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Asymptoticity of grafting and Teichmüller rays

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We show that any grafting ray in Teichmüller space determined by an arational lamination or a multicurve is (strongly) asymptotic to a Teichmüller geodesic ray. As a consequence the projection of a generic grafting ray to the moduli space is dense. We also show that the set of points in Teichmüller space obtained by integer (2π–) graftings on any hyperbolic surface projects to a dense set in the moduli space. This implies that the conformal surfaces underlying complex projective structures with any fixed Fuchsian holonomy are dense in the moduli space.

30F60; 32G15, 57M50

1 Introduction

A *complex projective structure* on a surface S_g of genus $g \geq 2$ is an atlas of charts of S_g mapping into $\mathbb{C}\mathrm{P}^1$ such that the transition maps are in $\mathrm{PSL}_2(\mathbb{C})$, and as this also determines a marked conformal structure, the space $\mathcal{P}(S_g)$ of such structures forms a bundle over the Teichmüller space $\mathcal{T}_g$. In particular, a hyperbolic structure on a surface can be thought of as a complex projective structure with Fuchsian (or real) holonomy, and the operation of *projective grafting* on a simple closed curve deforms such a complex projective structure by inserting a projective annulus along a geodesic representative of that curve. By taking limits, this procedure extends to geodesic laminations and gives a geometric parametrization of $\mathcal{P}(S_g)$ (see for example Kamishima and Tan [22], Tanigawa [33] and McMullen [29]). In this paper we shall consider *conformal grafting rays*, which are the image in $\mathcal{T}_g$ of such deformations of complex projective structures, and establish a (strong) asymptoticity with Teichmüller geodesics (Theorem 1.1) that is used to show a density result concerning the set of complex projective structures with Fuchsian holonomy (Theorem 1.5).

The conformal grafting ray determined by a pair (X, λ) of a hyperbolic surface X and a measured geodesic lamination λ shall be denoted by $\mathrm{gr}_{t\lambda} X$, and is a real-analytic one-parameter family of conformal structures obtained, roughly speaking, by cutting along λ on X and inserting a Euclidean metric whose width increases along the ray (a more precise description is given later in Section 2). Associated to a pair (X, λ) is also

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a *Teichmüller geodesic ray* starting from X and in the "direction" λ (see Definition 2.2). This paper establishes a strong asymptoticity between the two.

Two rays Θ and Ψ in $\mathcal{T}_g$ are said to be asymptotic if the *Teichmüller distance* (defined in Section 2) between them goes to zero, after reparametrizing if necessary. More concisely, $\lim\limits_{t\to\infty} \inf\limits_{Z\in\Theta} d_{\mathcal{T}}(Z,\Psi(t)) = 0$.

Theorem 1.1 *Let $X \in \mathcal{T}_g$ and let $\lambda \in \mathcal{ML}$ be arational, or a multicurve. Then there exists a $Y \in \mathcal{T}_g$ such that the grafting ray determined by (X,λ) is asymptotic to the Teichmüller ray determined by (Y,λ).*

Here a measured lamination λ is said to be *arational* when it is both *maximal* (complementary regions are all triangular) and *irrational* (has a single minimal component that is not a closed geodesic). Such laminations are in fact of full measure in $\mathcal{ML}$; we refer to Section 2 for a fuller discussion of the structure theory of geodesic laminations. In a sequel to this paper [16] we generalize Theorem 1.1 to the case of a general lamination.

The following are immediate corollaries of Theorem 1.1 and the work of Masur [26; 28].

Corollary 1.2 *Let X, Y be any two hyperbolic surfaces and let λ be a maximal uniquely ergodic lamination. Then the grafting rays determined by (X,λ) and (Y,λ) are asymptotic.*

Corollary 1.3 *For every $X \in \mathcal{T}_g$ and almost every $\lambda \in \mathcal{ML}$ in the Thurston measure, the projection of the grafting ray determined by (X,λ) is dense in the moduli space $\mathcal{M}_g$.*

Let $\mathcal{S}$ be the set of integer-weighted multicurves on S_g. As a further application of the techniques in the proof of Theorem 1.1 we show:

Theorem 1.4 *For any $X \in \mathcal{T}_g$, the set of integer graftings $\{\pi(\mathrm{gr}_{2\pi\gamma} X) \mid \gamma \in \mathcal{S}\}$ is dense in $\mathcal{M}_g$.*

The fact that such integer-grafts preserve holonomy (see Goldman [15, Section 2.14]) then implies:

Theorem 1.5 *Let $\mathcal{P}_\rho$ be the set of complex projective structures on a surface S_g with a fixed holonomy $\rho \in \mathrm{Rep}(\pi_1(\mathrm{S}_g), \mathrm{PSL}_2(\mathbb{C}))$. Then for any Fuchsian representation ρ, the projection of $\mathcal{P}_\rho$ to $\mathcal{M}_g$ has a dense image.*

In [13] Faltings had first conjectured that this projection of the set $\mathcal{P}_\rho$ is infinite, and this result can be thought of as the strongest possible affirmation of that.

The asymptotic behavior (as in Theorem 1.1) of two Teichmüller rays with respect to the Teichmüller metric is better known (see [26], Ivanov [21] and Lenzhen and Masur [25]). In [26] Masur proved that if λ is uniquely ergodic, then for any two initial surfaces $X, Y \in \mathcal{T}_g$ the Teichmüller rays determined by (X, λ) and (Y, λ) are asymptotic. The comparison of grafting rays and Teichmüller rays has been less explored, however a recent result along these lines (see also Díaz and Kim [8]) is the following "fellow-traveling" result in Choi, Dumas and Rafi [7]:

Theorem (Choi–Dumas–Rafi) *For any $X \in \mathcal{T}_g$ and any unit-length lamination λ, the grafting ray determined by (X, λ) and the Teichmüller ray determined by (X, λ) are a bounded distance apart, where the bound depends only on the injectivity radius of X.*

Theorem 1.1 makes a finer but less uniform comparison involving the stronger notion of asymptoticity defined above.

A crucial difference between grafting and Teichmüller rays is that the latter are lines of a flow whereas the former are not, that is, if one reuniformizes the surface along a grafting ray to graft again, we have

$$\mathrm{gr}_{t\lambda} \circ \mathrm{gr}_{s\lambda} \neq \mathrm{gr}_{(t+s)\lambda}\,.$$

Our results show that if one waits until t is sufficiently large, however, it approximates the Teichmüller geodesic flow. We aim to discuss this and a more quantitative version of Theorem 1.1 in forthcoming work. In [17] we study a similar asymptoticity for complex earthquakes and Teichmüller disks.

The proof of Theorem 1.1 is achieved by constructing quasiconformal maps of small dilatation from sufficiently large grafted surfaces along the grafting ray, to singular flat surfaces that lie along a common Teichmüller ray. It involves a comparison of the *Thurston metric*, a hybrid of a hyperbolic and Euclidean metric underlying a complex projective surface on one hand, and a singular flat metric induced by a holomorphic quadratic differential on the other. The case when the lamination λ is arational is handled in Section 4, and the case when the lamination is a multicurve is dealt with in Section 5. The proof of Theorem 1.4 in Section 6 is obtained by careful approximation of an arational lamination with multicurves. An outline of the strategy of the proofs of both theorems is provided in Section 3.

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2 Preliminaries

Teichmüller space $\mathcal{T}_g$ For a closed oriented genus-g surface S_g, the Teichmüller space $\mathcal{T}_g$ is the space of marked conformal (or equivalently, complex) structures on S_g with the equivalence relation of isotopy (see Imayoshi and Taniguchi [20] and Hubbard [18] for a treatment of the subject). Note that although for this article, we assume the surfaces have no punctures, the results still hold for the punctured case with a slight modification of the arguments.

The distance between two points X and Y in $\mathcal{T}_g$ in the *Teichmüller metric* is defined to be

$$d_{\mathcal{T}}(X,Y) = \frac{1}{2}\inf_f \ln K_f,$$

where $f\colon X \to Y$ is a quasiconformal homeomorphism, and K_f is its quasiconformal dilatation. The infimum is realized by the *Teichmüller map* between the surfaces.

A thorough discussion of the definitions of a quasiconformal map and its dilatation (also referred to as its quasiconformal *distortion*) is provided in the appendix. It suffices to point out here that, roughly speaking, a quasiconformal map takes infinitesimal circles on the domain to infinitesimal ellipses, and the dilatation is a measure of the maximum eccentricity of the image ellipses. The "difference" or distance between two conformal structures is then measured in terms of the dilatation of the *least* distorted quasiconformal map between them.

A consequence of the above definition is that if there exists a map $f\colon X \to Y$ which is $(1+O(\epsilon))$–quasiconformal (ie $K_f = 1 + O(\epsilon)$), then

$$d_{\mathcal{T}}(X,Y) = O(\epsilon).$$

Notation Here, and throughout this article, $O(\alpha)$ refers to a quantity bounded above by $C\alpha$, where $C > 0$ is some constant depending only on genus g (which remains fixed), the exact value of which can be determined a posteriori.

Definition 2.1 For the ease of notation, an *almost conformal map* shall refer to a map that is $(1+O(\epsilon))$–quasiconformal.

Hyperbolic surfaces and geodesic laminations Any conformal structure on a surface of genus $g \geq 2$ has a unique hyperbolic structure (a Riemannian metric of constant negative curvature -1) in its conformal class via uniformization. Thus Teichmüller space is, equivalently, the space of marked hyperbolic structures and this gives rise to a rich interaction between two-dimensional hyperbolic geometry and complex analysis.

A *geodesic lamination* on a hyperbolic surface is a closed subset of the surface which is a union of disjoint simple geodesics. A *maximal* lamination is a geodesic lamination such that any component of its complement is an ideal hyperbolic triangle. There is a rich structure theory of geodesic laminations (see Casson and Bleiler [4] for example): in particular, any geodesic lamination λ is a disjoint union of sublaminations

$$\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_m \cup \gamma_1 \cup \gamma_2 \cdots \cup \gamma_k, \tag{2-1}$$

where the λ_i are minimal components (with each half-leaf dense in the component) which consist of uncountably many geodesics (a Cantor set cross-section) and the γ_j are isolated geodesics.

A *measured* geodesic lamination is equipped with a transverse measure μ, that is, a measure on arcs transverse to the lamination which is invariant under sliding along the leaves of the lamination. It can be shown that for the support of a measured lamination the isolated leaves in (2-1) above are weighted simple closed curves (ruling out the possibility of isolated geodesics spiralling onto a closed component). We call a lamination *arational* if every simple closed curve intersects it, and it can be shown that for a *measured* lamination this condition is equivalent to being maximal and *irrational*, which is that it consists of a single minimal component and no isolated leaves. A measured lamination is *uniquely ergodic* if such a measure is unique. A maximal, uniquely ergodic lamination is necessarily arational. The set of weighted simple closed curves is dense in $\mathcal{ML}$, the space of measured geodesic laminations equipped with the weak-$*$ topology. The space $\mathcal{ML}$ also has a piecewise-linear structure and a corresponding *Thurston measure*.

Train tracks A *train track* on a surface is a graph with a labeling of incoming and outgoing half-edges at every vertex, and an assignment of (nonnegative) weights to the edges (or *branches*) that are *compatible*, such that at every vertex the sum of the weights of the incoming edges is equal to the sum of the edges of the outgoing edges. This provides a convenient combinatorial encoding of a lamination (see, for example, Fathi, Laudenbach and Poénaru [14] or Thurston [35]); in particular, for an assignment of *integer* weights, one can place a number of strands along each branch equal to the weight, and the compatibility condition ensures that these can be "hooked" together to form a multicurve.

Quadratic differentials and Teichmüller rays Any measured geodesic lamination corresponds to a unique *measured foliation* of the surface, obtained by "collapsing" the complementary components. Conversely, any measured foliation can be "tightened" to a geodesic lamination. The space of measured foliations $\mathcal{MF}$ is homeomorphic to $\mathcal{ML}$ via this correspondence (see, for example, Kapovich [23]).

A holomorphic quadratic differential ϕ (see Strebel [32] for a treatment of the subject) is locally of the form $\phi(z)dz^2$, where $\phi(z)$ is a holomorphic function, and has a vertical foliation given by the level sets of $\mathrm{Im}(\int_0^z \sqrt{\phi(z)}dz)$, which has singularities at the zeroes of $\phi(z)$. These singularities are also the cone-points in the singular flat metric given by $|\phi(z)||dz|^2$.

For a hyperbolic surface X, the map that assigns the vertical measured foliation $\mathcal{F}_v(\phi)$ to a quadratic differential $\phi(X)$ defines a homeomorphism between the space of quadratic differentials $Q(X)$ and $\mathcal{MF}$; see Hubbard and Masur [19]. Composed with the previous correspondence between measured laminations and foliations, we get a homeomorphism

$$q_L\colon Q(X) \to \mathcal{ML}.$$

Definition 2.2 A *Teichmüller ray* from a point X in $\mathcal{T}_g$ and in a direction determined by a holomorphic quadratic differential ϕ is a subset $\{X_t\}_{t\geq 0}$ of $\mathcal{T}_g$, where X_t is obtained by starting with the surface X with the horizontal and vertical foliations $\mathcal{F}_h(\phi)$ and $\mathcal{F}_v(\phi)$ and the corresponding singular flat metric, and scaling the metric along the horizontal foliation by a factor of e^t and the vertical foliation by a factor of e^{-t}.

Remark By a conformal rescaling, this is equivalent to stretching along the horizontal direction by a factor e^{2t}, and keeping the vertical direction fixed.

This ray is geodesic in the Teichmüller metric (see, for example, [20]).

By the above correspondence, we define a *Teichmüller ray determined by a pair* $(X,\lambda)\in\mathcal{T}_g\times\mathcal{ML}$ to be the Teichmüller ray starting from X in the direction determined by $q_L^{-1}(\lambda)$.

Complex projective structures and grafting An excellent exposition of this material can be found in Dumas [10], the following is a brief summary.

A *complex projective structure* on a surface S_g is a pair (dev,ρ), where $\mathrm{dev}\colon \widetilde{\mathrm{S}}_g\to\mathbb{C}\mathrm{P}^1$ is a local homeomorphism which is the *developing* map of its universal cover and $\rho\colon \pi_1(\mathrm{S}_g)\to \mathrm{PSL}_2(\mathbb{C})$ is the *holonomy* representation that satisfies

$$\mathrm{dev}\circ\gamma = \rho(\gamma)\circ\mathrm{dev}$$

for each $\gamma\in\pi_1(\mathrm{S}_g)$.

Equivalently, a complex projective structure is a collection of charts of the surface that map to $\mathbb{C}\mathrm{P}^1$ such that overlapping charts differ by a projective (or Möbius) transformation. Since Möbius transformations are holomorphic, any complex projective structure has an underlying conformal structure. This gives rise to a forgetful map from the space of complex projective structures $p\colon \mathcal{P}(\mathrm{S}_g) \to \mathcal{T}_g$.

A hyperbolic surface arises from a *Fuchsian* representation $\rho\colon \pi_1(\mathrm{S}_g) \to \mathrm{PSL}_2(\mathbb{R})$ and thus has a canonical complex projective structure. *Grafting*, introduced by Thurston (see [22; 33], Scannell and Wolf [31] and Dumas and Wolf [11] for subsequent development) can be thought of as a way to deform the Fuchsian complex projective structure.

In the case of grafting along a simple closed geodesic γ with transverse measure (weight) s, the process can be described as follows. Embed the universal cover $\widetilde{X}$ of the hyperbolic surface X as the equatorial plane in the ball-model of $\mathbb{H}^3$. The hyperbolic Gauss map to $\partial\mathbb{H}^3 = \mathbb{C}\mathrm{P}^1$ (the inverse of the *nearest point retraction*) provides the developing map of the Fuchsian structure. The curve γ lifts to a collection of geodesic lines on the equatorial plane. Now we bend along these lifts equivariantly by an angle s such that one gets a convex pleated plane. Via the Gauss map, this corresponds to inserting crescent-shaped regions of angle s along the images of the lifts of γ (in Figure 11 this is shown in the upper half-plane model where the imaginary axis is a lift of γ). This defines the image of the developing map of the new projective structure. Of course, for an angle $s \geq 2\pi$, the image wraps around $\mathbb{C}\mathrm{P}^1$, and the developing map is not injective. The new holonomy is the $\mathrm{PSL}_2(\mathbb{C})$–representation compatible with this new developing map in the sense described above.

Grafting for a *general* measured lamination λ is defined by taking the limit of a sequence of approximations of λ by weighted simple closed curves, that is, a sequence $s_i\gamma_i \to \lambda$ in $\mathcal{ML}$. For each $s_i\gamma_i$, one can form a pleated plane by equivariant bending, and a corresponding complex projective structure as above. It follows from the foundational work of Epstein and Marden [12] and Bonahon [5] that

the convex pleated planes converge in the Gromov–Hausdorff sense

which implies that

the developing maps converge uniformly on compact sets, and
the corresponding holonomy representations converge algebraically,

and so, in particular, there is a limiting complex projective structure on the surface S_g.

Thurston observed [22; 33] that the map

$$(X, \lambda) \mapsto \mathrm{Gr}_\lambda X$$

is a homeomorphism of $\mathcal{T}_g \times \mathcal{ML}$ to $\mathcal{P}_g$, where $\mathrm{Gr}_{(\cdot)}(\cdot)$ refers to the *projective* surface obtained by the above operation.

In this paper we are concerned with *conformal grafting* gr: $\mathcal{T}_g \times \mathcal{ML} \to \mathcal{T}_g$, where we consider only the conformal structure underlying the complex projective structure (so that $\mathrm{gr} = p \circ \mathrm{Gr}$ where $p\colon \mathcal{P}_g \to \mathcal{T}_g$ is the usual projection). It is known that for any fixed lamination λ, the grafting map $(X \mapsto \mathrm{gr}_\lambda X)$ is a self-homeomorphism of $\mathcal{T}_g$ [31].

Thurston metric A complex projective structure on a surface S_g determines a pair consisting of the developing map dev from the universal cover $\widetilde{\mathrm{S}}_g$ to $\mathbb{C}\mathrm{P}^1$ and the holonomy representation ρ. There is a canonical stratification of the image of dev on $\mathbb{C}\mathrm{P}^1$ (see Kulkarni and Pinkall [24]), and in particular, one can speak of maximally embedded round disks there. One can in fact recover the locally convex pleated plane (that one had in the bending description above) by taking an envelope of the convex hulls (domes) over these disks.

Definition 2.3 The *projective metric* on the universal cover $\widetilde{\mathrm{S}}_g$ is defined by pulling back the Poincaré metric on the maximal disks, via the developing map dev. The projective metric descends to the *Thurston metric* on the surface, under the quotient by the action of $\rho(\pi_1(\mathrm{S}_g))$.

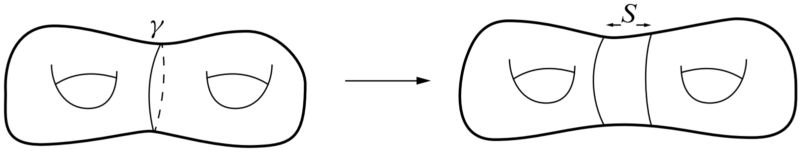

Figure 1: The grafting map along the weighted curve $s\gamma$

When γ is a simple closed curve, the equivariant collection of s–crescents obtained by bending the lifts of γ on the equatorial plane in $\mathbb{H}^3$ (see the previous section) descend to an annulus inserted at the closed geodesic γ (Figure 1). So in the Thurston metric on $\mathrm{gr}_{s\gamma} X$, the inserted annulus is flat (Euclidean), of width s, and the rest of the surface remains hyperbolic.

For a general lamination, the Thurston metric on $\mathrm{gr}_\lambda X$ is the limit of the Thurston metrics on $\mathrm{gr}_{s_i\gamma_i} X$, where $s_i\gamma_i \to \lambda$ is an approximating sequence of weighted simple closed curves (see [31]). The length in the Thurston metric on $\mathrm{gr}_\lambda X$ of an arc τ intersecting λ, is its hyperbolic length on X plus its transverse measure.

Definition 2.4 A *grafting ray* from a point X in $\mathcal{T}_g$ in the "direction" determined by a lamination λ is the 1–parameter family $\{X_t\}_{t\geq 0}$, where $X_t = \mathrm{gr}_{t\lambda}(X)$.

3 An overview of the proofs

The proofs of Theorems 1.1 and 1.5 involve understanding the geometry of the grafted surfaces $X_t = \mathrm{gr}_{t\lambda} X$ along the grafting ray determined by a pair (X, λ) for large t. By the definition of grafting, as described in Section 2, these surfaces carry a conformal metric which is Euclidean on the grafted region and hyperbolic elsewhere. A convenient way to picture this is to consider a thin "train track" neighborhood $\mathcal{T} \subset X$ that contains the lamination λ. The intuition is that as one grafts, the subsurface $\mathcal{T}$ widens in the transverse direction (along the "ties" of the train track), and conformally approaches a union of wide Euclidean rectangles.

The complement $X \setminus \lambda$ is unaffected by grafting: in the case when λ is *arational*, it comprises finitely many hyperbolic ideal triangles (the number depends only on genus) and for λ nonarational this complement might consist of ideal polygons or subsurfaces with moduli. The former case therefore is simpler and allows for more explicit constructions, and we focus on that first. For the latter case, this paper shall deal with the case when these subsurfaces have boundary components consisting of closed curves, and the general case is deferred to a subsequent paper. In either case, the underlying intuition is that this complementary hyperbolic part becomes negligible compared to the Euclidean part of the Thurston metric, for a sufficiently large grafted surface. (This has been exploited before in [9].)

3.1 The arational case

The surface $\widehat{X}_t$

For λ arational, one can consider the associated (singular) transverse horocyclic foliation which we denote by $\mathcal{F}$. This is obtained as follows. Since the lamination λ is maximal, it lifts to the universal cover of the surface to give a tessellation of the hyperbolic plane $\mathbb{H}^2$ by ideal hyperbolic triangles. Each ideal hyperbolic triangle has a partial foliation by horocyclic arcs belonging to horocycles tangent to each of the three ideal vertices (Figure 2). It can be shown (see Thurston [34]) that by some slight modification this can be extended across the ideal triangles and to the central region (missed by the horocyclic arcs) to form a foliation $\mathcal{F}$ of the surface which is transverse to λ, with "3–prong" singularities in the center of each ideal triangle, and the foliation being C^1 away from the singularities (the leaves are flowlines of a Lipschitz vector field).

The foliation $\mathcal{F}$ can be equipped with a transverse measure that is the hyperbolic distance along the leaves of λ. This measured foliation $\mathcal{F}$ persists under grafting along λ, with the leaves getting longer along a grafting ray, and the transverse measure remaining fixed.

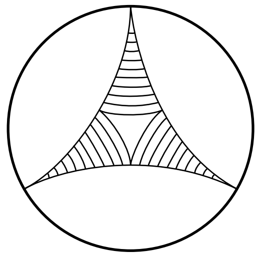

Figure 2: The partial horocyclic foliation of an ideal hyperbolic triangle; this can be modified to extend it to a singular foliation.

Then there is an associated Riemann surface $\widehat{X}_t$ with a singular flat metric obtained by collapsing the ideal triangle components of $X \setminus \lambda$ along the leaves of $\mathcal{F}$, but preserving the transverse measure of $\mathcal{F}$. The singularities of the flat metric on $\widehat{X}_t$ are 3–pronged conical singularities that arise by collapsing the central (unfoliated) region of each ideal triangle in the complement of the lamination.

There are a couple of ways one can think of the collapsed surface: one is to think of an explicit collapsing map (see [4]) that collapses the complement of the lamination (that is locally a Cantor set cross interval) by a function that is the Cantor function on each transverse cross section. The other is to think of $\widehat{X}_t$ as a singular flat surface obtained by gluing up Euclidean rectangles with the same combinatorics of the gluing as dictated by the structure of the lamination (or equivalently the corresponding train track $\mathcal{T}$) on X_t.

The singular flat surfaces $\widehat{X}_t$ have a horizontal foliation $\mathcal{F}$, and a vertical foliation that is measure-equivalent to $t\lambda$, since one keeps the vertical leaves (of λ) unchanged and scales the distance along the transverse (horizontal) direction by a factor of t. Hence as t varies the $\widehat{X}_t$ lie on a common Teichmüller ray (see Definition 2.2 and the remark following it). The "collapsing map" itself is far from being quasiconformal (it is not even a homeomorphism), and main idea behind the proof of Theorem 1.1 is to use the additional grafted region to "diffuse out" the collapsing to get a *quasiconformal* map from X_t to $\widehat{X}_t$, which is moreover almost conformal (see Definition 2.1).

Outline of the proof (arational case)

Step 1: The decomposition of X_t One first decomposes the grafted surface into rectangles and pentagonal pieces (Section 4.1) that essentially make up the train track neighborhood $\mathcal{T}$, and a slight thickening of the truncated ideal triangles in its complement, respectively. Lemmas 4.2 and 4.3 are concerned with the dimensions of the resulting pieces.

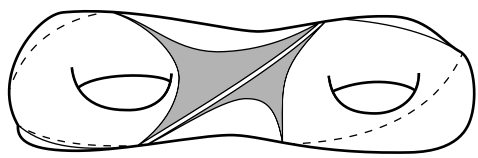

Figure 3: A partial picture of a maximal lamination, with two truncated ideal triangles in the complement shown shaded

Step 2: Mapping the pieces Next, one constructs quasiconformal maps that map the pieces in the decomposition to (singular) flat regions, by mapping the leaves of $\mathcal{F}$ in a suitable manner (Figures 3 and 4). The rectangular pieces of the grafted surface that "carries" the lamination need a finite approximation argument, which is carried out in Section 4.3, and culminates in Lemma 4.18. The maps for the pentagonal pieces (section Section 4.4) are constructed by first constructing maps of "truncated sectors." Much of these constructions depend on explicit constructions of C^1 maps of controlled dilatation that one can build between various hyperbolic or Euclidean regions, which are compiled in Section 4.2. The assumption of C^1–regularity is justified by the corresponding regularity of the Thurston metric (see Section 4.3) and the horocyclic foliation $\mathcal{F}$.

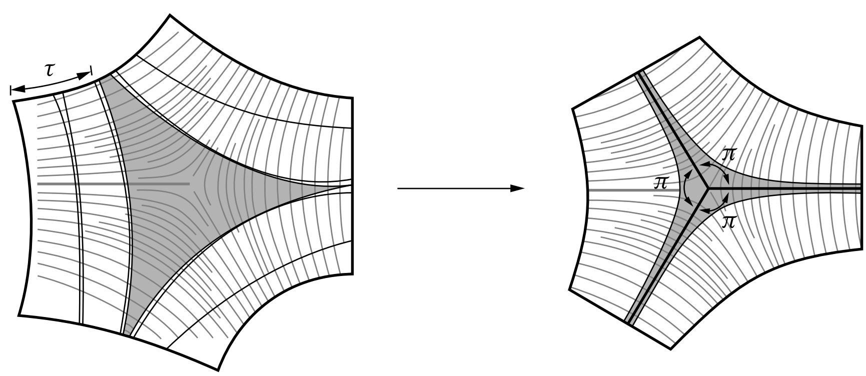

Figure 4: In Step 2, the maps of the pieces assemble to give a quasiconformal map of the portion of X_t shown on the left to the singular flat region on $\widehat{X}_t$ shown on the right. The (shaded) hyperbolic region is taken to a neighborhood of the central "tripod."

Step 3: Gluing the maps of the pieces To assemble the maps of the pieces to a map of the grafted surface X_t to the singular flat surface $\widehat{X}_t$, they need to be adjusted on the boundary. This is possible by an additional property of the maps called *almost isometry*

(Definition 4.9) which allows this adjustment to be made maintaining the almost conformality (see Lemma 4.13 in Section 4.2). At this stage one has a quasiconformal map that is almost conformal for *most* of the surface (Lemma 4.22).

Step 4: Adjusting to an almost conformal map The quasiconformal map from Lemma 4.22 is then adjusted to be almost conformal *everywhere*. This relies on the fact that the regions of no control of quasiconformal distortion are contained in portions of the surface surrounded by annuli of large modulus (Lemma 4.23), and a technical lemma on quasiconformal extensions (Lemma 4.24) whose proof we defer to the appendix.

3.2 The multicurve case

In contrast to the case of an arational (and therefore maximal) lamination, for a general lamination the complementary subsurface $X \setminus \mathcal{T}$ might contain an ideal polygon or nonsimply connected subsurface with a nontrivial parameter space (moduli) of conformal structures. Here $\mathcal{T}$ is a train track neighborhood of the lamination, as before. This subsurface remains is unaffected by the procedure of grafting.

In this article we shall consider the case when λ is a multicurve, and the general case shall be handled in a subsequent article [16]. The following is the outline of the argument for the special case when λ is a single nonseparating simple closed geodesic γ.

The surface X_t appearing along the grafting ray has a Euclidean cylinder of length t inserted at γ. As $t \to \infty$, the surface has a conformal limit X^∞, which is an "infinitely grafted" surface X^∞ obtained by gluing semi-infinite cylinders on the two boundary components of $X_t \setminus \gamma$ (see Figure 5). By abuse of notation we shall also think of X^∞ as the *compact* Riemann surface with two marked points obtained by filling in the punctures.

To find the singular flat surface $\widehat{X}_t$ to map the grafted surface X_t to, one starts by defining the singular flat surface Y^∞ that appears as the limit of the Teichmüller ray, that by the purported asymptoticity will be conformally equivalent to X^∞. This is obtained by prescribing a meromorphic quadratic differential on X^∞.

Namely, by a theorem of Strebel (Proposition 5.2), there is a meromorphic quadratic differential on X^∞ with two double poles and closed horizontal trajectories, that induces a singular flat metric which makes the surface isometric to two semi-infinite Euclidean cylinders glued along the boundary (we call this Y^∞ to distinguish this from X^∞, though they are conformally equivalent).

The singular flat surface $\widehat{X}_t$ is then obtained by truncating these infinite cylinders of Y^∞ at some "height" and gluing them along the truncating circles, and this lies

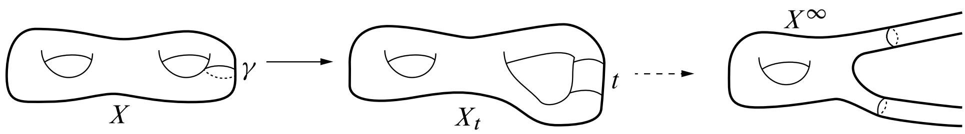

Figure 5: The surface X^∞ appears as a conformal limit as one grafts along a simple closed nonseparating curve γ.

along the Teichmüller ray determined by λ. It only remains to adjust the conformal map between X^∞ to Y^∞ to an *almost conformal* map between X_t and $\widehat{X}_t$; this is possible because a conformal map "looks" affine at small scales, or (in this case) like an isometry far out a cylindrical end (see Lemma 5.3).

3.3 Idea of the proof of Theorem 1.5

When the lamination λ is arational at least one of the weights on its train track representation is irrational. As described at the beginning of Section 3, the train track neighborhood $\mathcal{T}$ of λ widens along the grafting ray, and a typical rectangular piece (corresponding to a branch of the train track) looks more and more Euclidean, with its Euclidean width at time t equal to the initial weight times t.

A key observation is that since the switch conditions for the train track reduce to a system of linear equations with integer coefficients, there is always an assignment of *integer* weights that approximate those of the arational lamination, such that for each branch the integer weight is within a bound that depends only on the genus (Lemma 6.15). For sufficiently large t, this difference is small in proportion to the entire width, and this allows the construction of an almost conformal map from a surface along the grafting ray to a surface obtained by grafting along the multicurve corresponding to the integer solution (Lemma 6.19).

This construction together with a choice of λ that provides a *dense* grafting ray (Corollary 1.3 of Theorem 1.1) shows that integer graftings are dense in moduli space (Theorem 1.4), and Theorem 1.5 follows as they also preserve the Fuchsian holonomy.

4 The arational case

In this section we prove the following proposition, a special (and generic) case of Theorem 1.1.

Proposition 4.1 *Let $X \in \mathcal{T}_g$ and let λ be an arational (ie maximal and irrational) geodesic lamination. Then there exists a $Y \in \mathcal{T}_g$ such that the grafting ray determined by (X, λ) is asymptotic to the Teichmüller ray determined by (Y, λ).*

An outline of the proof was provided in Section 3.1, and we refer to that section for the notation used here.

4.1 A train track decomposition of the grafted surface

We begin by constructing a subsurface $\mathcal{T}_\epsilon \subset X$ containing the arational lamination λ, that is further decomposed into rectangles. This can be thought of as a physical realization of a train track carrying λ, and the rectangles correspond to the branches of the train track.

4.1.1 The return map From Section 3.1 recall that there is a *horocyclic foliation* $\mathcal{F}$ with C^1 leaves transverse to λ, obtained by integrating the Lipschitz line field along the horocyclic arcs of the ideal triangles in its complement. Choose an oriented segment τ from a leaf of $\mathcal{F}$ away from its singularities, such that the endpoints of τ are on leaves of the lamination λ which are isolated on the side away from τ. We shall choose τ so that its hyperbolic length is small enough, depending on ϵ; this shall be spelled out in Lemma 4.2.

In what follows we use the first return map of τ to itself (following leaves of λ) to form a collection of rectangles with vertical geodesic sides, and horizontal sides lying on τ. This is similar to the standard method constructing the suspension of interval exchange transformations, and in particular to the decomposition in [26] where it is done for a quadratic differential metric and the associated foliations.

The outcome of this decomposition shall be a collection of rectangles $R_1, R_2, \dots, R_n$ on the surface X whose union contains the lamination λ, and truncated ideal triangles $T_1, T_2, \dots, T_m$ in their complement.

For convenience, we lift the lamination to $\widetilde{\lambda}$ on the universal cover $\widetilde{X}$ of the surface (Figure 6), and pick $\widetilde{\tau}$, a choice of lift of the arc τ. Pick any $x \in \widetilde{\lambda} \cap \widetilde{\tau}$. The fact that λ is minimal implies that any half-leaf of $\widetilde{\lambda}$ emanating from x intersects another lift of τ. Consider such a segment l_x of a leaf of $\widetilde{\lambda}$ between the two lifts of τ. Each point of l_x is contained in a "flowbox"; a rectangle where the lamination $\widetilde{\lambda}$ is embedded as $K \times J$ for a Cantor set K and an interval J. Using the compactness of l_x, one can take a finite subcover by these flowboxes, and find a single rectangular flowbox $\widetilde{R}_x$ between the two lifts of τ that contains the entire segment l_x. This intersects $\widetilde{\tau}$ in an interval $\widetilde{I}_x$, and by using the compactness of $\widetilde{\tau}$, one can find a finite subcover $\widetilde{I}_1, \dots, \widetilde{I}_n$, and hence finitely many rectangles $\widetilde{R}_1, \dots, \widetilde{R}_n$ that contain all the leaf segments l_x where $x \in \widetilde{\lambda} \cap \widetilde{\tau}$. Here we assume that we merge two adjacent rectangles to a bigger rectangle whenever possible, so that the horizontal sides of these rectangles

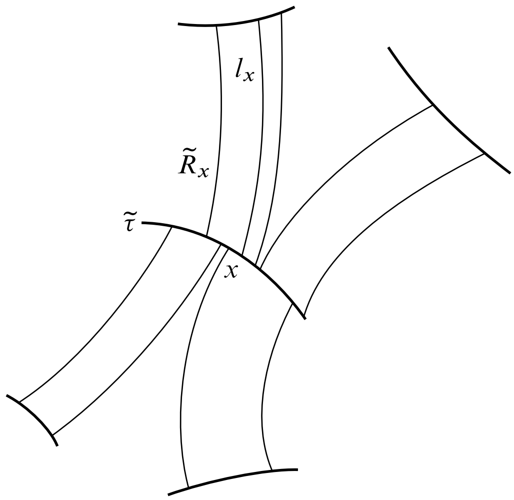

Figure 6: The lift of λ to the universal cover is decomposed into rectangular "flowboxes" by the lifts of the transversal arc τ.

are $\widetilde{\tau}$ and *distinct* lifts of τ, and we also assume that the vertical sides are geodesic segments of leaves of $\widetilde{\lambda}$.

These rectangles descend to the collection of rectangles $R_1, R_2, \dots, R_n$ on the surface X whose union contains the lamination λ. Their horizontal sides determine intervals $I_1, \dots, I_n$ on the arc τ. Since λ is arational, it has complementary regions $T_1, T_2, \dots, T_m$ which are ideal hyperbolic triangles truncated by subarcs of τ. Here n and m depend only on the genus: each rectangle is adjacent to two complementary regions, and there are exactly $4g-4$ ideal-triangular regions in the complement of λ.

We define

$$\mathcal{T}_\epsilon = R_1 \cup R_2 \cup \cdots \cup R_n. \tag{4-1}$$

Recall here that ϵ determines the choice of length of τ above, as shall be specified in Lemma 4.2.

Along the grafting ray the rectangles get wider, and one gets a decomposition of each grafted surface X_t with the same combinatorics of the gluing.

4.1.2 Labelling subarcs For later use, we denote the subarcs of the arc τ that are the horocyclic edges of $T_1, \dots, T_m$ by $J_1, \dots, J_{3m}$ (these in fact belong to the collection $\{J_i^{\pm}\}$) and the remaining subarcs in $\tau \setminus \{J_1 \cup \cdots \cup J_{3m}\}$ by $I'_1, \dots, I'_{3m+1}$ (see Figure 7). Note that for each $1 \le i \le n$ by choosing one of the two horizontal

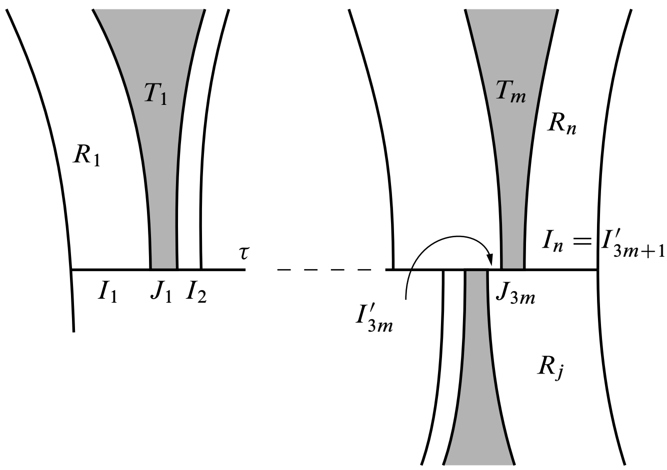

Figure 7: The train track decomposition and the labelling of the subarcs of τ

sides of R_i we can write

$$I_i = \bigcup_{k \in S_i} I'_k, \tag{4-2}$$

where S_i is a finite subset of $\{1, 2, \ldots, 3m+1\}$, and $S_i \cap S_j = \varnothing$ for $i \neq j$. (For example, in Figure 7 we have $I_j = I'_{3m} \cup I'_{3m+1}$ and $I_n = I'_{3m+1}$.)

4.1.3 Dimensions The *total width* of a rectangle R_i shall be the supremum of the lengths in the Thurston metric (see Definition 2.3) of the segments $l \cap R_i$, where l is a leaf of the transverse foliation $\mathcal{F}$.

The *hyperbolic width* of a rectangle R_i is defined to be the supremum of the hyperbolic lengths of the leaves of $\mathcal{F}$ that intersect R_i. The *Euclidean width* of a rectangle R_i is defined to be its total width *minus* its hyperbolic width. (This is also the transverse measure of the lamination λ across R_i.)

The *height* of the rectangle is the hyperbolic length of one of the vertical geodesic sides. (Since this hyperbolic length is the transverse measure of $\mathcal{F}$ that is preserved along its leaves, this induces a compatible notion of height on the other vertical side.)

Lemma 4.2 (Long, thin train track) *If the hyperbolic length of τ is sufficiently small, the height of each of the rectangles $R_1, R_2, \ldots, R_n$ is greater than $1/\epsilon^4$, and the hyperbolic width is less than ϵ.*

Proof Let the hyperbolic length of the arc τ be L. Then the two horizontal sides of a rectangle being intervals on the segment τ have length less than L. It follows from elementary hyperbolic geometry that on the (ungrafted) surface X the two vertical geodesic sides remain within L of each other, and each piecewise-horocyclic leaf of $\mathcal{F}$ between them has length at most $O(L)$. It is a fact that if the horocyclic edges of a truncated ideal triangle have length $O(L)$, then the length of each geodesic edge is at least $\ln(1/L)$. This is the height of the rectangle (see the preceding definition). Clearly $\ln(1/L) \to \infty$ as $L \to 0$, so we can choose $L < \epsilon$ small enough so that the statement of the lemma holds. □

Henceforth, we shall assume that τ was chosen short enough such that the conclusions of the above lemma hold. Our choice of lower bound of height shall be justified in Lemma 4.23.

Each rectangle has a Euclidean width equal to the transverse measure $\mu(t\lambda \cap R)$ which goes to infinity as the grafting time $t \to \infty$. Hence we also have:

Lemma 4.3 (Total width) *There is a $T > 0$ such that for all $t > T$, the total width of each rectangle in the above decomposition is greater than $1/\epsilon^4$.*

The decomposition $\mathcal{D}$ It will be useful to organize the above collection of rectangles and truncated ideal triangles into the following decomposition of a sufficiently grafted surface into rectangles and pentagons.

First decompose each ideal triangle T_j into three pentagons by including geodesic edges from its centroid p_j to the midpoints of the horocyclic sides (Figure 8). Each pentagon is thus a "$2\pi/3$–sector" of each truncated ideal hyperbolic triangle.

Each T_j is adjacent to rectangles from the collection $\{R_i\}$ on its three geodesic sides. By Lemma 4.3, for a surface sufficiently far along the grafting ray there is a grafted portion of Euclidean width much greater than 4 adjacent to each geodesic side. We thicken the pentagons by trimming subrectangles of Euclidean width 2 from the adjacent rectangles and appended to the truncated sectors of T_j (obtained above).

This trimming-and-appending results in the Euclidean (and total) widths of the rectangles $\{R_i\}$ decreasing by 4. By abuse of notation, we continue to denote these trimmed rectangles as $\{R_i\}_{1 \le i \le n}$, and we denote the pentagons (thickened sectors) as $\{P_j\}_{1 \le j \le 3m}$. These form the pieces of this new decomposition, which we shall refer to as the decomposition $\mathcal{D}$.

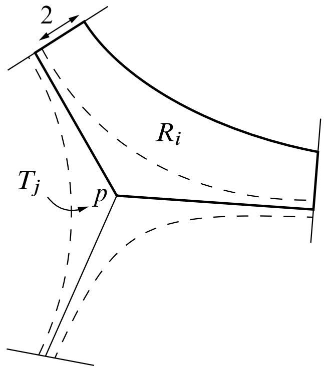

Figure 8: In the decomposition $\mathcal{D}$, each truncated ideal triangle T_j is divided into three sectors which are thickened to form pentagons, by appending a portion of the rectangle adjacent to the geodesic sides of T_j (shown dotted).

4.2 A compendium of quasiconformal maps

There are two types of pieces in the decomposition $\mathcal{D}$ of the grafted surface described in the previous section, rectangles R_i, and pentagons P_j, which can be further decomposed into truncated $2\pi/3$–sectors and width-2 rectangles. We shall eventually construct some quasiconformal maps of these pieces to the Euclidean plane which we shall glue to form a quasiconformal map of the grafted surface to the singular flat surface.

We first isolate as lemmas a few quasiconformal maps that will be useful at several steps of the actual construction. Since we would need some control on the quasiconformal dilatation of the final map, we take care to ascertain the distortion at all points of the domain.

We use repeatedly the following facts about quasiconformal maps (see, for example, Ahlfors [1]).

(A) If the partial derivatives of a C^1 map f between planar domains satisfy

$$\left|\frac{\|f_x\|}{\|f_y\|} - 1\right| < \epsilon \quad \text{and} \quad |\langle f_x, f_y\rangle| < \epsilon$$

at a point (where recall ϵ, as throughout this paper, is sufficiently small), its quasiconformal dilatation there is $1 + C\epsilon$ for some universal $C > 0$.

(B) If a map f is a homeomorphism onto its image, and it is quasiconformal on the domain except for a measure zero set (typically a collection of C^1 arcs), then f is quasiconformal everywhere on the domain.

(C) If a homeomorphism f is K–quasiconformal, so is its inverse f^{-1}.

Straightening map

Definition 4.4 An arc on the Euclidean plane is said to be ϵ–*almost vertical* if it is a portion of a graph $x = g(y)$, where g is a C^1–function and $|g'(y)| < \epsilon$.

We fix an $\epsilon > 0$ for the following discussion. The following lemma will be used in various contexts to map a rectangle with "almost vertical" sides to an actual rectangle by an almost conformal map that is also height-preserving. (Recall that an almost conformal map is $(1 + O(\epsilon))$–quasiconformal.)

Lemma 4.5 *Let R be a planar region that is bounded by two sides that are parallel horizontal line segments and two arcs which are the graphs $x = g_1(y)$ and $x = g_2(y)$ over the interval $0 \le y \le a$ on the y–axis, where g_1, g_2 are C^1–functions such that $g_2(y) > g_1(y) > 0$ for all $0 \le y \le a$. Then there is a height-preserving quasiconformal map f from R to a Euclidean rectangle of height a and width b, where $b = \sup_{0 \le y \le a} (g_2(y) - g_1(y))$. Moreover, if at any point $p \in R$ at height y we have*

(i) $|g_1'(y)| < \epsilon$

(ii) $|g_2'(y)| < \epsilon$

(iii) *the width ratio $A(y) = b/(g_2(y) - g_1(y))$ satisfies $|A(y) - 1| < \epsilon$*

then the quasiconformal distortion of f at p is $(1 + C\epsilon)$ for some universal constant $C > 0$.

Proof The map f one constructs is one that "stretches" horizontally the right amount at each height:

$$(x, y) \mapsto (A(y) \cdot (x - g_1(y)), y).$$

Computing the partial derivatives of f we get

$$f_x = \langle A(y), 0 \rangle,$$

$$f_y = \Big\langle -A(y) g_1'(y) - A(y)^2 (g_2'(y) - g_1'(y)) \Big(\frac{x - g_1(y)}{b} \Big), 1 \Big\rangle,$$

where $A(y) = b/(g_2(y) - g_1(y))$.

At a height y where the estimates (i)–(iii) hold, using them and the observation that $(x - g_1(y))/b \le 1$, we get

$$\left| \frac{\|f_x\|}{\|f_y\|} - 1 \right| < C\epsilon \quad \text{and} \quad |\langle f_x, f_y \rangle| < C\epsilon$$

for some universal constant $C > 0$, and the statement on quasiconformal distortion follows from property A above. □

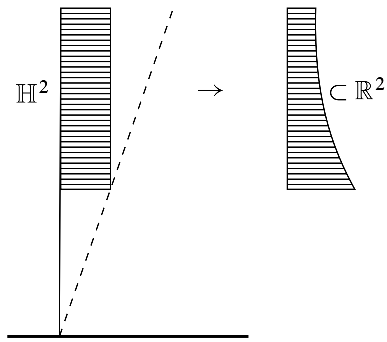

Figure 9: The map to $\mathbb{R}^2$ that straightens the horocyclic foliation of an ϵ–thin hyperbolic region (shown on the left in the upper half plane) is almost conformal (Lemma 4.6).

Maps straightening a horizontal foliation

Lemma 4.6 *Let R be a (Euclidean) rectangular region in $\mathbb{H}^2$ in the upper half-plane model bounded by two (vertical) geodesic sides and two (horizontal) horocyclic sides such that the hyperbolic width is ϵ (sufficiently small). Note that R is foliated by horocyclic segments.*

Then the map $f\colon R \to \mathbb{R}^2$ that

(i) *takes the left edge to a vertical segment preserving distance along it*

(ii) *maps each horocyclic leaf to a horizontal line*

(iii) *is distance-preserving along each horocyclic leaf*

is $(1 + C\epsilon)$–quasiconformal for some universal constant $C > 0$.

Moreover, the right edge of R is mapped to an almost vertical segment.

Proof Let R to be a rectangle in the upper-half plane model of $\mathbb{H}^2$ as described, lying above the line $y = 1$ and the left side lying on the y–axis (Figure 9). The map f is

$$(x, y) \mapsto \left(\frac{x}{y}, \ln y\right). \tag{4-3}$$

Note that the hyperbolic width of R being at most ϵ implies that

$$\left|\frac{x}{y}\right| = O(\epsilon) \tag{4-4}$$

for all points in R.

One can compute the derivatives of the above map to get the dilatation

$$\frac{\|f_x\|}{\|f_y\|} = \frac{\frac{1}{y}}{\sqrt{\frac{x^2}{y^4} + \frac{1}{y^2}}} = \frac{1}{\sqrt{\frac{x^2}{y^2} + 1}} = 1 + O(\epsilon), \tag{4-5}$$

$$|\langle f_x, f_y \rangle| = \frac{x}{y^3} = O(\epsilon), \tag{4-6}$$

by (4-4) and the fact that $y \geq 1$.

This computation also verifies that the image of the right edge is a graph over the y–axis of small derivative. □

A similar proof yields the following:

Lemma 4.7 (Inside-out version) *Let R be the region as in the previous lemma. Assume that the height of R is greater than 1. Then the map $f\colon R \to \mathbb{R}^2$ that takes the* right *edge to a vertical segment, and satisfies (ii) and (iii), is almost conformal.*

The following observation involves Euclidean regions one gets by grafting (see Figure 9 in Section 4.3).

Lemma 4.8 *Let R be the region on the upper-half plane consisting of all $z \in \mathbb{C}$ satisfying $a \leq |z| \leq b$, and $\alpha \leq \arg(z) \leq \pi/2$, for some $0 < \alpha < \pi/2$. We equip R with the metric $dz/|z|$. This region R is foliated by arcs $F_l = \{le^{i\theta} | \alpha \leq \theta \leq \pi/2\}$ for $a \leq l \leq b$. Then there is an isometry g from R to a rectangle in $\mathbb{R}^2$ of height $\ln(b/a)$ and width $\pi/2 - \alpha$, such that it maps each horizontal leaf of F to a horizontal line in a length-preserving way.*

Proof It can be checked that the conformal map $z \mapsto \pi/2 + i \ln z$ is the required isometry. □

Almost-isometries and quasiconformal extensions

Definition 4.9 Let L and L' be two intervals with a metric (eg, two line segments on the plane). Then a homeomorphism $f\colon L \to L'$ is said to be an ϵ*–almost isometry* if:

(1) f is C^1 with dilatation d (the derivative of f when L is parametrized by arclength) that satisfies $|d - 1| < \epsilon$.

(2) The lengths of any subinterval of L and its image in L' differ by an *additive* error less than ϵ.

Remark For brevity, we shall often use "ϵ–almost isometric" or just "almost isometric" to mean "$M\epsilon$–almost isometry" for some (universal) constant $M > 0$.

Here are some immediate observations, whose proofs we omit:

Lemma 4.10 *If $f\colon L \to L'$ and $g\colon L' \to L''$ are ϵ–almost isometries, then f^{-1} and $g \circ f$ are 2ϵ–almost isometries.*

Lemma 4.11 *If the difference of the lengths of the segments $|l(L) - l(L')| < \epsilon$ then the (orientation-preserving) affine map $f\colon L \to L'$ is an ϵ–almost isometry.*

Lemma 4.12 *If L is subdivided into subintervals $A_1, A_2, \ldots, A_N$ and L' into subintervals $A'_1, A'_2, \ldots, A'_N$ and the restrictions $f_{|A_i}\colon A_i \to A'_i$ are ϵ–almost isometries. Then $f\colon L \to L'$ is an $N\epsilon$–almost isometry.*

The following lemma shall be useful in our constructions.

Lemma 4.13 *Let R and R' be two planar rectangles of the same height $h > 1$ and moduli m, m' greater than 1. Suppose $f\colon \partial R \to \partial R'$ is a vertex-preserving homeomorphism that maps the left and right edges by an isometry and is an ϵ–almost isometry on the top and bottom edges. If ϵ is sufficiently small, then f can be extended to a $(1 + C\epsilon)$–quasiconformal map from R to R', where $C > 0$ is some universal constant.*

Proof Let S be a Euclidean rectangle of modulus m (recall that this equals the ratio of width by height). Then by mapping the rectangle to a unit disk and applying the Ahlfors–Beurling extension [2], any vertex-preserving piecewise-affine C^1 homeomorphism $f\colon \partial S \to \partial S$ of dilatation of the order of $1 + C\epsilon$ can be extended to a homeomorphism $f\colon S \to S$ which is $(1 + K\epsilon)$–quasiconformal, where K depends only on C and the modulus m of S. K gets worse (larger) for the modulus m very large or very small, but if m lies in a compact set, say $[1, 2]$, we get a uniform upper bound for K.

The strategy is to subdivide R into smaller rectangles of moduli between 1 and 2, and use the above fact.

Let p and q be the top and bottom corners on the left side. Choose a collection of points $p_1, p_2, \ldots, p_n$ on the top side, and $q_1, q_2, \ldots, q_n$ on the bottom such that:

(i) $l(\overline{pp_i}) = l(\overline{qq_i})$ for each $1 \leq i \leq n$.

(ii) On each side the points subdivide it into subintervals having lengths between h and $2h$; this is possible because the modulus $m > 1$.

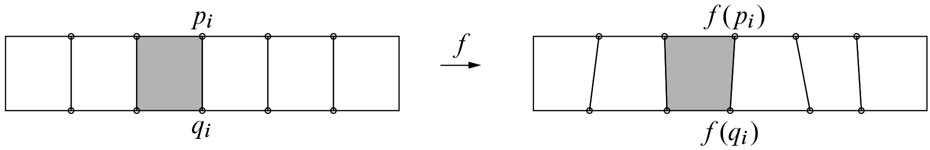

Figure 10: A boundary map that is "almost isometric" can be extended to an almost conformal map, by subdividing into smaller rectangles (Lemma 4.13).

Consider the rectangles $R_1, R_2, \ldots, R_n$ obtained by connecting each pair p_i, q_i by a straight line (Figure 10). By (ii) above, the modulus of each R_i is between 1 and 2.

Consider the images $f(p_i)$ and $f(q_i)$ on the top and bottom sides of $\partial R'$. By (i) above, property (2) of the definition of almost isometry, and the height $h > 1$, we have that the straight line joining $f(p_i)$ and $f(q_i)$ is ϵ–almost vertical for each i. We call the resulting collection of almost rectangles $R'_1, R'_2, \ldots, R'_n$. We extend the boundary map f to map each line from p_i to q_i to the line from $f(p_i)$ and $f(q_i)$ by an affine stretch (of dilatation $1 + O(\epsilon)$).

By Lemma 4.5 and a horizontal affine scaling we have an almost conformal map h from each R_i to R'_i. To correct for this map differing from the map f on ∂R_i, we consider the map $f \circ h^{-1}|_{\partial R'_i}\colon \partial R'_i \to \partial R'_i$. This has dilatation $1 + O(\epsilon)$ and can be extended to an almost conformal map g by the Ahlfors–Beurling extension (the moduli of R'_i also lie in a compact subset slightly larger than $[1, 2]$). The map $g \circ h\colon R_i \to R'_i$ agrees with f on ∂R_i.

These almost conformal maps of each R_i to R'_i piece together to give an almost conformal map of R to R'. (The property of almost conformality extends across the intermediate arcs.) □

Remark Such an almost conformal extension may fail to exist for a boundary map that is merely C^1 with small dilatation, without the additional condition (2) of Definition 4.9.

4.3 Map for a rectangular piece

Consider a typical rectangle $R = R_i$ in our decomposition of the grafted surface X_t. Such an R is bounded by geodesics on each vertical side and by leaves of the transverse horocyclic foliation on each horizontal side. This horocyclic foliation $\mathcal{F} \cap R$ gives a C^1 foliation of the rectangle. Geodesic arcs belonging to the lamination λ cut across the rectangle transverse to the foliation, and $R \setminus \lambda$ has countably many hyperbolic components, bounded by horizontal horocyclic arcs and vertical geodesic arcs. The goal of the section is to construct a quasiconformally equivalent "Euclidean" model for R.

Working in the universal cover We shall work in the universal cover $\widetilde{X}_t$ of X_t, where we consider a (fixed) lift $\widetilde{R}$ of R. Moreover we shall assume that the developing map dev: $\widetilde{X}_t \to \mathbb{C}\mathrm{P}^1$ of the complex projective structure on X_t is injective (and a homeomorphism) on $\widetilde{R}$. It is injective whenever the transverse measure across R is sufficiently small, so this condition can be ensured by subdividing R vertically. The map for R having arbitrary transverse width can then be obtained by piecing these divisions together: properties of quasiconformal extension tell us that if the map for each piece is almost conformal, so is the concatenated map.

By abuse of notation, we shall identify $\widetilde{R}$ with its homeomorphic image on $\mathbb{C}\mathrm{P}^1$, and consider it a planar domain (since it is a proper subset of $\mathbb{C}\mathrm{P}^1$ it lies in an affine chart).

The horizontal foliation $\mathcal{F}_{|R}$ lifts to the universal cover and to $\widetilde{R}$ via the developing map. We denote it by $\widetilde{\mathcal{F}}$. The Thurston metric on $R \subset X_t$ is locally isometric to the projective metric on $\widetilde{R}$ via $\mathrm{dev} \circ u^{-1}$ where u: $\widetilde{X}_t \to X_t$ is the universal covering (see Definition 2.3).

A finite approximation The developing image of the universal cover $\widetilde{X}_t$ (identified as a domain of $\mathbb{C} \subset \widehat{\mathbb{C}}$) is thought of as obtained by grafting the upper half plane identified as the universal cover of X, along the lifted measured lamination $\widetilde{\lambda}$. The grafting locus consists of a collection of infinitely many geodesics that can be approximated by a sequence of *finite* weighted subsets that produce approximations $\widetilde{X}_i$. (This can be thought of as approximating the Borel measure induced by λ on $S^1 \times S^1 \setminus \Delta$ by a sequence of sums of Dirac measures). We can further assume that these finite approximations

(i) always include the geodesics γ_l, γ_r that form the left and right edge of $\widetilde{R}$ on $\widetilde{X}$,

(ii) are maximal in the sense that the complement consists of ideal triangles.

By (ii), there is a piecewise-horocyclic foliation $\widetilde{\mathcal{F}}_i$ on each finite approximation $\widetilde{X}_i$.

Notation In what follows, given a subarc s of a leaf of $\mathcal{F}$, we shall denote its *hyperbolic* length as $l_h(s)$, its *total* length in the Thurston metric as $l(s)$, and its *Euclidean* length as $l_e(s)$, which is defined as the difference $l(s) - l_h(s)$.

Let $s \subset \gamma_l$ denote the left edge of $\widetilde{R}$. Then by (i) we can define a rectangle $\widetilde{R}_i$ on $\widetilde{X}_i$ as having s as the left edge, leaves of the foliation $\widetilde{\mathcal{F}}_i$ as the two horizontal edges, and a segment on γ_r as the right edge. Note that for all i, we can define the leaf $l_i^y \in \widetilde{\mathcal{F}}_i \cap \widetilde{R}_i$ at "height y" to be the leaf intersecting s at a distance of y from the lower endpoint of s. We also define l^y to be the leaf of $\widetilde{\mathcal{F}} \cap \widetilde{R}$ at height y.

Remark Since the holonomy along $\widetilde{\mathcal{F}}_i$ (and $\widetilde{\mathcal{F}}$) preserves the hyperbolic length along leaves of $\widetilde{X}_i$, this definition of "height" is well defined, that is, the leaf l_i^y (and l^y) are at height y all along $\widetilde{R}_i$ (and $\widetilde{R}$).

Definition 4.14 A *conformal metric* on a Riemann surface Σ is a metric given by $\rho(z)|dz|^2$ in each local coordinate chart, for some function ρ (the *conformal factor*) on Σ. It is said to be of class $C^{1,1}$ if ρ is differentiable with Lipschitz derivatives, in which case its $C^{1,1}$*–norm* is defined to be $\|\rho\|_{1,1}$. A family of conformal metrics are said to *converge pointwise* if the corresponding conformal factors converge pointwise.

The following lemma states the known convergence results that have also been mentioned in Section 2 while describing grafting for general laminations.

Lemma 4.15 *For the above sequence of finite approximations $\widetilde{X}_i$ the following are true.*

(i) *The Thurston (or projective) metric on $\widetilde{X}_i$ is a conformal metric of class $C^{1,1}$. They converge pointwise to the Thurston metric on $\widetilde{X}_t$. Moreover, the $C^{1,1}$ norms remain bounded as $i \to \infty$.*

(ii) *The horocyclic foliations $\widetilde{\mathcal{F}}_i \to \widetilde{\mathcal{F}}$ in the sense that for all $0 \le y \le l(s)$ we have $l_i^y \to l^y$ in the Hausdorff metric on compact subsets of $\mathbb{C}$.*

(iii) *The rectangles $\widetilde{R}_i \to \widetilde{R}$ in the Hausdorff metric on compact subsets of $\mathbb{C}$.*

Proof Part (i) involving the regularity of the Thurston metrics is a standard result (see [24] or [31, Lemma 2.3.1]). Part (ii) follows from part (i) and the "bending" description of grafting (see Section 2): in our choice of approximates the corresponding locally convex pleated planes corresponding to $\widetilde{F}_i$ converge in the Gromov–Hausdorff sense to the pleated plane corresponding to $\widetilde{F}$ by work of Epstein and Marden [12] and Bonahon [5], and the developing maps (which are the hyperbolic Gauss maps from these pleated planes) converge uniformly on compact sets. This implies that as $i \to \infty$ the leaf segments l_i^y on $\widetilde{R}_i$ converge to *some* horizontal leaf segment of $\widetilde{R}$ in the Hausdorff metric, and part (i) implies that the height of the latter is also y.

Part (iii) is now an immediate consequence, since the rectangles $\widetilde{R}_i$ are defined in terms of the vertical left edge γ_l and the segments l_i^y for $0 \le y \le l(s)$, which converge to those of $\widetilde{R}$. □

Map for the finite approximation

Lemma 4.16 *If the hyperbolic width of $\widetilde{R}_i$ is ϵ (sufficiently small) then there exists a $(1+C\epsilon)$–quasiconformal map $\widetilde{f}_i$ from $\widetilde{R}_i$ to the Euclidean plane, for some universal constant $C>0$, that satisfies the following.*

(i) *It takes the lower endpoint of the left vertical geodesic side s to the origin.*

(ii) *It is an isometry of s onto a segment on the y–axis.*

(iii) *Each leaf of $\widetilde{\mathcal{F}} \cap \widetilde{R}_i$ is mapped to a horizontal line in a length-preserving way.*

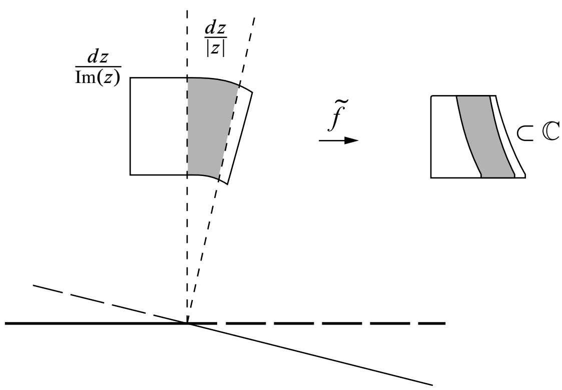

Figure 11: The map for the finite approximation case; the figure on the left shows a rectangle $\widetilde{R}_i$: the hyperbolic part (unshaded) is mapped by the straightening map of Lemma 4.6, and the Euclidean part (shown shaded) is then spliced in by the map from Lemma 4.8.

Proof The map $\widetilde{f}_i$ is uniquely determined and injective by conditions (i)–(iii), and is also C^1 since the grafted metrics are C^1 and the foliation is C^1, the horizontal leaves being integral curves of a nowhere-zero Lipschitz vector field. In particular it is a quasiconformal map, and it remains to show that the dilatation is $1+O(\epsilon)$, and it is enough to check that almost everywhere.

Recall that the projective metric (Definition 2.3) is the Poincaré metric on the maximal disk at every point. If one starts with a maximal disk in the upper half plane model with metric $dz/\,\mathrm{Im}(z)$, and the $x=0$ axis is the lift of the grafting curve, then grafting introduces a sector with the (Euclidean) metric $dz/|z|$ (Figure 11).

The rectangle $\widetilde{R}_i$ consists of regions that alternately lie in the hyperbolic part (of width $O(\epsilon)$) and the Euclidean part (see the figure) with finitely many separating geodesic arcs.

Collapse the Euclidean regions of $\widetilde{R}_i$ to get a rectangle $\widetilde{R}'$ of hyperbolic width $O(\epsilon)$. By Lemma 4.6 there is a map f_0 of $\widetilde{R}'$ to $\mathbb{R}^2$ satisfying (i)–(iii) above. Note that the proof of Lemma 4.6 shows that separating geodesic arcs are mapped to almost vertical arcs on the plane.

Starting with f_0 we now inductively splice in each Euclidean region to $\widetilde{R}'$ and extend the map already constructed, to the larger domain, such that (iii) is satisfied ((i) and (ii) are automatically satisfied for all these extensions). By the above observation on the interface arcs being almost vertical, and Lemma 4.8, these extensions are almost conformal on each region.

Since the interface arcs are of measure zero, the final map $f_n = \widetilde{f}_i$ thus constructed is $(1 + O(\epsilon))$–quasiconformal almost everywhere, as required. □

Taking a limit By the above lemma we now have a sequence

$$\widetilde{f}_i \colon \widetilde{R}_i \to \mathbb{R}^2$$

of almost conformal maps.

Lemma 4.17 *The maps $\widetilde{f}_i$ converge uniformly to an almost conformal map $\widetilde{f} \colon \widetilde{R} \to \mathbb{R}^2$ that satisfies the conditions (i)–(iii) in the above lemma.*

Proof The uniform convergence follows from parts (i) and (ii) of Lemma 4.15: part (ii) says the each leaf of $\widetilde{R}_i$ converges *as a set* to the corresponding leaf of $\widetilde{R}$, and by part (i) the *lengths* also converge, and then the uniform convergence follows from the definition of f (distance preserving along the leaves). To get the statement on almost conformality we employ a trick of considering the sequence of *inverse* maps $\widetilde{g}_i = \widetilde{f}_i^{-1}$. For an arbitrary $x \in \widetilde{R} \setminus \partial \widetilde{R}$ there is, by part (iii) of Lemma 4.15, an open neighborhood $\widetilde{U} \subset \mathbb{R}^2$ containing $\widetilde{f}(x)$, such that $\widetilde{U}$ is contained in $\widetilde{f}_i(\widetilde{R}_i)$ for sufficiently large i. The sequence $\widetilde{g}_i|_{\widetilde{U}}$ are a uniformly converging sequence of $(1 + O(\epsilon))$–quasiconformal maps from a fixed domain $\widetilde{U}$ to $\mathbb{C}$. The limit $\widetilde{g} = \widetilde{f}^{-1}$ is hence $(1 + O(\epsilon))$–quasiconformal on $\widetilde{U}$, and so is $\widetilde{f} = \widetilde{g}^{-1}$ in a neighborhood of x.

Note that f is height-preserving: this follows from parts (ii) and (iii) of Lemma 4.16 (which are preserved in the limit). □

The almost conformal model By the previous lemma we have obtained an almost conformal map $\widetilde{f}$ from $\widetilde{R}$ to $\mathbb{R}^2$. Together with the local isometry $\mathrm{dev} \circ p^{-1}$ this gives an almost conformal map of $R \subset X_t$ to a planar domain. We conclude the construction of a quasiconformal model for R by noting that this planar domain can be almost conformally straightened to a rectangle.

Notation Recall that the total width W of the rectangle R is the supremum of the lengths of the leaves of $\mathcal{F} \cap R$, and the Euclidean width W_e of R is the supremum of the *Euclidean* lengths of the leaves of $\mathcal{F} \cap R$.

By construction (see Lemma 4.2), the hyperbolic width of R is less than ϵ, and its height h is greater than 1. We also assume that its total width W is greater than 1, that is, one has grafted enough for Lemma 4.3 to hold.

Lemma 4.18 (Map for a rectangle) *With the above assumptions, there exists a height-preserving $(1+C\epsilon)$–quasiconformal map $\bar{f}$ from R to a Euclidean rectangle of width W_e and height h. (Here $C > 0$ is some universal constant.)*

Proof By the previous lemma we have a height-preserving almost conformal map $f = \tilde{f} \circ \mathrm{dev} \circ p^{-1}$ from R to a planar region D. The Euclidean width $W_e(y)$ (total minus hyperbolic) of a leaf at height y is a constant independent of height (it only depends on the total transverse measure of R). Since the hyperbolic width $W_h(y)$ at height y is $O(\epsilon)$ (Lemma 4.2), $W/W(y) = 1 + O(\epsilon)$ where $W(y) = W_h(y) + W_e(y)$ is the total width of D (and also of R) at height y. Since the two vertical sides of R are geodesic segments of length $h > 1$ on the (ungrafted) hyperbolic surface at distance $O(\epsilon)$, it follows from hyperbolic geometry that $|dW_h(y)/dy| = O(\epsilon)$. Hence by an application of Lemma 4.5 we obtain a height-preserving almost conformal map from D to a Euclidean rectangle of width W_e, and the required map $\bar{f}$ from R to W_e is obtained by precomposing this with f. □

Corollary 4.19 (Almost-isometric) *Let L and $\bar{L}$ be the top and bottom edges of R. The map $\bar{f}$ constructed above is an ϵ–almost isometry on L and $\bar{L}$, and isometric on the other two sides.*

Proof Consider the top edge L. The map of Lemma 4.17 is an isometry of L and the map (from Lemma 4.5) used in straightening step in the proof of the previous lemma is affine on horizontal lines, hence the composition (the map $\bar{f}$) is affine on L. Since $l(L) = W(h)$ and $l(\bar{f}(L)) = W_e$ differ by $O(\epsilon)$ we can apply Lemma 4.11 to conclude that $\bar{f}$ is an ϵ–almost isometry on L. The proof for the bottom edge $\bar{L}$ is identical. The property of being "height-preserving" implies that $\bar{f}$ is isometric on the vertical (left and right) sides of R. □

4.4 Map for a pentagonal piece

The purpose of this section is to define a quasiconformal map of each pentagonal piece P of the decomposition $\mathcal{D}$ (Section 4.1.3) to a planar region that is almost conformal for "most" of P (Lemma 4.21). The map of is obtained by "straightening" leaves of a foliation through P. These maps together with those from the previous section, will be assembled in Section 4.5 to define the map of the grafted surface.

The foliation $\mathcal{F}$ An ideal hyperbolic triangle has a partial foliation by horocyclic arcs which restricts to give a partial foliation of the "$2\pi/3$–sector" $\widehat{S}$ (Figure 12). When realized in the upper half plane $\mathbb{H}^2$ with γ a vertical geodesic and p the point $i\sqrt{3}/2$, the leaves of *nonnegative height* are the horizontal segments starting at $y = 1$. In general, the *height* of a leaf is the logarithm of its y–coordinate of the point where it intersects γ. We shall work with a horizontal foliation (that we continue to denote by $\mathcal{F}$) that extends the horocyclic foliation to the whole of $\widehat{S}$ as follows.

Let a be the geodesic arc from p which is orthogonal to γ. This divides $\widehat{S}$ into two parts, and on one of them we define the modified C^1 foliation to be one that

(1) agrees with the horocyclic foliation for height greater than $D = \ln(1/\epsilon)$ (when the leaves have width less than ϵ),

(2) interpolates between the leaf at height D and the arc a in such a way that the lengths of the leaves is a decreasing C^1 function of (nonnegative) height,

(3) has each leaf orthogonal to γ.

The foliation on the other part of $\widehat{S}$ is obtained by reflecting across a. The above length function is C^1 except at height 0.

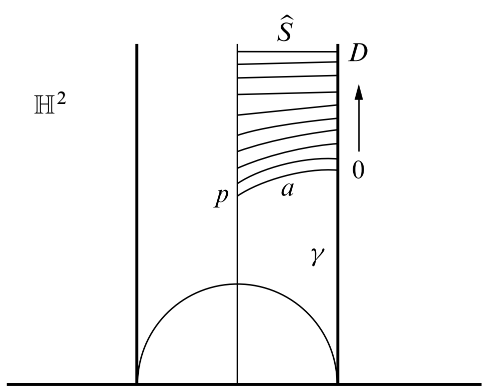

Figure 12: The map for the sector $\widehat{S}$ in Lemma 4.20 straightens the interpolating foliation between a and the horocyclic leaf at height D, and symmetrically for the other half.

Constructing the map Let S_H denote a "truncated $2\pi/3$–sector" truncated at leaves of $\widehat{\mathcal{F}}$ at height H (we take $H > D = \ln(1/\epsilon)$). Recall from Section 4.1.3 that a pentagonal piece P is made by appending a strip S of the adjacent grafted rectangles of (Euclidean) width 2 to the geodesic side of such an S_H coming one of the ideal hyperbolic triangles from the collection $\{T_j\}_{1\le j\le m}$. We have

$$P = S_H \cup S \quad \text{and} \quad S_H \cap S = \gamma.$$

Condition (3) above ensures that this foliation $\mathcal{F}$ on $\widehat{S}$ matches in a C^1 way with the partial foliation $\mathcal{F}$ on the grafted rectangles on the other side of γ, producing a "horizontal" foliation of P. Each leaf of $\mathcal{F}$ in P has a height (in the interval $[-H, H]$) obtained by following it to meet the geodesic γ, and considering the height of that point of intersection.

We define the map of the pentagonal piece P by defining it on the two pieces S_H and S: the next lemma deals with the former, and for the piece S we already have Lemma 4.18, these are put together in Lemma 4.21.

Lemma 4.20 (Straightening S_H) *There is a quasiconformal embedding $f\colon S_H \to \mathbb{R}^2$ such that:*

(i) *Each leaf of $\mathcal{F}$ is mapped isometrically to a horizontal line segment.*

(ii) *For all y, the left endpoint of the image of the leaf at height y is $(0, y)$.*

(iii) *The quasiconformal distortion of f is $1 + O(\epsilon)$ at all points of S_H at height $|h| > D = \ln(1/\epsilon)$.*

(iv) *γ is mapped to a graph of a function g over the y–axis that is C^1 except at 0, and is almost vertical and $|g(y)| = O(\epsilon)$ at points with $|y| > D$; moreover, $\sup_y g(y) = g(0) < 1$.*

Proof We map the part S_H^+ above a (of nonnegative height) and extend to the whole of S_H by reflection. We denote this map of S_H^+ by f_+.

Note that (i) and (ii) uniquely determines f_+, and ensures it is injective. The fact that the foliation is C^1 and the lengths of the leaves is C^1 in (positive) height ensures that f_+ is C^1, and is hence a homeomorphism to its image. (iii) follows from Lemma 4.7.

The fact that f_+ is isometric on the leaves implies that the function g that describes the image of γ as a graph is the length-function of the leaves. This is C^1, except at 0, by property (2) of the foliation $\mathcal{F}$. Part (iv) again follows from Lemma 4.7 (a calculation similar to Lemma 4.6). In fact, $|g(y)| \to 0$ exponentially as $|y| \to \infty$. The last statement of the lemma follows from the fact that the length function was chosen to be decreasing for increasing height (or decreasing height, by reflection) and the length at height 0 is the length of the geodesic arc a, which is $\ln(\sqrt{3}) \approx 0.54$. □

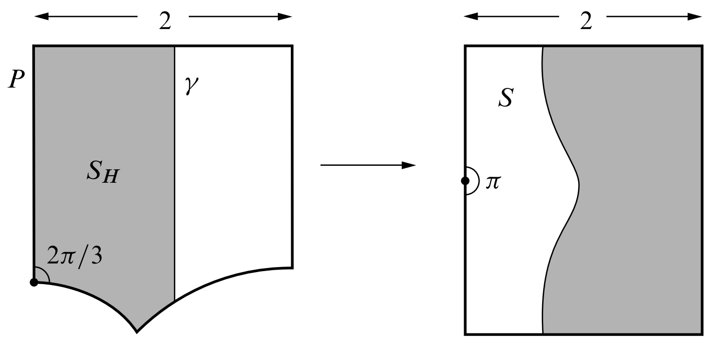

Figure 13: The map for a pentagonal piece (shown on the left) to a Euclidean rectangle; the region S_H to the left of γ is mapped by Lemma 4.20 and the region S to its right by Lemma 4.18, adjusted by an application of Lemma 4.5. The map on γ is the same as they both preserve height.

Lemma 4.21 (Map of a pentagonal piece) *There exists a quasiconformal map f from P to a Euclidean rectangle of height $2H$ and width 2, such that:*

(i) *f is height-preserving.*

(ii) *On the top and bottom sides, f is ϵ–almost isometric.*

(iii) *The quasiconformal distortion is $1 + O(\epsilon)$ on points of P outside of a D–neighborhood of the $2\pi/3$–angled vertex of P (recall $D = \ln(1/\epsilon)$).*

Proof The map f restricted to the truncated sector $S_H \subset P$ shall be the map from the previous lemma (Lemma 4.20); see Figure 13. Note that f satisfies (i), (ii) and (iii) on S_H. In fact it is isometric on the top and bottom horocyclic edges, which is stronger than (ii). Recall also that this map sends γ to a graph of a function g that satisfies:

(1) It is C^1 except at one point.

(2) It is almost vertical for height more than D.

(3) $|g(y)| = O(\epsilon)$ for $|y| > D$.

(4) $\sup_y g(y) = g(0) < 1$.

We can consider this image of γ as the graph of the function $w(y) = 2 - g(y)$ on a segment on the vertical line $y = 2$ (by abuse of orientation the "positive" side of the vertical segment is to its left). Now by Lemma 4.18 we map the grafted rectangle S to a Euclidean rectangle S' of height $2H$ and width 2 by an almost conformal height-preserving map f_1. By Lemma 4.5 there is a height-preserving quasiconformal map f_2

of this rectangle to the planar region bounded by a vertical segment on the line $y = 2$ and the graph of $w(y)$ over this segment (which lies to its left).

The map f restricted to $S \subset P$ shall be the composition $f_2 \circ f_1$. By properties (2) and (3) of the image of γ, the appropriate conditions of Lemma 4.5 are satisfied and f_2 is almost conformal for all points of height more than D. Since f_1 is almost conformal everywhere, the composition satisfies (iii). Since both f_1 and f_2 are height-preserving, (i) is satisfied. Finally, it can be checked that (ii) holds because of Corollary 4.19 and Lemmas 4.10 and 4.11.

Thus the map f is defined on S_H and S, and hence on their union P (since it is height-preserving on both they match along γ). Another application of Lemma 4.11 implies that it satisfies (ii). The usual property of quasiconformal maps (Property B at the beginning of Section 4.2) implies that it is quasiconformal everywhere, and (iii) is satisfied since it is satisfied on both S_H and S. □

4.5 Mapping the grafted surface

In this section we shall use the decomposition $\mathcal{D}$ of X_t into pentagons $\{P_j\}_{1\leq j\leq 3m}$ and rectangles $\{R_i\}_{1\leq i\leq n}$, as described at the end of Section 4.1. (We shall assume t is large enough such that the decomposition exists.)

Recall from Section 3.1 that $\widehat{X}_t$ is the singular flat surface obtained by collapsing the ideal triangle components of $X_t \setminus \lambda$ along the leaves of $\mathcal{F}$ and preserving the transverse measure (one could use a Cantor function, as in [4]). Recall that the triangular regions in the complement of the partial foliation $\mathcal{F}$ collapse to the singularities (cone points of angle 3π).

The pentagons in the decomposition of X_t are mapped by the collapsing map to Euclidean rectangles $\{S_j\}$ (of Euclidean width 2), which together with the collapsed images $\{R_i'\}$ of the trimmed-rectangles form a rectangular decomposition of $\widehat{X}_t$ with the same combinatorics as $\mathcal{D}$.

Let K denote the 1–skeleton of the decomposition $\mathcal{D}$ of X_t, and let $\widehat{K}$ the corresponding 1–skeleton on the singular flat surface $\widehat{X}_t$. An edge $E \subset K$ is either *horizontal* (if it is formed of segments of the horizontal foliation) or *vertical*, and we denote by K_H the collection of horizontal edges.

When the hyperbolic part is collapsed, a segment of a leaf of $\mathcal{F}$ of total width W collapses to a segment of width W_e (which, recall, is the *Euclidean* width of the segment). In particular, since the hyperbolic width is $O(\epsilon)$ for every horizontal edge E, the corresponding edge $\widehat{E}$ of $\widehat{K}$ differs in length by $O(\epsilon)$.

For each pentagonal piece P_j consider the map $f_j^P\colon P_j \to S_j$ which is the map $\overline{f}$ obtained from Lemma 4.21. Let $\overline{f}_j^P$ denote the restriction of the map to the horizontal edges of ∂P_j. These are ϵ–almost isometric (part (ii) of Lemma 4.21). Choose a map $\overline{g}\colon K_H \to \widehat{K}_H$ that is equal to $\overline{f}_j^P$ for each horizontal edge of a pentagonal piece, and affine on each remaining horizontal edge. By the above observation on the difference of lengths of the edges, and Lemma 4.11 and Lemma 4.12, $\overline{g}$ satisfies an $M\epsilon$–almost isometry condition on each horizontal edge of the 1–skeleton for some M that depends on the genus (the number of subedges of each horizontal edge is determined by the number of rectangles and pentagons in the decomposition that depends only on the genus). We can henceforth absorb that constant in the $O(\epsilon)$ term in the definition of almost isometry, and refer to the above map as being ϵ–almost isometric.

For each rectangle R_i in the decomposition, consider the map $f_i^R\colon R_i \to R_i'$ obtained from Lemma 4.18. Let $\overline{f}_i^R$ denote the restriction of the map to ∂R_i. Choose a map $\overline{h}_i\colon \partial R_i \to \partial R_i$ that is isometric on the vertical edges and agrees with $\overline{g} \circ (\overline{f}_i^R)^{-1}$ on the horizontal edges. By Corollary 4.19, and Lemma 4.10, this map is ϵ–almost isometric on the horizontal edges. The modulus of R_i' is greater than 1 by Lemma 4.3, so we can apply Lemma 4.13 and extend $\overline{h}_i$ to an almost conformal self-map h_i of R_i'. The map $h_i \circ f_i^R$ now maps R_i to R_i' such that the map agrees with $\overline{g}$ on K_H.

The maps $\{h_i \circ f_i^R\}$ and $\{f_i^P\}$ agree on each vertical edge E since they are height-preserving. Hence these maps agree on the 1–skeleton K and form a continuous map from X_t to $\widehat{X}_t$ that is quasiconformal on each piece P_i and R_i, and is hence quasiconformal everywhere.

Each map from the collection $\{h_i \circ f_i^R\}$ is almost conformal (since h_i is almost conformal from the above construction, and f_i^R is almost conformal by Lemma 4.18). Each map f_i^P is almost conformal away from a set of diameter $O(\ln(1/\epsilon))$, by part (iii) of Lemma 4.21.

We summarize this discussion in the following lemma.

Lemma 4.22 (Map of grafted surface) *There exists a $T > 0$ such that for all $t > T$ there is a quasiconformal homeomorphism $f\colon X_t \to \widehat{X}_t$ such that the quasiconformal distortion is $1 + O(\epsilon)$ away from finitely many simply connected subsets $K_1, K_2, \dots, K_m \subset X_t$ of diameter $O(\ln(1/\epsilon))$ in the Thurston metric. Moreover, each subset K_j is contained in a simply connected disk D_j of diameter at least $1/\epsilon^4$.*

Proof The construction of f was discussed above. Each $2\pi/3$–angled vertex of a pentagonal piece in the decomposition $\mathcal{D}$ is a centre of an ideal triangle complement of the lamination λ (in fact each such center is common to three pentagons). The subsets $K_1, K_2, \dots, K_m \subset X_t$ in the lemma (see Figure 14) are the $O(\ln(1/\epsilon))$–neighborhoods of the finitely many centres of the ideal triangle complements of λ, by part (iii) of Lemma 4.21. The last statement follows from Lemmas 4.2 and 4.3: the three rectangles adjacent to a truncated ideal triangle T_j are of height and width at least $1/\epsilon^4$ when the grafting time $t > T$ is sufficiently large, hence one can embed a disk D_j of diameter $1/\epsilon^4$ centered at the center of T_j, containing K_j. □

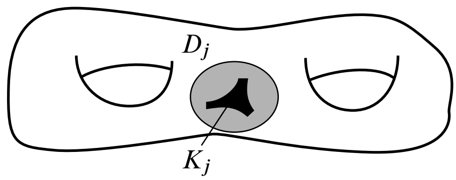

Figure 14: A typical pair (D_j, K_j) on the grafted surface; the map in Lemma 4.22 is almost conformal outside K_j (shaded darker). The annular region $D_j \setminus K_j$ (shaded lighter) has large modulus (Lemma 4.23).

Lemma 4.23 *For large enough t (as in Lemma 4.22), each annular region $D_j \setminus K_j$ for $1 \le j \le m$ where D_j and K_j are the simply connected subsets as above, has modulus greater than $\frac{2}{3\pi}\ln(1/\epsilon)$.*

Proof Consider the annular region that is the image of $f(D_j \setminus K_j)$ on the singular flat surface $\widehat{X}_t$. From the construction of f (in particular its property of being height-preserving and almost isometric along $\mathcal{F}$) it follows that one can embed a flat annulus A_j of large modulus in $f(D_j \setminus K_j)$. On a singular-flat plane comprising three half-planes around the origin which has a cone-angle 3π around it, the modulus of a round annulus with inner diameter R_{inn} and outer diameter R_{out} is

$$\operatorname{mod}(A_j) = \frac{1}{3\pi}\ln\frac{R_{\text{out}}}{R_{\text{inn}}}.$$

In our case, since the diameter of $f(D_j)$ is greater than $\frac{1}{\epsilon^4}$ and the diameter of $f(K_j)$ is at most $C'\ln(1/\epsilon)$ where $C' > 0$ is some constant, the modulus of A_j satisfies

$$\operatorname{mod}(A_j) \ge \frac{1}{3\pi}\ln\frac{1/\epsilon^4}{C'\ln(1/\epsilon)} > \frac{1}{3\pi}\ln\frac{1}{\epsilon^3} = \frac{1}{\pi}\ln\frac{1}{\epsilon}$$

for ϵ sufficiently small.

Since f is $(1+O(\epsilon))$–quasiconformal on $D_j \setminus K_j$, we have an embedded annulus $f^{-1}(A_j)$ of large modulus in $D_j \setminus K_j$ (in the Thurston metric). In particular, the modulus

$$\operatorname{mod}(D_j \setminus K_j) > \frac{1}{(1+O(\epsilon))}\cdot\frac{1}{\pi}\ln\frac{1}{\epsilon} > \frac{2}{3\pi}\ln\frac{1}{\epsilon}.$$

for sufficiently small ϵ. □

Remark Our choice of the lower bound for the modulus in the above lemma is justified by Corollary 4.26.

4.6 Modifying the map to almost conformality

In this section we modify the map $f\colon X_t \to \widehat{X}_t$ from Lemma 4.22 such that it is almost conformal. To do this we shall redefine the map in the subsets $D_1, D_2, \dots, D_m$. The crucial fact that allows this modification is that the lack of almost conformality for f is contained in the subsets K_j, $1 \le j \le m$ which have diameter much smaller than the diameter of D_j.

4.6.1 A quasiconformal extension lemma Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and let B_r be a closed ball of radius r about the origin.

The following result is probably well known to experts, however it does not seem to be readily available in the literature on the subject. A proof is provided in the Appendix.

Lemma 4.24 *For any $\epsilon > 0$ sufficiently small and any $0 \le r \le \epsilon$ if $f\colon \mathbb{D} \to \mathbb{D}$ satisfies*

(1) *f is a quasiconformal map*

(2) *the quasiconformal distortion is $(1 + C\epsilon)$ on $\mathbb{D} \setminus B_r$*

then the map f extends to a $(1+C'\epsilon)$–quasisymmetric map of the boundary, where C' is a constant depending only on C.

An immediate consequence of the Ahlfors–Beurling extension is the following.

Corollary 4.25 *Let $\epsilon > 0$ be sufficiently small, $r \le \epsilon$ and let $f\colon \mathbb{D} \to \mathbb{D}$ satisfy* (1) *and* (2) *as in the previous lemma. Then there exists an almost conformal map $g\colon \mathbb{D} \to \mathbb{D}$ such that $f_{|\partial\mathbb{D}} = g_{|\partial\mathbb{D}}$.*

We shall use the above for modifying a map between Riemann surfaces:

Corollary 4.26 *Let Σ, Σ' be homeomorphic Riemann surfaces. Let $f\colon \Sigma \to \Sigma'$ be a quasiconformal map and $K \subset D \subset \Sigma$ be concentric embedded disks such that:*

(i) *The modulus of the annulus $D \setminus K$ is at least $\frac{2}{3\pi} \ln \frac{1}{\epsilon}$.*

(ii) *f is $(1+\epsilon)$–quasiconformal on $\Sigma \setminus K$.*

Then there is a quasiconformal map $g\colon \Sigma \to \Sigma'$ such that $f|_{\Sigma\setminus D} = g|_{\Sigma\setminus D}$ and g is almost conformal on D.

Proof Let $\phi\colon D \to \mathbb{D}$ and $\psi\colon f(D) \to \mathbb{D}$ be uniformizing maps to the unit disk, normalized such that the centers are taken to $0 \in \mathbb{D}$.

Claim The images $\phi(K)$ and $\psi(f(K))$ have diameter $O(\epsilon)$.

Proof of claim Consider either image and let d be its diameter. By (i) and (ii) the modulus M of the image annulus in (either) case satisfies

$$M \geq \frac{1}{(1+\epsilon)} \cdot \frac{2}{3\pi} \ln \frac{1}{\epsilon} > \frac{1}{2\pi} \ln \frac{1}{\epsilon}$$

for small ϵ.

However by (A-18) in Lemma A.14) we also have

$$M < \frac{1}{2\pi} \ln \frac{16}{d}$$

which implies that in fact $d < 16\epsilon$. □

Now the map $g = \psi \circ f \circ \phi^{-1}\colon \mathbb{D} \to \mathbb{D}$ satisfy the requirements of Corollary 4.25 and hence can be replaced by an almost conformal map $h\colon \mathbb{D} \to \mathbb{D}$ that has the same map as g on $\partial\mathbb{D}$. We replace $f\colon D \to f(D)$ by the almost conformal map $\psi^{-1} \circ h \circ \phi\colon D \to f(D)$. This restricts to the same map as f on ∂D. Together with the map f on $\Sigma \setminus D$ it defines a continuous map of Σ to Σ'. This map is quasiconformal on D and $\Sigma \setminus D$, and is hence quasiconformal, since ∂D is a measure-zero set. □

4.7 Proof of Proposition 4.1

Proof of Proposition 4.1 We start with the map from Lemma 4.22 and modify it by applying Corollary 4.26 (taking $\Sigma = X_t$, $\Sigma' = \widehat{X}_t$, $D = D_j$, $K = K_j$), for each $1 \leq j \leq m$ in succession. This is possible since each annulus $D_j \setminus K_j$ has modulus at least $\frac{2}{3\pi} \ln \frac{1}{\epsilon}$ by Lemma 4.23 (so (i) of Corollary 4.26 holds). The final map $f\colon X_t \to \widehat{X}_t$ is almost conformal on $D_1 \cup D_2 \cup \cdots \cup D_m$ and agrees with the original f, and is hence almost conformal on the complement $X_t \setminus D_1 \cup D_2 \cup \cdots \cup D_m$. The property of almost conformality extends across the measure zero set consisting of the union of the ∂D_j.

This completes the construction of an almost conformal map $f\colon X_t \to \widehat{X}_t$, for all t sufficiently large.

The singular flat surface $\widehat{X}_1$ has a horizontal foliation, and a vertical foliation which is measure equivalent to λ. Now $\widehat{X}_t$ can be obtained from $\widehat{X}_1$ by scaling the lengths of the leaves of the horizontal foliation on $\widehat{X}_1$ by a factor of t, and keeping the vertical foliation λ the same. (This can also be described as scaling the transverse measure of λ to $t\lambda$.) As in the remark following Definition 2.2, this conformally equivalent to scaling the horizontal direction by a factor of $\sqrt{t}$ and the vertical direction by a factor of $1/\sqrt{t}$. The surface $\widehat{X}_t$ thus lies on the Teichmüller ray from $\widehat{X}_1$ determined by λ, at a distance of $\frac{1}{2}\ln t$.

Since our choice of $\epsilon > 0$ throughout was arbitrary, this shows the grafting ray based at $X = X_1$ determined by the lamination λ is asymptotic to the Teichmüller ray based at $Y = \widehat{X}_1$. □

The proofs of Corollaries 1.2 and 1.3 only require Theorem 1.1 in the arational case, hence we provide them here:

Proof of Corollary 1.3 Pick any Teichmüller ray determined by such an arational λ. By Proposition 4.1, the two grafting rays are both asymptotic to it, and hence to each other.

When λ is also uniquely ergodic, then for any other choice of basepoint Y, the Teichmüller ray Y_t determined by λ is asymptotic to the above Teichmüller ray, by the result of Masur [26], and hence by the triangle inequality, the grafting ray based at X is asymptotic to the Teichmüller ray Y_t. □

Proof of Corollary 1.3 Arational laminations form a full measure set in $\mathcal{ML}$ (with respect to the Thurston measure on $\mathcal{ML}$). It is known that for a generic choice of such an arational λ and any choice of basepoint the corresponding Teichmüller ray is dense in moduli space (this follows from the ergodicity of the Teichmüller geodesic flow, proved in [28]; see Masur [27] for explicit examples of such rays). By Proposition 4.1, a grafting ray determined by a generic λ is then asymptotic to a dense Teichmüller ray, and is hence itself dense. □

Remark Jayadev Athreya pointed out to the author by that it is easy to show that Theorem 1.1 implies that such a grafting ray is in fact equidistributed in moduli space. A subsequent article shall investigate some further dynamical and measure-theoretic features of the Teichmüller geodesic flow inherited by grafting flowlines.

5 The multicurve case

In this section we prove Theorem 1.1 in the case when λ is a multicurve (Proposition 5.4) following the outline in Section 3.2.

5.1 A quasiconformal interpolation

We start with a lemma about quasiconformal maps that shall be useful later. Throughout, $\mathbb{D}$ shall denote the unit closed disk on the complex plane, and B_r shall denote the closed disk of radius r centered at 0. We show that a conformal map defined on $\mathbb{D}$ can be adjusted to be the identity near 0, without too much quasiconformal distortion.

Lemma 5.1 *Let $g\colon \mathbb{D} \to g(\mathbb{D}) \subset \mathbb{C}$ be a univalent conformal map such that $g(0) = 0$ and $g'(0) = 1$. Then for any $\epsilon > 0$ (sufficiently small) there exists an $0 < s < 1$ and a map $G\colon \mathbb{D} \to g(\mathbb{D})$ such that:*

(1) *G is $(1+\epsilon)$–quasiconformal.*

(2) *G restricts to the identity map on B_s.*

(3) *$G|_{\partial\mathbb{D}} = g|_{\partial\mathbb{D}}$.*

Proof Since $g(0) = 0$ and g is conformal, there exists an expansion

$$g(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \psi(z),$$

where $a_i \in \mathbb{C}$ for $i \geq 2$.

We shall have $s = \epsilon$. Let $\phi_\epsilon\colon [0,1] \to [0,1]$ be a smooth bump function such that

(1) $\phi_\epsilon(t) = 0$ for $0 \leq t \leq \epsilon$

(2) $\phi_\epsilon(t) = 1$ for $2\epsilon \leq t \leq 1$

(3) $|\phi_\epsilon'(t)| = O(1/\epsilon)$ for all $t \in [\epsilon, 2\epsilon]$

and define

$$\psi_\epsilon(z) = \phi_\epsilon(|z|)\psi(z)$$

for all $z \in \mathbb{D}$.

Define the map $G\colon \mathbb{D} \to \mathbb{C}$ as

$$G(z) = z + \psi_\epsilon(z)$$

for $z \in \mathbb{D}$.

From the Koebe distortion theorem (see for example Pommerenke [30, Theorem 1.3]) we have

$$\frac{1}{(1+|z|)^2} \leq \left|\frac{g(z)}{z}\right| \leq \frac{1}{(1-|z|)^2}$$

and that implies

$$\left|\frac{g(z)}{z} - 1\right| \leq \frac{4|z|}{(1-|z|^2)^2} \quad \Longrightarrow \quad |\psi(z)| \leq \frac{4|z|^2}{(1-|z|^2)^2}.$$

For $\epsilon \leq |z| \leq 2\epsilon$ we therefore have

$$|\psi(z)| \leq C\epsilon^2$$

for some universal constant C (we know ϵ is sufficiently small).

From this and (3) above it is easy to check that

$$|\partial_{\bar{z}} G| = |\partial_{\bar{z}} \psi_\epsilon| = O(\epsilon),$$
$$|\partial_z G| = |1 + \partial_z \psi_\epsilon| = 1 + O(\epsilon),$$

for each $z \in \mathbb{D}$.

The map G is hence a diffeomorphism of quasiconformal dilatation of $1 + O(\epsilon)$ that restricts to the identity map on B_ϵ and to g on $\partial\mathbb{D}$ as required. □

Remark By conjugating by the dilation $z \mapsto (1/r)z$ the above result holds (for some $0 < s < r$) if the conformal map g is defined only on $B_r \subset \mathbb{D}$.

5.2 Proof of the multicurve case

Let λ be a multicurve, namely a collection of (weighted) disjoint simple closed geodesics $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ on the closed hyperbolic surface X, with weights $c_i > 0$ and lengths l_i, where $1 \leq i \leq n$.

Let X^∞ denote the infinitely grafted surface, obtained by cutting X along λ and gluing a semi-infinite Euclidean cylinder at each of the resulting $2n$ boundary components. This resulting Riemann surface can be thought of as the conformal limit of the grafting ray $X_t = \mathrm{gr}_{t\lambda} X$. (Recall that a time–$t$ grafting along a simple closed curve γ inserts a Euclidean cylinder of width t at γ).

We shall use the following result due to Strebel ([32]).

Proposition 5.2 *Let Σ be a Riemann surface of genus g, and $x_1, x_2, \ldots, x_m$ be marked points on Σ such that $2g-2+m > 0$. Then for any collection of real numbers $p_1, p_2, \ldots, p_m$ there exists a quadratic differential ϕ on Σ such that:*

(i) *ϕ is holomorphic on $\Sigma \setminus \{x_1, \ldots, x_m\}$.*

(ii) *ϕ has a double pole at x_i with residue $(\frac{p_i}{2\pi})^2$ for each $1 \leq i \leq m$.*

(iii) *All vertical trajectories of ϕ are closed, and they foliate m annular domains which are disks with punctures at $x_1, \ldots, x_m$.*

Applying the above proposition with $\Sigma = X^{\infty}$, we can obtain such a *Jenkins–Strebel differential* ϕ on a Riemann surface conformally equivalent (as marked conformal structures in $\mathcal{T}_{g,2n}$) to X^{∞}, with n pairs of marked points, each pair having residue $(l_i/2\pi)^2$. We denote this surface equipped with the quadratic differential metric as Y^{∞}: this then is a singular flat surface comprising n pairs of infinite Euclidean cylinders, each pair having circumference l_i. Let g be the conformal map from X^{∞} to Y^{∞} that preserves the marking (Figure 15).

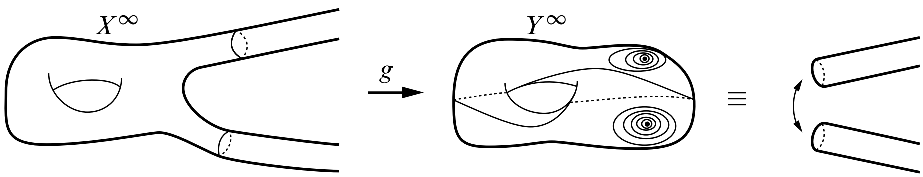

Figure 15: The surface X^{∞} on the left is the "infinitely" grafted surface. The conformally equivalent singular flat surface Y^{∞} has a quadratic differential metric with a pair of double poles, and metrically it is equivalent to two semi-infinite Euclidean cylinders glued along the boundary.

Let Y denote the surface obtained from Y^{∞} by truncating the infinite cylinders and gluing up the pairs (of matching lengths) so that they give Euclidean cylinders of circumference l_i and height c_i, in the homotopy class of γ_i, for each $1 \leq i \leq m$. This will be the basepoint of the Teichmüller ray Y_t that we shall show is asymptotic to the grafting ray X_t. Recall that the surface Y_t is obtained from Y (which is also Y_0 in our notation) by stretching along the horizontal foliation, and in particular the Euclidean cylinders on Y_t have height $c_i e^{2t}$.

Notation For a semi-infinite cylinder (homeomorphic to $S^1 \times [0,\infty)$) denoted by $\mathcal{C}$, we denote by $\mathcal{C}_{\geq h}$ (resp. $\mathcal{C}_{\leq h}$) the infinite subcylinder of all points of $\mathcal{C}$ of height greater (resp. lesser) than h.

Lemma 5.3 *Let $\mathcal{C}$ be a semi-infinite Euclidean cylinder, and let $f\colon \mathcal{C} \to \mathcal{C}$ be a conformal map which is a homeomorphism onto its image. Then for any $\epsilon > 0$ and $H_0 > 0$ there exists an $H_1 > H_0$ and a map $F\colon \mathcal{C} \to f(\mathcal{C}) \subset \mathcal{C}$ such that:*

(1) *F is $(1+\epsilon)$–quasiconformal.*

(2) *F is isometric on $\mathcal{C}_{\geq H_1}$.*

(3) *F restricts to f on $\mathcal{C}_{\leq H_0}$.*

(Here H_1 depends on ϵ and $H_1 \to \infty$ as $\epsilon \to 0$.)

Proof The cylinder $\mathcal{C}$ is conformally equivalent to the punctured unit disk $\mathbb{D}^*$ by a conformal map ϕ that takes ∞ to 0, and maps the round circle at height h (for each $0 \leq h < \infty$) to a circle of radius $r(h) = e^{-2\pi h}$ centered at the origin. The subcylinder $\mathcal{C}_{\geq H_0}$ corresponds to a subdisk that we denote by B. The conformal map f conjugates to a conformal map $g = \phi \circ f \circ \phi^{-1}$ from the punctured disk to itself, which can be extended to a conformal map of $\mathbb{D}$ into itself, such that $g(0) = 0$ and $g'(0) = c$ where $c \in \mathbb{C}$.

We can now apply Lemma 5.1 to the rescaled conformal map $(1/c)g$ restricted to the subdisk B to obtain an almost conformal map $G\colon B \to G(B) \subset \mathbb{D}$ that restricts to a dilatation $z \mapsto cz$ on $B_s \subset B$ for some sufficiently small s (that depends on ϵ) and agrees with g on ∂B. Since a dilatation conjugates back to an isometric translation of the semi-infinite cylinders, the map $F_0 = \phi^{-1} \circ G \circ \phi$ defined on $\mathcal{C}_{\geq H_0}$ satisfies (1) and (2), where $H_1 = \frac{1}{2\pi} \ln \frac{1}{s}$ and moreover, agrees with f on $\partial \mathcal{C}_{\geq H_0}$. Thus F_0 extended by f on $\mathcal{C}_{\leq H_0}$ defines a map on the entire $\mathcal{C}$, which is our desired F. □

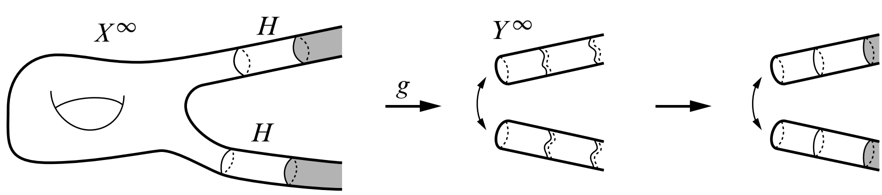

Figure 16: Lemma 5.3 allows one to adjust g such that it is isometric on the circle at height H. Discarding the shaded regions and gluing up along the truncating circles gives a grafted surface X_t on the left and the surface along the Teichmüller ray on the right.

Proposition 5.4 *For any (sufficiently small) $\epsilon > 0$ there exists a (sufficiently large) $T > 0$ such that for every $t > T$ there is a $(1+\epsilon)$–quasiconformal map from the grafted surface X_t to a singular flat surface Y_s for some s.*

Proof We start with the conformal map $g\colon X^\infty \to Y^\infty$ (Figure 16). Consider its restriction to one of the semi-infinite Euclidean cylinders $\mathcal{C}$, and let $\mathcal{C}'$ be the corresponding infinite cylinder on Y^∞ such that $g(\mathcal{C}) \cap \mathcal{C}' \neq \phi$. By properness, for H_0 sufficiently large, the image under g of the $\mathcal{C}_{\geq H_0}$ will be strictly contained in $\mathcal{C}'$. Applying Lemma 5.3 we have some $H_1(\epsilon) > H_0$ and a map F_C on the truncated cylinder $\mathcal{C}_{\leq H_1}$ that agrees with g on $\mathcal{C}_{\leq H_0}$ and is isometric on the circle at height H_1.

Repeating this for each infinite cylinder on X^∞, we obtain truncations of each and almost conformal maps such that together we have an almost conformal map F to Y^∞, with each of its cylinders also truncated at a circle at some height. The map F is isometric on the boundary circles, and in particular its restrictions to truncated paired cylinders agree on the truncating round circles. On gluing these maps we obtain an almost conformal map from the truncated X^∞ glued along the boundaries of the paired cylinders, to the truncated Y^∞ glued along the boundaries of its paired cylinders. By definitions of the surfaces, the latter is Y_s for some s, and the former is X_T for some T. (For an additional discussion of adjusting for unwanted "twists" see [17, Section 4].) By choosing to truncate at higher heights (greater than $H_1(\epsilon)$ as above) one can obtain a $(1+\epsilon)$–quasiconformal maps from X_t for any $t > T$, to a surface along the Teichmüller ray starting at Y. □

Recall that the surfaces X_t form the grafting ray determined by the multicurve λ, and the surfaces Y_s lie along the Teichmüller ray determined by λ and with basepoint $Y = Y_0$. Thus together with Proposition 4.1 this concludes the proof of Theorem 1.1.

6 Proof of Theorem 1.4

In this section we prove the following theorem stated in Section 1.

Theorem 1.4 *For any $X \in \mathcal{T}_g$, the set of integer graftings $\{\pi(\mathrm{gr}_{2\pi\gamma} X) \mid \gamma \in \mathcal{S}\}$ is dense in $\mathcal{M}_g$.*

As discussed in Section 1, this then immediately implies Theorem 1.5.

Plan of the proof Fix any $X \in \mathcal{T}_g$ and pick an arbitrary $Y \in \mathcal{M}_g$ and a sufficiently small $\epsilon > 0$. To establish Theorem 1.4 it is enough to show that for this choice we have:

Proposition 6.1 *There exists a $\gamma \in \mathcal{S}$ such that $d_{\mathcal{T}}(\pi(\mathrm{gr}_{2\pi\gamma} X), Y) < 2\epsilon$.*

By Corollary 1.3 we have a $\lambda \in \mathcal{ML}$ such that:

(1) λ is arational.

(2) The projection of the grafting ray determined by (X, λ) to moduli space is dense.

In particular, we have a sequence of times $t_i \to \infty$ such that

$$d_{\mathcal{T}}(\pi(\mathrm{gr}_{2\pi t_i \lambda}(X)), Y) < \epsilon. \tag{6-1}$$

The argument for the proof of Proposition 6.1 carried out in Sections 6.1–6.5 consists of choosing an appropriate approximation of each $2\pi t_i \lambda$ by a multicurve and showing that the corresponding grafted surface is close to the surface obtained by grafting along this multicurve (Lemma 6.19). Proposition 6.1 then follows easily from the triangle inequality (Section 6.6).

6.1 Almost-conformal constructions

In Section 4.2 we developed the notion of "almost isometries" (see Definition 4.9) and some related constructions of almost conformal maps. In this section, we weaken the definition by introducing a further bounded additive error A (see Definitions 6.2 and 6.6). This additive error shall be small relative to the other dimensions, however, and will still permit the construction of almost conformal maps (as in Lemma 6.7).

Definition 6.2 A homeomorphism f between two C^1–arcs on a conformal surface is an (ϵ, A)*–almost isometry* if f is continuously differentiable with dilatation d (the supremum of the derivative of f over the domain arc) that satisfies $|d-1| \leq \epsilon$ and such that the lengths of any subinterval and its image differ by an additive error of at most A.

Remark As before, we may say "(ϵ, A)–almost isometric" to mean "$(M\epsilon, A)$–almost isometric for some (universal) constant $M > 0$".

The following are analogues of Lemmas 4.10, 4.11 and 4.12, and we omit their (easy) proofs.

Lemma 6.3 *Let I_1, I_2 be arcs of lengths l_1 and l_2 such that $|l_1 - l_2| < A$ and $A/l_1 < \epsilon$. Then the orientation-preserving affine (stretch) map $f\colon I_1 \to I_2$ is (ϵ, A)–almost isometric.*

Lemma 6.4 *Let $f, g\colon I \to I$ be maps of an arc I that are (ϵ, A)–almost isometric and (ϵ, A')–almost isometric respectively. Then f^{-1} is (ϵ, A)–almost isometric, and $f \circ g$ is $(\epsilon, A + A')$–almost isometric.*

Lemma 6.5 *Let $I = I_1 \cup I_2 \cup \cdots \cup I_N$ be a partition of the arc I into subarcs with disjoint interior. Then any continuously differentiable map $f\colon I \to I$ with (ϵ, A)–almost isometric restrictions to $I_1, \ldots, I_N$ is (ϵ, NA)–almost isometric on I.*

Conversely, the restriction of an (ϵ, A)–almost isometry to a subarc is also an (ϵ, C)–almost isometry to its image.

Definition 6.6 A map f between two rectangles is (ϵ, A)*–good* if it is isometric on the vertical sides and (ϵ, A)–almost isometric on the horizontal sides.

Remark In what follows "rectangle" in the above definition shall mean a quadrilateral on the Euclidean plane, or the hyperbolic plane, or a complex-projective surface with the Thurston metric, having four arcs intersecting at right angles (the spaces mentioned have conformal metrics), and pairs of nonadjacent sides having equal length (we label one as *vertical* and the other *horizontal*).

6.1.1 Almost-conformal extension The following lemma is a slight generalization of Lemma 4.13.

Lemma 6.7 *Let R_1 and R_2 be two Euclidean rectangles with vertical sides of length h and horizontal sides of lengths l_1 and l_2 respectively, such that $l_1, l_2 > h$ and $|l_1 - l_2| < A$, where $A/h \leq \epsilon$. Then any (ϵ, A)–good map $f\colon \partial R_1 \to \partial R_2$ has a $(1 + C\epsilon)$–quasiconformal extension $F\colon R_1 \to R_2$ for some (universal) constant $C > 0$.*

Proof The proof follows by rescaling by a factor of $1/h$ and applying Lemma 4.13 to the resulting map between the resulting pair of rectangles. □

6.1.2 Finitely grafted rectangle Let R be a region in the hyperbolic plane bounded by two "vertical" geodesic sides of length l and two "horizontal" horocyclic sides of length w. Assume henceforth that $l > 1/\epsilon$ and $w < \epsilon$.

Let $a_1, a_2, \ldots, a_k$ be a finite collection of geodesic arcs with endpoints on the horizontal sides, with corresponding weights $w_1, w_2, \ldots, w_k$. Then one can obtain a *finitely grafted* rectangle R' by inserting Euclidean rectangles in the shape of truncated "crescents" (see Figure 17) of widths $w_1, w_2, \ldots, w_k$ at the arcs $a_1, a_2, \ldots, a_k$ respectively.

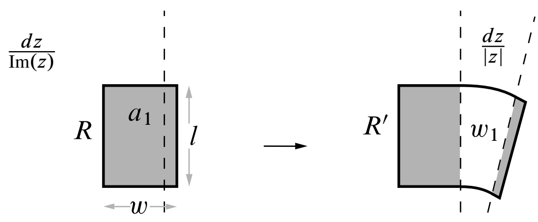

Figure 17: Grafting a rectangle R (shown on the upper half plane model) across a single weighted geodesic arc a_1 gives a finitely grafted rectangle R' (Section 6.1.2).

Lemma 6.8 *There is an almost conformal map f from R' to a Euclidean rectangle of vertical height l and horizontal width $w_1 + w_2 + \cdots + w_k$. Moreover, f is (ϵ, ϵ)–good on the boundary.*

Proof We give a sketch of the argument, and refer to Sections 4.2 and 4.3 for details and similar constructions. We always work in the upper half-plane model of the hyperbolic plane.

First, we can map the (ungrafted) rectangle R to the Euclidean plane by a map that "straightens" the horocyclic foliation across R. Since $w < \epsilon$, this straightening map is almost conformal (Lemma 4.6). It also follows from some elementary hyperbolic geometry that the hyperbolic "width" between the geodesic arcs on R is a C^1–function of the "height" with ϵ–small derivatives, and so their images under the straightening map are ϵ–almost vertical.

Next, the truncated "crescents" are spliced in: their straightening maps to the plane are in fact conformal with rectangular images (Lemma 4.8) and hence can be adjusted by almost conformal maps (Lemma 4.5) to fit with the almost vertical image arcs above.

This gives a composite map that is almost conformal with image a rectangle of height l and width $w + w_1 + w_2 + \cdots + w_k$. Since $w < \epsilon$ and $w_1 + w_2 + \cdots + w_k > 1$, one can finally compose by an almost conformal horizontal affine stretch to a rectangle of width $w_1 + w_2 + \cdots + w_k$ as required.

The statement about the almost isometry of the sides follows from Lemma 6.3 since prior to the final affine dilatation the map is isometric on the horizontal sides, and the final affine stretch is to a rectangle of width differing by $w < \epsilon$. □

6.1.3 Smoothing the horizontal sides In the finitely grafted rectangle R' above, the horizontal sides may not be differentiable arcs since the geodesic arcs $a_1, \ldots, a_k$ may not intersect the horizontal sides of R at right angles. However, since the rectangle R prior to grafting is thin ($w < \epsilon$) and long ($l > 1/\epsilon$) some elementary hyperbolic geometry implies that the geodesic arcs intersect the horizontal sides at an angle that differs from $\pi/2$ by a quantity bounded by $C\epsilon$ (for some universal constant $C > 0$).

In R', the horizontal sides can then be "smoothed" to be C^1 by "trimming" the horizontal sides of each grafted Euclidean "truncated crescent:" each such horizontal segment is replaced by a C^1 arc whose derivatives are ϵ–small, and have specified values at the endpoints that make the entire arc C^1. Denote the resulting new "smoothed" rectangle by R'' (Figure 18).

The following lemma can be thought of as a "vertical" version of the straightening lemma from Section 4.1 (Lemma 4.5).

Lemma 6.9 *Let $S = [0, w] \times [0, l]$ be a Euclidean rectangle and let S' be the region enclosed by the two parallel vertical line segments of S and two arcs which are the graphs $y = g_1(x)$ and $y = g_2(x)$ over the interval $0 \le x \le w$ on the x–axis, where g_1 and g_2 are C^1–functions such that*

(1) $g_2(x) > g_1(x) > 0$

(2) $|g_1'(x)| < \epsilon$ and $|g_1'(x)| < \epsilon$

(3) $|(l/(g_2(x) - g_1(x))) - 1| < \epsilon$

for all $0 \le x \le w$. Then there exists a $(1 + C\epsilon)$–quasiconformal map from S to S' which is (ϵ, ϵ)–good on the boundary. (Here $C > 0$ is some universal constant.)

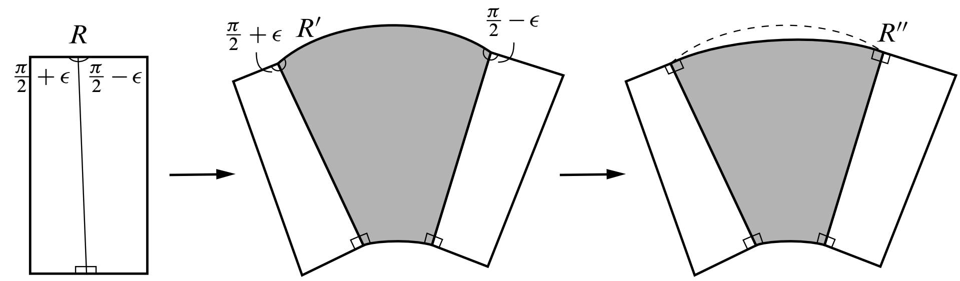

Figure 18: Grafting along a geodesic arc intersecting almost, but not quite at right angles gives a finitely grafted rectangle R' with the top edge having "corners" (the grafted Euclidean region is shown shaded.) This can be smoothed to a C^1–arc together with an almost conformal map from R' to the resulting rectangle R'' (Lemma 6.10).

We omit the proof, which is similar to Lemma 4.5, involving a map that stretches vertically by the right factor along the width of S'.

By a repeated application of the above lemma on each of the grafted strips, where the graphs g_1 and g_2 for each are determined by the C^1–"trimming," we have:

Lemma 6.10 *There exists an almost conformal map from R' to R'' which is (ϵ,ϵ)–good on the boundary.*

Moreover, by precomposing with the almost conformal map f of Lemma 6.8, we have:

Corollary 6.11 *There exists an almost conformal map from R'' to a Euclidean rectangle of vertical height l and horizontal width $w_1 + w_2 + \cdots + w_k$ which is (ϵ,ϵ)–good on the boundary.*

6.2 Thickening the train track

Recall from Section 4.1 that we have the subsurface $\mathcal{T}_\epsilon \subset X$ containing the arational lamination λ which is its "ϵ"–train track neighborhood. From that section, we have a decomposition of the surface into a collection of rectangles $R_1, R_2, \ldots, R_n$, and complementary regions which are truncated ideal hyperbolic triangles $T_1, T_2, \ldots, T_m$.

We now describe a "thickening" to ensure that λ is contained *properly* in $\mathcal{T}_\epsilon$. For each $T_1, T_2, \ldots, T_m$, choose thin strips adjacent to the geodesic sides and bounded by another geodesic segment "parallel" to the sides and append them to the rectangles adjacent to the sides. We continue to denote the collection of this slightly thickened rectangles by $R_1, R_2, \ldots, R_n$, and their union by $\mathcal{T}_\epsilon$.

6.3 Approximating λ by multicurves

Let $w_1, w_2, \dots, w_n$ be the *weights* of the train track $\mathcal{T}_\epsilon$, that is, w_i denotes the total transverse measure of the rectangle R_i. By our assumption of the maximality of λ, these weights are all positive reals.

By the above construction of the train track it follows that the subarcs $I'_1, \dots, I'_{3m+1}$ (see (4-2) in Section 4.1) also have a positive transverse measures $w'_1, \dots, w'_{3m+1}$. This follows from the minimality of λ: the only way such a subarc will carry no measure is if λ intersected it only at the endpoints, but the leaf of λ passing through an endpoint of one of these subarcs is isolated on the complementary side (it is part of the boundary of one of $T_1, \dots, T_m$) and cannot be isolated inside the subarc also.

For each $1 \le i \le n$ the weights are given by

$$w_i = \sum_{k \in S_i} w'_k, \tag{6-2}$$

where S_i is a finite subset of $\{1, 2, \dots, 3m+1\}$ as in (4-2).

Definition 6.12 A $3m+1$–tuple of (nonnegative) real numbers $(c_1, c_2, \dots, c_{3m+1})$ is an *admissible weighting* of $\mathcal{T}_\epsilon$ if the corresponding weights on the train track given by Equation (6-2) satisfy the switch conditions for the train track.

We begin with the following observation in elementary linear algebra.

Lemma 6.13 *Let* S *be a homogeneous system of linear equations in N variables, with all coefficients in the set $\{0, 1, -1\}$. Then there exists a constant $D > 0$ depending only on N (and not on* S*) such that for any N–tuple $(x_1, x_2, \dots, x_N)$ of real numbers that satisfies* S *there is an integer N–tuple $(k_1, k_2, \dots, k_N)$ with $|x_i - k_i| < D$ for each $1 \le i \le N$, which is also a solution.*

Proof Since the coefficients of the linear system S are integers, by Gauss–Jordan elimination there is a basis $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_M}\}$ of the vector space $\mathcal{V}$ of solutions such that each $\vec{v_i}$ is a vector with *rational* entries. Let L be the integer that is the least common multiple of all the denominators of the rational entries, such that $\vec{w_i} = L\vec{v_i}$ is an integer vector for each $1 \le i \le M$. Then this set of M linearly-independent integer vectors W spans a lattice in $\mathcal{V}$. Let D be the diameter of the torus $\mathbb{T}^M$ that is a quotient of $\mathcal{V}$ by the action of W, or equivalently, the radius of a fundamental domain in $\mathcal{V}$. Clearly then, any solution in $\mathcal{V}$ is less than D away from an integer vector.

The constant D at this point depends on the linear system S, but notice that since there are N variables, and each coefficient is from the finite set $\{0, 1, -1\}$, there are only finitely many possible choices of S (depending only on N), and hence we can choose D to be the maximum value as we vary over all of them. □

Corollary 6.14 *Let* S *be a homogeneous system as above, and let* $\vec{x} = (x_1, x_2, \ldots, x_N)$ *be a solution where each entry is a positive real number. Then there exists a* $T_0 > 0$ *such that for any* $t > T_0$*, there is an integer solution* $(k_1, k_2, \ldots, k_N)$ *with each entry positive, such that* $|tx_i - k_i| < D$ *for each* $1 \leq i \leq N$*.*

Proof Note that since S is homogenous the vector $t\vec{x}$ is also a solution. Each entry of this vector will be greater than D when $t > T_0 = D/\min_{1 \leq i \leq N} x_i$. Let $(k_1, k_2, \ldots, k_N)$ be the integer solution close to $t\vec{x}$ that the previous lemma guarantees. Since for each i, we have

$$|tx_i - k_i| < D \quad \text{and} \quad tx_i > D \Longrightarrow k_i > 0;$$

we see that each entry of this integer solution is positive. □

We now apply this to our setting:

Lemma 6.15 *There exists a* $D > 0$ *and* $T_0 > 0$ *such that for any* $t > T_0$ *there is a tuple* $\vec{k} = (k_1, k_2, \ldots, k_{3m+1})$ *of positive integers such that:*

(1) $\vec{k}$ *is an admissible weighting of* $\mathcal{T}_\epsilon$*.*

(2) $|tw'_j - k_j| < D$ *for each* $1 \leq j \leq 3m+1$*.*

Moreover, D *is a constant that is independent of* ϵ*.*

Proof The admissible weights on the train track $\mathcal{T}_\epsilon$ satisfy a linear system $\mathcal{S}$ in $3m+1$ variables corresponding to the switch conditions and Equations (6-2), which have coefficients in the set $\{0, 1, -1\}$. Hence one can apply Corollary 6.14 with $\vec{x}$ being the positive solution $(w_1, w_2, \ldots, w_{3m+1})$ corresponding to the transverse measures of the lamination λ, and this yields (1) and (2); see Figure 19. Also by the lemma, D depends only on m, which in turn depends only on the topology of the surface (see Section 4.1), and hence is independent of ϵ. □

Definition 6.16 For $t > T_0$, let γ_t denote the geodesic multicurve corresponding to the admissible integer weighting $\vec{k}$ on the train track $\mathcal{T}_\epsilon$ satisfying (2) of Lemma 6.15.

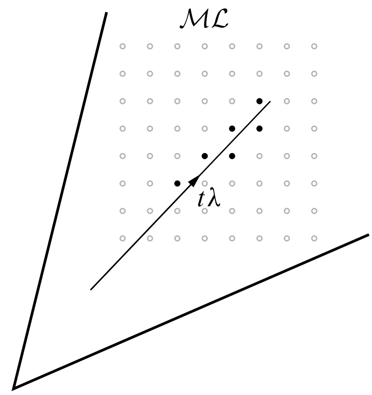

Figure 19: The train track weights give a coordinate chart for measured lamination space. A ray in this convex cone is never far from an integer lattice point (Lemma 6.15).

Lemma 6.17 *There exists a $T_1 > T_0$ such that for any $t > T_1$ the multicurve γ_t is contained in $\mathcal{T}_\epsilon \subset X$.*

Proof Notice that the induced weights $\overline{k}_1, \dots, \overline{k}_n$ on the branches $R_1, \dots, R_n$ of the train track $\mathcal{T}_\epsilon$ (obtained from Equation (6-2)) satisfy

$$|t w_i - \overline{k}_i| < (3m+1)D \tag{6-3}$$

for each $1 \le i \le n$.

This implies that $[\gamma_t] \to [\lambda]$ in $\mathcal{PML}$ as $t \to \infty$, and hence by the compactness result of Canary, Epstein and Green [6, Proposition 4.1.7] the corresponding geodesic representatives on the surface converge to λ in the Hausdorff topology, after passing to a subsequence. (The maximality of λ is used here too, since in general it is only true that the supports $|\gamma_t| \to |\lambda'| \supset |\lambda|$.) Since λ is a proper subset of the closed set $\mathcal{T}_\epsilon$ (this uses the "thickening" defined in Section 6.2), so is γ_t for large enough t. $\square$

6.4 Model rectangles

Recall that the train track decomposition of X (described in Section 4.1) persists as we graft along λ, with the *total width* of the rectangles $R_1, R_2, \dots, R_n$ increasing along the λ–grafting ray. We denote the grafted rectangles on $\mathrm{gr}_{2\pi t\lambda} X$ by $R_1^t, R_2^t, \dots, R_n^t$.

Here, the *total width* $w_i(t)$ is the maximum width of the rectangle R_i^t in the Thurston metric on $\mathrm{gr}_{2\pi t\lambda} X$, and we have

$$|w_i(t) - 2\pi t w_i| < \epsilon \tag{6-4}$$

since the initial *hyperbolic* widths of the rectangles on X is less than ϵ by Lemma 4.2.

We shall use the construction of an almost conformal *Euclidean model rectangle* for a rectangular piece R^t from the collection $\{R^t_1, R^t_2, \dots, R^t_n\}$ proved in Section 4.3. We restate the results of that section as follows:

Lemma 6.18 (Lemma 4.18 and Corollary 4.19) *For any $t > 1$, there is a $(1 + C\epsilon)$–quasiconformal map from R^t to a Euclidean rectangle of width $2\pi t w_i$ which is (ϵ, ϵ)–good on the boundary. (Here $C > 0$ is some universal constant.)*

We recapitulate the proof briefly: one first approximates the uncountable collection of geodesic arcs $R^t \cap \lambda$ by a sequence of *finite, weighted* collections of arcs. For each such finite approximation, one can show that if t is large in proportion to the hyperbolic width the map to the complex plane that straightens the transverse foliation is an almost conformal map, and then one takes a limit.

6.5 Grafted surfaces are close

Let $T_1 > 0$ be as in Lemma 6.17. The goal of this section is to prove:

Lemma 6.19 *There exists a (sufficiently large) $T_2 > T_1$ such that for any $t > T_2$, we have that $d_{\mathcal{T}}(\mathrm{gr}_{2\pi t\lambda} X, \mathrm{gr}_{2\pi\gamma_t} X) < \epsilon$.*

Here is a brief summary of the proof prior to the details.

On the initial (ungrafted) surface X one has a train track $\mathcal{T}_\epsilon$ containing the lamination λ which is decomposed into rectangles (corresponding to the branches) by a choice of transverse arc τ. The multicurve approximation γ_t is also contained in $\mathcal{T}_\epsilon$.

On the grafted surface $\mathrm{gr}_{2\pi t\lambda} X$ the rectangles in the train track decomposition widen to have more (Euclidean) width. Similarly on $\mathrm{gr}_{2\pi\gamma_t} X$ the rectangles in the initial decomposition are wider, though not with C^1–boundary as the arcs of γ_t might intersect τ at an angle slightly off $\pi/2$. The arc τ is then replaced by its "smoothed" arc which gives the correct rectangle decomposition on $\mathrm{gr}_{2\pi\gamma_t} X$.

Each rectangle on $\mathrm{gr}_{2\pi t\lambda} X$ is then mapped almost conformally to the corresponding one on $\mathrm{gr}_{2\pi\gamma_t} X$ via their Euclidean "models," by first mapping the boundary and then using the almost conformal extension lemma (Lemma 6.7). The complement of the train tracks are isometric as grafting leaves them unaffected, and these put together give the required almost conformal map between the two surfaces.

Proof of Lemma 6.19 Since $t > T_1$ we have that $\gamma_t \subset \mathcal{T}_\epsilon$ by Lemma 6.17. We let $R'_1, R'_2, \dots, R'_n$ be the rectangles obtained by grafting $R_1, R_2, \dots, R_n$ along $2\pi\gamma_t$. Note that $\gamma_t \cap R_i$ are a finite collection of geodesic arcs, and grafting along them gives *finitely* grafted rectangles as in Section 6.1.2.

Recall that all horizontal sides of the rectangles $R_1, R_2, \dots, R_n$ (on the surface X) lie on the arc τ which is a segment of a leaf of the horocyclic foliation. We chose the arc τ to be sufficiently small (see the comment following Lemma 4.2) and we can assume that its hyperbolic length w is less than ϵ.

After grafting along $2\pi t\lambda$, this arc is converted to an arc on $\mathrm{gr}_{2\pi t\lambda} X$ of length $w + 2\pi t w_1 + 2\pi t w_2 + \cdots + 2\pi t w_n$ which we denote by τ_λ (recall that as one grafts, the horocyclic foliation extends to a foliation on the grafted surface). After grafting along $2\pi\gamma_t$, the same arc τ is converted to an arc τ' on $\mathrm{gr}_{2\pi\gamma_t} X$ of length $w + 2\pi\overline{k}_1 + 2\pi\overline{k}_2 + \cdots + 2\pi\overline{k}_n$ which we can smooth to a C^1 arc (see Section 6.1.3) on the grafted surface that we denote by τ_γ. This simultaneously "smooths" the finitely grafted rectangles $R'_1, R'_2, \dots, R'_n$ to a new collection $R''_1, R''_2, \dots, R''_n$ which we use for the rest of the construction.

Recall also that there are the subarcs $J_1, J_2, \dots, J_{3m}$ of τ that are the horocyclic edges of the complementary regions $T_1, \dots, T_n$. These remain isometrically embedded in the arcs τ_λ and τ', and also in τ_γ (smoothing of τ' affects only the segments lying in the grafted part).

Claim For sufficiently large t there is an $(\epsilon, 50mD)$–almost isometry h from τ_γ to τ_λ that restricts to an isometry between the subarcs corresponding to $J_1, J_2, \dots, J_{3m}$.

Proof of claim Recall that the subarcs in between the $J_1, \dots, J_{3m}$ are the subarcs $I'_1, \dots, I'_{3m+1}$ that had weights $w'_1, \dots, w'_{3m+1}$ on the surface X. On τ_γ and τ_λ, the lengths of these become $2\pi k_i$ and $2\pi t w'_i$ respectively, which differ by at most $2\pi D$ by (2) of Lemma 6.15. For large t, we have that by Lemma 6.3, the affine maps between these subarcs are $(\epsilon, 2\pi D)$–almost isometries. By Lemma 6.5 the concatenated map of these together with isometries between the subarcs corresponding to $J_1, J_2, \dots, J_{3m}$ give an $(\epsilon, 7m \cdot 2\pi D)$–almost isometry from the entire arc τ_γ to τ_λ. □

By Lemma 4.2, for t sufficiently large the height of the finitely grafted rectangle R'_i is sufficiently large and the width is sufficiently small so that one can apply Corollary 6.11 and obtain, for each $1 \le i \le n$, an almost conformal map f_i from R''_i to a Euclidean rectangle of width $2\pi k_i$ which is (ϵ, ϵ)–good on the boundary.

Recall (Section 6.4) that R^t_i denotes the rectangle R_i on X after grafting along $2\pi t\lambda$. By Lemma 6.18 for t sufficiently large there exists, for each $1 \le i \le n$, an almost conformal map g_i from R_i^t to a Euclidean rectangle of width $2\pi t w_i$ which is (ϵ, ϵ)–good on the boundary.

From the above claim, the map h_i from the rectangle ∂S_i on $\mathrm{gr}_{2\pi\gamma_t} X$ to the rectangle ∂R_i^t on $\mathrm{gr}_{2\pi t\lambda} X$ that is isometric on the vertical sides and restricts to the map h on the horizontal sides (which lie on the arc τ_γ) is $(\epsilon, 50mD)$–good.

Consider the composition $g_i|_\partial \circ h_i \circ f_i|_\partial^{-1}$, where $f_i|_\partial$ and $g_i|_\partial$ are the restrictions of f_i and g_i to the boundary of the rectangles where they are defined. By Lemma 6.4 this is an $(\epsilon, 50mD + 2\epsilon)$–good map between two Euclidean rectangles and hence by Lemma 6.7 (the height of these rectangles is sufficiently large when ϵ is sufficiently small) it extends to an almost conformal map H_i between them. The composition $g_i^{-1} \circ H_i \circ f_i \colon S_i \to R_i^t$ is an almost conformal map that restricts to $h\colon \tau_\gamma \to \tau_\lambda$ on the horizontal sides and is isometric on the vertical sides.

The collection $\{H_1, H_2, \dots, H_n\}$ give an almost conformal map from $S_1 \cup \dots \cup S_n \subset \mathrm{gr}_{2\pi\gamma_t} X$ to $R_1^t \cup \dots \cup R_n^t \subset \mathrm{gr}_{2\pi t\lambda} X$ that is isometric on the geodesic sides. Since grafting does not affect the surface X in the complement of $\mathcal{T}_\epsilon$, there is an isometry between $\mathrm{gr}_{2\pi\gamma_t} X \setminus S_1 \cup \dots \cup S_n$ and $\mathrm{gr}_{2\pi t\lambda} X \setminus R_1^t \cup \dots \cup R_n^t$. Together with the above collection of maps this isometry defines the almost conformal map between $\mathrm{gr}_{2\pi\gamma_t} X$ and $\mathrm{gr}_{2\pi t\lambda} X$. □

6.6 Completing the proof

Proof of Proposition 6.1 We choose a $t_i > T_2$ (where T_2 is as in Lemma 6.19) that satisfies Equation (6-1). By the triangle inequality,

$$d_{\mathcal{T}}(\pi(\mathrm{gr}_{2\pi\gamma_{t_i}} X), Y) \leq d_{\mathcal{T}}(\pi(\mathrm{gr}_{2\pi t_i\lambda}(X)), Y) + d_{\mathcal{T}}(\mathrm{gr}_{2\pi t_i\lambda} X, \mathrm{gr}_{2\pi\gamma_{t_i}} X).$$

The first term on the right is less than ϵ by Equation (6-1) and the second term is $O(\epsilon)$ by Lemma 6.19. □

From the discussion at the beginning of Section 6, this proves Theorem 1.4.

Appendix: Proof of Lemma 4.24

The purpose of this section is to provide a proof of the following:

Lemma 4.24 *For any $\epsilon > 0$ sufficiently small and any $0 \leq r \leq \epsilon$ if $f\colon \mathbb{D} \to \mathbb{D}$ satisfies*

(1) *f is a quasiconformal map*

(2) *the quasiconformal distortion is $(1 + C\epsilon)$ on $\mathbb{D} \setminus B_r$*

then the map f extends to a $(1+C'\epsilon)$–quasisymmetric map of the boundary, where C' is a constant depending only on C.

For $r = 0$ the above result is an easy consequence of the work of Ahlfors and Beurling that we recall as Lemma A.9 below.

We begin by recalling the definitions and relevant known results in Section A.1, and prove a lemma about moduli of quadrilaterals in Section A.2, from which the proof of the theorem follows.

A.1 Background

A starting point for the rich theory of quasiconformal mappings can be Ahlfors' lectures [1].

Let Γ be a family of (rectifiable) curves in $\mathbb{D}$, and let ρ be a nonnegative measurable function on the disk $\mathbb{D}$ satisfying

$$l_\gamma(\rho) = \int_\gamma \rho \geq 1 \tag{A-1}$$

for all $\gamma \in \Gamma$ and

$$A(\rho) = \iint_{\mathbb{D}} \rho^2 dz d\bar{z} \neq 0, \infty. \tag{A-2}$$

Note that ρ can be thought of as the conformal factor for a metric conformally equivalent to the standard metric on the unit disk.

Definition A.1 The *extremal* length of Γ is denoted by $\lambda(\Gamma)$ is defined as

$$\lambda(\Gamma) = \sup_\rho A(\rho)^{-1}, \tag{A-3}$$

where ρ varies over all nonnegative measurable functions satisfying (A-1) and (A-2). This is a conformal invariant.

Definition A.2 A *quadrilateral* Q is the unit disk $\mathbb{D}$ together with two disjoint arcs A and B on its boundary. There is a conformal map from Q to a rectangle in $\mathbb{R}^2$ of height 1 and length m that takes the two arcs to vertical sides. The positive real number m is the *modulus* of the quadrilateral, denoted $\mathrm{mod}(Q)$.

One of the basic results asserts:

Lemma A.3 (Grötzsch) *If Γ is the collection of all rectifiable curves in $\mathbb{D}$ joining the boundary arcs A and B, then $\lambda(\Gamma) = \operatorname{mod}(Q)$.*

If $S \subset \mathbb{D}$ is a closed subset containing the boundary $\partial\mathbb{D}$, then we can restrict Γ in the lemma above to a collection Γ' of curves that are contained in S. We denote the corresponding extremal length $\lambda_S(Q) = \lambda(\Gamma')$. We shall also call this the *extremal length restricted to S*. Note that by the above lemma $\lambda_{\mathbb{D}}(Q) = \operatorname{mod}(Q)$.

From the definition of extremal length, it is easy to check:

Lemma A.4 *If $S \subseteq \mathbb{D}$ then $\lambda_S(Q) \geq \lambda_{\mathbb{D}}(Q)$.*

Definition A.5 Let Ω, Ω' be two domains in $\mathbb{C}$. Then a homeomorphism $f\colon \Omega \to \Omega'$ is said to be K–quasiconformal if:

(1) f has locally integrable distributional derivatives.

(2) We have the ratio

$$\frac{|f_z| - |f_{\bar{z}}|}{|f_z| + |f_{\bar{z}}|} \leq K \tag{A-4}$$

almost everywhere in Ω.

The *quasiconformal distortion* at a point in Ω is defined to be the value of the left-hand side of Equation (A-4).

Definition A.6 A homeomorphism $g\colon \mathbb{R} \to \mathbb{R}$ is M–quasisymmetric if for every $x, t \in \mathbb{R}$, we have

$$\frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M.$$

Definition A.7 A homeomorphism $g\colon \partial\mathbb{D} \to \partial\mathbb{D}$ is M–quasisymmetric if $h \circ g \circ h^{-1}\colon \mathbb{H}^2 \to \mathbb{H}^2$ is M–quasisymmetric when restricted to $\mathbb{R}$, where h is a conformal map from the unit disk $\mathbb{D}$ to the upper half-plane $\mathbb{H}^2$.

The following lemma can be culled from the discussion in [2, Section 4].

For A, B two intervals in $\mathbb{R}$, let $\lambda(A, B)$ denote the extremal length of the set of rectifiable paths in $\mathbb{H}^2$ from A to B. (To use the above definition of extremal length we first map $\mathbb{H}^2$ conformally to $\mathbb{D}$.)

Lemma A.8 *If $g\colon \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that*

$$\frac{1}{m} \leq \frac{\lambda(g(A), g(B))}{\lambda(A, B)} \leq m$$

for all disjoint intervals A, B.

Then g is M–quasisymmetric, where $M = e^{A(m-1)}$ where $A \approx 0.228$ is a universal constant.

The following are the fundamental results of Ahlfors and Beurling [2] (for the version stated here see Bishop [3]). Briefly, quasisymmetric maps of the boundary circle extend to quasiconformal maps of the unit disk, and vice versa, with the distortion constants (with K and M as above) being close to 1 if one of them is.

Lemma A.9 *For any $K > 1$ there is an $M > 1$ such that if $f\colon \mathbb{D} \to \mathbb{D}$ is a K–quasiconformal map then it extends to an M–quasisymmetric homeomorphism of the boundary. Moreover, there is a $K_0 > 1$ and $C_0 < \infty$ such that if $K = 1 + \epsilon < K_0$ then we can take $M \leq 1 + C_0\epsilon$.*

Lemma A.10 *Any M–quasisymmetric homeomorphism of $\partial\mathbb{D}$ can be extended to a K–quasiconformal map of $\mathbb{D}$. Moreover, there is a $M_1 > 1$ and $C_1 < \infty$ such that if $M = 1 + \epsilon < M_1$ then we can take $K \leq 1 + C_1\epsilon$.*

Finally, we note the following standard consequence of quasiconformality (for example, see [1, Chapter II]):

Lemma A.11 *Let $\Omega, \Omega' \subset \mathbb{D}$ be domains. If $f\colon \Omega \to \Omega'$ is a K–quasiconformal map, then for any collection Γ of rectifiable curves in Ω, we have*

$$\frac{1}{K} \leq \frac{\lambda_{\Omega'}(f(\Gamma))}{\lambda_{\Omega}(\Gamma)} \leq K. \tag{A-5}$$

A.2 An extremal length lemma

Let $A, B \subset \mathbb{D}$ be two disjoint boundary arcs, and Γ the collection of rectifiable paths from A to B and let $E = B_r$ be the ball of radius $r < 1$.

In this section we show (Lemma A.13) that the extremal length of Γ changes by a multiplicative factor of $1 + O(r)$ when E is excised, that is, when we restrict to the family of curves joining A and B and avoiding E. The constant in the $O(r)$ term is independent of the arcs A, B.

The following consequence of the Koëbe distortion theorem is used in its proof.

Lemma A.12 *Let $E = B_r \subset \mathbb{D}$ and let $\phi\colon \mathbb{D} \to \mathbb{C}$ be a conformal embedding such that $\phi(0) = 0$. Then*

$$\operatorname{diam}(\phi(E)) < Cr \operatorname{dist}(\phi(E), \partial\phi(\mathbb{D})) \tag{A-6}$$

for all r sufficiently small, for some universal constant C.

Proof Let $\delta = \operatorname{dist}(0, \partial\phi(\mathbb{D}))$. By a consequence of the Koëbe distortion theorem (see [30, Corollary 1.4]) we have

$$|\phi'(0)| \leq 4\delta \tag{A-7}$$

and by the other direction of the distortion theorem [30, Theorem 1.3] we have

$$|\phi(z)| \leq |\phi'(0)| \frac{|z|}{(1-|z|)^2}, \tag{A-8}$$

which using (A-7) gives

$$|\phi(z)| < 4\delta r/(1-r)^2 < 8r\delta \tag{A-9}$$

for any $z \in E$, since then $|z| \leq r$ and r is sufficiently small.

Hence

$$\operatorname{diam}(\phi(E)) < 16r\delta. \tag{A-10}$$

Now for $\omega \in E$, let $d_w = \operatorname{dist}(\phi(\omega), \partial\phi(\mathbb{D})) = |\phi(\omega) - \phi(s)|$ for some $\phi(s) \in \partial\phi(\mathbb{D})$.

We have

$$\delta \leq |\phi(s)| \leq d_w + |\phi(\omega)| \leq d_w + 8r\delta,$$

where the last inequality is by (A-9) and the second the triangle inequality.

Rearranging, and taking an infimum over $\omega \in E$ on the left-hand side, we obtain

$$\delta(1 - 8r) \leq \operatorname{dist}(\phi(E), \partial\phi(\mathbb{D})) \tag{A-11}$$

which implies, for r sufficiently small,

$$\delta \leq (1 + C'r) \operatorname{dist}(\phi(E), \partial\phi(\mathbb{D})) \tag{A-12}$$

for some constant C'. The proof is complete on combining (A-10) and (A-12). □

Lemma A.13 *Let $E = B_r \subset \mathbb{D}$ and Q the quadrilateral defined by the two boundary arcs A, B as above. Then*

$$1 \leq \frac{\lambda_{\mathbb{D}\setminus E}(Q)}{\lambda_{\mathbb{D}}(Q)} \leq 1 + C'r, \tag{A-13}$$

where C' is a constant independent of Q.

Proof The first inequality of (A-13) follows from Lemma A.4 so we need only to prove the other inequality.

Let ϕ be a conformal map from $\mathbb{D}$ to a rectangle R of vertical height 1 and horizontal length $m = \mathrm{mod}(Q)$ such that the arcs A and B are taken to the left and right vertical sides respectively. We further require that $\phi(0) = 0$. Such a map can be defined using elliptic integrals (see for example [1, Chapter III]).

It is well known (see Definition A.2 and Lemma A.3) that this conformal domain realizes the extremal length of Q: the conformal metric $\rho \equiv 1$ on the rectangle pulled back via ϕ realizes the supremum in Definition A.1.

Let Γ be the set of all rectifiable paths in R between the vertical sides, and let Γ' be the subcollection of Γ of paths disjoint from $\phi(E)$.

We shall adapt the Grötzsch argument to show that ρ is close to being extremal for the collection Γ'.

Let S be a strip $S = [0, m] \times J$ of vertical (y–) height $|J| = \mathrm{diam}(\phi(E))$ and horizontal (x–) range m that contains $\phi(E)$. By Lemma A.12, we know that

$$\mathrm{diam}(\phi(E)) < Cr \tag{A-14}$$

for r small, since $\phi(E) \subset R$ implies $\mathrm{dist}(\phi(E), \partial\phi(\mathbb{D})) \le \min\{1, m\} \le 1$.

Let ρ' be a conformal factor for R that satisfies

$$l_\gamma(\rho') \ge 1$$

for all $\gamma \in \Gamma'$.

In particular,

$$\int_0^m \rho'(x, y)\, dx \ge 1 \tag{A-15}$$

for any y in $[0, 1] \setminus J$.

By integrating (A-15) over y ranging over $[0, 1] \setminus J$, we get

$$1 - Cr \le \int_{[0,1]\setminus J} 1 dy \le \int_{[0,1]\setminus J} \int_0^m \rho'(x, y)\, dx dy = \iint_{R\setminus S} \rho'(x, y)\, dx dy.$$

Squaring, and using the Cauchy–Schwarz inequality for the right-hand term, we get

$$(1 - Cr)^2 \le \iint_{R\setminus S} (\rho')^2\, dx dy \iint_{R\setminus S} 1^2\, dx dy \le \left(\iint_R (\rho')^2\, dx dy\right) m.$$

So

$$\left(\iint_R (\rho')^2\,dx\,dy\right)^{-1} \le m/(1-Cr)^2 \le m(1+O(r)) \tag{A-16}$$

for r sufficiently small.

Taking a supremum over ρ' as in Definition A.1 we get

$$\lambda_{\mathbb{D}\setminus E}(Q) = \lambda_{R\setminus\phi(E)}(Q) = \lambda(\Gamma') \le m(1+O(r)),$$

where the first equality holds since extremal length is a conformal invariant. Since $m = \lambda_{\mathbb{D}}(Q)$, this is the right hand equality of (A-13) and the proof is complete. $\square$

A.3 Proof of Lemma 4.24

Henceforth let $f\colon \mathbb{D} \to \mathbb{D}$ be the quasiconformal map with quasiconformal distortion $1 + C\epsilon$ off a small ball B_r, as in Lemma 4.24 (see Figure 20).

We need the following "quasiconformal" version of Lemma A.12 for maps of the unit disk:

Lemma A.14 *We have* $\operatorname{diam}(f(B_r)) = O(r^{1-\epsilon})$.

Proof Let $d = \operatorname{diam}(f(B_r))$. For convenience we shall assume $C = 1$.

It is well known (eg see [1, III.A]) that for an annular domain on the plane that contains $0, 1$ in the bounded component of its complement, and the interval $[c, \infty)$ in the unbounded component where $c > 1$, we have

$$\lambda(P) < \frac{1}{2\pi} \ln 16c, \tag{A-17}$$

where P is the set of rectifiable curves connecting the inner boundary component to the outer boundary component. (This extremal length $\lambda(P)$ is the definition of the modulus of this annular region.)

Since f is a homeomorphism, if $A = \mathbb{D} \setminus B_r$ and Γ the set of paths between the boundary components of A then $f(A)$ is topologically an annulus, and by rotation and scaling we see that we see that it satisfies the above condition with $c = 1/d$. So we have

$$\lambda(f(\Gamma)) < \frac{1}{2\pi} \ln \frac{16}{d}. \tag{A-18}$$

By Lemma A.11 and the fact that f is $(1+\epsilon)$–quasiconformal on $\mathbb{D} \setminus B_r$ we know that

$$(1-\epsilon)\lambda(\Gamma) \le \frac{1}{1+\epsilon}\lambda(\Gamma) \le \lambda(f(\Gamma)) \tag{A-19}$$

but since

$$\lambda(\Gamma) = \frac{1}{2\pi} \ln \frac{1}{r}$$

we obtain from (A-18) and (A-19) that

$$d \le 16 r^{1-\epsilon}. \qquad \square$$

Corollary A.15 *If* $r \le \epsilon$ *then* $\operatorname{diam}(f(B_r)) = O(\epsilon)$.

Proof The maximum of x^{-x} is $e^{1/e} \approx 1.44$. So $r^{1-\epsilon} \le \epsilon^{1-\epsilon} < 1.45\epsilon$. $\square$

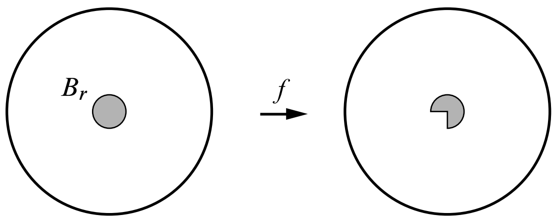

Figure 20: The image of B_r is of small diameter. By Lemma A.13, the extremal lengths change by a small factor when restricted to the complement of the shaded regions.

Proof of Lemma 4.24 By Lemma A.8 is enough to show that for any pair of disjoint arcs A and B on ∂D, we have

$$1 - O(\epsilon) \le \frac{\lambda_{\mathbb{D}}(f(Q))}{\lambda_{\mathbb{D}}(Q)} \le 1 + O(\epsilon), \tag{A-20}$$

where Q is the corresponding quadrilateral, and the constant in $O(\epsilon)$ is independent of Q.

This is because in the notation of Lemma A.8, if $m = 1 + O(\epsilon)$ then $M = e^{A(m-1)} = 1 + O(\epsilon)$.

Let $r \le \epsilon$. By Lemma A.13 we have

$$1 \le \frac{\lambda_{\mathbb{D}\setminus B_r}(Q)}{\lambda_{\mathbb{D}}(Q)} \le 1 + O(\epsilon)$$

and by Corollary A.15 and Lemma A.13 we have

$$1 \le \frac{\lambda_{\mathbb{D}\setminus f(B_r)}(f(Q))}{\lambda_{\mathbb{D}}(f(Q))} \le 1 + O(\epsilon).$$

Now by Lemma A.11, since f is almost conformal on $\mathbb{D} \setminus B_r$, we have

$$1 - O(\epsilon) \le \frac{\lambda_{\mathbb{D}\setminus f(B_r)}(f(Q))}{\lambda_{\mathbb{D}\setminus B_r}(Q)} \le 1 + O(\epsilon).$$

The required (A-20) follows easily from the above three inequalities. $\square$

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