_1-1$, $b_2=2u_2-1$ and $b_3=0$ in (3-7). This yields

$$\begin{aligned}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^0 &= \frac{1}{u_1!^2u_2!^2}\frac{1}{u_1+u_2+1},\ \langle\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^0\\ &= \frac{u_1u_2}{u_1!^2\,u_2!^2}\frac{1}{u_1+u_2}.\end{aligned} \tag{4-3}$$

As mentioned in the introduction, the one-point and two-point functions still satisfy Proposition 4.1 if we interpret degree -2 and -1 polynomials to mean the reciprocals of degree 2 and degree 1 polynomials.

We can now use (3-7), the string and divisor equations, to compute the initial step of the induction; the three point invariants

$$\begin{aligned}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_{2u_3}(\omega)\rangle^0 &= \frac{1}{u_1!^2\,u_2!^2\,u_3!^2},\\ \langle\tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3-1}(\omega)\rangle^0 &= \frac{u_2u_3}{u_1!^2\,u_2!^2\,u_3!^2}.\end{aligned} \tag{4-4}$$

Before we apply the inductive step, we need the following lemma.

Lemma 4.2 *Proposition 4.1 can be extended to include $\tau_0(1)$ terms.*

Proof This uses the string equation (3-5). Suppose Proposition 4.1 holds for the right-hand side of the string equation (3-5). Then we must check that the left-hand side is the required degree polynomial. Let $K=\{1,\dots,k\}$ and $J=\{k+1,\dots n\}$. The equation can be written as

$$\begin{aligned}&\left\langle\tau_0(1)\prod_{i=1}^{k}\tau_{2u_i}(\omega)\prod_{i=k+1}^{n}\tau_{2u_i-1}(\omega)\right\rangle^g\\ &\quad=\sum_{i=1}^{k}\frac{u_iu_{k+1}\dots u_n}{u_1!^2\dots u_n!^2}p^g_{n,k-1}(u_{K\setminus i},u_i,u_J)\\ &\qquad\qquad+\sum_{i=k+1}^{n}\frac{u_{k+1}\dots\widehat{u}_i\dots u_n}{u_1!^2\dots(u_i-1)!^2\dots u_n!^2}p^g_{n,k+1}(u_K,u_i-1,u_{J\setminus i})\\ &\quad=\frac{u_{k+1}\cdots u_n}{\prod_{i=1}^n u_i!^2}\left(\sum_{i=1}^{k}u_ip^g_{n,k-1}(u_{K\setminus i},u_i,u_J)+\sum_{i=k+1}^{n}u_ip^g_{n,k+1}(u_K,u_i-1,u_{J\setminus i})\right)\\ &\quad=:\frac{u_{k+1}\cdots u_n}{\prod_{i=1}^n u_i!^2}\widetilde{p}^g_{n,k}(u_K,u_J),\end{aligned}$$

where $\widehat{u}_i$ means to exclude the u_i term and we note that $p^g_{n,k\pm1}$ is a polynomial of degree $3g-3+n$, symmetric in the first $k\pm1$ and last $n-(k\pm1)$ variables. Thus $\widetilde{p}^g_{n,k}(u_K,u_J)$ has degree $3g-3+n+1$ and the required symmetries. □

4.1.2 Induction Suppose Proposition 4.1 is true for $g=0$ and $n'<n$. Apply

$$\frac{d^{n-3}}{dt_{b_4}\cdots dt_{b_n}}\bigg|_{t=0}$$

to (3-7) and let $\alpha_i=\omega$ to obtain the recursion

$$\begin{aligned}(4\text{-}5)\quad &\langle\tau_{b_1}(\omega)\cdots\tau_{b_n}(\omega)\rangle^0\\ &=\sum_{I\subset\{b_4,\dots,b_n\}}\big(\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_{b_2}(\omega)\tau_{b_3}(\omega)\tau_{CI}(\omega)\rangle^0\\ &\qquad+\langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_{b_2}(\omega)\tau_{b_3}(\omega)\tau_{CI}(\omega)\rangle^0\big)\end{aligned}$$

for $CI=\{b_4,\dots,b_n\}\setminus b_I$. We now wish to pull out the factors

$$\begin{aligned}&\frac{1}{u_i!^2}\quad\text{if } b_i=2u_i,\\ &\frac{u_i}{u_i!^2}\quad\text{if } b_i=2u_i-1,\end{aligned}$$

to be left with only polynomial terms, of degree up to $n-3$. By symmetry, we only need to show this for one of the b_i, so choose b_1.

Even If $b_1=2u_1$, then by induction for $|I|\neq 0,1$, both terms will look like

$$\frac{u_1}{u_1!^2}\,p(u_1)=\frac{1}{u_1!^2}\,[u_1p(u_1)],$$

where $u_1p(u_1)$ is a polynomial in u_1 of degree $|I|$.

Odd If $b_1=2u_1-1$, then by induction for $|I|\neq 0,1$, both terms will have the form

$$\frac{1}{(u_1-1)!^2}\,p(u_1)=\frac{u_1}{u_1!^2}\,[u_1p(u_1)],$$

where $u_1p(u_1)$ is a polynomial in u_1 of degree $|I|$.

Special cases We must be careful about the occurrences of one and two point invariants, as the inductive step begins at 3. These will occur in the first term when $|I|=0$ or $|I|=1$, and the second term when $|I|=0$. For the first term, application of the string equation leads to

$$\langle\tau_{b_1-2}(\omega)\rangle=\begin{cases}1/u_1!^2 & b_1=2u_1,\\ 0 & b_1=2u_1-1,\end{cases}$$

or

$$\langle\tau_{b_1-2}(\omega)\tau_{b_i}(\omega)\rangle^0+\langle\tau_{b_1-1}(\omega)\tau_{b_i-1}(\omega)\rangle^0 = \begin{cases} \frac{1}{(u_1-1)!^2u_i!^2}\frac{1}{u_1+u_i}+\frac{u_1u_i}{u_1!^2u_i!^2}\frac{1}{u_1+u_i}=\frac{u_1}{u_1!^2u_i!^2} & b_1=2u_1, b_i=2u_i,\\ \frac{(u_1-1)u_i}{(u_1-1)!^2u_i!^2}\frac{1}{u_1+u_i-1}+\frac{1}{(u_1-1)!^2(u_i-1)!^2}\frac{1}{u_1+u_i-1} & b_1=2u_1-1, b_i=2u_i-1,\\ \frac{u_1^2u_i}{u_1!^2u_i!^2} \end{cases}$$

and we still get the correct form. If $|I|=0$ the second term is only non-zero for $b_1=2u_1-1$ and we get

$$\langle\tau_0(\omega)\tau_{2u_1-2}(\omega)\rangle^0=\frac{1}{(u_1-1)!^2}\frac{1}{u_1}=\frac{u_1}{u_1!^2},$$

which is again correct.

Since $0\le|I|\le n-3$, adding terms on the right-hand side together gives the required degree of the polynomial part of the stationary Gromov–Witten invariant.

4.1.3 Genus-one case Initial case This time the induction begins from the one point function. If we set $\alpha_1=\omega$ and $F=1$ in (3-8) we get

$$\langle\tau_{b_1}(\omega)\rangle^1=\langle\tau_0(1)\tau_{b_1-1}(\omega)\rangle^0\langle\tau_0(\omega)\rangle^1+\langle\tau_0(\omega)\tau_{b_1-1}(\omega)\rangle^0\langle\tau_0(1)\rangle^1 + \tfrac{1}{12}\langle\tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\rangle^0.$$

The left-hand side is only non-zero if $b_1=2u_1$, which makes the second term on the right-hand side vanish for dimension reasons. Using the string equation, the initial terms of the genus-zero case and the value [16], $\langle\tau_0(\omega)\rangle^1=-\frac{1}{24}$. This reduces to

$$\langle\tau_{2u_1}(\omega)\rangle^1=-\frac{1}{24}\frac{1}{u_1!^2}+\frac{1}{12}\frac{1}{(u_1-1)!^2}\frac{1}{u_1}=\frac{1}{24u_1!^2}(2u_1-1). \tag{4-6}$$

4.1.4 Induction We have proven the theorem for genus zero and suppose it is true in genus one for $n'<n$. Let us apply

$$\frac{d^{n-1}}{dt_{b_2}\cdots dt_{b_n}}\bigg|_{t=0}$$

to (3-8) and let $\alpha_1=\omega$, $F=1$ to obtain the recursion

$$\begin{aligned}\langle\tau_{b_1}(\omega)\cdots\tau_{b_n}(\omega)\rangle^1=\sum_{I\subset\{b_2,\dots,b_n\}}&\big(\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_{CI}(\omega)\rangle^1\\ &+\langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_{CI}(\omega)\rangle^1\big)\\ &+\tfrac{1}{12}\langle\tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{b_2}(\omega)\cdots\tau_{b_n}(\omega)\rangle^0\end{aligned} \tag{4-7}$$

for $CI = \{b_2, \dots, b_n\} \setminus b_I$. As with genus zero, we wish to pull out the factors

$$\frac{1}{u_i!^2} \quad \text{if } b_i = 2u_i,$$
$$\frac{u_i}{u_i!^2} \quad \text{if } b_i = 2u_i - 1,$$

and be left with only polynomial terms, of degree up to n. Again by symmetry we only need to see this for one parameter, so look at b_1.

Even For $b_1 = 2u_1$ the first two terms will be

$$\frac{u_1}{u_1!^2} p(u_1) = \frac{1}{u_1!^2}[u_1 p(u_1)]$$

for $u_1 p(u_1)$ a polynomial in u_1 of degree $|I|$. The last term will look the same but this time $u_1 p(u_1)$ is a polynomial in u_1 of degree n.

Odd For $b_1 = 2u_1 - 1$ the first two terms will be

$$\frac{1}{(u_1-1)!^2} p(u_1) = \frac{u_1}{u_1!^2}[u_1 p(u_1)]$$

for $u_1 p(u_1)$ a polynomial in u_1 of degree $|I|$. The last term will look the same but this time $u_1 p(u_1)$ is a polynomial in u_1 of degree n.

Special cases We already saw in the genus-zero proof that application of the string equation to the two point genus-zero invariants gave the correct form.

This gives the correct form of all genus-one stationary Gromov–Witten invariants, and thus we have proven Proposition 4.1 for $g = 0, 1$. □

Remark Proposition 2.5 proved a polynomial form (2-12) for the coefficients of an expansion of the Eynard–Orantin invariants ω_n^g using the transform defined in Lemma 2.4. The transform is invertible so in particular any power series with coefficients having the polynomial form (2-12) continues analytically to a meromorphic multidifferential on the Riemann surface double covering the plane by $x = z + 1/z$. In particular, Proposition 4.1 proves that the generating functions Ω_n^g continue analytically to meromorphic multidifferentials over $x = z + 1/z$. This is weaker than Theorem 1.1, which identifies Ω_n^g with a known multidifferential.

4.2 String and dilaton equations for stationary Gromov–Witten invariants

It is easy to see that the divisor equation (3-4) restricts to a relationship between purely stationary invariants. It is subtler that the same is true for the string equation (3-5) and dilaton equation (3-6), which tell us how to remove a non-stationary term and a priori are not statements about stationary invariants alone. In the following we will use the expression $\tau_{-1}(\omega)$ as a device in formulae. It is common to use $\tau_{-1}(\omega)$ to represent zero in formulae but in our case it will be non-zero.

Proposition 4.3 *For $g = 0$ or 1, $\tau_0(1) ="\tau_{-1}(\omega)"$. More precisely,*

$$\left\langle \tau_0(1) \prod_{i=1}^{k} \tau_{2u_i}(\omega) \prod_{i=k+1}^{n-1} \tau_{2u_i-1}(\omega) \right\rangle^g = \frac{u_{k+1}\dots u_{n-1}}{\prod_{i=1}^{n-1} u_i!^2} p^g_{n,k}(u_1,\dots,u_{n-1},0),$$

where we have removed from (1-4) the factor $u_n/u_n!^2$ corresponding to an odd stationary class $\tau_{2u_n-1}(\omega)$ and set $u_n = 0$.

Proof We will use induction on n and the topological recursion (3-7).

Genus zero Let us begin with the initial cases. For dimension reasons, we need only check the following two cases whose expressions were computed in Section 4. Interpreting $u_i = 0$ to mean ignore the $u_i/u_i!^2$ factor before evaluating, gives

$$\begin{aligned}
\langle \tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^0\Big|_{u_1=0} &= \frac{u_1u_2}{u_1!^2\,u_2!^2}\frac{1}{u_1+u_2}\Big|_{u_1=0} \\
&= \frac{1}{u_2!^2} = \langle \tau_{2u_2-2}(\omega)\rangle^0 = \langle \tau_0(1)\tau_{2u_2-1}(\omega)\rangle^0,
\end{aligned}$$

$$\begin{aligned}
&\langle \tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3-1}(\omega)\rangle^0\Big|_{u_3=0} \\
&\qquad = \frac{u_2u_3}{u_1!^2\,u_2!^2\,u_3!^2}\Big|_{u_3=0} = \frac{u_2}{u_1!^2\,u_2!^2} \\
&\qquad = \frac{u_1u_2}{u_1!^2\,u_2!^2}\,\frac{1}{u_1+u_2} + \frac{1}{u_1!^2\,(u_2-1)!^2}\,\frac{1}{u_1+u_2} \\
&\qquad = \langle \tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^0 + \langle \tau_{2u_1}(\omega)\tau_{2u_2-2}(\omega)\rangle^0 \\
&\qquad = \langle \tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_0(1)\rangle^0,
\end{aligned}$$

so that the lemma is true for the smallest cases. Applying appropriate derivatives to (3-7) and setting $F = 1$ gives the recursion

$$\begin{aligned}
\text{(4-8)}\quad &\left\langle \prod_{i=1}^{3} \tau_{b_i}(\alpha_i)\cdot\tau_S(\omega)\right\rangle^0 \\
&\quad = \sum_{I\sqcup J=S} \big(\langle \tau_0(1)\tau_{b_1-1}(\alpha_1)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)\tau_J(\omega)\rangle^0 \\
&\qquad\qquad + \langle \tau_0(\omega)\tau_{b_1-1}(\alpha_1)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)\tau_J(\omega)\rangle^0 \big)
\end{aligned}$$

for $S = \{b_4, \dots, b_n\}$. We will show by induction that for $\alpha_1 = \alpha_2 = \alpha_3 = \omega$ and $b_2 = 2u_2 - 1$ an odd parity variable, the left-hand side evaluated at $u_2 = 0$ is equal to the left-hand side if $\alpha_1 = \alpha_3 = \omega$, $\alpha_2 = 1$, $b_2 = 0$. The induction will involve equating the right-hand sides.

RHS1 Let $\alpha_1 = \alpha_3 = \omega$, $\alpha_2 = 1$, $b_2 = 0$. Then after applying the divisor equation to the first term and the string equation to the second, the right-hand side becomes

$$\sum_{I \sqcup J = S} \bigg(\langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0 \left[\frac{|J| + b_3 + 1}{2} \right] \\ + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \big[\langle \tau_0(1)\tau_{b_3-1}(\omega)\tau_J(\omega)\rangle^0 + \langle \tau_0(1)\tau_{b_3}(\omega)\tau_{J-1}(\omega)\rangle^0 \big] \bigg),$$

where we have used the notation

$$\langle \tau_{J-1}(\omega)\rangle^g = \sum_{b_j \in J} \langle \tau_{J \setminus b_j}(\omega)\tau_{b_j-1}(\omega)\rangle^g.$$

RHS2 Let $\alpha_1 = \alpha_2 = \alpha_3 = \omega$ and $b_2 = 2u_2 - 1$. Then applying the divisor equation to the first term and the string equation to the second term, the right-hand side is

$$\sum_{I \sqcup J = S} \bigg(\langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_{2u_2-1}(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0 \left[\frac{|J| + b_3 + 2u_2 + 1}{2} \right] \\ + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \big[\langle \tau_{2u_2-2}(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0 + \langle \tau_{2u_2-1}(\omega)\tau_{b_3-1}(\omega)\tau_J(\omega)\rangle^0 \\ + \langle \tau_{2u_2-1}(\omega)\tau_{b_3}(\omega)\tau_{J-1}(\omega)\rangle^0 \big] \bigg).$$

By induction, the polynomial expressions for the first and final two terms are equal when we ignore the $u_2/u_2!^2$ factor and put $u_2 = 0$. It remains to consider the term $\langle \tau_{2u_2-2}(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0$. The u_2 dependence can be expressed as

$$\frac{1}{(u_2-1)!^2}\, p(u_2) = \frac{u_2}{u_2!^2}\, [u_2 p(u_2)]$$

for $u_2 p(u_2)$ a polynomial. When u_2 is set to zero in the polynomial component, this term will vanish and both right-hand side expressions are equal.

Genus one We shall proceed analogously. The smallest case consists of the two-point intersection numbers since the one-point function vanishes on an insertion of $\tau_{2u-1}(\omega)$ or $\tau_0(1)$. We may use the genus-one topological recursion (3-8), along with the initial computation in Section 4 to find an expression for $\langle \tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^1$. Let $b_1 = 2u_1 - 1$ and $\alpha_1 = \omega$. Taking a derivative to insert a $\tau_{2u_2-1}(\omega)$ term and discarding parts that are the wrong dimension gives

$$\begin{aligned}
&\langle\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^1 \\
&\qquad= \langle\tau_0(1)\tau_{2u_1-2}(\omega)\tau_{2u_2-1}(\omega)\rangle^0\langle\tau_0(\omega)\rangle^1 \\
&\qquad\qquad + \langle\tau_0(\omega)\tau_{2u_1-2}(\omega)\rangle^0\langle\tau_0(1)\tau_{2u_2-1}(\omega)\rangle^1 \\
&\qquad\qquad + \frac{1}{12}\langle\tau_0(1)\tau_0(\omega)\tau_{2u_1-2}(\omega_1)\tau_{2u_2-1}(\omega)\rangle^0 \\
&\qquad= -\frac{1}{24}\big(\langle\tau_{2u_1-3}(\omega)\tau_{2u_2-1}(\omega)\rangle^0 + \langle\tau_{2u_1-2}(\omega)\tau_{2u_2-2}(\omega)\rangle^0\big) \\
&\qquad\qquad + \langle\tau_0(\omega)\tau_{2u_1-2}(\omega)\rangle^0\langle\tau_{2u_2-2}(\omega)\rangle^1 \\
&\qquad\qquad + \frac{1}{12}\big(\langle\tau_0(\omega)\tau_{2u_1-3}(\omega_1)\tau_{2u_2-1}(\omega)\rangle^0 \\
&\qquad\qquad + \langle\tau_0(\omega)\tau_{2u_1-2}(\omega_1)\tau_{2u_2-2}(\omega)\rangle^0\big) \\
&\qquad= -\frac{1}{24}\left(\frac{(u_1-1)u_2}{(u_1-1)!^2\,u_2!^2}\frac{1}{u_1+u_2-1} + \frac{1}{(u_1-1)!^2\,(u_2-1)!^2}\frac{1}{u_1+u_2-1}\right) \\
&\qquad\qquad + \frac{1}{(u_1-1)!^2}\frac{1}{u_1}\frac{2u_2-3}{24(u_2-1)!^2} + \frac{1}{12u_1!^2\,u_2!^2}(u_1^2u_2(u_1-1)+u_1^2u_2^2) \\
&\qquad= \frac{u_1u_2}{24u_1!^2\,u_2!^2}(2u_1^2+2u_2^2+2u_1u_2-3u_1-3u_2)
\end{aligned}$$

so that

$$\begin{aligned}
\langle\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^1\Big|_{u_1=0} &= \frac{u_2}{24u_2!^2}(2u_2^2-3u_2) = \frac{1}{24(u_2-1)!^2}(2u_2-3) \\
&= \langle\tau_{2u_2-2}(\omega)\rangle^1 = \langle\tau_0(1)\tau_{2u_2-1}(\omega)\rangle^1
\end{aligned}$$

and we have verified the initial case. Applying appropriate derivatives to (3-8) and setting $\alpha_1=\omega,\ F=1$ gives the recursion

$$\begin{aligned}
\langle\tau_{b_1}(\omega)\tau_{b_2}(\alpha_2)\tau_S(\omega)\rangle^1 = \sum_{I\sqcup J=S}\big[&\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_{b_2}(\alpha_2)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_J(\omega)\rangle^1 \\
&+ \langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{b_2}(\alpha_2)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_J(\omega)\rangle^1 \\
&+ \langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_{b_2}(\alpha_2)\tau_J(\omega)\rangle^1 \\
&+ \langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_{b_2}(\alpha_2)\tau_J(\omega)\rangle^1\big] \\
&+ \tfrac{1}{12}\langle\tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{b_2}(\alpha_2)\tau_S(\omega)\rangle^0
\end{aligned}$$

for $S=\{b_3,\dots,b_n\}$. Now we may compare expressions.

RHS1 Let $\alpha_2 = 1$ and $b_2 = 0$.

$$\sum_{I \sqcup J = S} \big[\langle \tau_0(1)\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_J(\omega)\rangle^1 + \langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_0(1)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_0(1)\tau_J(\omega)\rangle^1 \big] + \tfrac{1}{12}\langle \tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_0(1)\tau_S(\omega)\rangle^0.$$

RHS2 Let $\alpha_2 = \omega$, $b_2 = 2u_2 - 1$.

$$\sum_{I \sqcup J = S} \big[\langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_{2u_2-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{2u_2-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_J(\omega)\rangle^1 + \langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_{2u_2-1}(\omega)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_{2u_2-1}(\omega)\tau_J(\omega)\rangle^1 \big] + \tfrac{1}{12}\langle \tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{2u_2-1}(\omega)\tau_S(\omega)\rangle^0.$$

Apply the string and divisor equations to the genus-one intersection numbers in RHS1 and RHS2. By induction, setting $u_2 = 0$ in the polynomial expressions for all terms in RHS2, we get equality with RHS1. (The induction is on n, but we have already shown all genus-zero to hold.) The application of the string and divisor equations is rather superficial, simply reducing the n–point genus-one intersection numbers to $(n-1)$–point genus-one intersection numbers to enable the induction to be on the number of insertions n. □

A similar strategy is required for the dilaton equation.

Proposition 4.4 *For $g = 0$ or 1, $\tau_1(1)$ classes can be evaluated in the expression (1-4) by removing the $1/u_i!^2$ factor from an even stationary class and setting $u_i = 0$ in the derivative:*

$$\left\langle \tau_1(1) \prod_{i=2}^{k} \tau_{2u_i}(\omega) \prod_{i=k+1}^{n} \tau_{2u_i-1}(\omega) \right\rangle^g = \frac{u_{k+1}\cdots u_n}{\prod_{i=2}^n u_i!^2} \frac{\partial}{\partial u_1} p^g_{n,k}(u_1,\dots,u_n)\bigg|_{u_1=0} \tag{4-9}$$

Proof We will use induction on n and the topological recursions (3-7), (3-8).

Genus zero Begin with the initial cases. Interpreting operations to mean "ignore the $1/u_i!^2$ factor" first gives

$$\begin{aligned}
\frac{\partial}{\partial u_2}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^0\Big|_{u_2=0} &= \frac{1}{u_1!^2}\frac{\partial}{\partial u_2}\frac{1}{u_1+u_2+1}\Big|_{u_2=0}\\
&= \frac{1}{u_1!^2}\frac{-1}{(u_1+1)^2} = -\langle\tau_{2u_1}(\omega)\rangle^0\\
&= \langle\tau_1(1)\tau_{2u_1}(\omega)\rangle^0,\\
\frac{\partial}{\partial u_3}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_{2u_3}(\omega)\rangle^0\Big|_{u_3=0} &= \frac{1}{u_1!^2u_2!^2}\frac{\partial}{\partial u_3}1\Big|_{u_3=0}\\
&= 0 = \langle\tau_1(1)\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^0,\\
\frac{\partial}{\partial u_3}\langle\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3}(\omega)\rangle^0\Big|_{u_3=0} &= \frac{u_1u_2}{u_1!^2u_2!^2}\frac{\partial}{\partial u_3}1\Big|_{u_3=0}\\
&= 0 = \langle\tau_1(1)\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^0,
\end{aligned}$$

so that the proposition holds for the smallest cases. Now apply appropriate derivatives to (3-7) to get the recursion

$$\begin{aligned}
\text{(4-10)}\quad \Big\langle\prod_{i=1}^{3}\tau_{b_i}(\alpha_i)\tau_S(\omega)\Big\rangle^0
&= \sum_{I\sqcup J=S}\big(\langle\tau_0(1)\tau_{b_1-1}(\alpha_1)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)\tau_J(\omega)\rangle^0\\
&\qquad + \langle\tau_0(\omega)\tau_{b_1-1}(\alpha_1)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_{b_2}(\alpha_2)\tau_{b_3}(\alpha_3)\tau_J(\omega)\rangle^0\big)
\end{aligned}$$

for $S=\{b_4,\dots,b_n\}$. We will show by induction that when $\alpha_1=\alpha_2=\alpha_3=\omega$ and $b_2=2u_2$ an even parity variable, if we ignore the $1/u_2!^2$ factor, take the derivative and set $u_2=0$, the left-hand side is the same as the left-hand side when $\alpha_1=\alpha_3=\omega$, $\alpha_2=1$ and $b_2=1$.

RHS1 Let $\alpha_1=\alpha_3=\omega$, $\alpha_2=1$, $b_2=1$. Then after applying the divisor equation to the first term and the string equation to the second, the right-hand side becomes

$$\begin{aligned}
\sum_{I\sqcup J=S}\Big(&\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\Big[\langle\tau_0(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0\\
&\qquad + \langle\tau_1(1)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0\Big[\frac{|J|+b_3+2}{2}\Big]\Big]\\
&+ \langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\big[\langle\tau_0(1)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0+\langle\tau_1(1)\tau_{b_3-1}(\omega)\tau_J(\omega)\rangle^0\\
&\qquad + \langle\tau_1(1)\tau_{b_3}(\omega)\tau_{J-1}(\omega)\rangle^0\big]\Big).
\end{aligned}$$

RHS2 Let $\alpha_1=\alpha_2=\alpha_3=\omega$, $b_2=2u_2$. After applying the divisor equation to the first term and the string equation to the second term, the right-hand side becomes

$$\sum_{I\sqcup J=S}\bigg(\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_{2u_2}(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0\bigg[\frac{|J|+b_3+2u_2+2}{2}\bigg]$$
$$+\langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\big[\langle\tau_{2u_2-1}(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0$$
$$+\langle\tau_{2u_2}(\omega)\tau_{b_3-1}(\omega)\tau_J(\omega)\rangle^0$$
$$+\langle\tau_{2u_2}(\omega)\tau_{b_3}(\omega)\tau_{J-1}(\omega)\rangle^0\big]\bigg).$$

By induction, pulling out $1/u_2!^2$, taking the derivative and setting $u_2=0$ gives equality with the last two terms of each right-hand side. For the first term, the product rule on u_2 and induction give equality with the first two terms, and all that remains is to check the third term. We may write the u_2 dependence in $\langle\tau_{2u_2-1}(\omega)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0$ as

$$\frac{u_2}{u_2!^2}\,p(u_2)=\frac{1}{u_2!^2}\,[u_2\,p(u_2)]$$

so that when the product rule is used, and u_2 subsequently set to zero, this is equivalent to ignoring a $u_2/u_2!^2$ factor and setting $u_2=0$ in the polynomial part. That is, by Proposition 4.3, $\langle\tau_0(1)\tau_{b_3}(\omega)\tau_J(\omega)\rangle^0$. Performing these evaluations gives an overall equality.

Genus one Begin with the initial case. Again interpreting $u_i=0$ to mean "ignore the $1/u_i!^2$ factor before performing any operations" gives

$$\frac{\partial}{\partial u_2}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^1\Big|_{u_2=0}=\frac{1}{u_1!^2}\frac{\partial}{\partial u_2}\frac{1}{24}(2u_1^2+2u_2^2+2u_1u_2-u_1-u_2)\Big|_{u_2=0}$$
$$=\frac{1}{24u_1!^2}(2u_1-1)=\langle\tau_1(1)\tau_{2u_1}(\omega)\rangle^1$$

and the proposition holds. Applying appropriate derivatives to (3-8) and setting $\alpha_1=\omega$, $F=1$ gives the recursion

$$\langle\tau_{b_1}(\omega)\tau_{b_2}(\alpha_2)\tau_S(\omega)\rangle^1=\sum_{I\sqcup J=S}\big[\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_{b_2}(\alpha_2)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_J(\omega)\rangle^1$$
$$+\langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{b_2}(\alpha_2)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_J(\omega)\rangle^1$$
$$+\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_{b_2}(\alpha_2)\tau_J(\omega)\rangle^1$$
$$+\langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_{b_2}(\alpha_2)\tau_J(\omega)\rangle^1\big]$$
$$+\tfrac{1}{12}\langle\tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{b_2}(\alpha_2)\tau_S(\omega)\rangle^0$$

for $S=\{b_3,\dots,b_n\}$. Now we may compare expressions.

RHS1 Let $\alpha_2 = 1$ and $b_2 = 1$.

$$\sum_{I \sqcup J = S} \big[\langle \tau_0(1)\tau_1(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_1(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_J(\omega)\rangle^1 + \langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_1(1)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_1(1)\tau_J(\omega)\rangle^1\big] + \tfrac{1}{12}\langle \tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_1(1)\tau_S(\omega)\rangle^0.$$

RHS2 Let $\alpha_2 = \omega$, $b_2 = 2u_2$.

$$\sum_{I \sqcup J = S} \big[\langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_{2u_2}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{2u_2}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_J(\omega)\rangle^1 + \langle \tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(\omega)\tau_{2u_2}(\omega)\tau_J(\omega)\rangle^1 + \langle \tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0 \langle \tau_0(1)\tau_{2u_2}(\omega)\tau_J(\omega)\rangle^1\big] + \tfrac{1}{12}\langle \tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{2u_2}(\omega)\tau_S(\omega)\rangle^0.$$

Given the proposition is true in genus zero, we need only consider terms three and four. Again we use the string and divisor equations superficially so we can apply induction on n. The inductive assumption on smaller terms tells us to ignore the $1/u_2!^2$ factor, take the derivative and evaluate at $u_2 = 0$ the polynomial expressions for RHS2, to get equality with RHS1. □

Remarks (1) Combining Proposition 4.3, respectively Proposition 4.4, with the string equation, respectively the dilaton equation, gives relations between stationary invariants alone. One might call these string and dilaton equations for stationary invariants.

(2) We expect Propositions 4.1, 4.3 and 4.4 to hold for the Gromov–Witten invariants of $\mathbb{P}^1$ for all genus g.

Usually one would use the topological recursion relations (3-7) and (3-8) together with the divisor, string (and dilaton) equations to determine all genus-zero and genus-one Gromov–Witten invariants. Using Propositions 4.1, 4.3 and 4.4, we can drop the need for the topological recursion relations, which is desirable since we do not have access to the topological recursion relations for the Eynard–Orantin invariants.

Theorem 4.5 *The divisor and string equations uniquely determine all genus-zero and genus-one stationary Gromov–Witten invariants from the initial cases* $\langle \tau_b(\omega)\rangle^1$ *and* $\langle \tau_{b_1}(\omega)\tau_{b_2}(\omega)\tau_{b_3}(\omega)\rangle^0$.

Proof The structure of proof is to uniquely determine the polynomials $p^g_{n,k}$ for $g=0,1$ from the string and divisor equations. For $g=1$ we need one further piece of information, and rather than using the dilaton equation we use knowledge of a top coefficient of $p^1_{n,k}$ supplied by the $g=0$ part of Theorem 4.5, which is proven first.

We begin with the genus-zero case and use the $g=0$ part of Proposition 4.1:

$$\left\langle \prod_{i=1}^{k} \tau_{2u_i}(\omega) \prod_{i=k+1}^{n} \tau_{2u_i-1}(\omega) \right\rangle^0 = \frac{u_{k+1}\cdots u_n}{\prod_{i=1}^n u_i!^2}\, p^0_{n,k}(u_1,\dots,u_n),$$

where $p^0_{n,k}(u_1,\dots,u_n)$ is a polynomial of degree $n-3$ in the u_i, symmetric in the first k and the last $n-k$ variables.

The divisor equation enables one to compute $p^g_{n,k}(0,u_2,\dots,u_n)$ from the simpler $p^g_{n-1,k-1}(u_2,\dots,u_n)$. By symmetry, this equates evaluation of any of the first k variables at 0 to known functions. Proposition 4.3 and the string equation enable one to compute $p^g_{n,k}(u_1,\dots,u_{n-1},0)$ from $p^g_{n-1,k}(u_1,\dots,u_{n-1})$ and $p^g_{n-1,k-1}(u_1,\dots,u_{n-1})$, which by symmetry gives evaluation of any of the last $n-k$ variables. Thus we can apply Lemma 2.8 to deduce the genus-zero case. If $k=0$, respectively $k=n$, then we require only the string equation, respectively the divisor equation.

The genus-one case relies on both the genus-zero and genus-one parts of Propositions 4.1 and 4.3 together with Proposition 2.5, which is used to determine one nonzero corresponding top coefficient of each quasi-polynomial $m^1_{n,k}$ associated to the Eynard–Orantin invariants, and the genus-zero part of Theorem 1.1, which identifies the Gromov–Witten and Eynard–Orantin invariants. Theorem 1.1 is a consequence of Theorem 4.5 but its genus-zero part only requires the genus-zero part of Theorem 4.5, which has already been proved.

Propositions 4.1 and 4.3, and Lemma 2.8 prove, using the argument as for the genus-zero case above, that the string and divisor equations determine the $p^1_{n,k}$ only up to a constant since $\deg p^1_{n,k}=n$. The constant is stored in the coefficient of $u_1\cdots u_n$. To determine $p^1_{n,k}$, and hence the genus-one stationary Gromov–Witten invariants, it remains to identify the coefficient of $u_1\cdots u_n$ in $p^1_{n,k}$.

Lemma 4.6 *The coefficient of* $u_1\cdots u_n$ *in* $p^1_{n,k}(u_1,\dots,u_n)$ *is* $\frac{1}{12}(n-1)!$.

Proof This would be a consequence of Theorem 1.2, except Theorem 1.2 has not been proved yet. Instead we derive it from genus-zero information. We use the genus-one topological recursion and the genus-zero part of Theorem 1.1 to determine this coefficient. Having taken the appropriate derivatives, the recursion (3-8) becomes

$$\langle\tau_{b_1}(\omega)\cdots\tau_{b_n}(\omega)\rangle^1 = \sum_{I\subset\{b_2,\dots,b_n\}} \big(\langle\tau_0(1)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(\omega)\tau_{CI}(\omega)\rangle^1 + \langle\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_I(\omega)\rangle^0\langle\tau_0(1)\tau_{CI}(\omega)\rangle^1\big) + \tfrac{1}{12}\langle\tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\omega)\tau_{b_2}(\omega)\cdots\tau_{b_n}(\omega)\rangle^0 \tag{4-11}$$

for $CI=\{b_2,\dots,b_n\}\backslash b_I$. For each $i=1,\dots,n$ let $b_i=2u_i$ or $2u_i-1$ depending on parity. Multiply (4-11) by the appropriate combinatorial factor $u_1!^2\cdots u_n!^2/u_{k+1}\cdots u_n$ and use Proposition 4.1 to get

$$p^1_{n,k}(u_1,\dots,u_n) = \sum_{I\subset\{u_2,\dots,u_n\}} \big(u_1 p^0_{|I|+2,k_1}(u_1,u_I,0)p^1_{|CI|+1,k_2}(0,u_{CI}) + u_1 p^0_{|I|+2,k_1+1}(0,u_1,u_I)p^1_{|CI|+1,k_2-1}(u_{CI},0)\big) + \tfrac{1}{12}u_1 p^0_{n+2,k}(0,u_1,\dots,u_n,0), \tag{4-12}$$

where $k_1+k_2=k$ or $k+2$ depending whether u_1 is even or odd, but this is unimportant since top coefficients are insensitive to k_i. The extra factor of u_1 in each term of the right-hand side comes from the change of parity of b_1: if $b_1=2u_1$, respectively $b_1=2u_1-1$, then the extra factor on the right-hand side is $u_1!^2\times u_1/u_1!^2=u_1$, respectively $u_1!^2/u_1\times 1/(u_1-1)!^2=u_1$.

The first two terms on the right-hand side of (4-12) do not contain $u_1\cdots u_n$ terms since the $|I|+1$ variables $\{u_1,u_I\}$ are contained in the first factors, whereas

$$\deg u_1 p^0_{|I|+2,k_1}(u_1,u_I,0) = |I| = \deg u_1 p^0_{|I|+2,k_1+1}(0,u_1,u_I)$$

so no u_1u_I term can exist. Hence only the third term on the right-hand side contributes to the coefficient of $u_1\cdots u_n$ and this is in genus zero so we can calculate it.

The monomial $u_2\cdots u_n$ appears in $p^0_{n+2,k}(0,u_1,\dots,u_n,0)$ as a top degree term and using the genus-zero equality with the Eynard–Orantin expansion (4-14),

$$p^0_{n+2,k}(0,u_1,\dots,u_n,0) = m^0_{n+2,k}(1,2u_1+1,\dots,2u_n,0).$$

Proposition 2.5 computes this coefficient to be

$$\frac{2}{2^n}\,\langle\tau_1^{n-1}\tau_0^3\rangle\,2^{n-1} = (n-1)!\,.$$

The extra factors of 2 come from the change of variables $b_i=2u_i$ or $2u_i+1$. Thus the coefficient of $u_1\cdots u_n$ in $p^1_{n,k}(u_1,\dots,u_n)$ is $\frac{1}{12}(n-1)!$ as required. □

Thus Theorem 4.5 is proven. □

We are finally in a position to prove Theorem 1.1.

Proof of Theorem 1.1 The stationary Gromov–Witten invariants are assembled into the sum

$$\Omega_n^g(x_1,\dots,x_n)=\sum_{b_1,\dots,b_n=0}^{\infty}\Big\langle\prod_{i=1}^{n}\tau_{b_i}(\omega)\Big\rangle^g\cdot\prod_{i=1}^{n}(b_i+1)!\,x_i^{-b_i-2}\,dx_i.$$

With respect to the local coordinate $x=z+1/z$ on a rational curve the Eynard–Orantin invariant ω_n^g has the following analytic expansion around the branch of $x=\infty$ defined by $z=\infty$:

$$\omega_n^g=:\sum_{b_1,\dots,b_n=1}^{\infty}\frac{M_n^g(b_1,\dots,b_n)}{x_1^{b_1+1}\cdots x_n^{b_n+1}}\,dx_1\cdots dx_n,$$

where M_n^g is quasi-polynomial and represented by the polynomials $M_{n,k}^g$ for k the number of odd b_i. We need to prove the identification

$$\Big\langle\prod_{i=1}^{n}\tau_{b_i}(\omega)\Big\rangle^g\cdot\prod_{i=1}^{n}(b_i+1)!=M_n^g(b_1+1,\dots,b_n+1).\tag{4-13}$$

The left-hand side of (4-13) satisfies the divisor equation (3-4) and string equation (for stationary invariants) (3-5) while the right-hand side of (4-13) satisfies its own divisor equation (2-14) and string equation (2-15). It is easy to see that the different divisor and string equations are equivalent as follows. Substitute (4-13) into (2-14) to get the divisor equation (3-4) for stationary Gromov–Witten invariants. Substitute (4-13) into (2-15) to get the string equation (3-5) for stationary Gromov–Witten invariants where we recall how to make sense of $M_n^g(b_1+1,\dots,0)$ and $\tau_{-1}(\omega)$. Propositions 2.5 and 4.1 show that (4-13) factorises into

$$\begin{aligned}\prod_{i=1}^{n}\Big\lfloor\frac{b_i+2}{2}\Big\rfloor\binom{b_i+1}{\lfloor\frac12(b_i+1)\rfloor}p_n^g(b_1,\dots,b_n)&\\ =\prod_{i=1}^{n}\Big\lfloor\frac{b_i+2}{2}\Big\rfloor&\binom{b_i+1}{\lfloor\frac12(b_i+1)\rfloor}m_n^g(b_1+1,\dots,b_n+1),\end{aligned}$$

where $p_n^g(b_1,\dots,b_n):=p_{n,k}^g(\lfloor\frac12(b_1+1)\rfloor,\dots,\lfloor\frac12(b_n+1)\rfloor)$ is quasi-polynomial in the b_i, and represented by the polynomials $p_{n,k}^g$ for k giving the number of even b_i. Evaluation of the quasi-polynomials at $b_n=-1$ (and removal of the factor $\lfloor\frac12\rfloor$) is shown in Proposition 4.3 and (2-15) to give the respective string equations.

Hence the stationary Gromov–Witten invariants and Eynard–Orantin invariants, given by the two sides of (4-13), satisfy the same divisor and string equations. The divisor and string equations uniquely determine the stationary Gromov–Witten invariants and Eynard–Orantin invariants in genus zero, and in genus one when we also know a coefficient of the associated genus-one quasi-polynomials (Theorems 2.6 and 4.5). So all we need to do is check that the initial cases $(g,n)=(0,3)$ and $(1,1)$ match and a single top coefficient for each $(1,n)$ matches.

Genus zero We can explicitly calculate the $(g,n)=(0,3)$ case: Using the topological recursion for Gromov–Witten invariants we have already seen (4-4):

$$\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_{2u_3}(\omega)\rangle^0 = \frac{1}{u_1!^2\,u_2!^2\,u_3!^2}\,,$$
$$\langle\tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3-1}(\omega)\rangle^0 = \frac{u_2u_3}{u_1!^2\,u_2!^2\,u_3!^2}\,.$$

Using [15], we can compute the expansion of the $(g,n)=(0,3)$ Eynard–Orantin invariants for the curve (1-2):

$$\begin{aligned}M_3^0(2u_1+1,2u_2+1,2u_3+1) &= \prod_{i=1}^{3}(2u_i+1)\binom{2u_i}{u_i}\\ &= \prod_{i=1}^{3}(2u_i+1)!\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_{2u_3}(\omega)\rangle^0,\end{aligned}$$

$$\begin{aligned}M_3^0(2u_1+1,2u_2,2u_3) &= (2u_1+1)u_2u_3\prod_{i=1}^{3}\binom{2u_i}{u_i}\\ &= (2u_1+1)!\,(2u_2)!\,(2u_3)!\,\langle\tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3-1}(\omega)\rangle^0,\end{aligned}$$

and so the theorem is true in genus zero and we have the equality

$$p^0_{n,k}(u_1,\dots,u_n) = m^0_{n,k}(2u_1+1,\dots,2u_k+1,2u_{k+1},\dots,2u_n). \tag{4-14}$$

Genus one This time both sets of invariants are determined by the string and divisor equations and a coefficient of the top degree polynomial terms. We must check that the initial cases and the top coefficients agree. We already saw the Gromov–Witten invariant (4-6) is given by

$$\langle\tau_{2u}(\omega)\rangle^1 = \frac{1}{24u!^2}(2u-1).$$

Using [15] and Lemma 2.4 we can compute the expansion of the $(g,n)=(1,1)$ Eynard–Orantin invariant:

$$M_1^1(2u_1+1) = (2u_1+1)\binom{2u_1}{u_1}\frac{1}{24}(2u_1-1) = (2u_1+1)!\,\langle\tau_{2u_1}(\omega)\rangle^1,$$

which matches the Gromov–Witten invariants as required. Furthermore, the coefficient of $u_1\cdots u_n$ in $m^1_{n,k}(2u_1+1,\dots 2u_k+1,2u_{k+1},\dots,2u_n)$ is (independent of k and given by) $2^n\times 2^{1-n}\langle\tau_1^n\rangle^1=\frac{1}{12}(n-1)!$ by Proposition 2.5, where the factor of 2^n comes from the change of variable from b_i to u_i. This agrees with the coefficient of $u_1\cdots u_n$ in $p^1_{n,k}(u_1,\dots u_n)$ calculated in Lemma 4.6. Thus the theorem is true in genus one. □

The generating functions for the 1–point and 2–point genus-zero stationary Gromov–Witten invariants given in (1-3) use the explicit formulae (4-2) and (4-3) as follows.

The expansion of $\omega^0_1(z)+\ln x dx=\ln(1+1/z^2)dx$ at $x=\infty$ is obtained by taking the residues

$$\begin{aligned}\operatorname*{Res}_{z=\infty}\ln\Big(1+\frac{1}{z^2}\Big)x^m\,dx&=\operatorname*{Res}_{z=\infty}\frac{2\,dz}{z^2(z+1/z)}\frac{x^{m+1}}{m+1}\\&=2\operatorname*{Res}_{z=\infty}\frac{x^m}{m+1}\frac{dz}{z^2}=\begin{cases}-\frac{(2d-1)!}{d!^2} & m=2d-1,\\ 0 & m=2d,\end{cases}\end{aligned}$$

where the first equality is integration by parts and the final equality simply takes -2 times the coefficient of z in $x^m=(z+1/z)^m$. The residue is the same as $\operatorname{Res}_{x=\infty}\ln(1+1/z^2)x^m\,dx$, which gives the negative of the coefficient of $x^{-m-1}\,dx$ in $\ln(1+1/z^2)\,dx$. Since $\langle\tau_{2d-2}(\omega)\rangle^0\cdot(2d-1)!=(2d-1)!/d!^2$ this verifies that $\Omega^0_1(x)$ is an analytic expansion of $\omega^0_1(z)+\ln x\,dx$.

For the 2–point genus-zero generating function, again we can simply take residues at $z_1=\infty=z_2$ of $x_1^{m_1}x_2^{m_2}\,dz_1\,dz_2/(1-z_1z_2)^2$, which reduces the verification of (1-3) to the verification of a binomial identity. Instead, define $F_{0,2}(x_1,x_2)$ to be the primitive of $\Omega^0_2(x_1,x_2)$ so its expansion is

$$\begin{aligned}F_{0,2}(x_1,x_2)&=\sum_{u_i=0}\langle\tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^0\frac{(2u_1)!\,(2u_2)!}{x_1^{2u_1+1}x_2^{2u_2+1}}\\&\qquad+\sum_{u_i=1}\langle\tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^0\frac{(2u_1-1)!\,(2u_2-1)!}{x_1^{2u_1}x_2^{2u_2}}\\&=\sum_{u_i=0}\frac{(2u_1)!\,(2u_2)!}{u_1!^2u_2!^2}\frac{1}{u_1+u_2+1}\cdot\frac{1}{x_1^{2u_1+1}x_2^{2u_2+1}}\\&\qquad+\frac{1}{4}\sum_{u_i=1}\frac{(2u_1)!\,(2u_2)!}{u_1!^2u_2!^2}\frac{1}{u_1+u_2}\cdot\frac{1}{x_1^{2u_1}x_2^{2u_2}}.\end{aligned}$$

It satisfies the partial differential equation

$$
\begin{aligned}
\text{(4-15)}\quad & \Big(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\Big)F_{0,2}(x_1,x_2)\\
&= -2\sum_{u_i=0}\frac{(2u_1)!\,(2u_2)!}{u_1!^2\,u_2!^2}\cdot\frac{1}{x_1^{2u_1+1}x_2^{2u_2+1}}\\
&\qquad\qquad -\frac{1}{2}\sum_{u_i=1}\frac{(2u_1)!\,(2u_2)!}{u_1!^2\,u_2!^2}\cdot\frac{1}{x_1^{2u_1}x_2^{2u_2}}\\
&= -2\frac{z_1z_2}{(1-z_1^2)(1-z_2^2)}-\frac{1}{2}x_1x_2\Big(\frac{z_1}{z_1^2-1}-\frac{1}{x_1}\Big)\Big(\frac{z_2}{z_2^2-1}-\frac{1}{x_2}\Big)\\
&= \frac{-2(z_1z_2+1)}{(1-z_1^2)(1-z_2^2)},
\end{aligned}
$$

where we have used the elementary identity

$$\sum_{u=0}^{\infty}\binom{2u}{u}x^{-(2u+1)}=\frac{z}{z^2-1},\qquad x=z+\frac{1}{z}\,.$$

Since $x_1(\partial/\partial x_1)+x_2(\partial/\partial x_2)$ is the degree operator on functions analytic at $x_1=\infty=x_2$, its only non-trivial kernel consists of the degree-zero, or constant, functions. Hence any solution of (4-15) is unique up to a constant.

The function $G_{0,2}(z_1,z_2)=\ln(z_1-z_2)-\ln(x_1-x_2)$ satisfies

$$
\begin{aligned}
&\Big(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\Big)G_{0,2}(z_1,z_2)\\
&\quad=\frac{z_1^2+1}{z_1^2-1}\,\frac{z_1}{z_1-z_2}-\frac{x_1}{x_1-x_2}+\frac{z_2^2+1}{z_2^2-1}\,\frac{z_2}{z_2-z_1}-\frac{x_2}{x_2-x_1}=\frac{-2(z_1z_2+1)}{(1-z_1^2)(1-z_2^2)}\,.
\end{aligned}
$$

Hence $G_{0,2}(z_1,z_2)$ is the unique (up to a constant) solution of (4-15) so $\Omega_2^0(x_1,x_2)$ is an analytic expansion of its second derivative

$$\Omega_2^0(x_1,x_2)\sim dz_1\,dz_2\,\frac{\partial^2}{\partial z_1\partial z_2}G_{0,2}=\frac{dz_1\,dz_2}{(z_1-z_2)^2}-\frac{dx_1\,dx_2}{(x_1-x_2)^2}$$

as required.

Remark The identification of the coefficients M_n^g in the expansion of ω_n^g around $x_i=\infty$ with Gromov–Witten invariants raises the question of finding a similar geometric interpretation of N_n^g that is related to M_n^g via Lemma 2.3. The N_n^g are much simpler and contain the essential information of the M_n^g and hence the Gromov–Witten invariants.

A corollary of Theorem 1.1 is Theorem 1.2.

Proof of Theorem 1.2 For $g=0,1$, Theorem 1.1 allows us to identify

$$M^g_{n,k}=\prod_{i=1}^n(b_i+1)!\left\langle\prod_{i=1}^k\tau_{2u_i}(\omega)\prod_{i=k+1}^n\tau_{2u_i-1}(\omega)\right\rangle^g$$

under the substitution $b_i=2u_i(+1)$, and hence we can identify their polynomial parts $m^g_{n,k}(b_1,\dots,b_n)=p^g_{n,k}(u_1,\dots,u_n)$ defined in Propositions 2.5 and 4.1.

Proposition 2.5 gives the coefficient of $b_1^{\beta_1}\cdots b_n^{\beta_n}=b^\beta$ in $m^g_{n,k}$ to be $v_\beta=0$ or $v_\beta=2^{-2g+3-n}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\beta_1}\cdots\psi_n^{\beta_n}$ since $y'(1)=1$ and $y'(-1)=-1$ in (2-13).

Hence the top coefficients c_β of $u_1^{\beta_1}\cdots u_n^{\beta_n}$, which satisfy $c_\beta=v_\beta\cdot 2^{3g-3+n}$, are given by

$$c_\beta=2^g\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\beta_1}\cdots\psi_n^{\beta_n} \tag{4-16}$$

for $|\beta|=3g-3+n$. □

5 Virasoro constraints

The Gromov–Witten invariants of $\mathbb{P}^1$ satisfy the following recursions for each $k>0$, known as Virasoro constraints,

$$\begin{aligned}(k+1)!\,&\langle[\tau_{k+1}(1)+2c_{k+1}\tau_k(\omega)]\tau_{b_S}(\omega)\rangle^g\\ &-\sum_{j=1}^n\frac{(k+b_j+1)!}{b_j!}\langle\tau_{k+b_j}(\omega)\tau_{b_1}(\omega)\cdots\widehat{\tau}_{b_j}(\omega)\cdots\tau_{b_n}(\omega)\rangle^g\\ =\sum_{m=0}^{k-2}&(m+1)!\,(k-m-1)!\Big[\langle\tau_m(\omega)\tau_{k-m-2}(\omega)\tau_{b_S}(\omega)\rangle^{g-1}\\ &+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\langle\tau_m(\omega)\tau_{b_I}(\omega)\rangle^{g_1}\langle\tau_{k-m-2}(\omega)\tau_{b_J}(\omega)\rangle^{g_2}\Big]\end{aligned}$$

for $c_k=1+1/2+\cdots+1/k$ and $\tau_{b_K}(\omega)=\prod_{j\in K}\tau_{b_j}(\omega)$.

In terms of the generating functions, the Virasoro constraints become

$$\eta^g_{n+1}(x,x_S)=\Omega^{g-1}_{n+2}(x,x,x_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\Omega^{g_1}_{|I|+1}(x,x_I)\Omega^{g_2}_{|J|+1}(x,x_J)+\sum_{i=1}^{n}\frac{dx\,dx_i}{(x-x_i)^2}\,\Omega^g_n(x,x_{S\backslash i}),\tag{5-1}$$

where non-stationary invariants are stored in the generating function

$$\eta^g_{n+1}(x,x_S):=\sum_{b_i\geq 0}\prod_{j=1}^{n}\frac{(b_j+1)!}{x_j^{b_j+2}}\,dx_j\,dx^2\sum_{k=0}^{\infty}\frac{(k+1)!}{x^{k+2}}\langle[\tau_{k+1}(1)+2c_{k+1}\tau_k(\omega)]\tau_{b_S}(\omega)\rangle^g.$$

A consequence of (2-1) is the following set of *loop equations*, also known as Virasoro constraints, proven in [9], satisfied by the Eynard–Orantin invariants. The loop equations express the fact that the sum over the fibres of x of a combination of the Eynard–Orantin invariants cancels the poles at the zeros of dx. Explicitly, the following function $P^g_{n+1}(x,z_S)$ has no poles at the zeros of dx:

$$P^g_{n+1}(x,z_S)dx(z)^2=\frac{1}{2}\sum_{x(z)=x}\Bigg[\omega^{g-1}_{n+2}(z,z,z_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\omega^{g_1}_{|I|+1}(z,z_I)\omega^{g_2}_{|J|+1}(z,z_J)\Bigg].\tag{5-2}$$

Equivalently the right-hand side vanishes to order two at each zero of dx. Note the sum now includes $(0,1)$ terms. The sum of differentials over fibres of x is to be understood via a common trivialisation of the cotangent bundle supplied by dx. The statement of the loop equations is unchanged if we replace $y(z)$ by $y_N(z)$ for $N\geq 6g-4+2n$. This is because each $\omega^{g'}_{n'}$ in the equation stabilises in this range, except for $\omega^0_1(z)$. If $y_N(z)\mapsto y_N(z)+a(1-z^2)^{N+1}$ then

$$\omega^0_1(z)\omega^g_{n+1}(z,z_S)\mapsto\omega^0_1(z)\omega^g_{n+1}(z,z_S)+(1-z^2)^2h(z)$$

for h analytic at $z=\pm 1$ since $a(1-z^2)^{N+1}$ cancels the poles of ω^g_{n+1}. Hence

$$P^g_{n+1}(x,z_S)\,dx(z)^2\mapsto P^g_{n+1}(x,z_S)\,dx(z)^2+z^2h(z)\,dx(z)^2,$$

which still has no poles at $z=\pm 1$. The proof of (5-2) uses the fact that the recursion (2-1) is retrieved from

$$0=\sum_{\alpha}\operatorname*{Res}_{z=\alpha}K(z_0,z)\cdot P^g_{n+1}(x,z_S)\,dx(z)^2$$

together with the identity $\sum_{x(z)=x} \omega_n^g(z, z_S) = 0$ (which has the effect of converting some z to $\hat{z}$).

For $x = z + 1/z$, the involution that swaps branches is given by $\hat{z} = 1/z$ and

$$\omega_n^g(1/z, z_1) = -\omega_n^g(z, z_1) + \delta_{g,0}\delta_{n,2}\frac{dx\, dx_i}{(x-x_i)^2}.$$

In particular

$$\begin{aligned} &P_{n+1}^g(x, z_S)\, dx^2 \\ &= \omega_{n+2}^{g-1}(z, z, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega_{|I|+1}^{g_1}(z, z_I)\omega_{|J|+1}^{g_2}(z, z_J) - \sum_{i=1}^n \frac{dx\, dx_i}{(x-x_i)^2}\, \omega_n^g(z, z_{S\setminus i}) \end{aligned}$$

and since there are no $\hat{z}$ terms this expression is almost defined and analytic at $z = \infty$ except for the term involving $\omega_1^0(z)$. We replace this term with $\omega_1^0 + \ln x_1\, dx_1 \sim \Omega_1^0(x_1)$ and define

$$\widetilde{P}_{n+1}^g(x, z_S)\, dx^2 = P_{n+1}^g(x, z_S)\, dx^2 + 2\omega_{n+1}^g(z, z_S) \ln x(z)\, dx(z),$$

which has an expansion around $z = \infty$, $z_i = \infty$ given by

$$\begin{aligned} \widetilde{P}_{n+1}^g(x, z_S)\, dx^2 \sim \Omega_{n+2}^{g-1}(x, x, x_S) &+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \Omega_{|I|+1}^{g_1}(x, x_I)\Omega_{|J|+1}^{g_2}(x, x_J) \\ &+ \sum_{i=1}^n \frac{dx\, dx_i}{(x-x_i)^2}\, \Omega_n^g(x, x_{S\setminus i}). \end{aligned}$$

Note that $\omega_2^0(z, z_i) - dx\, dx_i/(x-x_i)^2 \sim \Omega_2^0(x, x_i)$ so the second sum in P_{n+1}^g contributes a term $2(dx\, dx_i/(x-x_i)^2)\, \omega_n^g(z, z_{S\setminus i})$ to P_{n+1}^g, which leads to the change in sign in front of $(dx\, dx_i/(x-x_i)^2)\omega_n^g(z, z_{S\setminus i})$.

Thus the Virasoro constraints would imply the loop equations and hence give a proof of Theorem 1.1 for general genus if one could show that

$$\eta_{n+1}^g(x, x_S) - 2\Omega_{n+1}^g(x, x_S) \ln x(z)\, dx(z)$$

is analytic at each zero of dx and vanishes to order two there. Although the Eynard–Orantin recursion has a Virasoro structure, it seems difficult to prove the theorem this way. It is believed that the Virasoro constraints do not determine the stationary invariants and instead enable one to calculate non-stationary invariants from stationary invariants [17].

6 A matrix integral proof of Theorem 1.1 for all genus

The Eynard–Orantin invariants come from matrix integrals. In good cases, the expansion of the invariants ω_n^g around $\{\mathbf{x}_i = \infty\}$ coincides with the expectation value with respect to a measure on the space of Hermitian matrices of the product of resolvents

$$W_n^g(x_1, \dots, x_n) := \left\langle \prod_{i=1}^n \mathrm{Tr} \frac{1}{x_i - M} \right\rangle_{\mathrm{conn}}^g.$$

The right-hand side denotes the connected genus- g part of the perturbative expansion of the integral, which is expanded over a set of fatgraphs that naturally have genus. The space of matrices may be a variant of the space of Hermitian matrices.

Plancherel measure There is a natural measure on partitions given by the Plancherel measure, using the dimension of irreducible representations of S_N, labelled by partitions λ and satisfying $\sum_{|\lambda|=N} \dim(\lambda)^2 = N!$. We can use Eynard–Orantin techniques to study expectation values of the partition function

$$Z_N(Q) = \sum_{l(\lambda) \le N} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 Q^{2|\lambda|}.$$

The asymptotic expansion of Z_N as $Q \to \infty$,

$$\ln Z_N(Q) = \sum_g Q^{2-2g} F^g,$$

can be solved using the normalisation of the Plancherel measure. For $N \to \infty$, $\exp(-Q^2) Z_N(Q) \to 1$ so

$$F^g = \delta_{g,0}.$$

Expectation values of Z_N can be generated by the spectral curve [5]

$$C = \begin{cases} x = z + 1/z, \\ y = \ln z. \end{cases}$$

In particular, this leads to a heuristic proof that coefficients $M_n^g(b_1, \dots, b_n)$ of the resolvent $W_n^g(x_1, \dots, x_n)$, which should be the coefficients of the Eynard–Orantin invariant ω_n^g in the expansion about $x = \infty$, can be expressed as stationary Gromov–Witten invariants.

Heuristic proof of Theorem 1.1 We use the expression of Okounkov and Pandharipande (3-9) that relates Gromov–Witten invariants to the Plancherel measure:

$$\left\langle \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle_d^{\bullet} = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^n \frac{\mathbf{p}_{b_i+1}(\lambda)}{(b_i+1)!}$$

for

$$\mathbf{p}_k(\lambda)=\sum_{i=1}^{\infty}\left[(\lambda_i-i+\tfrac{1}{2})^k-(-i+\tfrac{1}{2})^k\right]+(1-2^{-k})\zeta(-k).$$

In [5], it is shown that the Plancherel measure can be written in the large N limit as a matrix integral

$$\sum_{l(\lambda)\le N}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2 Q^{2|\lambda|}=\frac{Q^{N^2}}{N!}\int_{H_N(\mathcal{C})}e^{-Qtr(V(X))}\,dX, \tag{6-1}$$

where $QV(x)=\ln(\Gamma(Qx))-\ln(\Gamma(-Qx))+i\pi Qx+\ln(Qx)-Qx\ln Q+QA_0$ for some constant A_0, $\mathcal{C}$ is a contour in the complex plane surrounding all of the positive integers and $H_N(\mathcal{C})$ is the set of normal $N\times N$ matrices whose eigenvalues lie on the contour $\mathcal{C}$.

$$H_N(\mathcal{C})=\{X\mid X=U^T\Lambda U,UU^T=\mathrm{Id}_N,\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_N),\lambda_i\in\mathcal{C}\}.$$

It was also found that this matrix model has a rational spectral curve given by

$$\widetilde{C}=\begin{cases}x=\frac{N-1/2}{Q}+z+1/z,\\ y=\ln(z).\end{cases} \tag{6-2}$$

Thus the $\widetilde{M}_n^g$ of $\widetilde{C}$ correspond to expectation values in this integral, or equivalently, expectation values of the Plancherel measure. If the h_i represent the $\pi/4$ rotated partitions, $h_i=\lambda_i-i+N$, then

$$\begin{aligned}
&W_n^g(x_1,\dots,x_n)\\
&:=\Big\langle\prod_{i=1}^n\sum_j\frac{1}{x_i-h_j/Q}\Big\rangle_{\text{conn}}^g=\sum_{b_1,\dots,b_n=1}^{\infty}\frac{\widetilde{M}_n^g(b_1,\dots,b_n)}{x_1^{b_1+1}\cdots x_n^{b_n+1}}\\
&=\sum_{b_1,\dots,b_n=1}^{\infty}\frac{1}{x_1^{b_1+1}\cdots x_n^{b_n+1}}\Bigg[\sum_{l(\lambda)\le N}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2 Q^{2|\lambda|-\sum b_i}\prod_{i=1}^n\sum_j h_j^{b_i}\Bigg]_{\text{conn}}^g\\
&=\sum_{b_1,\dots,b_n=1}^{\infty}\frac{1}{x_1^{b_1+1}\cdots x_n^{b_n+1}}\Bigg[\sum_{l(\lambda)\le N}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2 Q^{2|\lambda|-\sum b_i}\prod_{i=1}^n\sum_j(\lambda_j-j+N)^{b_i}\Bigg]_{\text{conn}}^g.
\end{aligned}$$

Since Eynard–Orantin invariants do not change when x changes by a constant, we can consider the curve

$$C_2=\begin{cases}x'=z+1/z,\\ y=\ln(z).\end{cases} \tag{6-3}$$

The ω_n^g are the same, but the expansion around $x'=\infty$ are different, giving new M_n^g:

$$W_n^g(x_1,\dots,x_n)=\left\langle\prod_{i=1}^n\sum_j\frac{1}{x_i'+(N-1/2)/Q-h_j/Q}\right\rangle_{\mathrm{conn}}^g$$
$$=\sum_{b_1,\dots,b_n=1}^{\infty}\frac{M_n^g(b_1,\dots,b_n)}{x_1'^{b_1+1}\cdots x_n'^{b_n+1}},$$

where

$$\begin{aligned}
&M_n^g(b_1,\dots,b_n)\\
&=\Bigg[\sum_{l(\lambda)\le N}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2Q^{2|\lambda|-b_1-\cdots-b_n}\prod_{i=1}^n\sum_j\big(h_j-N+\tfrac12\big)^{b_i}\Bigg]_{\mathrm{conn}}^g\\
&=\Bigg[\sum_d Q^{2d-\sum b_i}\sum_{|\lambda|=d}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2\prod_{i=1}^n\sum_j\big(\lambda_j-j+\tfrac12\big)^{b_i}\Bigg]_{\mathrm{conn}}^g\\
&=\Bigg[\sum_d Q^{2d-\sum b_i}\sum_{|\lambda|=d}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2\\
&\qquad\times\prod_{i=1}^n\Big(\mathbf{p}_{b_i}(\lambda)+\sum_j\big(-j+\tfrac12\big)^{b_i}-(1-2^{-b_i})\zeta(-b_i)\Big)\Bigg]_{\mathrm{conn}}^g\\
&=\Bigg[\sum_d Q^{2d-\sum b_i}\prod_{i=1}^n b_i!\left\langle\prod_{i=1}^n\tau_{b_i-1}(\omega)\right\rangle_d^{\bullet}\Bigg]_{\mathrm{conn}}^g+0.
\end{aligned}$$

Using the fact that

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-j+\frac12)^kz^k}{k!}=\sum_{j=1}^{\infty}e^{z(-j+\frac12)}=e^{z/2}\left(\frac{1}{1-e^{-z}}-1\right)$$
$$=\frac{2}{\sinh(z/2)}=\sum_{k=0}^{\infty}\frac{(1-2^{-k})\zeta(-k)}{k!}z^k$$

and comparing coefficients. Note that these extra components of $\mathbf{p}_k$ are only used in [16] so that evaluations can be made for finite partitions without the need to evaluate infinite series. In an expectation value they will have no effect. Since $2g-2+2d=\sum_{i=1}^n(b_i-1)$ defines the degree, taking the genus-g component involves taking only one term, and we extract the coefficient of Q^{2-2g-n}. The connected part then gives connected Gromov–Witten invariants:

$$M_n^g(b_1,\dots,b_n)=\prod_{i=1}^n b_i!\left\langle\prod_{i=1}^n\tau_{b_i-1}(\omega)\right\rangle^g. \qquad\square$$

Remark The main idea of the heuristic proof is that stationary Gromov–Witten invariants of $\mathbb{P}^1$, which are expectation values of the Plancherel measure by rigorous work of Okounkov and Pandharipande [16], can be collected neatly as resolvents of Plancherel measure. Eynard–Orantin invariants are heuristically resolvents of measures from matrix models. The Plancherel measure is heuristically represented by a matrix model with a potential that gives rise to the desired spectral curve. The failure of the proof to be rigorous lies in the heuristic ideas that the Eynard–Orantin invariants are resolvents of matrix models and that a matrix model retrieves the Plancherel measure. Even if some of this could be made precise, the Eynard–Orantin recursion is expected to coincide with Virasoro constraints and we saw in the previous section that the Virasoro constraints require non-stationary terms.

7 Formulae

The following values for $N^g_{n,k}$ were computed with the method of [15] and using Lemma 2.4 we can compute the corresponding $m^g_{n,k}$. We can use Theorem 1.1, Table 1 at the end of this section and the divisor equation (2-14) to compute the following expressions for stationary Gromov–Witten invariants of $\mathbb{P}^1$.

- Genus zero two-point invariants:

$$\langle \tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^{g=0} = \frac{1}{u_1!^2\, u_2!^2}\,\frac{1}{(u_1+u_2+1)}$$
$$\langle \tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^{g=0} = \frac{u_1u_2}{u_1!^2\, u_2!^2}\,\frac{1}{(u_1+u_2)}$$

- Genus zero three-point invariants:

$$\langle \tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_{2u_3}(\omega)\rangle^{g=0} = \frac{1}{u_1!^2\, u_2!^2\, u_3!^2}$$
$$\langle \tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3-1}(\omega)\rangle^{g=0} = \frac{u_2u_3}{u_1!^2\, u_2!^2\, u_3!^2}$$

- Genus zero four-point invariants:

$$\left\langle \prod_{i=1}^{4} \tau_{2u_i}(\omega)\right\rangle^{g=0} = \frac{1}{\prod_{i=1}^4 u_i!^2}(u_1+u_2+u_3+u_4+1)$$
$$\left\langle \prod_{i=1}^{2} \tau_{2u_i}(\omega)\prod_{i=3}^{4} \tau_{2u_i-1}(\omega)\right\rangle^{g=0} = \frac{u_3u_4}{\prod_{i=1}^4 u_i!^2}(u_1+u_2+u_3+u_4)$$

$$\left\langle \prod_{i=1}^{4} \tau_{2u_i-1}(\omega) \right\rangle^{g=0} = \prod_{i=1}^{4} \frac{u_i}{u_i!^2}(u_1+u_2+u_3+u_4)$$

- Repeatedly applying the divisor equation gives the even, genus-zero n point invariants:

$$\left\langle \prod_{i=1}^{n} \tau_{2u_i}(\omega) \right\rangle^{g=0} = \frac{1}{\prod_{i=1}^{n} u_i!^2}\left(\sum_{i=1}^{n} u_i+1\right)^{n-3}$$

- Genus one one-point invariants:

$$\langle \tau_{2u}(\omega)\rangle^{g=1} = \frac{1}{24u!^2}(2u-1)$$

- Genus one two-point invariants:

$$\langle \tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\rangle^{g=1} = \frac{1}{24u_1!^2u_2!^2}\,(2u_1^2+2u_2^2+2u_1u_2-u_1-u_2)$$

$$\langle \tau_{2u_1-1}(\omega)\tau_{2u_2-1}(\omega)\rangle^{g=1} = \frac{u_1u_2}{24u_1!^2u_2!^2}\,(2u_1^2+2u_2^2+2u_1u_2-3u_1-3u_2)$$

- Genus one three-point invariants:

$$\langle \tau_{2u_1}(\omega)\tau_{2u_2}(\omega)\tau_{2u_3}(\omega)\rangle^{g=1} = \frac{1}{24\prod_{i=1}^{3} u_i!^2}\left(\sum_{i=1}^{3} 2u_i^3-u_i^2+\sum_{i\neq j} u_iu_j(4u_i-1)+4u_1u_2u_3\right)$$

$$\langle \tau_{2u_1}(\omega)\tau_{2u_2-1}(\omega)\tau_{2u_3-1}(\omega)\rangle^{g=1} = \frac{u_2u_3}{24\prod_{i=1}^{3} u_i!^2}\left(\sum_{i=1}^{3} 2u_i^3-5u_i^2+3u_i + \sum_{i\neq j} u_iu_j(4u_i-3)+2u_1^2-3u_1-2u_2u_3+4u_1u_2u_3\right)$$

- Genus two one-point invariants:

$$\langle \tau_{2u}(\omega)\rangle^{g=2} = \frac{1}{2^7 3^2 5 u!^2}u^2(2u-3)(10u-17)$$

- Genus three one-point invariants:

$$\langle \tau_{2u}(\omega)\rangle^{g=3} = \frac{1}{2^{10}3^4 5 7 u!^2}\,u^2(u-1)^2(2u-5)(140u^2-784u+1101)$$

g	n	k	$N^g_{n,k}(b_1,\dots,b_n)$	$m^g_{n,k}(b_1,\dots,b_n)$
0	3	0,2	0	0
0	3	1,3	1	1
1	1	0	0	0
1	1	1	$\frac{1}{48}(b_1^2-3)$	$\frac{1}{24}(b_1-2)$
0	4	0	$\frac{1}{4}(b_1^2+b_2^2+b_3^2+b_4^2)$	$\frac{1}{2}(b_1+b_2+b_3+b_4)$
0	4	1,3	0	0
0	4	2	$\frac{1}{4}(b_1^2+b_2^2+b_3^2+b_4^2-2)$	$\frac{1}{2}(b_1+b_2+b_3+b_4-2)$
0	4	4	$\frac{1}{4}(b_1^2+b_2^2+b_3^2+b_4^2)$	$\frac{1}{2}(b_1+b_2+b_3+b_4-2)$
1	2	0	$\frac{1}{384}(b_1^2+b_2^2-8)(b_1^2+b_2^2)$	$\frac{1}{48}(b_1^2+b_2^2+b_1b_2 -3(b_1+b_2))$
1	2	1	0	0
1	2	2	$\frac{1}{384}(b_1^2+b_2^2-6)(b_1^2+b_2^2-2)$	$\frac{1}{48}(b_1^2+b_2^2+b_1b_2 -4(b_1+b_2)+5)$
1	3	0,2	0	0
1	3	1	$\frac{1}{4608}\big(\sum_{i=1}^3 b_i^6-20b_i^4+94b_i^2 +6\sum_{i\neq j} b_i^2b_j^2(b_i^2-5) +12b_1^2b_2^2b_3^2+3b_1^4-63b_1^2-15\big)$	$\frac{1}{96}\big(\sum_{i=1}^3 b_i^3-7b_i^2+14b_i +\sum_{i\neq j} b_ib_j(2b_i-5) +2b_1b_2b_3+b_1^2-5b_1-4\big)$
1	3	3	$\frac{1}{4608}\big(\sum_{i=1}^3 b_i^6-17b_i^4+103b_i^2 +6\sum_{i\neq j} b_i^2b_j^2(b_i^2-5) +12b_1^2b_2^2b_3^2-129\big)$	$\frac{1}{96}\big(\sum_{i=1}^3 b_i^3-8b_i^2+23b_i +2\sum_{i\neq j} b_ib_j(b_i-3) +2b_1b_2b_3-26\big)$
2	1	0	0	0
2	1	1	$\frac{(b_1^2-1)^2}{2^{16}3^3 5}(5b_1^4-186b_1^2+1605)$	$\frac{1}{2^9 3^2 5}(b-1)^2(b-4)(5b-22)$
3	1	0	0	0
3	1	1	$\frac{1}{2^{25}3^6 5^2 7}(b^2-1)^2(b^2-3)^2 (5b^6-649b^4+27995b^2-394695)$	$\frac{1}{2^{14}3^4 5 7}(b-1)^2(b-3)^2(b-6) (35b^2-462b+1528)$

Table 1

References

[1] **G Borot**, **B Eynard**, **M Mulase**, **B Safnuk**, *A matrix model for simple Hurwitz numbers, and topological recursion*, J. Geom. Phys. 61 (2011) 522–540 MR2746135

[2] **V Bouchard**, **A Klemm**, **M Mariño**, **S Pasquetti**, *Remodeling the B–model*, Comm. Math. Phys. 287 (2009) 117–178 MR2480744

[3] **V Bouchard**, **M Mariño**, *Hurwitz numbers, matrix models and enumerative geometry*, from: "From Hodge theory to integrability and TQFT tt*–geometry", (R Y Donagi, K Wendland, editors), Proc. Sympos. Pure Math. 78, Amer. Math. Soc. (2008) 263–283 MR2483754

[4] **R Dijkgraaf**, **H Fuji**, **M Manabe**, *The volume conjecture, perturbative knot invariants, and recursion relations for topological strings*, Nuclear Phys. B 849 (2011) 166–211 MR2795276

[5] **B Eynard**, *All order asymptotic expansion of large partitions*, J. Stat. Mech. Theory Exp. (2008) P07023, 34 MR2439683

[6] **B Eynard**, *Recursion between Mumford volumes of moduli spaces*, Ann. Henri Poincaré 12 (2011) 1431–1447 MR2855174

[7] **B Eynard**, **M Mulase**, **B Safnuk**, *The Laplace transform of the cut-and-join equation and the Bouchard–Mariño conjecture on Hurwitz numbers*, Publ. Res. Inst. Math. Sci. 47 (2011) 629–670 MR2849645

[8] **B Eynard**, **N Orantin**, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. 1 (2007) 347–452 MR2346575

[9] **B Eynard**, **N Orantin**, *Topological recursion in enumerative geometry and random matrices*, J. Phys. A 42 (2009) 293001, 117 MR2519749

[10] **J D Fay**, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics 352, Springer, New York (1973) MR0335789

[11] **E Getzler**, *Topological recursion relations in genus* 2, from: "Integrable systems and algebraic geometry", (M-H Saito, Y Shimizu, K Ueno, editors), World Sci. Publ. (1998) 73–106 MR1672112

[12] **E Getzler**, *The Toda conjecture*, from: "Symplectic geometry and mirror symmetry", (K Fukaya, Y-G Oh, K Ono, G Tian, editors), World Sci. Publ. (2001) 51–79 MR1882327

[13] **M Mariño**, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. (2008) 060, 34 MR2391060

[14] **P Norbury**, *String and dilaton equations for counting lattice points in the moduli space of curves*, Trans. Amer. Math. Soc. 365 (2013) 1687–1709 MR3009643

[15] **P Norbury**, **N Scott**, *Polynomials representing Eynard–Orantin invariants*, Q. J. Math. 64 (2013) 515–546 MR3063520

[16] **A Okounkov**, **R Pandharipande**, *Gromov–Witten theory, Hurwitz theory, and completed cycles*, Ann. of Math. 163 (2006) 517–560 MR2199225

[17] **A Okounkov**, **R Pandharipande**, *Virasoro constraints for target curves*, Invent. Math. 163 (2006) 47–108 MR2208418

[18] **A Okounkov**, **R Pandharipande**, *Gromov–Witten theory, Hurwitz numbers, and matrix models*, from: "Algebraic geometry, Part 1", (D Abramovich, A Bertram, L Katzarkov, R Pandharipande, M Thaddeus, editors), Proc. Sympos. Pure Math. 80, Amer. Math. Soc. (2009) 325–414 MR2483941

[19] **A N Tjurin**, *Periods of quadratic differentials*, Uspekhi Mat. Nauk 33 (1978) 149–195, 272 MR526014 In Russian; translated in Russian Math. Surveys 33 (1978) 169–221

[20] **H J H Tuenter**, *Walking into an absolute sum*, Fibonacci Quart. 40 (2002) 175–180 MR1902756

[21] **E Witten**, *Two-dimensional gravity and intersection theory on moduli space*, from: "Proceedings of the conference on geometry and topology held at Harvard University, April 27–29, 1990", (S T Yau, H B Lawson, editors), Surveys in differential geometry 1, Int. Press, Somerville, MA (1991) 243–310 MR1144529

Department of Mathematics and Statistics, University of Melbourne
Victoria 3010, Australia

pnorbury@ms.unimelb.edu.au, nickidge@gmail.com

http://www.ms.unimelb.edu.au/~pnorbury

Proposed: Lothar Göttsche Received: 18 July 2011
Seconded: Jim Bryan, Yasha Eliashberg Revised: 6 December 2013