Tetrahedra of flags, volume
and homology of SL(3)

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In the paper we define a "volume" for simplicial complexes of flag tetrahedra. This generalizes and unifies the classical volume of hyperbolic manifolds and the volume of CR tetrahedral complexes considered in Falbel [6], and Falbel and Wang [8]. We describe when this volume belongs to the Bloch group and more generally describe a variation formula in terms of boundary data. In doing so, we recover and generalize results of Neumann and Zagier [18], Neumann [16] and Kabaya [15]. Our approach is very related to the work of Fock and Goncharov [9; 10].

1 Introduction

Let M be a complete hyperbolic 3–manifold. The holonomy of the geometric representation $\rho_{\text{geom}}\colon \pi_1(M) \to \text{PGL}(2, \mathbb{C})$ is faithful and has discrete image. If $d\colon \widetilde{M} \to \mathbb{H}^3$ is a developing map for ρ_{geom} we may define the volume of ρ_{geom} as the integral of the pull-back by d of the hyperbolic volume form over a fundamental domain for M. It follows from Mostow's rigidity theorem that the volume of ρ_{geom} is a topological invariant of M. If the complete hyperbolic manifold M has cusps, Thurston showed that one could obtain complete hyperbolic structures on manifolds obtained from M by Dehn surgery by gluing a solid torus with a sufficiently long geodesic. Thurston framed his results for general representations $\rho\colon \pi_1(M) \to \text{PGL}(2, \mathbb{C})$, which do not need to be injective (or even discrete). The corresponding hyperbolic structures on M are not complete; still, there is a well-defined notion of volume of ρ. For a representation ρ associated to a Dehn surgery on M, the volume of ρ is the volume of the metric completion of the corresponding hyperbolic structure on M. Neumann and Zagier [18] and afterwards Neumann [16] provided a deeper analysis of these deformations of ρ_{geom} and their volumes. In particular, they showed that the variation of the volume depends only on the geometry of the boundary and they gave a precise formula for that variation in terms of the boundary holonomy. They work in the natural setting of

decorated triangulations. In this introduction we first briefly recall the classical work of Neumann and Zagier in order to motivate our main results, which are $\mathrm{PGL}(3, \mathbb{C})$ analogues of theirs.

1.1 Ideal triangulations

An *ordered simplex* is a simplex with a fixed vertex ordering. Recall that an orientation of a set of vertices is a numbering of the elements of this set up to even permutation. The face of an ordered simplex inherits an orientation. We call *abstract triangulation* a pair $\mathcal{T} = ((T_\nu)_{\nu=1,\dots,N}, \Phi)$, where $(T_\nu)_{\nu=1,\dots,N}$ is a finite family of abstract ordered simplicial tetrahedra and Φ is a matching of the faces of the T_ν reversing the orientation. For any simplicial tetrahedron T, we define $\mathrm{Trunc}(T)$ as the tetrahedron truncated at each vertex. The space obtained from $\mathrm{Trunc}(T_\mu)$ after matching the faces will be denoted by K.

We call *triangulation* – or rather *ideal triangulation* – of a compact 3–manifold M with boundary an abstract triangulation $\mathcal{T}$ and an oriented homeomorphism

$$K = \bigsqcup_{\nu=1}^{N} \mathrm{Trunc}(T_\nu)/\Phi \to M.$$

1.2 Neumann–Zagier bilinear relations

Thurston has proposed to parametrize the set of conjugacy classes of representations of $\pi_1(M)$ in $\mathrm{PGL}(2, \mathbb{C})$ by solutions of a system of polynomial equations – called *gluing equations* – associated to the combinatorial data $\mathcal{T}$. Indeed, an ideal tetrahedron T of $\mathbb{H}^3$ is described completely (up to isometry) by a single complex number $z \in \mathbb{C} - \{0, 1\}$. The numbers z, $1-1/z$ and $1/(1-z)$ give the same tetrahedron; to specify z uniquely we must pick an edge of T. Making such a choice for each T_ν and letting $z_\nu = z(T_\nu)$, we end up with one of the three complex numbers z_ν, $1-1/z_\nu$ and $1/(1-z_\nu)$ attached to each edge of T_ν. The necessary and sufficient condition that gluing these ideal tetrahedra gives a (not necessarily complete) hyperbolic manifold is that at each 1–cell e of $\mathcal{T}$ the tetrahedra T_ν abutting to e "close up" as one goes around e; see [18, page 312]. This may be encoded in an equation of the form

$$(1.2.1) \qquad\qquad \prod_\nu z_\nu^{r'_{j\nu}} (1-z_\nu)^{r''_{j\nu}} = \pm 1,$$

where $R' = (r'_{j\nu})$ and $R'' = (r''_{j\nu})$ are matrices with integer entries, whose columns are parametrized by the simplices of $\mathcal{T}$ and whose lines are parametrized by the 1–cells of K. For simplicity in this introduction we shall assume that the boundary of M is a

disjoint union of a finite collection of 2–dimensional tori. The most important family of examples is the compact 3–manifolds whose interior carries a complete hyperbolic structure of finite volume. By a simple Euler characteristic count we then have that the number of 1–cells of K is equal to the number N of tetrahedra. In particular R' and R'' are square matrices of size N. Solving these gluing equations gives rise to an efficient algorithm for constructing hyperbolic structures, which has been implemented in SnapPea. One key feature of the gluing equations is the symplectic property of the matrix $R = (R' \mid R'')$ obtained as the concatenation of the matrices R' and R'' in (1.2.1). Namely, denoting by

$$J_{2N} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

the standard symplectic matrix on $\mathbb{C}^{2N}$, we have:

(1) The rows of R Poisson commute, ie $R J_{2N}{}^t R = 0$.

(2) Denoting by $[R]$ the subspace of $\mathbb{C}^{2N}$ generated by row vectors and by $[R]^{\perp}$ its orthogonal with respect to the symplectic form associated to J_{2N}, we have a symplectic isomorphism

$$[R]^{\perp}/[R] \cong H_1(\partial M),$$

where the right-hand side is equipped with twice the intersection product on each boundary torus.

The proofs of these two facts are due to Neumann and Zagier [18, Theorem 2.2]; it is the fundamental ingredient from which all the results of [18] follow.

1.3 Overview of the paper

Here we consider representations ρ of $\pi_1(M)$ in PGL(3, $\mathbb{C}$). This framework includes both hyperbolic structures, CR structures on 3–manifolds (that is, $(\mathbb{S}^3, \mathrm{PU}(2, 1))$–geometric structures) and 3–dimensional real flag structures (that is, $(\mathrm{SL}(3, \mathbb{R})/B,$ SL(3, $\mathbb{R}$))–geometric structures, where B is the Borel subgroup of upper-triangular matrices). One motivation is to build an efficient algorithm for constructing geometric structures on 3–manifolds; we shall explain in Section 10 that our work can indeed be effectively implemented in a large number of cases to build new representations of $\pi_1(M)$ into PU(2, 1) and PGL(3, $\mathbb{R}$).

So consider a representation ρ of the fundamental group of M in PGL(3, $\mathbb{C}$). We first note that ρ is *parabolic*: the peripheral holonomies preserve a flag in $\mathbb{C}^3$. Recall that a flag in $\mathbb{C}^3$ is a line in a plane of $\mathbb{C}^3$. An *affine flag* in $\mathbb{C}^3$ is the data of a non-zero

point $x \in \mathbb{C}^3$ and a non-zero linear form $f\colon \mathbb{C}^3 \to \mathbb{C}$ such that $f(x) = 0$. We say that ρ is *unipotent* if the peripheral holonomies preserve an *affine* flag in $\mathbb{C}^3$. The fact that ρ is always parabolic links us to the work of Fock and Goncharov [9; 10]; we indeed make intensive use of their combinatorics on the space of representations of surface groups in $\mathrm{SL}(3, \mathbb{R})$.

Following Thurston's approach in the $\mathrm{PGL}(2, \mathbb{C})$ case, rather than working with representations of $\pi_1 M$ into $\mathrm{PGL}(3, \mathbb{C})$, we study decorations of simplicial complexes by cross-ratios associated to tetrahedral configurations of flags. Our paper is then divided into two parts. The first is purely local and deals with configurations of flags and decorations of tetrahedra. The second part is global and deals with decoration of tetrahedral complexes. We give a more precise overview in the two paragraphs below.

1.3.1 The local picture In Section 2 we describe flags and configuration of flags. We associate to a tetrahedron of flags (resp. of affine flags) a set of 16 complex z–coordinates (resp. a–coordinates); 12 coordinates associated to the edges of the tetrahedron (one for each oriented edge) and four coordinates associated to the faces; see Figure 3. Those are, in the Fock and Goncharov setting, the a– and z–coordinates on the boundary of each tetrahedron, namely a four-holed sphere. Similar coordinates have been independently considered by Garoufalidis, Goerner, Thurston and Zickert [11; 12] who refer to them as respectively shape and Ptolemy coordinates. These data define a *decorated tetrahedron*. Note that there are a lot of relations between the different z–coordinates; these are studied in Section 2. In particular we prove (Proposition 2.4.1) that a decorated tetrahedron is parametrized by the 4 coordinates associated to two opposite edges.

In Section 3 we use these coordinates to define the *volume* of a decorated tetrahedron, generalizing and unifying the volume of hyperbolic tetrahedra and CR tetrahedra. This volume is "natural" in the sense that when extended to define the volume of a simplicial complex of flag tetrahedra it is invariant under a change of triangulation of the simplicial complex (2-3 moves). In other words, it is natural to first define a map from the space of tetrahedra to the pre-Bloch group $\mathcal{P}(\mathbb{C})$, which is defined as the abelian group generated by all the points in $\mathbb{C} \setminus \{0, 1\}$ quotiented by the 5–term relations (see Section 3 for definitions and references). Eventually we obtain a well-defined volume

$$\mathrm{Vol} = \tfrac{1}{4} D \circ \beta \colon H_3(\mathcal{F}l) \to \mathbb{R}$$

(see Definition 3.2.1). Here $H_3(\mathcal{F}l)$ is the degree-3 homology group of the space of flags, β is a map from $H_3(\mathcal{F}l)$ to the pre-Bloch group $\mathcal{P}(\mathbb{C})$ and $D\colon \mathcal{P}(\mathbb{C}) \to \mathbb{R}$ is the dilogarithm.

We moreover relate our work to Suslin's work on K_3, showing that our volume map is essentially the Suslin map S from the (discrete) homology group $H_3(\mathrm{SL}(3,\mathbb{C}))$ to the pre-Bloch group $\mathcal{P}(\mathbb{C})$ (see Suslin [20]). Indeed: composing β with the natural projection π_* from $H_3(\mathrm{SL}(3,\mathbb{C}))$ to $H_3(\mathcal{Fl})$, we end up with a "volume map" $\beta \circ \pi_*\colon H_3(\mathrm{SL}(3,\mathbb{C})) \to \mathcal{P}(\mathbb{C})$.

Theorem (see Theorem 3.5.1) *The "volume map"* $\frac{1}{4}\beta \circ \pi_*\colon H_3(\mathrm{SL}(3,\mathbb{C})) \to \mathcal{P}(\mathbb{C})$ *coincides with Suslin's map* S.

This gives a geometric and intuitive construction of the latter. Here we are very close to the work of Zickert on the extended Bloch group [23].

Note that the volume function on hyperbolic manifolds was already extended in [6; 8] in order to deal with Cauchy–Riemann (CR). The definition there is valid for "cross-ratio structures" (which include hyperbolic and CR structures). It turns out to be a coordinate description of decorated triangulations and the invariant in $\mathcal{P}(\mathbb{C})$ coincides with the one defined before up to a multiple of four.

The Bloch group $\mathcal{B}(\mathbb{C})$ is a subgroup of the pre-Bloch group $\mathcal{P}(\mathbb{C})$. It is defined as the kernel of the map

$$\delta\colon \mathcal{P}(\mathbb{C}) \to \mathbb{C}^\times \wedge_{\mathbb{Z}} \mathbb{C}^\times$$

given by $\delta([z]) = z \wedge (1-z)$. The volume and the Chern–Simons invariant can then be seen through a function (the Bloch regulator)

$$\mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}.$$

The imaginary part is related to the volume and the real part is related to the Chern–Simons CS mod $\mathbb{Q}$ invariant.

In Section 4 we associate to a decorated tetrahedron T the element $\delta(\beta(T)) \in \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$, where $\beta(T) \in \mathcal{P}(\mathbb{C})$ is the "volume" defined in the previous section, and compute it using both a–coordinates and z–coordinates. The motivations for such computations range over three different directions. At first, the algebraic meaning of the map δ in relation with the Bloch group justifies by itself such a computation. It will allow (see Section 6) a geometric interpretation of elements in the Bloch group and prove that the Suslin map takes values in that group. Second it gives a grasp on our volume map, as it gives a formula for its derivative. The third direction of interest lies more in the proofs of our formulas: we link the quantity $\delta(\beta(T))$ to properties of natural 2–forms on the spaces of coordinates. This lays down the foundations for the remainder of the paper where we focus on these 2–forms and relate them to a well-known two-form, namely the Weil–Petersson form for surfaces (see Section 7).

We compute $\delta(\beta(T))$ in two different ways. The first uses a–coordinates associated to some lift of T as a tetrahedron of affine flags. In that respect we mainly follow Fock and Goncharov. The second way directly deals with z–coordinates and is related to the work of Neumann and Zagier. In the global part it will turn out to be fruitful to mix those two approaches.

1.3.2 The global picture This local work being done, we move on in Section 5 to the framework of decorated simplicial complexes. The decoration consists of a–coordinates or z–coordinates associated to each tetrahedron and satisfying appropriate compatibility conditions along edges and faces. This defines parabolic and unipotent decorations of the pair $(M, \mathcal{T})$. Parabolic decorations provide a generalization to $\mathrm{PGL}(3, \mathbb{C})$ of Thurston's work described above: to each tetrahedron of $\mathcal{T}$ we now associate a set of 16 non-zero complex coordinates. As in the $\mathrm{PGL}(2, \mathbb{C})$ case, these coordinates are subject to consistency relations after gluing by Φ. We give a complete description of these *gluing equations*.

Solving the gluing equations gives rise to an efficient algorithm for constructing representations $\pi_1(M) \to \mathrm{PGL}(3, \mathbb{C})$. We provide an explicit computation of the holonomy representation associated to such a decoration. In particular we list the remaining compatibility equations needed for the decoration to be *unipotent*. In Section 10 this is used to describe all unipotent decorations on the complement of the figure-eight knot. A systematic computation of all unipotent decorations of triangulated low complexity hyperbolic manifolds was undertaken in Falbel, Koseleff and Rouillier [7] where details of the computation for the figure eight-knot are described. We refer to this paper for further examples. The natural question of the rigidity of unipotent representation is investigated in Bergeron, Falbel and Guilloux [1] (see also Genzmer [13]).

Associated to a general decorated complex K we have a "volume" element

$$\beta = \beta(K) := \sum_{\nu=1}^{N} \beta(T_\nu) \in \mathcal{P}(\mathbb{C}).$$

The main result of this paper (Theorem 5.5.1) is an explicit computation of $\delta(\beta(K))$, which turns out to depend only on boundary data. The general result is a bit too technical to be stated in this introduction. Let us note that even in the $\mathrm{PGL}(2, \mathbb{C})$ case it appears to be new in this generality; see Kabaya [15] and Bonahon [3].

The proof of Theorem 5.5.1 occupies three sections. We first give in Section 6 a proof of Theorem 5.5.1 in a special case, namely when the decoration is unipotent. This will give a new proof that the Suslin map takes values in the Bloch group. We then deal with the proof of the general case in Sections 7 and 8. In doing so we have to

develop a generalization of the Neumann–Zagier symplectic property to the PGL$(3, \mathbb{C})$ case. To do so we heavily use the point of view developed in Section 4 and the 2–form constructed there. Theorem 5.5.1 finally yields a variational formula for the volume. This is addressed in Section 11.1.

Recall that the 16 complex coordinates associated to each tetrahedron of $\mathcal{T}$ are subject to relations. These relations are linearized in Section 4 so that we may think of the set of all coordinates of the simplicial complex K as an element

$$z \in \mathrm{Hom}_{\mathbb{Z}}(J, \mathbb{C}^{\times}),$$

where J is a $\mathbb{Z}$–module of dimension $8N$ and $\mathbb{C}^{\times}$ is seen as a $\mathbb{Z}$–module. In Section 6 we linearize the gluing equations:

Proposition (Lemma 6.1.1) *An element* $z \in \mathrm{Hom}_{\mathbb{Z}}(J, \mathbb{C}^{\times})$ *satisfies the gluing equations if and only if it is trivial on the image of the linear map denoted by* F' *in Section 7.3.*

It will be enough in this introduction to note that the domain of F' is a $\mathbb{Z}$–module of dimension $4N$. The image of F' is the direct analog of the space $[R]$ in the work of Neumann–Zagier alluded to above. As in Neumann [16] this subspace arises here very naturally.

In Section 4 the $\mathbb{Z}$–module J is naturally equipped with a non-degenerate skew-symmetric form Ω. This form turns out to be the natural generalization of Neumann–Zagier symplectic form. The following proposition is the direct analog of the first part of Neumann–Zagier bilinear relations.

Proposition (see (7.3.3)) *The subspace* $\mathrm{Im}(F')$ *is contained in* $\mathrm{Im}(F')^{\perp \Omega}$.

In fact we introduce a larger $\mathbb{Z}$–module J^2 corresponding to the space of all 16 coordinates on each tetrahedron. The module J is then a quotient of J^2. Now J^2 is naturally equipped with a skew-symmetric form Ω^2 that was already considered by Fock and Goncharov in a somewhat different context. The form Ω is induced by Ω^2 on the quotient. Everything being set up this way the proof of the above proposition turns out to be a tautology.

A quite similar generalization of the first part of Neumann–Zagier bilinear relations has been independently worked out by Garoufalidis, Goerner and Zickert [11] in the general PGL$(n, \mathbb{C})$ case. What is more subtle is the second part of Neumann–Zagier bilinear relations.

From the above proposition it follows that Ω induces a skew-symmetric form on the quotient

$$\mathcal{H}(J) = \mathrm{Im}(F')^{\perp \Omega} / \mathrm{Im}(F').$$

In this setting, our main result implies that Ω is non-degenerate on the quotient and relates the quotient symplectic space with the "Goldman–Weil–Petersson" form wp on $H^1(\partial M, \mathbb{C}^2)$. Recall that restricted to a torus component of ∂M the form wp is defined as the coupling of the cup product on H^1 with the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}^2$ defined by

$$\left\langle \binom{n}{m}, \binom{n'}{m'} \right\rangle = \tfrac{1}{3}(2nn' + 2mm' + nm' + n'm);$$

see Section 7.1.2.

Theorem (Corollary 7.3.2) *The form Ω induces a non-degenerate skew-symmetric form on $\mathcal{H}(J)$. Moreover, tensored with $\mathbb{C}$ the dual symplectic space $(\mathcal{H}(J)^*, \Omega^*)$ is symplectically isomorphic to $(H^1(\partial M, \mathbb{C}^2), \text{wp})$.*

The core of the proof of this theorem goes along the same lines as the homological proof of Neumann [16] of the Neumann–Zagier bilinear relations. We nevertheless believe that the use of the combinatorics of Fock and Goncharov sheds some light on Neumann's work even in the classical PGL$(2, \mathbb{C})$ case. The two theories fit well together, allowing a new understanding, in particular, of the Neumann–Zagier symplectic form. Indeed, an interesting point of our proof is that the Goldman–Weil–Petersson form for peripheral tori naturally arises.

Finally in Section 11, we describe applications of Theorem 5.5.1. First, we follow again Neumann–Zagier and obtain an explicit formula for the variation of the volume function that only depends on boundary data. Then, relying on remarks of Fock and Goncharov, we describe a 2–form on the space of representations of the boundary of our variety, which coincides with Weil–Petersson form in some cases (namely for hyperbolic structures and unipotent decorations). We should also mention that the Theorem above – generalizing Neumann–Zagier bilinear relations – has been used in [1] to prove new rigidity results for representations of $\pi_1(M)$ into PGL$(3, \mathbb{C})$.

For the ease of reading we have added at the end of the paper an index of the most important symbols used in the text.

Acknowledgements

2 Configurations of flags and cross-ratios

We consider in this section the flag variety $\mathcal{F}l$ and the affine flag variety $\mathcal{AF}l$ of SL$(3, k)$ for a field k. We define coordinates on the configurations of 4 flags (or affine flags) that are very similar to the coordinates used by Fock and Goncharov [9].

2.1 Flags, affine flags and their spaces of configuration

We set up notation here for our objects of interest.

2.1.1 The spaces of flags and affine flags Let k be a field and $V = k^3$. A flag in V is usually seen as a line and a plane, the line belonging to the plane. We give, for commodity reasons, the following alternative description using the dual vector space V^* and the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$:

We define the spaces of *affine flags* $\mathcal{AF}l(k)$ and *flags* $\mathcal{F}l(k)$ by

(2.1.1) $\qquad \mathcal{AF}l(k) = \{(x, f) \in (V \setminus \{0\}) \times (V^* \setminus \{0\}) \text{ such that } f(x) = 0\},$

(2.1.2) $\qquad \mathcal{F}l(k) = \{([x], [f]) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \text{ such that } f(x) = 0\}.$

The space of flags $\mathcal{F}l(k)$ is identified with the homogeneous space PGL$(3, k)/B$, where B is the Borel subgroup of upper-triangular matrices in PGL$(3, k)$. Similarly, the space of affine flags $\mathcal{AF}l(k)$ is identified with the homogeneous space SL$(3, k)/U$, where U is the subgroup of unipotent upper-triangular matrices in SL$(3, k)$. When k is fixed, we simply denote $\mathcal{F}l$ and $\mathcal{AF}l$ the spaces of flags or affine flags.

2.1.2 Configuration modules Given a G–space X, we classically define the configuration module of ordered points in X as follows. For $n \geq 0$, let $C_n(X)$ be the free abelian group generated by the set

$$(p_0, \ldots, p_n) \in X^{n+1}$$

of all ordered $(n + 1)$ set of points in X. The group G acts on X and therefore also acts diagonally on $C_n(X)$ giving it a left G–module structure.

We define the differential $d_n \colon C_n(X) \to C_{n-1}(X)$ by

$$d_n(p_0, \ldots, p_n) = \sum_{i=0}^{n} (-1)^i (p_0, \ldots, \widehat{p}_i, \ldots, p_n).$$

Then we can check that every d_n is a G–module homomorphism and $d_n \circ d_{n+1} = 0$. Hence we have the G–complex

$$C_\bullet(X) \colon \cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X).$$

The augmentation map $\epsilon\colon C_0(X) \to \mathbb{Z}$ is defined on generators by $\epsilon(p) = 1$ for each $p \in X$. If X is infinite, the augmentation complex is exact.

For a left G–module M, we denote by M_G its group of coinvariants, that is,

$$M_G = M/\langle gm - m \text{ for all } g \in G, m \in M \rangle.$$

Taking the coinvariants of the complex $C_\bullet(X)$, we get the induced complex

$$C_\bullet(X)_G\colon \cdots \to C_n(X)_G \to C_{n-1}(X)_G \to \cdots \to C_0(X)_G,$$

with differential $\bar{d}_n\colon C_n(X)_G \to C_{n-1}(X)_G$ induced by d_n. We call $H_\bullet(X)$ the homology of this complex.

We now let $G = \mathrm{PGL}(3,k)$ and $X = \mathcal{F}l$. For every integer $n \geq 0$, the $\mathbb{Z}$–module of *coinvariant configurations of $n+1$ ordered flags* is defined by

$$\mathcal{C}_\bullet(\mathcal{F}l) = C_\bullet(\mathcal{F}l)_G.$$

The natural projection $\pi\colon \mathrm{SL}(3,k) \to \mathrm{PGL}(3,k) \to \mathrm{PGL}(3,k)/B = \mathcal{F}l$ gives a map

$$\pi_*\colon H_3(\mathrm{SL}(3,k)) \to H_3(\mathcal{F}l).$$

We will study in this paper the homology groups $H_3(\mathrm{SL}(3,k))$ (which is the third group of discrete homology of $\mathrm{SL}(3,k)$), $H_3(\mathcal{A}\mathcal{F}l)$ and $H_3(\mathcal{F}l)$.

It is useful to consider a subcomplex of $\mathcal{C}_\bullet(\mathcal{F}l)$ of generic configurations. We leave to the reader the verification that indeed the definition below gives rise to subcomplexes of $C_3(\mathcal{F}l)$ and $\mathcal{C}_3(\mathcal{F}l)$.

Definition 2.1.1 A *generic configuration of flags* $([x_i], [f_i])$, $1 \leq i \leq n+1$, is given by $n+1$ points $[x_i]$ in general position and $n+1$ lines Ker f_i in $\mathbb{P}(V)$ such that $f_j(x_i) \neq 0$ if $i \neq j$. We will denote $C_n^r(\mathcal{F}l) \subset C_n(\mathcal{F}l)$ and $\mathcal{C}_n^r(\mathcal{F}l) \subset \mathcal{C}_n(\mathcal{F}l)$ the corresponding module of configurations and its coinvariant module by the diagonal action by $\mathrm{SL}(3,k)$.

A configuration of ordered points in $\mathbb{P}(V)$ is said to be in *general position* when they are all distinct and no three points are contained in the same line. Observe that the genericity condition of flags does not imply that the lines are in a general position.

2.2 Coordinates for a triangle of flags

Since G acts transitively on $C_1^r(\mathcal{F}l)$, we see that $C_n^r(\mathcal{F}l)_G = \mathbb{Z}$ if $n \leq 1$, and the differential $\bar{d}_1\colon C_1^r(\mathcal{F}l)_G \to C_0^r(\mathcal{F}l)_G$ is zero.

In order to describe $C_2^r(\mathcal{F}l)$, consider a configuration of 3 generic flags

$$([x_i], [f_i])_{1 \leq i \leq 3} \in C_2^r(\mathcal{F}l),$$

called a *triangle* of flags. One can then define a projective coordinate system of $\mathbb{P}(\mathbb{C}^3)$: take the one where the point x_1 has coordinates $[1:0:0]^t$, the point x_2 has coordinates $[0:0:1]^t$, the point x_3 has coordinates $[1:-1:1]^t$ and the intersection of $\mathrm{Ker}(f_1)$ and $\mathrm{Ker}(f_2)$ has coordinates $[0:1:0]^t$. The line $\mathrm{Ker}(f_3)$ then has coordinates $[z:z+1:1]$, where

$$z = \frac{f_1(x_2)f_2(x_3)f_3(x_1)}{f_1(x_3)f_2(x_1)f_3(x_2)} \in k^\times$$

is the *triple ratio*. We have $C_2^r(\mathcal{F}l) = \mathbb{Z}[k^\times]$. Moreover the differential $\bar{d}_2 \colon C_2^r(\mathcal{F}l)_G \to C_1^r(\mathcal{F}l)_G$ is given on generators $z \in k^\times$ by $\bar{d}_2(z) = 1$ and therefore $H_1(\mathcal{F}l) = 0$.

We denote by z_{123} the triple ratio of a cyclically oriented triple of flags $([x_i], [f_i])_{i=1,2,3}$. Note that $z_{213} = 1/z_{123}$.

2.3 Coordinates for a tetrahedron of flags

We call a generic configuration of 4 flags a *tetrahedron of flags*. The coordinates we use for a tetrahedron of flags are the same as those used by Fock and Goncharov [9] to describe a flip in a triangulation. We may see it as a blow-up of the flip into a tetrahedron. They also coincide with coordinates used in [6] to describe a *cross-ratio structure* on a tetrahedron (see also Section 3.4).

Let $([x_i], [f_i])_{1 \leq i \leq 4}$ be an element of $C_3(\mathcal{F}l)$. Let us dispose symbolically these flags on a tetrahedron 1234 (see Figure 1). We define a set of 12 coordinates on the edges of the tetrahedron (1 for each oriented edge) and a set of 4 coordinates associated to the faces.

2.3.1 Edge coordinates To define the coordinate z_{ij} associated to the edge ij, we first define k and l such that the permutation $(1, 2, 3, 4) \mapsto (i, j, k, l)$ is even. The pencil of (projective) lines through the point x_i is a projective line $\mathbb{P}_1(k)$. We naturally have four points in this projective line: the line $\mathrm{ker}(f_i)$ and the three lines through x_i and one of the $x_{i'}$ for $i' \neq i$. We define z_{ij} as the cross-ratio[1] of these four points,

(2.3.1) $z_{ij} := [\mathrm{ker}(f_i), (x_ix_j), (x_ix_k), (x_ix_l)].$

We may rewrite this cross-ratio thanks to the following useful lemma.

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[1]Note that we follow the convention (different from the one used by Fock and Goncharov) that the cross-ratio of four points x_1, x_2, x_3, x_4 on a line is the value at x_4 of a projective coordinate taking value ∞ at x_1, 0 at x_2, and 1 at x_3. So we employ the formula $(x_1 - x_3)(x_2 - x_4)/((x_1 - x_4)(x_2 - x_3))$ for the cross-ratio.
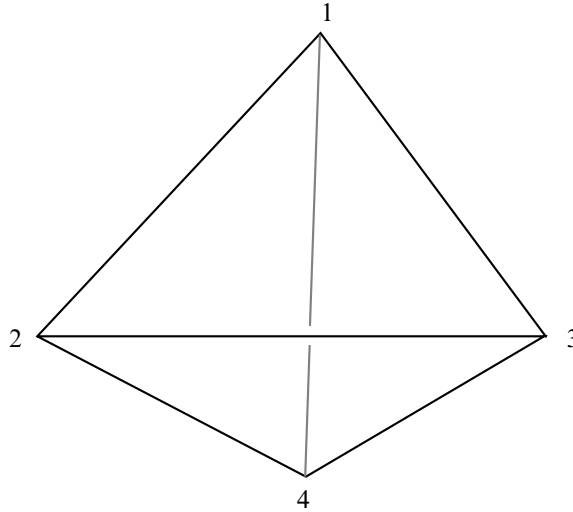
Figure 1: An ordered tetrahedron

Lemma 2.3.1 *We have*

$$z_{ij} = \frac{f_i(x_k) \det(x_i, x_j, x_l)}{f_i(x_l) \det(x_i, x_j, x_k)}.$$

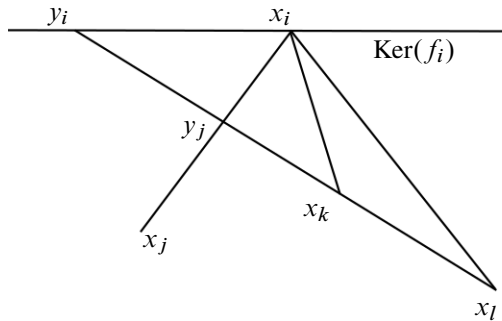*Here the determinant is taken with respect to the canonical basis on V.*

Figure 2: Cross-ratio

Proof Consider Figure 2. By duality, z_{ij} is the cross-ratio between the points y_i, y_j and x_k, x_l on the line $(x_k x_l)$. Now, f_i is a linear form vanishing at y_i and $\det(x_i, x_j, \cdot)$ is a linear form vanishing at y_j. Hence, on the line $(x_k x_l)$, the linear form $f_i(x)$ is proportional to $(\cdot - y_i)$ and $\det(x_i, x_j, \cdot)$ is proportional to $(\cdot - y_j)$. This proves the formula.   □

2.3.2 Face coordinates   Each face (ijk) inherits a canonical orientation as the boundary of the tetrahedron (1234). Hence, to the face (ijk), we associate the 3–ratio of the corresponding cyclically oriented triple of flags:

$$(2.3.2) \qquad z_{ijk} = \frac{f_i(x_j) f_j(x_k) f_k(x_i)}{f_i(x_k) f_j(x_i) f_k(x_j)} \, .$$

Observe that if the same face (ikj) (with opposite orientation) is common to a second tetrahedron then

$$z_{ikj} = \frac{1}{z_{ijk}} \, .$$
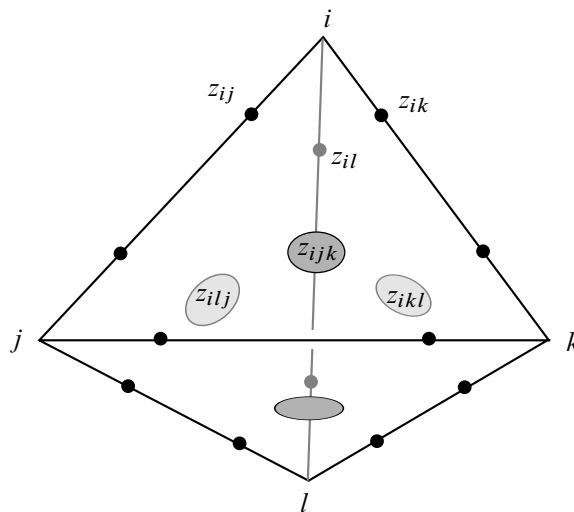
Figure 3 displays the coordinates.

Figure 3: The z–coordinates for a tetrahedron

2.3.3 Relations between coordinates   There are relations between the whole set of coordinates. Fix an even permutation (i, j, k, l) of $(1, 2, 3, 4)$. First, for each face (ijk), the 3–ratio is the opposite of the product of all cross-ratios "leaving" this face:

$$(2.3.3) \qquad z_{ijk} = -z_{il} z_{jl} z_{kl} \, .$$

Second, the three cross-ratios leaving a vertex are algebraically related by

$$(2.3.4) \qquad z_{ik} = \frac{1}{1-z_{ij}} \, , \qquad z_{il} = 1 - \frac{1}{z_{ij}} \, .$$

Relations (2.3.4) are directly deduced from the definition of the coordinates z_{ij}, while relation (2.3.3) is a consequence of Lemma 2.3.1.

2.4 A choice of parameters

At this point, we choose four coordinates, one for each vertex: z_{12}, z_{21}, z_{34}, z_{43}. The next proposition shows that a tetrahedron is uniquely determined by these four numbers, up to the action of $\mathrm{PGL}(3, k)$. It also shows that the space of *cross-ratio structures* on a tetrahedron of flags defined in [6] coincides with the space of generic tetrahedra as defined above.

Proposition 2.4.1 *A tetrahedron of flags is parametrized by the 4–tuple* $(z_{12}, z_{21}, z_{34}, z_{43})$ *of elements in* $k \setminus \{0, 1\}$. *In other terms, we have*

$$C_3^r(\mathcal{F}l) \simeq \mathbb{Z}\big[(k \setminus \{0, 1\})^4\big].$$

Proof Let e_1, e_2, e_3 be the canonical basis of V and (e_1^*, e_2^*, e_3^*) its dual basis. Up to the action of $\mathrm{PGL}(3, k)$, an element $([x_i], [f_i])$ of $C_3^r(\mathcal{F}l)$ is uniquely given, by a slight abuse of notation, as:

- $x_1 = (1, 0, 0)$, $f_1 = (0, z_1, -1)$
- $x_2 = (0, 1, 0)$, $f_2 = (z_2, 0, -1)$
- $x_3 = (0, 0, 1)$, $f_3 = (z_3, -1, 0)$
- $x_4 = (1, 1, 1)$, $f_4 = z_4(1, -1, 0) + (0, 1, -1)$

Observe that $z_i \neq 0$ and $z_i \neq 1$ by the genericity condition. Now we compute, using Lemma 2.3.1 for instance, that $z_{12} = \frac{1}{1-z_1}$, $z_{21} = 1 - z_2$, $z_{34} = z_3$, $z_{43} = 1 - z_4^{-1}$, completing the proof. □

We note that one can then compute $\bar{d}_3 \colon C_3^r(\mathcal{F}l)_G \to C_2^r(\mathcal{F}l)_G$ on the generators of $C_3^r(\mathcal{F}l)_G$ to be

$$\bar{d}_3(z_{12}, z_{21}, z_{34}, z_{43}) = [z_{123}] - [z_{124}] + [z_{134}] - [z_{234}].$$

2.5 Coordinates for affine flags

We will also need coordinates for a tetrahedron of affine flags (the a–coordinates in Fock and Goncharov [9]). Let $(x_i, f_i)_{1 \leq i \leq 4}$ be an element of $C_3(\mathcal{A}\mathcal{F}l)$. We also define a set of 12 coordinates on the edges of the tetrahedron (one for each oriented edge) and four coordinates associated to the faces.

We associate to the edge ij the number $a_{ij} = f_i(x_j)$ and to the face ijk (oriented as the boundary of the tetrahedron) the number $a_{ijk} = \det(x_i, x_j, x_k)$.

There are some relations between them, but they will not be of interest for us.

2.6 From affine flags to flags

By definition there is a natural projection $\mathcal{A}\mathcal{F}l \to \mathcal{F}l$: it consists in projectivizing the flags. It extends to a map $C_3^r(\mathcal{A}\mathcal{F}l) \to C_3^r(\mathcal{F}l)$. In other terms, when you give coordinates (a_{ij}, a_{ijk}) for a tetrahedron of affine flags, you also have a tetrahedron of flags and hence coordinates (z_{ij}, z_{ijk}).

We remark that the z–coordinates are ratios (up to a sign) of the affine coordinates:

$$(2.6.1) \qquad z_{ij} = -\frac{a_{ik}a_{ilj}}{a_{il}a_{ijk}} \quad \text{and} \quad z_{ijk} = \frac{a_{ij}a_{jk}a_{ki}}{a_{ik}a_{ji}a_{kj}}.$$

3 Tetrahedra of flags and volume

In this section we define the *volume* of a tetrahedron of flags, generalizing and unifying the volume of hyperbolic tetrahedra (see Section 3.3) and CR tetrahedra (see [6] and Section 3.4). Via Proposition 2.4.1, it coincides with the volume function on cross-ratio structures on a tetrahedron as defined in [6].

3.1 The pre-Bloch group, the dilogarithm and the volume

We define a volume for a tetrahedron of flags by constructing an element of the pre-Bloch group and then taking the dilogarithm map.

The *pre-Bloch group* $\mathcal{P}(k)$ is the quotient of the free abelian group $\mathbb{Z}[k \setminus \{0, 1\}]$ by the subgroup generated by the 5–term relations

$$(3.1.1) \qquad [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \quad \text{for all } x, y \in k \setminus \{0, 1\}.$$

For a tetrahedron of flags T, let $z_{ij} = z_{ij}(T)$ and $z_{ijk} = z_{ijk}(T)$ be its coordinates.

For each tetrahedron of flags, define the element

$$\beta(T) = [z_{12}] + [z_{21}] + [z_{34}] + [z_{43}] \in \mathcal{P}(\mathbb{C})$$

and extend it – by linearity – to a function

$$(3.1.2) \qquad \beta\colon C_3^r(\mathcal{F}l) \to \mathcal{P}(\mathbb{C}).$$

We emphasize here that $\beta(T)$ depends on the ordering of the vertices of the tetrahedron of flags[2] T. The following proposition implies that β is well defined on $H_3(\mathcal{F}l)$.

---

[2]This assumption may be removed by averaging β over all orderings of the vertices. In any case if c is a chain in $C_3(\mathcal{F}l)$ representing a cycle in $\mathcal{C}_3(\mathcal{F}l)$ we can represent c by a closed 3–cycle K together with a numbering of the vertices of each tetrahedron of K (see Section 5.2.2).

Proposition 3.1.1 *The map β vanishes on the boundary space $\overline{d}_4(\mathcal{C}_4^r(\mathcal{F}l))$.*

Proof We have to show that $\mathrm{Im}(\overline{d}_4)$ is contained in the subgroup generated by the 5–term relations. This is proven by computation and is exactly the content of [6, Theorem 5.2].[3] □

3.2 Dilogarithm and volume

We assume in this subsection that k is a subfield of $\mathbb{C}$. The *Bloch–Wigner dilogarithm* function is

$$D(x) = \arg(1-x)\log|x| - \mathrm{Im}\left(\int_0^x \log(1-t)\frac{dt}{t}\right),$$

$$= \arg(1-x)\log|x| + \mathrm{Im}(\ln_2(x)).$$

Here $\ln_2(x) = \int_0^x \log(1-t)\frac{dt}{t}$ is the dilogarithm function. The function D is well-defined and real analytic on $\mathbb{C} - \{0,1\}$ and extends to a continuous function on $\mathbb{P}_1(\mathbb{C})$ by defining $D(0) = D(1) = D(\infty) = 0$. It satisfies the 5–term relation and therefore, for k a subfield of $\mathbb{C}$, gives rise to a well-defined map

$$D: \mathcal{P}(k) \to \mathbb{R}$$

given by linear extension as

$$D\left(\sum_{i=1}^k n_i[x_i]\right) = \sum_{i=1}^k n_i D(x_i).$$

It is known that the function D is related to the volume of hyperbolic ideal tetrahedra; see Section 3.3.

We finally define the *volume map* on $\mathcal{C}_3^r(\mathcal{F}l)$ via the dilogarithm (the constant will be explained in the next section):

Definition 3.2.1 When k is a subfield of $\mathbb{C}$, the volume map $\mathrm{Vol}: \mathcal{C}_3^r(\mathcal{F}l) \to \mathbb{C}$ is defined by

$$\mathrm{Vol} = \tfrac{1}{4}D \circ \beta.$$

From Proposition 3.1.1, Vol is well defined on $H_3(\mathcal{F}l)$.

From the previous definition, we see how closely is β related to Vol. We will occasionally abuse notation and call β the volume map.

---

[3]Another point of view is given by Dimofte, Gabella and Goncharov [4].

3.3 The hyperbolic case

We briefly explain here how the hyperbolic volume for ideal tetrahedra in the hyperbolic space $\mathbb{H}^3$ fits into the framework described above.

An ideal hyperbolic tetrahedron is given by 4 points on the boundary of $\mathbb{H}^3$, ie $\mathbb{P}_1(\mathbb{C})$. Up to the action of SL$(2, \mathbb{C})$, these points are in homogeneous coordinates $[0, 1]$, $[1, 0]$, $[1, 1]$ and $[1, t]$; the complex number t being the cross-ratio of these four points. Its volume is then $D(t)$ (see eg Zagier [22]).

Here we present how $\mathbb{P}_1(\mathbb{C})$ naturally embeds into $\mathcal{F}l$ in such a way that our map Vol coincide with the hyperbolic volume. For that purpose, let us identify $\mathbb{C}^3$ with the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We then have the adjoint action of SL$(2, \mathbb{C})$ on $\mathbb{C}^3$ preserving the quadratic form given by the determinant on $\mathfrak{sl}(2, \mathbb{C})$. In usual coordinates, it is given by $xz - y^2$. The group SL$(2, \mathbb{C})$ preserves the isotropic cone of this form. The projectivization of this cone is identified to $\mathbb{P}_1(\mathbb{C})$ via the Veronese map (in canonical coordinates)

$$h_1 \colon \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_2(\mathbb{C}), \quad [x, y] \mapsto [x^2, xy, y^2].$$

The first jet of that map gives a map h from $\mathbb{P}_1(\mathbb{C})$ to the variety of flags $\mathcal{F}l$. A convenient description of that map is obtained thanks to the identification between $\mathbb{C}^3$ and its dual given by the quadratic form. Denote by $\langle \cdot , \cdot \rangle$ the bilinear form associated to the determinant. Then we have

$$h \colon \mathbb{P}_1(\mathbb{C}) \to \mathcal{F}l, \quad p \mapsto (h_1(p), \langle h_1(p), \cdot \rangle).$$

Let T be the tetrahedron of flags $h([0, 1])$, $h([1, 0])$, $h([1, 1])$ and $h([1, t])$. An easy computation gives its coordinates:

$$z_{12}(T) = t, \quad z_{21}(T) = t, \quad z_{34}(T) = t, \quad z_{43}(T) = t.$$

It implies that $\beta(T) = 4[t]$ and that Vol coincides with the hyperbolic volume:

$$\mathrm{Vol}(T) = D(t).$$

Remark Define an involution σ on the z–coordinates by

$$\sigma(z_{ijk}) = \frac{1}{z_{ijk}}$$

on the faces and

$$\sigma(z_{ij}) = \frac{z_{ji}(1 + z_{ilj})}{z_{ilj}(1 + z_{ijk})} \quad \text{and} \quad \sigma(z_{ji}) = \frac{z_{ij}(1 + z_{ijk})}{z_{ijk}(1 + z_{ilj})}$$

on edges. The set of fixed points of σ correspond exactly with the hyperbolic tetrahedra.

3.4 The CR case

CR geometry is modeled on the sphere $\mathbb{S}^3$ equipped with a natural $\mathrm{PU}(2, 1)$ action (see Jacobowitz [14] for an introduction). More precisely, consider the group $\mathrm{U}(2, 1)$ preserving the Hermitian form defined on $\mathbb{C}^3$ by

$$\langle z, w \rangle = w^* \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} z.$$

Define the following cones in $\mathbb{C}^3$:

$$V_0 = \{z \in \mathbb{C}^3 - \{0\} : \langle z, z \rangle = 0\},$$
$$V_- = \{z \in \mathbb{C}^3 : \langle z, z \rangle < 0\}.$$

Let $\pi \colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$ be the canonical projection. Then $\mathbb{H}^2_{\mathbb{C}} = \pi(V_-)$ is the complex hyperbolic space and its boundary is

$$\partial \mathbb{H}^2_{\mathbb{C}} = \mathbb{S}^3 = \pi(V_0) = \{[x, y, z] \in \mathbb{CP}^2 \mid x\bar{z} + |y|^2 + z\bar{x} = 0\}.$$

The group of biholomorphic transformations of $\mathbb{H}^2_{\mathbb{C}}$ is then $\mathrm{PU}(2, 1)$, the projectivization of $\mathrm{U}(2, 1)$. It acts on $\mathbb{S}^3$ by CR transformations.

An element $x \in \mathbb{S}^3$ gives rise to an element $([x], [f]) \in \mathcal{F}l(\mathbb{C})$ where $[f]$ corresponds to the unique complex line tangent to $\mathbb{S}^3$ at x. As in the hyperbolic case we may consider the inclusion map

$$h_1 \colon \mathbb{S}^3 \to \mathbb{P}_2(\mathbb{C})$$

and the first complex jet of that map gives a map

$$h \colon \mathbb{S}^3 \to \mathcal{F}l(\mathbb{C}),$$
$$x \mapsto (h_1(x), \langle \cdot, h_1(x) \rangle).$$

As in the case of hyperbolic geometry, one can characterize CR tetrahedra (that is, four flags that are the image of four points in $\mathbb{S}^3$ by the map h) by conditions on the coordinates z_{ij}. Moreover one can obtain them as the fixed point set of an involution on the space of flag tetrahedra. We will not make explicit these conditions as they will not be used in this paper and refer the reader to Falbel [5].

As a final remark, the definition given in [6] (up to multiplication by 4) of the volume of a CR tetrahedron T_{CR} agrees with our definition, that is, $\mathrm{Vol}(T_{CR}) = \frac{1}{4} D \circ \beta(T_{CR})$.

3.5 Relations with the work of Suslin on the homology of GL(n)

We show here how our map β allows a new and more geometric way to interpret the Suslin map $S\colon H_3(\mathrm{SL}(3,k)) \to \mathcal{P}(k)$ (see [20]). First of all, recall that the natural projection $\pi\colon \mathrm{SL}(3,k) \to \mathcal{F}l = \mathrm{PGL}(3,k)/B$ gives a map $\pi_*\colon H_3(\mathrm{SL}(3,k)) \to H_3(\mathcal{F}l)$.

Theorem 3.5.1 *The Suslin map* $S\colon H_3(\mathrm{SL}(3,k)) \to \mathcal{P}(k)$ *may be interpreted in term of the volume map* β. *Indeed, we have*

$$\beta \circ \pi_* = 4S.$$

Proof Let T be the subgroup of diagonal matrices (in the canonical basis) of $\mathrm{SL}(3,k)$. Recall that $\mathrm{SL}(2,k)$ is seen as a subgroup of $\mathrm{SL}(3,k)$ via the adjoint representation (as in Section 3.3). We find in the work of Suslin the following three results:

(1) $H_3(\mathrm{SL}(3,k)) = H_3(\mathrm{SL}(2,k)) + H_3(T)$ [20, page 227]

(2) S vanishes on $H_3(T)$ [20, page 227]

(3) S coincides with the cross-ratio on $H_3(\mathrm{SL}(2,k))$ [20, Lemma 3.4]

So we just have to understand the map $\beta \circ \pi_*$ on T and $\mathrm{SL}(2,k)$. As T is a subgroup of B, the map $\beta \circ \pi_*$ vanishes on T. And we have seen in Section 3.3 that, on a hyperbolic tetrahedron, β coincide with 4 times the cross-ratio.

This proves the theorem. $\square$

Remark After writing this section we became aware of Zickert's paper [23]. In it (see Section 7.1) Zickert defines a generalization – denoted $\hat{\lambda}$ – of Suslin's map. When specialized to our case his definition coincides with $\frac{1}{4}\beta \circ \pi_*$. We believe that the construction above sheds some light on the "naturality" of this map.

3.6 The Bloch group

We recall in this section the definition of the Bloch group $\mathcal{B}(k) \subset \mathcal{P}(k)$ as the kernel of a map δ and relate the derivative of the dilogarithm to δ.

We use wedge $\wedge_{\mathbb{Z}}$ for skew symmetric product on Abelian groups. Consider $k^{\times} \wedge_{\mathbb{Z}} k^{\times}$, where $k^{\times}$ is the multiplicative group of k. Recall that it is the abelian group generated by the set $x \wedge_{\mathbb{Z}} y$ factored by the relations

$$xy \wedge_{\mathbb{Z}} z = x \wedge_{\mathbb{Z}} z + y \wedge_{\mathbb{Z}} z \quad \text{and} \quad x \wedge_{\mathbb{Z}} y = -y \wedge_{\mathbb{Z}} x.$$

In particular, $1 \wedge_{\mathbb{Z}} x = 0$ for any $x \in k^{\times}$, and

$$x^n \wedge_{\mathbb{Z}} y = n(x \wedge_{\mathbb{Z}} y) = x \wedge_{\mathbb{Z}} y^n.$$

The *Bloch group* $\mathcal{B}(k)$ is defined as the kernel of the homomorphism

(3.6.1) $$\delta \colon \mathcal{P}(k) \to k^{\times} \wedge_{\mathbb{Z}} k^{\times},$$

which is defined on generators of $\mathcal{P}(k)$ by $\delta([z]) = z \wedge_{\mathbb{Z}} (1-z)$.

Computing $\delta(\beta)$ we will give another proof that the Suslin map $S = \beta \circ \pi_*$ takes its values in the Bloch group $\mathcal{B}(k)$; see Section 6.

Let us now compute the derivative of the Bloch–Wigner dilogarithm $D(z)$ using this map δ. Assume once again that $k \subset \mathbb{C}$. Then the derivatives of $D(z)$ are elementary functions:

(3.6.2) $$\frac{\partial D}{\partial z} = \frac{i}{2}\left(\frac{\log|1-z|}{z} + \frac{\log|z|}{1-z}\right), \quad \frac{\partial D}{\partial \bar{z}} = -\frac{i}{2}\left(\frac{\log|1-z|}{\bar{z}} + \frac{\log|z|}{1-\bar{z}}\right).$$

Assume that the parameter $z \in \mathbb{C}^*$ is varying in dependence on a single variable t. Then

$$\frac{d}{dt}D(z_t) = \frac{i}{2}\left[\left(\frac{\log|1-z|}{z} + \frac{\log|z|}{1-z}\right)\frac{dz}{dt} - \left(\frac{\log|1-z|}{\bar{z}} + \frac{\log|z|}{1-\bar{z}}\right)\frac{d\bar{z}}{dt}\right]$$
$$= \mathrm{Im}\left(\left(\frac{d}{dt}\log(z)\right)\log|1-z| - \left(\frac{d}{dt}\log(1-z)\right)\log|z|\right).$$

Here is how to interpret this computation using δ. Consider $\mathcal{F}(k^{\times})$ the space of algebraic functions on $k^{\times}$ and $\Omega^1(k^{\times})$ the space of 1–forms. Consider the map

$$\mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log) \colon \mathcal{F}(k^{\times}) \wedge_{\mathbb{Z}} \mathcal{F}(k^{\times}) \to \Omega^1(k^{\times})$$

defined by

$$\mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log)(f \wedge_{\mathbb{Z}} g) = \mathrm{Im}(\log|g| \cdot d(\log f) - \log|f| \cdot d(\log g)).$$

Then we have:

Proposition 3.6.1 *For the Bloch–Wigner dilogarithm function D (Section 3.2), the 1–form dD is the composition of $\mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log)$ and of the function δ (3.6.1), namely*

$$dD = \mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log)(\delta).$$

This proposition will in part motivate the study of the map $\delta(\beta)$: it represents the variation of the volume of a tetrahedron of flags when the flags are varying. This is also the point of Section 11.1.

4 Decoration of a tetrahedron and the pre-Bloch group

In this section we let T be an ordered tetrahedron of flags and compute the quantity $\delta(\beta(T)) \in \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$ we have defined in the previous section.

The first computation in this section uses a–coordinates associated to some lift of T as a tetrahedron of affine flags. In that respect we mainly follow Fock and Goncharov. The second way directly deals with z–coordinates and follows the approach of Neumann and Zagier. We will see in the remaining of the paper how fruitful it is to mix those two approaches.

All over this section, we denote by T a tetrahedron of flags $([x_i], [f_i])$ and by T_a a tetrahedron of affine flags (x_i, f_i) lifting T. Associated to T is a set of z–coordinates z_{ij} and z_{ijk} and associated to T_a is a set of a–coordinates a_{ij} and a_{ijk} (defined in Section 2).

4.1 A first formula via affine flags

4.1.1 The $\mathbb{Z}$–module J_T^2 associated to a tetrahedron

All our computations will go through considerations of $\mathbb{Z}$–modules equipped with a 2–form. We first define such a $\mathbb{Z}$–module, which will be very important to us, and then explain some constructions and results about $\mathbb{Z}$–modules. Let $J_T^2 = \mathbb{Z}^I$ be the 16–dimensional abstract free $\mathbb{Z}$–module where (see Figure 4)

$$I = \{\text{vertices of the (red) arrows in the 2–triangulation of the faces of } T\}.$$

We write the canonical basis $\{e_\alpha\}_{\alpha \in I}$ of J_T^2. It contains *oriented edges* e_{ij} (edges oriented from j to i) and *faces* e_{ijk}. Given α and β in I we set

$$\varepsilon_{\alpha\beta} = \#\{\text{oriented (red) arrows from } \alpha \text{ to } \beta\} - \#\{\text{oriented (red) arrows from } \beta \text{ to } \alpha\}.$$
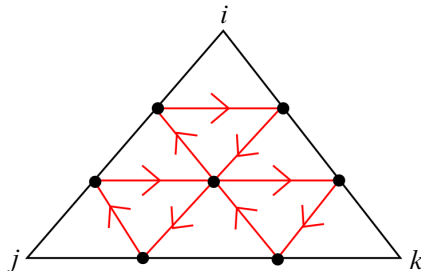

Figure 4: Combinatorics of W

This allows to define the bilinear skew-symmetric form Ω^2 on J_T^2 by[4]

(4.1.1) $$\Omega^2(e_\alpha, e_\beta) = \varepsilon_{\alpha\beta}.$$

Remark that we have one a–coordinate a_α for each $\alpha \in I$. So the set of a–coordinates $(a_\alpha)_{\alpha \in I}$ associated to our tetrahedron of affine flags T_a naturally defines an element

$$\sum_{\alpha \in I} a_\alpha e_\alpha \in k^\times \otimes_{\mathbb{Z}} J_T^2,$$

where k is any field that contains all the a–coordinates. At this point, we will review some notions and results on $\mathbb{Z}$–modules with such a 2–form.

4.1.2 $\mathbb{Z}$–modules and 2–forms Let V be a $\mathbb{Z}$–module equipped with a bilinear product

$$B\colon V \times V \to \mathbb{Z}.$$

Then its dual $\mathbb{Z}$–module V^* is the $\mathbb{Z}$–module $V^* = \mathrm{Hom}(V, \mathbb{Z})$. If V is a finitely generated free module, by the classical definition of the tensor product, we get $\mathbb{Z}$–modules

$$k^\times \otimes_{\mathbb{Z}} V \cong \mathrm{Hom}(V^*, k^\times) \quad \text{and} \quad k^\times \otimes_{\mathbb{Z}} V^* \cong \mathrm{Hom}(V, k^\times).$$

This going back and forth with dual modules when considering tensoring will be repeatedly used throughout this paper and should be kept in mind.

Now, we consider the bilinear product B. We define on the $\mathbb{Z}$–module $k^\times \otimes_{\mathbb{Z}} V$ the bilinear product

$$\wedge_B\colon (k^\times \otimes_{\mathbb{Z}} V) \times (k^\times \otimes_{\mathbb{Z}} V) \to k^\times \wedge_{\mathbb{Z}} k^\times$$

defined on generators by

$$(z_1 \otimes v_1) \wedge_B (z_2 \otimes v_2) = B(v_1, v_2)(z_1 \wedge_{\mathbb{Z}} z_2).$$

A key feature of this definition is that it is natural: it is preserved by mappings preserving the bilinear products. Indeed, as a direct consequence of the definitions, we have:

Lemma 4.1.1 *If $\phi\colon V \to W$ is a homomorphism of $\mathbb{Z}$–modules with bilinear forms B and b such that $\phi^*(b) = B$ then the induced map $\phi\colon k^\times \otimes_{\mathbb{Z}} V \to k^\times \otimes_{\mathbb{Z}} W$ satisfies*

$$\phi^*(\wedge_b) = \wedge_B.$$

Or, in other terms, for any element $x \in k^\times \otimes_{\mathbb{Z}} V$, we have $x \wedge_B x = \phi(x) \wedge_b \phi(x)$.

---

[4]Observe in particular that $\Omega^2(e_{ji}, e_{ijk}) = 1$ and so on, the logic being that the vector e_{ijk} is the outgoing vector on the face ijk and the vector e_{ji} (oriented from i to j) turns around it in the positive sense.

We conclude this abstract subsection with another piece of notation: we will need in numerous points to tensor our $\mathbb{Z}$–modules by $\mathbb{Z}[\frac{1}{2}]$. We prefer to define a notation for it:

Notation 4.1.1 For a $\mathbb{Z}$–module V, we define $V[\frac{1}{2}] := V \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$.

4.1.3 A first formula Recall that we associated to the tetrahedron of affine flags T_a a set of coordinates $(a_\alpha)_{\alpha \in I}$ and then in turn an element

$$a := \sum_{\alpha \in I} a_\alpha e_\alpha \in k^\times \otimes_{\mathbb{Z}} J_T^2 \,.$$

As a is an element of $k^\times \otimes_{\mathbb{Z}} J_T^2$, we may apply the construction of the product $\wedge_{\Omega^2}$ to get

$$a \wedge_{\Omega^2} a = \sum_{\alpha, \beta \in I} \varepsilon_{\alpha\beta} a_\alpha \wedge_{\mathbb{Z}} a_\beta \in k^\times \wedge_{\mathbb{Z}} k^\times.$$

The first step to link $\delta(\beta(T))$ with bilinear products is done in the following lemma:

Lemma 4.1.2 *Let T be a tetrahedron of flags, T_a a tetrahedron of affine flags lifting T and $a \in k^\times \otimes_{\mathbb{Z}} J_T^2$ the element associated to T_a. Then the element $\delta(\beta(T)) \in k^\times \wedge_{\mathbb{Z}} k^\times$ is computed in terms of a as*

(4.1.2) $$\delta(\beta(T)) = \tfrac{1}{2} a \wedge_{\Omega^2} a.$$

Proof To each ordered face (ijk) of T we associate the element

(4.1.3) $$W_{ijk} = a_{ijk} \wedge \frac{a_{ki} a_{jk} a_{ij}}{a_{ik} a_{kj} a_{ji}} + a_{ij} \wedge a_{ik} + a_{ki} \wedge a_{kj} + a_{jk} \wedge a_{ji} \in k^\times \wedge_{\mathbb{Z}} k^\times.$$

The proof in the CR case of [8, Lemma 4.9] leads to[5]

$$\delta(\beta(T)) = W_{143} + W_{234} + W_{132} + W_{124}.$$

Finally one checks that

$$W_{143} + W_{234} + W_{132} + W_{124} = \frac{1}{2} \sum_{\alpha, \beta \in I} \varepsilon_{\alpha\beta} a_\alpha \wedge_{\mathbb{Z}} a_\beta. \qquad \square$$

We let $W(T) = W_{143} + W_{234} + W_{132} + W_{124}$.

---

[5]Alternatively we may think of T as a geometric realization of a mutation between two triangulations of the quadrilateral (1324) and apply [9, Corollary 6.15].

Remarks   (1) The element $W(T)$ coincides with the W invariant associated by Fock and Goncharov to the triangulation by a tetrahedron of a sphere with 4 punctures (the orientation of the faces being induced by the orientation of the sphere).

(2) Whereas T – being a tetrahedron of flags – only depends on the flag coordinates, each W associated to the faces depends on the *affine* flag coordinates.

In the next paragraph we make the second remark more explicit by computing $\delta(\beta(T))$ using the z–coordinates.

4.2   The Neumann–Zagier symplectic space and a second formula

In this section, we focus directly on the tetrahedron of flags T and its associated z–coordinates $(z_\alpha)_{\alpha \in I}$. In order to get a formula for $\delta(\beta(T))$, we construct a $\mathbb{Z}$–module with a 2–form in which those coordinates live (after tensoring by $k^\times$). And then we relate this $\mathbb{Z}$–module to J_T^2. The 2–form we will construct is an extension of Neumann–Zagier symplectic form, already introduced by J Genzmer [13], in the space of z–coordinates associated to the edges of a tetrahedron. We reinterpret her definitions in our context of flag tetrahedra.

Recall that the z–coordinates are subject to the relations (2.3.3) and (2.3.4). The first one is $z_{ijk} = -z_{il}z_{jl}z_{kl}$ and the second one implies

(4.2.1) $$z_{ij}z_{ik}z_{il} = -1.$$

4.2.1   Finding a space for z–coordinates: The $\mathbb{Z}$–module J_T^*   Denote by $(e_\alpha^*)_{\alpha \in I}$ the basis of $(J_T^2)^*$ dual to $(e_\alpha)_{\alpha \in I}$. We associate to T the element

$$z := \sum_{\alpha \in I} z_\alpha e_\alpha^* \in k^\times \otimes_{\mathbb{Z}} (J_T^2)^*.$$

At this point, the $\mathbb{Z}$–module of interest seems to be $(J_T^2)^*$. But it is not clear which 2–form should be defined on it. Instead, we use the relations recalled above to find a submodule of $(J_T^2)^*$, equipped with a natural 2–form, which "contains" z.

Consider the map

$$p \colon J_T^2 \to (J_T^2)^*$$

defined by $p(v) = \Omega^2(\,\cdot\,, v)$. Its image $\mathrm{Im}(p)$ consists exactly of the forms in $(J_T^2)^*$ vanishing on the kernel of Ω^2. This kernel is the subspace generated by elements of the form

$$\sum_{\alpha \in I} b_\alpha e_\alpha$$

for all $\{b_\alpha\} \in \mathbb{Z}^I$ such that $\sum_{\alpha \in I} b_\alpha \varepsilon_{\alpha\beta} = 0$ for every $\beta \in I$. Equivalently it is the subspace generated by $e_{ij} + e_{ik} + e_{il}$ and $e_{ijk} - (e_{il} + e_{jl} + e_{kl})$. We will rather use as generators the elements

$$v_i = e_{ij} + e_{ik} + e_{il} \quad \text{and} \quad w_i = e_{ji} + e_{ki} + e_{li} + e_{ijk} + e_{ilj} + e_{ikl};$$
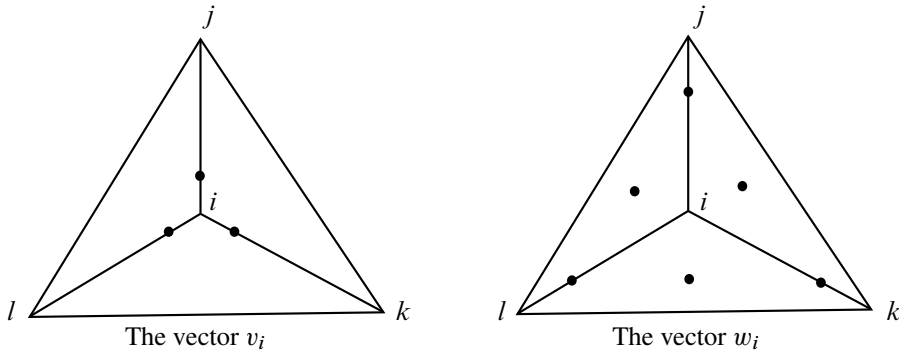
see Figure 5.

Figure 5: The vectors v_i and w_i in $\mathrm{Ker}(\Omega^2)$

We then define two $\mathbb{Z}$–modules $J_T^* = \mathrm{Im}(p) \subset (J_T^2)^*$ and its dual $J_T = J_T^2/\mathrm{Ker}(\Omega^2)$. Both are 8–dimensional. Remark that, by construction, the form Ω^2 induces a non-degenerate form, denoted Ω, on J_T. Moreover, on J_T^*, there is the natural dual 2–form, well-defined by

$$\Omega^*(p(v), p(v')) = \Omega^2(v, v') \quad \text{for } v, v' \in J_T^2$$

These constructions are justified by the following proposition: the element $z \in (J_T^2)^*$ almost belongs to $k^\times \otimes_\mathbb{Z} J_T^*$ (recall from Notation 4.1.1 that $J_T^*[\frac{1}{2}] = J_T^* \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{2}]$).

Proposition 4.2.1 *Let T be a tetrahedron of flags, and z its associated element in $k^\times \otimes_\mathbb{Z} (J_T^2)^*$. Then z belongs to $k^\times \otimes_\mathbb{Z} J_T^*[\frac{1}{2}]$.*

Proof Because of relations (2.3.3) and (4.2.1) the image of the kernel of Ω^2 by z is the (torsion) subgroup $\{\pm 1\} \subset k^\times$ (that is easily checked on v_i and w_i). We conclude that the element $z \in k^\times \otimes (J_T^2)^*[\frac{1}{2}]$ in fact belongs to $k^\times \otimes J_T^*[\frac{1}{2}]$. $\qquad\square$

4.2.2 Relation between a– and z–coordinates and the second formula On the basis (e_α) and its dual basis (e_α^*), we can write

$$p(e_\alpha) = \sum_\beta \varepsilon_{\alpha\beta} e_\beta^*.$$

We define accordingly the dual map

$$p^*\colon \operatorname{Hom}((J_T^2)^*, k^\times) \simeq k^\times \otimes_{\mathbb{Z}} J_T^2 \to k^\times \otimes_{\mathbb{Z}} (J_T^2)^* \simeq \operatorname{Hom}(J_T^2, k^\times).$$

Observe that if $a \in k^\times \otimes_{\mathbb{Z}} J_T^2$ and $z \in k^\times \otimes_{\mathbb{Z}} (J_T^2)^*$ are the elements associated to the a– and z–coordinates of T, then

$$p^*(a) = z \quad \text{in } k^\times \otimes J_T^*\left[\tfrac{1}{2}\right].$$

Indeed,

$$p^*(a)(e_\alpha) = a(p(e_\alpha)) = a\left(\sum_\beta \epsilon_{\alpha\beta} e_\beta^*\right) = \prod_\beta a_\beta^{\epsilon_{\alpha\beta}}.$$

In particular, we recover the formula

$$p^*(a)(e_{ij}) = \frac{a_{ik} a_{ilj}}{a_{il} a_{ijk}}.$$

So, according to (2.6.1), we have

$$p^*(a)(e_{ij}) = -z_{ij}.$$

There is therefore a sign missing here and $p^*(a) = z$ only holds modulo 2–torsion. This link is enough to get our second formula for $\delta(\beta(T))$, this time only in terms of z–coordinates:

Lemma 4.2.1 *Let T be a tetrahedron of flags and $z \in k^\times \otimes_{\mathbb{Z}} J_T^*\left[\tfrac{1}{2}\right]$ the element associated to T. Then the element $\delta(\beta(T)) \in k^\times \wedge_{\mathbb{Z}} k^\times$ is computed in terms of z as*

$$(4.2.2) \qquad \delta(\beta(T)) = \tfrac{1}{2} z \wedge_{\Omega^*} z.$$

Proof By definition of Ω^*, one has $p^*(\Omega^*) = \Omega^2$. It then follows from Lemma 4.1.1 that

$$(4.2.3) \qquad a \wedge_{\Omega^2} a = z \wedge_{\Omega^*} z.$$

One concludes using the formula $\delta(\beta(T)) = \tfrac{1}{2} a \wedge_{\Omega^2} a$ given in Lemma 4.1.2. □

4.3 Another point of view

One can present another point of view on this construction, which is easier to relate to the Neumann–Zagier bilinear form. The space J_T^* is 8–dimensional and we may associate to 8 oriented edges (two pointing at each vertex) of T a basis $\{f_{ij}\}$. Using this basis, the element $z \in k^\times \otimes_{\mathbb{Z}} J_T^*\left[\tfrac{1}{2}\right]$ is written $z = \sum z_{ij} f_{ij}$.

We then note that (up to eventually adding a root of -1 to k)

(4.3.1) $\quad \delta(\beta(T)) = z_{ij} \wedge_{\mathbb{Z}} (1 - z_{ij}) + z_{ji} \wedge_{\mathbb{Z}} (1 - z_{ji})$

$$+ z_{kl} \wedge_{\mathbb{Z}} (1 - z_{kl}) + z_{lk} \wedge_{\mathbb{Z}} (1 - z_{lk})$$

$$= \tfrac{1}{2} z \wedge_{\mathbb{Z}} Hz,$$

where H is the linear map $J_T^* \to J_T^*$, which on generators of J_T^* is given by $H(f_{ij}) = f_{ik}$ and $H(f_{ik}) = -f_{ij}$. It yields a linear map $H \colon k^\times \otimes_{\mathbb{Z}} J_T^* \to k^\times \otimes_{\mathbb{Z}} J_T^*$. We note that in coordinates we get

$$(Hz)_{f_{ij}} = \frac{1}{z_{ik}} \quad \text{and} \quad (Hz)_{f_{ik}} = z_{ij}.$$

The choice of the basis $\{f_{ij}\}$ of J_T^* and the choice of the map H are not canonical but they define the natural symplectic form

(4.3.2) $$\Omega^*(\cdot, \cdot) = \langle H\cdot, \cdot \rangle$$

on J_T^*, where $\langle \cdot, \cdot \rangle$ is the scalar product associated to the basis $\{f_{ij}\}$. Such a symplectic space was first considered by Neumann and Zagier [16; 18] in the PGL$(2, \mathbb{C})$ context.

5 Decoration of a tetrahedral complex and its holonomy

In the previous sections we defined coordinates for a single tetrahedron of flags and affine flags and defined its invariant β in $\mathcal{P}(k)$, directly related to the volume.

We study here how one may decorate a complex of tetrahedra with these coordinates, compute the holonomy of its fundamental group and define the generalized invariant β (in the pre-Bloch group) associated to the decorated complex of tetrahedra. We eventually state the main theorem of the paper, Theorem 5.5.1, which computes $\delta(\beta)$ in terms of boundary data.

5.1 Quasi-simplicial manifolds

Let us begin with the definition of a quasi-simplicial complex (see eg [16]): A *quasi-simplicial complex* K is a cell complex whose cells are simplices with injective simplicial attaching maps, but no requirement that closed simplices embed in $|K|$ – the underlying topological space. A *tetrahedral complex* is a quasi-simplicial complex of dimension 3.

From now on we let K be a tetrahedral complex. The *(open) star* of a vertex $v \in K^{(0)}$ is the union of all the open simplices that have v as a vertex. It is an open neighborhood of v and is the open cone on a simplicial complex L_v called the *link* of v.

A *quasi-simplicial 3–manifold* is a compact tetrahedral complex K such that $|K| - |K^{(0)}|$ is a 3–manifold (with boundary). By an orientation of K we mean an orientation of this manifold. A *3–cycle* is a closed quasi-simplicial 3–manifold.

A quasi-simplicial 3–manifold is topologically a manifold except perhaps for the finitely many singular points $v \in |K^{(0)}|$ where the local structure is that of a cone on $|L_v|$; a compact connected surface (with boundary). We will soon require that for each vertex $v \in K^{(0)}$, $|L_v|$ is homeomorphic to either a sphere, a torus or an annulus. Let $K_s^{(0)}$, $K_t^{(0)}$ and $K_a^{(0)}$ be the corresponding subsets of vertices. We note that $|K| - |K_t^{(0)} \cup K_a^{(0)}|$ is an (open) 3–manifold with boundary that retracts onto a compact 3–manifold with boundary M. Note that ∂M is the disjoint union $T_1 \cup \cdots \cup T_\tau \cup S_1 \cup \cdots \cup S_\sigma$ where each T_i is a torus and each S_i a surface of genus $g_i \geq 2$. Moreover: each T_i corresponds to a vertex in $K_t^{(0)}$ and each S_i contains at least one simple closed essential curve each corresponding to a vertex in $K_a^{(0)}$; see Figure 6.

Given such a compact oriented 3–manifold with boundary M, we call a quasi-simplicial 3–manifold as above a *triangulation* of M.
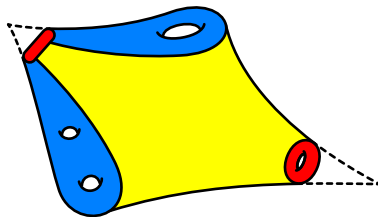

Figure 6: The retraction of a quasi-simplicial 3–manifold onto a compact 3–manifold with boundary

5.2 Decoration of a quasi-simplicial complex

Let M be a quasi-simplicial manifold triangulated by the complex K. Denote by $T_1, \ldots, T_\nu$ the tetrahedra of K. Let $\widetilde{K}$ be the universal covering of K: it is a triangulation of the universal covering $\widetilde{M}$ of M. As such, the fundamental group $\pi_1(M)$ acts on $\widetilde{K}$.

5.2.1   Decoration and coordinates

Definition 5.2.1   A *parabolic decoration* of the tetrahedral complex is the data of a flag for each vertex of $\widetilde{K}$ (equivalently a map from the 0–skeleton of the complex $\widetilde{K}$ to $\mathcal{F}l$) such that:

- For each tetrahedron of the complex, the corresponding tetrahedron of flags is in generic position.

- If two tetrahedra of $\widetilde{K}$ lift the same tetrahedron of K, then the two tetrahedra of flags define the same element in $\mathcal{C}_3(\mathcal{F}l)$.

Similarly, a *unipotent decoration* is the data of an affine flag for each vertex with the genericity condition and such that, if two tetrahedra of $\widetilde{K}$ lift the same tetrahedron of K, then the two tetrahedra of affine flags define the same element in $\mathcal{C}_3(\mathcal{A}\mathcal{F}l)$.

Let us make two comments on these definitions. First, a parabolic decoration – together with an ordering of the vertices of each 3–simplex – associates to each tetrahedron of K a well-defined configuration of 4 flags (ie an element of $\mathcal{C}_3(\mathcal{F}l)$). Hence, as seen in Section 2, it equips each tetrahedron of K with a set of z–coordinates (defined in Section 2.3). Second, a unipotent decoration induces a parabolic decoration via the canonical projection $\mathcal{A}\mathcal{F}l \to \mathcal{F}l$, so we get these z–coordinates, as well as a set of a–coordinates (see Section 2.5).

5.2.2   Representing elements of $H_3(\mathrm{PGL}(3, \mathbb{C}))$ by decorated 3–cycles   Neumann [17, Section 4] has proven that any element of $H_3(\mathrm{PGL}(3, \mathbb{C}))$ can be represented by an oriented 3–cycle K, together with an ordering of the vertices of each 3–simplex of K so that these orderings agree on common faces, and a decoration of K. Moreover, any class in $H_3(\mathrm{SL}(3, \mathbb{C}))$ can be represented by a unipotent decoration of K.

In other words: any class $\alpha \in H_3(\mathrm{PGL}(3, \mathbb{C}))$ can be represented as $f_*[K]$, where K is a quasi-simplicial complex such that $|K| - |K^{(0)}|$ is an oriented 3–manifold, $[K] \in H_3(|K|)$ is its fundamental class and $f: |K| \to \mathrm{B\,PGL}(3, \mathbb{C})$ is some map.

This motivates the study of decorated 3–cycles. From now on we fix K a decorated oriented quasi-simplicial 3–manifold together with an ordering of the vertices of each 3–simplex of K. Let N be the number of tetrahedra of K and denote by T_ν, $\nu = 1, \ldots, N$, these tetrahedra. We let $z_{ij}(T_\nu)$ be the corresponding z–coordinates. We now describe the consistency relations on these coordinates in order to be able to glue together the decorated tetrahedra.

5.3 Consistency relations

These relations are of two types (cf [6]): face relations and edge relations.

Let F be an internal face (2–dim cell) of K and T, and T' be the tetrahedron attached to F. In order to fix notation, suppose that the vertices of T are $1, 2, 3, 4$ and that the face F is 123. Let $4'$ be the remaining vertex of T'. The face F inherits two 3–ratios from the decoration: first $z_{123}(T)$ as a face of T and second $z_{132}(T')$ as a face of T'. But considering F to be attached to T or T' only changes its orientation, not the flags at its vertex. So these two 3–ratios are inverses. Hence we get the:

Face relation Let T and T' be two tetrahedra of K with a common face (ijk) (oriented as a boundary of T), then $z_{ijk}(T)z_{ikj}(T') = 1$.
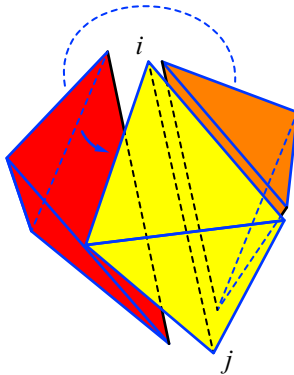

Figure 7: Tetrahedra sharing a common edge

We should add another compatibility condition to ensure that the edges are not singularities: we are going to compute the holonomy of a path in a decorated complex and we want it to be invariant under crossing the edges. One way to state the condition is the following one: let $T_1, \dots, T_\nu$ be a sequence of tetrahedra sharing a common edge (ij) and such that ij is an inner edge of the subcomplex composed by the T_μ (they are making looping around the edge; see Figure 7). Then we ask that the following be satisfied:

Edge relation $z_{ij}(T_1) \cdots z_{ij}(T_\nu) = z_{ji}(T_1) \cdots z_{ji}(T_\nu) = 1$

5.4 Holonomy of a decoration

From the definition of a decoration, we see that they define a holonomy representation of the fundamental group $\pi_1(M)$ (up to conjugation): Indeed, fix a tetrahedron T of $\widetilde{K}$ and pick an element γ in $\pi_1(M)$. Then $\gamma \cdot T$ is another tetrahedron of $\widetilde{K}$. Moreover, T

and $\gamma \cdot T$ lift the same tetrahedron of K. Hence the decoration transforms T and $\gamma \cdot T$ in two tetrahedra of flags that are congruent (they define the same element in $\mathcal{C}_3(\mathcal{F}l)$). As such, they are related by a well-defined element $\rho(\gamma) \in \mathrm{PGL}(3, k)$. It is easily checked that $\gamma \mapsto \rho(\gamma)$ is a representation. Starting with a different tetrahedron from T only conjugates the representation. We now compute this holonomy representation in terms of the z–coordinates associated to each tetrahedron.

5.4.1 Basis change in a tetrahedron Recall from Section 2.2 that, with a configuration of 3 generic flags $([x_i], [f_i])_{1 \leq i \leq 3} \in \mathcal{C}_2^r(\mathcal{F}l)$ with triple ratio X, we defined a projective coordinate system of $\mathbb{P}(\mathbb{C}^3)$ as the one where the point x_1 has coordinates $[1 : 0 : 0]^t$, the point x_2 has coordinates $[0 : 0 : 1]^t$, the point x_3 has coordinates $[1 : -1 : 1]^t$ and the intersection of $\mathrm{Ker}(f_1)$ and $\mathrm{Ker}(f_2)$ has coordinates $[0 : 1 : 0]^t$. The line $\mathrm{Ker}(f_3)$ then has coordinates $[X : X + 1 : 1]$.

Given an oriented face we therefore get 3 projective bases associated to the triples (123), (231) and (312). The cyclic permutation of the flags induces the coordinate change given by the matrix

$$T(X) = \begin{pmatrix} X & X+1 & 1 \\ -X & -X & 0 \\ X & 0 & 0 \end{pmatrix}.$$

Namely, if a point p has coordinates $[u : v : w]^t$ in the basis associated to the triple (123), it has coordinates $T(X)[u : v : w]^t$ in the basis associated to (231).

Lemma 5.4.1 *If we have a tetrahedron of flags $(ijkl)$ with its z–coordinates, then the coordinate system related to the triple (ijk) is obtained from the coordinate system related to the triple (ijl) by the coordinate change given by the matrix*

$$E(z_{ij}, z_{ji}) = \begin{pmatrix} z_{ji}^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{ij} \end{pmatrix}.$$

Beware that the orientation of (ijl) is not the one given by the tetrahedron.

Proof The matrix we are looking for fixes the flags $([x_i], [f_i])$ and $([x_j], [f_j])$ corresponding to the vertex i and j. In particular it should be diagonal. Finally it should send $[x_l]$ to $[x_k]$. But in the coordinate system associated to the triple (ijk), the point $[x_l]$ in the flag $([x_l], [f_l])$ corresponding to the vertex l has coordinates

$$x_l = [z_{ji} : -1 : z_{ij}^{-1}]^t.$$

This proves the lemma. $\qquad \square$

5.4.2 Computation of the holonomy From this we can explicitly compute the holonomy of a path in the complex. For that let us put three points in each face near the vertices, denoting by (ijk) the point in the face ijk near i. As we have said before, once the decoration is fixed, each of these points corresponds to a projective basis of $\mathbb{C}^3$. Each path can be deformed so that it decomposes in two types of steps (see Figure 8):

(1) A path inside an oriented face ijk from (ijk) to (jki)

(2) A path through a tetrahedron $ijkl$ from (ijk) to (ijl) (ie turning left around the edge ij oriented from j to i)
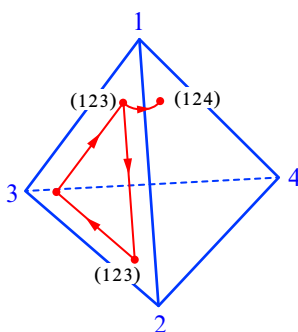

Figure 8: Two elementary steps for computing holonomy

Now the holonomy of the path is the coordinate change matrix so that: in case (1), you have to left multiply by the matrix $T(z_{ijk})$ and in case (2) by the matrix $E(z_{ij}, z_{ji})$.

In particular the holonomy of the path turning left around an edge, ie the path $(ijk) \rightarrow (ijl)$, is given by:

$$(5.4.1) \qquad\qquad L_{ij} = E(z_{ij}, z_{ji}) = \begin{pmatrix} z_{ji}^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{ij} \end{pmatrix}$$

As an example which we will use latter on, one may also compute the holonomy of the path turning right around an edge, ie, the path $(ilj) \rightarrow (ikj)$. We consider the sequence of coordinate changes (see Figure 9 for the path going from (231) to (241))

$$(ilj) \rightarrow (lji) \rightarrow (jil) \rightarrow (jik) \rightarrow (ikj).$$

The first two operations are cyclic permutations both given by the matrix $T(z_{ilj})$. It follows from Lemma 5.4.1 that the third is given by the matrix $E(z_{ji}, z_{ij})$. Finally
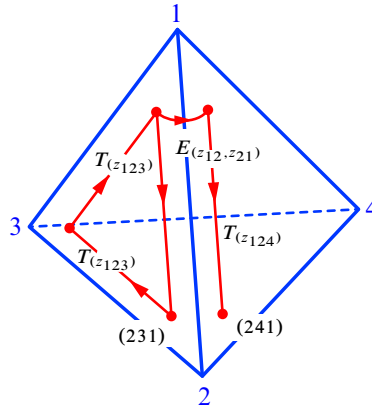
Figure 9: Turning right

the last operation is again a cyclic permutation given by the matrix $T(z_{ikj})$. The coordinate change from the basis (ilj) to (ikj) is therefore given by

$$T(z_{ikj})E(z_{ji}, z_{ij})T(z_{ilj})^2 = \begin{pmatrix} z_{ji}z_{ilj} & \star & \star \\ & z_{ikj} & \star \\ & & \frac{z_{ikj}}{z_{ij}} \end{pmatrix}.$$

Using $z_{ikj} = 1/z_{ijk}$, we get that the holonomy matrix, in PGL$(3, \mathbb{C})$, of the path turning right around an edge ij is

$$(5.4.2) \qquad R_{ij} = \begin{pmatrix} z_{ji}z_{ilj}z_{ijk} & \star & \star \\ & 1 & \star \\ & & \frac{1}{z_{ij}} \end{pmatrix}.$$

Remark Beware that $L_{ij}R_{ij}$ is not the identity in PGL$(3, \mathbb{C})$. This is due to the choices of orientations of the faces which prevents $L_{ij}R_{ij}$ to be a matrix of coordinate change. When computing the holonomy of a path we therefore have to avoid backtracking.

5.5 Decoration and the pre-Bloch group

Let k be a field containing all the z–coordinates of the tetrahedra T_ν, $\nu = 1, \dots, N$. To any of these (ordered) tetrahedra we have associated an element $\beta(T_\nu) \in \mathcal{P}(k)$ (see Section 3.1). Set

$$(5.5.1) \qquad \beta(K) = \sum_\nu \beta(T_\nu) \in \mathcal{P}(k).$$

As in Section 4, we will state a formula for the element $\delta(\beta(K))$; see Theorem 5.5.1. It will express this quantity in terms of boundary of M, which consists in the boundary

of the complex and its links. Before stating the theorem, we need to define coordinates associated to these boundaries.

5.5.1 Coordinates for the boundary of the complex The boundary Σ of the complex K is a triangulated punctured surface. As in Section 4 and in [9], we associate to Σ the set I_Σ of the vertices of the (red) arrows of the triangulation of Σ obtained using Figure 4. As in the preceding section we set $J_\Sigma^2 = \mathbb{Z}^{I_\Sigma}$ and consider the skew-symmetric form Ω_Σ^2 on J_Σ^2, introduced by Fock and Goncharov in [9], defined by the same formula as Ω^2 (see Section 4.1). Here again we let $J_\Sigma^* \subset (J_\Sigma^2)^*$ be the image of J_Σ^2 by the linear map $v \mapsto \Omega_\Sigma^2(v, \cdot)$.

The decoration of K yields a decoration of the punctures of Σ by flags, as in [9], and hence a point

$$z_\Sigma = (z_\alpha^\Sigma)_{\alpha \in I_\Sigma} \in k^\times \otimes_\mathbb{Z} J_\Sigma^*\left[\tfrac{1}{2}\right].$$

Here is a more descriptive point of view, using the holonomy of the decoration of K: It provides Σ with coordinates associated to each $\alpha \in I_\Sigma$. To each face ijk we associate the face 3–ratio z_{ijk} of the corresponding tetrahedra T of K:

$$z_{ijk}^\Sigma = z_{ijk}(T).$$

Moreover, to each oriented edge ij of the triangulation of Σ we associate the last eigenvalue of the holonomy of the path joining the two adjacent faces by turning left around ij in K. It is equal to the product $z_{ij}(T_1) \cdots z_{ij}(T_\nu)$, where $T_1, \ldots, T_\nu$ is the sequence of tetrahedra sharing ij as a common edge:

$$z_{ij}^\Sigma = z_{ij}(T_1) \cdots z_{ij}(T_\nu).$$

Note that when K has a unipotent decoration, then the punctures are decorated by affine flags. We immediately get an element $a_\Sigma \in k^\times \otimes_\mathbb{Z} J_\Sigma^2$ that projects onto z_Σ in $k^\times \otimes_\mathbb{Z} J_\Sigma^*\left[\tfrac{1}{2}\right]$. Here again we have

$$a_\Sigma \wedge_{\Omega_\Sigma^2} a_\Sigma = z_\Sigma \wedge_{\Omega_\Sigma^*} z_\Sigma.$$

The first expression is the W–element $W(\Sigma)$ associated to the decorated Σ by Fock and Goncharov.

5.5.2 Coordinates for the links We now define some coordinates for the links of K. From now on we assume that for each vertex $v \in K^{(0)}$, $|L_v|$ is homeomorphic to either a torus or an annulus. We fix symplectic bases (a_s, b_s) for each of the torus components and we fix c_r (resp. d_r) a generator of each homology group $H_1(L_r)$ (resp. $H_1(L_r, \partial L_r)$), where the L_r are the annulus boundary components. We furthermore assume that the algebraic intersection number $\iota(c_r, d_r) = 1$.

Each one of these homology elements may be represented as a path as in Section 5.4 that remains close to the associated vertex. So we may compute its holonomy using only matrices L_{ij} and R_{ij}; we will get an upper-triangular matrix. More conceptually, the path is looping around a vertex decorated by a flag, so must preserve the flag. So it may be conjugated to an upper-triangular matrix. Recall also that the diagonal part of a triangular matrix is invariant under conjugation by an upper-triangular matrix.

We define the *holonomy elements* A_s, B_s, C_r, D_r and A_s^*, B_s^*, C_r^*, D_r^* such that the holonomy matrices associated to a_s, b_s, c_r, d_r have the following form in a basis adapted to the flag decorating the link (see also Section 7.2 for a more explicit description):

$$\begin{pmatrix} \frac{1}{A_s^*} & * & * \\ 0 & 1 & * \\ 0 & 0 & A_s \end{pmatrix}.$$

5.5.3 A formula for $\delta(\beta(K))$ The following theorem computes $\delta(\beta(K))$ in terms of the coordinates z_Σ and the holonomy elements:

Theorem 5.5.1   *The invariant $\delta(\beta(K))$ only depends on the boundary coordinates z_Σ, A_s, B_s, C_r, D_r and A_s^*, B_s^*, C_r^*, D_r^*. Moreover:*

(1) *If the decoration of K is unipotent then $2\delta(\beta(K)) = z_\Sigma \wedge_{\Omega_\Sigma^*} z_\Sigma$.*

(2) *If K is closed, ie, $\Sigma = \varnothing$, and each link is a torus, we have the following formula for $3\delta(\beta(K))$:*

$$\sum_s \left(2A_s \wedge_\mathbb{Z} B_s + 2A_s^* \wedge_\mathbb{Z} B_s^* + A_s^* \wedge_\mathbb{Z} B_s + A_s \wedge_\mathbb{Z} B_s^* \right).$$

Remarks • Theorem 5.5.1 generalizes several results known in the SL(2, $\mathbb{C}$) case; see Neumann [16] – when K is closed – and Kabaya [15] – when all the connected components of Σ are spheres with 3 vertices. A related formula – still in the PGL(2, $\mathbb{C}$) case – is obtained by Bonahon [2; 3]. One may extract from our proof a formula for the general case. Though it should be related to the Weil–Petersson form on ∂M, we are not able yet to make this relation explicit.

• Thanks to Theorem 5.5.1, when the decoration of K is unipotent, the fact that β lies inside the Bloch group is a boundary condition (the only non-vanishing part is $\frac{3}{2}z_\Sigma \wedge_{\Omega_\Sigma^*} z_\Sigma$). As a consequence, if the boundary is empty, it will automatically belong to the Bloch group. Using the Suslin map, it allows us to construct geometrically any class in $K_3^{\mathrm{ind}}(k)$, supporting a remark of Fock and Goncharov; see [10, Proposition 6.16] and the following section.

- With the computation of the derivative of the dilogarithm D done in Proposition 3.6.1, this theorem yields a variational formula for the volume. It will be addressed in Section 11.1.

- The proof will run through the next three sections. We will heavily use the point of view developed in Section 4 and achieve it by constructing and relating 2–forms on different $\mathbb{Z}$–modules. These 2–forms are interesting on their own. For example, this leads to some rigidity results; see [1].

6 Some linear algebra and the unipotent case

The goal of this section is to prove Theorem 5.5.1 when K has a unipotent decoration. Along the way, we lay down the first basis for the homological proof in the general case.

6.1 Linearization of the consistency relations

Let (J^i, Ω^i) $(i = \varnothing, 2)$ denote the orthogonal sum of the spaces $(J^i_{T_\nu}, \Omega^i)$. We denote by e^μ_α the e_α–element in $J^i_{T_\mu}$.

As seen in Section 4, a decoration provides us with an element

$$z \in \operatorname{Hom}(J, k^\times)\left[\tfrac{1}{2}\right] \simeq k^\times \otimes_\mathbb{Z} J^*\left[\tfrac{1}{2}\right] = k^\times \otimes_\mathbb{Z} \operatorname{Im}(p^*)\left[\tfrac{1}{2}\right]$$

which satisfies the face and edge conditions.[6] We first translate these two consistency relations into linear algebra.

Let C^{or}_1 be the free $\mathbb{Z}$–module generated by the oriented internal[7] 1–simplices of K and C_2 the free $\mathbb{Z}$–module generated by the internal 2–faces of K. Introduce the map

(6.1.1) $$F\colon C^{\mathrm{or}}_1 + C_2 \to J^2$$

defined by, for an internal oriented edge $\overline{e}_{ij}$ of K,

$$F(\overline{e}_{ij}) = e^1_{ij} + \cdots + e^\nu_{ij},$$

where $T_1, \ldots, T_\nu$ is the sequence of tetrahedra sharing the edge $\overline{e}_{ij}$ such that $\overline{e}_{ij}$ is an inner edge of the subcomplex composed by the T_μ and each e^μ_{ij} gets identified with the *oriented* edge $\overline{e}_{ij}$ in K (recall Figure 7). And for a 2–face $\overline{e}_{ijk}$,

$$F(\overline{e}_{ijk}) = e^\mu_{ijk} + e^\nu_{ikj},$$

---

[6]Note that z moreover satisfies the non-linear equations $z_{ik}(T_\nu) = 1/(1 - z_{ij}(T_\nu))$.

[7]Recall that our complex may have boundary.

where μ and ν index the two 3–simplices having the common face $\overline{e}_{ijk}$. An element $z \in \text{Hom}(J^2, k^\times)$ satisfies the face and edge conditions if and only if it vanishes on $\text{Im}(F)$.

Let $(J^2_{\text{int}})^*$ be the subspace of $(J^2)^*$ generated by internal edges and faces of K.

The dual map $F^*\colon (J^2)^* \to C_1^{\text{or}} + C_2$ (here we identify $C_1^{\text{or}} + C_2$ with its dual by using the canonical basis) is the "projection map", which maps $(e_\alpha^\mu)^*$ to $\overline{e}_\alpha$ when $(e_\alpha^\mu)^* \in (J^2_{\text{int}})^*$ and maps $(e_\alpha^\mu)^*$ to 0 if $(e_\alpha^\mu)^* \notin (J^2_{\text{int}})^*$.

From the definitions we get the following:

Lemma 6.1.1 *An element $z \in k^\times \otimes_{\mathbb{Z}} (J^2)^*$ satisfies the face and edge conditions if and only if*

$$z \in k^\times \otimes_{\mathbb{Z}} \text{Ker}(F^*).$$

A decorated tetrahedral complex K thus provides us with an element

$$z \in k^\times \otimes (J^* \cap \text{Ker}(F^*))\left[\tfrac{1}{2}\right].$$

We focus now on the element $\delta(\beta(K)) \in k^\times \wedge_{\mathbb{Z}} k^\times$. By definition of $\beta(K)$, we have

$$\delta(\beta(K)) = \sum_1^\nu \delta(\beta(T_\nu)).$$

Moreover we have seen in Lemma 4.2.1 that

$$\delta(\beta(T_\nu)) = z(T_\nu) \wedge_{\Omega*} z(T_\nu).$$

Hence, by definition of J^* and Ω^*, we have

$$\delta(\beta(K)) = \tfrac{1}{2} z \wedge_{\Omega*} z.$$

6.2 Proof of Theorem 5.5.1 in the unipotent case

In this section we assume that K is equipped with a *unipotent* decoration. Consider an edge ij of K, and let $T_1, \ldots, T_\mu$ be the tetrahedra sharing it. The vertices i and j are equipped with affine flags (x_i, f_i) and (x_j, f_j). For any tetrahedron T sharing this edge, we have (see Section 2.5)

$$a_{ij}(T) = f_i(x_j).$$

In other terms it does not depend on the tetrahedron chosen and we have

$$a_{ij}(T_1)e_{ij}^1 + \cdots + a_{ij}(T_\mu)e_{ij}^\mu = f_i(x_j)F(\overline{e}_{ij}) \in k^\times \otimes_{\mathbb{Z}} J^2.$$

When looking at a face ijk between T and T', we get

$$a_{ijk}(T) = -a_{ikj}(T').$$

Killing the 2–torsion, we get

$$a_{ijk}(T)e_{ijk} + a_{ikj}(T')e_{ikj} = a_{ijk}(T)F(\overline{e}_{ijk}) \in k^{\times} \otimes_{\mathbb{Z}} J^2\left[\tfrac{1}{2}\right].$$

Going one step further to z–coordinates, we get that a unipotent decoration corresponds to a point $z \in k^{\times} \otimes (\mathrm{Im}(p \circ F))\left[\tfrac{1}{2}\right]$. In Section 5.5.1 we have defined a map

$$k^{\times} \otimes (\mathrm{Im}(p \circ F))\left[\tfrac{1}{2}\right] \to k^{\times} \otimes J_{\Sigma}^*\left[\tfrac{1}{2}\right].$$

The following proposition states that this map respects the 2–forms Ω^* and Ω_{Σ}^*.

Proposition 6.2.1 *In the unipotent case, Ω^* is the pullback of Ω_{Σ}^*.*

Proof We have seen that on each tetrahedron $p^*(\Omega^*(T)) = \Omega^2(T)$.

Since $\mathrm{Im}(p \circ F)$ is the image under p of the subspace $\mathrm{Im}(F)$ of J^2, each face f of T is an oriented triangle with a–coordinates, so we define a 2–form $\Omega^2(f, T)$ by the usual formula. If the face f is internal between T and T', we have $\Omega^2(f, T) = -\Omega^2(f, T')$ as the only difference is the orientation of the face (and hence of its red triangulation; see Figure 4).

Moreover $p^*(\Omega^*)$ is the sum of the $\Omega^2(T)$. Hence it reduces to the sum on external faces of $\Omega^2(f, T)$, that is, exactly $\Omega_{\Sigma}^2 = p^*(\Omega^*)$. □

Using the usual Lemma 4.1.1, we get the proof of Theorem 5.5.1 in the unipotent case:

$$\delta(\beta(K)) = z_{\Sigma} \wedge_{\Omega_{\Sigma}^*} z_{\Sigma}.$$

As a corollary, we see that the Suslin map $S = \beta \circ \pi_*$ (see Section 3.5) takes its value in the Bloch group $\mathcal{B}(k)$:

Corollary 6.2.1 *The Suslin map S sends $H_3(\mathrm{SL}(3, k))$ to the Bloch group $\mathcal{B}(k)$.*

Proof Via the projection $\mathrm{SL}(3, k) \to \mathcal{AF}l = \mathrm{SL}(3, k)/U$ (see Section 2) and the consideration in Section 5.2.2, an element of $H_3(\mathrm{SL}(3, k))$ is represented by a *closed* tetrahedral complex with a unipotent decoration. In other terms, $\pi_*(H_3(\mathrm{SL}(3, k)))$ is included in the unipotent decorations of closed tetrahedral complex.

Let K be such a closed tetrahedral complex with a unipotent decoration. Using the unipotent case of Theorem 5.5.1, $\delta(\beta(K))$ only depends on the boundary coordinates z_{Σ}. But, as Σ is empty, z_{Σ} is trivial. Hence δ is trivial on $\beta(K)$, which means that δ vanishes on $\beta \circ \pi_*(H_3(\mathrm{SL}(3, k)))$. As we showed that the Suslin map S equals $\beta \circ \pi_*$ and the Bloch group is the kernel of δ, the corollary is proven. □

6.3 The invariant W

It may be useful to give another point of view on this formula.

The boundary surface Σ is a union of ideally *triangulated* closed oriented[8] surfaces with punctures decorated by affine flags in the sense of Fock and Goncharov [9]: the triangles are decorated by *affine* flags coordinates in such a way that the edge coordinates on the common edge of two triangles coincide.

Each triangle being oriented, we may define the W–invariant

$$W(\Sigma) = \sum_{\Delta} W_{\Delta},$$

where W_{Δ} is defined by Equation (4.1.3).[9]

Recall from Section 5.5.1 that the unipotent decoration of Σ provides us with an element $a_{\Sigma} \in k^{\times} \otimes_{\mathbb{Z}} J_{\Sigma}^2$ that projects onto $z_{\Sigma} \in k^{\times} \otimes_{\mathbb{Z}} J_{\Sigma}^*\left[\frac{1}{2}\right]$. We have[10]

$$W(\Sigma) = \tfrac{1}{2} a_{\Sigma} \wedge_{\Omega_{\Sigma}^2} a_{\Sigma} = \tfrac{1}{2} z_{\Sigma} \wedge_{\Omega_{\Sigma}^*} z_{\Sigma}.$$

Hence, from the previous section, we deduce the following proposition:

Proposition 6.3.1 *In the unipotent case we have*

$$\delta(\beta(K)) = W(\Sigma).$$

Our goal is now to extend this result beyond the unipotent case; to this end we develop a theory analogous to the one of Neumann–Zagier but in the $\mathrm{PGL}(3, \mathbb{C})$ case. We first treat in detail the case where K is closed.

7 Neumann–Zagier bilinear relations for $\mathrm{PGL}(3, k)$

For this section and the next one, we assume that K is a closed tetrahedral complex.

A decoration of K provides us with and element $z \in k^{\times} \otimes (J^* \cap \mathrm{Ker}(F^*))\left[\frac{1}{2}\right]$; see Section 6.1. Moreover we have seen in that section that the invariant $\delta(\beta(K))$ is written $\frac{1}{2} z \wedge_{\Omega^*} z$. We will compute this last expression. But here we first describe the

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[8]The orientation being induced by that of K.

[9]Note that in the case of $K = T$ the boundary of T is a sphere with 4 punctures and the definition of $W(T)$ in Section 4 matches this one.

[10]Note in particular that $W(\Sigma)$ only depends on the flag z–coordinates; see also [10, Lemma 6.6]. Moreover, in case $K = T$, we recover Lemma 4.1.2.

right set-up to state the generalization of Proposition 6.2.1 to general – non-unipotent – decorations.

We construct a $\mathbb{Z}$–module together with a 2–form such that the "holonomy element" (see Section 5.5.2) belongs to this module after tensoring by $k^\times$. This module is a group of chains in a simplicial decomposition of the links. We then relate the 2–form (given by the intersection form) with our 2–forms Ω and Ω^*. It will relate closely Ω^* with the Weil–Petersson form on the cohomology of the tori. This leads to a more precise version of Theorem 5.5.1; see Corollary 7.3.2.

In Section 9, we will explain how to modify the definitions and proofs to deal with the general case.

7.1 Coordinates on the boundary

Denote by M the 3–manifold triangulated by K. As K is closed, its boundary Σ is empty and each $|L_v|$ is a torus. We first define coordinates for ∂M and a symplectic structure on these coordinates.

7.1.1 Two simplicial decompositions Each torus boundary surface S in the link of a vertex is triangulated by the traces of the tetrahedra; from this we build the CW–complex $\mathcal{D}$ whose edges consist of the inner edges of the first barycentric subdivision; see Figure 10. We denote by $\mathcal{D}'$ the dual cell division. Let $C_1(\mathcal{D}) = C_1(\mathcal{D}, \mathbb{Z})$ and $C_1(\mathcal{D}') = C_1(\mathcal{D}', \mathbb{Z})$ be the corresponding chain groups. Given two chains $c \in C_1(\mathcal{D})$ and $c' \in C_1(\mathcal{D}')$ we denote by $\iota(c, c')$ the (integer) intersection number of c and c'. This defines a bilinear form $\iota: C_1(\mathcal{D}) \times C_1(\mathcal{D}') \to \mathbb{Z}$ which induces the usual intersection form on $H_1(S)$. In that way $C_1(\mathcal{D}')$ is canonically isomorphic to the dual of $C_1(\mathcal{D})$.
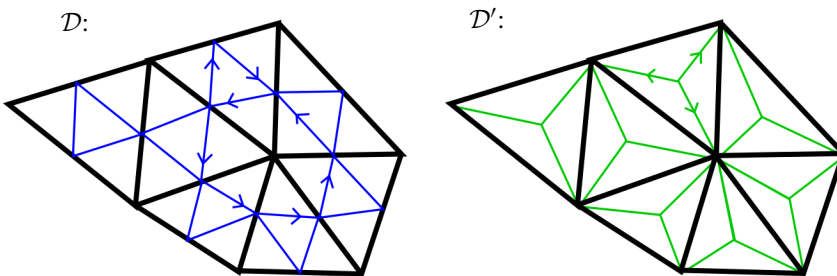

Figure 10: The two cell decompositions of the link

7.1.2 Goldman–Weil–Petersson form for tori   Here we equip

$$C_1(\mathcal{D}, \mathbb{R}^2) = C_1(\mathcal{D}) \otimes \mathbb{R}^2$$

with the bilinear form ω defined by coupling the intersection form ι with the scalar product on $\mathbb{R}^2$ seen as the space of roots of $\mathfrak{sl}(3, \mathbb{C})$ with its Killing form. We describe more precisely an integral version of this.

From now on we identify $\mathbb{R}^2$ with the subspace $V = \{(x_1, x_2, x_3)^t \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ via

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

We let $L \subset V$ be the standard lattice in V where all three coordinates are in $\mathbb{Z}$. We identify it with $\mathbb{Z}^2$ using the above basis of V. The restriction of the usual euclidean product of $\mathbb{R}^3$ gives a product, denoted $[\cdot, \cdot]$, on V (the "Killing form").[11] In other words, we have

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = 2 \quad \text{and} \quad \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = -1.$$

Identifying V with V^* using the scalar product $[\cdot, \cdot]$, the dual lattice $L^* \subset V^*$ becomes a lattice L' in V; an element $y \in V$ belongs to L' if and only if $[x, y] \in \mathbb{Z}$ for every $x \in L$.

We consider $C_1(\mathcal{D}, L)$ and define $\omega = \iota \otimes [\cdot, \cdot] : C_1(\mathcal{D}, L) \times C_1(\mathcal{D}', L') \to \mathbb{Z}$ by the formula

$$\omega(c \otimes l, c' \otimes l') = \iota(c, c')[l, l'].$$

This induces a (symplectic) bilinear form on $H_1(S, \mathbb{R}^2)$, which we still denote by ω. Note that ω identifies $C_1(\mathcal{D}', L')$ with the dual of $C_1(\mathcal{D}, L)$.

Remark The canonical coupling $C_1(\mathcal{D}, L) \times C^1(\mathcal{D}, L^*) \to \mathbb{Z}$ identifies $C_1(\mathcal{D}, L)^*$ with $C^1(\mathcal{D}, L^*)$. This last space is naturally equipped with the "Goldman–Weil–Petersson" form wp, dual to ω. Let $\langle \cdot, \cdot \rangle$ be the natural scalar product on V^* dual to $[\cdot, \cdot]$: letting $d : V \to V^*$ be the map defined by $d(v) = [v, \cdot]$, we have $\langle d(v), d(v') \rangle = [v, v']$. In coordinates, $d : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ 2y - x \end{pmatrix}.$$

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[11] In terms of roots of $\mathfrak{sl}(3)$, the chosen basis is, in usual notation, $e_1 - e_2$, $e_2 - e_3$.

Identifying V^* with $\mathbb{R}^2$ using the dual basis, we have

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \frac{2}{3} \quad \text{and} \quad \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \frac{1}{3}.$$

On $H^1(S, \mathbb{R}^2)$ the bilinear form wp induces a symplectic form – the usual Goldman–Weil–Petersson symplectic form – formally defined as the coupling of the cup-product and the scalar product $\langle \cdot, \cdot \rangle$.

7.2 Linearization of the holonomy elements

We now linearize the holonomy elements, ie, we explain how the computations of the eigenvalues of the holonomy of the torus may be done in our framework of $\mathbb{Z}$–modules.

7.2.1 The holonomy elements map To any decoration $z \in k^\times \otimes (J^* \cap \mathrm{Ker}(F^*))\left[\frac{1}{2}\right]$, we now explain how to associate an element

$$R(z) \in \mathrm{Hom}(H_1(S, L), k^\times)\left[\frac{1}{2}\right].$$

We may represent any class in $H_1(S, L)$ by an element $c \otimes \binom{n}{m}$ in $C_1(\mathcal{D}, L)$, where c is a closed path in S seen as the link of the corresponding vertex in the complex K. Using the decoration z we may compute the holonomy of the loop c, as explained in Section 5.4. This vertex being equipped with a flag stabilized by this holonomy, we write it as an upper-triangular matrix. Let $(1/C^*, 1, C)$ be the diagonal part. The function that maps $c \otimes \binom{n}{m}$ to $C^m(C^*)^n$ is the aforementioned element $R(z)$ of $k^\times \otimes H^1(S, L^*)\left[\frac{1}{2}\right]$.

7.2.2 Linearization for a torus In the preceding paragraph we have constructed a map

$$R: k^\times \otimes (J^* \cap \mathrm{Ker}(F^*))\left[\tfrac{1}{2}\right] \to \mathrm{Hom}(H_1(S, L), k^\times)\left[\tfrac{1}{2}\right].$$

As we have done before, for consistency relations, we now linearize this map.

Let $h: C_1(\mathcal{D}, L) \to J^2$ be the linear map defined on the elements $e \otimes \binom{n}{m}$ of $C_1(\mathcal{D}, L)$ by

(7.2.1) $$h\left(e \otimes \binom{n}{m}\right) = 2m e_{ij}^\mu + 2n e_{ji}^\mu + n(e_{ijk}^\mu + e_{ilj}^\mu).$$

Here we see the edge e as turning left around the edge (ij) in the link of the vertex i inside the tetrahedron $T_\mu = (ijkl)$; see Figure 11.
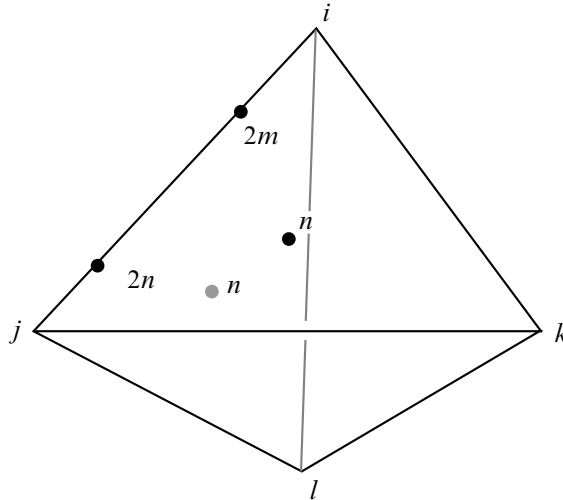
Figure 11: The map h

Lemma 7.2.1 Let $z \in k^{\times} \otimes (J^* \cap \mathrm{Ker}(F^*))\left[\frac{1}{2}\right]$. Viewing z as an element of $\mathrm{Hom}(J^2, k^{\times})\left[\frac{1}{2}\right]$, we have

$$z \circ h = R(z)^2.$$

Proof Let c be an element in $H_1(S)$. Recall that the torus is triangulated by the trace of the tetrahedra. To each triangle, there corresponds a tetrahedron T_μ and a vertex i of this tetrahedron. Now each vertex of the triangle corresponds to an edge ij of the tetrahedron T_μ oriented from the vertex j to i. Hence each edge of $\mathcal{D}$ may be canonically denoted by c_{ij}^μ: it is the edge in the link of i that turns left around the edge ij of the tetrahedron T_μ. We represent c as a cycle $\overline{c} = \sum \pm c_{ij}^\mu$. The cycle $\overline{c}$ turns left around some edges, denoted by e_{ij}^μ, and right around other edges, denoted by $e_{ij}^{\mu'}$. In other terms, we have $\overline{c} = \sum_\mu c_{ij}^\mu - \sum_{\mu'} c_{ij}^{\mu'}$. Then, using the matrices L_{ij}^μ (5.4.1) and $R_{ij}^{\mu'}$ (5.4.2), we see that the diagonal part of the holonomy of c is given by

$$(7.2.2) \qquad C = \frac{\prod z_{ij}^\mu}{\prod z_{ij}^{\mu'}} \quad \text{and} \quad C^* = \frac{\prod z_{ji}^\mu}{\prod z_{ji}^{\mu'} z_{ijk}^{\mu'} z_{ilj}^{\mu'}}.$$

Let us simplify the formula for C^* a bit. Recall the face relation: If T and T' share the same face ijk, we have $z_{ijk}(T)z_{ikj}(T') = 1$. Hence if our path c was turning right before a face F and continues after crossing F, the corresponding face coordinate simplifies in the product $\prod z_{ijk}^{\mu'} z_{ilj}^{\mu'}$. Let $\mathcal{F}$ be the set of faces (with multiplicity) at which c changes direction. For F in $\mathcal{F}$, let T be the tetrahedron containing F in

which α turns right. We consider F oriented as a face of T and denote z_F its 3–ratio. We then have

$$(7.2.3) \qquad C^* = \frac{\prod z_{ji}^{\mu}}{\prod z_{ji}^{\mu'} \prod_{\mathcal{F}} z_F}.$$

Now $h(c \otimes \binom{0}{1}) = 2 \sum e_{ij}^{\mu} - 2 \sum e_{ij}^{\mu'}$, as turning right is the opposite to turning left. It proves (with (7.2.2)) that

$$z \circ h\left(c \otimes \binom{0}{1}\right) = \left(\frac{\prod z_{ij}^{\mu}}{\prod z_{ij}^{\mu'}}\right)^2 = C^2.$$

We have to do a bit more rewriting to check it for $c \otimes \binom{1}{0}$. Indeed, we have

$$h\left(c \otimes \binom{1}{0}\right) = \sum_{\mu}(2e_{ji}^{\mu} + e_{ijk}^{\mu} + e_{ilj}^{\mu}) - \sum_{\mu'}(2e_{ji}^{\mu'} + e_{ijk}^{\mu'} + e_{ilj}^{\mu'}),$$

so that

$$z \circ h\left(c \otimes \binom{1}{0}\right) = \left(\frac{\prod z_{ji}^{\mu}}{\prod z_{ji}^{\mu'}}\right)^2 \frac{\prod z_{ijk}^{\mu} z_{ilj}^{\mu}}{\prod z_{ijk}^{\mu'} z_{ilj}^{\mu'}}.$$

For the same reason as before, the "internal faces" simplify in the products

$$\prod_{\mu} z_{ijk}^{\mu} z_{ilj}^{\mu} \quad \text{and} \quad \prod_{\mu'} z_{ijk}^{\mu'} z_{ilj}^{\mu'}.$$

Moreover, for $ijk = F \in \mathcal{F}$, we have

$$z_F = z_{ijk}^{\mu'} = \frac{1}{z_{ijk}^{\mu}}$$

as the orientation given to F is the one given by the tetrahedron in which c turns *right*. As z_{ijk}^{μ} appears at the numerator and $z_{ijk}^{\mu'}$ at the denominator, we get a factor $1/(z_F)^2$. So the last formula can be rewritten as

$$z \circ h\left(c \otimes \binom{1}{0}\right) = \left(\frac{\prod z_{ji}^{\mu}}{\prod z_{ji}^{\mu'}}\right)^2 \left(\frac{1}{\prod_{\mathcal{F}} z_F}\right)^2 = (C^*)^2,$$

which proves the lemma. $\qquad \square$

Let $h^*: (J^2)^* \to C_1(\mathcal{D}, L)^*$ be the map dual to h. Note that for any $e \in J^2$ and $c \in C_1(\mathcal{D}, L)$ we have

$$(7.2.4) \qquad (h^* \circ p(e))(c) = p(e)(h(c)) = \Omega^2(e, h(c)).$$

Now composing p with h^* and identifying $C_1(\mathcal{D}, L)^*$ with $C_1(\mathcal{D}', L')$ using ω we get a map

(7.2.5) $$g\colon J^2 \to C_1(\mathcal{D}', L')$$

and it follows from (7.2.4) that for any $e \in J^2$ and $c \in J_T$ we have

(7.2.6) $$\omega(c, g(e)) = \Omega^2(e, h(c)).$$

In the following we let $J_{\partial M} = C_1(\partial M, L)$ and $C_1(\partial M', L')$ be the orthogonal sum of the $C_1(\mathcal{D}, L)$'s and $C_1(\mathcal{D}', L')$'s for each torus link S. We abusively denote by $h\colon J_{\partial M} \to J^2$ and $g\colon J^2 \to C_1(\partial M', L')$ the product of the maps defined above on each T.

7.3 Homology of the complexes

We prove here that the maps defined above induce maps in homology and cohomology.

Consider the composition of maps

$$C_1^{\mathrm{or}} + C_2 \xrightarrow{F} J^2 \xrightarrow{p} (J^2)^* \xrightarrow{F^*} C_1^{\mathrm{or}} + C_2.$$

By inspection, one may check that $F^* \circ p \circ F = 0$. Here is a geometric way to figure this after tensoring by $k^\times \otimes \mathbb{Z}\left[\frac{1}{2}\right]$: First note that if $z = p^*(a)$, then a can be thought as a set of affine coordinates lifting of z. Now a belongs to the image of F exactly when these a–coordinates agree on elements of J^2 corresponding to common oriented edges (resp. common faces) of K (see Section 6). In such a case the decoration of K has a unipotent decoration lifting z. Finally the map F^* computes the last eigenvalue of the holonomy matrix of paths going through and back a face (face relations) and of paths going around edges (edge relations). In the case of a unipotent decoration these eigenvalues are trivial. This shows that $F^* \circ p \circ F = 0$.

In particular, letting $G\colon J \to C_1^{\mathrm{or}} + C_2$ be the map induced by $F^* \circ p$ and $F'\colon C_1^{\mathrm{or}} + C_2 \to J$ be the map F followed by the canonical projection from J^2 to J, we get a complex

(7.3.1) $$C_1^{\mathrm{or}} + C_2 \xrightarrow{F'} J \xrightarrow{G} C_1^{\mathrm{or}} + C_2.$$

Similarly, letting $G^* = p \circ F$ and $(F')^*$ be the restriction of F^* to $\mathrm{Im}(p) = J^*$, we get the dual complex

(7.3.2) $$C_1^{\mathrm{or}} + C_2 \xrightarrow{G^*} J^* \xrightarrow{(F')^*} C_1^{\mathrm{or}} + C_2.$$

We define the homology groups of these two complexes,

$$\mathcal{H}(J) = \mathrm{Ker}(G)/\mathrm{Im}(F') = \mathrm{Ker}(F^* \circ p)/(\mathrm{Im}(F) + \mathrm{Ker}(p)),$$

$$\mathcal{H}(J^*) = \mathrm{Ker}((F')^*)/\mathrm{Im}(G^*) = (\mathrm{Ker}(F^*) \cap \mathrm{Im}(p))/\mathrm{Im}(p \circ F).$$

We note that

(7.3.3) $$\mathrm{Ker}(G) = \mathrm{Im}(F')^{\perp\Omega} \quad \text{and} \quad \mathrm{Im}(G^*) = \mathrm{Ker}((F')^*)^{\perp\Omega^*}.$$

The symplectic forms Ω and Ω^* thus induce skew-symmetric bilinear forms on $\mathcal{H}(J)$ and $\mathcal{H}(J^*)$. These spaces are obviously dual spaces and the bilinear forms match through duality.

A decoration of K provides us with an element $z \in k^\times \otimes \mathrm{Ker}((F')^*)\left[\frac{1}{2}\right]$. We already have dealt with the subspace $k^\times \otimes \mathrm{Im}(p \circ F)\left[\frac{1}{2}\right]$ that corresponds to the unipotent decorations (see Section 6): in that case $\delta(\beta(K)) = 0$ as K is closed. We thus conclude that $\delta(\beta(K))$ only depends on the image of z in $k^\times \otimes_{\mathbb{Z}} \mathcal{H}(J^*)\left[\frac{1}{2}\right]$. We will describe this last space in terms of the homology of ∂M.

Let $Z_1(\mathcal{D}, L)$ and $B_1(\mathcal{D}, L)$ be the subspaces of cycles and boundaries in $C_1(\mathcal{D}, L)$. The following lemma is easily checked by inspection.

Lemma 7.3.1 *We have*

$$h(Z_1(\mathcal{D}, L)) \subset \mathrm{Ker}(F^* \circ p),$$

$$h(B_1(\mathcal{D}, L)) \subset \mathrm{Ker}(p) + \mathrm{Im}(F).$$

In particular h induces a map $\bar{h}\colon H_1(\mathcal{D}, L) \to \mathcal{H}(J)$ in homology. By duality, the map g induces a map $\bar{g}\colon \mathcal{H}(J) \to H_1(\mathcal{D}', L')$, as follows from the next lemma.

Lemma 7.3.2 *We have*

$$g(\mathrm{Ker}(F^* \circ p)) \subset Z_1(\mathcal{D}', L'),$$

$$g(\mathrm{Ker}(p) + \mathrm{Im}(F)) \subset B_1(\mathcal{D}', L').$$

Proof First of all, $Z_1(\mathcal{D}', L')$ is the orthogonal of $B_1(\mathcal{D}, L)$ for the coupling ω. Moreover, by definition of g, if $e \in \mathrm{Ker}(F^* \circ p)$, we have

$$g(e) \in Z_1(\mathcal{D}', L') \Leftrightarrow \omega(B_1(\mathcal{D}, L), g(e)) = 0$$

$$\Leftrightarrow \Omega^2(h(B_1(\mathcal{D}, L)), e) = 0.$$

The last condition is given by the previous lemma. The second point is similar. $\qquad\square$

Note that $H_1(\mathcal{D}, L)$ and $H_1(\mathcal{D}', L')$ are canonically isomorphic, so that we identify them (with $H_1(\partial M, L)$) in the following.

Theorem 7.3.1

(1) *The map $\bar{g} \circ \bar{h}$: $H_1(\partial M, L) \to H_1(\partial M, L)$ is multiplication by* 4.

(2) *Given $e \in \mathcal{H}(J)$ and $c \in H_1(\partial M, L)$, we have $\omega(c, \bar{g}(e)) = \Omega(e, \bar{h}(c))$.*

As a corollary, one understands the homology of the various complexes.

Corollary 7.3.1 *The map $\bar{h}$ induces an isomorphism from $H_1(\partial M, L)\left[\frac{1}{2}\right]$ to $\mathcal{H}(J)\left[\frac{1}{2}\right]$. Moreover we have $\bar{h}^* \Omega = -4\omega$.*

Corollary 7.3.2 *The form Ω^* on $k^\times \otimes J^* \cap \mathrm{Ker}(F^*)\left[\frac{1}{2}\right]$ is the pullback of* wp *on $H^1(\partial M, L^*)$ by the map R.*

Theorem 5.5.1 will follow from Corollary 7.3.2 and Lemma 4.1.1 (see Section 8.2 for an explicit computation). Corollary 7.3.2 is indeed the analog of Proposition 6.2.1 in the closed but non-unipotent case. We postpone the proof of Theorem 7.3.1 until the next section and, in the remaining part of this section, deduce Corollaries 7.3.1 and 7.3.2 from it.

7.4 Proof of Corollaries 7.3.1 and 7.3.2

We first compute the dimension of the spaces $\mathcal{H}(J)$ and $\mathcal{H}(J^*)$. Recall that l is the number of vertices in K.

Lemma 7.4.1 *The dimension of $\mathcal{H}(J)$ and $\mathcal{H}(J^*)$ is $4l$.*

Proof By the rank formula we have

$$\dim J^2 = \dim \mathrm{Ker}(F^* \circ p) + \dim \mathrm{Im}(F^* \circ p),$$

and by definition we have

$$\dim \mathrm{Ker}(F^* \circ p) = \dim(\mathrm{Ker}(p) + \mathrm{Im}(F)) + \dim \mathcal{H}(J).$$

We also have

$$\dim(\mathrm{Ker}(p) + \mathrm{Im}(F)) = \dim \mathrm{Ker}(p) + \dim \mathrm{Im}(F) - \dim(\mathrm{Ker}(p) \cap \mathrm{Im}(F)),$$
$$\dim \mathrm{Im}(F^* \circ p) = \dim \mathrm{Im}(F^*) - \dim(\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)).$$

The map F is injective and therefore F^* is surjective. We conclude that

$$\dim \mathrm{Im}(F) = \dim \mathrm{Im}(F^*) = \dim C_1^{\mathrm{or}} + \dim C_2.$$

But $\dim J^2 = 16N$, $\dim \mathrm{Ker}(p) = 8N$, $\dim C_2 = 2N$. Moreover, a classical computation proves that the number of edges of K is N, since the Euler characteristic of M is 0 [18]. So we have $\dim C_1^{\mathrm{or}} = 2N$. We are therefore reduced to prove that $\dim(\mathrm{Ker}(p) \cap \mathrm{Im}(F)) = 2l$. Restricted to a single tetrahedron T_μ, the kernel of p is generated by the elements

$$v_i^\mu = e_{ij}^\mu + e_{ik}^\mu + e_{il}^\mu \quad \text{and} \quad w_i^\mu = e_{ji}^\mu + e_{ki}^\mu + e_{li}^\mu + e_{ijk}^\mu + e_{ilj}^\mu + e_{ikl}^\mu$$

in $J^2(T_\mu)$ for i a vertex of T_μ (see Section 4.2).

In $\mathrm{Im}(F)$, all the coordinates of e_{ij}^μ that project on the same edge $\overline{e}_{ij}$ must be equal, as are the two coordinates of e_{ijk}^μ and $e_{ikj}^{\mu'}$ projecting on the same face. Hence, $\mathrm{Im}(F) \cap \mathrm{Ker}(p)$ is generated by the vectors $F(v_i)$ and $F(w_i)$, where

$$v_i = \sum_{\substack{\overline{e}_{ij} \text{ an edge} \\ \text{oriented away from } i}} \overline{e}_{ij}, \qquad w_i = \sum_{\substack{\overline{e}_{ji} \text{ an edge} \\ \text{oriented toward } i}} \overline{e}_{ji} + \sum_{\substack{\overline{e}_{ijk} \text{ a face} \\ \text{containing } i}} \overline{e}_{ijk}.$$

One verifies easily that these vectors are free, proving the lemma. $\qquad\square$

Since it follows from Theorem 7.3.1(1) that $\overline{h}$ has an inverse after tensoring by $\mathbb{Z}\left[\frac{1}{2}\right]$ we conclude from Lemma 7.4.1 that $\mathcal{H}(J)\left[\frac{1}{2}\right]$ and $H_1(\partial M, L)\left[\frac{1}{2}\right]$ are isomorphic. Now Theorem 7.3.1(2) implies that $\overline{h}$ and $\overline{g}$ are adjoint maps with respect to the forms ω on $H_1(\partial M, L)\left[\frac{1}{2}\right]$ and Ω on $\mathcal{H}(J)\left[\frac{1}{2}\right]$. Corollary 7.3.1 follows.

The second Corollary 7.3.2 is merely a dual statement: recall from Section 7.2 that the map R^2 is induced by the map $h^*\colon J^* \to C^1(\mathcal{D}, L^*)$ dual to h. Now the map $c' \mapsto \omega(\,\cdot\,, c')$ induces a symplectic isomorphism between $(H_1(\partial M, L'), \omega)$ and $(H^1(\partial M, L^*), \mathrm{wp})$. It therefore follows from Corollary 7.3.1 that the symplectic form Ω^* on $\mathcal{H}(J^*)$ is four times the pullback of wp by the map $\mathcal{H}(J^*) \to H^1(\partial M, L^*)$ induced by h^*. Remembering that h^* induces the *square* of R, the statement of Corollary 7.3.2 follows.

8 Homologies and symplectic forms

In this section we first prove Theorem 7.3.1 (in the closed case). We then explain how to deduce Theorem 5.5.1 from it and its Corollary 7.3.2.

8.1 Proof of Theorem 7.3.1

We first compute $g \circ h \colon C_1(\mathcal{D}, L) \to C_1(\mathcal{D}', L')$ using Equation (7.2.6).

8.1.1 A computation in a single tetrahedron We first work in a fixed tetrahedron and therefore forget about the μ's. We denote by c_{ij} the edge of $\mathcal{D}$ corresponding to a (left) turn around the edge e_{ij} and we denote by c'_{ij} its dual edge in $\mathcal{D}'$; see Figure 10. The following computations are straightforward:

$$\Omega\left(h\left(c_{ij} \otimes \binom{n}{m}\right), h\left(c_{ik} \otimes \binom{n'}{m'}\right)\right) = 2\left[\binom{n}{m}, \binom{n'}{m'}\right]$$

$$\Omega\left(h\left(c_{ij} \otimes \binom{n}{m}\right), h\left(c_{jk} \otimes \binom{n'}{m'}\right)\right) = -2\left[\binom{(n+2m)/3}{(2n+m)/3}\binom{n'}{m'}\right]$$

$$\Omega\left(h\left(c_{ij} \otimes \binom{n}{m}\right), h\left(c_{ji} \otimes \binom{n'}{m'}\right)\right) = 0$$

$$\Omega\left(h\left(c_{ij} \otimes \binom{n}{m}\right), h\left(c_{ki} \otimes \binom{n'}{m'}\right)\right) = 2\left[\binom{(n+2m)/3}{(2n+m)/3}, \binom{n'}{m'}\right]$$

and so on. Since it follows from Equation (7.2.6) that

$$\omega\left(c \otimes \binom{n'}{m'}, g \circ h\left(c_{ij} \otimes \binom{n}{m}\right)\right) = \Omega\left(h\left(c_{ij} \otimes \binom{n}{m}\right), h\left(c \otimes \binom{n'}{m'}\right)\right)$$

we conclude that the element $g \circ h\left(c_{ij} \otimes \binom{n}{m}\right)$ in $C_1(\mathcal{D}', L')$ is

$$g \circ h\left(c_{ij} \otimes \binom{n}{m}\right)$$
$$= 2(c'_{ik} - c'_{il}) \otimes \binom{n}{m} + 2(c'_{ki} - c'_{kj} + c'_{jl} - c'_{jk} + c'_{lj} - c'_{li}) \otimes \binom{(n+2m)/3}{(2n+m)/3}.$$

8.1.2 Gluing tetrahedra Let us come back to the complex. Consider a cycle $c = \sum_\mu c_{ij}^\mu$. We let the index μ be implicit in the following formulas and compute

$$g \circ h\left(c \otimes \binom{n}{m}\right) = \left(2\sum_\mu c'_{ik} - c'_{il}\right) \otimes \binom{n}{m}$$
$$+ \left(2\sum_\mu c'_{ki} - c'_{kj} + c'_{jl} - c'_{jk} + c'_{lj} - c'_{li}\right) \otimes \binom{(n+2m)/3}{(2n+m)/3}.$$

Interestingly, we are now reduced to a problem in the homology of ∂M and the lattice L does not play any role here.

8.1.3 A homological lemma Indeed, the first assertion in Theorem 7.3.1 follows
from the following lemma. The second assertion of Theorem 7.3.1 then follows from
Equation (7.2.6).

Lemma 8.1.1

- *The path* $\sum_\mu c'_{ik} - c'_{il}$ *is homologous to* $2c$ *in* $H^1(\partial M)$.

- *The path* $\sum_\mu c'_{ki} - c'_{kj} + c'_{jl} - c'_{jk} + c'_{lj} - c'_{li}$ *vanishes in* $H^1(\partial M)$.

This lemma is already proven by Neumann [16, Lemma 4.3]. The proof is a careful
inspection using Figures 12 and 13. The first point is quite easy: the path $\sum c'_{ik} - c'_{il}$ is
the boundary of a regular neighborhood of c. The second part is the "far from the cusp"
contribution in Neumann's paper. We draw on Figure 13 four tetrahedra sharing an
edge (the edges are displayed in dotted lines). The blue path is the path c in the upper
link. The collection of green paths are the relative $\sum_\mu c'_{ki} - c'_{kj} + c'_{jl} - c'_{jk} + c'_{lj} - c'_{li}$
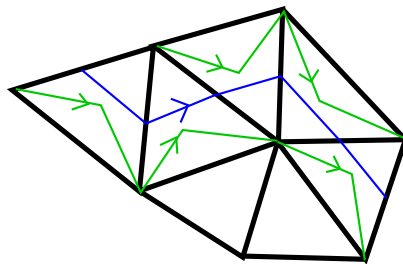in the other links. It consists in a collection of boundaries.

Figure 12: What happens inside the cusp: c in blue and $g \circ h(c)$ in green

8.2 Proof of Theorem 5.5.1 in the closed case

Theorem 5.5.1 is now a corollary. Indeed, if $z \in \mathcal{H}(J^*)$, we have from Lemma 4.1.1
and Corollary 7.3.2 that

$$3\delta(\beta(z)) = \tfrac{3}{2} z \wedge_{\Omega^*} z = \tfrac{3}{2} R(z) \wedge_{\mathrm{wp}} R(z).$$

It remains to compute the last quantity. Recall from the previous section the definition
of $R(z)$: if a loop c represents a class in homology, let $(1/C^*, 1, C)$ be the diagonal
part of its holonomy. Then $R(z)$ applied to $c \otimes \binom{n}{m}$ equals $C^m (C^*)^n$. In other terms,
denoting $[a_s]$ and $[b_s]$ the classes dual to a_s and b_s, we have (see Section 7.1.2)

$$R(z) = [a_s] \otimes \begin{pmatrix} A_s^* \\ A_s \end{pmatrix} + [b_s] \otimes \begin{pmatrix} B_s^* \\ B_s \end{pmatrix}.$$
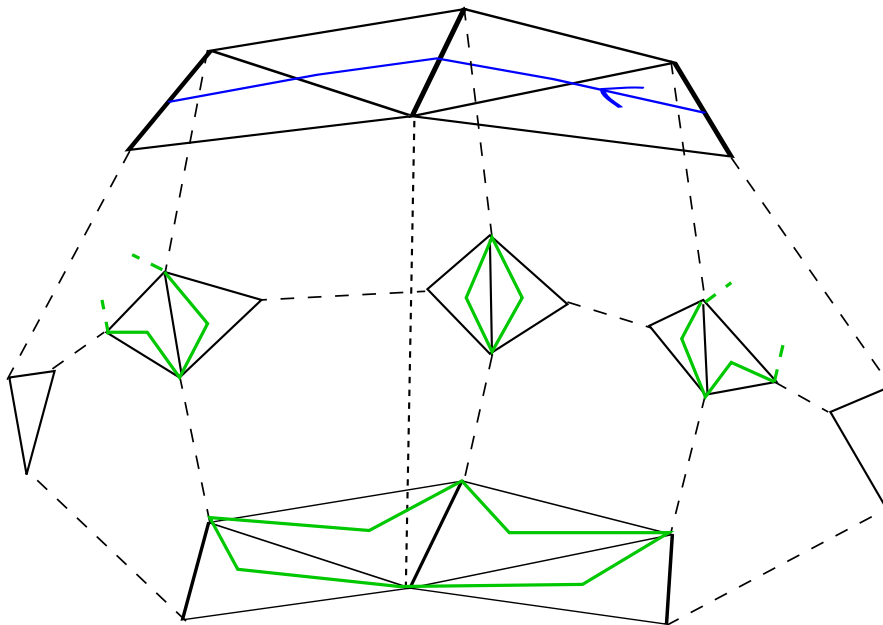
Figure 13: What happens far from the cusp

Recall from Section 7.1.2 that the form wp is the coupling of the cup product and the scalar product $\langle \cdot\, ; \cdot \rangle$ on $\mathbb{Z}^2$. Hence we conclude by

$$
\begin{aligned}
3\delta(\beta(z)) &= \sum_s 3 \big\langle \left(A_s^*, A_s \right), \left(B_s^*, B_s \right) \big\rangle \\
&= \sum_s 2A_s \wedge B_s + 2A_s^* \wedge B_s^* + A_s^* \wedge B_s + A_s \wedge B_s^*.
\end{aligned}
$$

9 Extension to the general case

We consider now the case of a complex K with boundary and explain how the preceding proof of Theorem 5.5.1 shall be adapted to deal with it. Recall that the boundary of $K - K^{(0)}$ decomposes as the union of a triangulated surface Σ and the links. The latter are further decomposed as torus links S_s and annulus links L_r. We proceed as in the closed case and indicate the modifications to be done. For simplicity we suppose that $k = \mathbb{C}$.

9.1 Coordinates for the boundary of the complex

We denote by $C_1^{\mathrm{or}} + C_2$ the $\mathbb{Z}$–module generated by *internal* (oriented) edges and faces. Recall from Section 5.5.1 that a parabolic decoration of K gives a parabolic decoration of Σ, ie, an element $z_\Sigma \in k^\times \otimes_\mathbb{Z} (J_\Sigma^2)^*[\frac{1}{2}]$, whose interpretation is that one may glue the decorated surface Σ to the decorated complex fulfilling the consistency relations. More precisely, if e_α is a basis vector of J_Σ^2, one defines the e_α^* component of z_Σ by

$$z_\alpha^\Sigma \prod_\nu z_\alpha^\nu = 1,$$

where the product is over all the e_α^ν identified with e_α. As usual we will rather consider the corresponding linear map

$$h_\Sigma \colon J_\Sigma^2 \to J^2, \quad e_\alpha \mapsto -\sum_\nu e_\alpha^\nu.$$

as well as the dual map $h_\Sigma^* \colon (J^2)^* \to (J_\Sigma^2)^*$. Note that if e_α^ν in J^2 corresponds to an *internal* edge or face then $h^*((e_\alpha^\nu)^*) = 0$ whereas if it corresponds to a boundary element $e_\alpha \in J_\Sigma^2$ we have $h^*((e_\alpha^\nu)^*) = -e_\alpha^*$. In particular one easily checks that the following diagram is commutative:

(9.1.1)
$$
\begin{array}{ccc}
J^2 & \xrightarrow{\ p\ } & (J^2)^* \\
{\scriptstyle h}\big\uparrow & & \big\downarrow{\scriptstyle h^*} \\
J_\Sigma^2 & \xrightarrow{\ p_\Sigma\ } & (J_\Sigma^2)^*
\end{array}
$$

Recall that $J_\Sigma = J_\Sigma^2 / \operatorname{Ker}(p_\Sigma)$.

The cell decomposition $\mathcal{D}$ is now defined for every cusp, each of which is either a torus or an annulus. In the latter case we may consider cycles relative to the boundary. We denote by $Z_1^{\mathrm{rel}}(\mathcal{D}, L)$, resp. $Z_1^{\mathrm{rel}}(\mathcal{D}', L')$, the subspace of relative cycles in $C_1(\mathcal{D}, L)$, resp. $C_1(\mathcal{D}', L')$. It is the orthogonal of $B_1(\mathcal{D}', L')$, resp. $B_1(\mathcal{D}, L)$, with respect to the form ω defined as above; see Section 7.1.2.

We now set

$$J_{\partial M}^2 = J_\Sigma^2 \oplus C_1(\mathcal{D}, L), \quad (J_{\partial M}^2)' = J_\Sigma^2 \oplus C_1(\mathcal{D}', L'),$$

and let

$$\Omega_{\partial M}^2 \colon J_{\partial M}^2 \times (J_{\partial M}^2)' \to \mathbb{Z}$$

be the bilinear coupling obtained as the orthogonal sum of Ω_Σ^2 and ω. As above it corresponds to these data the map $p_{\partial M} \colon J_{\partial M}^2 \to ((J_{\partial M}^2)')^*$, $p_{\partial M}(c) = \Omega_{\partial M}^2(c, \cdot)$,

as well as the spaces

$$J_{\partial M} = J^2_{\partial M} / \operatorname{Ker}(p_{\partial M}) = J_\Sigma \oplus C_1(\mathcal{D}, L),$$
$$(J'_{\partial M})^* = \operatorname{Im}(p_{\partial M}) = J^*_\Sigma \oplus C_1(\mathcal{D}', L').$$

The bilinear coupling induces a canonical perfect coupling

$$\Omega_{\partial M} \colon J_{\partial M} \times J'_{\partial M} \to \mathbb{Z},$$

which identifies $J^*_{\partial M}$ with $J'_{\partial M}$.

9.2 Complexes and homologies

As in the closed case (see Section 7.2) the linearization of the holonomy yields an extension of h_Σ to a map $h \colon J^2_{\partial M} \to J^2$. We then have the following diagram:

$$C_1^{\mathrm{or}} + C_2 \xrightarrow{\ F\ } J^2 \xrightarrow{\ p\ } (J^2)^* \xrightarrow{\ F^*\ } C_1^{\mathrm{or}} + C_2$$

$$h \Big\uparrow \qquad\qquad h^* \Big\downarrow$$

$$J^2_{\partial M} \qquad\qquad (J^2_{\partial M})$$

Now it follows from Equation (9.1.1) that the image of $h^* \circ p$ is contained in $J^*_{\partial M}$. Identifying it with $J'_{\partial M}$ using $\Omega_{\partial M}$ we get a map $g \colon J^2 \to J'_{\partial M}$. As in the closed case, for any $c \in J^2_{\partial M}$ and $e \in J^2$, we have[12]

(9.2.1) $$\Omega_{\partial M}(c, g(e)) = \Omega^2(e, h(c)).$$

We moreover have the inclusions

$$h(J^2_\Sigma \oplus Z_1^{\mathrm{rel}}(\mathcal{D}, L)) \subset \operatorname{Ker}(F^* \circ p),$$
$$h(J^2_\Sigma \oplus B_1(\mathcal{D}, L)) \subset \operatorname{Im}(F) + \operatorname{Ker}(p).$$

Denoting

$$\mathcal{H}_{\partial M} = (J^2_\Sigma \oplus Z_1^{\mathrm{rel}}(\mathcal{D}, L)) / h^{-1}(\operatorname{Im}(F) + \operatorname{Ker}(p)),$$
$$\mathcal{H}'_{\partial M} = (J_\Sigma \oplus Z_1^{\mathrm{rel}}(\mathcal{D}', L')) / g(\operatorname{Im}(F) + \operatorname{Ker}(p)),$$

we conclude that the maps h and g induce maps

$$\bar{h} \colon \mathcal{H}_{\partial M} \to \mathcal{H}(J) \quad \text{and} \quad \bar{g} \colon \mathcal{H}(J) \to \mathcal{H}'_{\partial M}.$$

It furthermore follows from Equation (9.2.1) that $\Omega_{\partial M}$ induces a bilinear coupling

$$\bar{\Omega}_{\partial M} \colon \mathcal{H}_{\partial M} \times \mathcal{H}'_{\partial M} \to \mathbb{Z}.$$

---

[12] Here we abusively use the same notation for c and its image in $J_{\partial M}$.

Lemma 9.2.1 *The bilinear coupling $\bar{\Omega}_{\partial M}$ is non-degenerate.*

Proof Denote by $\partial M \setminus \Sigma$ the union of the links (tori and annuli). The quotient J_Σ of J_Σ^2 naturally identifies with the quotient of $\text{Im}(F) + h(J_\Sigma^2)$ by $\text{Im}(F) + \text{Ker}(p)$. Note that the former identifies with the image of the $\mathbb{Z}$–module generated by *all* (oriented) edges and faces of K into J^2. We then have two short exact sequences

$$0 \longrightarrow J_\Sigma \longrightarrow \mathcal{H}_{\partial M} \longrightarrow H_1^{\text{rel}}(\partial M \setminus \Sigma, L) \longrightarrow 0,$$

$$0 \longrightarrow H_1(\partial M \setminus \Sigma, L') \longrightarrow \mathcal{H}'_{\partial M} \longrightarrow J_\Sigma \longrightarrow 0.$$

These are in duality with respect to $\Omega_{\partial M}$. Moreover this duality yields Ω_Σ on the product $J_\Sigma \times J_\Sigma$ and the intersection form, coupled with $[\cdot, \cdot]$, on $H_1^{\text{rel}}(\partial M \setminus \Sigma, L) \times H_1(\partial M \setminus \Sigma, L)$. Since both are non-degenerate, this proves the lemma. $\square$

It now follows from (9.2.1) that $\Omega_{\partial M}(\cdot, g \circ h(\cdot)) = \Omega^2(h(\cdot), h(\cdot))$. And computations similar to Section 8.1 show that the right-hand side has a trivial kernel on $\mathcal{H}_{\partial M}$. The coupling $\bar{\Omega}_{\partial M}$ being non-degenerate, we conclude that $\bar{h}$ is injective. As in the closed case, we may furthermore compute the dimension of $\mathcal{H}(J)$. Let ν_t be the number of tori and ν_a be the number of annuli. Then, computing the Euler characteristic of the double of K along Σ, the proof of Lemma 7.4.1 yields the following:

Lemma 9.2.2 *The dimension of $\mathcal{H}(J)$ is $4\nu_t + 2\nu_a + \dim(J_\Sigma)$.*

This is easily seen to be the same as both the dimensions of $\mathcal{H}_{\partial M}$ and $\mathcal{H}'_{\partial M}$; see the proof of Lemma 9.2.1. Over $\mathbb{C}$ the maps $\bar{h}$ and $\bar{g}$ are therefore invertible and we conclude that the form Ω on J induces a form $\bar{\Omega}$ on $\mathcal{H}(J)$ such that

$$\bar{\Omega}_{\partial M}(c, \bar{g}(e)) = \bar{\Omega}(e, \bar{h}(c)).$$

In particular $\bar{\Omega}$ is determined by $\Omega_{\partial M}$ and the invariant $\delta(\beta(K))$ only depends on the boundary coordinates. This concludes the proof of Theorem 5.5.1.

10 Examples

In this section we describe the complement of the figure eight knot obtained by gluing two tetrahedra. In the case of hyperbolic geometry, the discrete faithful unipotent representations into $\text{PSL}(2, \mathbb{C})$ were first obtained by Riley (see the interesting account in [19]). The gluing of tetrahedra used here is Thurston's description [21]. We will describe in this paper only the unipotent solutions of the gluing equations and leave an analysis of the full solution variety to a future paper. More details on the solutions are

described in [7] where other hyperbolic manifolds are analyzed. It turns out that all solutions that are not hyperbolic for the figure-eight knot were already obtained in [5] as they correspond to CR decorations.

Let z_{ij} and w_{ij} be the coordinates associated to the edge ij of the two tetrahedra as shown in Figure 14.
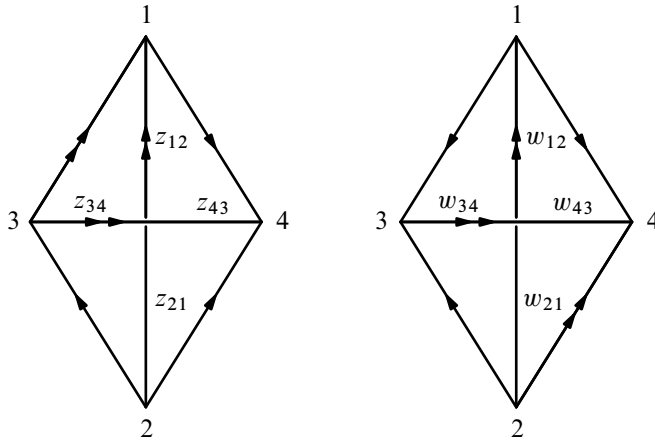

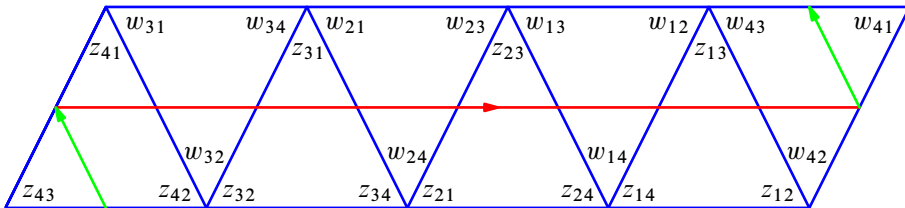Figure 14: The figure-eight knot represented by two tetrahedra

Figure 15: The link at the boundary for the figure-eight knot. The horizontal line corresponds to the generator with eigenvalues A and A^* the other line corresponds to the generator with eigenvalues B and B^*.

The edge equations are

$$z_{12}w_{12}z_{13}w_{43}z_{43}w_{42} = 1, \qquad z_{21}w_{21}z_{31}w_{34}z_{34}w_{24} = 1,$$
$$z_{42}w_{32}z_{32}w_{31}z_{41}w_{41} = 1, \qquad z_{24}w_{23}z_{23}w_{13}z_{14}w_{14} = 1.$$

The face equations are

$$z_{13}z_{43}z_{23}w_{14}w_{34}w_{24} = 1, \qquad z_{14}z_{24}z_{34}w_{21}w_{41}w_{31} = 1,$$
$$z_{12}z_{42}z_{32}w_{13}w_{43}w_{23} = 1, \qquad z_{21}z_{31}z_{41}w_{12}w_{32}w_{42} = 1.$$

And the holonomies are (see Section 7.2)

$$A = z_{41}\frac{1}{w_{32}}z_{31}\frac{1}{w_{24}}z_{23}\frac{1}{w_{14}}z_{13}\frac{1}{w_{41}}, \qquad\qquad B = z_{43}\frac{1}{w_{41}},$$

$$A^* = \frac{1}{z_{14}}\frac{w_{14}w_{41}}{w_{32}}\frac{1}{z_{13}}\frac{w_{13}w_{31}}{w_{24}}\frac{1}{z_{32}}\frac{w_{23}w_{32}}{w_{14}}\frac{1}{z_{31}}\frac{w_{31}w_{13}}{w_{42}}, \qquad B^* = \frac{1}{z_{34}}\frac{w_{23}w_{32}}{w_{41}}.$$

If $A = B = A^* = B^* = 1$, the solutions of the equations correspond to unipotent structures. The complete hyperbolic structure on the complement of the figure-eight knot determines a solution of the above equations. In fact, in that case, if $\omega = (1 + i\sqrt{3})/2$ then

$$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \omega$$

is a solution the equations as obtained in [21].

The three spherical CR decorations with unipotent boundary holonomy were obtained in [5] as the following solutions (up to complex conjugation):

$$z_{12} = \overline{z}_{21} = z_{34} = \overline{z}_{43} = w_{12} = \overline{w}_{21} = w_{34} = \overline{w}_{43} = \omega,$$

$$z_{12} = \frac{5 - i\sqrt{7}}{4}, \quad z_{21} = \frac{3 - i\sqrt{7}}{8}, \quad z_{34} = \frac{5 + i\sqrt{7}}{4}, \quad z_{43} = \frac{3 + i\sqrt{7}}{8},$$

$$w_{12} = \frac{3 - i\sqrt{7}}{8}, \quad w_{21} = \frac{5 - i\sqrt{7}}{4}, \quad w_{34} = \frac{3 + i\sqrt{7}}{8}, \quad w_{43} = \frac{5 + i\sqrt{7}}{4},$$

and

$$z_{12} = \frac{-1 + i\sqrt{7}}{4}, \quad z_{21} = \frac{3 - i\sqrt{7}}{2}, \quad z_{34} = \frac{-1 - i\sqrt{7}}{4}, \quad z_{43} = \frac{3 + i\sqrt{7}}{2},$$

$$w_{12} = \frac{3 + i\sqrt{7}}{2}, \quad w_{21} = \frac{-1 - i\sqrt{7}}{4}, \quad w_{34} = \frac{3 - i\sqrt{7}}{2}, \quad w_{43} = \frac{-1 + i\sqrt{7}}{4}.$$

The first solution above corresponds to a discrete representation of the fundamental group of the complement of the figure eight knot in $PU(2, 1)$ with faithful boundary holonomy. Moreover, its action on complex hyperbolic space has limit set the full boundary sphere. The other solutions give rise to spherical CR structures in the complement of the figure eight knot.

11 Applications

11.1 Volumes of decorated tetrahedral complex

We assume in this section that $k = \mathbb{C}$ and that K is a closed tetrahedral complex. A decoration of K provides us with an element $z \in \mathbb{C}^\times \otimes_{\mathbb{Z}} J^*\left[\frac{1}{2}\right]$ that satisfies the face

and edge conditions as well as the non-linear equations

$$z_{ik}(T_v) = \frac{1}{1 - z_{ij}(T_v)}\,.$$

Let $X = \mathbb{C}^\times \otimes_{\mathbb{Z}} J^*\big[\tfrac{1}{2}\big]$; this is a complex variety.

Following Section 3.2 we define the *volume* of K as

(11.1.1) $$\mathrm{Vol}(K) = \tfrac{1}{4} D(\beta(K)).$$

This defines a real analytic function on X

$$\mathrm{Vol}\colon X \to \mathbb{C}.$$

Let $\mathcal{F}(X)^\times$ be the group of invertible real analytic functions on X and $\Omega^1(X)$ the space of real analytic 1–form on X. The holonomy elements A_s, A_s^* and B_s, B_s^* of Theorem 5.5.1 define elements in $\mathcal{F}(X)^\times$. Now, as in Section 3.6, there is a map $\mathrm{Im}(d \log \wedge_{\mathbb{Z}} \log)\colon \mathcal{F}(X)^\times \wedge_{\mathbb{Z}} \mathcal{F}(X)^\times \to \Omega^1(X)$ defined by

$$\mathrm{Im}(d \log \wedge_{\mathbb{Z}} \log)(f \wedge_{\mathbb{Z}} g) = \mathrm{Im}\big(\log|g| \cdot d(\log f) - \log|f| \cdot d(\log g)\big).$$

Following Neumann and Zagier [18], we want to compute the variation of $\mathrm{Vol}(K)$ as we vary $z \in X$. Equivalently we compute $d\,\mathrm{Vol} \in \Omega^1(X)$ using holonomy elements:

Proposition 11.1.1 *The derivative of the volume* Vol *depends only on the holonomy elements:*

$$d\,\mathrm{Vol} = \frac{1}{12} \sum_s \mathrm{Im}(d \log \wedge_{\mathbb{Z}} \log)(2A_s \wedge_{\mathbb{Z}} B_s + 2A_s^* \wedge_{\mathbb{Z}} B_s^* + A_s^* \wedge_{\mathbb{Z}} B_s + A_s \wedge_{\mathbb{Z}} B_s^*).$$

Proof Paraphrasing Section 3.6, we can write with the notation above: $dD(\beta) = \mathrm{Im}(d \log \wedge_{\mathbb{Z}} \log)(\delta(\beta))$. Proposition 11.1.1 follows from Theorem 5.5.1 and (11.1.1). $\square$

Remark Specializing to the hyperbolic case, we recover the result of Neumann and Zagier [18]; see also Bonahon [3, Theorem 3].

11.2 Weil–Petersson forms

Let k be an arbitrary field. The Milnor group $K_2(k)$ is the cokernel of $\delta\colon \mathcal{P}(k) \to k^\times \wedge_{\mathbb{Z}} k^\times$.

Let $X_\Sigma = \mathbb{C}^\times \otimes_{\mathbb{Z}} J_\Sigma^*\big[\tfrac{1}{2}\big]$; it is a complex manifold. As above we may consider the field $\mathcal{F}(X_\Sigma)^\times$; we let $\Omega^2_{\mathrm{hol}}(X_\Sigma)$ denote the space of holomorphic 2–forms on

X_Σ. The element z_Σ defines an element in $\mathcal{F}(X_\Sigma)^\times$. We still denote the projection of $z_\Sigma \wedge_{\Omega_\Sigma^*} z_\Sigma$ into $K_2(\mathcal{F}(X_\Sigma)^\times)$.

Now, since $d \log \wedge_\mathbb{Z} d \log((1-f) \wedge_\mathbb{Z} f) = 0$, there is a group homomorphism

$$d \log \wedge_\mathbb{Z} d \log: K_2(\mathcal{F}(X)_\Sigma^\times) \to \Omega^2(X_\Sigma), \quad f \wedge_\mathbb{Z} g \mapsto d \log(f) \wedge_\mathbb{Z} d \log(g).$$

In the hyperbolic case and when the decoration is unipotent, Fock and Goncharov [9] prove that

$$\tfrac{1}{2} d \log z_\Sigma \wedge_{\Omega_\Sigma^*} d \log z_\Sigma = d \log \wedge_\mathbb{Z} d \log(W(\Sigma))$$

is the Weil–Persson form. Although expected, the analogous statement in the $\mathrm{PGL}(3, \mathbb{C})$ case seems to be open. In any case, Theorem 5.5.1 implies that this form vanishes, equivalently the "Weil–Persson forms" corresponding to the different components of Σ add up to zero.

Main symbols used

k is an arbitrary field

$\mathcal{AFl}(k)$ (or just $\mathcal{AFl}$ as k is fixed) is the space of affine flags; see Equation (2.1.1)

$\mathcal{Fl}(k)$ (or just $\mathcal{AFl}$ as k is fixed) is the space of flags; see Equation (2.1.2)

z_{ij} denotes an edge coordinate; see Equation (2.3.1)

z_{ijk} denotes a face coordinate; see Equation (2.3.2)

$\mathcal{P}(k)$ denotes the pre-Bloch group; see Section 3.1

$\beta: \mathcal{C}_3(\mathcal{Fl}) \to \mathcal{P}(\mathcal{C})$ is a "volume map"; see Equation (3.1.2). See also Equation (5.5.1)

D is the Bloch–Wigner dilogarithm; see Section 3.2

$\mathcal{B}(k)$ denotes the Bloch group; see Section 3.6

$\wedge_\mathbb{Z}$ is the skew symmetric product on Abelian groups; see Section 3.6

$\delta: \mathcal{P}(k) \to k^\times \wedge_\mathbb{Z} k^\times$ is defined in Equation (3.6.1)

J_T^2 is a 16–dimensional $\mathbb{Z}$–module associated to a tetrahedron T; see Section 4.1.1

Ω^2 is a skew-symmetric form on J_T^2; see Equation (4.1.1).

$V[\tfrac{1}{2}] := V \otimes_\mathbb{Z} \mathbb{Z}[\tfrac{1}{2}]$; see Notation 4.1.1

$\wedge_B$ denotes a bilinear product on a $k^\times$–modules; see Section 4.1.2

$p: J_T^2 \to (J_T^2)^*$ is the linear map defined by $p(v) = \Omega^2(\cdot, v)$; see Section 4.2.1

$J_T = J_T^2 / \operatorname{Ker}(\Omega^2)$ and $J_T^* = \operatorname{im}(p) \subset (J_T^2)^*$ are 8–dimensional $\mathbb{Z}$–modules; see Section 4.2.1

Ω and Ω^* are non-degenerate skew-symmetric forms on resp. J_T and J_T^*; see Section 4.2.1

K is a quasi-simplicial complex; see Section 5.1

M is a quasi-simplicial manifold triangulated by K; see Section 5.1

J_Σ^2 is a $\mathbb{Z}$–module associated to a boundary Σ of K; see Section 5.5.1

J_Σ and J_Σ^* are $\mathbb{Z}$–modules associated to a boundary Σ of K; see Section 5.5.1

z_α^Σ are coordinates associated to a boundary Σ of K; see Section 5.5.1

(J^2, Ω^2) and (J, Ω) are associated to K; see Section 6.1

C_1^{or} denotes the free $\mathbb{Z}$–module generated by the oriented internal 1–simplices of K; see Section 6.1

C_2 denotes the free $\mathbb{Z}$–module generated by the internal 2–faces of K; see Section 6.1

$F \colon C_1^{\mathrm{or}} + C_2 \to J^2$ is a linearization of the consistency relations; see Equation (6.1.1)

$\mathcal{D}$ and $\mathcal{D}'$ are simplicial complexes defined in Section 7.1.1

R is the holonomy map defined in Section 7.2.1

$h \colon C_1(\mathcal{D}, L) \to J^2$ is the linearization of R defined in Equation (7.2.1)

$g \colon J^2 \to C_1(\mathcal{D}', L')$ is a linear map defined in Equation (7.2.5)

$F' \colon C_1^{\mathrm{or}} + C_2 \to J$ is the linear map obtained by composing F with the projection $J^2 \to J$; see Section 7.3

$G \colon J \to C_1^{\mathrm{or}} + C_2$ is the map induced by $F^* \circ p$; see Section 7.3

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